PHY 396L: Quantum Field Theory II Notes

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Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. Also there will also be a lot of stuff in here from Prof. Maloney's QFT I and QFT II course here since I like how he explains things. If you have any comments let me know at `hi@delonshen.com`.

Maloney QFT II Lecture 1: Systematics of Renormalization

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence lets consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \to \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$

Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges(TODO Maloney says "logarithmically divergent"[1]). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2 \log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2} \log s/\Lambda^2 + \dots$$

This seems like a disaster since as the cutoff $\Lambda \to 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

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[1]Okay so googling leads to this. $dU/U = d\log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase $\mathcal{M}$ in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0.

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2}\log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}\log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for $\mathcal{M}$ above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \to 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2. So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in $\mathcal{L}$ we introduce a countert-erm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

   Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

   Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

Now if we want to shfit this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

(a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$

(b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$

(c) Z_g fixed by g = physical 3-pt funct at some energy

(d) Z_m fixed by the $(\text{mass})^2$ of a one particle state should be m^2. The physical mass is not necessarily m^2.

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g. For example Y is fixed by the cancellation of the one point function. The existance of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.
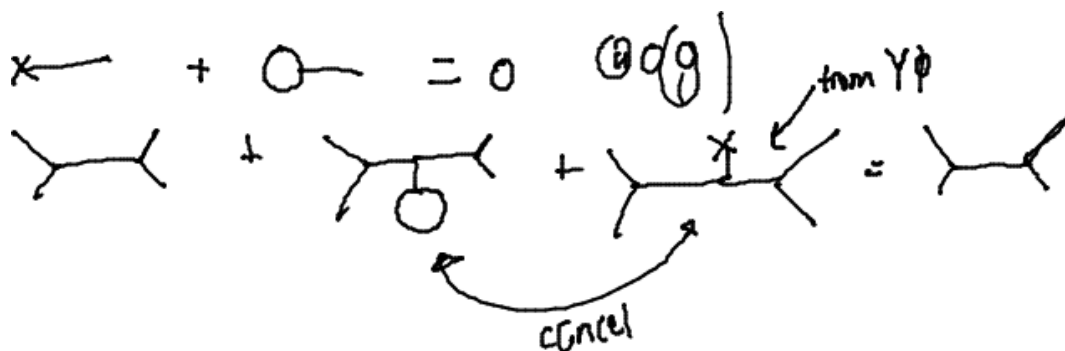
For other counterterms we have to do some computations.

Maloney QFT I Lecture 12 Part 1: Interactions

A free QFT where $S[\phi]$ is quadratice in ϕ has no interactions. Consider some sort of free theory with two kinds of particles a and b. Schematically lets have some Hamiltonian

$$H = a^\dagger a + b^\dagger b \Rightarrow a^\dagger |0\rangle \xrightarrow{e^{iHt}} a^\dagger |0\rangle \times \text{phase}$$

Basically for a free theory if you start with particle a, no matter how much time passes you'll still always have that particle a. For an interacting theory

$$H = a^\dagger a + b^\dagger b + \lambda(a(b^\dagger)^2 + a^\dagger b^2)$$

Where λ is small. The term in the parenthesis is the interaction term where the first term destroys the a particle and creates two b particle and the second term destroys the two b particles and gives us back the a particle(hermitian means process is reversible which is why we have both terms instead of just one term.) Solving this exactly should be a disaster but we can use approximation methods. First just by taylor expanding we can see that

$$a^\dagger |0\rangle \to e^{iHt}(a^\dagger |0\rangle) \approx a^\dagger |0\rangle + i\lambda t(b^\dagger)^2 |0\rangle + \dots$$

Basically there's some probably per unit time where a particle becomes two b particles. What we wanted to show here is that the non-quadratic terms that create interactions. There should be some things we should take away

(a) The vacuum state of $|0\rangle_{\text{free}}$ no longer works since. We defined this state as the state that's annihilated by lowering operators. However this is no longer the case. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$. The true vacuum state $|\Omega\rangle$ is more complex in interacting theories.

