REPORT

Zajęcia: Analog and digital electronic circuits

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Lab 9

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Topic: "IIR filtering "

Variant: 13

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1 semestr,

Gr.1a

1. Abstract

The objective was to investigate IIR filtering technique for different parameters of filtering.

2. Theoretical introduction

The transfer function of digital filters can be generally expressed in the z-domain as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}} = \frac{b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_M z^{-M}}{a_0 z^0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}}$$

with input X(z) and output Y (z). Real input signals x[k] that should end up as real output signals y[k] (in terms of signal processing fundamentals this is a special case, though most often needed in practice) require real coefficients b, a ∈ R. This is only achieved with

- single or multiple real valued
- single or multiple complex conjugate pairs of zeros and poles.

Furthermore, in practice we most often aim at (i) causal and (ii) bounded input, bound output (BIBO) stable LTI systems, which requires (i) M ≤ N and (ii) poles inside the unit circle. If all poles *"and"* zeros are *"inside"* the unit circle then the system is *"minimum-phase"* and thus H(z) is straightforwardly *"invertible"*.

Further concepts related to the transfer function are:

Analysis of the transfer characteristics is done by the DTFT H(z = ejΩ), i.e. evaluation on the unit circle.

We use a0 = 1 according to convention in many textbooks.

The convention for arraying filter coefficients is straightforward with

Python index starting at zero: b_0 = b[0], b_1 = b[1], b_2 = b[2], ..., a_0 = a[0] = 1, a_1 = a[1], a_2 = a[2].

2.1. Filtering Process

A non-recursive system with a_1, a_2, ..., a_N = 0 always exhibits a finite impulse response (FIR), note: a_0 = 1 for output though. Due to the finite length impulse response, a non-recursive system is always stable.

The output signal of a non-recursive system in practice can be calculated by linear convolution:

$$y[k] = \sum_{m=0}^{M} h[m]x[-m+k]$$

of the finite impulse response h[m] = [b_0, b_1, b_2, ..., b_M] and the input signal x[k].

A recursive system exhibits at least one $a_n \geq 1 \neq 0$. Because of the feedback of the output into the system, a potentially infinite impulse response (IIR) and a potentially non-stable system results.

For a recursive system, in practice the difference equation needs to be implemented:

$$y[k] = b_0x[k]+b_1x[k-1]+b_2x[k-2]+...+b_Mx[k-M]-a_1y[k-1]-a_2y[k-2]-a_3y[k-3]-...-a_Ny[k-N]$$

A pure non-recursive system is obtained by ignoring the feedback paths, i.e. setting a_1, a_2, ..., a_N = 0. - A pure recursive system is obtained by ignoring the forward paths, i.e. setting b_0, b_1, b_2, ..., b_M = 0. Then, the values of the state variables z^{-1}, z^{-2}, ..., z^{-M} alone determine how the system starts to perform at k = 0, since the system has no input actually. This system type can be used to generate (damped) oscillations.

A recursive system can have a finite impulse response, but this is very rarely the case. Therefore, literature usually refers to

- an FIR filter when dealing with a non-recursive system
- an IIR filter when dealing with a recursive system

2.2. Signal Flow Chart of Direct Form I

For example, the signal flow for a second order (M = N = 2), system with

- a non-recursive part (feedforward paths, left z−-path)
- a recursive part (feedback paths, left z−-path)

is depicted below (graph taken from Wikimedia Commons) as straightforward direct form I, i.e. directly following the difference equation.

2.3. IIR Filter

In the following 2nd order IIR filters shall be discussed. The transfer function is with usual convention a_0 = 1

$$H(z) = \frac{\sum_{m=0}^{2} b_m z^{-m}}{\sum_{n=0}^{2} a_n z^{-n}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

Very simple pole / zero placements are given in order to demonstrate the principle and the impact of zeros and poles. Once this is understood, more complicated filter characteristics and filter design methods can be approached.

