# REPORT

Zajęcia: Analog and digital electronic circuits

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**Lab 10**

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**Topic: "Design of Non-Recursive Filters using the Window Method"**

Variant: 13

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1 semestr,

Gr.1a

## 1. Abstract

The objective is to design non-recursive filters using window technique.

## 2. Theoretical introduction

### 2.1. Causal Filters

Let's assume that the desired frequency characteristic of the discrete filter is given by its continuous frequency response Hd($e^{j\Omega}$) in the discrete-time Fourier domain. Its impulse response is given by inverse discrete-time Fourier transform (inverse DTFT) of the frequency response:

$$h_\mathrm{d}[k] = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} H_\mathrm{d}(\mathrm{e}^{\,\mathrm{j}\,\Omega})\, \mathrm{e}^{\,\mathrm{j}\,\Omega\, k}\; \mathrm{d}\Omega$$

In the general case, $h_d[k]$ will not be a causal FIR. The Paley-Wiener theorem, were the transfer function is zeros over an interval of frequencies. The basic idea of the window method is to truncate the impulse response $h_d[k]$ in order to derive a causal FIR filter. This can be achieved by applying a window w[k] of finite length N to $h_d[k]$

$$h[k] = h_\mathrm{d}[k] \cdot w[k]$$

where h[k] denotes the impulse response of the designed filter and w[k] = 0 for $k < 0 \wedge k \geq N$. Its frequency response H($e^{j\Omega}$) is given by the multiplication theorem of the discrete-time Fourier transform (DTFT)

$$H(\mathrm{e}^{\,\mathrm{j}\,\Omega}) = \frac{1}{2\pi}\, H_\mathrm{d}(\mathrm{e}^{\,\mathrm{j}\,\Omega}) \circledast W(\mathrm{e}^{\,\mathrm{j}\,\Omega})$$

where W($e^{j\Omega}$) denotes the DTFT of the window function w[k]. The frequency response H($e^{j\Omega}$) of the filter is given as the periodic convolution of the desired frequency response $H_d$($e^{j\Omega}$) and the frequency response of the window function W($e^{j\Omega}$). The frequency response H($e^{j\Omega}$) is equal to the desired frequency response $H_d$($e^{j\Omega}$) only if W($e^{j\Omega}$) = $2\pi \cdot \delta(\Omega)$. This would require that w[k] = 1 for $k = -\infty, \ldots, \infty$. Hence for a window w[k] of finite length, deviations from the desired frequency response are to be expected. In order to investigate the effect of truncation on the frequency response H($e^{j\Omega}$), a particular window is considered. A straightforward choice is the rectangular window w[k] = rectN [k] of length N . Its DTFT is given as

$$W(\mathrm{e}^{\,\mathrm{j}\,\Omega}) = \mathrm{e}^{-\mathrm{j}\,\Omega\,\frac{N-1}{2}} \cdot \frac{\sin(\frac{N\,\Omega}{2})}{\sin(\frac{\Omega}{2})}$$

The frequency-domain properties of the rectangular window have already been discussed for the leakage effect. The rectangular window features a narrow main lobe at the cost of relative high sidelobe level. The main lobe gets narrower with increasing length N . The convolution of the desired frequency response with the frequency response of the window function effectively results in smoothing and ringing. While the main lobe will smooth discontinuities of the desired transfer function, the sidelobes result in undesirable ringing effects. The latter can be alleviated by using other window functions. Note that typical window functions decay towards their ends and are symmetric with respect to their center. This may cause problems for desired impulse responses with large magnitudes towards their ends.

## 2.2. Example - Causal approximation of ideal low-pass

The design of an ideal low-pass filter using the window method is illustrated in the following. For $|\Omega| < \pi$ the transfer function of the ideal low-pass is given as

$$H_\mathrm{d}(\mathrm{e}^{\,\mathrm{j}\,\Omega}) = \begin{cases} 1 & \text{for } |\Omega| \leq \Omega_\mathrm{c} \\ 0 & \text{otherwise} \end{cases}$$

where $\Omega_c$ denotes the cut frequency of the low-pass. An inverse DTFT of the desired transfer function yields

$$h_\mathrm{d}[k] = \frac{\Omega_\mathrm{c}}{\pi} \cdot \mathrm{sinc}(\Omega_\mathrm{c}\, k)$$

The impulse response $h_d[k]$ is not causal nor FIR. In order to derive a causal FIR approximation, a rectangular window w[k] of length N is applied

$$h[k] = h_\mathrm{d}[k] \cdot \mathrm{rect}_N[k]$$

The resulting magnitude and phase response is computed numerically in the following.

## 2.3. Zero-Phase Filters

Let's assume a general zero-phase filter with transfer function $H_d(e^{j\Omega}) = A(e^{j\Omega})$ with magnitude $A(e^{j\Omega}) \in R$. Due to the symmetry relations of the DTFT, its impulse response $h_d[k] = F^{-1} * \{Hd(ej\Omega\}$ is conjugate complex symmetric

$$h_d[k] = h_d^*[-k]$$

A zero-phase filter of length N > 1 is not causal as a consequence. The anti-causal part could simply be removed by windowing with a Heaviside signal. However, this will result in large deviations between the desired transfer function and the designed filter. This explains the findings from the previous example, that an ideal-low pass cannot be realized very well by the window method. The reason is that an ideal-low pass has zero-phase, as most of the idealized filters.

