PHY 396L: Quantum Field Theory II Notes

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Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. Also there will also be a lot of stuff in here from Prof. Maloney's QFT I and QFT II course here. If you have any comments let me know at `hi@delonshen.com`.

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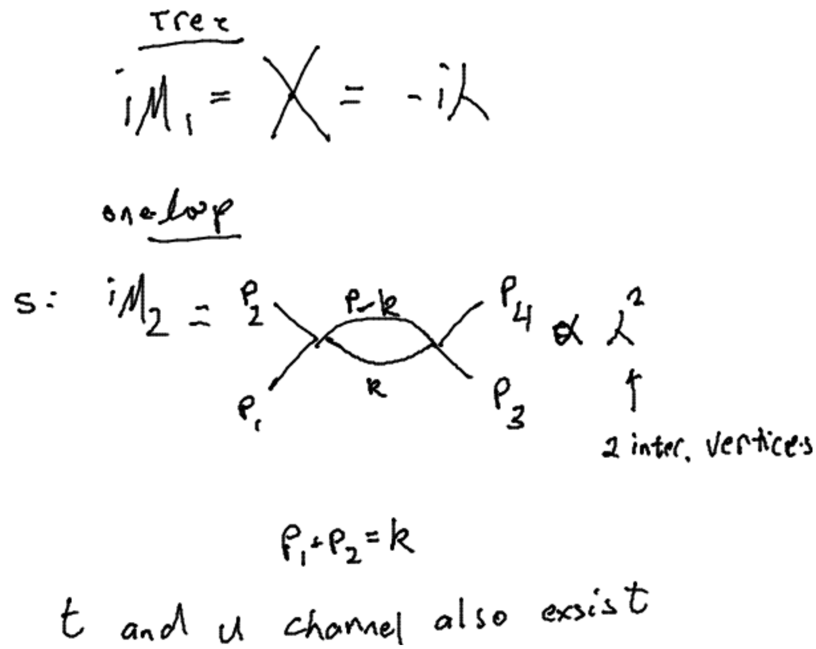
Maloney QFT II Lecture 1: Systematics of Renormalization

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence lets consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \to \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$



Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges(TODO Maloney says "logarithmically divergent"[1]). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2\log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

This seems like a disaster since as the cutoff $\Lambda \to 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

[1] Okay so googling leads to this. $dU/U = d\log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase $\mathcal{M}$ in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0.

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2}\log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}\log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for $\mathcal{M}$ above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \to 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2. So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in $\mathcal{L}$ we introduce a counterterm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

Now if we want to shfit this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

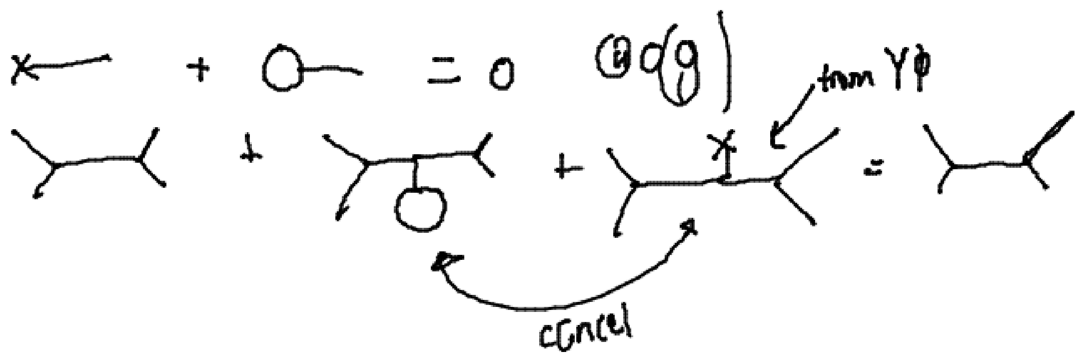
(a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$

(b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$

(c) Z_g fixed by g = physical 3-pt funct at some energy

(d) Z_m fixed by the (mass)^2 of a one particle state should be m^2. The physical mass is not necessarily m^2.

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g. For example Y is fixed by the cancellation of the one point function. The existance of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.



For other counterterms we have to do some computations.

Maloney QFT I Lecture 12 Part 1: Interactions

A free QFT where $S[\phi]$ is quadratice in ϕ has no interactions. Consider some sort of free theory with two kinds of particles a and b. Schematically lets have some Hamiltonian

$$H = a^\dagger a + b^\dagger b \Rightarrow a^\dagger|0\rangle \xrightarrow{e^{iHt}} a^\dagger|0\rangle \times \text{phase}$$

Basically for a free theory if you start with particle a, no matter how much time passes you'll still always have that particle a. For an interacting theory

$$H = a^\dagger a + b^\dagger b + \lambda(a(b^\dagger)^2 + a^\dagger b^2)$$

Where λ is small. The term in the parenthesis is the interaction term where the first term destroys the a particle and creates two b particle and the second term destroys the two b particles and gives us back the a particle(hermitian means process is reversible which is why we have both terms instead of just one term.) Solving this exactly should be a disaster but we can use approximation methods. First just by taylor expanding we can see that

$$a^\dagger|0\rangle \to e^{iHt}(a^\dagger|0\rangle) \approx a^\dagger|0\rangle + i\lambda t(b^\dagger)^2|0\rangle + \dots$$

Basically there's some probably per unit time where a particle becomes two b particles. What we wanted to show here is that the non-quadratic terms that create interactions. There should be some things we should take away

(a) The vacuum state of $|0\rangle_{\text{free}}$ no longer works since. We defined this state as the state that's annihilated by lowering operators. However this is no longer the case. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$. The true vacuum state $|\Omega\rangle$ is more complex in interacting theories.

(b) We shouldn't think of a and $a^\dagger$ as creating/destroying particles anymore.

To make things explicit lets consider a typical interacting theory

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \mathcal{L}_{\text{int}} \Leftrightarrow \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Expanding $V(\phi)$ around $\phi = 0$ we get

$$V(\phi) = V_0 + V_1\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

V_0 is ignorable. We can also let $\phi \to \phi + \text{const}$ so that $V_1 = 0$ Consider $\phi(x^\mu)$ where ϕ is slowly varying of $\mathbf{x}$.

$$\Rightarrow \mathcal{L} \approx \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Energy is minimized when ϕ sits at the minimum of $V(\phi)$. This means the ground state or "vacuum" where $V'(\phi_*) = 0$. So if $V_1 \neq 0$ then the field will realx to a value of $V(\phi)$ with $V'(\phi) = 0$ and if we want to study the properties near this vacuum we have a new field $\phi' = \phi - \phi_*$. So for now lets just choose vacuum so that in the vacuum $\phi = 0$. e.g. $\langle\Omega|\phi|\Omega\rangle = 0$.