3. Input data (Variant)

This report was created with base of variant 13:

M	b_0	b_1	b_2
2	0	1	1

GitHub repository:

https://github.com/Delisolara/AaDEC

4. Course of actions

```
import numpy as np
import matplotlib.pyplot as plt
from matplotlib.markers import MarkerStyle
from matplotlib.patches import Circle
from scipy import signal

np.set_printoptions(precision=16)
```

Picture 1. Uploaded libraries

```
def zplane_plot(ax, z, p, k):
    # draw unit circle
    Nf = 2**7
    Om = np.arange(Nf) * 2*np.pi/Nf
    plt.plot(np.cos(Om), np.sin(Om), 'C7')

    try:  # TBD: check if this pole is compensated by a zero
        circle = Circle((0, 0), radius=np.max(np.abs(p)),
                        color='C7', alpha=0.15)
        plt.gcf().gca().add_artist(circle)
    except ValueError:
        print('no pole at all, ROC is whole z-plane')

    zu, zc = np.unique(z, return_counts=True)  # find and count unique zeros
    for zui, zci in zip(zu, zc):  # plot them individually
        plt.plot(np.real(zui), np.imag(zui), ms=8,
                 color='C0', marker='o', fillstyle='none')
        if zci > 1:  # if multiple zeros exist then indicate the count
            plt.text(np.real(zui), np.imag(zui), zci)

    pu, pc = np.unique(p, return_counts=True)  # find and count unique poles
    for pui, pci in zip(pu, pc):  # plot them individually
        plt.plot(np.real(pui), np.imag(pui), ms=8,
                 color='C0', marker='x')
        if pci > 1:  # if multiple poles exist then indicate the count
            plt.text(np.real(pui), np.imag(pui), pci)
```

Picture 2. Implemented code

```
    plt.text(0, +1, 'k={0:f}'.format(k))
    plt.text(0, -1, 'ROC for causal: white')
    plt.axis('square')
    plt.xlabel(r'$\Re\{z\}$')
    plt.ylabel(r'$\Im\{z\}$')
    plt.grid(True, which="both", axis="both",
             linestyle="-", linewidth=0.5, color='C7')
```

Picture 3. Implemented code

```
def bode_plot(b, a, N=2**10, fig=None):  # for IIR if length of b and a are the same
    if fig is None:
        fig = plt.figure()

    z, p, gain = signal.tf2zpk(b, a)
    W, Hd = signal.freqz(b, a, N, whole=True)

    # print('number of poles:', len(p), '\npole(s) at:', p,
    #       '\nnumber of zeros:', len(z), '\nzero(s) at:', z)

    gs = fig.add_gridspec(2, 2)
    # magnitude
    ax1 = fig.add_subplot(gs[0, 0])
    ax1.plot(W/np.pi, np.abs(Hd), "C0",
             label=r'$|H(\Omega)|$)',
             linewidth=2)
    ax1.set_xlim(0, 2)
    ax1.set_xticks(np.arange(0, 9)/4)
    ax1.set_xlabel(r'$\Omega \,/\, \pi$', color='k')
    ax1.set_ylabel(r'$|H|$', color='k')
    ax1.set_title("Magnitude response", color='k')
    ax1.grid(True, which="both", axis="both",
             linestyle="-", linewidth=0.5, color='C7')
```

Picture 4. Implemented code

```
    # phase
    ax2 = fig.add_subplot(gs[1, 0])
    ax2.plot(W/np.pi, (np.angle(Hd)*180/np.pi), "C0",
             label=r'$\mathrm{angle}(H('r'\omega))$',
             linewidth=2)
    ax2.set_xlim(0, 2)
    ax2.set_xticks(np.arange(0, 9)/4)
    ax2.set_xlabel(r'$\Omega \,/\, \pi$', color='k')
    ax2.set_ylabel(r'$\angle(H)$ / deg', color='k')
    ax2.set_title("Phase response", color='k')
    ax2.grid(True, which="both", axis="both",
             linestyle="-", linewidth=0.5, color='C7')

    # zplane
    ax3 = fig.add_subplot(gs[0, 1])
    zplane_plot(ax3, z, p, gain)

    # impulse response
    N = 2**4  # here specially chosen for the examples below
    k = np.arange(0, N)
    x = np.zeros(N)
    x[0] = 1
    h = signal.lfilter(b, a, x)
    ax4 = fig.add_subplot(gs[1, 1])
    ax4.stem(k, h, linefmt='C0', markerfmt='C0o',
             basefmt='C0:')
    ax4.set_xlabel(r'$k$')
    ax4.set_ylabel(r'$h[k]$')
    ax4.set_title('Impulse Response')
    ax4.grid(True, which="both", axis="both", linestyle="-",
             linewidth=0.5, color='C7')

# some defaults for the upcoming code:
figsize = (12, 9)
```