(b) We shouldn't think of a and $a^\dagger$ as creating/destroying particles anymore.

To make things explicit lets consider a typical interacting theory

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \mathcal{L}_{\text{int}} \Leftrightarrow \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Expanding $V(\phi)$ around $\phi = 0$ we get

$$V(\phi) = V_0 + V_1\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

V_0 is ignorable. We can also let $\phi \to \phi + \text{const}$ so that $V_1 = 0$ Consider $\phi(x^\mu)$ where ϕ is slowly varying of **x**.

$$\Rightarrow \mathcal{L} \approx \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Energy is minimized when ϕ sits at the minimum of $V(\phi)$. This means the ground state or "vacuum" where $V'(\phi_*) = 0$. So if $V_1 \neq 0$ then the field will realx to a value of $V(\phi)$ with $V'(\phi) = 0$ and if we want to study the properties near this vacuum we have a new field $\phi' = \phi - \phi_*$. So for now lets just choose vacuum so that in the vacuum $\phi = 0$. e.g. $\langle\Omega|\phi|\Omega\rangle = 0$.

  Question: what interaction terms matter? Why should we care about ϕ^3 and not ϕ^{100}? We can answer this question with dimensional analysis

$$S = \int d^4x((\partial\phi)^2 - m^2\phi^2 + \cdots + \lambda_{ij}\partial^j\phi^i)$$

First we know in natural units $[x] = E^{-1}$ and thus $[\partial/\partial x] = E$. First action is dimensionless. Thus $[\phi] = E$ ($[d^4x\partial_x^2] = E^{-2}$). We can also see that $[m] = E$ and $[\lambda_{ij}] = E^{4-i-j}$. Now imagine we're doing some experiment which probes the theory at scale E. For example $\phi\phi \to \phi\phi$ with COM energy $\approx E$. What physical effects would come from our λ_{ij} term? well $\lambda_{ij}E^{i+j-4} = \lambda_{ij}E^{-[\lambda_{ij}]}$. What this means is that at low energies. we should only care about $[\lambda] > 0$ since if $[\lambda < 0]$ then E is raised to a positive power and thus is exponentially decreasing.

(a) $[\lambda] > 0$ is relevant

(b) $[\lambda] < 0$ is irrelevant

(c) $[\lambda] = 0$ is marginal

For a scalar theory at low energies the only relevant terms are kinetic terms and ϕ^3 and ϕ^4. Here's an idea from Landau: To study a system at low energy we can follow a recipe

(a) Guess DOF

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(b) Guess symmetries

(c) Write down the most general theory (e.g. action)that satisfies the symmetries

(d) Study the relevant interaction terms.

To constrain our theory futher we should also ask what sort of symmetries we have? For example for the symmetry $\phi \to -\phi$ then there can't be a ϕ^3 term. We call this theory $\lambda\phi^4$ theory.

Maloney QFT I Lecture 13: Path Integrals

Thought I should take a look at this since QFT I at UT didn't approach Feynmann diagrams from path integrals.

We have some QM system with DOF q_i (e.g. for a field theory $\phi(\mathbf{x})$). We want to compute the transition from some inital time t_i with some configuration q_i to the final state (q_f, t_f).

$$\langle q_f, t_f | q_i, t_i \rangle$$

So we use the fact that we can always insert a complete set of basis states $|q, t\rangle$ at some intermediate time t where $t \in (t_i, t_f)$. So what happens? Let $|q_i, t_i\rangle = q_1, t_1$ and $|q_f, t_f\rangle = |q_n, t_n\rangle$. This gives us

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq \, \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle$$
$$= \int dq_2 \ldots dq_{n-1} \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \ldots \langle q_2, t_2 | q_1, t_1 \rangle$$

Basically we're integrating over all possible paths (including unphysical paths) from the inital state to the final state. So we can think of this transition amplitude as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq(t)$$

