REPORT

Zajęcia: Windowing

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Lab 3

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Topic: "Random signals"

Variant: 13

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Informatyka II stopień,

stacjonarne,

2 semestr,

Gr.1a

1. Abstract

The objective is to get an idea on:

- the probability density function
- what is a sample function of a random process
- first / second order ensemble averages (moments)
- the concept of stationarity and ergodicity
- the concept of temporal average vs. ensemble average
- the autocorrelation / cross correlation as as a higher order ensemble and temporal averages

2. Theoretical introduction

2.1. First-Order Ensemble Averages

For a probability density function (PDF) $p_x(\theta,k)$ which describes a random process of "drawing" signal amplitudes θ for the *n-th* sample function $x_n[k]$ over time k we can define the following expectation:

$$E\{f(x[k])\} = \int_{-\infty}^{\infty} f(\theta)\, p_x(\theta, k)\, \mathrm{d}\theta$$

$$E\{f(x[k])\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n[k])$$

using the operator or mapping function $f(\cdot)$.

Most important are the following first-order ensemble averages, also called univariate moments, named so, since one random process is involved.

2.1.1. Linear mean / 1st raw moment

for mapping function $f(\theta) = \theta^1$:

$$\mu_x[k] = E\{x[k]\} = \int_{-\infty}^{\infty} \theta\, p_x(\theta, k)\, \mathrm{d}\theta$$

$$\mu_x[k] = E\{x[k]\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n[k]$$

2.1.2. Quadratic mean / 2nd raw moment

for mapping function $f(\theta) = \theta^2$:

$$E\{x^2[k]\} = \int_{-\infty}^{\infty} \theta^2 \, p_x(\theta, k) \, \mathrm{d}\theta$$

$$E\{x^2[k]\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} x_n^2[k]$$

2.1.3. Variance / 2nd centralized moment

for mapping function $f(\theta) = (\theta - \mu_x[k])^2$:

$$\sigma_x^2[k] = E\{(x[k] - \mu_x[k])^2\} = \int_{-\infty}^{\infty} (\theta - \mu_x[k])^2 \, p_x(\theta, k) \, \mathrm{d}\theta$$

$$\sigma_x^2[k] = E\{(x[k] - \mu_x[k])^2\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (x_n[k] - \mu_x[k])^2$$

These three moments are generally linked as:

$$E\{x^2[k]\} = \mu_x^2[k] + \sigma_x^2[k],$$

which reads quadratic mean is linear mean plus variance.

For stationary processes these ensemble averages are not longer time-dependent, but rather $\mu_x[k] = \mu_x = const$, etc. holds. This implies that the PDF describing the random process is not changing over time.

2.2. Second-Order Ensemble Averages

The second-order ensemble averages, also called bivariate moments (because two random processes are involved) can be derived from:

$$E\{f(x[k_x], y[k_y])\} = \iint_{-\infty}^{\infty} f(\theta_x, \theta_y) \, p_{xy}(\theta_x, \theta_y, k_x, k_y) \, \mathrm{d}\theta_x \, \mathrm{d}\theta_y$$

$$E\{f(x[k_x], y[k_y])\} = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(x_n[k_x], y_n[k_y])$$

using appropriate mapping functions $f(\cdot)$.

For stationary processes only the difference $k = k_x - k_y$ is relevant as bivariate PDF

$$p_{xy}(\theta_x, \theta_y, k_x, k_y) = p_{xy}(\theta_x, \theta_y, \kappa).$$

For **stationary processes** two important cases lead to fundamental tools for random signal processing:

- Case 1: $k = 0$, i.e $k = k_x = k_y$
- Case 2: $k \neq 0$

2.2.1. Case 1 for Stationary Process

The general linear mapping functions

$$\begin{aligned}
\text{for raw (1,1)-bivariate moment:} \quad & f(\theta_x, \theta_y) = \theta_x^1 \cdot \theta_y^1, \\
\text{for centralized (1,1)-bivariate moment:} \quad & f(\theta_x, \theta_y) = (\theta_x - \mu_x[k_x])^1 \cdot (\theta_y - \mu_y[k_y])^1 \\
\text{for standardized (1,1)-bivariate moment:} \quad & f(\theta_x, \theta_y) = \left(\frac{\theta_x - \mu_x[k_x]}{\sigma_x[k_x]}\right)^1 \cdot \left(\frac{\theta_y - \mu_y[k_y]}{\sigma_y[k_y]}\right)^1.
\end{aligned}$$

simplify under the assumption of stationary processes and considering case 1: $k = 0$, i.e $k = k_x = k_y$

The resulting expectations $E\{\cdot\}$ then are:

- the raw (1,1) - bivariate moment known as cross-power P_{xy}
- the centralized (1,1) - bivariate moment known as co-variance σ_{xy}
- the standardized (1,1) - bivariate moment known as correlation coefficient ρ_{xy}

2.2.2. Case 2 for Stationary Process

For $k = k_x - k_y \neq 0$ the raw and centralized moments are of special importance:

$$\begin{aligned}
\text{raw :} \quad & \varphi_{xy}[k_x, k_y] = \varphi_{xy}[\kappa] = E\{x[k] \cdot y[k - \kappa]\} = E\{x[k + \kappa] \cdot y[k]\} \\
\text{centralized :} \quad & \psi_{xy}[\kappa] = \varphi_{xy}[\kappa] - \mu_x \mu_y
\end{aligned}$$

The raw moment is known as cross-correlation function $\varphi_{xy}[k]$ the centralized moment is known as cross-covariance function $\varphi_{xy}[k]$

If for the second process *y* we consider the process *x* so that $x[k] = y[k]$

$$\begin{aligned}
\text{raw :} \quad & \varphi_{xx}[\kappa] = E\{x[k] \cdot x[k - \kappa]\} = E\{x[k + \kappa] \cdot x[k]\} \\
\text{centralized :} \quad & \psi_{xx}[\kappa] = \varphi_{xx}[\kappa] - \mu_x^2
\end{aligned}$$

the auto-correlation function $\varphi_{xx}[k]$ and auto-covariance function $\varphi_{xx}[k]$ are obtained.

The auto- and cross-correlation functions are of fundamental importance for random signal processing, as these are linked to LTI system signal processing.

2.3. Ergodic Processes

Averaging over time is equal to ensemble averages:

$$\overline{f(x_n[k], x_n[k-\kappa_1], x_n[k-\kappa_2], \dots)} = E\{f(x[k], x[k-\kappa_1], x[k-\kappa_2], \dots)\} \;\; \forall n.$$

2.3.1. Wide-Sense Ergodic

ergodicity holds for linear mapping

$$\overline{x_n[k] \cdot x_n[k-\kappa]} = E\{x[k] \cdot x[k-\kappa]\} \;\; \forall n$$

$$\overline{x_n[k]} = E\{x[k]\} \;\; \forall n.$$

2.3.2. Important Temporal Averages

The linear mean as temporal average of the nnn sample function fffff s for instance given by

$$\overline{x_n[k]} = \lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x_n[k].$$

Furthermore:

The **quadratic mean** from simple quadratic mapping is given as

$$\lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x_n^2[k],$$

the **variance** is given as

$$\lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} (x_n[k] - \overline{x_n[k]})^2,$$

the **cross-correlation** as

$$\lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x[k] \cdot y[k-\kappa],$$

and the **auto- correlation** as

$$\lim_{K \to \infty} \frac{1}{2K+1} \sum_{k=-K}^{K} x[k] \cdot x[k-\kappa].$$

These equations hold for power signals, i.e. the summation yields a finite value.

3. Input data (Variant)

This report was created with base of variant 13:

f	*A*	*B*	*N*
500	500.25	499.75	1800

GitHub repository:
https://github.com/Delisolara/AaDEC

4. Course of actions

4.1. Importing Libraries

```
import numpy as np
import matplotlib as mpl
from matplotlib import pyplot as plt
from numpy.random import Generator, PCG64
from scipy import signal
from scipy import stats
```

Picture 1. Uploaded libraries

4.2. Cross correlation between x and y

```
def my_xcorr(x, y):
    N, M = len(x), len(y)
    kappa = np.arange(N+M-1) - (M-1)
    ccf = signal.correlate(x, y, mode='full', method='auto')
    return kappa, ccf
```

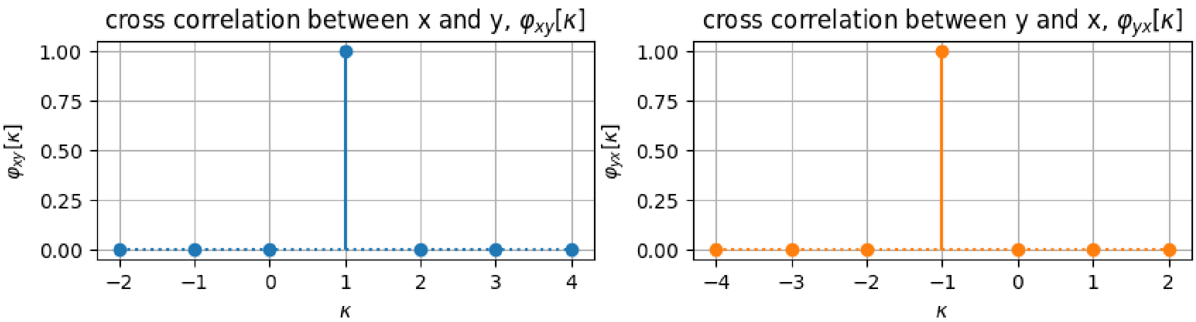
Picture 2. Implemented code

```
if True:  # test my_xcorr with simple example
    x = np.array([0, 1, 0, 0, 0])
    y = np.array([1, 0, 0])
    # plot my_xcorr(x, y) vs. my_xcorr(y, x)
    plt.figure(figsize=(10, 2))
    plt.subplot(1, 2, 1)
    kappa_xy, ccf_xy = my_xcorr(x, y)
    plt.stem(kappa_xy, ccf_xy,
             basefmt='C0:',
             linefmt='C0',
             markerfmt='C0o')
```