Question: what interaction terms matter? Why should we care about ϕ^3 and not ϕ^{100}? We can answer this question with dimensional analysis

$$S = \int d^4x((\partial\phi)^2 - m^2\phi^2 + \cdots + \lambda_{ij}\partial^j\phi^i)$$

First we know in natural units $[x] = E^{-1}$ and thus $[\partial/\partial x] = E$. First action is dimensionless. Thus $[\phi] = E$ ($[d^4x\partial_x^2] = E^{-2}$). We can also see that $[m] = E$ and $[\lambda_{ij}] = E^{4-i-j}$. Now imagine we're doing some experiment which probes the theory at scale E. For example $\phi\phi \to \phi\phi$ with COM energy $\approx E$. What physical effects would come from our λ_{ij} term? well $\lambda_{ij}E^{i+j-4} = \lambda_{ij}E^{-[\lambda_{ij}]}$. What this means is that at low energies. we should only care about $[\lambda] > 0$ since if $[\lambda < 0]$ then E is raised to a positive power and thus is exponentially decreasing.

(a) $[\lambda] > 0$ is relevant

(b) $[\lambda] < 0$ is irrelevant

(c) $[\lambda] = 0$ is marginal

For a scalar theory at low energies the only relevant terms are kinetic terms and ϕ^3 and ϕ^4. Here's an idea from Landau: To study a system at low energy we can follow a recipe

(a) Guess DOF

(b) Guess symmetries

(c) Write down the most general theory (e.g. action)that satisfies the symmetries

(d) Study the relevant interaction terms.

To constrain our theory futher we should also ask what sort of symmetries we have? For example for the symmetry $\phi \to -\phi$ then there can't be a ϕ^3 term. We call this theory $\lambda\phi^4$ theory.

Maloney QFT I Lecture 13: Path Integrals

Thought I should take a look at this since QFT I at UT didn't approach Feynmann diagrams from path integrals.

We have some QM system with DOF q_i (e.g. for a field theory $\phi(\mathbf{x})$). We want to compute the transition from some inital time t_i with some configuration q_i to the final state (q_f, t_f).

$$\langle q_f, t_f | q_i, t_i \rangle$$

So we use the fact that we can always insert a complete set of basis states $|q,t\rangle$ at some intermediate time t where $t \in (t_i, t_f)$. So what happens? Let $|q_i, t_i\rangle = q_1, t_1$ and $|q_f, t_f\rangle = |q_n, t_n\rangle$. This gives us

$$\begin{aligned} \langle q_f, t_f | q_i, t_i \rangle &= \int dq \, \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \\ &= \int dq_2 \ldots dq_{n-1} \, \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \ldots \langle q_2, t_2 | q_1, t_1 \rangle \end{aligned}$$

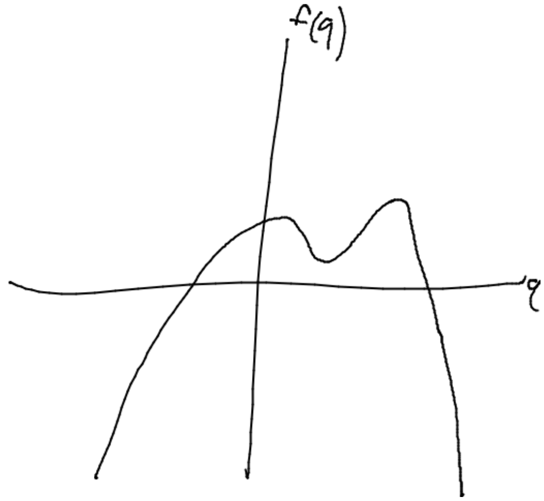
Basically we're integrating over all possible paths (including unphysical paths) from the inital state to the final state. So we can think of this transition amplitude as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq(t)$$

To make this idea precise we need to figure out how to integrate over a space of functions. The next thing we need to do is figure out what the fuck we're integrating. First lets just guess. In the limit $\hbar \to 0$ the integrand should be a function only of the classical solution of the EOM $\delta S/\delta q = 0$. First lets take a detour into approximating finite dimensional integrals which can be generalized to infinite dimensional integrals: the method of saddle points. Consider the one dimensional integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

And we want the integral to converge so f should look something like.



And we want to perform this integral when $\lambda \to \infty$. In this limit the integral should be dominated by the local maxima of $f(q)$. Namely by the q_* where $f'(q_*) = 0$. And what is this function Z in the limit where λ is large?

$$Z = \sum_{q_*} e^{\lambda f(q_*)}$$

It turns out there are a bunch of subleading corrections which are

$$Z = \sum_{q_*} e^{\lambda f(q_*)} \left(\sqrt{\frac{2\pi}{\lambda f''(q_*)}} + \dots \right)$$

(I proved first term in the next subsection). The higher order corrections are basically from Feynmann diagrams. Comparing this method of saddle point with the problem of path integrals we can guess that the integrand should be a function of $\frac{1}{\hbar}S$ (since the integrand does to infinity as $\hbar \to 0$ meaning that only things that solve $\delta S/\delta Q = 0$ contribute like in the method of steepest descent) and we can guess that

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq(t) \exp\left\{ \frac{i}{\hbar} S[q(t)] \right\}$$

Lets just check our work. Consider $H = \frac{1}{2}p^2 + V(q)$. Then

$$e^{-iH\delta t} = e^{-i(p^2/2+V(q))t} = e^{-i\frac{1}{2}p^2\delta t} e^{-iV(q)\delta t} e^{O(\delta t^2)}$$

Where the second equality comes from Cambell-Backer-Houstouder?? formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]+\dots}$$

Now lets insert an identity again in the following.

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q)\delta t} | q_{n-1} \rangle$$

Now we can use that $|p_{n-1}\rangle$ is a momentum eigenstate and $|q_{n-1}\rangle$ is a position eigenstate to replace the operators in the exponential with eigenvalues

$$\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle = \int dp_{n-1}\,\langle q_n|e^{-i\delta t p_{n-1}^2/2}|p_{n-1}\rangle\langle p_{n-1}|e^{-iV(q_{n-1})\delta t}|q_{n-1}\rangle$$

Taking out the identity we now just have

$$\begin{aligned}\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))}\langle q_n|p_{n-1}\rangle\langle p_{n-1}|q_{n-1}\rangle \\ &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))}e^{ip_{n-1}(q_n-q_{n-1})} \\ &= \int dp_{n-1} e^{-i\delta t H(q_{n-1},p_{n-1})}e^{ip_{n-1}(q_n-q_{n-1})}\end{aligned}$$

So what we see is that

$$\begin{aligned}\langle q_n,t_n|q_i,t_i\rangle &= \int dq_2\dots dq_{n-1}\,\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle\dots\langle q_2|e^{-iH\delta t}|q_1\rangle \\ &= \int dq_2\dots dq_{n-1}dp_2\dots dp_{n-1}\prod_{i=2}^{n-1} e^{-i\delta t H(q_i,p_i)}e^{ip_i(q_{i+1}-q_i)} \\ &= \int dq_2\dots dq_{n-1}dp_2\dots dp_{n-1}\exp\left\{i\sum_{i=2}^{n-1}\left(\frac{p_i}{\delta t}(q_{i+1}-q_i)-H(q_i,p_i)\right)\delta t\right\}\end{aligned}$$

Now in the limit when $n\to\infty$ then $\delta t\to 0$. This gives us

$$\langle q_n,t_n|q_i,t_i\rangle = \int DqDp\exp\left\{i\int dt\,(p\dot{q}-H=L)\right\}$$

This is a path integral over phase space, not just position(configuration) space. We can reduce this to an integral in configuration space by noting that the integral over phase space is a gaussian integral. Note that

$$\int dp_i e^{i(-p_i^2/2-p_i(q_i+1-q_i)/\delta t)\delta t} = \text{const}\times e^{i\dot{q}_i^2/2}$$

Some comments about this derivation:

(a) $\int Dq$, the integral over a space of functions is not a well defined objectet

(b) This derivation makes manifest ideas of interference and symmetries (e.g. double slit experiment)

(c) Useful in perturbation theory.