The impulse response of a stable system, in the sense of the bounded- input/bounded-output (BIBO) criterion, has to be absolutely summable. Which in general is given when its magnitude decays by tendency with increasing time-index k. This observation motivates to shift the desired impulse response to the center of the window in order to limit the effect of windowing. This can be achieved by replacing the zero-phase with a linear-phase, as is illustrated below.

## 2.4. Causal Linear-Phase Filters

The design of a non-recursive causal FIR filter with a linear phase is often desired due to its constant group delay. Let's assume a filter with generalized Slinear phase. For $|\Omega| < \pi$ its transfer function is given as

$$H_\mathrm{d}(\mathrm{e}^{\mathrm{j}\,\Omega}) = A(\mathrm{e}^{\mathrm{j}\,\Omega}) \cdot \mathrm{e}^{-\mathrm{j}\alpha\Omega+\mathrm{j}\beta}$$

where A ($e^{j\Omega}$) ∈ R denotes the amplitude of the filter, α the linear slope of the phase and β a constant phase offset. Such a system can be decomposed into two cascaded systems: a zero-phase system with transfer function A($e^{j\Omega}$) and an all-pass with phase $\phi(\Omega) = -\alpha\Omega + \beta$. The linear phase term $-\alpha\Omega$ results in the constant group delay $tg(\Omega) = \alpha$.

The impulse response h[k] of a linear-phase system shows a specific symmetry which can be deduced from the symmetry relations of the DTFT for odd/even symmetry of Hd($e^{j\Omega}$) as

$$h[k] = \pm h[N-1-k]$$

for k = 0, 1, . . . , N − 1 where N ∈ N denotes the length of the (finite) impulse response. The transfer function of a linear phase filter is given by its DTFT

$$H_\mathrm{d}(\mathrm{e}^{\mathrm{j}\,\Omega}) = \sum_{k=0}^{N-1} h[k]\,\mathrm{e}^{-\mathrm{j}\,\Omega\,k}$$

These relations have to be considered in the design of a causal linear phase filter. Depending on the desired magnitude characteristics A($e^{j\Omega}$) the suitable type is chosen. The odd/even length N of the filter and the phase (or group delay) is chosen accordingly for the design of the filter.

## 2.5. Example - Causal linear-phase approximation of ideal low-pass

We aim at the design of a causal linear-phase low-pass using the window technique. According to the previous example, the desired frequency response has an even symmetry A($e^{j\Omega}$) = A $e^{-j\Omega}$) with A($e^{j0}$) = 1. This could be realized by a filter of type 1 or 2. We choose type

1 with β = 0, since the resulting filter exhibits an integer group delay of $tg(\Omega) = \frac{N-1}{2}$ samples. Consequently the length of the filter N has to be odd.

The impulse response $h_d[k]$ is given by the inverse DTFT of $Hd(e^{j\Omega})$ incorporating the linear phase:

$$h_\mathrm{d}[k] = \frac{\Omega_\mathrm{c}}{\pi} \cdot \operatorname{sinc}\left(\Omega_\mathrm{c}\left(k - \frac{N-1}{2}\right)\right)$$

The impulse response fulfills the desired symmetry for k = 0, 1, . . . , N −1. A causal FIR approximation is obtained by applying a window function of length N to the impulse response hd[k]

$$h[k] = h_\mathrm{d}[k] \cdot w[k]$$

Note that the window function w[k] also has to fulfill the desired symmetries.

As already outlined, the chosen window determines the properties of the transfer function $H(e^{j\Omega})$. The spectral properties of commonly applied windows have been discussed previously. The width of the main lobe will generally influence the smoothing of the desired transfer function $H_d(e^{j\Omega})$, while the sidelobes influence the typical ringing artifacts.

## 3. Input data (Variant)

This report was created with base of variant 13:

| N | $\Omega_c$ |
|---|---|
| 80 | π/14 |

GitHub repository:

https://github.com/Delisolara/AaDEC

## 4. Course of actions

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as sig
%matplotlib inline
```

*Picture 1. Uploaded libraries*