Picture 3. Implemented code

```
plt.xlabel(r'$\kappa$')
plt.ylabel(r'$\varphi_{xy}[\kappa]$')
plt.title(r'cross correlation between x and y, $\varphi_{xy}[\kappa]$')
plt.grid(True)
plt.subplot(1, 2, 2)
kappa_yx, ccf_yx = my_xcorr(y, x)
plt.stem(kappa_yx, ccf_yx,
         basefmt='C1:',
         linefmt='C1',
         markerfmt='C1o')
plt.xlabel(r'$\kappa$')
plt.ylabel(r'$\varphi_{yx}[\kappa]$')
plt.title(r'cross correlation between y and x, $\varphi_{yx}[\kappa]$')
plt.grid(True)
```

Picture 4. Implemented code



Picture 5. The result

4.3. Histogram as PDF Estimate, First-Order Ensemble Averages

```python
# set seed for reproducible results
seed = 1234
stats.norm.random_state = Generator(PCG64(seed))

# create random process based on normal distribution
Ns = 2**10  # number of sample functions for e.g. time instance k=0
loc, scale = 5, 3  # mu, sigma

theta = np.arange(-15, 25, 0.01)  # amplitudes for plotting PDF
# random process object with normal PDF
rv = stats.norm(loc=loc, scale=scale)
# get random data from sample functions
x = stats.norm.rvs(loc=loc, scale=scale, size=Ns)
```

Picture 6. Implemented code

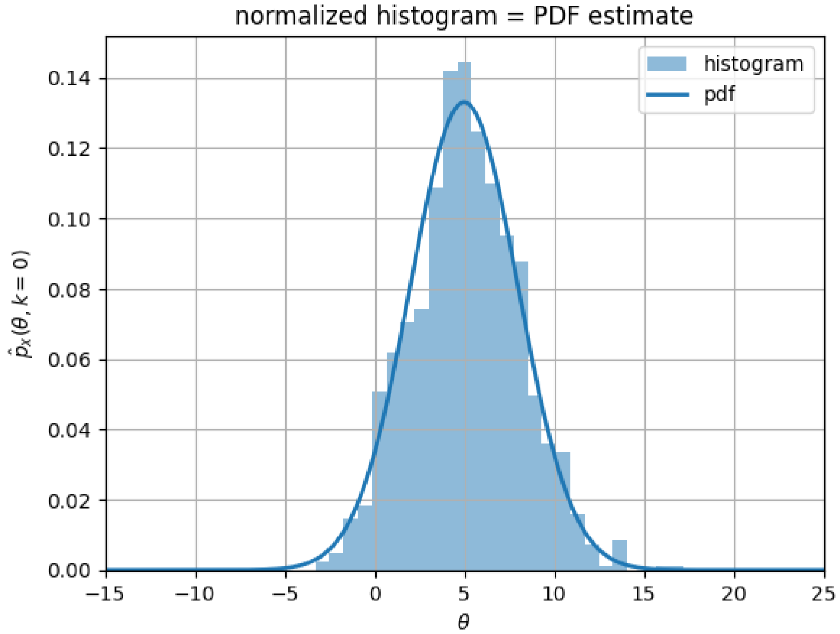
```python
# plot
fig, ax = plt.subplots(1, 1)
hist_estimate = ax.hist(x, bins='auto', density=True, histtype='bar',
                        color='C0', alpha=0.5, label='histogram')
ax.plot(theta, rv.pdf(theta), 'C0-', lw=2, label='pdf')
ax.set_xlabel(r'$\theta$')
ax.set_ylabel(r'$\hat{p}_x(\theta,k=0)$')
ax.set_title('normalized histogram = PDF estimate')
ax.set_xlim(-15, 25)
ax.legend()
ax.grid(True)
```