(d) Not useful for calculations. Not really necessary in QM but necessary in QFT.

So how do we compute expectation values? Let $Q(t)$ operator that measure $q(t)$.

$$\begin{aligned}\langle q_n, t_n|Q(t)|q_1, t_1\rangle &= \int dq q(t)\langle q_n, t_n|q, t\rangle\langle q, t|q_1, t_1\rangle \\ &= \int_{q(t_1)=q_1}^{q(t_n)=q_n} Dq e^{iS[q]}q(t)\end{aligned}$$

Now what happens if weant to compute a two point function

$$\int Dq e^{iS[q]}q(t)q(t') = \Theta(t'-t)\langle q_n, t_n|Q(t')Q(t)|q_1, t_1\rangle + \Theta(t-t')\langle q_n, t_n|Q(t)Q(t')|q_1, t_1\rangle$$

Namely the thing that the path integral computes is time ordered expectation values. Thus note that if we have many observables

$$\int Dq e^{iS[q]}q_1 \dots q_n = \langle|\mathbf{T}q_1 \dots q_n|\rangle$$

So far we've considered position eigenstates. In a more general state

$$\psi = \int dq\, |q\rangle\langle q|\psi\rangle$$

To compute VEV:

$$\langle 0|\mathbf{T}(Q_1 \dots Q_n)|0\rangle = \lim_{\substack{t_f\to-\infty \\ t_i\to\infty}} \int dq_i dq_f \psi_0(q_i)\psi_0^*(q_f)\int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS}q_1 \dots q_n$$

Theres' a trick to avoid this

$$|q, t\rangle = \sum_{n=0}^{\infty}|n\rangle\langle n|q, t=0\rangle e^{iE_n t}$$

Now let $t \to (1-i\epsilon)t$ and $t \to -\infty$ (very early times). Then $e^{iE_n t} \to e^{i(1-i\epsilon)tE_n}$. Thus we will pick out terms near the vacuum energy. Thus

$$|q, t\rangle \to \langle 0|q, 0\rangle|0\rangle$$

This analytic continuation projects onto the goround state for ket at $t \to -\infty$ and for bra this projects onto the ground state for $t \to \infty$. So then we can compute the VEV as

$$\langle 0|\mathbf{T}Q_1 \dots Q_n|0\rangle = \int Dq \exp\{iS[q]\}q_1 \dots q_n$$

But where $S = \int d\tilde{t}L$ where $\tilde{t} = (1-i\epsilon)t$. Sometimes we take this analytical continuation where $\tilde{t} = it$. This is a "Euclidean continuation." The reason that this is called "euclidean" comes from the following. Consider the invariant interval in special relativity. Under that analytical continuation

$$ds^2 = dt^2 - d\mathbf{x}^2 \to -(d\tilde{t}^2 + d\mathbf{x}^2)$$

It's useful in finite temperature physics.

Some Points About the Method of Saddle Points

Based on Problem Set 7. We're considering that integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

To derive the first correction we'll expand $f(q)$ around the saddle points $\{q_*\}$ where $f'(q_*)$ is zero.

$$f(q) \approx f(q_*) + (f'(q_*)(q-q_*) = 0) + \frac{f''(q_*)}{2}(q-q_*)^2 + \dots$$

This gives us the integral (after we note that at the saddle point $f''(q_*)$ is negative)

$$Z \approx e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{\lambda f''(q_*)(q-q_*)^2/2} dq = e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{-\lambda |f''(q_*)|(q-q_*)^2/2} dq$$

OwO whats this? *Notices your Gaussian integral.* We can do a change of variables

$$u = \sqrt{\frac{\lambda |f''(q_*)|}{2}}(q-q_*) = C(q-q_*) \Rightarrow du = C dq$$

Now we can evaluate the integral with the standard $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\boxed{Z \approx \sqrt{\frac{2\pi}{\lambda |f''(q_*)|}} e^{\lambda f(q_*)}}$$

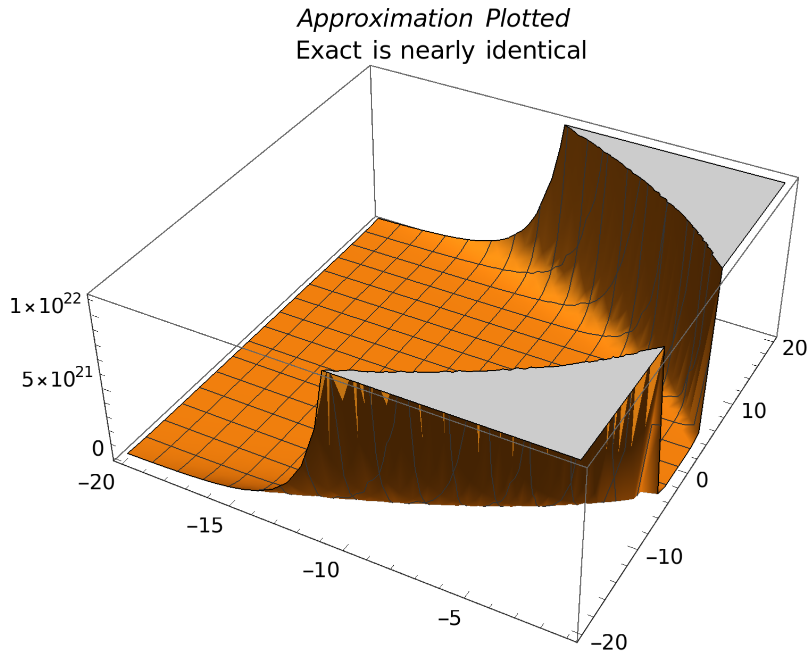
This prefactor we just defined in the context of quantum mechanics is the one-loop correction. Lets see our boy in action. Consider the Gaussian integral where $a < 0$

$$\int_{-\infty}^{\infty} dq\ e^{aq^2+bq+c} = \sqrt{\frac{\pi}{|a|}} e^{c-b^2/4a}$$

We can find the approximation with our saddle point method

$$f'(q_*) = 2aq_* + b = 0 \Rightarrow q_* = -\frac{-b}{2a} \Rightarrow \int_{-\infty}^{\infty} dq\ e^{aq^2+bq+c} \approx \sqrt{\frac{\pi}{|a|}} e^{c+bq+aq^2}$$

Lets fix $c = 42$ and plot both the approximate and the exact answer for some values of a and b just to get a feel for this approximation



It's a good approximation!

