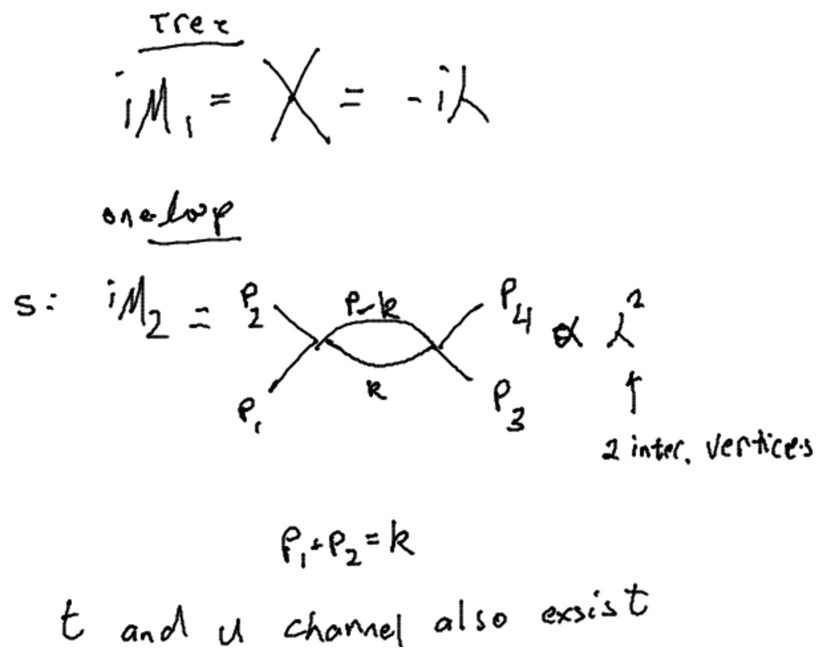
PHY 396L: Quantum Field Theory II

Delon Shen

Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. The lilac sections are better put together and proofread than other parts of the notes since they're from me working through problems on my own or reading notes and not from me furiously trying to live-tex notes in vim. Also there will be some stuff in here from Prof. Maloney's QFT I and QFT II course. If you have any comments let me know at `hi@delonshen.com`.

Maloney QFT II Lecture 1: Systematics of Renormalization

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence lets consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \to \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$



Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges(TODO Maloney says "logarithmically divergent"[1]). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2 \log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2}\log s/\Lambda^2 + \ldots$$

This seems like a disaster since as the cutoff $\Lambda \to 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

[1] Okay so googling leads to this. $dU/U = d\log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase $\mathcal{M}$ in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0.

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2}\log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}\log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for $\mathcal{M}$ above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \to 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2. So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in $\mathcal{L}$ we introduce a counterterm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

Now if we want to shfit this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

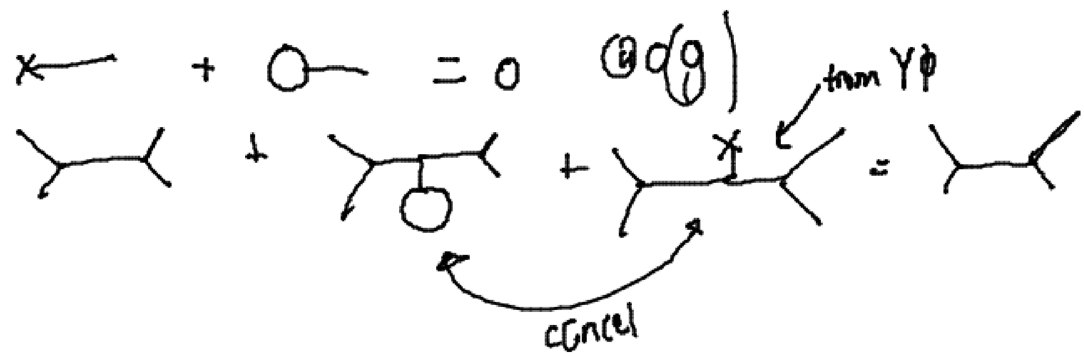
(a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$

(b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$

(c) Z_g fixed by g = physical 3-pt funct at some energy

(d) Z_m fixed by the (mass)^2 of a one particle state should be m^2. The physical mass is not necessarily m^2.

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g. For example Y is fixed by the cancellation of the one point function. The existance of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.



For other counterterms we have to do some computations.

Maloney QFT I Lecture 12 Part 1: Interactions

A free QFT where $S[\phi]$ is quadratice in ϕ has no interactions. Consider some sort of free theory with two kinds of particles a and b. Schematically lets have some Hamiltonian

$$H = a^\dagger a + b^\dagger b \Rightarrow a^\dagger|0\rangle \xrightarrow{e^{iHt}} a^\dagger|0\rangle \times \text{phase}$$

Basically for a free theory if you start with particle a, no matter how much time passes you'll still always have that particle a. For an interacting theory

$$H = a^\dagger a + b^\dagger b + \lambda(a(b^\dagger)^2 + a^\dagger b^2)$$

Where λ is small. The term in the parenthesis is the interaction term where the first term destroys the a particle and creates two b particle and the second term destroys the two b particles and gives us back the a particle(hermitian means process is reversible which is why we have both terms instead of just one term.) Solving this exactly should be a disaster but we can use approximation methods. First just by taylor expanding we can see that

$$a^\dagger|0\rangle \to e^{iHt}(a^\dagger|0\rangle) \approx a^\dagger|0\rangle + i\lambda t(b^\dagger)^2|0\rangle + \ldots$$

Basically there's some probably per unit time where a particle becomes two b particles. What we wanted to show here is that the non-quadratic terms that create interactions. There should be some things we should take away

(a) The vacuum state of $|0\rangle_{\text{free}}$ no longer works since. We defined this state as the state that's annihilated by lowering operators. However this is no longer the case. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$. The true vacuum state $|\Omega\rangle$ is more complex in interacting theories.

(b) We shouldn't think of a and $a^\dagger$ as creating/destroying particles anymore.

To make things explicit lets consider a typical interacting theory

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \mathcal{L}_{\text{int}} \Leftrightarrow \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Expanding $V(\phi)$ around $\phi = 0$ we get

$$V(\phi) = V_0 + V_1\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

V_0 is ignorable. We can also let $\phi \to \phi + \text{const}$ so that $V_1 = 0$ Consider $\phi(x^\mu)$ where ϕ is slowly varying of $\mathbf{x}$.

$$\Rightarrow \mathcal{L} \approx \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Energy is minimized when ϕ sits at the minimum of $V(\phi)$. This means the ground state or "vacuum" where $V'(\phi_*) = 0$. So if $V_1 \neq 0$ then the field will realx to a value of $V(\phi)$ with $V'(\phi) = 0$ and if we want to study the properties near this vacuum we have a new field $\phi' = \phi - \phi_*$. So for now lets just choose vacuum so that in the vacuum $\phi = 0$. e.g. $\langle\Omega|\phi|\Omega\rangle = 0$.

Question: what interaction terms matter? Why should we care about ϕ^3 and not ϕ^{100}? We can answer this question with dimensional analysis

$$S = \int d^4x((\partial\phi)^2 - m^2\phi^2 + \cdots + \lambda_{ij}\partial^j\phi^i)$$

First we know in natural units $[x] = E^{-1}$ and thus $[\partial/\partial x] = E$. First action is dimensionless. Thus $[\phi] = E$ ($[d^4x\partial_x^2] = E^{-2}$). We can also see that $[m] = E$ and $[\lambda_{ij}] = E^{4-i-j}$. Now imagine we're doing some experiment which probes the theory at scale E. For example $\phi\phi \to \phi\phi$ with COM energy $\approx E$. What physical effects would come from our λ_{ij} term? well $\lambda_{ij}E^{i+j-4} = \lambda_{ij}E^{-[\lambda_{ij}]}$. What this means is that at low energies. we should only care about $[\lambda] > 0$ since if $[\lambda < 0]$ then E is raised to a positive power and thus is exponentially decreasing.

(a) $[\lambda] > 0$ is relevant

(b) $[\lambda] < 0$ is irrelevant

(c) $[\lambda] = 0$ is marginal

For a scalar theory at low energies the only relevant terms are kinetic terms and ϕ^3 and ϕ^4. Here's an idea from Landau: To study a system at low energy we can follow a recipe

(a) Guess DOF

(b) Guess symmetries

(c) Write down the most general theory (e.g. action)that satisfies the symmetries

(d) Study the relevant interaction terms.

To constrain our theory futher we should also ask what sort of symmetries we have? For example for the symmetry $\phi \to -\phi$ then there can't be a ϕ^3 term. We call this theory $\lambda\phi^4$ theory.

Maloney QFT I Lecture 13: Path Integrals

Thought I should take a look at this since QFT I at UT didn't approach Feynmann diagrams from path integrals.

We have some QM system with DOF q_i (e.g. for a field theory $\phi(\mathbf{x})$). We want to compute the transition from some inital time t_i with some configuration q_i to the final state (q_f, t_f).

$$\langle q_f, t_f | q_i, t_i \rangle$$

So we use the fact that we can always insert a complete set of basis states $|q,t\rangle$ at some intermediate time t where $t \in (t_i, t_f)$. So what happens? Let $|q_i, t_i\rangle = q_1, t_1$ and $|q_f, t_f\rangle = |q_n, t_n\rangle$. This gives us

$$\begin{aligned}\langle q_f, t_f | q_i, t_i \rangle &= \int dq \, \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \\ &= \int dq_2 \dots dq_{n-1} \, \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \dots \langle q_2, t_2 | q_1, t_1 \rangle\end{aligned}$$

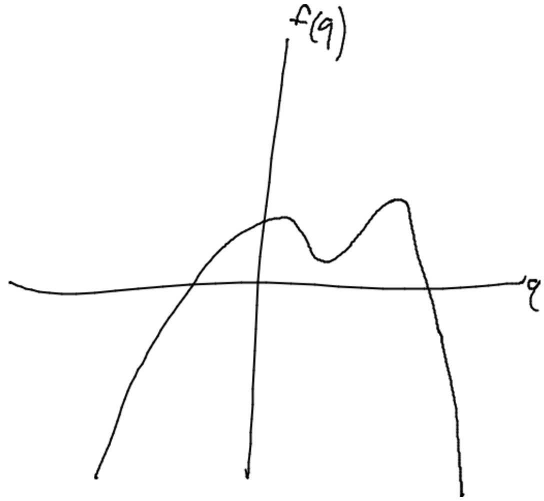
Basically we're integrating over all possible paths (including unphysical paths) from the inital state to the final state. So we can think of this transition amplitude as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq(t)$$

To make this idea precise we need to figure out how to integrate over a space of functions. The next thing we need to do is figure out what the fuck we're integrating. First lets just guess. In the limit $\hbar \to 0$ the integrand should be a function only of the classical solution of the EOM $\delta S/\delta q = 0$. First lets take a detour into approximating finite dimensional integrals which can be generalized to infinite dimensional integrals: the method of saddle points. Consider the one dimensional integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

And we want the integral to converge so f should look something like.



And we want to perform this integral when $\lambda \to \infty$. In this limit the integral should be dominated by the local maxima of $f(q)$. Namely by the q_* where $f'(q_*) = 0$. And what is this function Z in the limit where λ is large?

$$Z = \sum_{q_*} e^{\lambda f(q_*)}$$

It turns out there are a bunch of subleading corrections which are

$$Z = \sum_{q_*} e^{\lambda f(q_*)} \left(\sqrt{\frac{2\pi}{\lambda f''(q_*)}} + \dots \right)$$

(I proved first term in the next subsection). The higher order corrections are basically from Feynmann diagrams. Comparing this method of saddle point with the problem of path integrals we can guess that the integrand should be a function of $\frac{1}{\hbar}S$ (since the integrand does to infinity as $\hbar \to 0$ meaning that only things that solve $\delta S/\delta Q = 0$ contribute like in the method of steepest descent) and we can guess that

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq(t) \exp\left\{ \frac{i}{\hbar} S[q(t)] \right\}$$

Lets just check our work. Consider $H = \frac{1}{2}p^2 + V(q)$. Then

$$e^{-iH\delta t} = e^{-i(p^2/2+V(q))t} = e^{-i\frac{1}{2}p^2\delta t} e^{-iV(q)\delta t} e^{O(\delta t^2)}$$

Where the second equality comes from Cambell-Backer-Houstouder?? formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]+\dots}$$

Now lets insert an identity again in the following.

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q)\delta t} | q_{n-1} \rangle$$

Now we can use that $|p_{n-1}\rangle$ is a momentum eigenstate and $|q_{n-1}\rangle$ is a position eigenstate to replace the operators in the exponential with eigenvalues

$$\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle = \int dp_{n-1}\,\langle q_n|e^{-i\delta t p_{n-1}^2/2}|p_{n-1}\rangle\langle p_{n-1}|e^{-iV(q_{n-1})\delta t}|q_{n-1}\rangle$$

Taking out the identity we now just have

$$\begin{aligned}\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle &= \int dp_{n-1}e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))}\langle q_n|p_{n-1}\rangle\langle p_{n-1}|q_{n-1}\rangle\\ &= \int dp_{n-1}e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))}e^{ip_{n-1}(q_n-q_{n-1})}\\ &= \int dp_{n-1}e^{-i\delta t H(q_{n-1},p_{n-1})}e^{ip_{n-1}(q_n-q_{n-1})}\end{aligned}$$

So what we see is that

$$\begin{aligned}\langle q_n,t_n|q_i,t_i\rangle &= \int dq_2\ldots dq_{n-1}\,\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle\ldots\langle q_2|e^{-iH\delta t}|q_1\rangle\\ &= \int dq_2\ldots dq_{n-1}dp_2\ldots dp_{n-1}\prod_{i=2}^{n-1}e^{-i\delta t H(q_i,p_i)}e^{ip_i(q_{i+1}-q_i)}\\ &= \int dq_2\ldots dq_{n-1}dp_2\ldots dp_{n-1}\exp\left\{i\sum_{i=2}^{n-1}\left(\frac{p_i}{\delta t}(q_{i+1}-q_i)-H(q_i,p_i)\right)\delta t\right\}\end{aligned}$$

Now in the limit when $n\to\infty$ then $\delta t\to 0$. This gives us

$$\langle q_n,t_n|q_i,t_i\rangle = \int DqDp\exp\left\{i\int dt\,(p\dot{q}-H=L)\right\}$$

This is a path integral over phase space, not just position(configuration) space. We can reduce this to an integral in configuration space by noting that the integral over phase space is a gaussian integral. Note that

$$\int dp_i e^{i(-p_i^2/2-p_i(q_i+1-q_i)/\delta t)\delta t} = \text{const}\times e^{i\dot{q}_i^2/2}$$

Some comments about this derivation:

(a) $\int Dq$, the integral over a space of functions is not a well defined objectet

(b) This derivation makes manifest ideas of interference and symmetries (e.g. double slit experiment)

(c) Useful in perturbation theory.

(d) Not useful for calculations. Not really necessary in QM but necessary in QFT.

So how do we compute expectation values? Let $Q(t)$ operator that measure $q(t)$.

$$\begin{aligned}\langle q_n, t_n|Q(t)|q_1, t_1\rangle &= \int dq q(t)\langle q_n, t_n|q, t\rangle\langle q, t|q_1, t_1\rangle \\ &= \int_{q(t_1)=q_1}^{q(t_n)=q_n} Dqe^{iS[q]}q(t)\end{aligned}$$

Now what happens if weant to compute a two point function

$$\int Dqe^{iS[q]}q(t)q(t') = \Theta(t'-t)\langle q_n, t_n|Q(t')Q(t)|q_1, t_1\rangle + \Theta(t-t')\langle q_n, t_n|Q(t)Q(t')|q_1, t_1\rangle$$

Namely the thing that the path integral computes is time ordered expectation values. Thus note that if we have many observables

$$\int Dqe^{iS[q]}q_1 \dots q_n = \langle|\mathbf{T}q_1 \dots q_n|\rangle$$

So far we've considered position eigenstates. In a more general state

$$\psi = \int dq\,|q\rangle\langle q|\psi\rangle$$

To compute VEV:

$$\langle 0|\mathbf{T}(Q_1 \dots Q_n)|0\rangle = \lim_{\substack{t_f\to-\infty \\ t_i\to\infty}} \int dq_i dq_f \psi_0(q_i)\psi_0^*(q_f)\int_{q(t_i)=q_i}^{q(t_f)=q_f} Dqe^{iS}q_1 \dots q_n$$

Theres' a trick to avoid this

$$|q, t\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|q, t=0\rangle e^{iE_n t}$$

Now let $t \to (1-i\epsilon)t$ and $t \to -\infty$ (very early times). Then $e^{iE_n t} \to e^{i(1-i\epsilon)tE_n}$. Thus we will pick out terms near the vacuum energy. Thus

$$|q, t\rangle \to \langle 0|q, 0\rangle|0\rangle$$

This analytic continuation projects onto the goround state for ket at $t \to -\infty$ and for bra this projects onto the ground state for $t \to \infty$. So then we can compute the VEV as

$$\langle 0|\mathbf{T}Q_1 \dots Q_n|0\rangle = \int Dq\exp\{iS[q]\}q_1 \dots q_n$$

But where $S = \int d\tilde{t}L$ where $\tilde{t} = (1-i\epsilon)t$. Sometimes we take this analytical continuation where $\tilde{t} = it$. This is a "Euclidean continuation." The reason that this is called "euclidean" comes from the following. Consider the invariant interval in special relativity. Under that analytical continuation

$$ds^2 = dt^2 - d\mathbf{x}^2 \to -(d\tilde{t}^2 + d\mathbf{x}^2)$$

It's useful in finite temperature physics.

Some Points About the Method of Saddle Points

Based on Problem Set 7. We're considering that integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

To derive the first correction we'll expand $f(q)$ around the saddle points $\{q_*\}$ where $f'(q_*)$ is zero.

$$f(q) \approx f(q_*) + (f'(q_*)(q-q_*) = 0) + \frac{f''(q_*)}{2}(q-q_*)^2 + \dots$$

This gives us the integral (after we note that at the saddle point $f''(q_*)$ is negative)

$$Z \approx e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{\lambda f''(q_*)(q-q_*)^2/2} dq = e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{-\lambda |f''(q_*)|(q-q_*)^2/2} dq$$

OwO whats this? *Notices your Gaussian integral.* We can do a change of variables

$$u = \sqrt{\frac{\lambda |f''(q_*)|}{2}}(q-q_*) = C(q-q_*) \Rightarrow du = C dq$$

Now we can evaluate the integral with the standard $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\boxed{Z \approx \sqrt{\frac{2\pi}{\lambda |f''(q_*)|}} e^{\lambda f(q_*)}}$$

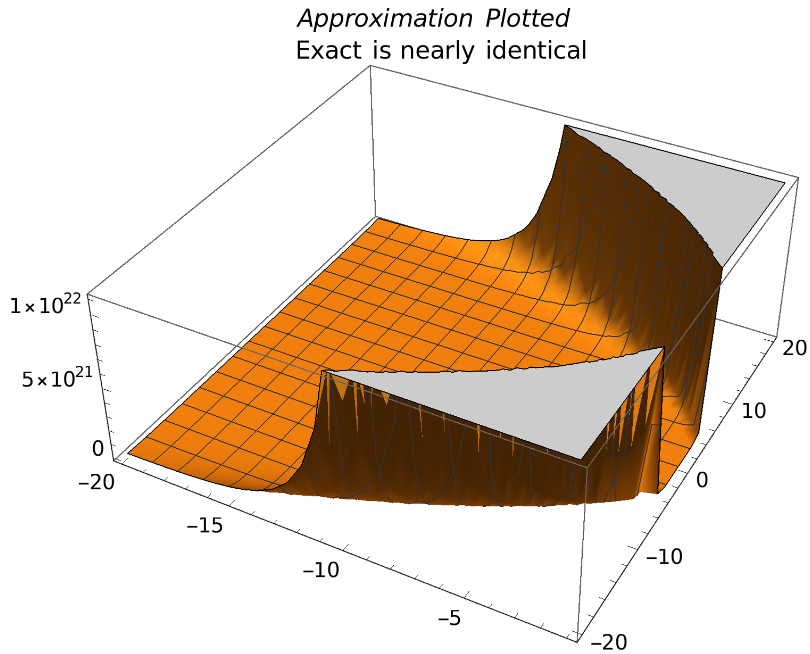
This prefactor we just defined in the context of quantum mechanics is the one-loop correction. Lets see our boy in action. Consider the Gaussian integral where $a < 0$

$$\int_{-\infty}^{\infty} dq \; e^{aq^2+bq+c} = \sqrt{\frac{\pi}{|a|}} e^{c-b^2/4a}$$

We can find the approximation with our saddle point method

$$f'(q_*) = 2aq_* + b = 0 \Rightarrow q_* = -\frac{-b}{2a} \Rightarrow \int_{-\infty}^{\infty} dq \; e^{aq^2+bq+c} \approx \sqrt{\frac{\pi}{|a|}} e^{c+bq+aq^2}$$

Lets fix $c = 42$ and plot both the approximate and the exact answer for some values of a and b just to get a feel for this approximation



It's a good approximation!

Lecture 1: Amputating Bad Legs and Feynman's Cool Trick to Evaluate Integrals

January 19, 2021

Lets start by considering $\lambda\phi^4$ theory and a $2 \to 2$ scattering. The corresponding Feynman diagrams up to 1-loop level is found in Figure 1 Why are the bad diagrams bad? Well consider one of the bad diagrams

TODO Figure

Well $q_2 = p_1$ for any q_1. So the integral becomes

TODO Integral, tl;dr divergence

Namely all of them have bad propagators. Going further we can look at the two-loop example and see that the exact same thing happens. These bad features are called *external leg bubbles.* All diagrams with external leg bubbles are bad, the propagator is frozen on shell and thus blows up.

Lets look at a practical solution. He'll give us a half-assed justification for this solution and then will talk about what's really going on next week. The solution is: *amputate all the external leg bubbles.* To do this we need to figure out how we find these external leg bubbles and the carefully figure out what do we need to amputate. If cutting 1 propagator at a connected diagram breaks it into two disconnected pieces and if one of the pieces has just one external leg

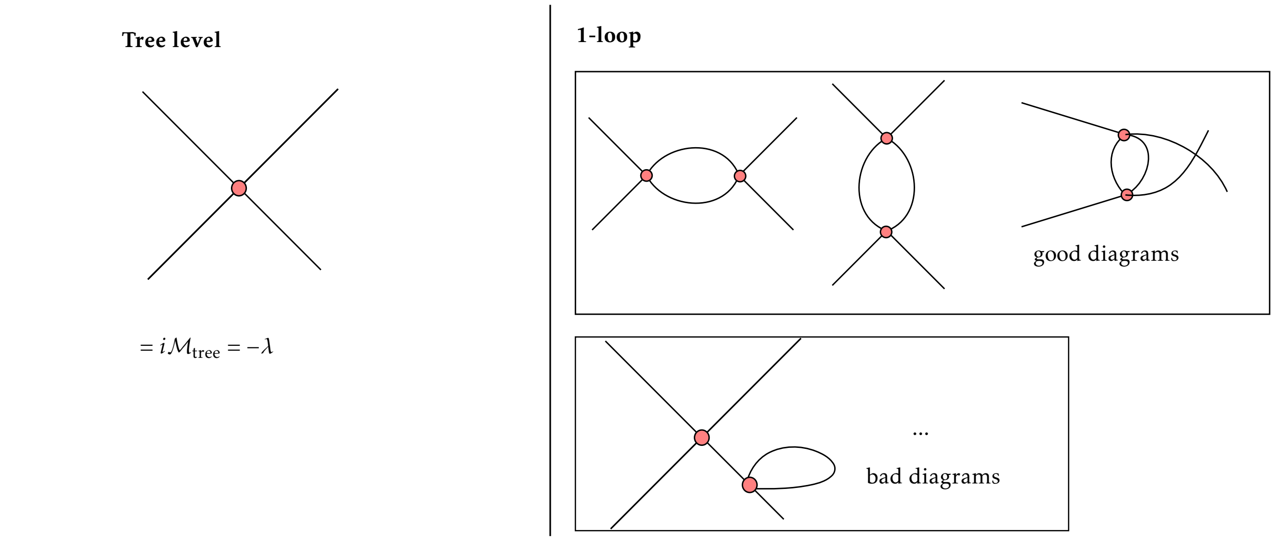
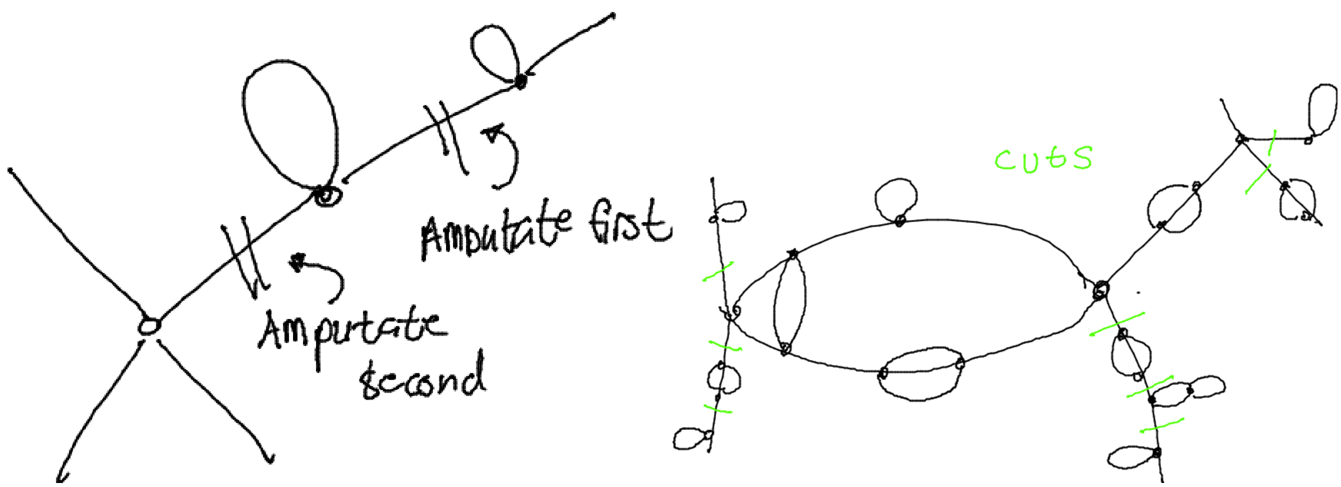


Figure 1: Illustraction of good and bad diagrams in a $2 \to 2$ proceess for $\lambda\phi^4$ theory.

then amputate that piece.



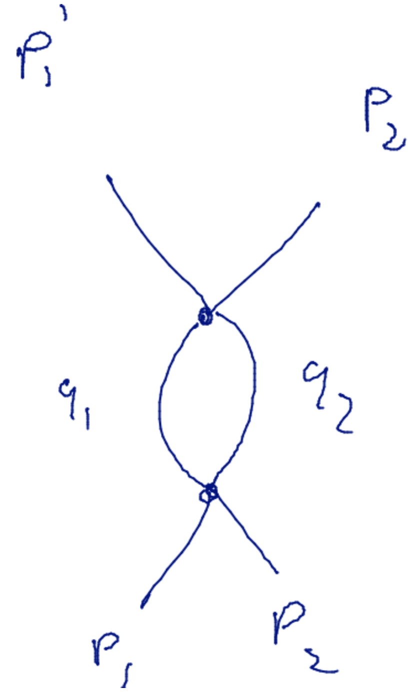
The bottom line is that

$$i\mathcal{M}_{\text{scattering}} = \sum \text{amputated connected diagrams}$$

So where does this rule come from. Well last semester when we were looking at S-matrix elements we were given a formula that was unjustified

$$\langle p_1', p_2', \ldots | \hat{S} | p_1, p_2, \ldots \rangle = \text{Limit regulazation going away}\left[C_{\text{vac}} \times \prod_{\text{ext legs}} F(\text{leg}) \times \langle 0 | a_{p_1'} a_{p_2'} \hat{S} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle \right]$$

Now note that free matrix elemnts is the sum over all Feynmann diagrams. The C_{vac} cancels all vacuum bubbles. What we want to show soon is that $F(\text{ext leg})$ cancels all leg bubbles in that leg. So in the end we're left with diagrams that are unproblematic. This is the "justification" of the no external leg bubble rule. Now the right way to derive this rule is to calculate the correlation function of multiple field, then get (some name) action formula, and then we'll get out result. It turns out that off shell the external leg bubbles are sometimes useful. So now lets go back to the very first diagrams we wrote down and figure out how to calculate loop diagrams.



$$\Rightarrow q_1 + q_2 = p_1 + p_2 = p_1' + p_2'$$

Depends only on the net $p_1 + p_2$ meaning that the matrix element $iF(s)$ only depends on $(p_1 + p_2)^2 = s$. Thus

$$i\mathcal{M}_{1-\text{loop}} = iF(s) + iF(t) + iF(u)$$

By crossing symmetry if we find one of them we find all of them. For example

$$iF(t) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \frac{i}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i\epsilon}$$

Where $q_{\text{net}} = p_1' - p_1 = p_2 - p_2'$ and $q_{\text{net}}^2 = t$. So how do we perform integrals like this? We'll use what's called the *Feynman Parameter Trick*.

Using Feynman's Parameter Trick

Consider the following integral

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

This is true because $(1-x)A + xB$ interpolates between A and B. This integral was know ages before Feynman. However Feynman made good use of it.

$$\frac{1}{q_1^2 - m^2 + i\epsilon} \times \frac{1}{q_1^2 - m^2 + i\epsilon} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)]^2}$$

So looking inside the square bracket we can see that

$$\begin{aligned}
[(1-x)(q_1^2-m^2+i\epsilon)+x(q_2^2-m^2+i\epsilon)] &= (1-x)q_1^2+xq_2^2-m^2+i\epsilon \\
&= (1-x)q_1^2+xq_1^2-2x(q_{\text{net}}q_1)+xq_{\text{net}}^2-m^2+i0 \\
&\text{complete the square in the } 2x \text{ term} \\
&= (q_1-xq_{\text{net}})^2-x^2q_{\text{net}}^2+xq_{\text{net}}^2+m^2+i0 \\
&= (q_1-xq_{\text{net}})^2-\Delta(x)+i0
\end{aligned}$$

Where we define $\Delta(x)=m^2+(x^2-x)q_{\text{net}}^2=m^2+(x^2-x)t$. This means that

$$\frac{1}{q_1^2-m^2+i0}\times\frac{1}{q_2^2-m^2+i0}=\int_0^1\frac{dx}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2}$$

So now we can write $iF(t)$ more cleanly

$$\begin{aligned}
iF(t) &= \frac{1}{2}\lambda^2\int\frac{d^4q_1}{(2\pi)^4}\int_0^1\frac{dx}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2} \\
&= \frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4q_1}{(2\pi)^4}\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2}
\end{aligned}$$

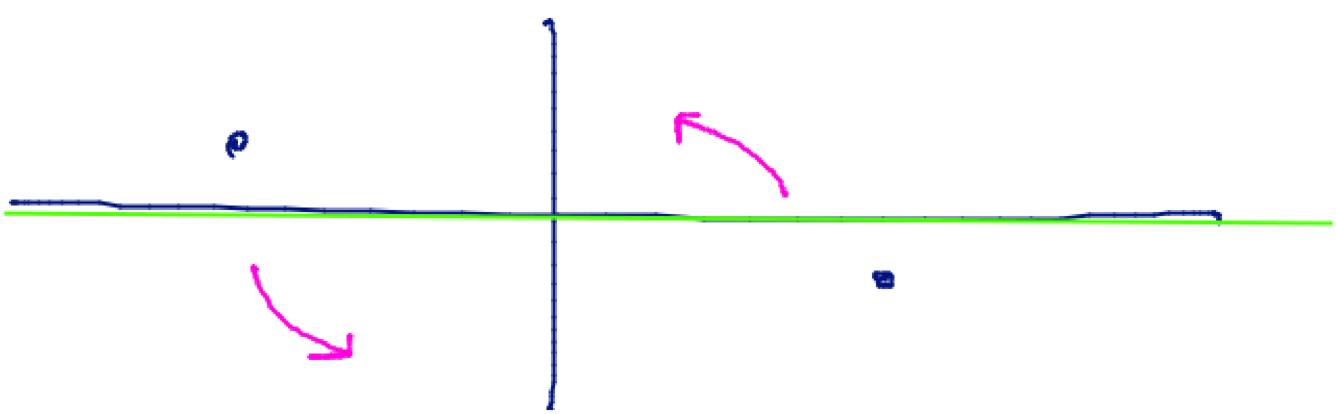
Changing the order of integration is a bit suspect here but at this level it turns out this actually works. Now once we've changed the order of integration for each x change momentum integration variable form q_1 to $k=q_1-xq_{\text{net}}$. This gives us

$$\boxed{iF(t)=\frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4k}{(2\pi)^2}\frac{1}{[k^2-\Delta(x)+i0]^2}\qquad\text{where}\qquad\Delta(x)=m^2-tx(1-x)>0}$$

Now how do we perform this integral? Me thinks residual. Lets focus on the d^4k integral

$$\int\frac{d^4k}{(2\pi)^2}\frac{1}{[k^2-\Delta(x)+i0]^2}=\int d^3\mathbf{k}\int dk_0\frac{1}{[k_0^2-\mathbf{k}^2-\Delta(x)+i0]^2}$$

The $\int dk_0$ integral has double poles at $k_0=\pm[\sqrt{\mathbf{k}^2+\Delta}-i\epsilon]$. So lets deform the contour in the complex k_0 plane. Also note that as $k_0\to\infty$ the term is negligible since the integral is about proportional to k_0^{-4}. So we may change something? can't read hand writing. Now we can use Wick rotations.



Note that we rotate CCW since if we rotated CW then we'd hit the poles and they don't like being hit. So let $k_0 = ik_4$ for real k_4 running from $-\infty$ to ∞. So what does this give us?

$$k_\mu k^\mu - \Delta = k_0^2 - \mathbf{k}^2 - \Delta = \underline{-k_4^2 - \mathbf{k}^2} - \Delta$$

Notice the sign is the same in the underlined section. It becomes the "euclidean continuation?." So we can combine the 3-vector $\mathbf{k}$ and k_4 into a euclidean 4-vector $k_E = (k_1, k_2, k_3, k_4)$.So $k^2 = g_{\mu\nu}k^\mu k^\nu = k_0^2 - \mathbf{k}^2$ becomes $-k_E^2 = -k_4^2 - \mathbf{k}^2$. At the same time $dk_0 = idk_4$ so Minkowski $d^4k = id^4k_E$. And with this setup we get the integral

$$\begin{aligned}\int \frac{d^4k_{\text{minkowski}}}{[k^2 - \Delta + i0]^2} &= \int d^3\mathbf{k} \int \frac{dk_0}{\dots} \\ &= \int d^3\mathbf{k} \int \frac{idk_4}{[-k_4^2 - \mathbf{k}^2 - \Delta]} \\ &= i \int \frac{d^4k_E}{[k_E^2 + \Delta]^2}\end{aligned}$$

Where in the second equality since everything is negative we can get rid of the $i\epsilon$ perscription. The bottom line is

$$\boxed{F(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2}}$$

So the $SO^+(3,1)$ Lorentz symmetry of the Minkowski space analytically continues to $SO(4)$ rotation symmetry in $D = 4$ euclidean space under a wick rotation by $\pi/2$. We can go even futher and change variables from $d^4k_E = dk_E^{\text{real}} \times (k_E^{\text{rot?}})^3 \times d^3\Omega_k$ where Ω is solid angle which means $\int_{D=4} d^3\Omega_k = 2\pi^2$ and so $\int d^4k_E \to \int 2\pi^2 k_E^3 dk_E$ meaning that our integral is

$$\int \frac{d^4k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \frac{2\pi^2}{16\pi^4} \int_0^\infty \frac{k_E^3 dk_E}{[k_E^2 + \Delta]^2}$$

There is one subtlety however. AT $k_E \to \infty$ we know

$$\frac{k_E^3}{[k_E^2 + \Delta]^2} \approx \frac{1}{k_E} \Rightarrow \frac{dk_E}{k_E} = d\log k_E \text{ is logarithmically divergent}$$

Aside: Generalizations for Integral Used Feynman Parameter Trick

We proved some generalization of Feynman's Parameter Trick on the problem set. They're not too hard to prove for anyone with some free time. I'll put them here for reference.

$$\frac{1}{A^nB} = \int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} \tag{F.a}$$

$$\frac{1}{A^nB^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}} \tag{F.b}$$

$$\begin{aligned}\frac{1}{ABC} &= \int_0^1 dx \int_0^{1-x} \frac{2dy}{[xA+yB+(1-x-y)C]^3} \\ &= \int_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1) \times \frac{2}{[xA+yB+zC]^3}\end{aligned} \tag{F.c}$$

$$\begin{aligned}\frac{1}{A_1\ldots A_k} &= \int\limits_{x_1\ldots x_k\geq 0}\cdots\int d^kx\ \delta(x_1+\cdots+x_k-1)\times\frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^k} \\ &= \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \ldots \int\limits_0^{1-x_2-\ldots-x_{k-2}} dx_{k-1}\frac{(k-1)!}{[x_1A_1+\cdots+(1-x_2-\ldots-x_{k-1}A_k]^k}\end{aligned} \tag{F.d}$$

$$\frac{1}{A_1^{n_1}A_2^{n_2}\ldots A_k^{n_k}} = \frac{n_1+\cdots+n_k-1)!}{(n_1-1)!\ldots(n_k-1)!}\times \tag{F.e}$$

$$\times \int\limits_{x_1,\ldots,x_k\geq 0}\cdots\int d^kx\ \delta(x_1+\cdots+x_k-1)\times\frac{x_1^{n_1-1}\ldots x_k^{n_k-1}}{[x_1A_1+\cdots+x_kA_k]^{(n_1+\cdots+n_k)}} \tag{F.f}$$

(F.a) We wish to verify

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2}$$

Let $u = [xA+B-Bx]$ which gives us $du = (A-B)dx$. The bounds go from B to A. The integral then becomes

$$\boxed{\frac{1}{A-B}\int_B^A \frac{du}{u^2} = -\frac{1}{A-B}\left(\frac{B-A}{AB}\right) = \frac{1}{AB}}$$

(F.b) We wish to verify

$$\frac{1}{A^nB} = \int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}}$$

Lets do the same change of variables we did in (F.a). Also note that $x = (u-B)/(A-B)$.

$$\begin{aligned}
\int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} &= \frac{n}{(A-B)^n}\int_B^A \frac{(u-B)^{n-1}du}{u^{n+1}} \\
&= \frac{n}{(A-B)^n}\int_B^A \frac{du}{u^2}\left(1-\frac{B}{u}\right)^{n-1} \\
&\text{Let } w = \frac{1}{u} \Rightarrow dw = -u^{-2}du \\
&= -\frac{n}{(A-B)^n}\int_{1/B}^{1/A} dw\,(1-Bw)^{n-1} \\
&= \frac{1}{B(A-B)^n}(1-Bw)^n\Big|_{1/B}^{1/A} \\
&= \frac{1}{B(A-B)^n}(1-B/A)^n \\
&\boxed{\int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} = \frac{1}{A^nB}}
\end{aligned}$$

(F.c) We wish to verify

$$\frac{1}{A^nB^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}} = Z$$

Lets do the same change of variables again

$$\begin{aligned}
Z &= \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_B^A \frac{(u-B)^{n-1}(A-u)^{m-1}du}{u^{n+m}} \\
&= \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_B^A \frac{1}{u^2}(1-B/u)^{n-1}(A/u-1)^{m-1}du \\
&= -\frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_{1/B}^{1/A}(1-Bw)^{n-1}(Aw-1)^{m-1}dw
\end{aligned}$$

Now lets integrate by parts (using the tabular method)

$$\begin{array}{c|c}
(1-Bw)^{n-1} & (Aw-1)^{m-1} \\
-B(n-1)(1-Bw)^{n-2} & \frac{1}{Am}(Aw-1)^m \\
\vdots & \vdots \\
(-1)^{n-1}B^{n-1}(n-1)! & \frac{1}{A^{n-1}m(m+1)...(m+n-2)}(Aw-1)^{m+n-2}
\end{array}$$

Because of our bounds each non-integral term vanishes. Also note the sign of each term from integration by parts is determined by $(-1)^{n+1}$. Thus the final integral is positive.