Picture 7. Implemented code

```python
# get histogram data from ax.hist()
edges = hist_estimate[1]
freq = hist_estimate[0]

# simple ensemble averages by numeric integration
# over histogram data as a simple estimate of the pdf
theta_num = edges[:-1]
dtheta = np.diff(edges)
mu = np.sum(theta_num * freq * dtheta)  # mu estimate
qm = np.sum(theta_num**2 * freq * dtheta)  # quadratic mean estimate
sig2 = np.sum((theta_num-mu)**2 * freq * dtheta)  # sigma^2 estimate
print('ideal   ensemble average: mu = %5.2f, mu^2 = %5.2f, sigma^2 = %5.2f, mu^2 + sigma^2 = %5.2f' %
      (loc, loc**2, scale**2, loc**2+scale**2))
print('numeric ensemble average: mu = %5.2f, mu^2 = %5.2f, sigma^2 = %5.2f, mu^2 + sigma^2 = %5.2f' %
      (mu, mu**2, sig2, qm))
print('ideal sigma = %5.2f, numeric sigma = %5.2f' % (scale, np.sqrt(sig2)))
```

Picture 8. Implemented code

```
ideal   ensemble average: mu =  5.00, mu^2 = 25.00, sigma^2 =  9.00, mu^2 + sigma^2 = 34.00
numeric ensemble average: mu =  4.74, mu^2 = 22.46, sigma^2 =  9.14, mu^2 + sigma^2 = 31.60
ideal sigma =  3.00, numeric sigma =  3.02
```

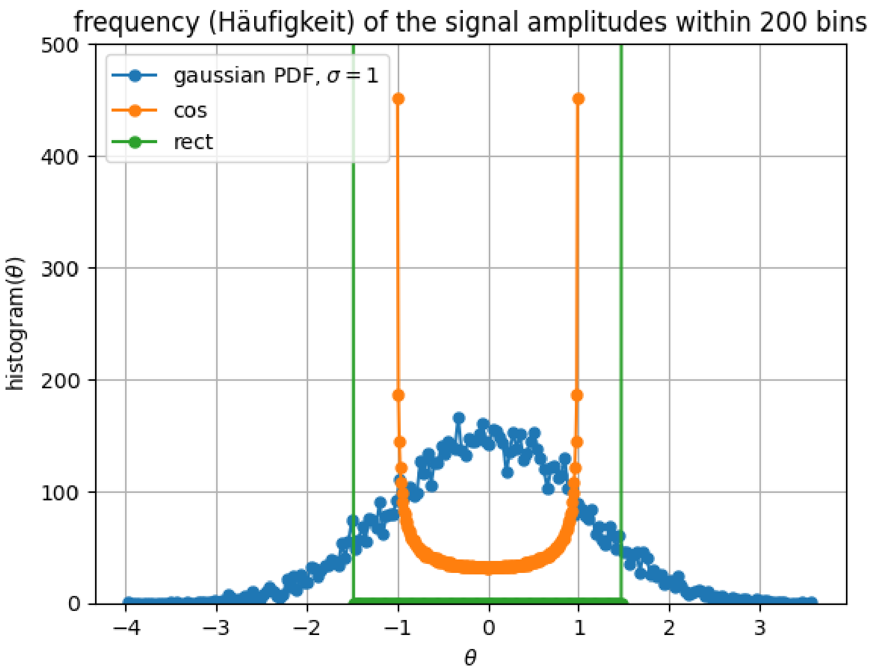


Picture 9. The result

4.4. Histogram of Gaussian Noise, Cosine and Rectangular Signal

```
bins = 200

Ns = 10000  # number of sample function
Nt = 1  # number of time steps per sample function

# normal pdf
x = np.random.normal(loc=0, scale=1, size=[Ns, 1])
pdf, edges = np.histogram(x[:, 0], bins=bins, density=False)
plt.plot(edges[:-1], pdf, 'o-', ms=5, label=r'gaussian PDF, $\sigma=1$')

# cosine signal with peak amplitude 1
x = np.cos(1 * 2*np.pi/Ns*np.arange(0, Ns))
pdf, edges = np.histogram(x, bins=bins, density=False)
plt.plot(edges[:-1], pdf, 'o-', ms=5, label='cos')

# rect signal with amplitude 1.5
x = np.cos(1 * 2*np.pi/Ns*np.arange(0, Ns))
x[x >= 0] = +1.5
x[x < 0] = -1.5
pdf, edges = np.histogram(x, bins=bins, density=False)
plt.plot(edges[:-1], pdf, 'o-', ms=5, label='rect')

plt.ylim(0, 500)
plt.xlabel(r'$\theta$')
plt.ylabel(r'histogram($\theta$)')
plt.title('frequency (Häufigkeit) of the signal amplitudes within 200 bins')
plt.legend()
plt.grid(True)
```