Lecture 1: Amputating Bad Legs and Feynman's Cool Trick to Evaluate Integrals

Lets start by considering $\lambda\phi^4$ theory and a $2 \to 2$ scattering. The corresponding Feynman diagrams up to 1-loop level is found in Figure 1 Why are the bad diagrams bad? Well consider

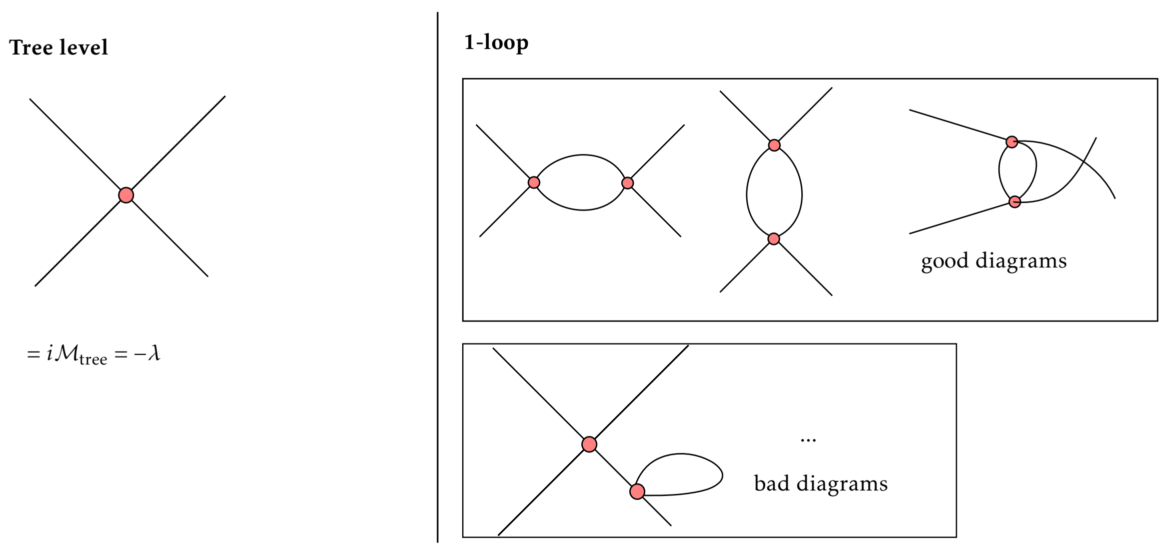


Figure 1: Illustration of good and bad diagrams in a $2 \to 2$ proceess for $\lambda\phi^4$ theory.

one of the bad diagrams

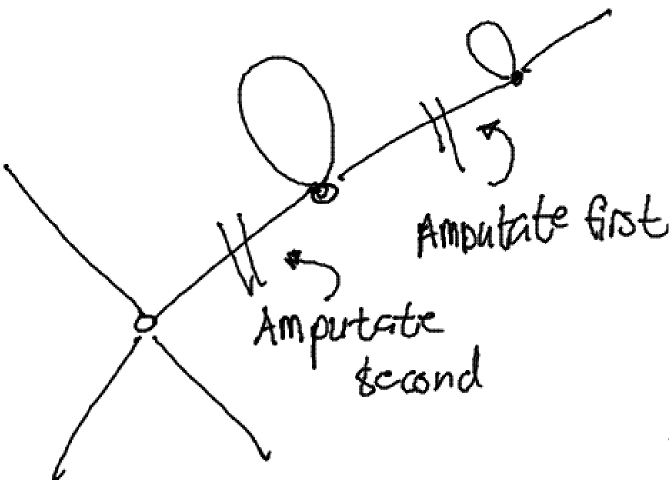
TODO Figure

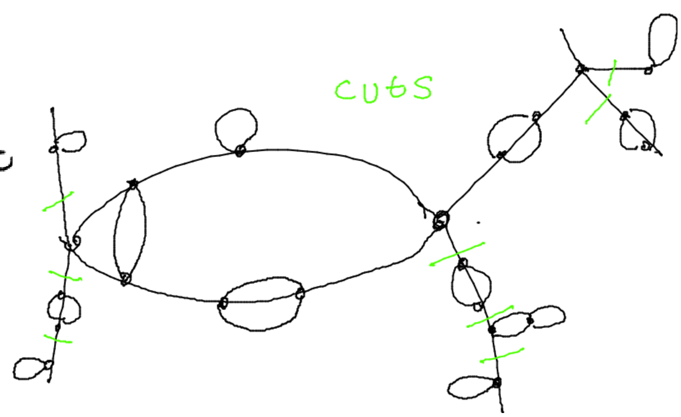
Well $q_2 = p_1$ for any q_1. So the integral becomes

TODO Integral, tl;dr divergence

Namely all of them have bad propagators. Going further we can look at the two-loop example and see that the exact same thing happens. These bad features are called *external leg bubbles*. All diagrams with external leg bubbles are bad, the propagator is frozen on shell and thus blows up.

Lets look at a practical solution. He'll give us a half-assed justification for this solution and then will talk about what's really going on next week. The solution is: *amputate all the external leg bubbles*. To do this we need to figure out how we find these external leg bubbles and the carefully figure out what do we need to amputate. If cutting 1 propagator at a connected diagram breaks it into two disconnected pieces and if one of the pieces has just one external leg then amputate that piece.





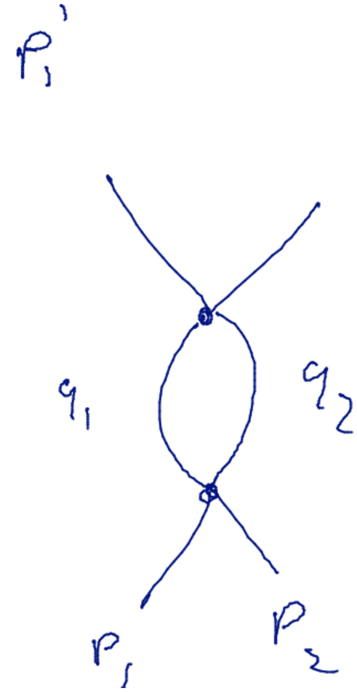
The bottom line is that

$$i\mathcal{M}_{\text{scattering}} = \sum \text{amputated connected diagrams}$$

So where does this rule come from. Well last semester when we were looking at S-matrix elements we were given a formula that was unjustified

$$\langle p_1', p_2', \ldots | \hat{S} | p_1, p_2, \ldots \rangle = \text{Limit regulazation going away} \left[C_{\text{vac}} \times \prod_{\text{ext legs}} F(\text{leg}) \times \langle 0 | a_{p_1'} a_{p_2'} \hat{S} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle \right]$$

Now note that free matrix elemnts is the sum over all Feynmann diagrams. The C_{vac} cancels all vacuum bubbles. What we want to show soon is that $F(\text{ext leg})$ cancels all leg bubbles in that leg. So in the end we're left with diagrams that are unproblematic. This is the "justification" of the no external leg bubble rule. Now the right way to derive this rule is to calculate the correlation function of multiple field, then get (some name) action formula, and then we'll get out result. It turns out that off shell the external leg bubbles are sometimes useful. So now lets go back to the very first diagrams we wrote down and figure out how to calculate loop diagrams.



$$\Rightarrow q_1 + q_2 = p_1 + p_2 = p_1' + p_2'$$

Depends only on the net $p_1 + p_2$ meaning that the matrix element $iF(s)$ only depends on $(p_1 + p_2)^2 = s$. Thus

$$i\mathcal{M}_{1-\text{loop}} = iF(s) + iF(t) + iF(u)$$

By crossing symmetry if we find one of them we find all of them. For example

$$iF(t) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \frac{i}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i\epsilon}$$

Where $q_{\text{net}} = p_1' - p_1 = p_2 - p_2'$ and $q_{\text{net}}^2 = t$. So how do we perform integrals like this? We'll use what's called the *Feynman Parameter Trick*.