This leaves us with

$$\begin{aligned}\int_{1/B}^{1/A}(1-Bw)^{n-1}(Aw-1)^{m-1}dw &= \frac{B^{n-1}(n-1)!}{A^{n-1}m(m+1)\dots(m+n-2)}\int_{1/B}^{1/A}(Aw-1)^{m+n-2}dw \\ &= -\frac{B^{n-1}(n-1)!}{A^n m(m+1)\dots(m+n-1)}(A/B-1)^{m+n-1} \\ &= -\frac{B^{n-1}(n-1)!}{A^nB^{m+n-1}m(m+1)\dots(m+n-1)}(A-B)^{m+n-1} \\ &= -\frac{(n-1)!(m-1)!}{A^nB^m(m+n-1)!}(A-B)^{m+n-1}\end{aligned}$$

Plugging this integral back in gives us

$$\boxed{\frac{(n+m-1)!}{(n-1)!(m-1)!}\times\int_0^1\frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}}=\frac{1}{A^nB^m}}$$

(F.d) We can evaluate this in a straightforward manner

$$\begin{aligned}Z &= -\frac{1}{B-C}\int_0^1 dx\Big\{[xA+(1-x)B]^{-2}-[xA+(1-x)C]^{-2}\Big\} \\ &= \frac{1}{B-C}\left\{\frac{1}{A-B}[xA+(1-x)B]^{-1}-\frac{1}{A-C}[xA+(1-x)C]^{-1}\right\}\Bigg|_0^1 \\ &= \frac{1}{B-C}\left\{\frac{1}{A(A-B)}-\frac{1}{A(A-C)}-\frac{1}{B(A-B)}+\frac{1}{C(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{BC(A-C)-BC(A-B)-AC(A-C)+AB(A-B)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{(A-C)(BC-AC)+(A-B)(AB-BC)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{C(A-C)(B-A)+B(A-B)(A-C)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{-C(A-B)+B(A-B)}{ABC(A-B)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{-C+B}{ABC}\right\} \\ &= \frac{1}{ABC}\end{aligned}$$

(F.e) We know the given integral is equivalent to

$$Z=\int_0^1 dx_1\int_0^{1-x_1}dx_2\dots\int_0^{1-x_1-\dots-x_{n-2}}dx_{n-1}\frac{(n-1)!}{[x_1A_1+\dots+(1-x_1-\dots-x_{n-1})A_n]^n}$$

From our result above we know that evaluating all the integrals will result in a factor of $1/(n-1)!$. This is because of the exponent in the dnominator. From examing (F.d) we also note that after each integral something of the following form

$$\int_0^{1-x_1-\ldots-x_{n-2}} dx_{n-1}\frac{(n-1)!}{[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-1})A_n]^n} = -\frac{(n-2)!}{(A_{n-1}-A_n)[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-2})A_{n-1}]^{n-1}} + \frac{(n-2)!}{(A_{n-1}-A_n)[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-2})A_n]^{n-1}} \tag{1}$$

I think we can analyze this with recursion. Lets define the following

$$Z(\{A_1,\ldots,A_{n-2},A_{n-1},A_n\},n) = \int_0^1 dx_1\int_0^{1-x_1} dx_2\ldots\int_0^{1-x_1-\ldots-x_{n-2}} dx_{n-1}\frac{(n-1)!}{[x_1A_1+\ldots x_{n-1}A_{n-1}+(1-x_1-\ldots-x_{n-1})A_n]^n} \tag{2}$$

We see from (1)

$$Z(\{A_1,\ldots,A_{n-1},A_n\},n) = \frac{-1}{A_{n-1}-A_n}\{Z(\{A_1,\ldots,A_{n-2},A_{n-1}\},n-1) - Z(\{A_1,\ldots,A_{n-2},A_n\},n-1)\}$$

Also from (F.d) we see that

$$Z(\{A_1,A_2\},2) = -\frac{1}{A_1-A_2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)$$

Now instead of being rigorous (I'm just trying to convince myself afterall) we'll just check this recursion relation with mathematica. Consider the following code which implements this recursive definition.

```
In[105]:= Clear[f]
          f[A_, 2] := -1 / (A[[1]] - A[[2]]) (1 / A[[1]] - 1 / A[[2]])
          f[A_, n_] := -1 / (A[[n - 1]] - A[[n]]) (f[A[[1 ;; n - 1]], n - 1] - f[Join[A[[1 ;; n - 2]], {A[[n]]}], n - 1])
          Table[f[Table["A" <> ToString[i], {i, 1, maxn}], maxn] // FullSimplify, {maxn, 2, 10}]
```

Out[108]= $\left\{\frac{1}{\text{A1 A2}}, \frac{1}{\text{A1 A2 A3}}, \frac{1}{\text{A1 A2 A3 A4}}, \frac{1}{\text{A1 A2 A3 A4 A5}}, \frac{1}{\text{A1 A2 A3 A4 A5 A6}}, \frac{1}{\text{A1 A2 A3 A4 A5 A6 A7}}, \frac{1}{\text{A1 A2 A3 A4 A5 A6 A7 A8}}, \frac{1}{\text{A1 A2 A3 A4 A5 A6 A7 A8 A9}}, \frac{1}{\text{A1 A10 A2 A3 A4 A5 A6 A7 A8 A9}}\right\}$

In fact we can do better than just check it with mathematica. We're given

$$\frac{1}{A_1\ldots A_k} = \int\ldots\int_{x_1,\ldots,x_k\geq 0} d^kx\,\delta(x_1+\cdots+x_k-1)\times\frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^k}$$

And we've proven a base case of $k=3$. From here we just need to prove that if the relationship is satisfied for k then it also satisfies $k+1$. Consider

$$\frac{1}{A_1\ldots A_{k+1}} = \int\ldots\int_{x_1,\ldots,x_k\geq 0} d^kx\,\delta(x_1+\cdots+x_k-1)\times\frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^kA_{k+1}}$$

We use (F.b) to get

$$\frac{1}{A_1 \dots A_{k+1}} = \underset{x_1,\dots,x_k \geq 0}{\int \dots \int} d^k x \int_0^1 dy\ \delta(x_1 + \cdots + x_k - 1) \times \frac{k! y^{k-1}}{[y(x_1 A_1 + \cdots + x_k A_k) + (1-y) A_{k+1}]^{k+1}}$$

Let $yx_i = z_i$ and $(1-y) = z_{k+1}$ meaning that $dx_i = dz_i/y$ and $dy = -dz_{k+1}$ and the limits in the innermost integral go from 1 to 0 allowing us to change sign in order to keep the prefactor positive.

$$\begin{aligned} \frac{1}{A_1 \dots A_{k+1}} = &\underset{z_1,\dots,z_k \geq 0}{\int \dots \int} d^k z \int_0^1 dz_{k+1}\ \delta(y^{-1}(z_1 + \cdots + z_k + z_{k+1} - 1)) \\ &\times \frac{k! y^{-1}}{[z_1 A_1 + \cdots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}} \end{aligned}$$

Now we use $\delta(ax) = \delta(x)/|a|$ to get

$$\begin{aligned} \frac{1}{A_1 \dots A_{k+1}} = &\underset{z_1,\dots,z_k \geq 0}{\int \dots \int} d^k z \int_0^1 dz_{k+1}\ \delta(z_1 + \cdots + z_k + z_{k+1} - 1) \\ &\times \frac{k!}{[z_1 A_1 + \cdots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}} \end{aligned}$$

The last thing we want to show is that the bounds can equivalently be stated as z_{k+1} is positive. First we note that the delta function asserts

$$z_1 + \cdots + z_{k+1} = 1$$

And the other bounds of the integral assert $z_1, \dots z_k \geq 0$. Now considere the situation when $z_{k+1} > 1$. Then we have

$$z_1 + \cdots + z_{k+1} > z_1 + \cdots + z_k + 1$$

From here we see that no combination of positive $z_1 \dots z_k$ will satisfy the condition that $z_1 + \cdots + z_{k+1} = 1$ meaning that there is no contribution for $z_{k+1} > 1$. Thus the bounds of the innermost integral can be restated as $z_{k+1} \geq 0$ since any contribution from $z_{k+1} > 1$ is killed by the delta function. This gives us

$$\frac{1}{A_1 \dots A_{k+1}} = \underset{z_1,\dots,z_{k+1} \geq 0}{\int \dots \int} d^{k+1} z \delta(z_1 + \cdots + z_{k+1} - 1) \frac{k!}{[z_1 A_1 + \cdots + z_{k+1} A_{k+1}]^{k+1}}$$

Therefore we've proven the inductive hypothesis thus completing the proof for $(F.e)$.

(F.f) You can just get this result by taking the partial derivative of $(F.e)$ a million times. Typing that out seem wasteful.

Maloney QFT I Lecture 14: Feynman Diagrams I

In this lecture we finally become men? To recap in QM we can compute observables with the following

$$\langle q_f, t_f | T Q(t_q) \dots Q(t_n) | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS/\hbar}\, q(t_i) \dots q(t_n)$$

So for example if we're doing a double slit experiment and want to measure which slit the the particle goes through we'll insert an operator q in the integral. We can also rephrase this integral as a integral through phase space

$$= \int DqDp \exp\left\{ i \int (p\dot{q} - H(p,q))dt \right\} q(t_1) \dots q(t_n)$$

In the end of class we introduced a trick to compute correlcation function instead of transition amplitudes. Since

$$e^{it(1-i\epsilon)H} |\psi\rangle = |0\rangle\langle 0|\psi\rangle \text{ at } t \to -\infty$$

Similarly we could apply this to a bra-vector

$$\langle\psi| e^{it(1-i\epsilon)H} = \langle\psi|0\rangle\langle 0| \text{ at } t \to \infty$$

This means if you want to calucate VEV

$$\langle 0|T Q(t_1) \dots Q(t_n)|0\rangle \propto \int DqDp \exp\left\{ i \int (\dot{q}p - (1-i\epsilon)H)dt \right\} q(t_1) \dots q(t_n)$$

Effectivley what we're doing is adding a small imaginary part to the energies and basically projects us onto the ground state. At the tree level this $i\epsilon$ factor isn't very important but when we do loop calculation this $i\epsilon$ factor becomes important. Something to emphasize the equation above is that we have propto instead of equality since we ignored some normalizations. To get equality we could do

$$\langle 0| \dots |0\rangle = \frac{\int Dq\, e^{iS/\hbar} q(t_1) \dots q(t_n)}{\int Dq\, e^{iS/\hbar}}$$

Where the denominator is there to keep $\langle 0|0\rangle = 0$.

To generalize this to many DOF $q_i(t) \to \int Dq_i(t)$. In a QFT where the DOf is a local $\phi(t,\mathbf{x}) \to \int D\phi(\mathbf{x},t)$ (integrate over all possible field configurations ϕ). Today our goal is to compute

$$\langle 0|T\phi(x_1) \dots \phi(x_n)|0\rangle = \frac{\int D\phi\, e^{iS} \phi(x_1) \dots \phi(x_n)}{\int D\phi\, e^{-S}}$$

To do this lets define the partition function or generating function of a QFT. We call J the source

$$Z[J(x)] = \int D\phi\, e^{iS + i\int d^4x\, J(x)\phi(x)}$$

It should be clear that if we computed the EOM, $J(x)$ would be a source term on the RHS of the EOM. An if we wanted to we can expand the exponential

$$Z[J(x)] = \int D\phi\, e^{iX}\left(1 + i\int d^4x J(x)\phi(x) - \int d^4x\, d^4y\, J(x)J(y)\phi(x)\phi(y) + \dots\right)$$

This partition function includes all possible expectation values that we care about. First lets consider how we could get expectation values from this partition function. Well lets remeber how we extract single terms from a power series. We take the derivative wrt the variable and then set that variable to zero. For example $e^x = c_0 + c_1 x + c_2 x^2 + \dots$, $de^x/dx = c_1 + 2c_2 x + \dots \Rightarrow (de^x/dx)(x =) = c_1$. This generalizes to what we have here. However here we need to take the functional derivative

$$\boxed{\text{Define } \frac{\delta}{\delta J(x)} \text{ as } \frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)}$$

More compliacted derivatives are defined using this formula and the chain rule

$$\frac{\delta V(\phi(y))}{\delta\phi(x)} = \frac{\partial V(\phi(y))}{\partial\phi(x)}\frac{\delta\phi(y)}{\delta\phi(x)} = V'(\phi(y))\delta^{(4)}(x-y)$$

Now we just need to put these two things together

$$Z[J] = \int D\phi\, e^{iS + i\int J(x)\phi(x)d^4x}$$

So if we want the VEV we compute

$$\boxed{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{(-i)^n}{Z}\frac{\delta}{\delta J(x_1)}\dots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0}}$$

Lets look at an example, free QFT $S = -\frac{1}{2}\int d^4x\, \phi(\partial^2 + m^2)\phi$. This means that

$$Z[J] = \int D\phi \exp\left\{-\frac{i}{2}\int d^4\, \phi(x)(\partial^2 + m^2)\phi(x) + i\int d^4x\, J(x)\phi(x)\right\}$$

Lets consider a simpler but illuminating problem

$$\int d\phi \exp(\phi A\phi + j\phi) \approx \exp\left(jA^{-1}j\right)$$

Where the second $\approx$ comes from shifting ϕ to complete the square. For multidimensional integral we get (TODO PSET)

$$Z[J]\int d\phi_i \exp\left(\phi_i A^{ij}\phi_j + j^i\phi_i\right) \approx \exp(j^i(A^{-1})_{ij}j^j)$$

Lets think of a discrete model of QFT (e.g. space is lattice). We can identify A^{ij} with $(\partial^2 + m^2)$. We also met that $A^{-1} = (\partial^2 + m^2)^{-1} = D_F$, the feynman propagator. To compute the integral we can use fourier transform. One thing we can note is that the kinetic operator $(\partial^2 + m^2)$ is matrix that is diagonal in momentum space. What does that mean? Lets take our $\phi(x)$ and think about it in momentum space

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx}\tilde{\phi}(k)$$

And lets change variables so that our path integral isn't over all configurations of $\phi(x)$ but all possible fourier components of $\tilde{\phi}(k)$.

$$\int D\phi(x) \to \int D\tilde{\phi}(k)$$

To think about this more clearly we can consider the integral over all field configurations of $\phi(x)$ as the integral over all possible values of x (π_x) in some one dimensional integral (? TODO huh)

$$\int \pi_x d\phi(x) \to \int \pi_k d\tilde{\phi}(k)$$

Whenever we do a change of variables we have a jacobian determinent. Here it's not too hard to compute since we're going from ϕ to a linear combination of ϕ. But we're just gonna ignore it since it's some overall constant for now. So what does the integrand look like in fourier space.

$$\begin{aligned}\int d^4x\ \phi(x)(\partial^2+m^2)\phi(x) &= \int d^4x \int d^4kd^4k'\ e^{-ikx}\tilde{\phi}(k)(\partial^2+m^2 = -k'^2+m^2)e^{-ik'x}\tilde{\phi}(k') \\ &= \int d^4k\ \tilde{\phi}(k)(-k^2+m^2)\tilde{\phi}(-k)\end{aligned}$$

This is what we mean when we say taht the kinetic operator is diagonal in mometnum space. We also have

$$\int d^4x\ J(x)\phi(x) = \int d^4k\tilde{J}(k)\tilde{\phi}(-k) = \frac{1}{2}\int d^4k\left(\tilde{J}(k)\tilde{\phi}(-k)+\tilde{J}(-k)\tilde{\phi}(k)\right)$$

This means we have

$$Z[J] = \int D\tilde{\phi}(k)\ \exp\left\{\frac{i}{2}\int d^4k(\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k)+\tilde{J}(k)\tilde{\phi}(-k)+\tilde{J}(-k)\tilde{\phi}(k)\right\}$$

We can separate out the integrand into the product of a bunch of one dimensional integrals over each independent momentum mode

$$Z[J] = \prod_k \int d\phi(k)\ \exp\left\{\frac{i}{2}\left(\tilde{\phi}(k)(k^2-m^2)\tilde{\phi}(-k)+\tilde{J}(k)\tilde{\phi}(-k)+\tilde{J}(-k)\phi(k)\right)\right\}$$

That's just a one dimensional gaussian integral. Let $\chi(k) = \tilde{\phi}(k)+\tilde{J}(k)/(k^2-m^2)$. this means

$$Z[J] = \prod_k \int d\chi(k)\ \exp\left\{\frac{i}{2}\chi(k)\chi(-k)(k^2-m^2)+\frac{i}{2}\tilde{J}(k)\tilde{J}(-k)/k^2-m^2)\right\}$$

So we can then write

$$Z[J] = Z[0]\exp\left\{\frac{i}{2}\int d^4k\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2}\right\}$$

If we then do the inverse fourier transform we now have

$$Z[J]=Z[0]\exp\left\{\frac{1}{2}\int d^4xd^4y\ J(x)D(x-y)J(y)\right\}$$

Where $D(x-y)=\int d^4k\ e^{ik(x-y)}\frac{i}{k^2-m^2+i\epsilon}$ is the feynman propagator. Now lets try to compute a two-point function

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle=(-i)^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\left(1-\frac{1}{2}\int d^4xd^4y\ J(x)J(y)D(x-y)+\dots\right)\Bigg|_{J=0}$$

Applying chain rule we just get

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle=D(x-y)$$

In fact this is a more general result for any free QFT which is

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle=\sum_{\text{pairings}}(\text{product of free propagators})$$

So for example

$$\langle 0|\phi(x_1)\dots\phi(x_4)|0\rangle=D(x_1-x_2)D(x_3-x_4)+D(x_1-x_3)D(x_2-x_4)+D(x_1-x_4)D(x_2-x_3)$$

This is Wick's theorem. Lets introduce a graphical notation to keep track of things. First recall

$$Z[J]=\exp\left(-\frac{1}{2}\int d^4xd^4y\ J(x)D(x-y)J(y)\right)$$

When we introduce interactions what we'll end up with is a more complex diagramatic set of

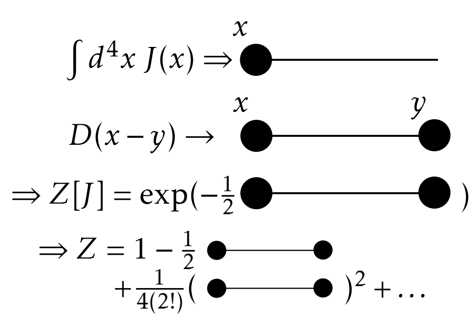


Figure 2: Graphical represntation of partition function

rules. e.g. for a ϕ^3 theory we ahve new sets of ingridients.

Lecture 2: Example of Renormalization

January 21, 2021

At the very end of the last lecture for the two scalar scaterring we found out that the momentum integral suffers from ultraviolet divergence. Divergences like that are all over the place in QFT. So people started to develop a renormalization procedure which deals with diffent physics couplings. All this machinary however has a nasty flavor of sweeping infinities under the carpet. It wasn't very clear what the hell was going on. Even though it worked perturbtivley well for $\lambda\phi^4$ and QED it didn't work for some other theories and doesn't work non-perturbativley. So in the 1950s there was a backlash which lead to alternates like the bootstrap program (aparently this name has something to do with lifting yourself up by your bootstraps?) What really comes to understanding renormalization is by undersatnding effective field theories. This paradigm came from two sides. First there were some people studying critical phenomena in condensed matter. When you're working near a critical point you have long distance correlations. Pretty soon those people realized the behavior at long distances doesn't really care about short distance details. It doesn't matter what kind of lattice we're dealing with (except for maybe some lattice contant). The knowledge of short distance could be summarized in a few parameters. So what Ken Wilson and friends figured out is that you want to worry about effective long distance theories and ignore short distance behavior. The other approach came from a completley different end: current algebra. People studying low-energy interaction between π-mesons and nucleons noticed that there was partial conservation of axial current(fancy spontaneously broken symmetry). They then wrote currents of this broken symmetry and then Weinberg and company figured out that the best way to understand what goes on with current algebra is use effective field theory for low energy(cut all meson zoo except pion.) And what came the big understanding: what we're doing in real life is an effective field theory. We know particles up to some GeV and we don't know anything above that scale. We don't know what kind of particles might occur at 10 TeV. If we had a full QFT then we'd need to include those particles. Even if we say something silly like "there is nothing beyond the standard model" we still have to still consider quantum gravity which becomes relevant at 10^{-19} GeV. So the only thing we can do is effective field theory, a theory which is only known up to some cutoff. So the best we can do is parametrize our ignorance of whats beyond what we know. And in practice this translates to is cutting off ultraviolet modes by saying "we don't know what happens before this" and just say there is literally nothing. And you emulate this nothing with a UV cutoff. And then we paramterize our ignorance by introducing bear(?) couplings that aren't actually measurable experimentailly but whatever gives us the right answer. Fortunatley there is only a few parameters we need to adjust. Before we do any examples lets revisit something we should all know. Lets consider the Debye model of solids.

Aside: Debye Model of Solids

What is relevant to low energies?

(a) For $|\mathbf{k}| \ll 1/(\text{lattice spacing})$ we know that $\omega(k) = c_s|\mathbf{k}|$ where c_s is the speed of sound.

(b) Finite volume of $\mathbf{k}$ space. In 3D we have $\mathbf{k}$ modulo inverse lattice times 2π and k belongs to a 3-torous.

So what did Debye do? He approximated $\omega(\mathbf{k}) = c_s|\mathbf{k}|$ exactly and k-space ball of radius θ/c_s (θ is the Debye temperature?) meaning that

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d^3k}{(2\pi)^3}$$

Up to $|\mathbf{k}| = \theta/c_s$. So θ acts like kind of a cutoff for the condensed matter theory. In QFT we introudce a UV cutoff λ that will shove all the gory details of the UV physics in some redefinition.

Back to 1-loop correction in $\lambda\phi^4$ theory

We found that for some diagram

$$F(t) = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \qquad \Delta(x) = m^2 - tx(1-x) > 0$$

We want to evaluate

$$\begin{aligned} I &= \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \\ &= \frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{dk_E\ k_E^3}{[k_E^2+\Delta]^2} \end{aligned}$$

Now we set UV cutoff: cut the integral at $|k_E| = \Lambda$ for $\Lambda \gg$ what?.

$$I_{\text{regulated}} = \frac{1}{2\pi^2}\int_0^\Lambda \frac{dk_E k_E^3}{[k_E^2+\Delta]^2}$$

Let $\nu = k_E^2 + \Delta$ meaning that

$$I_{\text{regulated}} = \frac{1}{8\pi^2}\times\frac{1}{2}\int_\Delta^{\Delta+\Lambda^2}\frac{d\nu(\nu-\Delta)}{\nu^2} = \frac{1}{16\pi^2}\left(\log\nu + \frac{\Delta}{\nu}\right)\Bigg|_\Delta^{\Delta+\Lambda^2} = \frac{1}{16\pi^2}\Big(\log[(\Delta+\Lambda^2)/\Delta] + \Delta/(\Delta+\Lambda^2) - 1\Big)$$

Now we take an assumption that $\Lambda^2 \gg m^2$ or q_{net}^2 and therefoer we can neglect all negative powers of Λ. So we can say

$$\log[(\Lambda^2+\Delta)/\Delta] = \log(\Lambda^2/\Delta) + \frac{\Delta}{\Lambda^2} - \frac{\Delta^2}{2\Lambda^2} + \cdots \approx \log(\Lambda^2/\Delta)$$

$$\frac{\Delta}{\Delta+\Lambda^2} - 1 \approx -1$$

This all gives us

$$I_{\text{regulated}} = \frac{1}{16\pi^2}\Big(\log(\Lambda^2/\Delta) - 1\Big) \Rightarrow F(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ (\log(\Lambda^2/\Delta(x)) - 1)$$

To do this integral we just do $\log(\Lambda^2/\Delta(x)) = \log(\Lambda^2/m^2) - \log(\Delta^2/m^2 = 1 - tx(1-x)/^2)$. This means

$$F(t) = \frac{\lambda^2}{32\pi^2}\Big[\log(\lambda^2/m^2) - 1 - J(\lambda/m^2)\Big] \qquad J(t/m^2) = \int_0^1 dx\ \log(1 - tx(1-x)/m^2)$$

There are three diagrams at the 1-loop level (s,t,u) shown in Figure 1 meaning that

$$\mathcal{M}_{1-\text{loop}} = F(t) + F(s) + F(u)$$

Which can be calculated with crossing symmetry.

$$\mathcal{M}(s,t,u) = -\lambda + \frac{\lambda^2}{32\pi^2?}\left[3\log(\lambda^2/m^2) - 3 - J(t/m^2) - J(u/m^2) - J(s/m^2)\right] + O(\lambda^3)$$

So how do we deal with infinite parameters. Here's an overview of the renormalization procedure. λ in the above formula is the *bare* coupling in the bare lagrangian of the theory. This λ_{bare} is not directly measured by any experiment. We adjust it as needed so tah the perturbative amplitudes we calulate fit experimental data. To renormalize our procdure goes like this

(1) Start by definiting physcia coupling λ_{phys} in terms of some scattering amptliude. For example $\lambda_{\text{phys}} = -\mathcal{M}_{\text{elastic}}$ at the threshold.

(2) Use perturbation theory (feynmann graphs?) to calculate $\lambda_{\text{physical}}$ as a power series in λ_{bare}.

$$\lambda_{\text{phys}} = \lambda_{\text{bare}} + A_1\lambda_b^2 + A_2\lambda_b^3 + A_3\lambda_b^4 + \dots$$

Note that $A_1, A_2, A_3, \dots$ depend on $\log\Lambda_{\text{UV}}$. Now formally assume that not only λ_{bare} is small but alos $\lambda_{\text{bare}} \times \log\Lambda_{\text{UV}}$ is small so this perturbative theory makes sense. Do formal perturbation theory in λ_{bare}.

(3) Reverse the power series for λ_{phys}

$$\Rightarrow \lambda_{\text{bare}} = \lambda_{\text{phys}} + B_1\lambda_{\text{phys}}^1 + B_2\lambda_{\text{p}}^3 + \dots$$

This tells us $B_1 = -A_1$ and $B_2 = 2A_1^2 - A^2$ and so on.

$$\Rightarrow \lambda_{\text{bare}} = \lambda_p - A_1\lambda_p^2 + (2A_1^2 - A_2)\lambda_p^3 + \dots$$

(4) For any interesting ampltiude $\mathcal{M}$(kinematical params), use feynman graphs to calculate $\mathcal{M}$

$$\mathcal{M}(\text{kine. para.}) = \lambda_b^n\mathcal{M}_0 + \lambda_b^{n+1}\mathcal{M}_1 + \lambda_b^{n+2}\mathcal{M}_2 + \dots$$

(5) Re-expand in terms of $\lambda_b = \lambda_{\text{phys}} + B_1\lambda_{\text{phys}}^2 + B_2\lambda_{\text{phys}}^3 + \dots$. This means that

$$\begin{aligned}
\mathcal{M} = \mathcal{M}_0 &\times \lambda_{\text{phys}}^n(1 + nB_1\lambda_{\text{phys}} + n(n-1)/2B_1^2\lambda_{\text{phys}}^2 + nB_2\lambda_{\text{phys}}^2 \dots) \\
&+ \lambda_{\text{phys}}^{n+1}(1 + (n+1)B_1\lambda_{\text{phys}} + \dots) \\
&+ \lambda_{\text{phys}}^{n+2}(1 + \dots) + \dots \\
&= \lambda_{\text{phys}}^n\mathcal{M}_0 + \lambda_{\text{phys}}^{n+1}[nB_1\mathcal{M}_0 + \mathcal{M}_1] \\
&+ \lambda_{\text{phys}}^2[(n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0 + (n+1)B_1\mathcal{M}_1 + \mathcal{M}_2] + \dots
\end{aligned}$$

The point of all this is to see that we get a power series in λ_{phys}.

(6) In the power series in λ_{phys} the dependence on $\log\Lambda_{\text{UV}}$ cancels out from each term. That is $B_1, B_2, B_3 \dots$ depend on $\log\Lambda$. $\mathcal{M}_1, \mathcal{M}_2, \dots$ depend onf $\log\Lambda$. But $\mathcal{M} - 1 + nB_1\mathcal{M}_0$ is independent of $\log\Lambda_{\text{UV}}$ and likewise $\mathcal{M}_2 + (n+1)B_1\mathcal{M}_1 + (n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0$ is indpendent of $\log\Lambda$.

And that's the renormalization procedure. Lets look at an example for elastic scattering

1-loop calculation for Elastic Scattering

First we define

$$\lambda_{\text{phys}} = -\mathcal{M}@\text{threshold } t = u = 0 \text{ and } s = 4m^2$$

And thus from EQREFHERE we get

$$\lambda_{\text{b}} = \lambda_{\text{phys}} + \frac{\lambda_b^2}{32\pi^2}\left(3\log\Lambda^2/m^2 - 3 - 2(J(0) = 0) - J(4)\right)$$

And when we rewrite $\mathcal{M}(s,t,u)$ in terms of λ_{phys} we get

$$\mathcal{M}(s,t,u) = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\left(J(4) - J(s/m^2) - J(t/m^2) - J(u/m^2)\right)$$

And that's how the renormalization theoyr works. Now he focused on λ but in reality there is also a renormalization of mass $m_b \neq m_{\text{phys}}$. Also nowaways they do perturbation theory with counterturm which directly reorganizes perturbation theory in terms of $\lambda_{\text{phys}}, m_{\text{phys}}^2, \ldots$ directly at the level of feynman graphs.

Lecture 3: UV Regularization Schemes

January 22, 2021

Here are some ways to regularize

(a) Wilson's hard edge cutoff: Used in condensed matter but not commonly used in particle physics

(b) Pauli Villars (1949): Commonly used today and one of the older ones

(c) Covariant Higher Derivatives: Used in supersymmetry.

(d) Dimensional Regularization: Most commonly used today. This basically is taking the dimension of ST $D = 4 - 2\epsilon$. This will be the main subject of lecture 4 if we don't get to it today.

(e) Lattice (discrete spacetime): the hardest and most physical cutoff. This is the only cutoff that works nonperturbativley

Wilson's Hard Edge Cutoff

Limit all euclidan momenta to $|k_E| \leq \Lambda$. Example

$$q_2 = q_{\text{net}} - q_1 \Rightarrow |q_{1E} - q_E^{\text{net}}| < \Lambda$$

This means that q_1 is the intersection of two balls in eucledian momentum space. In terms of $k = q_1 + xq_{\text{net}}$. We have

$$|k_E - xq_{\text{net}}| \leq \Lambda \qquad \text{and} \qquad |k_E - (x-1)q_E^{\text{net}}| \leq \Lambda$$

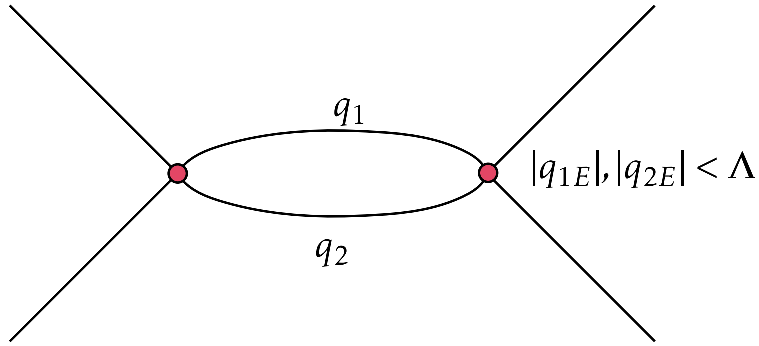


Figure 3: Setup for 1-loop diagram in Wilson's Hard Edge Cutoff

For $|q^{\text{net}}| \ll \Lambda$ we then have

$$|k_E| \leq \Lambda + (\text{direction-dependent } O(q^{\text{net}}))$$

Now for lograithmically divergent integrals

$$\Lambda + O(q_{\text{net}}) \text{ is as good as } \Lambda$$

$$\int_0^{\Lambda+O(q^{\text{net}})} \frac{2k_E^3 dk_E}{(k_E^2+\Delta)^2} = \log\left(\frac{(\Lambda+O(k_{\text{net}}))^2}{\Delta}\right) - 1 \approx \log \Lambda^2/\Delta - 1 + O(q_{\text{net}}/\Lambda)$$

But for worse divergences this is no good. For example if

$$\int = (\text{upper limit})^2 = \Lambda^2 + \underline{O(\Lambda q_{\text{net}})}$$

The underlined term is definitely trouble. That's not the only problem with the hard edge cutoff. Here are some other theories

(a) In gauge theories like QED or QCD hard edge cutoff breaks gauge invariance. Arbitrary phase change can only go up to momentum. In practice breaking gauge invariance means broken ward identites which is trouble.

(b) Hard edge cutoff changes analytical properties of the amplitudes. Thinking about this kind of stuff was all the rage in the 60's.

The bottom line is that the hard edge cutoff mostly works in perturbation theory but makes no physical sense as a complete non-perturbative theory. That's all we can say about the hard edge cutoff.

Pauli Villars regularization

In this case loop momenta q^μ are unlimited but the effects of high $q \geq O(\Lambda)$ is **cancelled** by similar-loops of very heavy particles(fields). So we have

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - \Lambda^2 + i0} \right\} \qquad q_2 = q_{\text{net}} - q_1$$

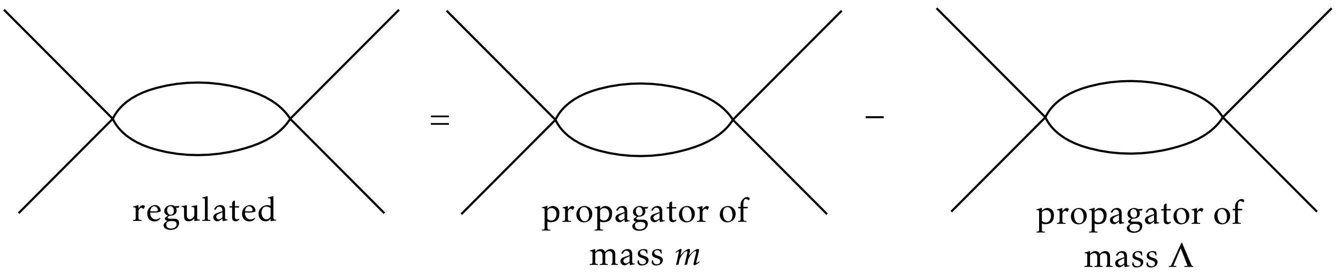


Figure 4: Visual representation of the Pauli Villars regularzation scheme

And notice that we subtract before integration. So for $q_1, q_2 \ll \Lambda$

$$Firstterm \approx \frac{1}{q_2^2},\ Secondterm = O(1/\Lambda^4)$$

Note that the second term is much less than the first term so physical first term. However for $q_1, q_2 \gg \Lambda$ the integrand I

$$I \approx \frac{1}{q_2^2} - \frac{1}{(q^2 - \Lambda^2)^2} \approx O(\Lambda^2/q^2)$$

And so the integral converges. Evaluating the integral is basically introducing feynman paramters and thus finding that it behavies like $\log(\Lambda/\Delta)$ plus some constant.

So to summarize Pauli Villars is good for pertubation theory. when $qp_{\text{external}} \ll \Lambda$. But this cannot be extened to a physical theory at all energies. This is because of the propagators in the compensating loops. On one side we have scalar propagators meaning that we need a scalar field of mass Λ. But on the other hand the minus sign of the loop means fermi statistics. Basically we have a scalar fermion which breaks spin-statistics. Spin-statics assuem relativity, positive energy of all particles, and positive norm on hilbert space. What we break is wrong sign of Hilbert space norm for the compensating scalar which is very unphysical. And this is why we cannot just incorporate that scalar in the theory and then just say "it's a physcial theorywith one fat boy." You cannot extend the theory to the energy where that particle can be produced. That would be unphyscial.