Picture 10. Implemented code



Picture 11. The result

4.5. Higher-Order Ensemble Averages

```
# create two random processes based on normal distribution
Ns = 2**10  # number of sample functions at certain time instant k
Nt = 1  # number of time steps per sample function
np.random.seed(1)

# 1st process:
locx, scalex = 1, 3
x = np.random.normal(loc=locx, scale=scalex, size=[Ns, Nt])

# 2nd process:
locy, scaley = 2, 4
y = np.random.normal(loc=locy, scale=scaley, size=[Ns, Nt])
```

Picture 12. Implemented code

```
crosspower = np.mean(x * y)
covariance = np.mean((x-np.mean(x)) * (y-np.mean(y)))
rho = np.mean((x-np.mean(x))/np.std(x) * (y-np.mean(y))/np.std(y))
print('crosspower = %4.3f, covariance = %4.3f,  correlation coefficient rho = %4.3f' %
      (crosspower, covariance, rho))
```

Picture 13. Implemented code

```
crosspower = 2.048, covariance = -0.256,  correlation coefficient rho = -0.021
```

Picture 14. The result

4.6. Ensemble Average vs. Temporal Average

```
# create random process based on normal distribution
Ns = 4000  # number of samples to set up an ensemble
Nt = 15000  # number of time steps to set up 'ensemble over time'-characteristics
np.random.seed(1)

s = np.arange(Ns)  # ensemble index (s to indicate sample function)
t = np.arange(Nt)  # time index

loc, scale = 5, 3  # mu, sigma
x = np.random.normal(loc=loc, scale=scale, size=[Ns, Nt])
```

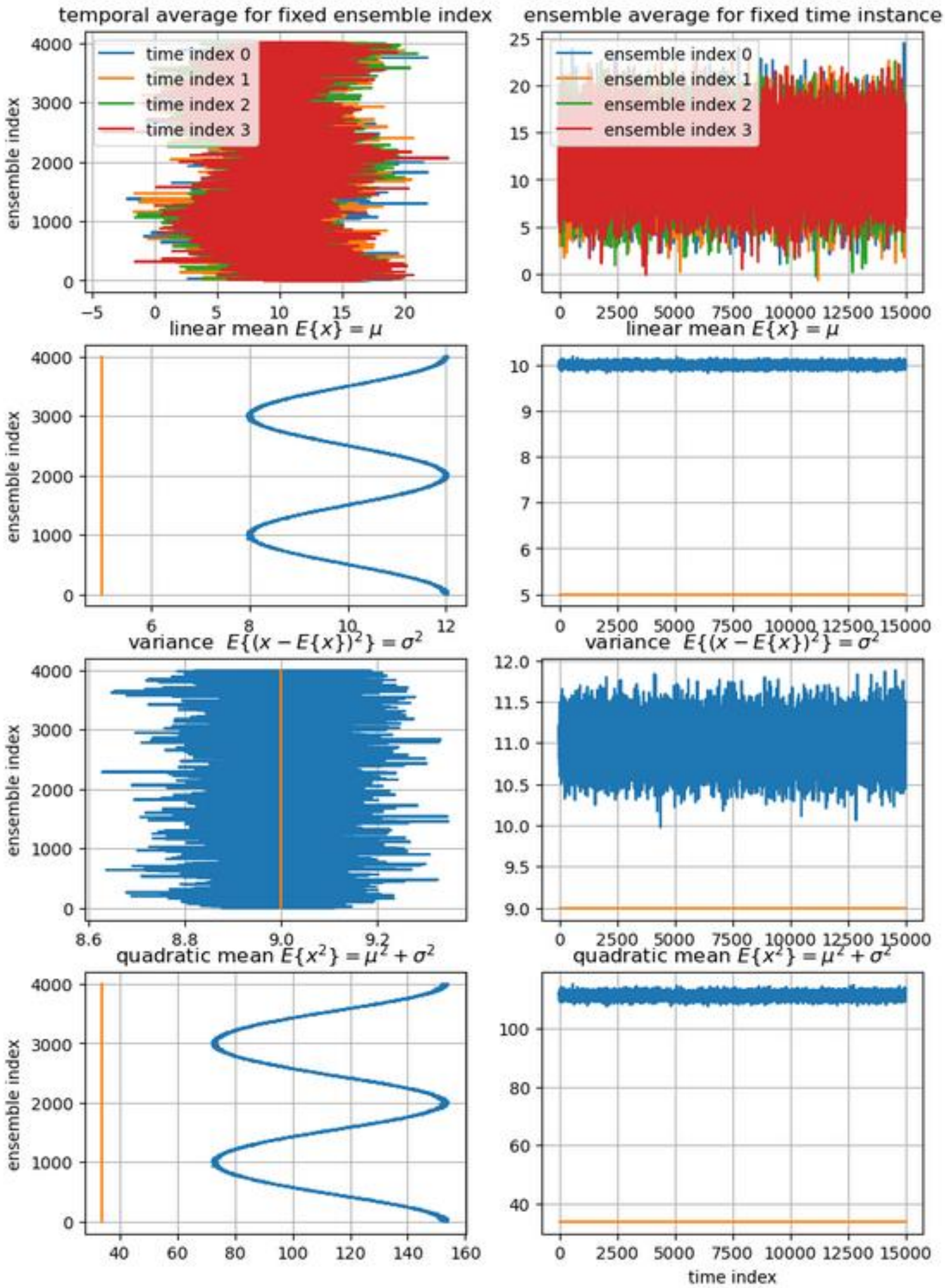
Picture 15. Implemented code

```
# we check the three cases:
# 1. simulate an ergodic process, i.e. ensemble average == temporal average
case_str = 'x'
# 2./3. very simple simulation of non-stationary process by changing the mean
case_str = 'cos_s'  # add cosine over ensemble equally for all time instances
# case_str = 'cos_t'  # add cosine over time equally for all ensembles

if case_str == 'x':  # use x directly == ergodic process
    tmp = 1  # dummy variable since nothing to do here
elif case_str == 'cos_s':  # add cosine over ensemble equally for all time instances
    tmp = 2*np.cos(2 * 2*np.pi/Ns * np.arange(0, Ns)) + 5
    x = x + np.transpose(np.tile(tmp, (Nt, 1)))
elif case_str == 'cos_t':  # add cosine over time equally for all ensembles
    tmp = 2*np.cos(2 * 2*np.pi/Nt * np.arange(0, Nt)) + 5
    x = x + np.tile(tmp, (Ns, 1))
```