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

This is true because $(1-x)A + xB$ interpolates between A and B. This integral was know ages before Feynman. However Feynman made good use of it

$$\frac{q_1^2 - m^2 + i\epsilon}{\times} \frac{1}{q_1^2 - m^2 + i\epsilon} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)]}$$

So looking inside the square bracket we can see that

$$\begin{aligned}
[(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)] &= (1-x)q_1^2 + xq_2^2 - m^2 + i\epsilon \\
&= (1-x)q_1^2 + xq_1^2 - 2x(q_{\text{net}} + q_1) + xq_{\text{net}}^2 - m^2 + i0 \\
&\text{complete the square in the } 2x \text{ term} \\
&= (q_1 - xq_{\text{net}})^2 - x^2 q_{\text{net}}^2 + xq_{\text{net}}^2 + m^2 + i0 \\
&= (q_1 - xq_{\text{net}})^2 - \Delta(x) + i0
\end{aligned}$$

Where we define $\Delta(x) = m^2 + (x^2 - x)q_{\text{net}}^2 = m^2 + (x^2 - x)t$. This means that

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2}$$

So now we can write $iF(t)$ more cleanly

$$\begin{aligned} iF(t) &= \frac{1}{2}\lambda^2 \int \frac{d^4q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2} \\ &= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4q_1}{(2\pi)^2} \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2} \end{aligned}$$

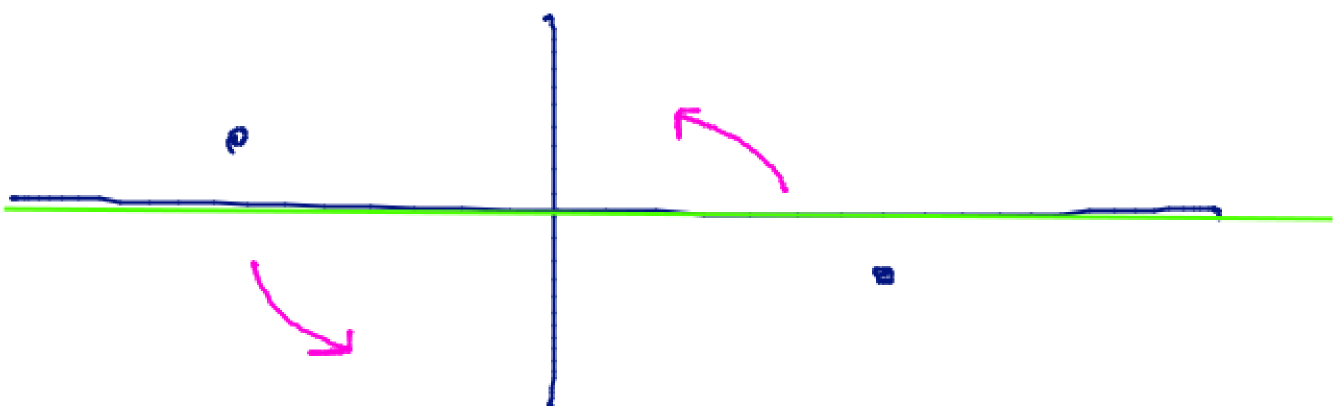
Changing the order of integration is a bit suspect here but at this level it turns out this actually works. Now once we've changed the order of integration for each x change momentum integration variable form q_1 to $k = q_1 - xq_{\text{net}}$. This gives us

$$\boxed{iF(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^2} \frac{1}{[k^2 - \Delta(x) + i0]^2} \qquad \text{where} \qquad \Delta(x) = m^2 - tx(1-x) > 0}$$

Now how do we perform this integral? Me thinks residual. Lets focus on the d^4k integral

$$\int \frac{d^4k}{(2\pi)^2} \frac{1}{[k^2 - \Delta(x) + i0]^2} = \int d^3\mathbf{k} \int dk_0 \frac{1}{[k_0^2 - \mathbf{k}^2 - \Delta(x) + i0]^2}$$

The $\int dk_0$ integral has double poles at $k_0 = \pm[\sqrt{\mathbf{k}^2 + \Delta} - i\epsilon]$. So lets deform the contour in the complex k_0 plane. Also note that as $k_0 \to \infty$ the term is negligible since the integral is about proportional to k_0^{-4}. So we may change something? can't read hand writing. Now we can use Wick rotations.



Note that we rotate CCW since if we rotated CW then we'd hit the poles and they don't like being hit. So let $k_0 = ik_4$ for real k_4 running from $-\infty$ to ∞. So what does this give us?

$$k_\mu k^\mu - \Delta = k_0^2 - \mathbf{k}^2 - \Delta = \underline{-k_4^2 - \mathbf{k}^2} - \Delta$$

Notice the sign is the same in the underlined section. It becomes the "euclidean continuation?." So we can combine the 3-vector $\mathbf{k}$ and k_4 into a euclidean 4-vector $k_E = (k_1, k_2, k_3, k_4)$.So $k^2 =$

$g_{\mu\nu}k^\mu k^\nu = k_0^2 - \mathbf{k}^2$ becomes $-k_E^2 = -k_4^2 - \mathbf{k}^2$. At the same time $dk_0 = idk_4$ so Minkowski $d^4k = id^4k_E$. And with this setup we get the integral

$$\begin{aligned}\int \frac{d^4k_{\text{minkowski}}}{[k^2-\Delta+i0]^2} &= \int d^3\mathbf{k}\int\frac{d^4k_E}{\dots}\\ &= \int d^3\mathbf{k}\int\frac{idk_4}{[-k_4^2-\mathbf{k}^2-\Delta]}\\ &= i\int\frac{d^4k_E}{[k_E^2+\Delta]^2}\end{aligned}$$

Where in the second equality since everything is negative we can get rid of the $i\epsilon$ perscription. The bottom line is

$$\boxed{F(t) = \frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2}}$$

So the $SO^+(3,1)$ Lorentz symmetry of the Minkowski space analytically continues to $SO(4)$ rotation symmetry in $D=4$ euclidean space under a wick rotation by $\pi/2$. We can go even futher and change variables from $d^4k_E = dk_E^{\text{real}}\times(k_E^{\text{rot?}})^3\times d^3\Omega_k$ where Ω is solid angle which means $\int_{D=4} d^3\Omega_k = 2\pi^2$ and so $\int d^4k_E \to \int 2\pi^2 k_E^3 dk_E$ meaning that our integral is

$$\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} = \frac{2\pi^2}{16\pi^4}\int_0^\infty\frac{k_E^2dk_E}{[k_E^2+\Delta]^2}$$