To make this idea precise we need to figure out how to integrate over a space of functions. The next thing we need to do is figure out what the fuck we're integrating. First lets just guess. In the limit $\hbar \to 0$ the integrand should be a function only of the classical solution of the EOM $\delta S/\delta q = 0$. First lets take a detour into approximating finite dimensional integrals which can be generalized to infinite dimensional integrals: the method of saddle points. Consider the one dimensional integral

$$Z = \int_{-\infty}^{\infty} dq \, e^{\lambda f(q)}$$

And we want the integral to converge so f should look something like.
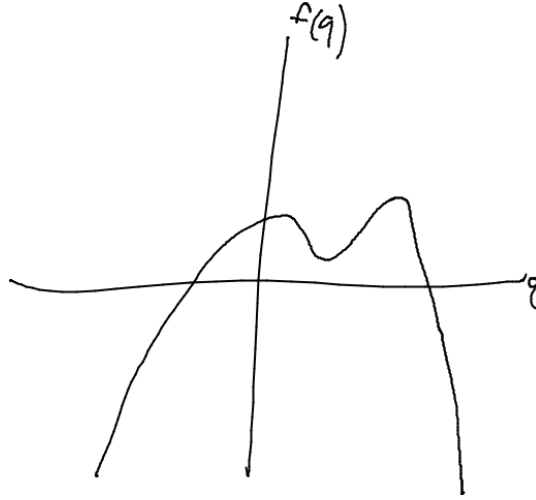
And we want to perform this integral when $\lambda \to \infty$. In this limit the integral should be dominated by the local maxima of $f(q)$. Namely by the q_* where $f'(q_*) = 0$. And what is this function Z in the limit where λ is large?

$$Z = \sum_{q_*} e^{\lambda f(q_*)}$$

It turns out there are a bunch of subleading corrections which are

$$Z = \sum_{q_*} e^{\lambda f(q_*)} \left(\sqrt{\frac{2\pi}{\lambda f''(q_*)}} + \cdots \right)$$

(TODO prove first term). The higher order corrections are basically from Feynmann diagrams. Comparing this method of saddle point with the problem of path integrals we can guess that the integrand should be a function of $\frac{1}{\hbar}S$ (since the integrand does to infinity as $\hbar \to 0$ meaning that only things that solve $\delta S/\delta Q = 0$ contribute like in the method of steepest descent) and we can guess that

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq(t) \exp\left\{ \frac{i}{\hbar} S[q(t)] \right\}$$

Lets just check our work. Consider $H = \frac{1}{2}p^2 + V(q)$. Then

$$e^{-iH\delta t} = e^{-i(p^2/2 + V(q))t} = e^{-i\frac{1}{2}p^2\delta t} e^{-iV(q)\delta t} e^{O(\delta t^2)}$$

Where the second equality comes from Cambell-Backer-Houstouder?? formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]+\cdots}$$

Now lets insert an identity again in the following.

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q)\delta t} | q_{n-1} \rangle$$

Now we can use that $|p_{n-1}\rangle$ is a momentum eigenstate and $|q_{n-1}\rangle$ is a position eigenstate to replace the operators in the exponential with eigenvalues

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p_{n-1}^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q_{n-1})\delta t} | q_{n-1} \rangle$$

Taking out the identity we now just have

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2 + V(q_{n-1}))} \langle q_n | p_{n-1} \rangle \langle p_{n-1} | q_{n-1} \rangle$$

$$= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2 + V(q_{n-1}))} e^{ip_{n-1}(q_n - q_{n-1})}$$

$$= \int dp_{n-1} e^{-i\delta t H(q_{n-1}, p_{n-1})} e^{ip_{n-1}(q_n - q_{n-1})}$$