Picture 16. Implemented code

```python
fig, axs = plt.subplots(4, 2, figsize=(9, 13))
# plot signals
for i in range(4):
    axs[0, 0].plot(x[:, i], s, label='time index '+str(i))
    axs[0, 1].plot(t, x[i, :], label='ensemble index '+str(i))
# plot means
axs[1, 0].plot(np.mean(x, axis=1), s)
axs[1, 1].plot(t, np.mean(x, axis=0))
axs[1, 0].plot([loc, loc], [0, Ns])
axs[1, 1].plot([0, Nt], [loc, loc])
# plot variance
axs[2, 0].plot(np.var(x, axis=1), s)
axs[2, 1].plot(t, np.var(x, axis=0))
axs[2, 0].plot([scale**2, scale**2], [0, Ns])
axs[2, 1].plot([0, Nt], [scale**2, scale**2])
# plot quadratic mean
axs[3, 0].plot(np.mean(x**2, axis=1), s)
axs[3, 1].plot(t, np.mean(x**2, axis=0))
axs[3, 0].plot([loc**2+scale**2, loc**2+scale**2], [0, Ns])
axs[3, 1].plot([0, Nt], [loc**2+scale**2, loc**2+scale**2])
# labeling
axs[3, 1].set_xlabel('time index')
for i in range(4):
    #axs[i,1].set_xlabel('time index')
    axs[i, 0].set_ylabel('ensemble index')
    for j in range(2):
        axs[i, j].grid(True)
axs[0, 0].set_title(r'temporal average for fixed ensemble index')
axs[0, 1].set_title(r'ensemble average for fixed time instance')
for i in range(2):
    axs[0, i].legend(loc='upper left')
    axs[1, i].set_title(r'linear mean $E\{x\} = \mu$')
    axs[2, i].set_title(r'variance  $E\{(x -E\{x\} )^2\} = \sigma^2$')
    axs[3, i].set_title(r'quadratic mean $E\{x^2\} = \mu^2+\sigma^2$')
```