There is one subtlety however. AT $k_E\to\infty$ we know

$$\frac{k_E^3}{[k_E^2+\Delta]^2}\approx\frac{1}{k_E}\Rightarrow\frac{dk_E}{k_E} = d\log k_E \text{ is logarithmically divergent}$$

Maloney QFT I Lecture 14: Feynman Diagrams

In this lecture we finally become men? To recap in QM we can compute observables with the following

$$\langle q_f,t_f|TQ(t_q)\dots Q(t_n)|q_i,t_i\rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS/\hbar}\, q(t_i)\dots q(t_n)$$

So for example if we're doing a double slit experiment and want to measure which slit the the particle goes through we'll insert an operator q in the integral. We can also rephrase this integral as a integral through phase space

$$= \int DqDp\,\exp\left\{i\int(p\dot{q}-H(p,q))dt\right\}q(t_1)\dots q(t_n)$$

In the end of class we introduced a trick to compute correlcation function instead of transition amplitudes. Since

$$e^{it(1-i\epsilon)H}|\psi\rangle = |0\rangle\langle 0|\psi\rangle \text{ at } t\to-\infty$$

Similarly we could apply this to a bra-vector

$$\langle\psi| e^{it(1-i\epsilon)H} = \langle\psi|0\rangle\langle 0| \text{ at } t\to\infty$$

This means if you want to calucate VEV

$$\langle 0|TQ(t_1)\ldots Q(t_n)|0\rangle \propto \int DqDp \exp\left\{i\int(\dot{q}p-(1-i\epsilon)H)dt\right\} q(t_1)\ldots q(t_n)$$

Effectivley what we're doing is adding a small imaginary part to the energies and basically projects us onto the ground state. At the tree level this $i\epsilon$ factor isn't very important but when we do loop calculation this $i\epsilon$ factor becomes important. Something to emphasize the equation above is that we have propto instead of equality since we ignored some normalizations. To get equality we could do

$$\langle 0|\ldots|0\rangle = \frac{\int Dq\; e^{iS/\hbar} q(t_1)\ldots q(t_n)}{\int Dq\; e^{iS/\hbar}}$$

Where the denominator is there to keep $\langle 0|0\rangle = 0$.

To generalize this to many DOF $q_i(t)\to\int Dq_i(t)$. In a QFT where the DOf is a local $\phi(t,\mathbf{x})\to\int D\phi(\mathbf{x},t)$ (integrate over all possible field configurations ϕ). Today our goal is to compute

$$\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{\int D\phi\; e^{iS}\phi(x_1)\ldots\phi(x_n)}{\int D\phi\; e^{-S}}$$

To do this lets define the partition function or generating function of a QFT. We call J the source

$$Z[J(x)] = \int D\phi\; e^{iS+i\int d^4x\; J(x)\phi(x)}$$

It should be clear that if we computed the EOM, $J(x)$ would be a source term on the RHS of the EOM. An if we wanted to we can expand the exponential

$$Z[J(x)] = \int D\phi\; e^{iX}\left(1+i\int d^4xJ(x)\phi(x)-\int d^4x\; d^4y\; J(x)J(y)\phi(x)\phi(y)+\ldots\right)$$

This partition function includes all possible expectation values that we care about. First lets consider how we could get expectation values from this partition function. Well lets remeber how we extract single terms from a power series. We take the derivative wrt the variable and then set that variable to zero. For example $e^x = c_0+c_1x+c_2x^2+\ldots$, $de^x/dx = c_1+2c_2x+\cdots\Rightarrow (de^x/dx)(x=) = c_1$. This generalizes to what we have here. However here we need to take the functional derivative

$$\boxed{\text{Define } \frac{\delta}{\delta J(x)} \text{ as } \frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)}$$

More compliacted derivatives are defined using this formula and the chain rule

$$\frac{\delta V(\phi(y))}{\delta\phi(x)} = \frac{\partial V(\phi(y))}{\partial\phi(x)}\frac{\delta\phi(y)}{\delta\phi(x)} = V'(\phi(y))\delta^{(4)}(x-y)$$

Now we just need to put these two things together

$$Z[J] = \int D\phi \; e^{iS+i\int J(x)\phi(x)d^4x}$$

So if we want the VEV we compute

$$\boxed{\langle 0|T\phi(x_1)\ldots\phi(x_n)|0\rangle = \frac{(-i)^n}{Z}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\Big|_{J=0}}$$

Lets look at an example, free QFT $S = -\frac{1}{2}\int d^4x\; \phi(\partial^2+m^2)\phi$. This means that

$$Z[J] = \int D\phi \exp\left\{-\frac{i}{2}\int d^4\; \phi(x)(\partial^2+m^2)\phi(x) + i\int d^4x\; J(x)\phi(x)\right\}$$

Lets consider a simpler but illuminating problem

$$\int d\phi \exp\left(\phi A\phi + j\phi\right) \approx \exp\left(jA^{-1}j\right)$$

Where the second $\approx$ comes from shifting ϕ to complete the square. For multidimensional integral we get (TODO PSET)

$$Z[J]\int d\phi_i \exp\left(\phi_i A^{ij}\phi_j + j^i\phi_i\right) \approx \exp(j^i(A^{-1})_{ij}j^j)$$

Lets think of a discrete model of QFT (e.g. space is lattice). We can identify A^{ij} with (∂^2+m^2). We also met that $A^{-1} = (\partial^2+m^2)^{-1} = D_F$, the feynman propagator. To compute the integral we can use fourier transform. One thing we can note is that the kinetic operator (∂^2+m^2) is matrix that is diagonal in momentum space. What does that mean? Lets take our $\phi(x)$ and think about it in momentum space

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\tilde{\phi}(k)$$

And lets change variables so that our path integral isn't over all configurations of $\phi(x)$ but all possible fourier components of $\tilde{\phi}(k)$.

$$\int D\phi(x) \to \int D\tilde{\phi}(k)$$

To think about this more clearly we can consider the integral over all field configurations of $\phi(x)$ as the integral over all possible values of x (π_x) in some one dimensional integral (? TODO huh)

$$\int \pi_x d\phi(x) \to \int \pi_k d\tilde{\phi}(k)$$

Whenever we do a change of variables we have a jacobian determinent. Here it's not too hard to compute since we're going from ϕ to a linear combination of ϕ. But we're just gonna ignore it since it's some overall constant for now. So what does the integrand look like in fourier space.

$$\begin{aligned}\int d^4x\; \phi(x)(\partial^2+m^2)\phi(x) &= \int d^4x \int d^4k d^4k'\; e^{-ikx}\tilde{\phi}(k)(\partial^2+m^2 = -k'^2+m^2)e^{-ik'x}\tilde{\phi}(k') \\ &= \int d^4k\; \tilde{\phi}(k)(-k^2+m^2)\tilde{\phi}(-k)\end{aligned}$$