Covariant Higher Derivatives

In this case we have softer propagators at $q > O(\Lambda)$ by adding higher derivatives terms to the lagrangian. For example for a scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 - \frac{1}{2\Lambda^2}(\partial^2\phi)^2$$

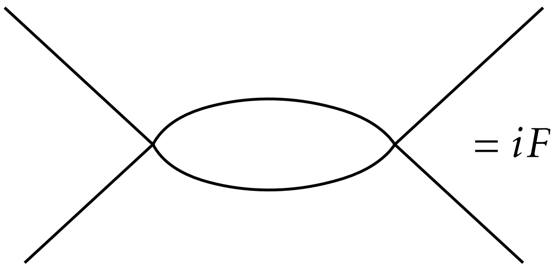
So free ϕ obeys $(\partial^2 + m^2 + \lambda^4/\Lambda^2)\phi = 0$ and that makes the greens function for a propagator

$$G = \frac{i}{q^2 - m^2 - q^4/\Lambda^2 + i\epsilon}$$

And that means for $q^2 \ll \Lambda^2$ the propagator is the usual thing and for large $q \gg \Lambda$ the propagator becomes

$$G \approx -\frac{i\Lambda^2}{q^4} \ll \frac{1}{q^2}$$

Basically this is supressed But for $q^2 \gg \Lambda^2$ we get $\int \frac{\Lambda^4}{q^8} d^4q$ which is convergent. Bottom line



$$= iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\frac{1}{q_1^2-m^2-q_1^4/\Lambda^2+i0} \times \frac{1}{q_2^2-m^2-q_2^4/\Lambda^2+i0}$$

Figure 5: Visual represntation of Higher Order Derivative regulariztion scheme

is that higher derivative regular is good for perturbation theory but does not work as a complete theory for energies about equal to Λ. Trouble with higer derivativve regulator: higher derivatives means that ϕ encodes several particles.

$$\frac{i}{q^2-m^2-q^4/\Lambda+i0} \approx \frac{i}{q^2-m^2+i0} - \frac{i}{q^2-\Lambda^2+i0}$$

We will learn soon that a pole in the propagator coresponds to a physical mass of the particle. So 2 scalar particles are encoded in this scalar field. Also for $q^2 = \Lambda^2$ residue of the pole has the wrong sign which causes with particles $m = \Lambda$ has negative Hilbert space norm. A unphysical ghost particle has appeared. This regulation scheme works for some things.

Aside: Using Pauli-Villars and Higher Derivative Reularization Scheme

We used hard edge to evaluate the loop diagram in lecture. In this section we use PV and HD.

(a) **Pauli-Villars:** In this regularization scheme we deal with the UV divergence by introducing a super heavy particle. Lets start by examinig the t-channel contribution. From out notes we have

$$iF(t) = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\left\{\frac{1}{q_1^2-m^2+i0}\times\frac{1}{q_2^2-m^2+i0} - \frac{1}{q_1^2-\Lambda^2+i0}\times\frac{1}{q_2^2-\Lambda^2+i0}\right\}$$

As perscribed in the lecture we can evaluate this integral by introducing Feynman parameters. So first we will consider

$$\frac{1}{q_1^2-m^2+i0}\times\frac{1}{q_2^2-m^2+i0} = \int_0^1 \frac{dx}{[(1-x)(q_1^2-m^2+i0)+x(q_2^2-m^2+i0)]^2}$$

The term in the square brackets we can expand out and simplify using $q_2 = q_{\text{net}} - q_1$ and mathematica

$$\begin{aligned}(1-x)(q_1^2 - m^2 + i0) + x(q_2^2 - m^2 + i0) &= i0 - m^2 + q1^2 - q1^2 x + q_2^2 x \\ &= i0 - m^2 + q_1^2(1-x) + xq_{\text{net}}^2 + xq_1^2 - 2xq_{\text{net}}q_1 \\ &= i0 - m^2 + xq_{\text{net}}^2 + q_1^2 - 2xq_{\text{net}}q_1 \\ \text{Complete the square} &= i0 - m^2 + xq_{\text{net}}^2 + (q_1 - xq_{\text{net}})^2 - x^2 q_{\text{net}}^2 \\ &= (q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0\end{aligned}$$

Where $\Delta(x)$ is defined implicitly above. So now we have

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2}$$

Meaning that

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} - \int_0^1 \frac{dy}{[(q_1 - yq_{\text{net}})^2 - \Delta(y,\Lambda) + i0]^2} \right\}$$

First let us consider the red term. We can change the order of integration

$$\begin{aligned}\int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} &= \int_0^1 dx \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} \\ \text{Let } k = q_1 - xq_{\text{net}} &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta(x,m) + i0]^2}\end{aligned}$$

Now focusing on the d^4k integral. Note in the k_0 integral we have poles when $k_0 = \pm[\sqrt{\mathbf{k}^2 + \Delta(x,y} - i0]$. Lets perform a wick rotation CCW by letting $k_0 = ik_4$ meaning that $k_\mu k^\mu = -k_4^2 - \mathbf{k}^2$. Namely the metric becomes euclidean. So we can define a $k_E = (k_1, k_2, k_3, k_4)$ and get the integral

$$\int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} = i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2}$$

However we can go even further. The integral only depends on k_E^2 so we have a $SO(4)$ symmetry. This means we can change $d^4 k_E = k_E^3 dk_E d\Omega^3$. This is in analogy to the spherical integral we're used to $d^3x = r^2 dr d\Omega$. In 4 dimensions $\int d\Omega^3 = 2\pi^2$. So now we have

$$i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2} = i \int_0^1 dx \int \frac{k_E^3 dk_E \, 2\pi^2}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2}$$

We can get a similar result for the blue term. So in all we have

$$F(t) = \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^\infty dk_E \times k_E^3 \left\{ \frac{1}{[k_E^2 + \Delta(x,m)]^2} - \frac{1}{[k_E^2 + \Delta(x,\Lambda)]^2} \right\}$$

Lets evaluate these integrals. First let $u = k_E^2 + \Delta(x,m)$ meaning that $du = 2k_E dk_E$ meaning that $k_E^3 dk_E = du(u - \Delta(x,m))/2$

$$\int_0^\infty \frac{k_E^3}{[k_E^2 + \Delta(x,m)]^2} dk_E = \int_\Delta^\infty \frac{du(u-\Delta(x,m))}{2u^2}$$
$$= \frac{1}{2}\left(\ln(u) + \Delta(x,m)u^{-1}\right)\Big|_\Delta^\infty$$
$$= \frac{1}{2}(\ln(\infty) - \ln(\Delta(x,m)) - 1)$$

Now adding the two contributaions gives us

$$F(t) = \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\{\ln(\infty) - \ln(\Delta(x,m)) - 1 - \ln(\infty) + \ln(\Delta(x,\Lambda)) + 1\}$$
$$= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Delta(x,\Lambda)}{\Delta(x,m)}\right)\right\}$$
$$= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}\right)\right\}$$

Take $\Lambda^2 \gg m^2, q_{\text{net}}^2$ giving us

$$= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}\right)\right\}$$
$$= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2}{m^2}\right) - \ln\left(\frac{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2}\right)\right\}$$

$$\boxed{F(t) = \frac{\lambda^2}{32\pi^2} \times \left\{\ln(\Lambda^2/m^2) - J(t/m^2)\right\}}$$

(b) **Higher Derivatives:** In this regularization scheme we deal with UV divergence by introducing higher derivative terms to the lagrangian. This gives us the propagator

$$iF = \frac{\lambda^2}{2}\int \frac{d^4 q_1}{(2\pi)^4}\left\{\frac{1}{q_1^2 - m^2 - q_1^4/\Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 - q_2^4/\Lambda^2 + i0}\right\}$$

In the problem set it is suggested we use

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i0} \approx \frac{i}{q^2 - m^2 + i0} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i0}$$

To see where this approximation comes fromes lets multiply the things together. The RHS becomes

$$\text{RHS} = \frac{i(-\Lambda^2)}{-0^2 - 0i\Lambda^2 - 0im^2 + L^2 m^2 + 2iq^2 0 - L^2 q^2 - m^2 q^2 + q^4}$$

Now we'll ignore higher order $i0$ terms and assume that $\Lambda^2 \gg m^2, q^2$ to get

$$\text{RHS} \approx \frac{i}{i0 - m^2 + q^2 - q^4/\Lambda^2} = LHS$$

Lets plug this prescribed approximation in

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\left\{\frac{1}{q_1^2-m^2+i0}\times\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\times\frac{1}{q_2^2-m^2+i0}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\right\}$$

The red terms are out of control so lets apply Feynman's very poggers trick to deal with them. From part (a) of this problem we have

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}$$

The problem again prescribes that we use

$$\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\approx\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\approx\frac{-\Lambda^2}{(q_1-xq_{\text{net}})^2-\Lambda^2+i0}$$

First lets try to understand the $(q_1-xq_{\text{net}})^2$ term. In the limit where $\Lambda^2 \gg q_1^2, q_2^2$ we know that the difference between q_1^2/Λ^2 and q_2/Λ^2 is negligible and so interpolating between q_1^2 and q_2^2 using $q_1-xq_{\text{net}})^2$ makes sense due to the fact that it's basically a constant as well. So for the low energy limit I kinda see what this trick is doing. Now for the $\Lambda^2 \ll q_1^2, q_2^2$ limit the whole term is $O(1/q^2)$ so I guess q_{net} is negligible so we can interpolate between the q_1 and q_2 values as above without too much trouble. This feels sketchy but we'll go along with it for now. If we plug this interpolation into our above iF we get

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}$$

Lets try to do the same wick rotation trick we did in the last part of this problem.

$$\begin{aligned}iF &= \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}\\ &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4q_1\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}\\ &\text{Let } k=q_1-xq_{\text{net}} \text{ giving us}\\ &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4k\left\{\frac{\Lambda^4}{[k^2-\Lambda^2+i0]^2}\times\frac{1}{[k^2-\Delta(x,m)+i0]^2}\right\}\end{aligned}$$

Focusing on the d^4k integral. We have four poles but we can avoid them in the same way with a Wick rotation $k_0 = ik_4$ which allows us to wrtie

$$iF = i\frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4k_E\left\{\frac{\Lambda^4}{[k_E^2+\Lambda^2]^2}\times\frac{1}{[k^2+\Delta(x,m)]^2}\right\}$$

Using the $SO(4)$ symmetry to write

$$iF = i\frac{\lambda^2}{2(2\pi)^4}\times 2\pi^2\int_0^1 dx\int_0^\infty dk_E\times k_E^3\left\{\frac{\Lambda^4}{[k_E^2+\Lambda^2]^2}\times\frac{1}{[k_E^2+\Delta(x,m)]^2}\right\}$$

Now how do we evaluate this? Excellent question! I have no idea. But mathematica does!

$$\int_0^\infty dk_E \times \ldots = \frac{\Lambda^4}{2\left(\Lambda^2-\Delta(x,m)\right)^3}\left(\frac{\Delta(x,m)\left(\Lambda^2-\Delta(x,m)\right)}{\Delta(x,m)+k_{\mathrm{E}}^2} - \log\left(k_{\mathrm{E}}^2+\Lambda^2\right)\left(\Delta(x,m)+\Lambda^2\right)\right.$$
$$\left.+\left(\Delta(x,m)+\Lambda^2\right)\log\left(\Delta(x,m)+k_{\mathrm{E}}^2\right)+\frac{\Lambda^4-\Lambda^2\Delta(x,m)}{k_{\mathrm{E}}^2+\Lambda^2}\right)\Bigg|_0^\infty$$
$$= \frac{\Lambda^4}{2(\Lambda^2-\Delta(x,m)^3)}\Big(0+(\log(\infty)-\log(\infty))(\Delta(x,m)+\Lambda^2)+0$$
$$-\Lambda^2+\Delta(x,m)+(\Delta(x,m)+\Lambda^2)\log(\Delta(x,m))-\log(\Lambda^2))-\Lambda^2+\Delta(x,m)\Big)$$
$$= \frac{\Lambda^4}{2(\Lambda^2-\Delta(x,m))^3}\Big(-2(\Lambda^2-\Delta(x,m))+(\Lambda^2-\Delta(x,m))\log(\Lambda^2/\Delta(x,m))\Big)$$
$$= \frac{\Lambda^4}{2(\Lambda^2-\Delta(x,m)^2}\Big(-2+\log(\Lambda^2/\Delta(x,m))\Big)$$

When we assert $\Lambda^2 \gg m^2, q_{\text{net}}^2$ we get that

$$\frac{\Lambda^4}{2(\Lambda^2-\Delta(x,m))^2} \approx \frac{\Lambda^4}{2\Lambda^4} = \frac{1}{2} \qquad \log\left(\frac{\Lambda^2}{\Delta(x,m)}\right) = \log\left(\frac{\Lambda^2}{m^2}\right) - \log\left(\frac{\Delta(x,m)}{m^2}\right)$$

$$\Rightarrow \int_0^1 dk_E \times \cdots = \frac{1}{2}\Big(\log(\Lambda^2/m^2) - 2 - \log(\Delta(x,m)/m^2)\Big)$$

Putting this all together gives us

$$\boxed{F = \frac{\lambda^2}{32\pi^2}\left\{\log\left(\frac{\Lambda^2}{m^2}\right) - J\left(\frac{t}{m^2}\right) - 2\right\}}$$

OPTICAL THEOREM

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IN QUANTUM MECHANICS

Mostly from L19 and L20 of Zwiebach's course on OCW. First lets review scattering in quantum mechanics. Lets say we want to study a potential. One way we can do this is sending particles in this potential and seeing how they scatter. In general we have a beam of particles going towards a target that get scattered to detectors. Lots of weird things can happen in these scattering experiments. For example

$$p + p \to p + p + \pi^0$$

Two protons turns into two protons and a pion. There is no notion of a conservation of particles. Recall this was one of our motivations for quantum field theory. However for right now we won't consider changes of particles in the process. Namely our processes will consider processes of the flavor

$$a+b \to a+b$$

We'll also restrict our discussion to elastic scattering(internal states do not changes.) An example of inelastic scattering is the Frank-Hertz experiment. Here will also assume no spin, non-relativistic, and interactions are of the form $V(\mathbf{r}_1 - \mathbf{r}_2)$ which allows us to analyze this problem like a central force scattering.

So what do we need to solve? First we have the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

and assume we have a energy eigenstate $\psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-\frac{iEt}{\hbar}} \Rightarrow$ We want to solve the time-independent Schrodinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

First we'll consider the potential to have finite volume. We'll assume that outgoing states are plane waves which have energy $E = \frac{\hbar^2 k^2}{2m}$. Putting this onto the RHS of our equation gives us

$$\left[-\frac{\hbar^2}{2m}(\nabla^2 + k^2) + V(\mathbf{r})\right]\psi(\mathbf{r}) = 0$$

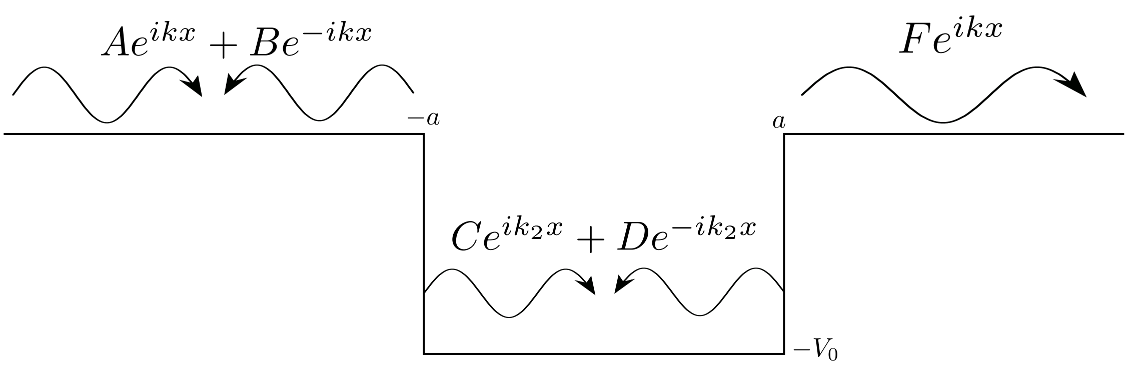

Figure 6: Illustration of a one dimensional potential where we decompose the solution into ingoing and outgoing waves for various regions. Intuition behind scattering

Before we continue lets remember how we dealt with one-dimensional potentials (e.g. Figure 6). We'd be throwing some wave into the potential and then use our intuition to write out a form of the solution in terms of ingoing and outgoing waves. This is what we'll be doing here in scattering. Lets consider the simplest case $V = 0$. Then $\psi = e^{i\mathbf{k}\cdot\mathbf{x}}$ are solutions. This is the same as saying that Ae^{ikx} in Figure 6 is a solution before we ever see the potential. From this lets define an incident wave function $\phi(\mathbf{x}) = e^{ikz}$. We say ϕ instead of ψ since ψ is usually used to denote the full solution whereas ϕ indicates that it might not be the full solution. We know that ϕ is a solution of the S.E. away from $V(r)$. To complete our solution we should expect a spherical outgoing waves. To write this spherical wave we write e^{ikr}. However there is a physical reason that this doesn't work very well. The solution doesn't fall off meaning that we're just accumulating probability as we go radially outwards. To see this from equations just note that $(\nabla^2 + k^2)e^{ikr} \neq 0$. It turns out the one we want is

$$(\nabla^2 + k^2)e^{ikr}/r = 0$$

So lets try to write the scattering solution as

$$\psi_{\text{scattering}}(\mathbf{x}) = \frac{e^{ikr}}{r}$$

But there is no reason that the solution doesn't also depend on θ and ϕ. So lets add a function to our solution

$$\psi_{\text{scattering}}(\mathbf{x}) = f(\theta, \phi)\frac{e^{ikr}}{r}$$

We'll see later that this is the leading term of a solution. So our full solution outside of the potential can be written as

$$\psi(\mathbf{r}) \approx \phi(\mathbf{r}) + \psi_{\text{scattering}}(\mathbf{r}, \theta, \phi) = e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r}$$

What we need to figure out is $f(\theta, \phi)$, the scattering amplitude. So here we have finished setting up the problem! We'll build to a computation of $f(\theta, \phi)$ using partial waves and phase shifts. But before we do that we need to talk about cross-sections. Lets say in some solid angle $d\Omega$ that corresponds to the detector we see n particles. we can associate a $d\sigma$ on the volume of the potential that captures n particles. What this means is that we get a idea of how the particle sees the particles.

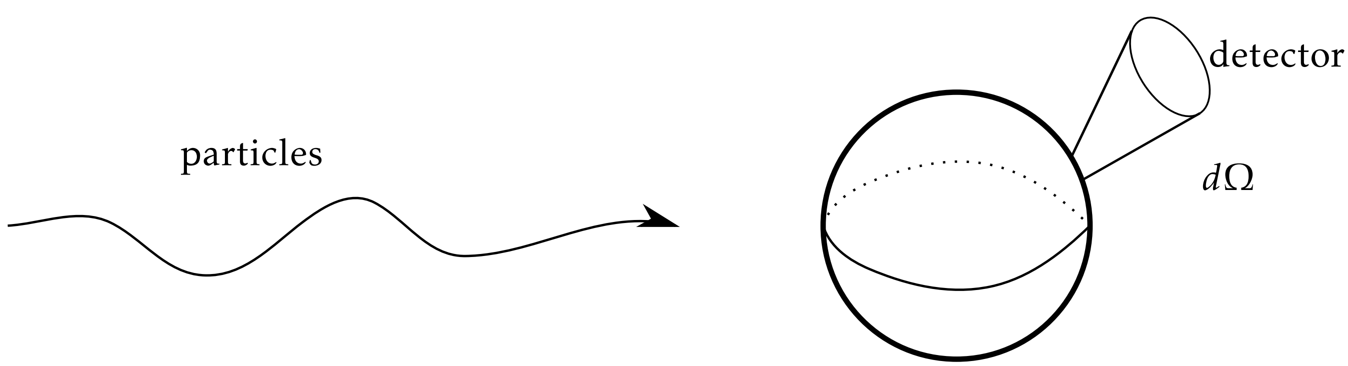

Figure 7: Visualization of scattering cross section.

$$d\sigma = \frac{\text{number of particles scattered per unit time into solid angle } d\Omega \text{ at } (\theta, \phi)}{\text{flux of incidnt particles}}$$

We can think of incdient flux IF (really the probability current) as

$$IF = \frac{\hbar}{m}\text{Im}\left(\phi(\mathbf{r})^*\nabla\phi(\mathbf{r})\right) = \frac{\hbar k}{m}\hat{\mathbf{z}}$$

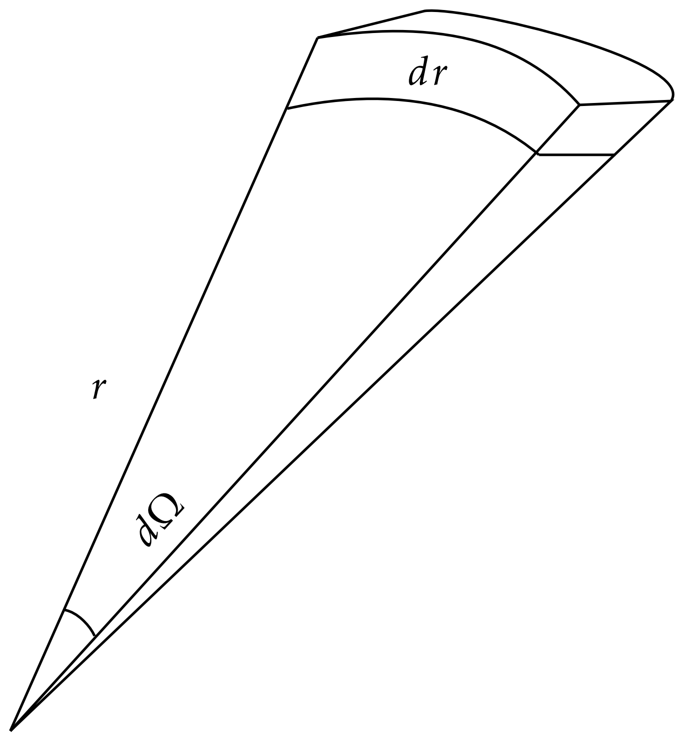

Figure 8: Illustration used in derivation of scattered particle density

Now lets try to find the number of particles in the small volume dr in Figure 8

$$dn = \text{number of particles in } dr$$

To find this we need to find the square of the wave function in the scattered region $\psi_{\text{scattering}}$. This gives us

$$dn = \left| f(\theta,\phi)\frac{e^{ikr}}{r} \right|^2 r^2 d\Omega dr = |f(\theta,\phi)|^2 d\Omega dr$$

Now all these particles will go through the box a infinitesmial $dt = \frac{dr}{v}$ which is

$$dt = \frac{mdr}{\hbar k}$$

Thus we have

$$\frac{dn}{dt} = \frac{\hbar k}{m}\left|f(\theta,\phi)\right|^2 d\Omega$$

And finally we get

$$d\sigma = \frac{\frac{\hbar k}{m}\left|f(\theta,\phi)\right|^2 d\Omega}{\frac{\hbar k}{m}} = |f(\theta,\phi)|^2 d\Omega$$

Which can be written as

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2}$$

And integrating this we can get a total cross section

$$\boxed{\sigma = \int d\Omega \ \left|f(\theta,\phi)\right|^2}$$

Now lets reduce to a central force. First will we assume that

$$V(\mathbf{r}) = V(r)$$

So we should expect only $f(\theta)$. Now lets consider the solution of the free particle solution in a spherically symmetric potential

$$\psi(\mathbf{r}) = \frac{\mathcal{U}_{E\ell}(r)}{r} Y_{\ell m}(\Omega)$$

And from this we can get the schordinger equation for $\mathcal{U}_{E\ell}$ as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right)\mathcal{U}_{E\ell}(r) = \frac{\hbar^2 k^2}{2m}\mathcal{U}_{E\ell} \Rightarrow \left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right)\mathcal{U}_{E\ell} = k^2\mathcal{U}_{E\ell}$$

Now we'll define a new variable $\rho = kr$ which makes the quation simpler

$$\left(-\frac{2}{\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right)\mathcal{U}_{E\ell} = \mathcal{U}_{E\ell}$$

These are solved by Bessel functions

$$\mathcal{U}_{E\ell} = A_\ell \rho j_\ell(\rho) + B_\ell \rho n_\ell(\rho)$$

Both of these behave nicely at infinity. Now we sould be able to write the plane wave solution e^{ikz} as a superposition of our solutions above. To do this first note that the plane wave isn't singular at the origin so we can ignore the spherical Neumann function. Furthermore we have azimuthal symmetry thus we can also ingore the m quantum number. All of this together we get

$$e^{ikz} = \sum_\ell a_\ell j_\ell Y_{\ell,0}$$

Now by nature of the spherical bessel function and the spherical harmonics we can find that

$$e^{ikz} = e^{ikr\cos\theta} = \sqrt{4\pi}\sum \ell\sqrt{2\ell+1}\, i^\ell Y_{\ell,0}(\theta) j_\ell(kr)$$

Lets take a closer look at the spherical bessel function. As we go towards infinitiy we have

$$j_\ell(kr) \to \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right)\frac{1}{2ik}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

The terms in the brackets are just ingoing and outgoing waves. This fact will be useful when we start trying to solve this problem. Lets plug this ins

$$e^{ikz} \approx \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, i^\ell Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2}}{r}\right\}$$

The "partial wave" refers to the fact that each ℓ term are independent. To understand how this works better lets consider the simplest case. $D = 1$. Lets say we have some hard wall with some potential $V(x)$ before we hit the hard wall. If $V = 0$ we have the incoming $\phi = \sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$. The full solution then

$$\psi(x) = \frac{1}{2i}\left(\cdots - e^{-ikx}\right)$$

So we have the same incoming wave but some undetermined outgoing wave that we must to solve for. What do we know about the outgoing wave? Well the probabilty current must be conserved and must have the same energy. So what can we write? Well we could write in a phase shift

$$\psi(x) = \frac{1}{2i}\left(e^{-ikx+2i\delta_k} - e^{-ikx}\right)$$

Which is valid for $x > a$. We can also assert that

$$\psi(x) = \phi(x) + \psi_{\text{scattered}}(x)$$

Solving for the scattered scattered wave

$$\psi_{\text{scattered}} = \frac{1}{2i}e^{ikx}\left(e^{2i\delta_k} - 1\right) = e^{ikx}e^{i\delta_k}\sin\delta_k$$

How do we extend this to three dimensions? Well consider

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\} + f_{\ell}(\theta)\frac{e^{ikr}}{r}$$

Where the hell is the incoming wave in this mess? There's only one and it's the red term. We can also write

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{??? - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

Namely we have some undetermined outgoing wave and the same incoming wave. Now we need some inspiration(whisky?) to figure out what this ??? term is. First we know that the probability must be conserved. Futhermore we know each ℓ works independently. So that means that the amplitude of the ??? term has to be the same as the incoming wave for each ℓ or else the probabiliy would not be conserved which would be a disaster. So using our intuition from the 1-d case we will write an outgoing wave with some phase shift

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)+2i\delta_\ell}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

The δ_ℓ parameterizes our ignorance. We can know solve for $f_\ell(\theta)$

$$f_\ell(\theta)\frac{e^{ikr}}{r} = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell0}\underbrace{\frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)}_{e^{i\delta}\sin\delta}\frac{e^{ikr}}{r}\underbrace{e^{-i\pi\ell/2}}_{(-i)^{\ell}}$$

And after a lot of happy cancellations we get

$$\boxed{f_\ell(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,Y_{\ell 0}(\theta)e^{i\delta_\ell}\sin\delta_\ell}$$

A lot of simple things come out of this. Recall

$$\sigma = \int |f_\ell(\theta)|^2 d\Omega = \int f_\ell(\theta)^* f_\ell(\theta) d\Omega$$

Now plugging in our formula for $f_\ell(\theta)$ give us

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell,\ell'}\sqrt{2\ell+1}\sqrt{2\ell'+1}e^{-i\delta_\ell}\sin\delta_\ell e^{i\delta_{\ell'}}\sin\delta_{\ell'}\int d\Omega\; Y^*_{\ell 0}(\Omega)Y_{\ell' 0}(\Omega) = \frac{4\pi}{k^2}\sum(2\ell+1)\sin^2\delta_\ell \tag{OT*}$$

There is one more important result that we should talk about which was the whole point of this song and dance. **Optical Theorem.** Lets say you have some detector surrounding some scattering experiment. The "shadow" of the scattering object equal to the scattered particles that you can detect. In EM the intuition for this shadow should be clear. You shine some light on a object and the "shadow" relates to the photons that are scattered off the surface. However here the intuition of the shadow is not so clear. We have to parts of the solution $\psi(\mathbf{r})$ that contribute to the shadow. So lets see how this works in quantum mechanics. First we need the result that

$$Y_{\ell 0}(\theta = 0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

We can then write

$$f_k(\theta = 0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \Rightarrow \text{Im } f_k(\theta=0) = \frac{1}{k}\sum_\ell(2\ell+1)\sin^2\delta_\ell$$

But compare this to (OT*). What we see is that

$$\boxed{\sigma = 4\pi \text{Im } f_k(\theta = 0)}$$

In Quantum Field Theory

See Lecture 4.

Lecture 4: More Regularization and Optical Theorem

January 26, 2021

One thing he did not finish is that in gauge theories the higher derivative regularization scheme becomes the **covariant higher derivative regularization scheme.** For example in QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\overline{\psi}(i\not{D} + e\not{A} - m)\psi) + \frac{1}{4\Lambda^2}(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu}) + \frac{1}{2\Lambda^2}\overline{\psi}(i\not{\partial} + e\not{A})^3\psi$$

If you look at the final term you find that we get a softer propagator

$$\text{three point vertex } = ie\gamma^{\nu}\left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)$$

For all multi loop graphs the propagator allows for UV regularization. But for one loop graphs and especially the graphs that look like two, three, and four legged spiders we get no regularization with the above lagrangian. So we need extra regulators like Pauli-Villars. So in practice this is used for proving all sorts of theorems. That's it's for higher derivatives.

There is a renormalization scheme that is a perfectly good non-perturbative equantum theory: lattice renormalization. We need to go through the path integrals to really talk about this so Prof. Kaplunovsky will come back to it after we discuss path integrals (after spring break.) For now lets put this aside and talk about dimensional regularization as the UV regular most commonly used (in high energy physics.)

Dimensional Regularization

What we want to do is generalize loop integrals to fractional (or even not rational) space time dimensions and then analytically continue back to $D = 4$.

$$\int_{\text{reg}} \frac{d^4k_E}{(2\pi)^4} f(k_E^{\nu}) = \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} f(k_E^{\nu})$$

Where μ is the reference energy scale at which the spherical momentum space shell dk_D^{rad} has the same volume in D dimensions as in 4-dimensions. Now why is this regulated? Lets consider the radial integral. We start with

$$d^4k_E \propto (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

Now in D dimensions we get

$$(k_E^{\text{rad}})^3 dk_E^{\text{rad}} \to \mu^{4-D} \times (k_E^{\text{rad}})^{D-1} dk_E^{\text{rad}} = \left(\frac{\mu}{k_E^{\text{rad}}}\right)^{4-D} \times (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

The red term is what regularizes things. Lets take a closer look at the red factor. Lets consider $D = 4 - 2\epsilon$

$$\left(\frac{\mu}{k_E^{\text{rad}}}\right)^{4-D=2\epsilon} = \left(\frac{k_E^2}{\mu^2}\right)^{-\epsilon} = \exp\left(-\epsilon \times \log\frac{k_E^2}{\mu^2}\right)$$

The exponential becomes small when the log term becomes proportional to $\frac{1}{\epsilon}$.

Lets consider a generic logarithmically divergent momentum integral. From our other regularization schemes we saw that

$$\text{regulated integral} = C \times \log\frac{\Lambda^2}{m^2} + \dots$$

So for dimensional regularization we expect

$$\text{regulated integral } = C \times \left(\frac{1}{\epsilon} + \log\frac{\mu^2}{m^2}\right) + \dots$$

TODO why do expect this??? Before we continue lets learn how to take integrals in arbitrary dimensions. Consider

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2)$$

For an integer dimension D we have $k_E^2 = k_1^2 + \cdots + k_D^2$. From this we get

$$\exp(-t k_E^2) = \prod_i \exp(-t k_i^2)$$

Using this allows us to rewrite the integral

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2) = \prod_i \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \exp(-t k_i^2) = (4\pi t)^{-\frac{D}{2}}$$

It's just a gaussian integral! Now lets just analytically continue this formula for non-integer D byt saying

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2) = (4\pi t)^{-\frac{D}{2}}$$

So what will we do for non-Gaussian momentum integrals? I would cry.

$$I = \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \int \cdots$$

Well first lets recall the gamma function integral

$$\int_0^{\infty} dt\ t^{n-1} \times \exp(-t(k_E^2 + \Delta)) = \frac{\Gamma(n)}{[k_E^2 + \Delta]^n}$$

For $n = 2$ we have

$$\frac{1}{[k_E^2 + \Delta]^2} = \frac{1}{\Gamma(2) = 1} \times \int_0^{\infty} dt\ t \times \exp(-t(k_E^2 + \Delta))$$

This gives us

$$\begin{aligned}
\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} &= \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} \int_0^{\infty} dt\ t \times \exp(-t(k_E^2 + \Delta)) \\
&= \int_0^{\infty} dt\ t e^{-t\Delta} \times \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} e^{-t k_E^2}
\end{aligned}$$

Now just using our Gaussian integral we get (μ shouldn't be there?)

$$\int \cdots = \int_0^{\infty} dt\ t e^{-t\Delta} \times (4\pi t)^{-\frac{D}{2}} = (4\pi)^{-\frac{D}{2}} \int_0^{\infty} dt\ t^{1-\frac{D}{2}} \times e^{-t\Delta}$$

Now as long as the power of $t > -1$ the integral converges. This means for $D < 4$ or we have a complex D, whenever $\mathrm{Re}(E) < 4$. What's the physcial meaning of this divergence? It's just the ultraviolet divergence of the original integral. In particular for $D = 4 - 2\epsilon$ we get

$$\mu^{4-D} \times \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} = \frac{(4\pi\mu^2)^{\epsilon}}{16\pi^2} \times \Gamma(\epsilon) \times \Delta^{-\epsilon}$$

From this we can extract the matrix element for the t-channel process

$$\mathcal{F}(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\,\Gamma(\epsilon)\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon}$$

This is actually an exact answer for $D = 4-2\epsilon$. Now how do we see that this diverges as $D \to 4$? Well the gamma function blows up as $\epsilon \to 0$ due to the poles at $\epsilon = 0$. How do we calculate these poles? Using $\Gamma(x+1) = x\Gamma(x)$ we can consider

$$\begin{aligned}\Gamma(\epsilon) &= \frac{\Gamma(\epsilon+1)}{\epsilon} = \frac{1}{\epsilon}\left(\Gamma(1)+\epsilon\times\Gamma'(1)+\frac{\epsilon^2}{2}\Gamma''(1)+\dots)\right) \\ &= \frac{1}{\epsilon}-\gamma_E+\frac{\pi^2+6\gamma_E^2}{12}\times\epsilon+O(\epsilon^2)\end{aligned}$$

Where γ_E is the Euler-Mascheroni constant. In this we can clearly see the pole at $\epsilon = 0$. Lets try expanding the term inside the parenthesis in the integral in powers of ϵ

$$\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon} = \exp\left(\epsilon\times\log\left(\frac{4\pi\mu^2}{\Delta(x)}\right)\right) = 1+\epsilon\times\log\frac{4\pi\mu^2}{\Delta(x)}+\dots$$

The point of all this is to show that we should be careful about taking $\epsilon \to 0$. What the hell do we do with this expression? For other regularization schemes the idea was that we toss out all negative powers of Λ. In this case we note that a positive power of ϵ corresponds to a negative power of Λ. This means the integrand over the feynman parameter becomes

$$\Gamma(\epsilon)\times\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^{\epsilon} \to \frac{1}{\epsilon}-\gamma_E+\log\frac{4\pi\mu^2}{\Delta(x)}$$

Giving us

$$\mathcal{F}_{\mathrm{DR}}(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{1}{\epsilon}-\gamma_E+\log\frac{4\pi\mu^2}{\Delta(x)}\right) = \frac{\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon}-\gamma_E+\log\frac{4\pi\mu^2}{m^2}-J\left(\frac{t}{m^2}\right)\right)$$

So far all our answers for regularizations have been of a similar form. They can be the exact same however if

$$\log\Lambda_{HE}^2-1 = \log\Lambda_{PV}^2 = \log\Lambda_{HD}^2-2$$

All regularization schemes are in the same ballpark and related by a numerical constant. We can extend this identification by letting

$$\frac{1}{\epsilon}-\gamma_E+\log(4\pi\mu^2) = \log\Lambda_{HE}^2-1 = \log\Lambda_{PV}^2 = \log\Lambda_{HD}^2-2$$

Optical Theorem

This theorem relates forward amplitude with total cross section. For example consider scattering forward scattering ampltiude for elastic scattering $f_{\mathrm{e}}(\theta=0)$. We have

$$\mathrm{Im}\ f_{\mathrm{e}}(\theta=0) = \frac{k_{\mathrm{reduced}}}{4\pi}\times\sigma_{\mathrm{total}}$$

Now in the partial wave analysis we decompose the the scattering function into eigenstates of angular momentum

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)\times\left(\frac{e^{2i\delta_l-1}}{2i} = e^{i\delta_l}\sin\delta_l\right)$$

Using this we can do all our gymanstics to get

$$\sigma_{\text{tota}} = \frac{4\pi}{k^2}\times\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Taking the imaginary part of forward scattering gives us

$$\text{Im}\, f(\theta=0) = \frac{1}{k}\times\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

Now lets talk about this in a relativistic notation. We rewrite our first statement as

$$\text{Im}\,\mathcal{M}_{\text{elastic,forward}} = 2E_1E_2|\mathbf{v}_1-\mathbf{v}_2|\times\sigma_{\text{total}}$$

The prefactor is invariant under the boosts along the axis of collision. If we go to the center of masss frame the prefactor evaluates to $4E_{\text{cm}}^{\text{net}}|\mathbf{p}_{\text{cm}}|$. Lets prove this formula today. On thursday we'll apply this formula to $\lambda\phi^4$ theory. This theorem follows from the unitarity of the S-matrix (or the scattering operator $S^\dagger S = 1$). Lets write S as $S = 1 + iT$. Thus

$$1 = S^\dagger S = 1 + iT - iT + T^\dagger T \Rightarrow iT^\dagger - iT = T^\dagger T$$

Now lets sandwith the above with diagonal matrix elements $\langle i|\ldots|i\rangle$. First ths LHS

$$\left\langle i|iT^\dagger - iT|i\right\rangle = i\langle i|T|i\rangle^* - i\langle i|T|i\rangle = 2\text{Im}\,\langle i|T|i\rangle$$

Wheras on the RHS we'll resolve the idnetity

$$\left\langle i|T^\dagger T|i\right\rangle = \sum_{|f\rangle}|\langle f|T|i\rangle|^2$$

Now we have the result

$$2\text{Im}\,\langle i|T|i\rangle = \sum_{|f\rangle}|\langle f|T|i\rangle|^2$$

Ther's something wrong here though. Lets factor out the energy-momentum conservatio nwhich gives us

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_f - p_i)\times\langle f|M|i\rangle$$

There will be troublesome δ functions of both sides. To resolve this consider a state $|i\rangle$ that is similar to $|i\rangle$ except for the momentum. We'll take $|i\rangle$ so that we can approximaite

$$\langle i'|M|i\rangle = \langle i|M|i\rangle$$

Now we can sandhiwhc our things between $|i\rangle'$ and i. This gives us

$$\left\langle i'|iT^\dagger - iT|i\right\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i)\times 2\text{Im}\langle i|M|i\rangle$$

And on the RHS

$$\left\langle i'|T^\dagger T|i\right\rangle = \sum_{|f\rangle}\langle f|T|i'\rangle^*\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i)\times\sum_{|f\rangle}|\langle f|M|i\rangle|\times(2\pi)^4\delta^{(4)}(p_f - p_i)$$

Which finally gives us the result

$$2\text{Im}\langle i|M|i\rangle = \sum_{\langle f|}|\langle f|M|i\rangle|^2\times(2\pi)^4\delta^{(4)}(p_f - p_i)$$

Now let $|i\rangle$ be an inital state with two particles and $|f\rangle$ be all possible final states. the sum sums overall possible reactions

$$\sum_{\langle f|} = \sum_{\text{channels}}\prod_{a=1}^{n}\left(\sum_{s'_a}\int\frac{d^3p'_a}{(2\pi)^3 2E'_a}\right)$$

Plugging this into our above result gives us a big boy that is just the phase space integral (with spin sums). This gives us

$$\begin{aligned}2\text{Im}\langle i|M|i\rangle &= \sum_{\text{channels}} 4E_1E_2|v_{12}^{\text{rel}}|\times\sigma_{\text{net}}(1+2\to 1'+\cdots+n')\\ &= 4E_1E_2|v_{12}^{\text{rel}}|\times\sigma_{\text{total}}(1+2\to\text{ anything}\end{aligned}$$

And from this we have our optical theorem

$$\boxed{\text{Im}\mathcal{M}_{\text{elastic,forward}} = 2E_1E_2|v_{12}^{\text{rel}}|\times\sigma_{\text{total}}(1+2\to\text{ anything})}$$

Lecture 5: Applying Optical Theorem and Beginning Correlation Functions

January 28, 2021

Lets see how optical theorem work in $\lambda\phi^4$ theory.