Picture 5. Implemented code

```
# taken from lecture's repository
def bilinear_biquad(B, A, fs):
    A0, A1, A2 = A
    B0, B1, B2 = B
    fs2 = fs**2

    a0 = A2 + 2*A1*fs + 4*A0*fs2
    b0 = B2 + 2*B1*fs + 4*B0*fs2

    b1 = 2*B2 - 8*B0*fs2
    a1 = 2*A2 - 8*A0*fs2

    b2 = B2 - 2*B1*fs + 4*B0*fs2
    a2 = A2 - 2*A1*fs + 4*A0*fs2

    b = np.array([b0, b1, b2]) / a0
    a = np.array([a0, a1, a2]) / a0

    return b, a

def f_prewarping(f, fs):
    return 2*fs*np.tan(np.pi*f/fs)
```

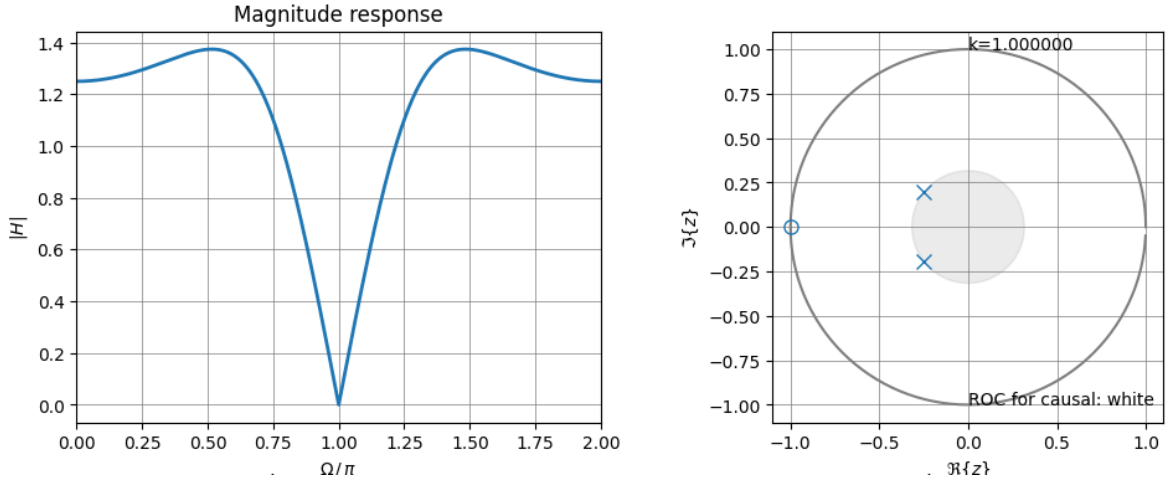
Picture 6. Implemented code

```
def pz_placement(zr, za, pr, pa):
    z = zr * np.exp(+1j*za)
    p = pr * np.exp(+1j*pa)

    b = [0,1,1]
    a = [1,0.5,0.1]

    bode_plot(b, a, fig=plt.figure(figsize=figsize))
```

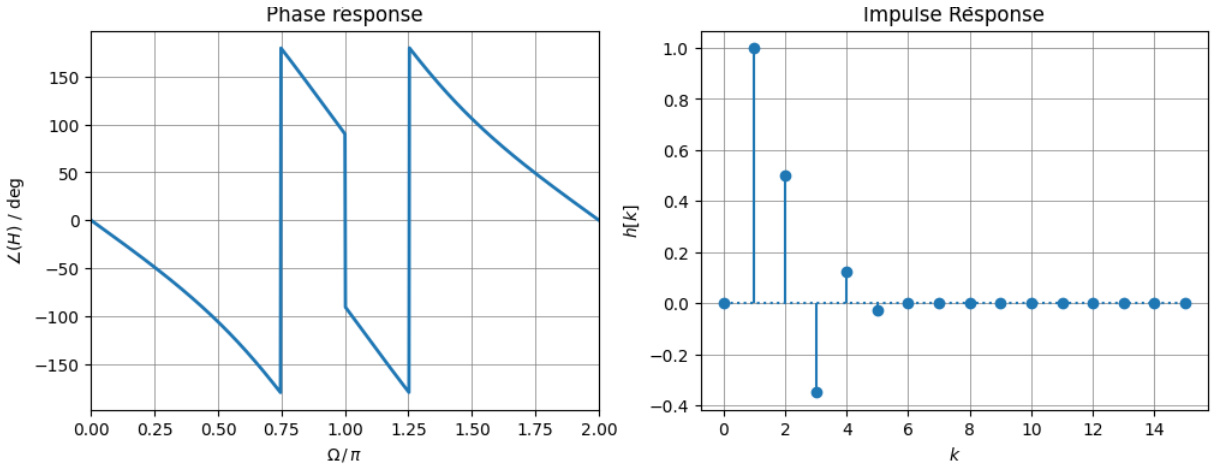
Picture 7. Implemented code

```
# no IIR actually, but rather FIR just to make a point:
# put poles into origin
pz_placement(zr=1/2, za=np.pi/2, pr=0, pa=0)
```