```python
N = 80  # length of filter
Omc = np.pi/14

# compute impulse response
k = np.arange(N)
hd = Omc/np.pi * np.sinc(k*Omc/np.pi)

# windowing
w = np.ones(N)
h = hd * w

# frequency response
Om, H = sig.freqz(h)
```

*Picture 2. Implemented code*

```python
# plot impulse response
plt.figure(figsize=(10, 3))
plt.stem(h )
plt.title('Impulse response')
plt.xlabel(r'$k$')
plt.ylabel(r'$h[k]$')
```

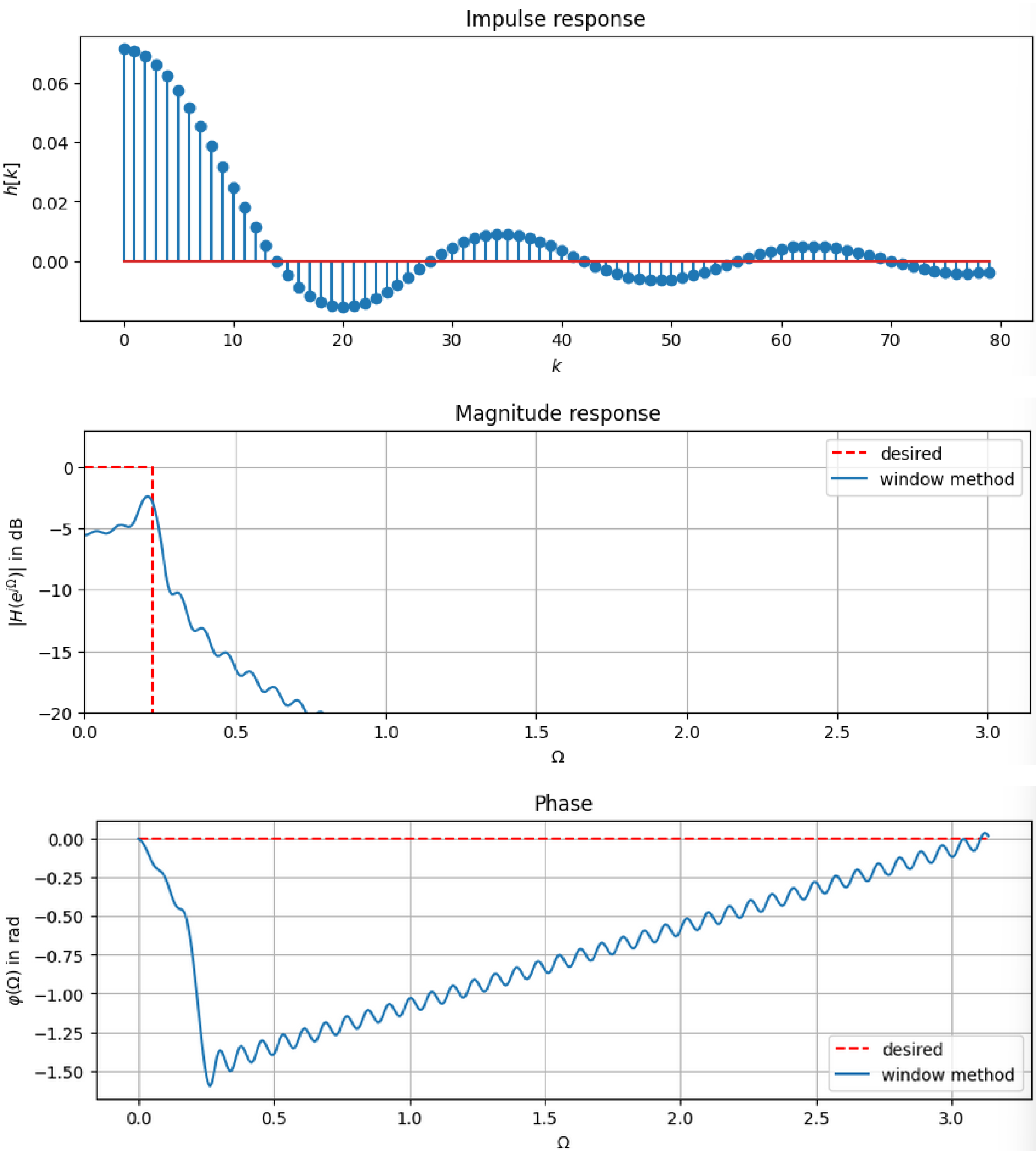
*Picture 3. Implemented code*

```python
# plot magnitude responses
plt.figure(figsize=(10, 3))
plt.plot([0, Omc, Omc], [0, 0, -100], 'r--', label='desired')
plt.plot(Om, 20 * np.log10(abs(H)), label='window method')
plt.title('Magnitude response')
plt.xlabel(r'$\Omega$')
plt.ylabel(r'$|H(e^{j \Omega})|$ in dB')
plt.axis([0, np.pi, -20, 3])
plt.grid()
plt.legend()
```

*Picture 4. Implemented code*

```python
# plot phase responses
plt.figure(figsize=(10, 3))
plt.plot([0, Om[-1]], [0, 0], 'r--', label='desired')
plt.plot(Om, np.unwrap(np.angle(H)), label='window method')
plt.title('Phase')
plt.xlabel(r'$\Omega$')
plt.ylabel(r'$\varphi (\Omega)$ in rad')
plt.grid()
plt.legend()
```

*Picture 5. Implemented code*




*Picture 6. The result*

## 5. Conclusions

In this lab, we explored the design of an ideal low-pass filter using the window method. The objective was to create a filter that meets specific frequency response characteristics and then evaluate its performance.