Applying Optical Theorem to $\lambda\phi^4$ Theory

At the tree level

$$\frac{d\sigma_{\text{elastic}}}{d\Omega_{cm}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s}$$

Integrating we can get the net elastic scattering cross section

$$\sigma_{\text{net}}^{\text{elastic}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s}\times\frac{4\pi}{2} = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

For inelastic processes $2 \to n$ where $n \geq 4$ we find that the tree amplitude is $O(\lambda^{\frac{n}{2}})$ and thus $\mathcal{M} = O(\lambda^{\frac{n}{2}})$. This means that the total scattering cross section

$$\sigma_{\text{total}} = \sigma_{\text{elastic}}^{\text{net}} + O(\lambda^4) = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

Now applying optical theorme to this we get

$$\text{Im}\, \mathcal{M}(\text{elastic,forward}) = 2E_1E_2|\mathbf{v}_{12}^{\text{rel}}| \times \frac{\lambda^2 + O(\lambda^3)}{32\pi s} = \frac{\lambda^2 v}{32\pi} + O(\lambda^3)$$

From this we can see that

$$\text{Im}\mathcal{M}_{\text{tree}}(\dots) = 0 \qquad \text{Im}\mathcal{M}_{1-\text{loop}}(\dots) = \frac{\lambda^2 v}{32\pi} > 0$$

Aside: You can apply exactly the same analysis to QED. Take some tree level process. Amplitude is of order E^2 and so the cross section is of order α^2 and by optical theorem if we take an amptliude in the forward direction with unchanged spin then the tree level amplitude must be real but the 1-loop amplitude has an imaginary part.

Lets verify the optical theorem. At tree level

$$\mathcal{M}_{\text{tree}}^{\text{elastic}} = -\lambda, \;\; \text{Im}\mathcal{M}_{\text{tree}}^{\text{elastic}} = 0$$

We'll see soon that

$$\forall \theta: \quad \text{Im}\mathcal{M}_{1-\text{loop}}^{\text{elastic}} = \frac{\lambda^2 v}{32\pi} \qquad v = \frac{|p|}{E} = \sqrt{1 - \frac{m^2}{E^2}} = sqrt 1 - \frac{4m^2}{s}$$

In an earlier lecture we hsoed that

$$\mathcal{M}_{1-\text{loop}}^{\text{elastic}} = \frac{\lambda^2}{32\pi^2}\Big(J(4) - J(t/m^2) - J(u/m^2) - J(s/m^2)\Big)$$

Recalling the form of the J function we note that the green term is imaginary. Or more specifically for $s > 4m^2$ the argument of the logarithm becomes negative at some values of x. Now recall from complex analysis

$$\log(z) = \log(|z|) + i\arg(z)$$

some complex analysis here

We then find that

$$\text{Im}J\left(\frac{s}{m^2}\right) = \int_0^1 dx\, \text{Im}\log\frac{m^2 - sx(1-x)}{m^2} = \pm\pi \times (x_2 - x_1) = \pm\pi v$$

The sign convention comes from the branch cut and how we choose to analytically continue. In the end we get

$$\text{Im}J\Big(\big(s + i\epsilon/m^2\big)\Big) = \mp\pi v$$

Plugging this into our $\mathcal{M}_{1-\text{loop}}^{\text{elastic}}$ we get

$$\text{Im}\mathcal{M}_{1-\text{loop}}^{\text{elastic}}(s \pm i\epsilon, t) = \pm\frac{\lambda^2 v}{32\pi}$$

Now comparing with our result from optical theorem we need to choose $s + i\epsilon$. It turns out that there is another way to extract the imaginary part of $\mathcal{M}$. A guy named Kolkovol(?) created a diagrammatic way to extract these imaginary parts.

Introduction to Correlation Functions

We define the n-point correlation function

$$\mathcal{F}_n(x_1,\dots,x_n) = \left\langle \Omega | T \hat{\Phi_H}(x_1)\dots\hat{\Phi_H}(x_n)|\Omega\right\rangle$$

Or for vector or tensor fields or something

$$\mathcal{F}_n^{a_1\dots a_n}(\dots) = \left\langle \Omega | T \hat{\Phi_H^{a_1}}(x_1)\dots\hat{\Phi_H^{a_n}}(x_n)|\Omega\right\rangle$$

Last semester we learend how to calcualte correleation functions of interacting theory by using perturabtion theory and free correlation function. To figure out how we relate the correlation function of an interacting theory with the correlation function we defined above we first consider

$$\Phi_H(\mathbf{x},t) = e^{iHt}\Phi_S(\mathbf{x})e^{-iHt} = e^{iHt}e^{-iH_0t}\Phi_I e^{iH_0t}e^{-iHt}$$

We can define a unitary operator

$$U_I(t,t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}$$

Using this we can get

$$U_I(t,0) = e^{iH_0t}e^{-iHt} \qquad \text{and} \qquad U_I(0,t) = e^{iHt}e^{-iH_0t}$$

$$\Rightarrow \Phi_H(x) = U_I(0,x^0)\Phi_I U_I(x^0,0)$$

Now lets consider two fields

$$\Phi_H(x)\Phi_H(y) = U_i(0,x^0)\Phi_I(x)U_I(x^0,y^0)\Phi_I(y)U_I(y^0,0)$$

The above expression is true becasue $U_I(x^0,0)U_I(0,y^0) = U_I(x^0,y^0)$. From here we can generalize this to

$$\Phi_H(x_1)\Phi_H(x_2)\dots\Phi_H(x_n) = U_I(0,x^0)\Phi_I(x_1)U_I(x_1^0,x_2^0)\Phi_I(x_2)\dots$$

Now we need to relate the free vacuum to a true vacuum $|\Omega\rangle$. To do this we consider the state $U_I(0,-T)|0\rangle$ for a complet T. Taking the limit $T \to (1-i\epsilon)\times\infty$. Basically the real part grows to infinity faster than the imaginary part grows to$-\infty$. Lets now consider (where $H_0|0\rangle = 0$)

$$U_I(0,-T)|0\rangle = e^{-iHT}e^{iH_0T}|0\rangle = e^{-iHT}|0\rangle$$

We can expand the vacuum state in terms of eigenstates of H denoted by $|Q\rangle$

$$|0\rangle = \sum_Q |Q\rangle\times\langle Q|0\rangle \Rightarrow e^{-iHT}|0\rangle = \sum_Q |Q\rangle\times e^{-iTE_Q}\langle Q|0\rangle$$

What happens is that we can pick out the lowest energy state because of how we defined T. What this means is that $U_I(0,-T)|0\rangle$ becomes $|\Omega\rangle\times e^{-iTE_\Omega}\langle\Omega|0\rangle$. And thus

$$|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty} U_I(0,-T)|0\rangle\times e^{iTE_\Omega}/\langle\Omega|0\rangle$$

In a similar way we can find with the same $T \to \dots$ trick

$$\langle\Omega| = \lim_{T\to(1-i\epsilon)\infty} \frac{e^{iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0||U_I(+T,0)$$

Now using these two results we can find that big thing inside the big correlation function becomes

$$\langle\Omega|\dots|\Omega\rangle = C(T)\langle 0|\dots|0\rangle$$
$$\dots = U_1(T,0)U_I(0,x^0)\Phi_I(x)U_I(x^0,y^0)\dots 19 \text{ in notes}$$

Time time ordering this gives us

$$= T\left(\Phi_I(x)\Phi_I(y) \times \exp\left(-\frac{i\lambda}{24}\int_{-T}^{T} dt \int d^3\mathbf{z}\phi_I^4(t,\mathbf{z})\right)\right)$$

Which follows from a Dyson series expansion of the evolution operator. This gives

$$\begin{aligned}\mathcal{F}_2(x,y) &= \langle\Omega|T\Phi_H(x)\Phi_H(y)|\Omega\rangle \\ &= \lim_{T\to(1-i\epsilon)\infty} C(T)\langle 0|\dots|0\rangle\end{aligned}$$

So far we have x is later than y. But what if x is earlier than y? Well we reverse somet things around.

Now what about the n-point function

$$\begin{aligned}\mathcal{F}_n(x_1,\dots,x_n) &= \langle\Omega|T\Phi\dots|\Omega\rangle \\ &= \lim_{T\to(1-i0)\infty} C(T) \times \left\langle 0\middle|T\left(\Phi_I(x_1)\dots\Phi_I(x_n)\times\exp\left\{\frac{-i\lambda}{24}\int d^4z\Phi_I^4(z)\right\}\right)\middle|0\right\rangle\end{aligned}$$

The coefficent $C(T)$ is the smae for all correlation function for the same n. What happens if $n=0$? Then $\mathcal{F}_0 = \langle\Omega|\Omega\rangle = 1$ meaning that

$$\lim_{T\to\dots} C(T)\times\langle 0|T(\exp\dots)|0\rangle = 1$$

Then we can eleminate the $C(T)$ factors

$$\mathcal{F}_n(x_1\dots) = \lim_{T} \frac{\langle 0|\dots|0\rangle}{\langle 0|T(\exp\dots)|0\rangle}$$

We can use this mess for perturbation theory!

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Using this perturbation theory we can write Feynman diagram for n-point correlation functions.

Fyenman rules for correlation function here

Just like for scattering amplitudes we can factorize things into vacuum bubble diagrams And other. The ratio we took above is what cancels out the vacuum bubbles.

$$\langle 0|T(\exp\dots)\rangle = \sum(\text{vacuum bubbles w/o external vertices})$$

What's good about killing these vacuum bubbles? Well the leading divergence cancels out when we do this. So we can set $T \to (1+i0)\infty$ diagram by digram. In practice this means when we integrate over a vertex instead of integrating from $-T$ to T we integrate from $-\infty$ to ∞. This is important since then we can go to momentum space with foureir transform and then get momentum space feynman rules.

Momentum Space Feynamn Rules here

When we did scattering amplitudes we learned that we only needed to consider connected diagrams (discconected meant basically no scattering.) For correlation functios like define connected correlation functions

$$\mathcal{F}_n^{\text{connected}} = \sum \cdots$$

We can use this to get the original $\mathcal{F}_n$ through (block expansions(?)) the following

$$\begin{aligned}
\mathcal{F}_2(x,y) &= \mathcal{F}_2^{\text{conn}}(x,y) \\
\mathcal{F}_4(x,y) &= \mathcal{F}_4^{\text{conn}}(x,y,z,w) + \mathcal{F}_2^{\text{conn}}(x,y) \times \mathcal{F}_2^{\text{conn}}(z,w) \\
&\quad + \mathcal{F}_2^{\text{conn}}(x,z) \times \mathcal{F}_2^{\text{conn}}(y,w) + \mathcal{F}_2^{\text{conn}}(x,w) \times \mathcal{F}_2^{\text{conn}}(y,z) \\
&\cdots
\end{aligned}$$

Connected 4-point, 6-point, and so on correltaion functions are related to the scattering amplitude wit the LSZ reduction formula. Next time we'll go more in depth into 2-point correlation functions which are important for renormalization and knowing what the hell is going on.

LECTURE 6: THE 2-POINT CORRELATION FUNCTION

Last lecture we learned about correlation functions for quantum fields and ended up with connected correlation function that can be defined recursivley. Today we're going to focus on the 2-point function. To do this we're ignore Feynman diagrams and worry about the actual physics. Consider $\mathcal{F}_2(x-y)$ where $x^0 > y^0$. We have

$$\mathcal{F}_2(x-y) = \langle \Omega | T \Phi_H(x) \Phi_H(y) | \Omega \rangle = \sum_{|\Psi\rangle} \langle \Omega | \Phi_H(x) | \Psi \rangle \times \langle \Psi | \Phi_H(y) | \Omega \rangle$$

The Ψ is all quantum states of the theory which can occur from the action of the field $\phi(y)$. For example for QED you could have a photon, or three photon, or electron positron paris. these pairs could be free and flying away or they could be bound like in hydrogen. Different types of states will have different parametrs. But all staets will have a net momentum. In addition we might have some other quantum states where we could just consider relative momentum. So for convinience we'll denote $|\Psi\rangle = |\psi, p^\mu\rangle$. With this we can define

$$\mathcal{F}_2(x-y) = \sum_\psi \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E(\mathbf{p}, M(\psi))} \times \langle \Omega | \Phi_H(x) | \psi, p \rangle \times \langle \psi, p | \Phi_H | \Omega \rangle$$

Lets try to take out the x dependence. To do this we use translational symmetry in all 4-dimensions which means that

$$\Phi_H(x) = \exp(i x_\mu P^\mu) \Phi_H \exp(-i x_\mu P^\mu)$$

If we take the vacuum to have energy 0 then

$$\langle\Omega|\exp(ix_\mu P^\mu) = \langle\omega| \qquad \exp(-ix_\mu P^\mu)|\psi,p\rangle = e^{-ix_\mu p^\mu}\times|\psi,p\rangle$$

Asserting all of this we can get

$$\langle\Omega|\Phi_H(x)|\psi,p\rangle\times\langle\psi,p|\Omega_H(y)|\Omega\rangle = e^{-ip(x-y)}\times\left|\langle\psi,p|\Phi_H(0)|\Omega\rangle\right|^2$$

Now since the state $\Phi_H(0)|\Omega\rangle$ is invariant under orthocronous Lorentz symmetry we have that the matrix element is the same for all $\mathbf{p}$ on the mass shell. From all this we can get

$$\mathcal{F}_2(x-y) = \sum_\psi\left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times D(x-y;M(\psi))$$

What happens if $x^0 < y^0$? If we work through this carefully we just see that x and y are ex hagned. To combine everything we just use the Feynman propagator.

$$\mathcal{F}_2(x-y) = \sum_\psi\left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times G_F(x-y,M(\psi))$$

The above equation can be rewritten more generally by redefining the one-point function as a measure. This equation is usually called the Kallen-Lehmann spectracl represtnation(TODO Kallen has some fancy dots in their name)

$$\mathcal{F}_2(x-y) = \int_0^\infty\frac{dm^2}{2\pi}\rho(m^2)\times\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2-m^2+i0}$$

$$\rho(m^2) = \sum_\psi\left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times(2\pi\delta(M^2(\psi)-m^2)$$

The second function is the *spectral density function.* Don't get distracted by the fancy math though. Lets talk about some features about this spectral density.

(a) Real and non-negative.

(b) In free field theory $\rho(m^2) = 2\pi\delta(m^2-M^2)$. This is because for a free theory the only thing you can make out of a vaccum with a field is a single particle.

The spectral density becomes more interesting when we consider a interacting theory. For example for $\lambda\phi^4$ theory we can make three particles and then can make a continuous spectrum. A more general kind of theory can have even funkier spectral densities (see Kaplunovsky's lecture notes page 13 for some examples.)

Lets move on from this spectral representation to its consequences on the two point function. The two point function only depends on x and y so lets do a fourier transform

$$\mathcal{F}_2(p) = \int_0^\infty\frac{dm^2}{2\pi}\rho(m^2)\times\frac{i}{p^2-m^2+i0}$$

How do features off the sepctral density function transform into analytically features of $\mathcal{F}_2(p)$ In general one particle states will give delta spike contribution while multparrtilce states give a continuum.

$$\rho(m^2) = Z \times 2\pi\delta(m^2 - M^2_{\text{particle}}) + \text{smooth continuum}$$

The amptliude of the delta function will have amptliude

$$Z = |\langle 1\text{particle}|\Phi_H(0)|\Omega\rangle|^2 > 0$$

$\sqrt{Z}$ is the strength with which that the quantum field creates the single particle state. With all this we have

$$\mathcal{F}_2(\rho^2) = \frac{iZ}{p^2 - M^2_{\text{particle}} + i\epsilon} + \text{smooth}(p^2)$$

An implication of this we'll find is that a pole in the two point function is at physical particle masses. Why do we care about all of this? We know that the pyscial and bare coupling are differnt by perturabtion theory. This implies that the physcial mass and the bare mass are different in perturbation theory. We can find the physcial mass in perturbation theory by looking at pole masses. We shall see tuesday that

$$M^2_{\text{pole}} = m^2_{\text{bare}} + \text{loop correction} = f(m^2_{\text{bare}}, \lambda_{\text{b}}, \Lambda_{UV})$$

We can then identify $M^2_{\text{pole}} = M^2_{\text{particle}}$ to solve equations. Something also to note is that the contiuum creates a branch cut running from the threshold to ∞. We can see this from

$$\mathcal{F}_2(p^2 + i0) - \mathcal{F}_2(p^2 - i0) = \frac{1}{2\pi i}\int_0^\infty \frac{dm^2\rho(m^2)}{m^2 - p^2 - i0} - \cdots = \frac{1}{2\pi i}\oint_{\text{around p}^2} \frac{dm^2\rho(m^2)}{m^2 - p^2} = \rho(p^2)$$

This means that there is a discontinuity accross the real axis meaning that there is a branch cut. First lets look at the real axis of this Riemann surface. For spacelike p^2, the two point function becomes

$$i\mathcal{F}_2(p^2) = \int_0^\infty \frac{dm^2}{2\pi}\frac{\rho(m^2)}{m^2 - p^2}$$

This is well behaved. Now lets go timelike p^2 and stay below the threshold of the branch cut our $i\mathcal{F}_2$ is still find and good. However once we reash the treshold we get a singularity which leads to a branch cut

$$(\text{above the branchcut}) \;\; i\mathcal{F}_2(p^2 \pm i0) = R \pm i\frac{\rho(p^2)}{2}$$

So the question is if we want to evalute the two point function above the branch cut threshold which $\pm$ sign do we take? The answer is we take $+i0$. This is because we want the thing to look like the Feynmann propagator(oh?). The physcial value is on the top side of the branch cut. This is something to remember: the physical side of the branch cut is the top side.

The RIemann surface of the 2point function has an ifnite number of Riemann sheets but only one physical sheet. So which one is the physcial sheet? According to the notes the physial sheet begin on the upper side of the branch cut, extend CCW to negatvie real axis and then back to posiive axis. TODO HUH? On the physical sheet the integral can be taken lietrally and all real poles correspond to physical particle states. We can have poles off-axis on the unphysical sheet. These poles off-axis actually correspond to resonances. Lets be careful in how we define these unphyscial poles. First define on the upper side of the branch cut

$$i\mathcal{F}_2(p^2+i0)$$

Once we have this lets analytically continue to complex p^2. If we go up i the complex plane $\mathrm{Im}(p^2)>0$. If we go below the branch cut we 're on the unphysical sheet. On this unphysical sheet we might hit a pole. For an example lets say we have a pole at $p^2=M^2-iM\Gamma$. Suppose this pole is close the real axis meaning that $\Gamma \ll 1$. Then when we stay on the real axis, this pole dominates over the known pole contributions. So for real p^2 in the viscinity of M^2 we have

$$\mathcal{F}(p^2)=\frac{iZ}{p^2-M^2+iM\Gamma}+\text{smooth}(p^2)$$

This is the Breit-Wigner resonance. By optical theorem we know that Γ is the total decay rate of the unstbale particle(resonances are unstable particles.) $\frac{1}{\Gamma}$ is the average lifetime of the particle.

Now lets turn back to perturbation theory. At the tree level we have the free feynman propagator. AT one loop we start multiplying by other factors. And it just keeps going on. So when you naivley calcuatle $\mathcal{F}_2$ with diagrams, you just get a lot of poles

$$\left(\frac{i}{p^2-m_b^2+i0}\right)^{\text{power}\geq 2}\times\text{othres}$$

This is unphysical, we should only have simple poles. The perturabtion theory, instead of shifting tree-level poles to physical mass, just creates a bunch of higher order poles. To resolve this we can re-sum the perturbation exapnsion so higher order poles add upt oa simple shifted pole. E.g.

$$\sum_{n=0}^{\infty}\left(\frac{i}{pr-m_b^2+i0}\right)^{n+1}\times(-i\Delta)^n=\frac{i}{p^2-m_b^2+i0}\times\frac{1}{1-\frac{\Delta}{p^2-m_b^2+i0}}=\frac{i}{p^2-(m_b^2+\Delta)+i0}$$

To see this graphically consider an arbitrary multi-lop feynman graph for two-point correlation function. We have a line with a bunch of (irrducible one-particle) subgraphs. Irreducible means that if you cut any one propagator then it stays connected. Technically is one-propagator irreducible but people call it one-particle. We'll skip the graph theory proof but all contributing feynmann graph that csontirbute to the 2-poitn correlator look like that line picture we should have in our head. So the formal sum over all such diagrams with N bubbles (then we'll have $N+1$ propagators) looks like

$$\mathcal{F}=\left(\frac{i}{p^2-m_b^2+i0}\right)^{N+1}\times\prod_{i=1}^{N}[1PI\text{ bubble}\#i]$$

We can reorganize this so that

$$\mathcal{F}=\left(\frac{i}{p^2-m_b^2+i0}\right)^{N+1}\times\{\text{sum over bubbles}\}^N$$

Now we reframe our perturbation theory as summing over bubble diagrams to some order to calculate the physical mass in perturbation theory

$$\mathcal{F}_2(p^2) = \frac{i}{p^2 - m_b^2 + i[\text{sum over bubbles}] + i0}$$

Lecture 7: Computing Two-Point Functions and Yukawa Theory

February 02, 2021

On friday we spent most of our time on analytical properties of the two point function, discussing the meaning of its poles and branch cuts. IN particular we learned that the position of the pole is the physcial mass of the particles. Furthermore the residue of the poles is the strength with which the quantum field is created from the vacuum. Today we'll talk more about two point functions

The first thing we notice is that if we add things order by order in perturbation theory you notice that we get higher order poles exactly at the bare mass. However higher order poles are unphysical. The only way to shift the pole is to resum a infinite number of diagrams. Ideally all of them but in practice a well defined subset of diagrams. So lets see what's going on. We did this on friday really quickly so lets do it slowly today. The basic idea is we look at the most general diagram contributing to the one-point function. They all have this form (equation (66) in notes). A bunch of propagators connecting one-particle irreducible (1PI) subgraph (if you cut a propagator in the subgraph the graph remains connected.) So we're summing over a bunch of bubbles

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0}\right)^{N+1} \times \prod_{i=1}^{N} [1PI \text{ bubble}\#i]$$

What all of this gives us is

$$\mathcal{F}_2(p^2) = \frac{i}{p^2 - m_b^2 - \sigma(p^2) + i0}$$

The thing that we forgot to address is what do we do in practice. The idea is we sum to a certain loop order. Once we have down that we go back to that arrangment (equation (69)) and we say instead of summing all such diagrams up to L loops altogether we sum up to L loops for each bubbles. Then we sum over arbitrary number of bubbles. We are summing over any number of bubbles and for each number of bubbles we go up to L loops. That's the partial resummation and how we'll reorganize our summation. That's the right way to do perturbation theory for the two point function.

Ok we get the red expression, what do we do with it? We know that the physical mass is where the pole is. This means that

$$p^2 - m_b^2 - \Sigma(p^2) = 0 \Rightarrow m_b^2 = M^2 - \Sigma(p^2 = M^2)$$

We need to solve for this bare mass parameter of the perturbation theory in order to get physically correct result. That was the bare mass versus the physcial mass. What about the field strengths factor Z. We can get that from the residue of the pole as well. Let $p^2 = M^2 + \delta p^2$. Expand $\Sigma(p^2)$

$$\Sigma(p^2) = \Sigma(M^2) + \delta p^2 \times \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} + \dots$$

All of this gives us a new form of the denominator

$$p^2 - m_b^2 - \Sigma(p^2) = M^2 + \delta p^2 - m_b^2 - \Sigma(M^2) - \delta p^2 \times \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} + \dots$$

We can make some cancellations if we're considering a pole giving us

$$p^2 - m_b^2 - \Sigma(p^2) = \delta p^2 \left(1 - \frac{d\Sigma}{dp^2} \dots\right)$$

What happens when we invert this? Well we get

$$\mathcal{F}_2(p^2 = M^2 + \delta p^2) = \frac{i}{p^2} \times \left(1 - \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2} + \dots\right)^{-1} = \frac{i}{\delta p^2} \times \frac{1}{1 - (d\Sigma/dp^2)A + \text{finite}}$$

All together we get that near the pole

$$mcF_2(p^2 \text{ near } M^2) = \frac{iZ}{p^2 - M^2 + i0} + \text{finite} \Rightarrow Z = \frac{1}{1 - \left.\frac{d\Sigma}{dp^2}\right|_{p^2=M^2}}$$

And that is how we calculate the field strength renoramlziation in perturbation theory

EXAMPLE 1: (1-LOOP EXAMPLE FOR FIELD STRENGTH RENORMALIZATION) $-i\Sigma(p^2)$ is the sum of all 1PI graphs with two external legs. Lets go up to L loops. We have

$$\Sigma(p^2) = \lambda \times f_1(p^2) + \lambda^2 \times f_2(p^2) + \dots + \lambda^L f_L(p^2) + \dots$$

Lets start with the one loop order. We only have one graph.

$$(86) TODO$$

Evalauting the above graph gives us

$$-i\Sigma_{1-\text{loop}}(p^2) = TODO$$

And thusto one-loop order we have a mass shift of

$$m_b^2 = M^2 - \Sigma$$

However we have no strength renormalization

$$\frac{d\Sigma_{1-\text{loop}}}{dp^2} = 0 \Rightarrow Z_{1-\text{loop}} = 1$$

Quick aside: The two loop diagrams are given in (91) of the lecture notes. The calculation of these graphs is on the next homework. Some hints. First we use the Feynmann parameter trick to get everything to a signal denominator. Then we have a denominator that dpeends on a linear combination of terms quadratic in the proagators. We can then diagonalzie this polynomial for some constants. Something to note here is that we need to keep unit jacobian. The next step is to take the derivative before we take an integral and express it in a specific term. This way will make things less divergent. And then we do what we learned in class. Wick rotate momenta, dimensinoal regularization. At the end of the day we get a huge bunch of shit(it's very long.) After all this we verify some Feynman paramter integrals aren't don't give divergences. And as an optinal exercise we actually calcualte the relevant integrals for $p^2 = M^2$ using mathematica. Finally we asseble to field strength renormalization.

Enough about the HW lets go back to the 1-loop case. We have the momenta integral

$$\int \frac{d^4q}{(2\pi)^4} \frac{i}{q^2 - m_b^2 + i0}$$

It diverges quadratically. Lets wick rotate to Euclidean momentum space $d^4q \to i d^4q_E$. This means

$$\frac{i}{q^2 - m_b^2 + i0} = \frac{i}{-q_E^2 - m_b^2} = \frac{-i}{q_E^2 + m_b^2}$$

Once we're in this form we no longer need $i0$. All of this gives us

$$\Sigma_{1-\text{loop}} = \frac{\lambda}{2} \int \frac{d^4q_E}{(2\pi)^4} \frac{1}{q_E^2 + m_b^2}$$

This integral diverges quadraitcally. Lets use Wilson's hard edge cutoff instead of dimensional regularization. We now have

$$\int_{\text{reg}} \frac{d^4q_E}{(2\pi)^4} \frac{1}{q_E^2 + m_b^2} = \int_0^\Lambda dq_E \times q_E^2 \frac{1}{q_E^2 + m_b^2} \times \int \frac{d^3\Omega(q_E^\mu)}{(2\pi)^4}$$

After doing our usual tricks we get

$$\Sigma_{1-\text{loop}} = \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m_b^2 \times \log\left(\frac{\Lambda^2}{m_b^2}\right)\right)$$

Anyhow what does this equation give us? Well according to (77) in Kaplunovsky's notes that's how the mass shifts

$$M_{\text{phys}}^2 = m_b^2 + \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m_b^2 \times \log \frac{\Lambda^2}{m_b^2}\right) + O(\lambda^2)$$

That is an example of mass shift calculation.

And now we have arrived at the fine-tuning problem. Lets say we want to put UV cutoff way above mass. Formally the prefactor $\frac{\lambda}{32\pi^2}$ is infinitesimal but in reality it's still significant. And if we put Λ^2 100 times the mass, we have a decent chance of overshooting. Take the Higgs particle. It has $M = 125$ GeV and self-coulling $\lambda \approx 0.25$. So the UV cutoff just above the LHC reach at 10 TeV we end up with a one-loop correction to the Higg's field of > 200 GeV. If we continue the standard model to the GUT scale of 10^{16} GeV and $\lambda_b = 0.3$ then the quantum correction to the particle mass would be about 10^{29} GeV^2. So we've gotta fine tune m_b to 25 significant digits. Moreover if we go to two loops we get $(\lambda/32\pi^2)^{\text{loops}}\Lambda^2$ so we would need to adjust the fine tuned m_b^2 order by order up to about ten-loop order. All of this is to say that fine tuning the bare mass feels kinda yucky. Lots of people have tried all kinds of ways to avoid this problem. Supersymmetry is one of them. Unforntunatley supersymmetric standard model is on life support because of the LHC. Composite higgs also doesn't seem to work. By the way people in condensed matter people also have fine tuning problem. Ok lets move on.

Lets address a technical issue. A regulator dependce on divergences. For logarithmic divergences we saw in HW1 that all wehave to do is rescale λ a little bit for each regularization so that they give same cross section . Unforntunatley this doesn't work for quadratic divergences. Recall we got

$$\Sigma = \frac{\lambda}{32\pi^2}\left(\Lambda_{\text{HE}}^2 - m_b^2 \times \log\frac{\Lambda_{\text{HE}}^2}{m_b^2}\right)$$

If we plug in some constant C we get

$$\Sigma = \frac{\lambda}{32\pi^2}\left(C \times \Lambda_{\text{HE}}^2 - m_b^2 \times \log\frac{\Lambda_{\text{HE}}^2}{m_b^2} - m_b^2\log(C)\right)$$

But if we use another cutoff we'd have

$$\Sigma = \frac{\lambda}{32\pi^2}\left(a\Lambda^2 - m_b^2 \times \log\frac{\Lambda^2}{m_b^2} - b \times m_b^2\right)$$

So not only do we have to adjust things order by order but we also we need to be very careful in specying the details of the regulator. Now that's the general problem. Dimensional regularization has a more particular problem. The green term is just zero in dimensional regularization terms. You might think is very fortunate, we don't need to fine tune. But all of this is BS. All of this regulator is just a cover-up for the unknown UV behavior. Regulators are just intermediate step. But generally we'd expect the green term to be non-zero and we'd have to worry about fine-tuning. Well, lets take a closer look at dimensional regularization.

> Example 2: (Dimensional Regularization and Mass Shift) Consider
>
> $$\Sigma(D) = \frac{\lambda}{2} \times \int \frac{\mu^{4-d}d^Dq_D}{(2\pi)^D}\frac{1}{q_E^2 + m_b^2}$$
>
> This diverges for $D \geq 2$ so we need low dimension $D < 2$. We can relate the numerator to a gaussian integral
>
> $$\frac{1}{q_E^2 + m_b^2} = \int_0^\infty dt\, \exp\left\{-t\left(q_E^2 + m_b^2\right)\right\}$$

All of this gives us

$$\Sigma(D) = \frac{\lambda}{2}\mu^{4-D}\int\frac{d^Dq_E}{(2\pi)^D}\int_0^\infty(\text{gaussian integral}) = \frac{\lambda}{2}\mu^{4-D}\int_0^\infty dt\,\exp\left\{-tm_b^2\right\}\times(4\pi t)^{-\frac{D}{2}}$$

If we anylitically continue the Σ to $D<2$ dimensions then the t integral evaluates toa gamma function (equation (105) in notes.) *missed some stuff here but what we get is that UV divergences is worse than logarithmic.*

Now lets analytically conitue the mass shift (105) to $D = 4-2\epsilon$. This gives us

$$\Sigma = \frac{\lambda}{32\pi^2}(4\pi\mu^2)^\epsilon(m_b^2)^{1-\epsilon}\Gamma(\epsilon-1)$$

We can rewrite the gamma function as

$$\Gamma(\epsilon-1) = \frac{\Gamma(\epsilon)}{\epsilon-1} = \frac{-1}{1-\epsilon}\times\left(\frac{1}{\epsilon}-\gamma_E+O(E)\right) = -\left(\frac{1}{\epsilon}-\gamma_E+1+O(\epsilon)\right)$$

The other term in Σ also becomes

$$\left(\frac{4\pi\mu^2}{m_b^2}\right)^\epsilon = \exp\left\{\epsilon\times\log\frac{4\pi\mu^2}{m_b^2}\right\} = 1+\epsilon\times\log\frac{4\pi\mu^2}{m_b^2}+\dots$$

So all of this gives us

$$\Gamma(\epsilon-1)\times(\dots)^\epsilon = -\left(\frac{1}{\epsilon}-\gamma_E+1+\log\frac{4\pi\mu^2}{m_b^2}+\dots\right)$$

Which altogether gives us

$$\Sigma = -\frac{\lambda m_b^2}{32\pi^2}\times\left(\frac{1}{\epsilon}-\gamma_E+1+\log\frac{4\pi\mu^2}{m_b^2}+\dots\right)$$

Which can be restated in terms of an effective Λ_{eff}

$$\Sigma = -\frac{\lambda m_b^2}{32\pi^2}\times\left(\log\frac{\Lambda_{\text{eff}}^2}{m_b^2}+\text{const}\right)$$

Note that what we completley lose here is the leading Λ^2 term. The dimensional regularization gave us a hint taht there is ap ole at $D=2$ that we should watch out for, that log divergence isn't the whole story. However dimensional regularization didn't tell us shit about what's actually going on. what a dick :(. Basically when we use dimensional regularization we should be careful about quadratic divergences.