Picture 8. Implemented code



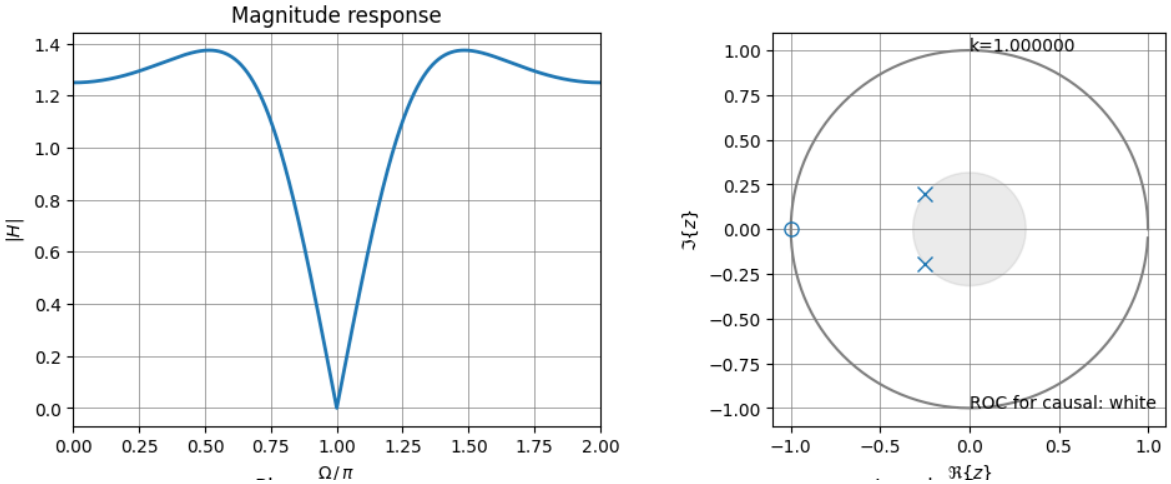
Picture 9. The result



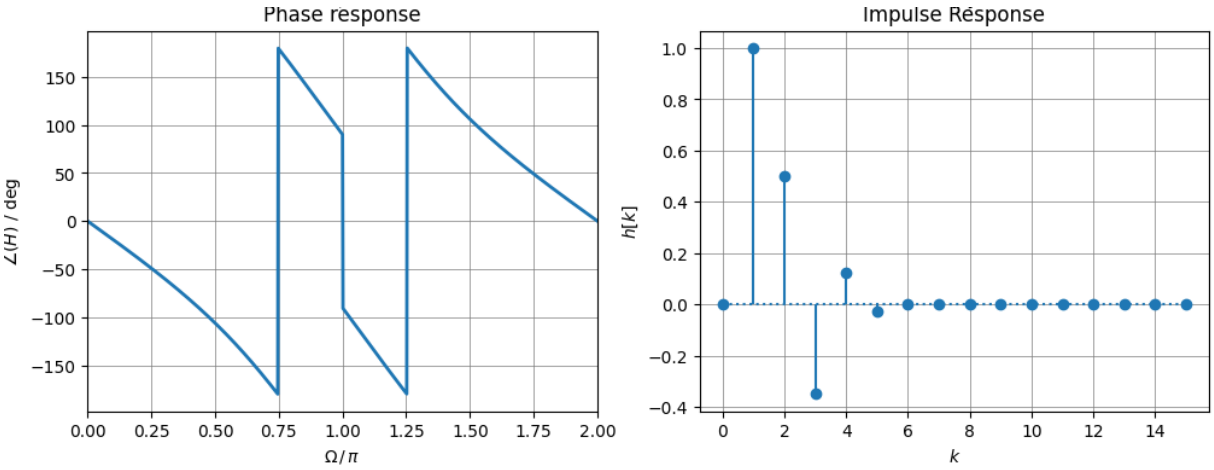
Picture 10. The result

```
# filter transfer function from above can be inverted
# this yields a stable IIR filter, since its poles are inside unit circle
# and thus white ROC includes the unit cirlce
pz_placement(zr=0, za=0, pr=1/2, pa=np.pi/2)
```

