PHY 396L: Quantum Field Theory II Notes

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Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. Also there will also be a lot of stuff in here from Prof. Maloney's QFT I and QFT II course here since I like how he explains things. If you have any comments let me know at `hi@delonshen.com`.

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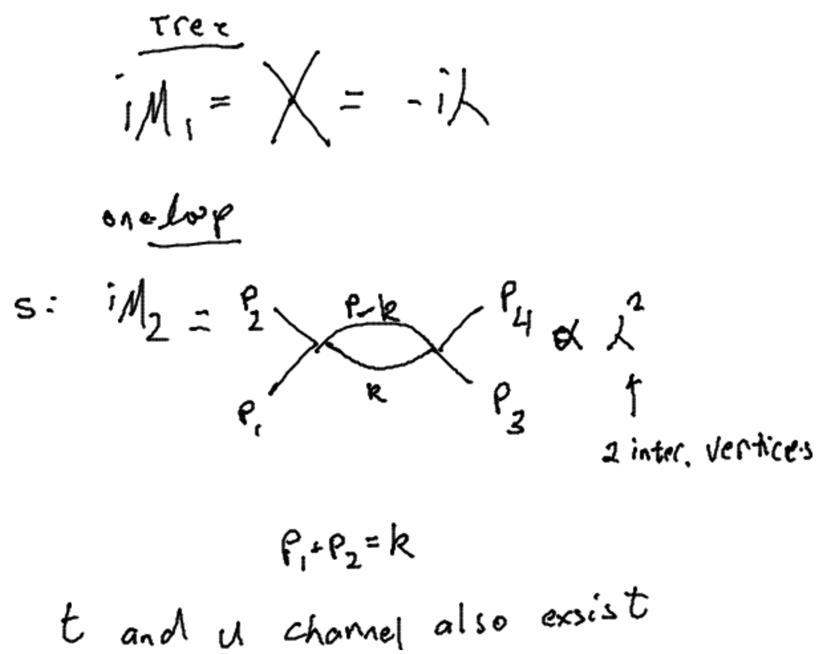
Maloney QFT II Lecture 1: Systematics of Renormalization

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence lets consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \to \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$



Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges(TODO Maloney says "logarithmically divergent"[1]). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2\log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2}\log s/\Lambda^2 + \ldots$$

This seems like a disaster since as the cutoff $\Lambda \to 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

[1]Okay so googling leads to this. $dU/U = d\log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase $\mathcal{M}$ in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0.

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2}\log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}\log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for $\mathcal{M}$ above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \to 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2. So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in $\mathcal{L}$ we introduce a counterterm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

Now if we want to shfit this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

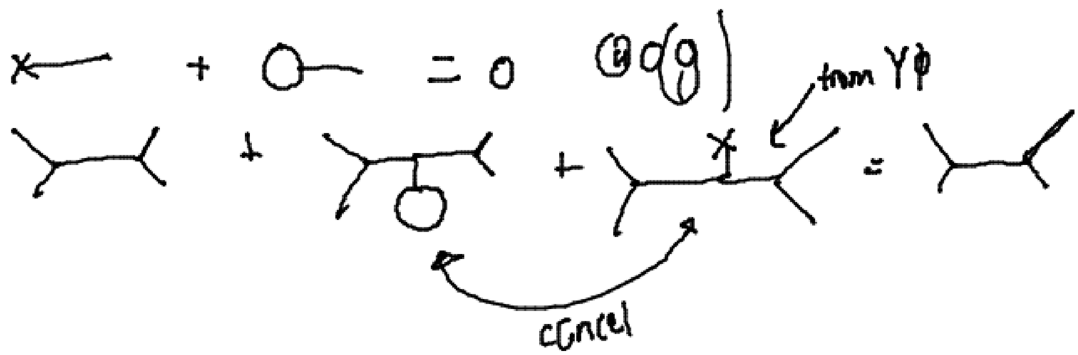
(a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$

(b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$

(c) Z_g fixed by g = physical 3-pt funct at some energy

(d) Z_m fixed by the (mass)^2 of a one particle state should be m^2. The physical mass is not necessarily m^2.

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g. For example Y is fixed by the cancellation of the one point function. The existance of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.



For other counterterms we have to do some computations.