Now lets talk about something else **Yukawa Theory.**

Example 3: (Yukawa Theory for nonunity field strength regularization) We have the lagrangian

$$\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m_f)\Psi + \frac{1}{2}(\partial_\mu \Phi)^2 - \frac{1}{2}m_s^2\Phi^2 + g\Phi\overline{\Psi}\Psi$$

Lets focus on the scalar two point correlation function and calculate it to one loop order.

$$\mathcal{F}_\phi(p^2) = \frac{i}{p^2 - m_s^2 - \Sigma_\Phi(p^2) + i0}$$

At the one loop level there is only one 1PI graph contirbuting to Σ_Φ. (114) which evlautes to

$$-i\Sigma_\Phi^{1-\text{loop}}(p^2) = -\int \frac{d^4 q_1}{(2\pi)^4}\text{Tr}\left(\frac{i}{q\!\!\!/_1 - m_f + i0}(-ig)\frac{i}{q\!\!\!/_2 - m_f + i0}(-ig)\right)$$

To calculate this we'll start by alcualting the trace

$$\text{Trace} = g^2\frac{\text{tr}((q\!\!\!/_1 + m_f)(q\!\!\!/_2 + m_f))}{(q_1^2 - m_f^2 + i0)(q_2^2 - m_f^2 + i0)} = g^2\frac{4q_1q_1 + 4m_f^2}{\cdots}$$

Now we use feynman paramter trick to bring the deonomitaor here to the form

$$\frac{1}{\text{denomiantor}} = \int_0^1 \frac{d\xi}{\mathcal{D}^2}$$

We have

$$\mathcal{D} = (1-\xi)\times(q_1^2 - m_f^2 + i0) + \xi\times(q_2^2 - m_f^2 + i0)$$

After some lots of algebra we can get this down to

$$\mathcal{D} = k^2 - \Delta(\xi) + i0 \qquad k = q_1 + \xi p \text{ and } \Delta(\xi) = m_f^2 - \xi(1-\xi)p^2$$

this also allows us to write

$$q_1 = k - \xi p \qquad q_2 = k + (1-\xi)p \Rightarrow 4(q_1q_2) + 4m_f^2 = 4k^2 + 4\Delta(\xi) + 4(1-2\xi)(kp)$$

Plugging everything back in we get

$$\Sigma_\Phi(p^2) = -4ig^2\int\frac{d^4q_1}{(2\pi)^4}\int_0^1 d\xi\frac{k^2 + \Delta + (1-\xi)(kp)}{[k^2 - \Delta + i0]^2}$$

Next we're going to switch order of integration. For a convergent integral this is not a problem. For a divergent we can only do this if we proprly implement the regulator. E.g. for hard edge we have a complicated lens shaped domain of integration. What we're going to do is use the dimensional regularization.

$$\Sigma_\Phi(p^2) = -4ig^2\int_0^1 d\xi\int\frac{d^4k}{(2\pi)^4}\frac{k^2 + \Delta + (1-2\xi)(kp)}{[k^2 - \Delta + i0]^2}$$

Now lets send $k \to -k$. What happens? Well the blue term vanishes since it's an odd function and if we integrate over a symmetric region the integral vanishes. So we're left with

$$\Sigma_\Phi = -4ig^2 \int_0^1 d\xi \int \frac{d^4k}{(2\pi)^4} \frac{k^2+\Delta}{[k^2-\Delta+i0]^2}$$

The trick that we learned today is to **sepearte even and odd parts of integral to get rid of the odd ones.** Now we do our usual wick rotation to eulidean momentum space

$$d^4k \to id^4k_E \qquad (k^2+\Delta) \to (\Delta - k_E^2) \qquad (k^2-\Delta+i0) \to -(k_E^2+\Delta)$$

This gives us

$$\Sigma_\Phi = 4g^2 \int_0^1 d\xi \int \frac{d^4k_E}{(2\pi)^4} \frac{\Delta - k_E 62}{(k_E^2+\Delta)^2}$$

This integral diverges. Lets try to maek this less divergent by taking the derivative

$$\frac{d\Sigma_\Phi}{dp^2}$$

The integral depends on the scalar monentum only through $\Delta = m_f^2 - \xi(1-\xi)p^2$. This means that

$$\frac{\partial(\text{integrand})}{\partial p^2} = -\xi(1-\xi)\partial(\text{integran})\Delta$$

So we get

$$\frac{d\Sigma_\Phi}{dp^2} = (130) \qquad \frac{d^2\Sigma_\Phi}{d(dp^2)^2} = (131)$$

And after doing some derivative magic in (133) we get

$$\frac{d\Sigma_\Phi}{dp^2} = -4g^2 \int_0^1 d\xi\, \xi(1-\xi) \times \int \frac{d^4k_E}{(2\pi)^4} \frac{3k_E^2 - \Delta}{(\Delta + k_E^2)^3}$$

This is only logarithmically divergent instead of quadrtaiclly divergent like it's daddy without a derivative. Now lets take another derivative which gives

$$\frac{d^2\Sigma_\Phi}{(dp^2)^2} = 4g^2 \int_0^1 d\xi\, \xi^2(1-\xi)^2 \times \int \frac{d^4k_E}{(2\pi)^4} \frac{2\Delta - 10k_E^2}{(\Delta + k_E^2)^4}$$

This is a finite function of p^2. What this tells us is that the first dierative is the sum of some divergent constant plus a finite function of p^2. This in turn tells us that

$$\Sigma_\Phi(p^2) = \text{divergent constnat 1} + \text{divergent constant 2} \times p^2 + \text{finite function}(p^2)$$

We also know that

$$\text{divergent constant 1} = O(\Lambda^2) \text{ and divergent constant 2} = O(\log \Lambda^2)$$

It turns out that is a completley general behavior for any scalar field in any renormalizable theory to any loop order(that's a really general behavior OwO.) We'll see this again in $\lambda\phi^4$ theory when we calcualte things to two loop order.

We'll finish this example next time.

CALCULATING FIELD STRENGTH RENORMALIZATION TO 2-LOOP LEVEL FOR $\lambda\phi^4$ THEORY

Based on problemset

(a) For this problem we have

$$\frac{(-i\lambda)^2}{3!}\iint \frac{dq_1^\mu}{(2\pi)^4}\frac{dq_2^\mu}{(2\pi)^4}\frac{i}{q_1^2-m_b^2+i0}\frac{i}{q_2^2-m_b^2+i0}\frac{i}{(p-q_1-q_2)^2-m_b^2+i0}$$

Where the $(-i\lambda)^2$ terms comes from the coupling constant and the 3! comes from the interchange of the three propagator. Recall the Feynman parameter trick we proved a while ago

$$\frac{1}{ABC}=\int_0^1 dx\int_0^{1-x}\frac{2dy}{[xA+yB+(1-x-y)C]^3}=\iiint\limits_{x,y,z\geq 0} dxdydz\,\delta(x+y+z-1)\times\frac{2}{[xA+yB+zC]^3}$$

Using this we can write

$$\begin{aligned}
&(\text{Product of three propagators})=\\
&=\iiint\limits_{x,y,z\geq 0} dxdydz\,\delta(x+y+z-1)\frac{2}{[x(q_1^2-m_b^2+i0)+y(q_2^2-m_b^2+i0)+z(\{p-q_1-q_2\}^2-m_b^2+i0)]^3}\\
&\qquad=\iiint\limits_{x,y,z\geq 0} dxdydz\,\delta(x+y+z-1)\frac{2}{[q_1^2x+q_2^2y+(q_1+q_2-p)^2z+(i0-m_b^2)(x+y+z)]^3}
\end{aligned}$$

From the dirac delta function we know $x+y+z=1$ allowing up to simplify a little bit

$$\begin{pmatrix}\text{Product of}\\ \text{Three Propagators}\end{pmatrix}=\iiint\limits_{x,y,z\geq 0} dxdydz\,\delta(x+y+z-1)\frac{2}{[q_1^2x+q_2^2y+(q_1+q_2-p)^2z+(i0-m_b^2)]^3}$$

(b) Now we want to change variables so that the cubed term becomes

$$\alpha\times k_1^2+\beta\times k_2^2+\gamma\times p^2-m^2+i0$$

$$k_1=q_1+c_{11}\times q_2+c_{12}\times p \qquad k_2=q_2+c_2\times p$$

To achieve this lets expand everything explicitly

$$i0-m_b^2+p^2z-2pq_1z-2pq_2z+q_1^2x+q_2^2y+q_1^2z+q_2^2z+2q_1q_2z$$

To try to get things to meet in the middle lets try explicitly expanding out the red term.

$$\begin{aligned}
\begin{pmatrix}\text{Red}\\ \text{Term}\end{pmatrix}&=\alpha c_{12}^2p^2+\beta c_2^2p^2+2\alpha c_{12}pq_1+2\alpha c_{11}c_{12}pq_2\\
&+2\beta c_2pq_2+\alpha c_{11}^2q_2^2+2\alpha c_{11}q_1q_2+i0-m_b^2+\gamma p^2+\alpha q_1^2+\beta q_2^2
\end{aligned}$$

On comparing the red term wit the blue term we see that

$$\begin{aligned}\alpha c_{12}^2 + \beta c_2^2 + \gamma &= z \\ 2\alpha c_{12} &= -2z \\ 2\alpha c_{11} c_{12} + 2\beta c_2 &= -2z \\ \alpha c_{11}^2 + \beta &= y + z \\ 2\alpha c_{11} &= 2z \\ \alpha &= x + z\end{aligned}$$

Lets let mathematica do algebra. Hading this system to mathematica gives us

$$\alpha = x + z \quad \beta = y + \frac{xz}{x+z} \quad \gamma = \frac{xyz}{yz + x(y+z)}$$

$$c_{11} = \frac{z}{x+z} \quad c_{12} = -\frac{z}{x+z} \quad c_2 = -\frac{xz}{yz + x(y+z)}$$

So we have

$$\boxed{k_1 = q_1 + \frac{z}{x+z} \times q_2 - \frac{z}{x+z} \times p \qquad k_2 = q_2 - \frac{xz}{yz + xy + xz} \times p}$$

Which in turn gives us

$$\boxed{(x+z) \times k_1^2 + \left(y + \frac{xz}{x+Z}\right) \times k_2^2 + \frac{xyz}{yz + xy + xz} \times p^2 - m^2 + i0}$$

Now notice that the jacobian

$$\frac{\partial k_i}{\partial q_j} = \begin{bmatrix} 1 & \frac{z}{x+z} \\ 0 & 1 \end{bmatrix} \Rightarrow \det\left(\frac{\partial k_i}{\partial q_j}\right) = 1$$

Therefore we don't need to multiply by some constant in the integral when we change variables. Our integral in total then becomes

$$\Sigma(p^2) = -\frac{\lambda^2}{3} \times \frac{1}{(2\pi)^8} \iint dk_1 dk_2 \iiint_{x,y,z \geq 0} dxdydz\, \delta(x+y+z-1) \frac{1}{[\alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m_b^2 + i0]^3}$$

Lets let $\mathcal{D}$ equal the term we are cubing in the denominator

(c) Lets take the derivative

$$\frac{d}{d(p^2)} \frac{1}{[\mathcal{D}]^3} = -\frac{3\gamma}{[\mathcal{D}]^4}$$

This gives us

$$\frac{d\Sigma(p^2)}{d(p^2)} = \frac{\lambda^2}{(2\pi)^8} \iint dk_1 dk_2 \iiint_{x,y,z \geq 0} dxdydz\, \delta(x+y+z-1) \times \gamma \times \frac{1}{[\mathcal{D}]^4}$$

(d) To evaluate the momentum integral lets wick rotate to the euclidean momentum space $k_i^0 = ik_i^4$. This gives us

$$\frac{d\Sigma(p^2)}{d(p^2)} = -\lambda^2 \iint \frac{dk_{1E}}{(2\pi)^4}\frac{dk_{2E}}{(2\pi)^4} \iiint_{x,y,z\geq 0} dxdydz\,\delta(x+y+z-1)\times\gamma\times\frac{1}{[\alpha\times k_{1E}^2+\beta\times k_{2E}^2-\gamma\times p^2+m_b^2]^4}$$

Now we'll use dimensional regularization to take care of the momentum integral

$$\text{Red Stuff} = \iint \frac{\mu_1^{4-D}\mu_2^{4-D}d^Dk_{1E}d^Dk_{2E}}{(2\pi)^{2D}}\frac{1}{[\alpha\times k_{1E}^2+\beta\times k_{2E}^2-\gamma\times p^2+m_b^2]^4}$$

Now recall the Gamma function integral

$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

We can use this to our advantage. Consider some function f which doesn't depend on t. We can write another integral

$$\int_0^\infty t^{n-1}\times\exp\{-tf\}dt$$

We can do a change of variables $u = tf \Rightarrow du = f\,dt$ giving us

$$\frac{1}{f^n}\int_0^\infty u^{n-1}\times\exp(-u)du = \frac{\Gamma(n)}{f^n} = \int_0^\infty t^{n-1}\times\exp\{-tf\}dt$$

This means we can write when $n = 4$

$$\frac{1}{[\alpha\times k_{1E}^2+\beta\times k_{2E}^2-\gamma\times p^2+m_b^2]^4} = \frac{1}{\Gamma(4)=3!=6}\times\int_0^\infty dt\ t^3\times\exp\left\{-t\left(\alpha\times k_{1E}^2+\beta\times k_{2E}^2-\gamma\times p^2+m_b^2\right)\right\}$$

Which means that

$$\text{Red Stuff} = \iint \frac{\mu_1^{4-D}\mu_2^{4-D}d^Dk_{1E}d^Dk_{2E}}{(2\pi)^{2D}}\times\frac{1}{6}\int_0^\infty dt\ t^3\times\exp\left\{-t\left(\alpha\times k_{1E}^2+\beta\times k_{2E}^2-\gamma\times p^2+m_b^2\right)\right\}$$
$$= \frac{\mu_1^{4-D}\mu_2^{4-D}}{6}\int_0^\infty dt\ t^3\exp\left\{-t(m_b^2-\gamma p^2)\right\}\times\int\frac{d^Dk_{1E}}{(2\pi)^D}\exp\left\{-t\alpha k_{1E}^2\right\}\times\int\frac{d^Dk_{2E}}{(2\pi)^D}\exp\left\{-t\beta k_{2E}^2\right\}$$

the blue and green terms are just gaussian integrals that are analytically continued to arbitrary dimensions. We can see that

$$\text{Blue Stuff} = (4\pi\alpha t)^{-\frac{D}{2}} \qquad \text{Green Stuff} = (4\pi\beta t)^{-\frac{D}{2}}$$

All together we get

$$\text{Red Stuff} = \frac{(4\pi)^{-D}(\alpha\beta)^{-D/2}\mu_1^{4-D}\mu_2^{4-D}}{6}\int_0^\infty dt\ t^{3-D}\exp\left\{-t(m_b^2-\gamma p^2)\right\}$$

Now let $D = 4-\epsilon$ giving us

$$\text{Red Stuff} = \frac{(4\pi)^{-4+\epsilon}(\alpha\beta)^{-2+\epsilon/2}\mu_1^\epsilon\mu_2^\epsilon}{6}\int_0^\infty dt\ t^{-1+\epsilon}\exp\left\{-t(m_b^2-\gamma p^2=f)\right\}$$

Inside the integral we let $u = tf \Rightarrow du = f\,dt$ giving us

$$\text{Red Stuff} = \frac{(4\pi)^{-4+\epsilon}(\alpha\beta)^{-2+\epsilon/2}\mu_1^\epsilon\mu_2^\epsilon}{6(m_b^2-\gamma p^2)^\epsilon}\int_0^\infty dt\ u^{-1+\epsilon}\exp\{-u\} = \left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)^\epsilon \times \frac{(\alpha\beta)^{-2}}{6(4\pi)^4}\times\Gamma(\epsilon)$$

Now lets expand the gamma function and the parenthesized term

$$\Gamma(\epsilon) = \frac{1}{\epsilon}-\gamma_E+\dots \qquad \underbrace{\exp\left\{\epsilon\times\log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)\right\}}_{\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)^\epsilon} = 1+\epsilon\times\log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)+\dots$$

Throwing out positive powers of ϵ leaves us with

$$\text{Purple Stuff} = \frac{1}{\epsilon}-\gamma_E+\log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)$$

Note that μ is some reference energy scale to maintain momentum in phase space. Thus $\mu_1=\mu_2=\mu$. We have finally finished evaluating the momentum integral after taking $\epsilon\to 0$

$$\text{Momentum Integral} = \frac{1}{6(4\pi)^4(\alpha\beta)^2}\times\left(\frac{1}{\epsilon}-\gamma_E+\log\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m_b^2-\gamma p^2}\right)\right)$$

(e) It's just pluggin in (d) to (c)

(f) Lets make sure that the Feynmann parameter integrals converge. The δ function asserts that $x+y+z=1$ meaning that we're just looking at the section of a plane in a octant of $\mathbb{R}^3$. So just a triangle. On examining the given form of the integral

$$\iiint\limits_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3}$$

We see that everything seems kosher except for the vertices. However using mathematica we find that in the limit $x\to 0$ and $y\to 0$ and $z\to 1$ the integrand just vanishes so we won't have a divergence there. By symmetry the same thing happens for the other two corners. Similarly for

$$\iiint\limits_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1)\times\frac{xyz}{(xy+xz+yz)^3}\times\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz(p^2/m^2))^2}$$

We have the same sketchy points at the vertices. We'll evaluate things on-shell meaning $p^2/m^2=1$. Doing a similar analysis with mathematica I think we're fine. Pardon the hand-wavyness of this part of the problem *(g)* From our notes we know that the field strength renormalization is calculated as

$$Z = \left(1 - \frac{d\Sigma}{dp^2}\bigg|_{p^2=m^2}\right)^{-1}$$

So at last we need to calculate the whole integral. Plugging the red stuff back in we get

$$\frac{d\Sigma}{dp^2} = -\lambda^2 \times \frac{1}{6(4\pi)^4} \iiint\limits_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1) \times \frac{\gamma}{(\alpha\beta)^2} \times \left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right)\right)$$

First lets rewrite the log term to be nicer

$$\log\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m_b^2(1-p^2/m_b^2\gamma)}\right) = \log\left(\frac{4\pi\mu^2}{m_b^2}\right) + \log\left(\frac{\sqrt{\alpha\beta}}{(1-p^2/m_b^2\gamma)}\right)$$

Now we can note that

$$\frac{\alpha\beta}{(1-p^2/m^2\gamma)^2} = \frac{(yz+xy+xz)^3}{(yz+xy+xz-xyzp^2/m^2)^2}$$

Giving us

$$= \log\left(\frac{4\pi\mu^2}{m_b^2}\right) + \frac{1}{2}\log\left(\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz(p^2/m^2))^2}\right)$$

Something else to note is that

$$\frac{\gamma}{(\alpha\beta)^2} = \frac{xyz}{(yz+xy+xz)^3}$$

Plugging stuff back in gives us (and also soves part (e))

$$\frac{d\Sigma}{dp^2} = -\lambda^2 \times \frac{1}{6(4\pi)^4} \iiint\limits_{x,y,z\geq 0} dxdydz\delta(x+y+z-1) \times$$
$$\times \frac{xyz}{(xy+xz+yz)^3} \times \left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m_b^2}\right) + \frac{1}{2}\log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz(p^2/m^2))^2}\right)$$

Now how in the fucking world do we evaluate this? I have no clue. But we're given in the problem set that

$$\iiint\limits_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1) \times \frac{xyz}{(xy+xz+yz)^3} = \frac{1}{2}$$

$$\iiint\limits_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1) \times \frac{xyz}{(xy+xz+yz)^3} \times \log\frac{(xy+xz+yz)^3}{(xy+xz+yz-xyz)^2} = -\frac{3}{4}$$

Note that the second integral is evaluated on shell. So now we get

$$\frac{d\Sigma}{dp^2}\bigg|_{p^2=m_b^2} = -\frac{\lambda^2}{3072\pi^4}\left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m_b^2}\right) - \frac{3}{4}\right)$$

Now plugging this into Z and expanding for $\lambda \ll 1$ we get

$$\boxed{Z \approx 1 + \frac{\lambda^2}{6144\pi^4}\left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2}{m_b^2}\right) - \frac{3}{4}\right)}$$

Lecture 8: Finishing up Yukawa and Starting Counterterms

February 04, 2021

Whoever is late is late >:3c. Last time we talked about what goes in perturbation theory for the two-point function. Then he showed us how in $\lambda\phi^4$ theory there is a mass shift for 1-loop but there is no field strength renormalization until we get to two loop. Lets get back to the example we were doing last time

Example 4: (Yukawa Theory Continued) Lets get back to the computations. We'll use dimensional regularization to evaluate $\Sigma_\Phi(p^2)$. The integrand

$$\frac{\Delta - k_E^2}{(k_E^2+\Delta)^2} = \frac{2\Delta}{(k_E^2+\Delta)^2} - \frac{1}{(k_E^2+\Delta)} = \int_0^\infty dt(2\Delta \times t - 1)\times e^{-t(k_E^2+\Delta)}$$

This means that the dimensionally regulated integral becoemes

$$\mu^{4-D}\int \frac{d^D k_E}{(2\pi)^D}\frac{\Delta - k_E^2}{(k_E^2+\Delta)^2} = \mu^{4-D}(4\pi)^{-\frac{D}{2}}\int_0^\infty dt\ e^{-t\Delta}(2\Delta t^{-1-(D/2)} - t^{-(D/2)})$$

As $t \to 0$ then the integral diverges. So to evaluate this integral we analytically this integral some dimension $D<2$ and then once we evalaute things we analytically continue back. Well for $D<2$

$$Integral = 2\Delta \times \Gamma(2-D/2)\times\Delta^{(D/2)-2} - \Gamma(1-D/2)\times\Delta^{(D/2)-1}$$

All of this simplifies to

$$Integral = \Delta^{(D/2)-1}\times\frac{2D-2}{D-2}$$

This has poles at at $D=2$. Getting everyone back together gives us

$$\Sigma_\Phi = 4g^2\mu^{4-D}(4\pi)^{-D/2}\Gamma\left(2-\frac{D}{2}\right)\frac{2D-2}{D-2}\times\int_0^1 d\xi[\Delta(\xi)]^{(D/2)-1}$$

Analytically continuing this to $D = 4-2\epsilon$ we get

$$\Sigma_\Phi = \frac{g^2}{4\pi^2}\Gamma(\epsilon)\frac{6-4\epsilon}{2-2\epsilon}\times\int_0^1 d\xi\Delta\times\left(\frac{4\pi\mu^2}{\Delta(\xi)}\right)^\epsilon$$

As we take $\epsilon \to 0$ we get

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon),\ \frac{6-4\epsilon}{2-2\epsilon} = 3\epsilon + O(\epsilon^2),\ldots$$

Thus giving us

$$\Sigma_\Phi(p^2) = \frac{3g^2}{4\pi^2}\int_0^1 d\xi\Delta(\xi)\times\left(\frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log\frac{4\pi\mu^2}{\Delta}\right)$$

Lets reflect a bit on the example we just did. The divergent term $\frac{1}{\epsilon}\times$stuff has exactly the form $a + b\times p^2$ which we expected from our analysis last time. Now what about the finite part of the integral $\Delta\times\log$? Can it be imaginary? Well as long as Δ is positive then the log is real and thus Σ is real. The imaginary part happens when Δ becomes negative and then the log gets an imaginary part. The actual calculation is part of our homework. In the same homework problem we will find that

$$\text{Im}\Sigma_{\Phi}^{1-\text{loop}} = -m_s\Gamma^{\text{tree}}(\Phi \to \overline{\Psi} + \Psi)$$

Where does thie formula come from? The optical theorem for decay rate

DEFINITION 1: (OPTICAL THEOREM FOR DECAY RATES)

$$\text{Im}\langle 1|\mathcal{M}|1\rangle = M \times \Gamma_{\text{total}}(1 \to \text{anything})$$

In the language of Feynman digrams the particle going to itself is just a 1PI bubble with two external legs. This is exactly $-i\Sigma_{\Phi}(p^2)$. So we can also say

$$\text{Im}\Sigma_{\Phi}(p^2 = M_s^2) = -M_s\Gamma_{\text{total}}(\Phi \to \text{anything})$$

Now at the 1-loop level just as he argued earlier that tree level scattering attiude translates to imaginary part of 1-loop. Finally lets take a derivate of our result to get teh field strength renormalization. It's actually better to take the derivative before we analatycailly continue to $D \to 4$. So we have

$$\frac{d\Sigma_{\Phi}}{dp^2} = (156)$$

Going to the $D = 4 - 2\epsilon$

$$4g^2\mu^{4-D}(D-1)\Gamma(2-D/2) \times \Delta^{(D/2)-2} \to \frac{3g^2}{4\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log\frac{4\pi\mu^2}{\Delta} + O(\epsilon)\right)$$

Plugging everything to (156) and evlaulating the integral gives us

$$\frac{d\Sigma_{\Phi}^{1-\text{loop}}}{dp^2} = \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log\frac{4\pi\mu^2}{m_f^2} - I(p^2/m_f^2)\right)$$

What is this derivative good for? Well evaluating at $p^2 = M_s^2$ and this determines the field sterngth factor

$$Z_{\phi} = |\langle 1\text{scalarparticel}|\mathcal{M}|\Omega\rangle|^2$$

And thus

$$Z_{\Phi} = 1 - \frac{g^2}{8\pi^2}\left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log\frac{4\pi\mu^2}{m_f^2} - I(M_s^2/m_f^2)\right) + O(g^4)$$

And this is the end of this calculation.

Okay! The next subject is counterterm and perturbation theory. The way he talked about perturbation theory earlier it looked like we start with bare mass and bare coupling and we have to express everything in terms of those parametrs and then relate those paramters to the physical mass and physical coupling. It looks like a bunch of steps there. there's a way to rphrase perturbation theory directly without all this nonsense.

Counterterms

Well lets start with a $\lambda\phi^4$ theory. we have the bare lagargian

$$\mathcal{L}_{\text{bare}} = \frac{1}{2}(\partial_\mu\phi_{\text{b}})^2 - \frac{1}{2}m_{\text{b}}^2\phi_{\text{b}}^2 - \frac{1}{24}\lambda_{\text{b}}\phi_{\text{b}}^4$$

We also know that

$$\langle 1 \text{ particle}|\phi|\text{vaccume}\rangle = \sqrt{z}e^{ipx}$$

Lets define the renormalized field

$$\phi_r(x) = \frac{1}{\sqrt{z}}\phi_b$$

So then of course

$$\langle 1\text{-particle}|\phi_r|\text{vaccume}\rangle = e^{ipx}$$

To simplify our notation

$$\phi_r = \phi(x) \qquad \phi_b = \sqrt{z}\phi(x)$$

Then the lagrangian is

$$\mathcal{L}_{\text{b}} = \frac{1}{2}z\left(\partial_\mu\phi\right)^2 - \frac{1}{2}zm_b^2\phi^2 - \frac{1}{24}z^2\lambda_b\phi^4$$

Which we can compre to the physical lagrangain

$$\mathcal{L}_{\text{physical}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_{\text{physical}}^2\phi^2 - \frac{1}{24}\lambda_{\text{physical}}\phi^4$$

So consider the difference.

$$\mathcal{L}_{\text{b}} = \mathcal{L}_{\text{phy}} + \text{counterterms}$$

And this leads us to

$$\mathcal{L}_{\text{counterterms}} = \frac{1}{2}\delta_x(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{1}{24}\delta_\lambda\phi^4$$

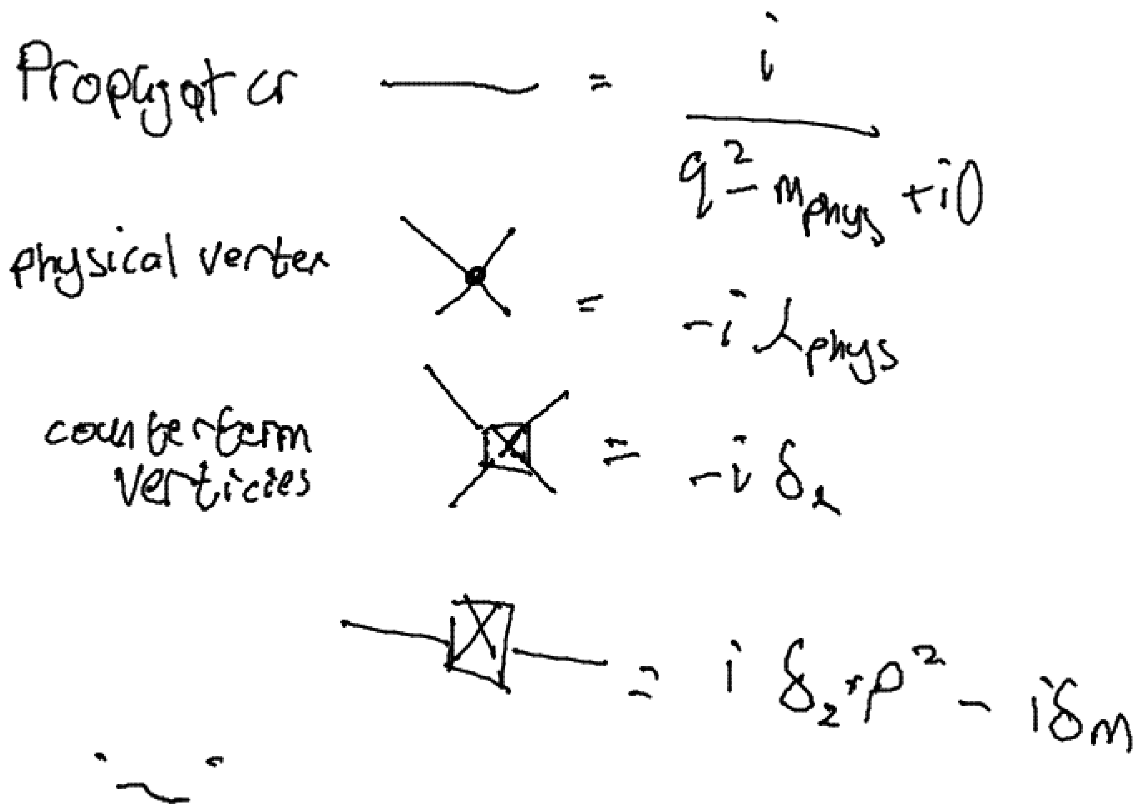
Where

$$\delta_z = z - 1, \; \delta_m = zm_b^2 - m_{\text{phys}}^2, \; \delta_\lambda = z^2\lambda_b - \lambda_{\text{phys}}$$

So far all we've done is algebra. The fun part comes in the next setp in defining the Feynman rules. We are going to treat all counterterms as perturbations. In other words we take

$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m_{\text{b}}^2\phi^2 \qquad \mathcal{L}_{\text{pert}} = -\frac{1}{24}\lambda_{\text{phys}}\phi^4 + \text{corrections}$$

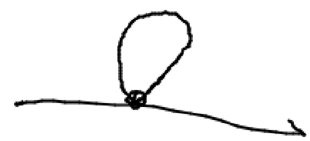
So far we've treated everything quadratic as free and everything else as perturbations. Here we treat some quadratic terms as perturbations as well. So here are the Feynman rules



So what are counterterms goods for? Well the coutnerterm couplings δ_i are adjusted order by order in perturbation theory so as to keep all ampltidues finite. In other words we cancel all divergences. Well lets look at some examples. We'll learn to generalize after these examples.

Example 5: (An Example Using Counterterms) Take an example

$-i\Sigma(p^2) =$ 

$=$  + higher loops

$+$ 

counterterm

$$\Rightarrow \boxed{\Sigma_{net}(p^2) = \Sigma_{loops}(p^2) - \delta_z \times p^2 + \delta_m}$$

We know to 1-loop order

$$\Sigma_{1-\text{loop}} = \frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2 \log\frac{\Lambda^2}{m^2}\right)$$

And threfore we're going to set δ^m to cancel the first loop contribution.

$$\delta^m = -\frac{\lambda}{32\pi^2}\left(\Lambda^2 - m^2 \log\frac{\Lambda^2}{m^2}\right) + O(\lambda^2)$$

We also know that $\Sigma_{1-\text{loop}}$ is p^2 independent so

$$\delta_z = 0 + \underbrace{O(\lambda^2)}_{\text{homework}}$$

Thus we see

$$\Sigma_{\text{net}}(p^2) \text{ is finite.}$$

More generally, fnite parts of δ_m and δ_z are set so that

$$\Sigma_{\text{net}}(p^2 = m^2_{\text{phys}}) = 0 \qquad \left.\frac{d\Sigma_{\text{net}}}{dp^2}\right|_{p^2=m^2_{\text{phys}}} = 0$$

This is because the two point function

$$\mathcal{F}_2(p) = \frac{i}{p^2 - m_{\text{phys}}^2 - \Sigma_{\text{net}}(p^2) + i0}$$

HAs unshifted pole at $p^2 = m_{\text{phys}}^2$ meaning that $\Sigma_{\text{net}}(p^2) = 0$. Also we don't want to modify the residue. So we also don't want to modify the derivative of the denominator meaning that the derivative of Σ_{net} has to be zero. And what does all this mean for the counteterms?

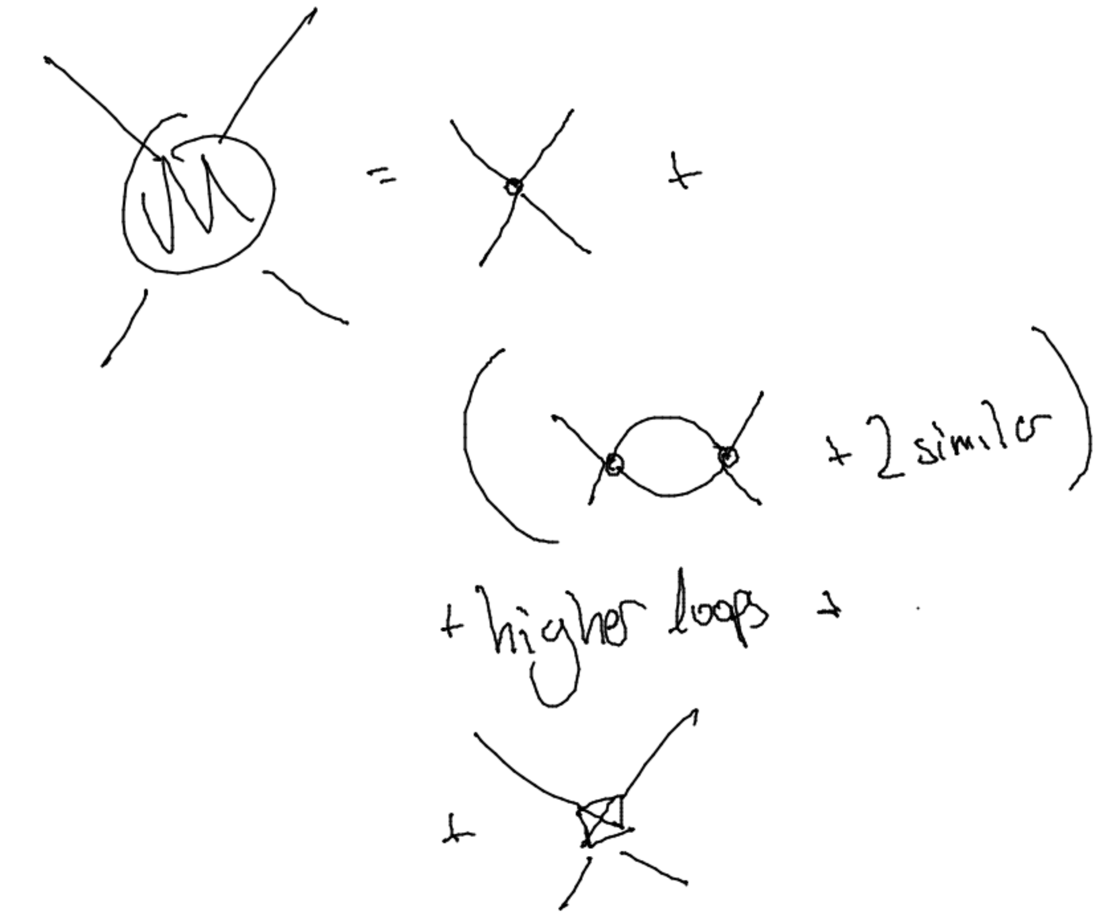
$$\Sigma_{\text{net}} = \Sigma_{1-\text{loop}} - \delta_z \times p^2 + \delta_m$$

And that means that at $p^2 = m^2$

$$\Sigma_{\text{loop}} - \delta_z \times m^2 + \delta_m = 0 \qquad \frac{d\Sigma_{\text{loops}}}{dp^2} - \delta_z = 0$$

And this is what gives precise values of δ_z and δ_m. Now lets take another example.