Picture 11. Implemented code



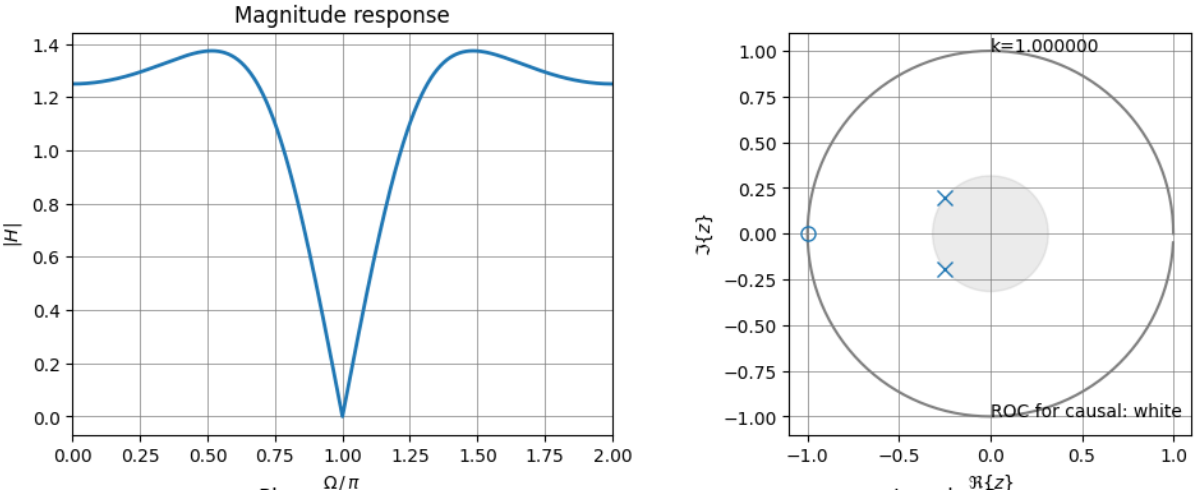
Picture 12. The result



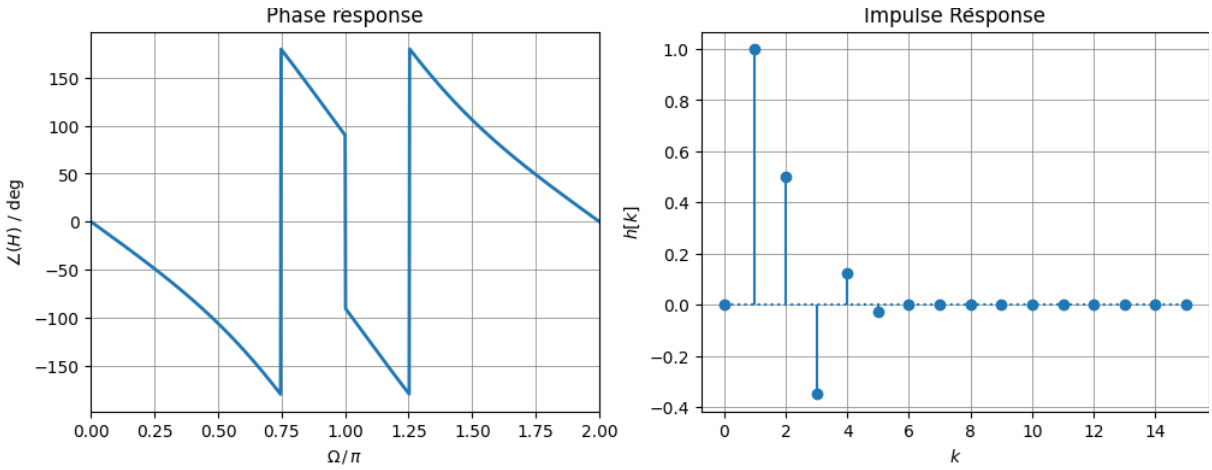
Picture 13. The result

```
# shift zeros closer to poles
# less ripple in magnitude response
# note that y-axis has changed in comparison to above example
pz_placement(zr=1/3, za=np.pi/2, pr=1/2, pa=np.pi/2)
```