This is what we mean when we say taht the kinetic operator is diagonal in mometnum space. We also have

$$\int d^4x\, J(x)\phi(x) = \int d^4k \tilde{J}(k)\tilde{\phi}(-k) = \frac{1}{2}\int d^4k\left(\tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right)$$

This means we have

$$Z[J] = \int D\tilde{\phi}(k)\,\exp\left\{\frac{i}{2}\int d^4k(\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right\}$$

We can separate out the integrand into the product of a bunch of one dimensional integrals over each independent momentum mode

$$Z[J] = \prod_k \int d\phi(k)\,\exp\left\{\frac{i}{2}\left(\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\phi(k)\right)\right\}$$

That's just a one dimensional gaussian integral. Let $\chi(k) = \tilde{\phi}(k) + \tilde{J}(k)/(k^2-m^2)$. this means

$$Z[J] = \prod_k \int d\chi(k)\,\exp\left\{\frac{i}{2}\chi(k)\chi(-k)(k^2-m^2) + \frac{i}{2}\tilde{J}(k)\tilde{J}(-k)/k^2-m^2)\right\}$$

So we can then write

$$Z[J] = Z[0]\text{exp}\left\{\frac{i}{2}\int d^4k\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2}\right\}$$

If we then do the inverse fourier transform we now have

$$Z[J] = Z[0]\text{exp}\left\{\frac{1}{2}\int d^4xd^4y\; J(x)D(x-y)J(y)\right\}$$

Where $D(x-y) = \int d^4k\; e^{ik(x-y)}\frac{i}{k^2-m^2+i\epsilon}$ is the feynman propagator. Now lets try to compute a two-point function

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = (-i)^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\left(1 - \frac{1}{2}\int d^4xd^4y\; J(x)J(y)D(x-y) + \dots\right)\Bigg|_{J=0}$$

Applying chain rule we just get

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D(x-y)$$

In fact this is a more general result for any free QFT which is

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \sum_{\text{pairings}}(\text{product of free propagators})$$

So for example

$$\langle 0|\phi(x_1)\dots\phi(x_4)|0\rangle = D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3)$$

This is Wick's theorem. Lets introduce a graphical notation to keep track of things. First recall

$$Z[J] = \exp\left(-\frac{1}{2}\int d^4xd^4y\; J(x)D(x-y)J(y)\right)$$

When we introduce interactions what we'll end up with is a more complex diagramatic set of rules. e.g. for a ϕ^3 theory we ahve new sets of ingridients.

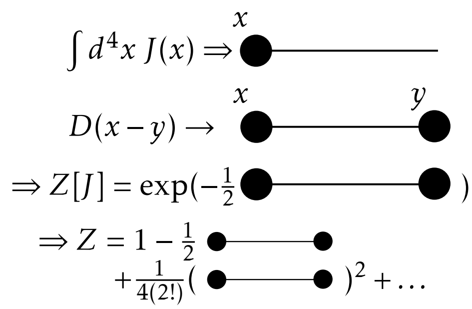


Figure 2: Graphical represntation of partition function

Lecture 2:

At the very end of the last lecture for the two scalar scaterring we found out that the momentum integral suffers from ultraviolet divergence. Divergences like that are all over the place in QFT. So people started to develop a renormalization procedure which deals with diffent physics couplings. All this machinary however has a nasty flavor of sweeping infinities under the carpet. It wasn't very clear what the hell was going on. Even though it worked perturbtivley well for $\lambda\phi^4$ and QED it didn't work for some other theories and doesn't work non-perturbativley. So in the 1950s there was a backlash which lead to alternates like the bootstrap program (aparently this name has something to do with lifting yourself up by your bootstraps?) What really comes to understanding renormalization is by undersatnding effective field theories. This paradigm came from two sides. First there were some people studying critical phenomena in condensed matter. When you're working near a critical point you have long distance correlations. Pretty soon those people realized the behavior at long distances doesn't really care about short distance details. It doesn't matter what kind of lattice we're dealing with (except for maybe some lattice contant). The knowledge of short distance could be summarized in a few parameters. So what Ken Wilson and friends figured out is that you want to worry about effective long distance theories and ignore short distance behavior. The other approach came from a completley different end: current algebra. People studying low-energy interaction between π-mesons and nucleons noticed that there was partial conservation of axial current(fancy spontaneously broken symmetry). They then wrote currents of this broken symmetry and then Weinberg and company figured out that the best way to understand what goes on with current algebra is use effective field theory for low energy(cut all meson zoo except pion.) And what came the big understanding: what we're doing in real life is an effective field theory. We know particles up to some GeV and we don't know anything above that scale. We don't know what kind of particles might occur at 10 TeV. If we had a full QFT then we'd need to include those particles. Even if we say something silly like "there is nothing beyond the standard model" we still have to still consider quantum gravity which becomes relevant at 10^{-19} GeV. So the only thing we can do is effective field theory, a theory which is only known up to some cutoff. So the best we can do is parametrize our ignorance of whats beyond what we know. And in practice this translates to is cutting off ultraviolet modes by saying "we don't know what happens before this" and just say there is literally nothing. And you emulate this nothing with a UV cutoff. And then we paramterize our ignorance by introducing bear(?) couplings that aren't actually measurable experimentailly but whatever gives us the right answer. Fortunatley there is only a few parameters we need to adjust. Before we do any examples lets revisit something we should all know. Lets consider the Debye model of solids.

Aside: Debye Model of Solids

What is relevant to low energies?

(a) For $|\mathbf{k}| \ll 1/(\text{lattice spacing})$ we know that $\omega(k) = c_s|\mathbf{k}|$ where c_s is the speed of sound.

(b) Finite volume of $\mathbf{k}$ space. In 3D we have $\mathbf{k}$ modulo inverse lattice times 2π and k belongs to a 3-torous.

So what did Debye do? He approximated $\omega(\mathbf{k}) = c_s|\mathbf{k}|$ exactly and k-space ball of radius θ/c_s (θ is the Debye temperature?) meaning that

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d^3k}{(2\pi)^3}$$

Up to $|\mathbf{k}| = \theta/c_s$. So θ acts like kind of a cutoff for the condensed matter theory. In QFT we introudce a UV cutoff λ that will shove all the gory details of the UV physics in some redefinition.