Example 6: (Four-point amptliude) consider



$$\mathcal{M} = -\lambda_{phys} + \frac{\lambda_{phys}^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 3 - J(t/m^2) - J(u/m^2) - J(s/m^2)\right) + O(\lambda^3) - \delta_\lambda$$

First of all δ_λ cancel the UV divergence. Thus

$$\delta_\lambda = \frac{\lambda_p^2}{32\pi^2}(3\log\frac{\Lambda^2}{m^2} + \text{constants}) + O(\lambda^3)$$

And how do we obtain the finite constant here? Well we need to define what we mean by physical λ. Well if we define $-\lambda_{\text{phys}} = m$ at the threshold where $s = (4)m^2$ $t = u = 0$. We get

$$-\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2 - 3 - J(4) - 2J(0)}\right) - \delta_\lambda = \lambda_{\text{phys}}$$

So we find that

$$\delta_\lambda = \frac{\lambda_{\text{phys}}}{32\pi^2}\left(3\log\frac{\Lambda^2}{m^2} - 3 - J(4) - 2J(0)\right) + O(\lambda_{\text{phys}}^3)$$

And when we evaluate thing

$$J(0) = 0 \qquad J(4) = -2$$

And if we plug all this into a formula for $\mathcal{M}$ we have

$$\mathcal{M} = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\left(J(r) - J(s/m^2) - J(u/m^2) - J(t/m^2)\right)$$

So we have seen how coutnerterms coutner some divergences. The qeusetion is are they good enough to cancel all divergences. The answer is yes. The 3 countereterms δ_z δ_m and δ_λ are adjusted order by order in perturbation theory can cancel all the divergences of the $\lambda\phi^4$ theory. He'll outline for us this proof for the few minutes left today. Here's a clue as to what's going on. We need to classify divergences and divergent ampltiudes. LEts start with the superficial degree of divergence. Take a generic connected Feynman diagram. It will have l loops, $\mathcal{P}$ propagators and v vertices and e external legs. You remember the drill from last semseter . So we get

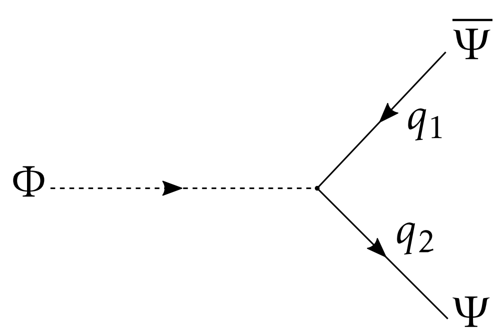
$$I = \int d^4 q_1 \ldots \int d^4 q_l (-i\lambda)^v \prod_{j=1}^{\mathcal{P}} \frac{i}{(\text{linear combination of } q\text{'s and } p^2_{\text{external}}) - m^2 + i0}$$

If we have some superficial degree of divergence $\mathcal{D} = 4L - 2P$ the naively for $\mathcal{D} < 0$ the integral I converges. For $\mathcal{D} = 0$ I diverges as $\log \Lambda_{\text{UV}}$. And for $\mathcal{D} > 0$ the integral diverges as $\Lambda_{\text{UV}}^{\mathcal{D}}$. Why did we say naively? This is because $\mathcal{D}$. We must check for divergent subgraphs because we may look at the graph and say we have $\mathcal{D} < 0$ but we have a negative subgraph. The man is now showing us some graphs in his notes

Confirming Optical Theorem for Decay Rates in Yukawa Theory

(based on problemset from course)

(a) At tree level we only have one process that contributes



We know that the decay rate is just the matrix element times a phase space factor.

$$\Gamma = \mathcal{P} \times |\mathcal{M}|^2$$

Where $|\mathcal{M}|^2 = \sum_{s_1,s_2} |\mathcal{M}|^2$ is the spin-summed matrix element. First we can read off the matrix elemnt using Feynmann rules for Yukawa Theory (P& S pg 118)

$$i\mathcal{M} = \overline{u}^{s_2}(q_2) \times (-ig) \times v^{s_1}(q_1)$$

We want to evaluate the spin sum[2]

$$
\begin{aligned}
\sum_{s_1,s_2}|\mathcal{M}|^2 &= g^2\mathrm{Tr}\left(\sum_{s_1,s_2}\overline{u}^{s_2}v^{s_1}\times\overline{v}^{s_1}u^{s_2}\right)\\
&= g^2\mathrm{Tr}\left(\sum_{s_1}\overline{v}^{s_1}(q_1)v^{s_1}(q_1)\times\sum_{s_2}\overline{u}^{s_2}(q_2)u^{s_2}(q_2)\right)\\
&= g^2\mathrm{Tr}\left((\not{q}_1-m_f)(\not{q}_2+m_f)\right)\\
\text{Kill odd number of slashed momentum}\quad &= g^2\mathrm{Tr}\left(\not{q}_1\not{q}_2-m_f^2\right)\\
&= g^2\left(4q_1q_2-4m_f^2\right)
\end{aligned}
$$

Thus we get the spin sum

$$
|\mathcal{M}|^2 = 4g^2(q_1q_2-m_f^2)
$$

We can simplify this a bit with

$$
2q_1q_2 = \underbrace{(q_1+q_2)^2}_{M_s^2}-\underbrace{\left(q_1^2+q_2^2\right)}_{2m_f^2}\Rightarrow|\mathcal{M}|^2 = 2g^2\left(M_s^2-4m_f^2\right)
$$

We need to calculate the phase space factor. From Kaplunovksy's notes on phase sapce factor we know that there is a pre-integral factor of $1/2M_s$

$$
\begin{aligned}
2M_s\mathcal{P} &= \iint\frac{d^3q_1}{(2\pi)^3 2E_1}\frac{d^3q_2}{(2\pi)^3 2E_2}(2\pi)^4\delta^{(3)}(p-q_1-q_2)\delta(E-E_1-E_2)\\
&= \int\frac{d^3q_1}{(2\pi)^3 2E_1}\frac{1}{(2\pi)^3 2E_2}(2\pi)^4\delta(E-E_1-E_2)
\end{aligned}
$$

The angular integral just gives a factor of 4π leaving us with

$$
2M_s\mathcal{P} = \int_0^\infty\frac{4\pi(2\pi)^4|q_1|^2d|q_1|}{(2\pi)^3 2E_1 2E_2(2\pi)^3}\delta(M_s-E_1-E_2)
$$

By (17) and (19) of Kaplunovsky's notes on phase space we're left with

$$
2M_s\mathcal{P} = \left(\frac{|q_1|^2}{4E_1E_2\pi}\right)\times\underbrace{\left(\frac{d(E_1+E_2)}{dq_1}\right)^{-1}}_{\frac{E_1E_2}{q_1M_s}}\Rightarrow\mathcal{P} = \frac{|q_1|}{8\pi M_s^2}
$$

All together we have

$$
\Gamma = \frac{|q_1|\times g^2(M_s^2-4m_f^2)}{4\pi M_s^2}
$$

[2]No prefactor since incoming scalar particle has no spin

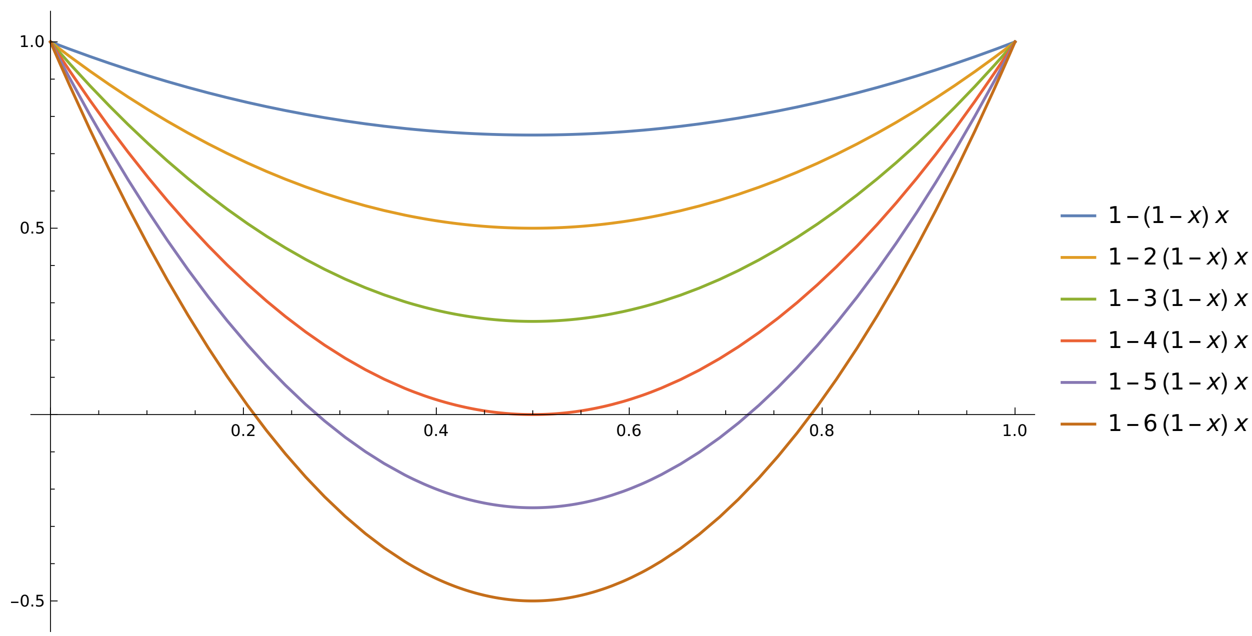
Now we can assert that $|q_1| = \sqrt{M_s^2/4 - m_f^2} = \beta M_s/2$ where $\beta = \sqrt{1 - 4m_f^2/M_s^2}$. This gives us

$$\boxed{\Gamma = \frac{g^2 M_s \beta^3}{8\pi}}$$

(b) If $\Delta(\xi)$ becomes negative then we will gain an imaginary part. Recall the definition of Δ.

$$\Delta(x) = m_f^2 - \xi(1-\xi)p^2$$

Lets plot $\Delta(x)/m_f^2$ for various values of p



We can see then that once $p^2 > 4m_f^2$, the logarithm dips below zero for a certain range of x. We now want to evaluate the integral. First note that $\log(-a) = \log(a) \pm i\pi$ if a is positive and real. Now lets consider

$$\text{Im}\left[\log(4\pi\mu^2) - \log\Delta\right] = -\text{Im}\left[\underbrace{\log(\Delta)}_{\text{real}\pm i\pi}\right] = \mp\pi$$

From this we can extract the imaginary part of $\Sigma_\Phi^{1-\text{loop}}$

$$\text{Im}\left[\Sigma_\Phi^{1-\text{loop}}\right] = \mp\frac{12g^2}{16\pi}\int_a^b dx\, \Delta(x) =$$

Now we need the bounds which occur when $\Delta < 0$. This means

$$m_f^2 = x(1-x)M_s^2 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4m_f^2/M_s^2}}{2} = \frac{1 \pm \beta}{2}$$

So we integrate

$$\frac{12g^2}{16\pi}\int_{(1-\beta)/2}^{(1+\beta)/2} dx\,\Delta(x) = \frac{12g^2}{16\pi}\left(\beta\left(m_f^2 - M_s^2/4\right) + \frac{1}{12}\beta^3 M_s^2\right)$$

We note that

$$m_f^2 - M_s^2/4 = -\beta^2 M_s^2/4$$

Thus getting

$$\mathrm{Im}\left[\Sigma_\Phi^{1-\text{loop}}\right] = \pm\frac{g^2 M_s^2 \beta^3}{8\pi}$$

All that is left to do is find sign arg of Δ. If we bump up $p^2 = \text{real} + i\epsilon$ then will we rotate CW or CCW to reach Δ. This will give us the arg.

$$\mathrm{Im}\left[\Delta(p^2 = \text{real} + i\epsilon)\right] = \epsilon(x^2 - x) < 0 \text{ since } 0 < x < 1$$

This means that we go through $-\pi$ to get to Δ meaning that

$$\boxed{\mathrm{Im}\left[\Sigma_\Phi^{1-\text{loop}}\right] = -\frac{g^2 M_s^2 \beta^3}{8\pi}}$$

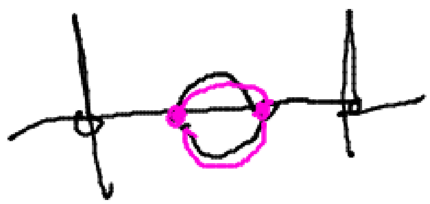
(c) We see that

$$\mathrm{Im}\Sigma_\Phi^{1-\text{loop}} = \frac{-g^2 M_s^2 \beta^3}{8\pi} = -M_s \times \frac{g^2 M_s \beta^3}{8\pi} = -M_s \times \Gamma(S \to f + \overline{f})$$

Lecture 9: Counterterms and Subgraphs

February 05, 2021

We'll start with a notice. The last semester's lecture are posted on the class website. Now lets continue on what we started last lecture. At the very end we went through pretty fast on superficial divergences. Take ϕ^4 theory. When we write an amplitude we'll have L loops and $\mathcal{P}$ propagotrs. So the superficial degree of divergence is $\mathcal{D} = 4L - 2\mathcal{P}$ If this superficial degree of divergence is negtaive then it'll converge. $\mathcal{D} = 0$ is logarithmically divergence and for $\mathcal{D} > 0$ we have $\Lambda^{\mathcal{D}}$ divergence. But as we said, we start with this supeficial degree of divergence but weshould also cehck for subgraphs.



whole graph

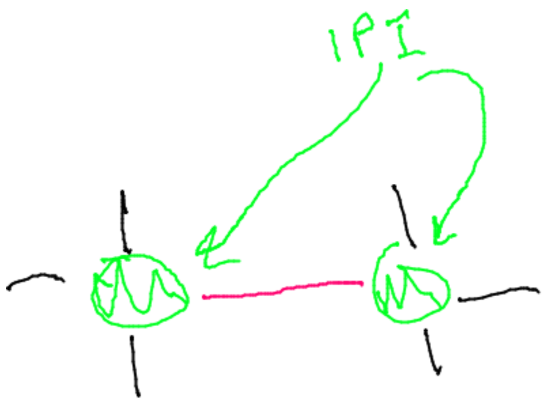
$L=2 \quad P=5 \quad D<0$

subgraphs

$L=2 \quad P=3 \quad D=2>0$

diverges as Λ^2

So the general rule is to check $\mathcal{D}$ for all the 1PI subgraphs. The rason is as follows



q^μ depends only on ϕ_{ext} not q_{ext}

$\Rightarrow \Pi(\text{propagator}) \sim (q_{loop})^{-2(P-1)}$

$\Rightarrow D_{eff} = 4L-2(P-1) = D+2$

$\Rightarrow D$ doesn't tell us anything because of D_{eff}

So now that we know what to chekc lets see what kinds of graphs are divergent and which aren't. Can we relate superficial degree of freedom to things like the number of external legs.

So lets take a one particle irrdicuble graph or subgraph. It will have v vertices, l-loops and $\mathcal{P}$ propagators and E external to the subgraph. So a propagtor to the big graph may be an external line to the subgraph. So we're treating a subgraph as a feynmann diagram of its own. And then what do I know? I know the Euler theorem that says that $L - P + V = 1$. And I also know t hat if I count line ends we get $2P + E = 4V$. So if we count the supericial degree of divergence

$$\mathcal{D} = 4L - 2P = 4(1 + P - V) - 2P + (0 = 4V - 2P - E)$$

And now for we can do some cancellations to end up with

$$\boxed{\mathcal{D} = 4 - E}$$

So the subgraphs can be determined to be divergent or not just by counting how many lines are going into it. Okay! Lets move on. All divergent graphs or subgraphs have $E \leq 4$. That really limits then. Moreover we have $\phi \to -\phi$ symmetry which means that E must be even because any subgraph with odd number of external legs will have odd number of ϕ fields with is just 0 by symmetry. Or in graph theory if all vertices have even valence all graphs have even number of external legs. And that means that divergent graphs or subgraphs have $E = 0, 2, 4$. In other words, what do we have?

$E = 0$  vacuum bubble $\to$ ignore.

$E = 2$  contributes to $-i\Sigma(p^2)$

$E = 4$  contributes to $i\mathcal{V}(p_1, p_2, p_3, p_4)$

$\Sigma(p^2)$ and $\mathcal{V}(p_1, p_2, p_3, p_4)$ are the only divergent amplitudes

If we UV-regulate these divergent subgraphs and cancel their UV divergences then all amplitudes become finite. This is because any other ampltidue with bigger external legs with come from $\mathcal{D} < 0$. So divergence can only come from subgraphs. And if we somehow manage to cancels the divergences of these subgraphs then eveything is kosher. So lets start with $-i\Sigma(p^2)$.

$-i\Sigma(p^2) =$  $\mathcal{D} = 4 - E = +2$

$\Rightarrow O(\Lambda^2)$ divergence

To reduce divergence lets take derivatives

$$\frac{\partial \Sigma}{\partial p^2} : O(\Lambda^1) \text{ divergence}$$

How do we see that? Well schematically

$$-i\Sigma \propto \int d^{4L}q \prod_{j=1}^{\mathcal{P}} \frac{1}{q_j^2 - m^2 + i0}$$

where $q_1 \ldots q_l$ are looop momenta and $q_{l+1} \ldots q_{\mathcal{P}}$ are lienar commbinations of $q_1 \ldots q_l$ and p_{ext}. So when we write

$$\frac{d\Sigma}{dp^\mu} = \int d^{4L}q \sum_{j=l+1}^{\mathcal{P}} \frac{-2(\partial q_j/\partial p)q_j^\mu}{[q_j^2 - m^2 + i0]^2} \times \prod(\text{other})$$

Now if we look at the numerator $-q_j(\partial q_j/\partial p)$ this has degree 1 in $q_1 \ldots q_l$. The denomiantor will have degree $4 + 2(\mathcal{P} - 1) = 2\mathcal{P} + 2$. So o

$$\mathcal{D}[\partial \Sigma p] = 4L + 1 - (2P + 2) = 4L - 2P - 1 = \mathcal{D}[\Sigma] - 1 = +1$$

From thsi we get that

$$\frac{\partial \Sigma}{\partial p^\mu} = O(\Lambda)$$

By the same token $\partial^2\Sigma/\partial p^\mu \partial p^\nu$ has

$$\mathcal{D}[\Sigma] - 2 = 0$$

This diverges as $O(\log \Lambda)$. And finally

$$\frac{\partial^3 \Sigma}{\partial p^\mu \partial p^\nu \partial p^\sigma} \text{ has } \mathcal{D} = -1 \Rightarrow \text{converges}$$

The third derivative of Σ is finite. This means that the inifite part of $\Sigma(p^2)$ is quadratic polynomial in p^μ.

$$A^{\mu\nu} p_\mu p_\nu + B^\mu p_\mu + C$$

And by lorentz symmetry $A^\nu = A \times y^{\mu\nu}$ and $B^\mu = 0$(I think he just means isotropy so we can't have odd power terms?) . This means that

$$\Sigma(p^2) = C + A \times p^2 + \text{ finite } f(p^2)$$

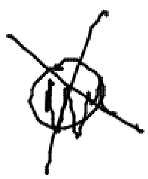
This gives us

$$C = O(\Lambda^2),\ A = O(\log\Lambda)$$

Now why is that important? because in a counterterm perturbation theory $\Sigma^{\text{net}}(p^2) = \Sigma^{\text{loops}}(p^2) + \delta_m - \delta_z p^2$. So comparing everything to each other we see that

$$\delta_z = A \qquad \delta_m = -C \Rightarrow \Sigma^{\text{net}}(p^2) \text{ is finite}$$

What about the other divergent amplitude.



$iV(p_1 \ldots p_4)$

$D = 4 - E = 0$

$\Rightarrow O(\log\Lambda)$ divergent

From the same logic above we have all derivatives of the above subgraph as finite. This means that

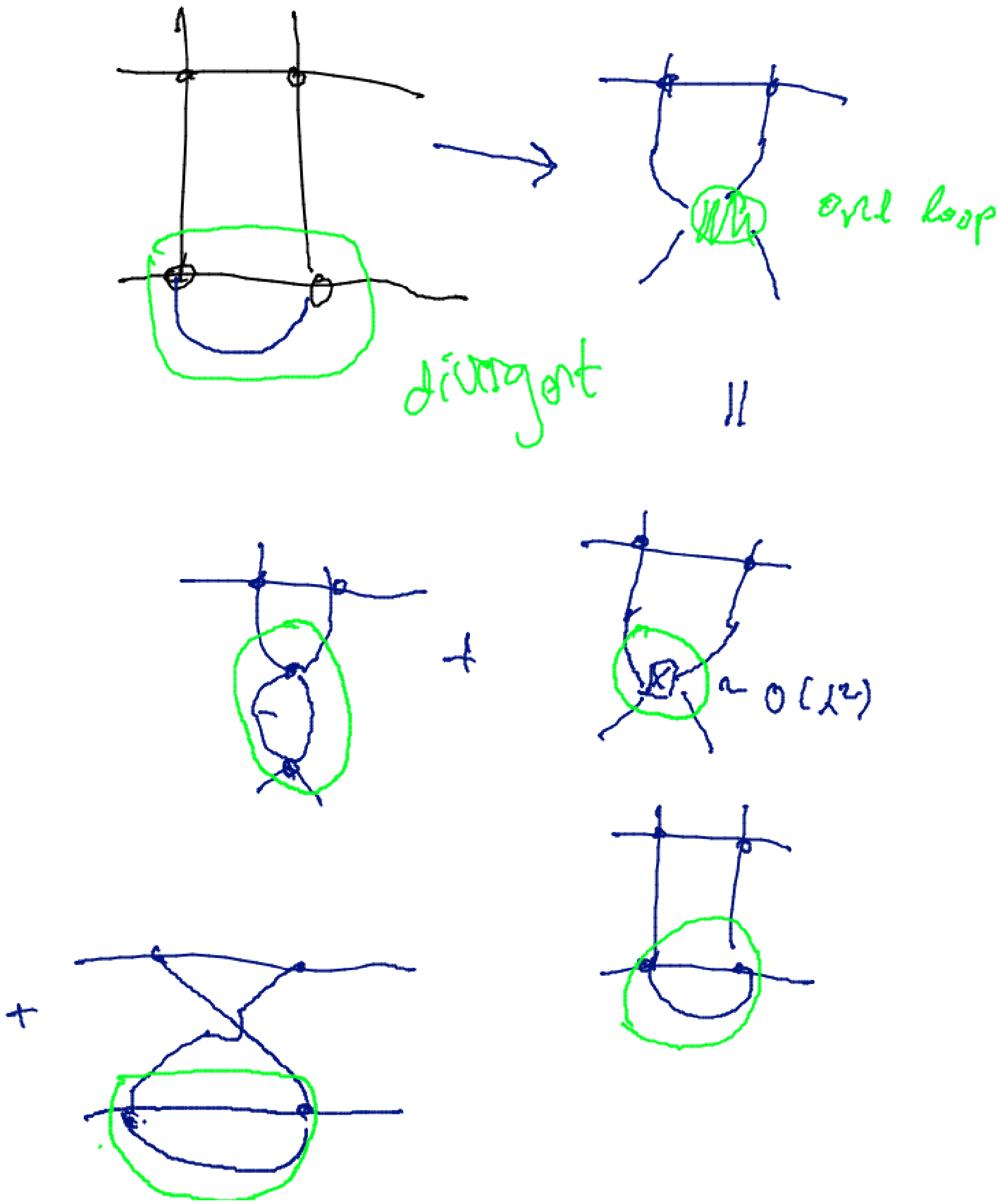
$$V = \text{divergent constant} + \text{ finite } f(p_1, \ldots, p_4)$$

And we have

$$\Sigma V^{\text{net}}(p_1, \ldots, p_4) = V^{\text{loops}}(p_1 \ldots p_4) - \delta_\lambda$$

Where δ_λ conceals the divergent constant of V_{loops}. this makes $V_{\text{net}}(p_1 \ldots p_4)$ finite. All other subgraphs are finite.

Here's the bottom line. The 3 countter terms $\delta_m, \delta_z, \delta_\lambda$ are adjusted order by order in perturbation theory together cancel all the UV divergences of all amplitudes. Well! We have started a little bit late so lets let him go for another five minutes. So far we've only talked about graphs that diverge by themselves. What about subgraph divergences?



What all this is saying is that at large q the combined one loop circle is finite and of the order $O(\log q^2)$ and then the whole graph is going to be proportional to

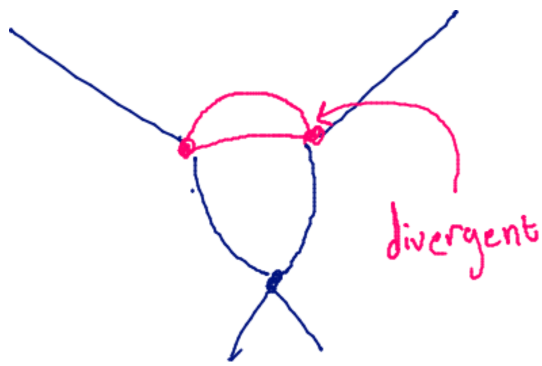
$$\propto \int \frac{d^4q}{(q^2)^3} \times O(\log(q^2)) \text{ which is finite}$$

So the bottom line here is that sure we can have a divergent subgraph but if we combine that divergent subgraph that could replace it in the same order in λ then all together the net green block becomes finite and when we plug things into the whole integral weget a fininte answer. To summarize the counterterms canel UV divergences of subgraphs *in situ*(right in their place). Okay we're out of time. We'll make a break at this point. The one thing we did not finish today is that we did not discuss nested or overlapping divergences. We'll talk about such divergences a bit on Tuesday and we'll assign the rest of the story as a reading assignment for the next homework and switch onto the general subject of renormalizability and non-renormalizability.

Lecture 10: Nested/Overlapping Divergences and Beginning of Renormalizability (plus some dimensional analysis!)

February 09, 2021

Today we're going to continue talking about the issue of renormalization and renormalizability. Last lecture we saw that in $\lambda\phi^4$ theory the counterterms δ_m, δ_z and δ_λ cancel overall UV divergences and divergent subgraphs. Theere is one potential problem with subgraph divergences which is the nested and overlapping divergences. For example



Also whole graph divergent

So we have for large q

$$\text{subgraph: } \log\Lambda^2 + \text{finite f(momenta)}$$

where $f \propto \log q^2$

$$\text{Whole graph: } \int \frac{d^4q}{(2\pi)^4}\frac{1}{q^4}\left(\log\Lambda^2 + \text{finite(something} \times q^2)\right)$$

Which eventually turns into

$$\text{Whole graph: } \int \cdots == \left(\log\Lambda^2\right)^2 + (\log\Lambda^2)\times \text{momentumdependent} + \text{finite}$$

The red term is a potential problem. Now if you look at the counterterm we get

$$\delta_\lambda = \log \Lambda^2 + \text{const}$$

Which means we have

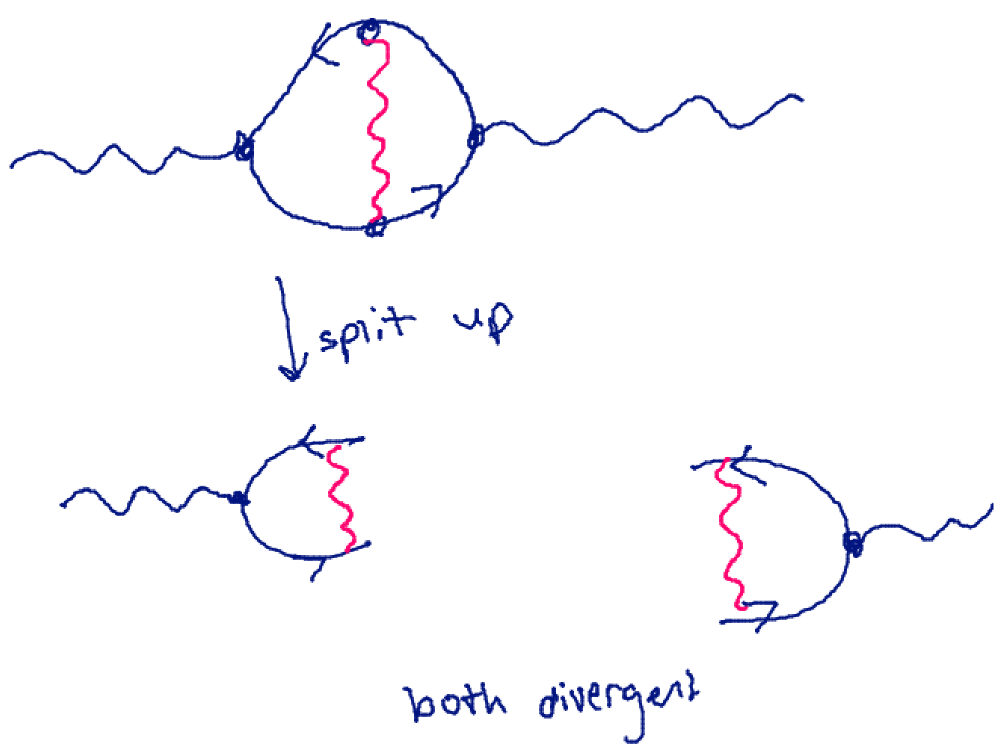
$$\text{whole graph} = (\log \Lambda^2 + \text{const}) \times (\log \Lambda^2 + \text{finite(momenta)})$$

So the whole term will have the eform

$$\text{whole graph} = \left(\log \Lambda^2\right)^2 + \log \lambda^2 + \text{momentum dependent} + \text{finite}$$

And the red term will be trouble. However, we have a miracle of miracles. Net graphs of the type we had above (the upper loop stands for net oneloop + δ_λ), the whole shmeer $\log \Lambda^2 \times$ (mom. dep.) terms cancel out. And so the net amplitude is a divergent constant plus finite where the divergent term is cancelled by $\delta_\lambda^{2-\text{loop}}$. He's not going to do this kind of analysis in class. Instead it's for homework (Peskin and Schroeder 10.5). So that's our example of a nested divergence.

Now lets take a look at a overlapping divergence example.



However there is a theorem that says if you can handle overall divergence then all the subgraph divergences will take care of themselves (BPHZ(Bogolyubov, Parasyuk, Hess, Zimmerman) Theorem). To restate, the theorem roughly says if you can cancel all the overall divergences then the subgraph divergences including nested and overlapping divergences are okay. This is a very rough formulation. Ok. Let us happily put aside the specifics of $\lambda\phi^4$ theory and just tell us that this theory is an example of a renormalizable theory. Now lets consider a general renormalizable QFT.

Definition 2: (Renormalizable QFT) In a renormalizable QFT all UV divergences are cancelled by a finite number of counterterms but their values need to be adjusted order by order to all orders of perturbation theory. A key feature of these theories is the fact that there is an infinite number of divergent graphs (because divergence happens to all ordes of loop amplitude) but only a finite number of divergent amplitudes. Furthermore divergent part of amplitudes are polynomial in external momenta. $\lambda\phi^4$ theory is a nice example of a renormalizable theory.

We'll learn next lecture that QED is a renormalizable theory but QED if more complicated. However not all theories are renormalizable. Some of them are super-renormalizable and some of them are non-renormalizable. Lets get some examples here. (We'll only get one example of super-renormalizable but an *infinite* number of non-renormalizable theories.)

Definition 3: (Super-renormalizable QFT) A super-renormalizable QFT has only a finite number of divergent diagrams. Divergences stop after so many loops. And that means that all divergences cancelled by a few counterterms determined at a fininte order of perturbation theory. So the difference between this and a normal renormalizable QFT is that here you don't need to recalculate counterms for infinity, you can stop after a finite number of orders.

Definition 4: (Nonrenormalizable QFT) A nonrenormalizable of QFT needs a infinite set of counterterms. There are an infinite number of divergent amplitudes and in fact, all amplitudes diverge at high enough order of perturbation theory. So these are okay as effective theories but doing renormalization is pointless cause you'll need a counterterm for basically everything.

We'll now start to look at more general examples. Lets look at a scalar theory with $c\phi^k$ interactions.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 - \sum_{k=3}^{\infty}\frac{c_k}{k!}\phi^k$$

Now we don't have to have all the k's. But we're trying to consider a completley general setup. What this lagrangian tells us is that we have vertices of valence $k = 3, 4, \ldots$. And now lets do loop counting or degree of divergence counting. So recall that the superficial degree of divergence is

$$\mathcal{D} = 4L - 2P$$

Lets consider the fact that

$$L - P + V_{\text{net}} = 1 \rightarrow L = 1 + P - \sum_k V_k$$

On the other hand if you look at the number of line ends, that's just counting lines meaning that

$$\text{number of line ends } = 2P + E = \sum_k kV_k$$

So the superifical degree of divergence is

$$\mathcal{D} = 4(1 + P - \sum_k V_k) - 2P + \underbrace{\sum_k kV_k - 2P - E}_{=0}$$

After cancelling some thing we get

$$\boxed{\mathcal{D} = 4 - E + \sum_k (k-4)V_k}$$

Now lets get specific. If we take $\kappa\phi^3$ interactions only. That is $k = 3$ only. In this case we get

$$\mathcal{D} = 4 - E - V$$

And ealier we learned that for a cubic t heory $V = E - 2 + 2L$ which means that

$$\mathcal{D} = 6 - 2E - 2L$$

So we can see that the more loops the less divergence. And the only divergent amplitudes will be $E + L \leq 3$. So if $E = 0$ that's just a vacuum bubble which diverges for $L \leq 3$ only but we don't really care. Now if we take $E = 1$ we get a tadpole. Also $L \leq 2$ means 2 divergnt graphs. A baloon and a Baymax. For $E = 2$ we get the stick figure sticking arm out from the top view. This is it! No more divergent graphs. Wow, it's super-renormalizable. What about the counterterms needed to cancel out these divergences?

$\boxed{\text{counterterms}}$

$\mathcal{D}=0 \Rightarrow \Sigma(p^2)$ is div. const + finite

$\Rightarrow$ need δ_m to cancel but no δ_Z

$= -i\delta_m$ only

furthermore δ_m fixed at 1-loop order

tadpoles cancelled by bare lagrangian

$\mathcal{L}_{\text{bare}} = -\delta_j \phi$

$= -i\delta_j$. so δ_j is fixed at 2-loop order

so $\chi\phi^3$ theory is super-renormalizable

Next lets consider $V = \frac{\chi}{3}\phi^3 + \frac{\lambda}{4!}\phi^4$. In this case the superifical degree of divergence is

$$\mathcal{D} = 4 - E - V_3$$

Notice that it's independent of V_4. All this means is that finite number of divergent amplitudes ($E \leq 4$) but infinite number of divergent graphs. Or more concretley any number of loops that allows for any V^4. This too is a renormalizable theory.

$$\mathcal{L}_{\text{bare}} = \frac{\delta_z}{2}(\partial_\mu\phi)^2 - \frac{\delta_m}{2}\phi^2 - \delta_j\phi - \frac{\delta_\chi}{6}\phi^3 - \frac{\delta_\lambda}{24}\phi^4$$

Well that was the good news. If we allow any kind of quartic or cubic coupling it looks like this theory. As long as the number of fields is finite then it is a perfectly renormalizable theory. But what if we allow $k > 4$? For example suppose we have just $k = 5$ and $V = \frac{c_5}{120}\phi^5$. In this case the superficial degree of divergence is

$$\mathcal{D} = 4 - E + V$$

This grows with V so the higher order of perturbation theory the worse the divergence of any amplitude. So take any amplitude with any number of external legs. At high enough order of perturbation theory $\mathcal{D} > 0$ which means there is at least a constant divergent term. Lets look at some examples. Consider 1-loop for the ϕ^5 theory. First consider $E = 6$.

one loop
$E=6$

: $\mathcal{D}=0 \Rightarrow$ log divergence $\to$ need $\delta_6\phi^6$ counterterm to cancel

2 loop
$E=7$

$\mathcal{D}=0 \to O(\log\Lambda)$
need $\delta_7\phi^7$ counterterm.

if we keep going we eventually need all $\delta_k\phi^k$ counterterms for all $k=5,6,7,8,\dots$

even worse. consider

= 2 loops $E=4$, $\mathcal{D}=+2$
$\Rightarrow O(\Lambda^2)$ divergence $\to$ amplitude is going to be = div. cons. + (div. cons.)$(p_1+p_2)^2$ + finite

need $\delta^* \phi^2(\partial_\mu\phi)^2$

At higher loops we'll need all products of ϕ and derivatives at all powers. The point of all this is to show that we have a whole zoo of divergences. ϕ^5 needs an infinite family of counterterms more and more at each loop order. This is an example of a non-renormalizable theory. Like any ϕ^k coupling with $k > 4$ makes QFT non-renormalizable. What about theories that have more than just scalars? Back in november he gave us an idea of dimensiona analysis and the fact that negative dimensions are trouble. We can now be more technical about it and relate dimensional analysis to renormalizabliyty. Lets recap very quickly. The basic idea is that we use units where $\hbar = c = 1$ so there is just one dimensionful units, units of energy. $[m] = [p^\mu] = E^{+1}$ and $[x^\mu] = E^{-1}$. The action is dimensionaless meaning that $[\mathcal{L}] = E^4$. For example for a scalr field theory $\Phi(x)$ has

$$\mathcal{L} = \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2$$

This means that $[\Phi] = E$. And in the same way all bosons have dimensions E^{+1} (mostly because of the kinetic term) for fermions the kintec terms looks like

$$\mathcal{L}_{\text{free}} = \overline{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

So fermions have dimeension $TODO$. In pertubration theory the copuling parametrs has dimension

$$[C_n] = [\mathcal{L}]/[\Phi^n] = E^{4-n}$$

This turns out to be a general rule

(a) All couplings of a renomralizable thoery must have non-negative energy dimensions

(b) coupling of theory has striclty posiive energy dimensions then the theory is super-renormalizable

(c) if any coupling has negatie energy dimension then the theory is non-renromalizable.

So those are the rules. Lets let him explain us where this comes from. Take a generic interaction term. It will have n_b bosonic fields, n_f fermionic fields, n_d spacetime deritavies acting on all thes fields. Thus we have

$$[\text{field product}] = E^{n_b+\frac{3}{2}n_f+n_d}$$

This means that the coupling dimension is

$$[g] = E^{\Delta} \text{ for } \Delta = 4 - n_b - \frac{3}{2}n_f - n_d$$

Now lets count the superifical degree of divergence. consider some L loop amplitude

$$\int d^{4L}q \prod(\text{propagator}) \times \prod(\text{vertices})$$

Well bosonic propgator go like $\frac{1}{q^2}$ and femrionic propagators go like $\frac{1}{q}$. But if there are derivaes we get more q terms. Alltogether we get

$$\int d^{4L}q \frac{1}{q^{2P_b+P_f}} \times \prod_{v}^{\text{vertices}} q^{n_d(v)}$$

So the superifical degree of divergence is

$$\mathcal{D} = 4L - 2P_b - P_f + \sum_{v=1}^{V} n_d(v)$$

If we use Euler theorem then

$$L - P_{\text{net}} + V = 1 \Rightarrow L = 1 + P_b + P_f - V$$

All togehter this means that

$$\mathcal{D} = 4 + 2P_b + 3P_f + \sum_{v=1}^{V}(n_d - 4)$$

If we count the line ends we can also see that

$$2P_b + E_b = \sum_v n_b \qquad 2P_f + E_f = \sum_v n_f(v)$$

Thus giving us

$$2P_b + 3P_f = \sum_v (n_b + \frac{3}{2}n_f) - E_b - \frac{3}{2}E_f$$

And consequently we get

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f + \sum_{v=1}^{V} \underbrace{\left(n_b + \frac{3}{2}n_f + n_d - 4\right)}_{\text{energy dimension of field product}}$$

So we get the important relation

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f - \sum_{v=1}^{V} \Delta(g_v)$$

and that is why coupling constant dimension is so important for renormalizabliity. If we have some dimensionless constnat while the rest are positive then theoyry has a infinite number of divergent diagrams but all diagrams have $E_b + \frac{3}{2}E_F \leq 4$ which tells us that there is only a finite numbere of divergent amlitudes. So we only need a finite number of counterterms. If some term has negative dimension energy dimension then we can go to higher orderss of loop order and get more and more non-renormalizable theories[3].

so now we know why we care about coupling dimensions.

$$\Delta = 4 - n_b - \frac{3}{2}n_f - n_d$$

To see a zoo of what kinds of fields we can have look at diman2.pdf. I can't type as fast as he's going through these notes.