Picture 14. Implemented code



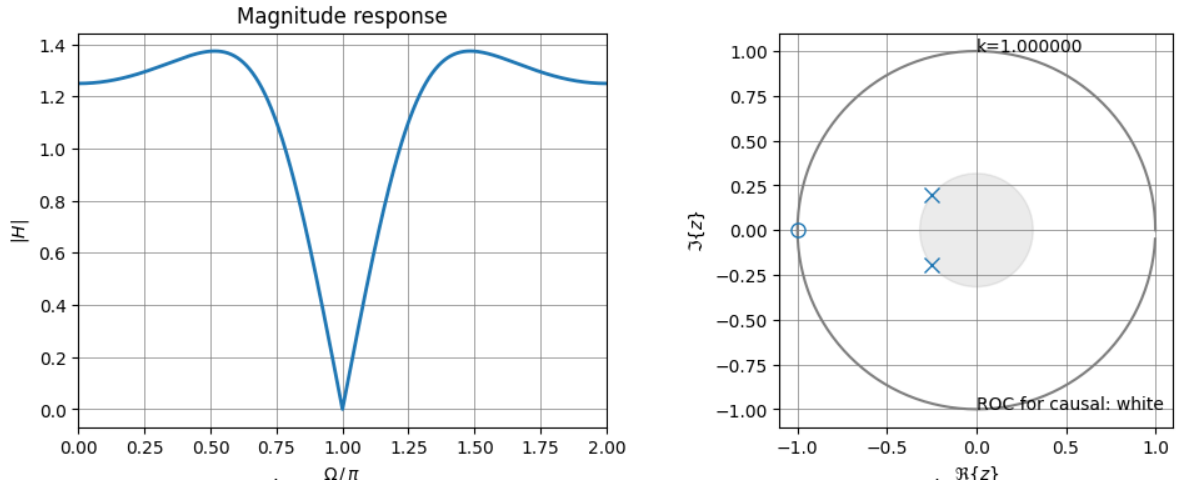
Picture 15. The result



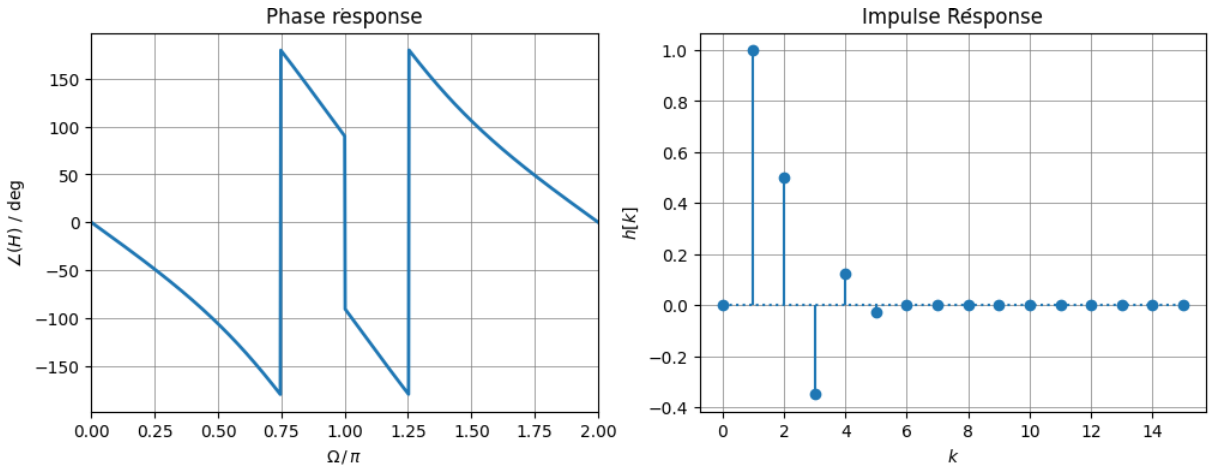
Picture 16. The result

```
# 2nd order lowpass filter
# special here is:
# zero at -1 thus amplitude 0 at fs/2, phase -180 deg at fs/2
# two real poles at same location
pz_placement(zr=1, za=np.pi, pr=1/2, pa=0)
```