Picture 17. Implemented code



Picture 18. The result

4.7. Higher-Order Temporal Averages

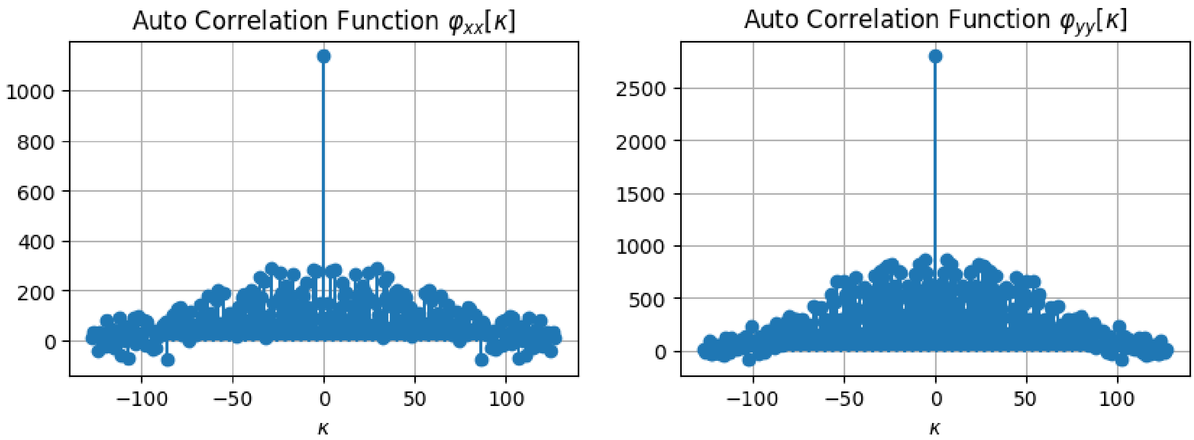
```
# create two random processes based on normal distribution
Ns = 1  # number of sample functions at certain time instant k
Nt = 2**7  # number of time steps per sample function
np.random.seed(1)

# 1st process:
locx, scalex = 1, 3
x = np.random.normal(loc=locx, scale=scalex, size=[Ns, Nt])

# 2nd process:
locy, scaley = 2, 4
y = np.random.normal(loc=locy, scale=scaley, size=[Ns, Nt])
```

Picture 19. Implemented code

4.8. Auto Correlation Function (ACF)

```
plt.figure(figsize=(10, 3))
plt.subplot(1, 2, 1)
kappa, ccf = my_xcorr(x[0, :], x[0, :])
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'Auto Correlation Function $\varphi_{xx}[\kappa]$')
plt.grid(True)
plt.subplot(1, 2, 2)
kappa, ccf = my_xcorr(y[0, :], y[0, :])
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'Auto Correlation Function $\varphi_{yy}[\kappa]$')
plt.grid(True)

# check the axial symmetry, why is the peak always at kappa=0
```

Picture 20. Implemented code

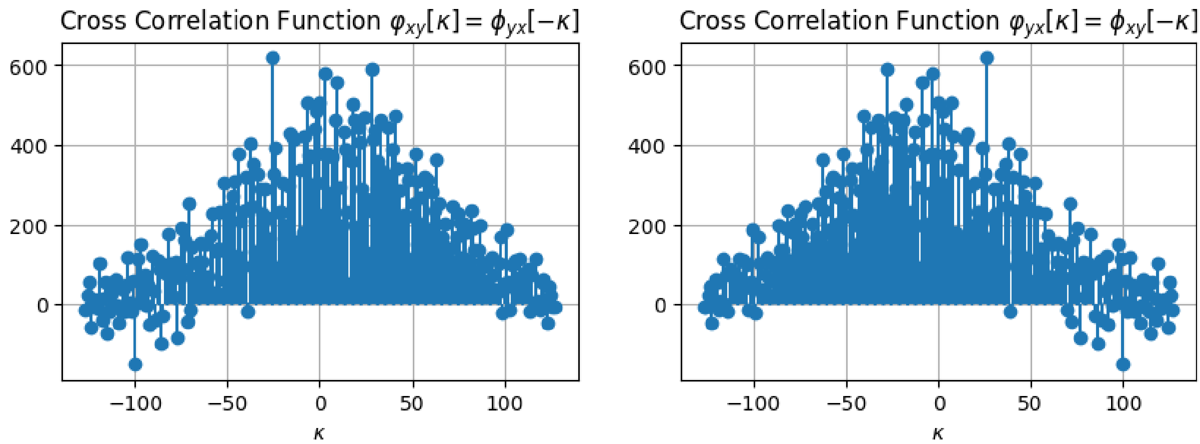


Picture 21. The result

4.9. Cross Correlation Function (CCF)

```
plt.figure(figsize=(10, 3))
plt.subplot(1, 2, 1)
kappa, ccf = my_xcorr(x[0, :], y[0, :])
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(
    r'Cross Correlation Function $\varphi_{xy}[\kappa]=\phi_{yx}[-\kappa]$')
plt.grid(True)
plt.subplot(1, 2, 2)
kappa, ccf = my_xcorr(y[0, :], x[0, :])
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(
    r'Cross Correlation Function $\varphi_{yx}[\kappa]=\phi_{xy}[-\kappa]$')
plt.grid(True)
```