LECTURE 11: RENORMALIZING QED AND WARD IDENTITIES

February 11, 2021

Lets start! Last lecture we started talking about renormalizaiblility of some field theories and the outlined the kinds of allowable couplings in $D = 4$. Now we're going to talk about the best know QFT that is practical, QED. And we'll take a look at renormalizing QED. So for simplicity we're going to look at the simplest version of QED, photon and electrons. We all know how to add other fields to the theory

$$\mathcal{L}_{\text{phys}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\Psi}(i\not{D} - m_e)\Psi = -\frac{1}{4}F_{\mu\nu}^2 + \overline{\Psi}(i\not{\partial} - m)\Psi + eA_\mu\overline{\Psi}\gamma^\mu\Psi$$

In the bare lagrangina we replace e with bare couling and bare mass. By convention Z_3 and Z_2 correspond to EM and electron field while Z_1 is for the electric charge. This consideration vies us

$$A^\mu_{\text{bare}}(x) = \sqrt{Z_e} \times A^\mu \quad \Psi_{\text{b}} = \sqrt{Z_2} \times \Psi$$

[3]this paragraph is really bad. Look at notes diman2.pdf

All together

$$\mathcal{L}_{\text{b}} = -\frac{Z_3}{4}F_{\mu\nu}F^{\mu\nu} + Z_2\overline{\Psi}(i\partial\!\!\!/ - m_{\text{b}})\Psi + Z_1 e \times A_\mu \overline{\Psi}\gamma^\mu\Psi$$

Where

$$Z_1 \times e = Z_2\sqrt{Z_3} \times e_{\text{bare}} \quad \mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{phys}} + \mathcal{L}^{\text{counter}}_{\text{terms}}$$

In counterterm perturbation theory we can also wrtie

$$\mathcal{L}^{\text{counter}}_{\text{terms}} = -\frac{\delta_3}{4} \times F_{\mu\nu}F^{\mu\nu} + \delta_2 \times \overline{\Psi}i\partial\!\!\!/\Psi - \delta_m\overline{\Psi}\Psi + e\delta_1 A_\mu\overline{\Psi}\gamma^\mu\Psi$$

Where we can solve for δ_i. For now we need to find the effects of the gauge averaging by adding a gauge symmetry breaking term $\mathcal{L}_G$ where

$$\mathcal{L}_{\text{G}} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

Where ξ paramterizes a speicifc gauge (e.g. $\xi = 1$ is Feynman gauge.) So when we do counterterm perturbation theory all the counterterms are perturbations. In the notes we're given feynman rules for the rules we should remember from last semster as well as the feynmann rules for the counterterms which is the only new bunch of Feynmann rules.

Lets go to power counting, my favorite kind of counting. QED has only one physical coupling e that dimensionless. Thus we should think that QED is a renormalizable theory (we saw this last lecture from dimensional analysis.) The problem with the pwoer counting analysis is if you ask how many counterterms we need to cancel out divergences it turns out we need more counterterms than we actually have to keep gauge symmetry. Fortuantely Ward-Takahashi identities show that you don't actually need the missing counterterms, if you sum up each diagrams in each order then you have enough counterterms. Anyways that for tomorrow. Today lets do the power counting. The supericial degree of divergence is

$$\mathcal{D} = 4 - \frac{3}{2}E_e - E_\gamma$$

So we need

$$\frac{3}{2}E_e + E_\gamma \leq 4$$

And since the number of electronic lines is always even we have either $E_e = 0$ and $E_\gamma \leq 4$ or $E_e = 2$ and $E_\gamma \leq 1$. All other graphs will not contribute to a superifcial divergence. So lets lit out all the possiblites

(a) $E_e = 0 = E_\gamma$, just a vacuum bubble, we ignore

(b) $E_e = 0$ and $E_\gamma = 1$, tadpole which vanishes by lorentz symmetry acting on the μ index of a zero-momentum photon [4]

(c) $E_e = 0$ and $E_\gamma = 2$. Two photonic legs. This diverges quadratically and contributes to $i\Sigma^\gamma_{\mu\nu}(k)$.

[4] what the fuck?

(d) $E_e = 0$ and $E_\gamma = 3$ (looks like someone's OC Sharingan). These diagrams form pairs that cancel each other out. This is *Furry's theorem*[5] which comes from charge conjugation.

(e) $E_e = 0$ and $E_\gamma = 4$. This one is also non-vanishing and we need to worry about it since it contributes to 4-photon amplitude $iV_{\kappa\lambda\mu\nu}$

(f) $E_e = 2$ and $E_\gamma = 0$ which contirbutes to $-i\Sigma^e_{\alpha\beta}(p)$ where $\alpha\beta$ are dirac indices. By symmetry (lorentz+partiy) we have

$$\Sigma^e_{\alpha\beta}(p^\mu) = A(p^2) \times \delta_{\alpha\beta} + B(p^2) \times p_\mu \gamma^\mu_{\alpha\beta}$$

We can exand this as a powoer sieres in p^2 and use $p^2 = (\not{p})^2$ so we can also turn the term above into a power series of p slashed. Because of this we can also write $\Sigma^e(\not{p})$.

(g) $E_e = 2$ and $E_\gamma = 1$. This is just the same vertex we had above but dressed with e-γ vertex. Apparently this is a real pain.

We can now look at the momentum dependence of the divergences. Lets keep going. First we'll do it in reverse order. We'll first consider (g). The dressed vertex has

$$\mathcal{D}[\Gamma^\mu] = 0 \Rightarrow \text{log divergences}$$

And the derivative of the dressed vertex wrt electron or photons momentum has $\mathcal{D} = -1$ thus we know that

$$\Gamma^\mu_{\alpha\beta}(p', p) = [O(\log\Lambda)\text{constant}] \times \gamma^\mu_{\alpha\beta} + \text{finite}(p', p)$$

Furthermore we can see by lorentz symmetry that the constant array must be proprotial to the gamma matrix and that gives the result above. By counterterm perturbation theory we also have

$$\Gamma^\mu_{\text{net}}(p', p) = (\Gamma^\mu_{\text{tree}} = \gamma^\mu) + \Gamma^\mu_{\text{loops}} + \delta_1 \times \gamma_\mu$$

So if we pick the correct γ_μ then we can counter the divergence.

Next we'll consider the electron propagatorl (g). The superificial degreee of divergence is

$$\mathcal{D} = 1$$

So it diverges like p^2 and the taking derivatives reduces superifical degree of divergence by one. So we have

$$\Sigma^e = [O(\Lambda)\text{ constant}] + [O(\log\Lambda)\text{ constant}] \times \not{p} + \text{finite}(\not{p})$$

And in counterterm perturbation theory we have

$$\Sigma^e_{\text{net}}(\not{p}) = \Sigma^e_{\text{loops}}(\not{p}) + \delta_m - \delta_2 \times \not{p}$$

So again the coutnerterms cancel out the divergences.

[5] 0w0

Now for some bad news. First the four-photon amplitude (e) in the list above. First superifical degree of divergence is $\mathcal{D} = 0$. So log divergence. By lorentz symmetry and bose symmetry we need

$$V_{\kappa\lambda\mu\nu}(\{k_i\}) = [O(\log\Lambda) \times (\text{a bunch of metrics } g_{\alpha\beta}) + \text{finite}_{\kappa\lambda\mu\nu}$$

How the fuck do we deal with the the term that is proprotinal to the product of metrics. This is our first problem.

Now lets consider (b). For this one the third derivative is finite. This means that the divergent part of $\Sigma^{\gamma}_{\mu\nu}$ has to be some polynomial in k and for lrentz symmetry we need a metric

$$A \times g_{\mu\nu} + B \times k^2 \times g_{\mu\nu} + C \times k_\mu k_\nu$$

Where A is $O(\Lambda^2)$ and B, C is $O(\log\Lambda)$. Naivley we could say that the counter term is

$$-i\delta_{3A} k^2 g_{\mu\nu} + i\delta_{3B} \times k_\mu k_\nu - i\delta_{m\gamma} \times g_{\mu\nu}$$

But this will create terms in the lagrangian that don't existts, namely the photon mass term.

$$\frac{\delta_{m\gamma}}{2} A_\nu A^\nu$$

All togehter we know that QED has a finite number of divergent amplitudes which can be cancelled by a finite number of counterterms but the gauge symmetry of QED stop some counterterms from exsisting. In QCD we have a similar situation. For QED there is something we can do in this class to show that we don't actually need those counterterms. In the end they cancel eachother out and that miracle happens because of Ward-Takahashi identies. There's a lot of these idneitites and they all stem from electric current conservation and the one we care about right now is the identity relating to purely photonic amltitudes. For a diagram with n photonic lines we have

$$\text{diagram} = i\mathcal{M}^{\mu_1\cdots\mu_n}(k_1, \ldots, k_n)$$

The ward identity says that for any i

$$(k_i)_{\mu_i} \times \mathcal{M}^{\mu_1\cdots\mu_n}(k_1, \ldots, k_n) = 0$$

A caveat is that this doesn't work any one diagram, you have to add up all diagrams in a particular group. But once you do that the above relation holds. Lets apply this to the 4-photon amplitude. We saw a few minutes ago that

$$\mathcal{V}_{\kappa\lambda\mu\nu}(k_1, k_2, k_3, k_4) = C \times (\text{bunch of metrics}) + \text{finite}$$

Where C is $O(\log\Lambda)$ constnat. By Ward identity

$$0 = k_1^\kappa \times \mathcal{V}_{\kappa\lambda\mu\nu} = C \times (\text{bunch of } k_1 \text{ times metrics}) + k_1^\kappa \times \text{finite}_{\kappa\lambda\mu\nu}$$

The finite terms cannot cancel so we can say that $C = 0$. In other word the ward idneitty for the 4-photon ampltiude does not give an ultraviolet divergence. So that takes care of that. Lets move on the two photon ampltiude. We have

$$\Sigma^{\gamma}_{\mu\nu} = \Xi(k^2) \times g_{\mu\nu} + \Pi(k^2) \times k_\mu k_\nu$$

For scalar function Ξ and Π. By the ward identity we have

$$0 = k^\mu \times \Sigma^\lambda_{\mu\nu} = \Xi(k^2) \times k_\nu - \Pi(k^2) \times k^2 k_\nu$$

$$0 = k^\nu \times \Sigma^\lambda_{\mu\nu} = \Xi(k^2) \times k_\mu - \Pi(k^2) \times k^2 k_\mu$$

And thus

$$\Xi(k^2) = \Pi(k^2) \times k^2 \Rightarrow \Sigma^\gamma_{\mu\nu} = \Pi(k^2) \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

A few minutes ago we had that the divergent part of the 2-phoonto ampltiude had some form (above.). So again lets shove a ward identity into this

$$0 = A \times k_\nu + B \times k^2 k_\nu + C \times k^2 k_\nu$$

From this we can read off $A = 0 \quad C = -B$ giving us

$$[\Sigma^\gamma_{\mu\nu}(k)]_{\text{divergent}} = B \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

This is exactly what we had above. This means for the entire amplitude

$$\Sigma^\gamma_{\mu\nu}(k) = \Pi(k^2)(k^2 g_{\mu\nu} - k_\mu k_\nu) \text{ where } \Pi(k^2) \text{ is a log divergent constnat plus something finite}$$

We have to be careful here. This is only true for the netamplitude or for the next combination for the whole loop order. For QED itself at the one-loop order there is only one diagram. However at higher loop orders individual diagrams may disobey the above rule but summing them all together satisfies the identity. But once you get you can ask what kind of counterterm do you need to cancel things out? Well if we add the counterterms

$$[\Sigma^\gamma_{\mu\nu}]^{\text{net}} = [\Sigma^\gamma_{\mu\nu}]^{\text{loops}} - \delta_3 \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

So we can canel the divergence.

To summarize, the ward identites alow us to ignore the counterterms that we seem to be missing since the bad counterterms aren't needed when we add everything together.

So now we know how to use the counterterms to cancel the UV idvergent loop graphs. However how do we set the finite part of the counterterms. These follow from *renormailzation conditions* (we need pole at physicl mass with unit ampltiude). Lets try to do the electron's dressed propagator, the fourier transform of the 2-point function.

$$\mathcal{F}^e_2(p) = \int d^4(x-y) e^{ip(x-y)} \mathcal{F}^e_2(x-y) = \frac{i}{\not{p} - m + i0} + \frac{i}{\not{p} - m + i0}(-i\Sigma^3(\not{p}))\frac{i}{\not{p} - m + i0} + \dots$$

So everythign here is a 4x4 matrix with dirac indices so we need to be careful how we multiply. Thankfully the 1PI amptliudes are powerseries in $\not{p}$ so it commmutes with $\not{p}$ and function of $\not{p}$ so we can write the N-bubble term as

$$\left(\frac{i}{\not{p} - m + i0}\right)^{N+1} \times (-i\Sigma^e(\not{p}))^N \frac{i}{\not{p} - m + i0} \times \left(\frac{\Sigma^e(\not{p})}{\not{p} - m + i0}\right)^N$$

Giving us

$$\mathcal{F}^e_2(p) = \frac{i}{\not{p} - m - \Sigma^e(\not{p}) + i0}$$

Something about poles for the free propagator here. Now lets talk about the dressed propagator. It should have a pole at $\not p$ = physical electron mass and residue is the electron field strength factor. Namely the pole should be at the exact same position as the free propagator with unit residue. This means that

$$\mathcal{F}_2^3(p) = \frac{i}{\not p - m - \Sigma^3(\not p) + i0} = \frac{i}{\not p - m + i0} + \text{smooth}$$

So if we expand the denominatir in powers of $(\not p - m)$ we get

$$\not p - m - \Sigma^3(\not p) = -\Sigma^e @ (\not p = m) + \left(1 - \frac{d\Sigma^3}{d\not p} @ (\not p = m)\right) \times (\not p - m) + O((\dots)^2)$$

This means the renormalizaiton condiiton is

$$\text{at } \not p = m \text{ both } \Sigma^e = 0 \text{ and } \frac{d\Sigma^3}{d\not p} = 0$$

This means that

$$\Sigma^e_{\text{net}}(\not p) = \Sigma^e_{\text{loop}} + \delta_m - \delta_2 \times \not p$$

All together this means that

$$\Sigma^e_{\text{loops}} \text{ at } \not p = m + \delta_m - \delta_2 \times m = 0 \qquad \frac{d\Sigma^e_{\text{loops}}}{d\not p} @ (\not p = m) - \delta_2 = 0$$

These are the renormalization conditiosn that fix the finite parts of the counterterms

Lecture 12:

February 12, 2021

Last lecture we talked about QED and why it's renormalizable. Towards the end we explained how to set finite parts with renormalizablity conditions. Today we're going to do some calculations. Lets talk about the dressed photon propagator. So there's a lot of lorentz indices floating around. The best way to deal with this is treat things as matrices. Also to keep matrix multiplation clear we'll have $\text{prop}^\mu{}_\nu$. We have for the free propagator

$$\text{prop}^\mu{}_\nu = \frac{-i}{k^2 + i0}\left[\delta^\mu_\nu + (\xi - 1)\frac{k^\mu k_\nu}{k^2 + i0}\right] \qquad [i\Sigma^\gamma(k)]^\mu{}_\nu = i\Pi(k^2) \times (k^2\delta^\mu_\nu - k^\mu k_\nu)$$

Given our matrix notation we can write

$$|\mathcal{F}_{2\gamma}| = \text{prop} + \text{prop} \times (i\Sigma^\gamma) \times \text{prop} + \dots$$

The easiest way to evaluate these matrices is ti introduce projector matrices

$$[P_\parallel(k)]^\mu{}_\nu = \frac{k^\mu k_\nu}{k^2 + i0}$$

this projects 4-vectors onto their components in the k^μ direction. We simillarly have a perpendicular projection

$$[P_\perp(k)]^\mu{}_\nu = \delta^\mu_\nu - \frac{k^\mu k_\nu}{k^2 + i0}$$

This allows us to write the photon propagator as

$$\text{prop} = \frac{-i}{k^2+i0}\times\left(P_\perp + \xi P_\parallel\right) \quad [i\Sigma^\gamma(k)]^\mu{}_\nu = i\Pi(k^2)\times k^2 P_\perp$$

Now using the properties of the projection we can say that everything commutes. All of this gives us

$$\text{N}^{\text{th}}\text{term} = (\text{prop})^{N+1}\times(i\Sigma^\gamma)^N$$

After some algebra we can get

$$T(N) = \text{N}^{\text{th}}\text{term} = \frac{-i}{k^2+i0}\times(\Pi(k^2))^N\times\begin{cases} P_\perp & N>0 \\ P_\perp + \xi P_\parallel & N=0\end{cases}$$

So we can sum things up to give us

$$|\mathcal{F}_{2\gamma}| = \sum_{N=0}^{\infty} T(N) = \frac{-i}{k^2+i0}\times\frac{1}{1-\Pi(k^2)}\times(P_\perp + \tilde{\xi}P_\parallel) \text{ where } \tilde{\xi} = \xi(1-\Pi(k^2))$$

So now we have our dressed propagator. Now the most salient feature of this propagator is that it automatically has a pole at $k^2=0$. This magic comes from the fact that Σ is proportional to k^2. Basically it's the ward identity that keeps things automatically massless. And that's why we don't need a counterterm for the photon mass. Lets now talk about the residue of the pole. We can just evaluate things to get

$$\text{Residue} = \frac{1}{1-\Pi(k^2=0)}$$

Physically this residue is the field strength renormalization and that means taht this residue should be one meaning that we can guess that Π vanishes at $k^2=0$.

$$\Pi^{\text{net}}(k^2) = \Pi^{\text{loops}}(k^2) - \delta_e \to 0 \text{ for } k^2 = 0$$

This determines the finite part of δ_3. Now we'll move on to the one-loop calculation. We cant to calculate the phton two point insertion and verify the ward identities.

$$k_\mu\Sigma^{\mu\nu}(k) = 0 \quad k_\nu\Sigma^{\mu\nu}(k) = 0$$

And due to Lorentz symmetires we should have

$$\Sigma^{\mu\nu} = \Pi(k^2)\times(g^{\mu\nu}\times k^2 - k^\mu k^\nu)$$

In QED there is a single 1-loop diagram with 2 photonic legs thats given in the lecture notes. This gives us the following

$$i\Sigma^{\mu\nu}_{1-\text{loop}} = -\int\frac{d^4p_1}{(2\pi)^4}\text{tr}\left[(ie\gamma^\mu\frac{i}{\not{p}_1 - m + i0}(ie\gamma^\nu)\frac{i}{\not{p}_2 - m + i0}\right]$$

We can write the thing in side the trace as

$$e^2\times\text{tr}\left[\gamma^\nu\times\frac{\not{p}_1+m}{p_1^2-m^2+i0}\times\gamma^\mu\times\frac{\not{p}_1+m}{p_2^2-m^2+i0}\right] = \frac{e^2\mathcal{N}^{\mu\nu}}{\mathcal{D}}$$

After some tracelogy we have

$$\mathcal{N}^{\mu\nu} = 4p_1^\mu p_2^\nu + 4p_1^\nu p_2^\mu - 4(p_1 p_2)g^{\mu\nu} + 4m^2 g^{\mu\nu}$$

And using Feynman parameters we can write

$$\frac{1}{\mathcal{D}} = \frac{1}{(p_1^2 - m^2 + i0)\times(p_2^2 - m^2 + i0)} = \int_0^1 dx \times \frac{1}{[(1-x)\times(p_1^2 - m^2 + i0) + x\times(p_2^2 - m^2 + i0)]^2}$$

After a lot more algebra we can write the trace

$$\text{tr} = \int_0^1 dx \times \frac{e^2 \mathcal{N}^{\mu\nu}}{[p^2 - \Delta(x) + i0]^2} \text{ where } p = p_1 + xk \text{ and } \Delta(x) = m^2 - x(1-x)k^2$$

We're going to use dimensional regularization today to evaluate

$$\Sigma^{\mu\nu}_{1-\text{loop}} = ie^2 \int_0^1 dx \int \frac{d^4 p}{(2\pi)^4} \frac{\mathcal{N}^{\mu\nu}}{[p^2 - \Delta + i0]^2}$$

We can define

$$p_1 = p - x\times k \quad p_2 = p_1 + k = p + (1-x)k$$

After a lot of algebra we find that we can split ut $\mathcal{N}^{\mu\nu} = \mathcal{N}^{\mu\nu}_{\text{good}} + \mathcal{N}^{\mu\nu}_{\text{bad}} + \mathcal{N}^{\mu\nu}_{\text{odd}}$ where "good", "bad"m and "odd". "Good" has the terms that have what we ant for the ward identity, "bad" and "bad" contribute to the momentum integral. So what's the point of this reorgnaiation? WEll we wnat Σ to be proprotiona to the "good" so we will argue that the bad and odd parts integrate to zero. Lets start with the odd term. It should be clear that these integrate to zero since they are odd and we're integrating over all space(and integral over all momentum space is invariant.) We've done this trick before. Showing that the "bad" integrates to zero is a bit more effore. First we wick rotate toe euclidean momentum space. This means

$$(\text{bad}) = -\frac{d^4 p_E}{(2\pi)^4} \frac{8p_E^\mu p_E^{|} nu + 4g^{\mu\nu}\times(\Delta + p_E^2)}{[\Delta + p_E^2]^2}$$

The angular integral is evalauted using the $SO(4)$ symmetry

$$\int \frac{d^3\Omega_p}{(2\pi)^2} p_E^i p_E^j = \delta^{ij} \times \frac{p_E^2}{4}$$

Similarly we have

$$\int d^4 p_E \times f(p_E^2) \times p_E^i p_E^j = \delta^{ij} \times \int d^4 p_E \times f(p_E^2) \times \frac{p_E^2}{4}$$

If we use minkowski indices in arbitrary dimension we'll have

$$\int d^D p_E f(p_E^2) \times p_E^\mu p_E^\nu = -g^{\mu\nu} \times \int d^D p_E \times f(p_E^2) \times \frac{p_E^2}{D}$$

Using this trick we can plug some things into (bad) to get

$$(\text{bad}) = -g^{\mu\nu} \times \mu^{4-D} \times \int \frac{d^D p_E}{(2\pi)^D} \frac{4\Delta + (4 - 8/D)\times p_E^2}{[\Delta + p_E^2]^2}$$

Now the next thing to do is express everything in terms of a gaussian integral. This will give us

$$\frac{4\Delta + (4-8/D)\times p_E^2}{[\Delta + p_E^2]^2} = \int_0^\infty dt \left(\left(4 - \frac{8}{D}\right) + \frac{8}{D}\Delta \times t\right) \times \exp\left\{-t(\Delta + p_E^2)\right\}$$

Plugging this integral into the (bad) integral, changing the order of integration, and then evalauting the gaussian momentum integral we get an tinegral taht vanishes for $D < 2$. Because of this we'll say wehn we analytically continue things to $D = 4$ we still have a vanishing integral. So what we did today is verify the ward identy for the one-loop propagator correction.

Lecture 13: More 1-loop QED Renormalization

February 24, 2021

It's been 12 days since the last class :pensive:. Lets review what we did last time. We talked about QED renormalization at one loop level. At one loop level there is only one diagram with two external photonic legs. (look at last lecture notes, i'm not typing it again). We ended last time with

$$\Sigma^{\mu\nu}_{1-\text{loop}}(k) = (g^{\mu\nu}k^2 - k^\mu k^\nu) \times \Pi_{1-\text{loop}}(k^2) \text{ where } \Pi_{1-\text{loop}}(k^2) = 8e^2 \int_0^1 dx\, x(1-x) \times \int \frac{d^4p}{(2\pi)^4} \frac{i}{[p^2 - \Delta + i0]^2}$$

We can evaluate $\Pi_{1-\text{loop}}$ in the normal way (see peskin and schroeder problem 10.2. We did something like this a million times.) In the end we get

$$\Pi_{1-\text{loop}}(k^2) = -\frac{e^2}{12\pi^2} \times \left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2} + I(k^2/m^2)\right) \text{ where } I \text{ is defined in notes}$$

Now in counter-term perturbation theory we have

$$\delta_3 = -\frac{e^2}{12\pi^2} \times \left(\frac{1}{\epsilon}\right)$$

Where the finite part can be set by requiring

$$\Pi^{\text{net}}(k^2 = 0) = 0 \text{ for field strength renormalization}$$

This condition gives us

$$\delta_3 = -\frac{e^2}{12\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m_e^2}\right)$$

And then the remaining is

$$\Pi(k^2) = -\frac{e^2}{12\pi^2} \times I\left(\frac{k^2}{m_e^2}\right)$$

What is the consequence of this renormalization? Consider processes with virtual photons. The dressed photon propagator has the form

$$= \frac{-i}{k^2 + i0} \times \frac{1}{1 - \Pi_{\text{net}}(k^2)} \times (g^{\mu\nu} + (\tilde{\xi} - 1)\frac{k^\mu k^\nu}{k^2 + i0}$$

Where this part is nontrivial. Now consider coulomb scattering of heavy and slow charged particles but with dressed propagators (see (50) for feynmann diagram) This means that

$$m_e \ll |\mathbf{p}_i|, |\mathbf{q}| \ll M_1, M_2$$

In this limit this turns into a tree level interaction vertes with means that the vertes $\approx (-iQ) \times 2M\delta^{\nu 0}$. Why do we want heavy as well as slow? We want this because we want momentum transfer to be larger than electron mass. With all these we get

$$\mathcal{M} = \frac{4M_1M_2Q_1Q_2}{t} \times \frac{1}{1-\Pi(t)}$$

Using the bron formula we can write the fourier transform of the effective potential

$$\tilde{V}_{\text{eff}}(\mathbf{q}) = \int d^3\mathbf{x}\, e^{-i\mathbf{q}\mathbf{x}} V_{\text{eff}}(\mathbf{x}) = \left(\frac{Q_1Q_2}{4\pi\mathbf{q}^2}\right)_{\text{tree}} \times \frac{1}{1-\Pi(-\mathbf{q}^2)}$$

What's the physical meaning of the $1/1-\Pi$ factor? It makes the charges slowly momentum dependent on momentum transfer.

tl;dr: α the copuling constant is momentum dependent.

Peskin and Schroeder Problem 10.2(a,b)

Started: February 12, 2021. Finished: February 24, 2021

(a) Lets look at the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + \overline{\psi}(i\partial\!\!\!/ - M)\psi - ig\overline{\psi}\gamma^5\psi\phi$$

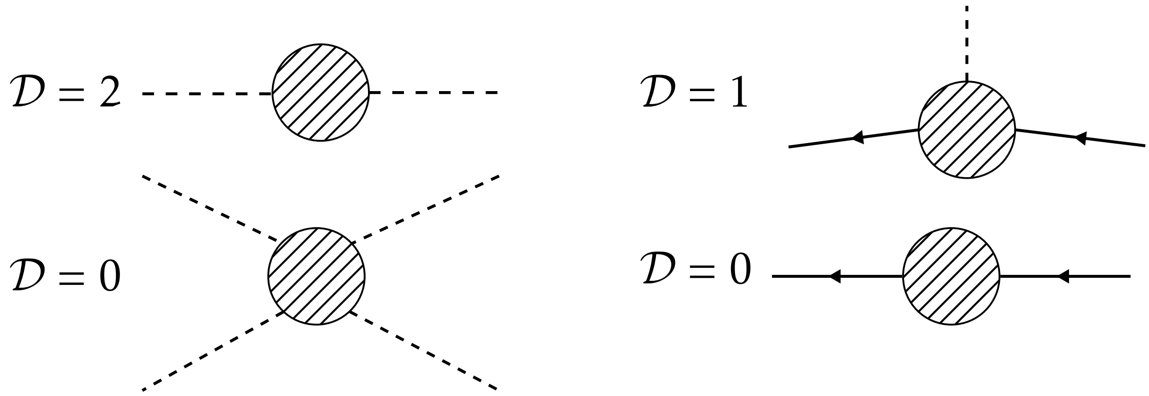
We know that $[\mathcal{L}] = E^4$ and that $[\partial_\mu] = E$ implying that $[\phi] = +E$ $[\psi] = E^{3/2}$ and thus $[g] = E^{4-3-1} = 0$. From our rules derived in lecture we have

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f \geq 0 \Rightarrow 4 \geq E_b + \frac{3}{2}E_f$$

Due to parity symmetry we know that $E_b = 0, 2, 4$. This gives us the following possibilities

$$\begin{bmatrix} 0,1,\frac{5}{2} & 0,2,1 & 1,0,3 \\ 1,1,\frac{3}{2} & 1,2,0 & 2,0,2 \\ 2,1,\frac{1}{2} & 3,0,1 & 4,0,0 \end{bmatrix}$$

Where we have a matrix of $E_b, E_f, \mathcal{D}$. First we ignore the $E_B = 3$ and $E_b = 1$ terms since they have a odd number of external scalar particles which vanishes under parity symmetry. Also we can ignore the fractional $\mathcal{D}$ terms. So we have four diagrams



From this we see the need for a scalar self interaction $\delta\mathcal{L} = \lambda/4! \times \phi^4$ in the theory for things to be renormalizable. We have the counter term lagrangian

$$\mathcal{L}_{\text{ct}} = \frac{1}{2}\delta_{Z_\phi}(\partial_\mu\phi)^2 - \frac{1}{2}\delta_m\phi^2 + i\delta_{Z_\psi}\overline{\psi}\partial\!\!\!/\psi - \delta_M\overline{\psi}\psi - i\delta_g\overline{\psi}\gamma^5\psi\phi - \frac{1}{4!}\delta_\lambda\phi^4$$

From our notes we also have

$$\mathcal{L}_{\text{b}} = \mathcal{L}_{\text{phys}} + \mathcal{L}_{\text{ct}}$$

Where

$$\mathcal{L}_{\text{phys}} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2_{\text{phys}}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - M_{\text{phys}})\psi - ig_{\text{phys}}\overline{\psi}\gamma^5\psi\phi - \frac{1}{4!}\lambda_{\text{phys}}\phi^4$$

$$\mathcal{L}_b = \frac{1}{2}Z_\phi(\partial_\mu\phi)^2 - \frac{1}{2}m_b^2 Z_\phi\phi^2 + Z_\psi\overline{\psi}(i\partial\!\!\!/ - M_b)\psi - ig_b Z_\psi\sqrt{Z_\phi}\overline{\psi}\gamma^5\psi\phi - \frac{1}{4!}Z_\phi^2\lambda_b\phi^4$$

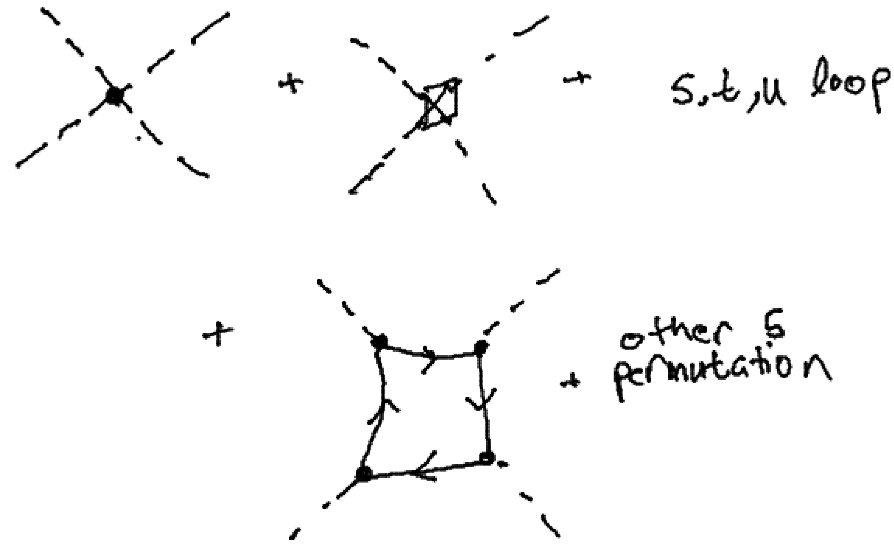
Note that we implicitly rewrote the bare ϕ_b and ψ_b fields in terms of the field strength renormalizations $\sqrt{Z_\phi}\phi = \phi_b$ and $\sqrt{Z_\psi}\psi = \psi_b$. On comparing $\mathcal{L}_{\text{ct}}$ with $\mathcal{L}_b$ and $\mathcal{L}_{\text{phys}}$ we have

$$\delta_{Z_\phi} = Z_\phi - 1 \quad \delta_m = m_b^2 Z_\phi - m^2_{\text{phys}} \quad \delta_{Z_\psi} = Z_\psi - 1 \quad \delta_M = Z_\psi M_b - M_{\text{phys}}$$

$$\delta_g = g_b Z_\psi\sqrt{Z_\phi} - g_{\text{phys}} \quad \delta_\lambda = Z_\phi^2\lambda_b - \lambda_{\text{phys}}$$

(b) Lets start knocking these guys down. First lets consider the 4-scalar vertex. At one loop

level we've got



Note that the s,t,u loop contribution we calculated in class (equaiton 25 of DR.pdf) and got that the contribution for all three diagrams is

$$\mathcal{F}_{\text{DR}} = \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log\left(4\pi\mu^2/m^2\right)\right)$$

Lets try to take a crack at the big square loop. First we'll label the external legs momentum as k_i and internal legs momentum as p_i. Our Feynmann rules gives us

$$\begin{aligned} &= -g^4 \int \frac{d^4 p_1}{(2\pi)^4} \text{Tr}\Bigg(\gamma^5 \times \frac{i}{\not{p}_1 - m + i0} \times \gamma^5 \times \frac{i}{\not{p}_1 + \not{k}_1 - m + i0} \\ &\times \gamma^5 \times \frac{i}{\not{p}_1 + \not{k}_1 + \not{k}_2 - m + i0} \times \gamma^5 \times \frac{i}{\not{p}_1 + \not{k}_1 + \not{k}_2 + \not{k}_3 - m + i0}\Bigg) \end{aligned}$$

On the eproblem set we are given the hint that the infinite part does not depend on the scalar momenta so we can let $k_i = 0$. So our above expression reduces to

$$= -g^4 \int \frac{d^4 p_1}{(2\pi)^4} \text{Tr}\left(\left[\frac{\gamma^5}{\not{p}_1 - m + i0}\right]^4\right)$$

But wait, it gets even better! We can use

$$\frac{\not{p}_1 + m}{p_1^2 - m^2 + i0} = \frac{1}{\not{p}_1 - m + i0}$$

To get the following expression for the divergent amplitude

$$= -g^4 \int \frac{d^4 p_1}{(2\pi)^4} \frac{\text{Tr}\left(\gamma^5(\not{p}_1 + m)\gamma^5(\not{p}_1 + m)\gamma^5(\not{p}_1 + m)\gamma^5(\not{p}_1 + m)\right)}{[p_1^2 - m^2 + i0]^4}$$

But wait, it gets even more better! Using the anticommutation of γ^5 and γ^μ and the fact that $(\gamma_5)^2 = 1$ we get

$$
\begin{aligned}
&= -g^4 \int \frac{d^4p_1}{(2\pi)^4} \frac{\mathrm{Tr}\Big((-\not{p}_1 + m)(\not{p}_1 + m)(-\not{p}_1 + m)(\not{p}_1 + m)\Big)}{[p_1^2 - m^2 + i0]^4} \\
&= -g^4 \int \frac{d^4p_1}{(2\pi)^4} \frac{\mathrm{Tr}\Big((\not{p}_1^2 - m^2)^2\Big)}{[p_1^2 - m^2 + i0]^4} \\
&= -g^4 \int \frac{d^4p_1}{(2\pi)^4} \frac{4 \times \Big((\not{p}_1^2 - m^2)^2\Big)}{[p_1^2 - m^2 + i0]^4} \\
&= -4g^4 \int \frac{d^4p_1}{(2\pi)^4} \frac{1}{[p_1^2 - m^2 + i0]^2}
\end{aligned}
$$

Lets evaluate this with dimensional regularization(I swear we did this in lecture at one point....) We'll wick rotate to the euclidean momentum space getting

$$= -4g^4 \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{[p_E^2 + m^2]^2}$$

Now moving to arbitrary dimension

$$= -4g^4 \int \frac{\mu^{4-D} d^D p_E}{(2\pi)^D} \frac{1}{[p_E^2 + m^2]^2}$$

We proved in our notes before that

$$\frac{\Gamma(n)}{f^n} = \int_0^\infty t^{n-1} \times \exp\{-tf\}\, dt$$

This means that we can write

$$\frac{1}{[p_E^2 + m^2]^2} = \frac{1}{\Gamma(2) = 1} \times \int_0^\infty t \times \exp\Big\{-t\Big(p_E^2 + m^2\Big)\Big\}\, dt$$

Which means that we can write our momentum integral becomes

$$= -4g^4\mu^{4-D} \int \frac{\mu^{4-D} d^D p_E}{(2\pi)^D} \times \int_0^\infty t \times \exp\Big\{-t\Big(p_E^2 + m^2\Big)\Big\}\, dt = -4g^4 \int dt\, t \times \exp\Big\{-tm^2\Big\} \times (4\pi t)^{-D/2}$$