Picture 17. Implemented code



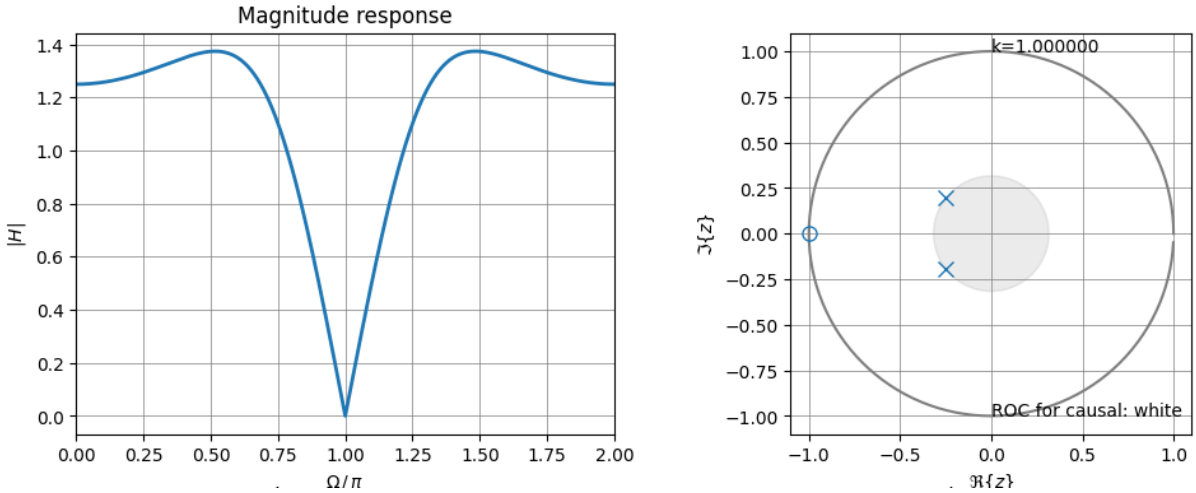
Picture 18. The result



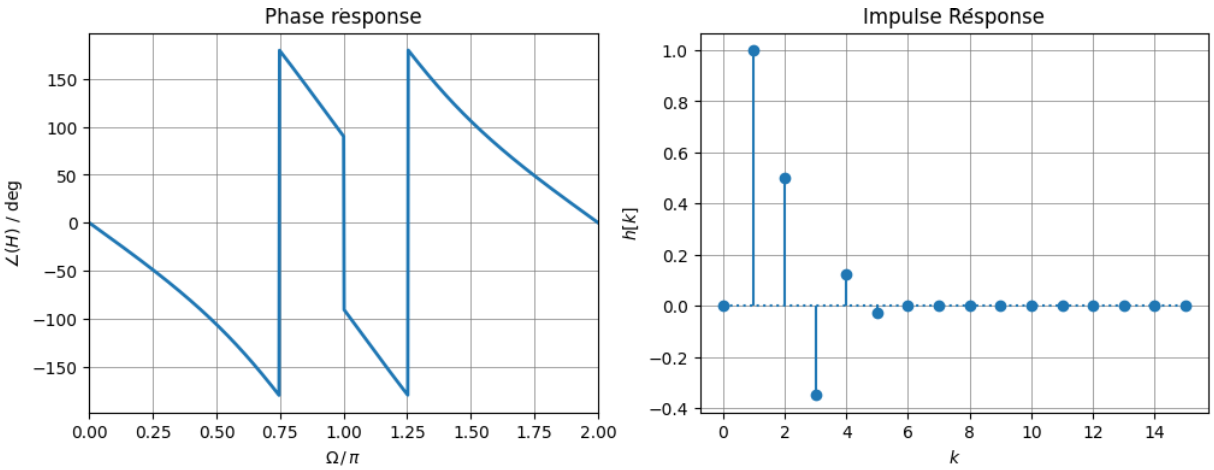
Picture 19. The result

```
# 2nd order lowpass filter
# zero at -1 thus amplitude 0 at fs/2
# this time complex conjugate pole pair to yield about the same magnitude
# at DC
# check the differences between the two filters
# you might create an own plot where both filters can be overlayed
pz_placement(zr=1, za=np.pi, pr=3/5, pa=np.pi/8)
```

Picture 20. Implemented code



Picture 21. The result



Picture 22. The result

5. Conclusions

In this lab, we conducted an in-depth analysis of Infinite Impulse Response (IIR) filters by examining various parameters provided in the table. We generated and plotted the magnitude response, poles, phase response, and impulse response for these IIR filters.