Picture 22. Implemented code

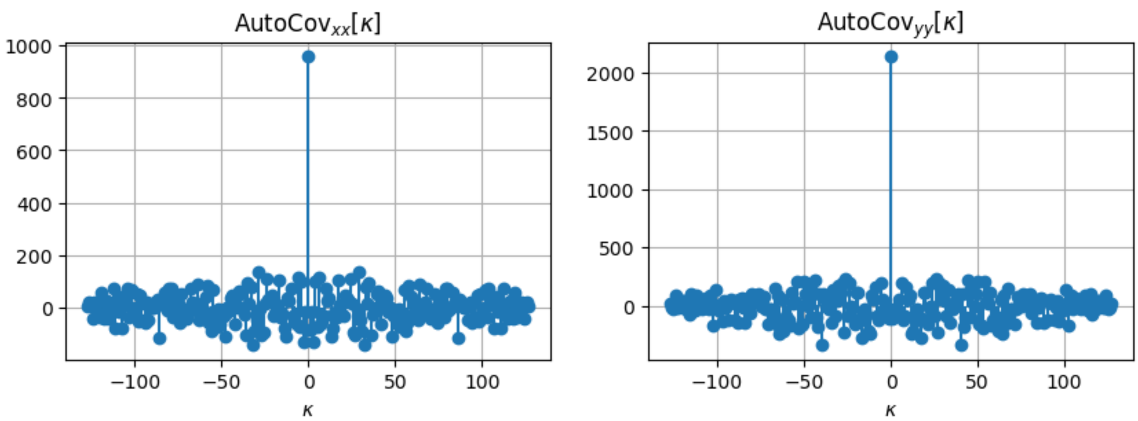


Picture 23. The result

4.10. Auto Covariance Function

```
plt.figure(figsize=(10, 3))
plt.subplot(1, 2, 1)
kappa, ccf = my_xcorr(x[0, :]-np.mean(x[0, :]), x[0, :]-np.mean(x[0, :]))
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'AutoCov$_{xx}[\kappa]$')
plt.grid(True)
plt.subplot(1, 2, 2)
kappa, ccf = my_xcorr(y[0, :]-np.mean(y[0, :]), y[0, :]-np.mean(y[0, :]))
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'AutoCov$_{yy}[\kappa]$')
plt.grid(True)
```

Picture 24. Implemented code

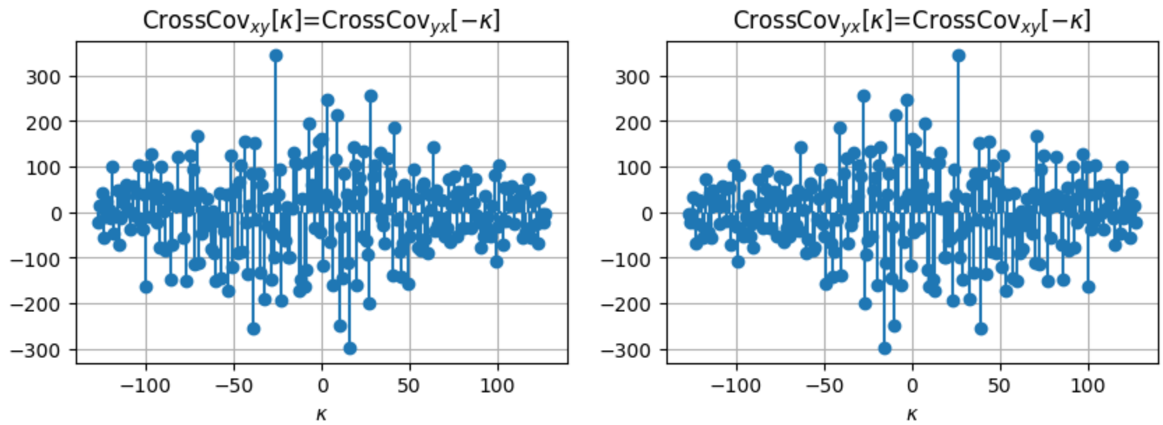


Picture 25. The result

4.11. Cross Covariance Function

```
plt.figure(figsize=(10, 3))
plt.subplot(1, 2, 1)
kappa, ccf = my_xcorr(x[0, :]-np.mean(x[0, :]), y[0, :]-np.mean(y[0, :]))
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'CrossCov$_{xy}[\kappa]$=CrossCov$_{yx}[-\kappa]$')
plt.grid(True)
plt.subplot(1, 2, 2)
kappa, ccf = my_xcorr(y[0, :]-np.mean(y[0, :]), x[0, :]-np.mean(x[0, :]))
plt.stem(kappa, ccf, basefmt='C0:', use_line_collection=True)
plt.xlabel(r'$\kappa$')
plt.title(r'CrossCov$_{yx}[\kappa]$=CrossCov$_{xy}[-\kappa]$')
plt.grid(True)
```

Picture 26. Implemented code



Picture 27. The result

5. Conclusions

In this lab, we explored random signals, covering topics such as probability density functions (PDFs), ensemble averages, and concepts like stationarity and ergodicity. These concepts provide valuable insights into the statistical properties of signals, enabling us to analyze and interpret stochastic phenomena effectively.