Back to 1-loop correction in $\lambda\phi^4$ theory

We found that for some diagram

$$F(t) = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \qquad \Delta(x) = m^2 - tx(1-x) > 0$$

We want to evaluate

$$\begin{aligned} I &= \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \\ &= \frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{dk_E\ k_E^3}{[k_E^2+\Delta]^2} \end{aligned}$$

Now we set UV cutoff: cut the integral at $|k_E| = \Lambda$ for $\Lambda \gg$ what?.

$$I_{\text{regulated}} = \frac{1}{2\pi^2}\int_0^\Lambda \frac{dk_E k_E^2}{[k_E^2+\Delta]^2}$$

Let $\nu = k_E^2 + \Delta$ meaning that

$$I_{\text{regulated}} = \frac{1}{8\pi^2}\times\frac{1}{2}\int_\Delta^{\Delta+\Lambda^2}\frac{d\nu(\nu-\Delta)}{\nu^2} = \frac{1}{16\pi^2}\left(\log\nu + \frac{\Delta}{\nu}\right)\Bigg|_\Delta^{\Delta+\Lambda} = \frac{1}{16\pi^2}\Big(\log[(\Delta+\Lambda^2)/\Delta] + \Delta/(\Delta+\Lambda^2) - 1\Big)$$

Now we take an assumption that $\Lambda^2 \gg m^2$ or q_{net}^2 and therefoer we can neglect all negative powers of Λ. So we can say

$$\log[(\Lambda^2+\Delta)/\Delta] = \log(\Lambda^2/\Delta) + \frac{\Delta}{\Lambda^2} - \frac{\Delta^2}{2\Lambda^2} + \cdots \approx \log(\Lambda^2/\Delta)$$

$$\frac{\Delta}{\Delta+\Lambda^2}-1\approx -1$$

This all gives us

$$I_{\text{regulated}}=\frac{1}{16\pi^2}\left(\log(\Lambda^2/\Delta)-1\right)\Rightarrow F(t)=\frac{\lambda^2}{32\pi^2}\int_0^1 dx\ (\log(\Lambda^2/\Delta(x))-1)$$

To do this integral we just do $\log(\Lambda^2/\Delta(x))=\log(\Lambda^2/m^2)-\log(\Delta/m^2=1-tx(1-x)/^2)$. This means

$$F(t)=\frac{\lambda^2}{32\pi^2}\left[\log(\lambda^2/m^2)-1-J(\lambda/m^2)\right]\qquad J(t/m^2)=\int_0^1 dx\ \log(1-tx(1-x)/m^2)$$

There are three diagrams at the 1-loop level (s,t,u) shown in Figure 1 meaning that

$$\mathcal{M}_{1-\text{loop}}=F(t)+F(s)+F(u)$$

Which can be calculated with crossing symmetry.

$$\mathcal{M}(s,t,u)=-\lambda+\frac{\lambda^2}{32\pi^2?}\left[3\log(\lambda^2/m^2)-3-J(t/m^2)-J(u/m^2)-J(s/m^2)\right]+O(\lambda^3)$$

So how do we deal with infinite parameters. Here's an overview of the renormalization procedure. λ in the above formula is the *bare* coupling in the bare lagrangian of the theory. This λ_{bare} is not directly measured by any experiment. We adjust it as needed so tah the perturbative amplitudes we calulate fit experimental data. To renormalize our procdure goes like this

(1) Start by definiting physcia coupling λ_{phys} in terms of some scattering amptliude. For example $\lambda_{\text{phys}}=-\mathcal{M}_{\text{elastic}}$ at the threshold.

(2) Use perturbation theory (feynmann graphs?) to calculate $\lambda_{\text{physical}}$ as a power series in λ_{bare}.

$$\lambda_{\text{phys}}=\lambda_{\text{bare}}+A_1\lambda_b^2+A_2\lambda_b^3+A_3\lambda_b^4+\ldots$$

Note that $A_1,A_2,A_3,\ldots$ depend on $\log\Lambda_{\text{UV}}$. Now formally assume that not only λ_{bare} is small but alos $\lambda_{\text{bare}}\times\log\Lambda_{\text{UV}}$ is small so this perturbative theory makes sense. Do formal perturbation theory in λ_{bare}.

(3) Reverse the power series for λ_{phys}

$$\Rightarrow \lambda_{\text{bare}}=\lambda_{\text{phys}}+B_1\lambda_{\text{phys}}^1+B_2\lambda_{\text{p}}^3+\ldots$$

This tells us $B_1=-A_1$ and $B_2=2A_1^2-A^2$ and so on.

$$\Rightarrow \lambda_{\text{bare}}=\lambda_p-A_1\lambda_p^2+(2A_1^2-A_2)\lambda_p^3+\ldots$$

(4) For any interesting ampltiude $\mathcal{M}$(kinematical params), use feynman graphs to calculate $\mathcal{M}$

$$\mathcal{M}(\text{kine. para.})=\lambda_b^n\mathcal{M}_0+\lambda_b^{n+1}\mathcal{M}_1+\lambda_b^{n+2}\mathcal{M}_2+\ldots$$

(5) Re-expand in terms of $\lambda_b = \lambda_{\text{phys}} + B_1\lambda_{\text{phys}}^2 + B_2\lambda_{\text{phys}}^3 + \dots$. This means that

$$\begin{aligned}\mathcal{M} = \mathcal{M}_0 \times \lambda_{\text{phys}}^n&(1 + nB_1\lambda_{\text{phys}} + n(n-1)/2B_1^2\lambda_{\text{phys}}^2 + nB_2\lambda_{\text{phys}}^2 \dots)\\ &+ \lambda_{\text{phys}}^{n+1}(1 + (n+1)B_1\lambda_{\text{phys}} + \dots)\\ &+ \lambda_{\text{phys}}^{n+2}(1 + \dots) + \dots\\ &= \lambda_{\text{phys}}^n\mathcal{M}_0 + \lambda_{\text{phys}}^{n+1}[nB_1\mathcal{M}_0 + \mathcal{M}_1]\\ &+ \lambda_{\text{phys}}^2[(n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0 + (n+1)B_1\mathcal{M}_1 + \mathcal{M}_2] + \dots\end{aligned}$$

The point of all this is to see that we get a power series in λ_{phys}.

(6) In the power series in λ_{phys} the dependence on $\log\Lambda_{\text{UV}}$ cancels out from each term. That is $B_1, B_2, B_3\dots$ depend on $\log\Lambda$. $\mathcal{M}_1, \mathcal{M}_2, \dots$ depend onf $\log\Lambda$. But $\mathcal{M} - 1 + nB_1\mathcal{M}_0$ is independent of $\log\Lambda_{\text{UV}}$ and likewise $\mathcal{M}_2 + (n+1)B_1\mathcal{M}_1 + (n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0$ is indpendent of $\log\Lambda$.

And that's the renormalization procedure. Lets look at an example for elastic scattering

1-loop calculation for Elastic Scattering

First we define

$$\lambda_{\text{phys}} = -\mathcal{M}\text{@threshold } t = u = 0 \text{ and } s = 4m^2$$

And thus from EQREFHERE we get

$$\lambda_{\text{b}} = \lambda_{\text{phys}} + \frac{\lambda_b^2}{32\pi^2}\Big(3\log\Lambda^2/m^2 - 3 - 2(J(0) = 0) - J(4)\Big)$$

And when we rewrite $\mathcal{M}(s,t,u)$ in terms of λ_{phys} we get

$$\mathcal{M}(s,t,u) = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\Big(J(4) - J(s/m^2) - J(t/m^2) - J(u/m^2)\Big)$$

And that's how the renormalization theoyr works. Now he focused on λ but in reality there is also a renormalization of mass $m_b \neq m_{\text{phys}}$. Also nowaways they do perturbation theory with counterturm which directly reorganizes perturbation theory in terms of $\lambda_{\text{phys}}, m_{\text{phys}}^2, \dots$ directly at the level of feynman graphs.