So what we see is that

$$\langle q_n, t_n | q_i, t_i \rangle = \int dq_2 \ldots dq_{n-1} \langle q_n | e^{-iH\delta t} | q_{n-1} \rangle \ldots \langle q_2 | e^{-iH\delta t} | q_1 \rangle$$

$$= \int dq_2 \ldots dq_{n-1} dp_2 \ldots dp_{n-1} \prod_{i=2}^{n-1} e^{-i\delta t H(q_i, p_i)} e^{ip_i(q_{i+1} - q_i)}$$

$$= \int dq_2 \ldots dq_{n-1} dp_2 \ldots dp_{n-1} \exp\left\{ i \sum_{i=2}^{n-1} \left(\frac{p_i}{\delta t}(q_{i+1} - q_i) - H(q_i, p_i) \right) \delta t \right\}$$

Now in the limit when $n \to \infty$ then $\delta t \to 0$. This gives us

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq Dp \exp\left\{ i \int dt \, (p\dot{q} - H = L) \right\}$$

This is a path integral over phase space, not just position(configuration) space. We can reduce this to an integral in configuration space by noting that the integral over phase space is a gaussian integral. Note that

$$\int dp_i e^{i(-p_i^2/2 - p_i(q_i + 1 - q_i)/\delta t)\delta t} = \text{const} \times e^{i\dot{q}_i^2/2}$$

Some comments about this derivation:

(a) $\int Dq$, the integral over a space of functions is not a well defined objectet

(b) This derivation makes manifest ideas of interference and symmetries (e.g. double slit experiment)

(c) Useful in perturbation theory.

(d) Not useful for calculations. Not really necessary in QM but necessary in QFT.

So how do we compute expectation values? Let $Q(t)$ operator that measure $q(t)$.

$$\langle q_n, t_n | Q(t) | q_1, t_1 \rangle = \int dq q(t) \langle q_n, t_n | q, t \rangle \langle q, t | q_1, t_1 \rangle$$

$$= \int_{q(t_1)=q_1}^{q(t_n)=q_n} Dq e^{iS[q]} q(t)$$

Now what happens if weant to compute a two point function

$$\int Dq e^{iS[q]} q(t)q(t') = \Theta(t'-t)\langle q_n, t_n|Q(t')Q(t)|q_1, t_1\rangle + \Theta(t-t')\langle q_n, t_n|Q(t)Q(t')|q_1, t_1\rangle$$

Namely the thing that the path integral computes is time ordered expectation values. Thus note that if we have many observables

$$\int Dq e^{iS[q]} q_1 \dots q_n = \langle |\mathbf{T} q_1 \dots q_n|\rangle$$

So far we've considered position eigenstates. In a more general state

$$\psi = \int dq\, |q\rangle\langle q|\psi\rangle$$

To compute VEV:

$$\langle 0|\mathbf{T}(Q_1 \dots Q_n)|0\rangle = \lim_{\substack{t_f \to -\infty \\ t_i \to \infty}} \int dq_i dq_f \psi_0(q_i)\psi_0^*(q_f) \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS} q_1 \dots q_n$$

Theres' a trick to avoid this

$$|q, t\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|q, t=0\rangle e^{iE_n t}$$

Now let $t \to (1-i\epsilon)t$ and $t \to -\infty$ (very early times). Then $e^{iE_n t} \to e^{i(1-i\epsilon)tE_n}$. Thus we will pick out terms near the vacuum energy. Thus

$$|q, t\rangle \to \langle 0|q, 0\rangle|0\rangle$$

This analytic continuation projects onto the goround state for ket at $t \to -\infty$ and for bra this projects onto the ground state for $t \to \infty$. So then we can compute the VEV as

$$\langle 0|\mathbf{T}Q_1 \dots Q_n|0\rangle = \int Dq \exp\{iS[q]\} q_1 \dots q_n$$

But where $S = \int d\tilde{t} L$ where $\tilde{t} = (1-i\epsilon)t$. Sometimes we take this analytical continuation where $\tilde{t} = it$. This is a "Euclidean continuation." The reason that this is called "euclidean" comes from the following. Consider the invariant interval in special relativity. Under that analytical continuation

$$ds^2 = dt^2 - d\mathbf{x}^2 \to -(d\tilde{t}^2 + d\mathbf{x}^2)$$

It's useful in finite temperature physics.