Where the red term comes from a gaussian integral

$$\text{red stuff} = \int \frac{d^D p_E}{(2\pi)^D} \exp\Big\{-tp_E^2\Big\}$$

So all together we have

$$= -4g^4\mu^{4-D}(4\pi)^{-D/2} \times \int_0^\infty t^{1-D/2} \times \exp\Big\{-tm^2\Big\}$$

Now let $D = 4 - 2\epsilon$ giving us

$$= -4g^4\mu^{2\epsilon}(4\pi)^{-2+\epsilon} \times \int_0^\infty t^{-1+\epsilon}\exp\left\{-tm^2\right\}$$

Now using the gamma function integral again we get

$$= -4g^4 \times \frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \frac{\Gamma(\epsilon)}{m^{2\epsilon}} = -\frac{4g^4}{16\pi^2}\left(\frac{4\pi\mu^2}{m^2}\right)^\epsilon \times \Gamma(\epsilon)$$

We'll take the first few powers in the expansion of ϵ

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \ldots$$

As well as the first few powers of ϵ in the series expansion of the lavendar term

$$\text{Lavendar} = 1 + \left(\log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right)\right)\epsilon$$

Throwing out positive powers of ϵ then leaves us with

$$= -\frac{4g^4}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right)\right)$$

With this we can multiply by 6 (to account for permutations of the big square loop) and now write the renormalization condiition by first writing

$$\mathcal{V}(\{k_i = 0\}) = -\lambda - \delta_\lambda + \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right)\right) - \frac{24g^4}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{m^2}\right)\right)$$

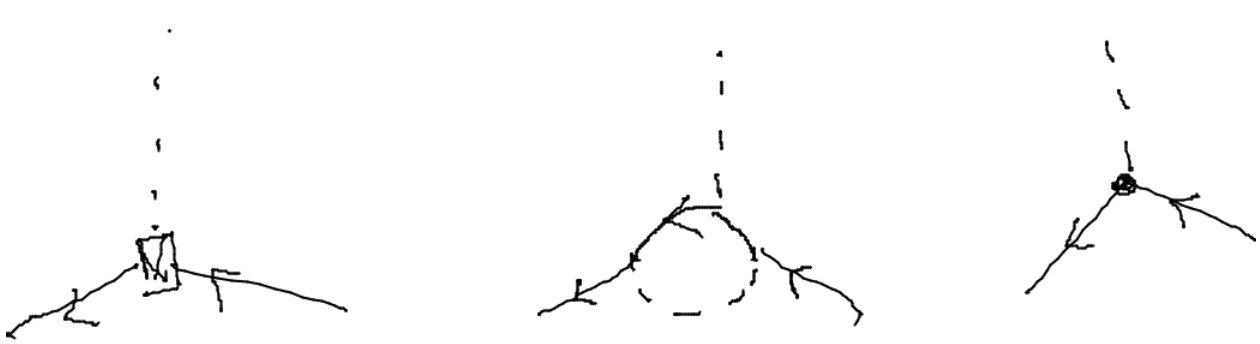
Thus we can read off that

$$\boxed{\delta_\lambda = \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right)\right) - \frac{24g^4}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{m^2}\right)\right) + \text{finite.}}$$

Where we adopted the convetion

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi) = \frac{1}{\bar{\epsilon}}$$

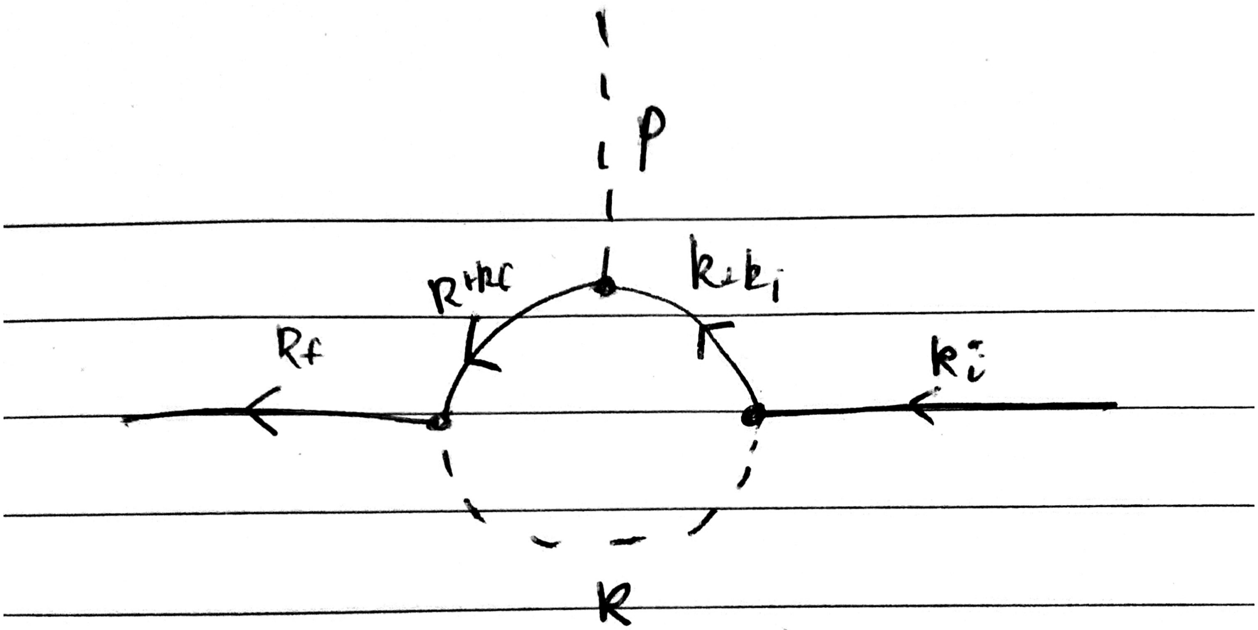
Now lets look at the $\mathcal{D} = 1$ 3 vertex interaction comming from the Yukawa coupling. The diagrams that contribute are



We need to cancel the logarithmic divergence with a counterm meaning that the three-point function must satisfy

$$-i\Gamma(k_f,k_i) = g\gamma^5 - i\mathcal{V}(k_f,k_i) + \delta_g\gamma^5 \text{ where } -i\Gamma(0,0) = g\gamma^5 - i\mathcal{V}(0,0) + \delta_g\gamma^5 = g\gamma^5 \Rightarrow \delta_g\gamma^5 = -i\mathcal{V}(0,0)$$

The only integral we need to do here is the one-loop contribution. Consider the following picture



With our Feynmann rules we can write (setting the external momenta to zero since the divergence does not depend on those)

$$-i\mathcal{V}(k_f,k_i=0) = -(-g)^3\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i0}\times\gamma^5\times\frac{i}{\not{k}-M+i0}\times\gamma^5\times\frac{i}{\not{k}-M+i0}\times\gamma^5$$

Lets focus on the fermionic part of the integral. Similar to the 4-vertex term we can write

$$\begin{aligned}
\text{blue stuff} &= -\left\{\gamma^5\times\frac{(\not{k}+M)}{k^2-M^2+i0}\times\gamma^5\times\frac{\not{k}+M}{k^2-M^2+i0}\times\gamma^5\right\} \\
&= -\left\{\gamma^5\times\frac{(-\not{k}+M)(\not{k}+M)}{(k^2-M^2+i0)^2}\right\} \\
&= \frac{\gamma^5}{(k^2-M^2+i0)}
\end{aligned}$$

So our integral tehn becomes

$$-i\mathcal{V}(k_f,k_i=0) = g^3\gamma^5\int\frac{id^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i0)(k^2-M^2+i0)}$$

Lets rewrite the momentum integral by wick rotating to the euclidean metric $k^0 = ik^4$ thus giving us

$$-i\mathcal{V} = g^3\gamma^5 \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{(k_E^2+m^2)(k_E^2+M^2)}$$

Now using Feynamnn's trick we get

$$= g^3\gamma^5 \int \frac{d^4k_E}{(2\pi)^4}\int_0^1 \frac{dx}{[x(k_E^2+m^2)+(1-x)(k_E^2+M^2)]^2}$$

Now lets dimreg the momentum integral

$$\text{mom} = \text{mom. integral} = \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[x(k_E^2+m^2)+(1-x)(k_E^2+M^2)]^2}$$

Going to arbitrary dimensions

$$\text{mom} = \int \frac{d^Dk_E \times \mu^{4-D}}{(2\pi)^D}\frac{1}{[x(k_E^2+m^2)+(1-x)(k_E^2+M^2)]^2}$$

Use gamma function integral

$$\text{mom} = \int \frac{d^Dk_E \times \mu^{4-D}}{(2\pi)^D} \times \int_0^\infty t \times \exp\left\{-t[x(k_E^2+m^2)+(1-x)(k_E^2+M^2)]\right\} dt$$

Now we evaluate the gaussian integrals

$$\int \frac{d^Dk_E}{(2\pi)^D}\exp\left\{-tk_E^2\right\} \times \exp\left\{-t(xm^2+(1-x)M^2\right\} = (4\pi t)^{-D/2} \times \exp\left\{-t(xm^2+(1-x)M^2)\right\}$$

Which all together gives us

$$\text{mom} = (4\pi)^{-D/2} \times \mu^{4-D} \times \int t^{1-D/2}\exp\left\{-t(xm^2+(1-x)M^2)\right\} dt$$

Taking $D = 4-2\epsilon$ then gives

$$\text{mom} = \frac{1}{16\pi^2}(4\pi\mu^2)^\epsilon \times \int t^{\epsilon-1} \times \exp\left\{-t(xm^2+(1-x)M^2)\right\}$$

Using the gamma integral then gives us

$$\text{mom} = \frac{1}{16\pi^2}\left(\frac{4\pi\mu^2}{xm^2+(1-x)M^2}\right)^\epsilon \times \Gamma(\epsilon)$$

Now expanding in powers of ϵ and neglecting positive powers of ϵ gives us

$$\text{mom} = \frac{1}{16\pi^2}\left(\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2(1-x)+m^2x}\right)\right)$$

So now we get

$$-i\mathcal{V}(k_f, k_i = 0) = \frac{g^3}{16\pi^2}\gamma^5 \int_0^1 \left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2(1-x) + m^2 x}\right)\right) dx$$

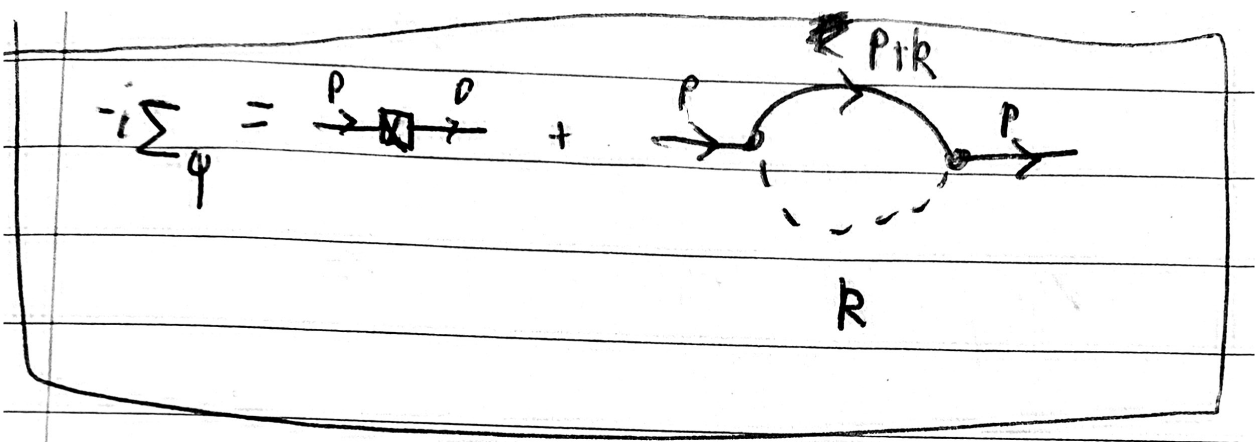
Letting Mathematica chew through this integral then gives us

$$-i\mathcal{V} = \frac{g^3}{16\pi^2}\gamma^5\left(1 + \frac{1}{\bar{\epsilon}} + 2\log\mu + \frac{2M^2\log M - 2m^2\log m}{m^2 - M^2}\right)$$

With this we can now read off the counterterm

$$\gamma^5\delta_g = \frac{g^3}{16\pi^2}\gamma^5\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right)\right) + \text{finite} \Rightarrow \boxed{\delta_g = \frac{g^3}{16\pi^2}\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right)\right) + \text{finite}}$$

Next we're tackling the $\mathcal{D} = 0$ fermion propagator. The diagrams that contribute are



From our Feynmann rules we can evaluate the one-loop diagram

$$-i\Sigma(k) = g^2\int\frac{d^4k}{(2\pi)^4}\gamma^5 \times \frac{i}{(k^2 - m^2 + i0)} \times \gamma^5 \times \frac{i}{(\not{k} + \not{p} - M + i0)}$$

Like in some of our previous divergent diagrams we can rewrite the above as

$$= -g^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^5(\not{k} + \not{p} + M)\gamma^5}{(k^2 - m^2 + i0)((k+p)^2 - M^2 + 0)}$$

First lets evalaute the numerator with our favorite parameter trick

$$\frac{1}{\text{denominator}} = \int_0^1 dx \times \frac{1}{[(1-x)(k^2 - m^2 + i0) + x((k+p)^2 - M^2 + i0)]^2}$$

Now we can define $\ell = k + xp$ and $\Delta = xM^2 + (1-x)m^2 - x(1-x)p^2$ to get rewrite the above more succintly

$$= \int_0^1 dx \times \frac{1}{[\ell^2 - \Delta + i0]^2}$$

Now we can put this back into our integral to get

$$-i\Sigma(k) = -g^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \times \frac{(-\not{\ell} + M - \not{p}(1-x))}{[\ell^2 - \Delta + i0]^2}$$

Notice since the integrand is odd in ℓ we have that the $\not{\ell}$ term vanishes thus leaving us with

$$= -g^2 \int_0^1 dx(M - \not{p}(1-x)) \times \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 - \Delta + i0)^2}$$

We've evaluated the momentum integral before for sure. Quoting the result from out notes gives us

$$\text{momentum integral} = \frac{i}{16\pi^2} \times \left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon} \times \Gamma(\epsilon) \approx \frac{i}{16\pi^2}\left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{\Delta}\right)\right)$$

So our total integral becomes

$$-i\Sigma(k) = -\frac{ig^2}{16\pi^2} \int_0^1 dx(M - \not{p}(1-x)) \times \left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{\Delta}\right)\right)$$

From here we can now calculate the finite parts of the counter-terms. Note that the total twopoint function has the form

$$i\Gamma(\not{p}) = i(\not{p} - M) - i\Sigma(k) + i(\delta_{Z_\psi}\not{p} - \delta_M)$$

And we require that $\Sigma(0) + \delta_M = 0$ as well as $\frac{d}{d\not{p}}\Sigma\Big|_{\not{p}=0} = \delta_{Z_\psi}$. This gives us

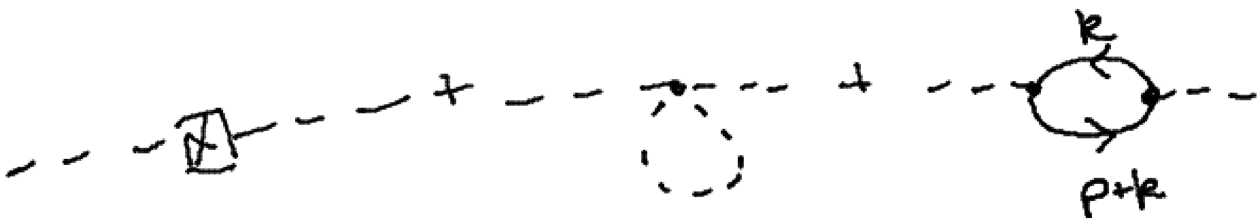
$$\begin{aligned}
\delta_M &= -\frac{g^2}{16\pi^2}\left[\int_0^1 dx\ (M - \not{p}(1-x)) \times \left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{\Delta}\right)\right)\right]\Bigg|_{\not{p}=0} \\
&= -\frac{g^2}{16\pi^2}\left[\int_0^1 dx\ M \times \left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{xM^2 + (1-x)m^2}\right)\right)\right]\Bigg|_{\not{p}=0} \\
&= -\frac{g^2 M}{16\pi^2}\left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right)\right) + \text{finite}
\end{aligned}$$

$$\begin{aligned}
\delta_{Z_\psi} &= \frac{d}{d\not{p}}\Sigma\Big|_{\not{p}=0} \\
&= \left(-\frac{g^2}{16\pi^2}\int_0^1 dx\ (1-x) \times \left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{\Delta}\right)\right)\right)\Bigg|_{\not{p}=0} \\
&= \left(-\frac{g^2}{16\pi^2}\int_0^1 dx\ (1-x) \times \left(\frac{1}{\overline{\epsilon}} + \log\left(\frac{\mu^2}{xM^2 + (1-x)m^2}\right)\right)\right) \\
&= -\frac{g^2}{16\pi^2}\left(\frac{1}{2\overline{\epsilon}} + \kappa\right)
\end{aligned}$$

Where κ is some manifestly finite term.

$$\kappa = \frac{m^4 - 4m^2M^2 + 3m^4 - 4(m^4 - 2m^2M^2)\log m - 4M^4\log M + 4(m^2 - M^2)^2\log\mu}{4(m^2 - M^2)^2}$$

Finally we have the (hopefully) easiest divergent scalar propagator. The diagrams that contribute are.



The second diagram we write as

$$\mathcal{F}_2 = \frac{-i\lambda}{2}\int\frac{d^4q}{(2\pi)^4}\frac{i}{q^2 - m_b^2 + i0}$$

We can evaluate this with dim-reg. First going to euclideann momentum space

$$= \frac{i\lambda}{2}\int\frac{d^4q_E}{(2\pi)^4}\frac{1}{q_E^2 + m_b^2}$$

Then going to arbitrary dimension

$$= \frac{i\lambda\mu^{4-D}}{2}\int\frac{d^Dq_E}{(2\pi)^D}\frac{1}{q_E^2 + m_b^2}$$

Using gamma function integral

$$= \frac{i\lambda\mu^{4-D}}{2}\int\frac{d^Dq_E}{(2\pi)^D}\int_0^\infty \exp\left\{-t[q_E^2 + m_b^2]\right\}dt$$

Evaluating the Gaussian integral

$$= \frac{i\lambda\mu^{4-D}(4\pi)^{-D/2}}{2}\int dt\ (t)^{-D/2}\exp\left\{-tm_b^2\right\} = \frac{i\lambda\mu^{4-D}}{2(4\pi)^{D/2}}(m^2)^{-1+D/2}\Gamma(1 - D/2)$$

Now let $D = 4 - 2\epsilon$ and expanding in powers of ϵ we get

$$= \frac{-im^2\lambda}{32\pi^2}\left(\frac{1}{\epsilon} + 1 + \log\left(\frac{\mu^2}{m^2}\right)\right)$$

The fermionic loop diagram can be written as

$$\mathcal{F}_3 = \int\frac{d^4k}{(2\pi)^4}\mathrm{tr}\left\{(-ig\gamma^5)\times\frac{i((\not{p} + \not{k}) + M)}{(p+k)^2 - M^2 + i0}(-ig\gamma^5)\frac{i(\not{k} + M)}{k^2 - M^2 + i0}\right\}$$

Lets throw around some gamma matrices to simplify

$$= -ig^2 \int \frac{id^4k}{(2\pi)^4} \frac{\mathrm{tr}\left\{(M - \not{p} - \not{k})(\not{k} + M)\right\}}{((p+k)^2 - M^2 + i0)(k^2 - M^2 + i0)}$$

Like the fermionic propagator we'll use feynamnn parameter

$$\text{denominator} = \int_0^1 dx \times \frac{1}{[(1-x)(k^2 - M^2 + i0) + x((k+p)^2 - M^2 + i0)]^2}$$

Now let $\ell = k + xp$ and $\Delta = M^2 - x(1-x)p^2$ giving us

$$\text{denomiantor} = \int dx \times \frac{1}{[\ell^2 - \Delta + i0]^2}$$

We also want to rewrite the numerator in terms of the shifted momentum ℓ. The first thing to do is evaluate the trace. First odd powers of momentum cancel leaving us with

$$\mathrm{tr}\left\{(M - \not{p} - \not{k})(\not{k} + M)\right\} = 4M^2 - 4k(k+p) = 4(M^2 - l^2 + p^2 x(1-x)) = 4(2M^2 - \Delta - \ell^2)$$

Where we got rid of the terms with ℓ^1 which vanish due to $\ell \to -\ell$ symmetry. The integral then becomes

$$\mathcal{F}_3 = -4ig^2 \int_0^1 dx \times \int \frac{id^4\ell}{(2\pi)^4} \times \frac{-(\ell^2 + \Delta) + 2M^2}{[\ell^2 - \Delta + i0]^2}$$

Now going to the euclidean momentum space

$$= -4ig^2 \int_0^1 dx \times \int \frac{d^4\ell_E}{(2\pi)^4} \frac{\ell_E^2 - \Delta + 2M^2}{[\ell_E^2 + \Delta]^2}$$

First lets split this up into two integrals I_1 and I_2. We have

$$I_2 = 2M^2 \int \frac{d^4\ell_E}{(2\pi)^4} \frac{1}{[\ell_E^2 + \Delta]^2} = 2M^2 \times \frac{1}{16\pi^2} \left(\frac{1}{\bar{\bar{\epsilon}}} + \log\left(\frac{\mu^2}{\Delta} \right) \right)$$

We've evaluated an integral like I_2 a million times now at this point. For I_1 we have

$$I_1 = \int \frac{d^4\ell_E}{(2\pi)^4} \frac{\ell_E^2 - \Delta}{[\ell_E^2 + \Delta]^2}$$

Dim reg

$$I_1 = \mu^{4-D} \int \frac{d^D\ell_E}{(2\pi)^D} \frac{\ell_E^2 - \Delta}{[\ell_E^2 + \Delta]^2}$$

Gamma integral

$$\text{mom} = \mu^{4-D} \int_0^\infty dt \times \int \frac{d^D\ell_E}{(2\pi)^D} t \times \left(\ell_E^2 - \Delta\right) \exp\left\{-t\left[\ell_E^2 + \Delta\right]\right\}$$

Rearrange

$$\text{mom} = -\mu^{4-D} \int_0^\infty dt \times t \times \exp\{-t\Delta\} \times \left(\frac{\partial}{\partial t} + \Delta \right) \times \int \frac{d^D\ell_E}{(2\pi)^D} \exp\left\{-t\ell_E^2\right\}$$

Gaussian integral

$$\text{mom} = -\mu^{4-D}\int_0^\infty dt \times t \times \exp\{-t\Delta\} \times \left(\frac{\partial}{\partial t} + \Delta\right) \times (4\pi t)^{-D/2}$$

Using mathemeatica to evaluate integral, let $D = 4 - 2\epsilon$ and expand for small ϵ

$$\text{mom} = -\frac{3\Delta}{16\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log(4\pi) + \log\left(\frac{\mu^2}{\Delta}\right) + \frac{1}{3}\right)$$

So all together we now have

$$\mathcal{F}_3 = -4ig^2\int_0^1 dx \times \frac{1}{16\pi^2}\left(\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{\Delta}\right)\left(2M^2 - 3\Delta\right)\right) - \Delta\right)$$

From this we can read off the counterterms from the condiitons $\Sigma(0) = -\delta_m$ and $\frac{d}{dp^2}\Sigma(0) = \delta_{Z_\phi}$

$$\begin{aligned}
\delta_m &= \frac{m^2\lambda}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + 1 + \log\left(\frac{\mu^2}{m^2}\right)\right) + 4g^2\int_0^1 dx \times \frac{1}{16\pi^2}\left(\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right)\right)\left(2M^2 - 3M^2\right) - M^2\right) \\
&= \frac{m^2\lambda}{32\pi^2}\left(\frac{1}{\bar{\epsilon}} + 1 + \log\left(\frac{\mu^2}{m^2}\right)\right) + \frac{g^2}{4\pi^2}\left(-M^2\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{M^2}\right) - M^2\right)\right) \\
&\approx \left(\frac{m^2\lambda}{32\pi^2} - \frac{M^2 g^2}{4\pi^2}\right)\frac{1}{\bar{\epsilon}} + \text{finite}
\end{aligned}$$

$$\delta_{Z_\phi} = -4g^2\int_0^1 dx \times \frac{1}{16\pi^2}\left(\left(\frac{1}{\bar{\epsilon}} + \log\left(\frac{\mu^2}{-x(1-x)}\right)\right)(3x(1-x)) + x(1-x)\right) = -\frac{g^2}{8\pi^2} \times \frac{1}{\bar{\epsilon}} + \text{finite.}$$

LECTURE 14: WARD IDENTITIES

February 25, 2021

Last time we talk about renormalization of electric charge to 1-loop theory. Now we'll prove Ward-Takahashi identities. On the homework we'll look at the diagrammtic proof. Today we'll look at the proof from current conservation. It turns out there is not just one identity but a whole series of identities. The Ward identity for a spider looking kinda guy (1) in notes is

$$(k_i)_{\mu_i} \times V_N^{\mu_1 \dots \mu_N}(k_1, \dots, k_N) = 0$$

The most general ward identity can be found from this. Consider any QFT with any charged and neutral field. We require that this QFT includes A^μ coupled to a conserved electric current $\partial_\mu J^\mu = 0$. Now consider an amplitude with N photons and M particles of any other kind which we'll denote by $S_{N,M}$. We have that the diagram equals

$$= iS_{N,M}^{\mu_1,\dots,\mu_N}(p_1, \dots, p_M; k_1, \dots, k_N)$$

Now we contract this ampliutde with the momentum of one of the photons

$$(k_i)_{\mu_i} S^{\mu_1\ldots\mu_N}_{N,M} = -\sum_{j=1}^{M} Q_J \times S_{N-1,M}(p_1,\ldots,{\color{red}p_j+k_1},\ldots,p_M;k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_N)$$

Other Ward identities are just special cases of the above equation. Lets try to prove the above equation with current conservation. We'll consider a field theory with a $U(1)$ phase symmetry. In quantum theory if we have a current operator J^μ we have

$$\partial_\mu J^\mu - 9$$

But operator identities aren't what we measur. What we measure are correlation function. So what we need to measure is

$$G_n^{a_1,\ldots,a_n;\mu}(x_i;y) = \langle\Omega|\mathbf{T}\phi^{a_1}(x_1)\ldots\phi^{a_n}(x_n)\times J^\mu(y)|\Omega\rangle$$

In QED the ϕ's are just ψ or ϕ or $\overline{\phi}$. We would expect that the operator identity $\partial_\mu J^\mu = 0$ translated to $\partial_{y^\mu} G = 0$. But since time ordering and partial deritaives wrt time don't commute this isn't so simple. To fix this consider

$$\frac{\partial}{\partial y^0}\mathbf{T}\left(A(x)\times B(y)\right) = \mathbf{T}\left(A(x)\times\frac{\partial}{\partial y^0}B(y)\right) + \delta(x^0-y^0)\times[A(x),B(y)]$$

The second term comes from reordering when $x^0 = y^0$. In particular we get

$$\frac{\partial}{\partial y^\mu}\mathbf{T}(\phi(x)\times J^\mu(y)) = \mathbf{T}(\phi(x)\times\partial_\mu J^\mu) + \delta(x^0-y^0)\times[\phi(x), J^0(y)]$$

The second term is singular when $x = y$ which can be seen from the commutation relation

$$[\phi^a(x),Q] = -Q(a)\times\phi^a(x) \Rightarrow [\phi^a(x),J^0(y)] = -Q(a)\times\delta^{(3)}(\mathbf{x}-\mathbf{y})\times\phi^a(x)$$

$$\Rightarrow \frac{\partial}{\partial y^\mu}\mathbf{T}(\phi^a(x)\times J^\mu(y)) = 0 + \delta^{(4)}\times(-Q(a)\times\phi^a(x))$$

Now if we have a bunch of fields we have from chain rule

$$\frac{\partial}{\partial y^\mu}\mathbf{T}(\phi^{a_i}(x_i)\times J^\mu(y) = \mathbf{T}(\phi^{a_i}(x_i))\times\sum_{j=1}^{n}(-Q(a_j))\times\delta^{(4)}(x_j-y_j)$$

This allows us to compute the derivative of the correlation function G

$$\frac{\partial}{\partial y^\mu}G_n^{a_i;\mu}(x_i;y) = -\mathcal{F}_n^{\{a_i\}}(\{x_i\})\times\underbrace{\sum_{j=1}^{n}Q(a_j)\times\delta^{(4)}(x_j-y)}_{\text{contact terms}}$$

$$\text{where } \mathcal{F}_n^{\{a_i\}}(\{x_i\}) = \langle\Omega|\mathbf{T}\{\phi^{a_i}(x_i\}|\Omega\rangle$$

The ward identites come from contact terms. Before we continue lets do some fourier transforms of contact terms. First consider

$$F(p,k) = \int d^4x\, e^{ipx} \times \int d^4y\, e^{iky} \times f(x)\delta^{(4)}(x-y) = f(p+k)$$

What was a delta function is coordinate space is the sum of two momentum in momentum space. Likewise if we have $f(\{x_i\}) \times \delta^{(4)}(x_j - y)$ we then have $f(p_1, \ldots, p_j + k, \ldots, p_n)$. This means that the fourier transform of everything gives us

$$k_\mu G_n^{\{a_i\};\mu}(\{p_i\}, k) = -i \sum_{j=1}^{n} Q(a_j) \times \mathcal{F}_n^{\{a_i\}}(p_1, \ldots, p_j + k, \ldots, p_n)$$

what follows in thelecture is a lot of diagrams which I'm not going to attempt to type up

Now lets talk about one more ward-identity that actually follows from identities we have already proven. We want to show that Z_1 the electron vertex r enormalizaiotn factor and Z_2 the electron field renormalization favor we have

$$Z_1 = Z_2 \Rightarrow \delta_1 = \delta_2$$

Sometimes people call this the Ward identity. First consider the 2-electron 1-photon amplitude

$$k_\mu S_1^\mu(p', p; k) = e_b S_0(p' - k = p, p) - e_b S_0(p', p_k = p')$$

We notice that $S_0(p' = p)$ is the electron propagator which we've shown

$$S_0(p' = p) = \mathcal{F}_2(p) = \frac{i}{\not{p} - m_b - \Sigma(\not{p}) + i0}$$

Notice that we can also write S_1^μ as

$$S_1^\mu(p', p) = S_0(p') \times ie_b \Gamma^\mu(p', p) \times S_0(p)$$

Where we let Γ be the 1PI dressed vertex. Plugging things in gives us[6]

$$S_0(p') \times ie_b k_\mu \Gamma^\mu \times S_0(p) = e_b S_0(p) - e_b S_0(p') \Rightarrow ie_b k_\mu \Gamma^\mu = \frac{e_b}{S_0(p)} - \frac{e_b}{S_0(p')} = -ie_b(\not{p} - m_b - \Sigma(\not{p})) + ie_b(\not{p}' - m_b - \Sigma(\not{p}'))$$

Lets take the limit where $k = p' - p \to 0$ and keep terms on both sides that are 0 and first order with $\not{p}, \not{p}' \to M_{\text{phys}}$. In this limit we have

$$(\not{p} - m_b - \Sigma(\not{p})) \to \frac{(\not{p} - M_{\text{phys}})}{Z_2} + O((\not{p} - M_{\text{phys}})^2)$$

To see this explicitly

$$\mathcal{F}_2 = \frac{i}{\not{p} - m_b - \Sigma(\not{p})} = \frac{iZ_2}{\not{p} - M_{\text{phys}}} + \text{finite.} = \frac{iZ_2}{\not{p} - M_{\text{phys}} + O((\ldots)^2)}$$

[6] apparently the signs are wonky in his lecture notes somewhere

So it follows that

$$\not{p} - m_b - \Sigma(\not{p}) = \frac{\not{p} - M_{\text{phys}}}{Z_2} + O((\not{p} - M)^2)$$

Thus we get

$$(\not{p}' - m_b - \Sigma(\not{p}')) - (\not{p} - m_b - \Sigma(\not{p})) \to \frac{\not{k}}{Z_2}$$

Thus all together we get

$$k_\mu \Gamma^\mu = \frac{k_\mu \gamma^\mu}{Z_2} \Rightarrow \boxed{\Gamma^\mu = \frac{\gamma^\mu}{Z_2}}$$

What we need to do now is tie this up to Z_1. To do this we consider a process where an on-shell electron emits a zero-momentum photon which has amplitude

$$\mathcal{M} = e_{\text{phys}} \times \epsilon_\mu u' \gamma^\mu u$$

We can use the LSZ reduction formula we can write the matrix elemnt

$$i\mathcal{M} = Z_2\sqrt{Z_3} \times \left(\sum \binom{\text{amputated}}{\text{diagrams}}\right) \times \binom{\text{spin/polarization}}{\text{factors } \epsilon_\mu, \overline{u}', u}$$

The amputated diagrams with 3 external legs (the process we consider here) is equal to $ie_b\Gamma^\mu$ in our chosen momentum limit. Thus all togther we have

$$e_{\text{phys}}\gamma^\mu = Z_2\sqrt{Z_3} \times e_{\text{b}} \times \Gamma^\mu$$

Furhtermore electric charge renromalizaition gives us

$$Z_2\sqrt{Z_3} \times e_{\text{b}} = Z_1 \times e_{\text{phys}}$$

Meaning that in our chosen momentum limit whihc generalizes we have

$$\Gamma^\mu = \frac{\gamma^\mu}{Z_1}$$

One final thing we wanted to talk about today. The ward identities gives us

$$e_{\text{phys}} = \sqrt{Z_3} \times e_{\text{b}}$$

Thus in bare perturbation theory we can write

$$e_{\text{phys}} A^\mu_{\text{phys}} = e_{\text{b}} A^\mu_{\text{b}}$$

This means that the covariant derivative works with both the bare of physical field meaning that the guage-covairant kinetic term in the physical lagrangian becomes

$$\mathcal{L}_{\text{phys}} = \overline{\Psi}(i\gamma^\mu D_\mu)\Psi + \cdots \Rightarrow \mathcal{L}_{\text{b}} = Z_2 \times \overline{\Psi}(i\gamma^\mu D_\mu)\Psi + \ldots$$

Simialrly for a QED with multiple fermions we have $\forall i\ Z_i^1 = Z_i^2$. This means the renormalization factor $\sqrt{Z_3}$ is the same for all charged particles in a theory. This even generalizes to vectors and scalars and so on. In counterterm perturbation theory this means that $\delta_1 = \delta_2$ and so on.

Lecture 15: Form Factors

February 26, 2021

Whoever is late is late :3c. Yesterday we talked about Ward identities and how they are derived from current conservation as well as a application of the ward identties to field strength renormalization. Today we'll talk about vertex renormalization and form factors. The idea of form factors come from scattering electrons off a nucleus. If we're considering a point particle then we have coulomb scattering. For a nucleus we have a distribution of charge that we're scattering off of. From QM we know that the scatttering ampltiude is

$$f(\mathbf{q}) = \frac{em_e}{2\pi} \times A^0(\mathbf{q}) \text{ where } A^0(\mathbf{q}) = \frac{\rho(\mathbf{q})}{\mathbf{q}^2} = \frac{eZ}{\mathbf{q}^2} \times F(\mathbf{q}^2) \text{ where } F \text{ is the form factor}$$

The form factor allws us to write the scattering amplitude as

$$f(\mathbf{q}^2) = \frac{e^2 Z m_e}{2\pi \mathbf{q}^2} \times F(\mathbf{q}^2)$$

Experimentally we measure scattering, fit the form factor, then fourier transform to get a notion of the charge distribution.

Lets consider scattering an electron off a proton. At tree-level in QED we have

$$i\mathcal{M} = \frac{i}{q^2} \times \overline{u}(e')(ie\gamma_\mu)u(e) \times \overline{u}(p')(-ie\Gamma^\mu)u(p)$$

Where $-ie\Gamma^\mu$ is the photon-proton-proton vertex. This Γ factor is kind of a form factor. For on shell protons and zero photon momentum we know that $\Gamma \to \gamma$. For non-zero photon momentum q things get spicy. We know by lorentz symmetry that Γ^μ must have the form

$$\Gamma^\mu(p', p) = A \times p^\mu + B \times p'^\mu + C \times \gamma^\mu$$

Where A, B, and C are functions of lorentz invariants(scalars?). Notice that this means that these can be functions of $\not{p}$ and $\not{p}'$ meaning that ordering matters very much. However we can circumvent this with the normal gamma matrix relations by bringing all products of $\not{p}^{(\prime)}$ and γ^μ by bringing $\not{p}$ to the right and $\not{p}'$ to the left. Now to continue lets consider where form factors come from. We were considering on shell protons so lets assert that $p^2 = p'^2 = M^2$. This means we can write Γ^μ in terms of 16 variables of one function. To go even futher we use the fact that in the matrix element we have Γ^μ in the context of

$$\overline{u}(p') \times \Gamma^\mu \times u(p)$$

So in this context we can use $\not{p} \times u(p) = Mu(p)$ and similar for primed. So we can write in the context of on shell spinoor sandwhihc

$$\Gamma^\mu = A(q^2)p^\mu + B(q^2) + p'^\mu + C(q^2) \times \gamma^\mu$$

Where A B and C are polynomials of M. We can go EVEN FURTHER by using ward identities to write

$$q_\mu \times \overline{u}(p')\Gamma^\mu u(p) = 0$$

This implies that we need $A(q^2) = B(q^2)$. To see this we can grind through some algebra (17-21 of FormFactors.pdf) t owrite

$$q_\mu \times \overline{u}(p')\Gamma^\mu u(p) = \frac{B-A}{2} \times q^2 \times \overline{u}(p')u(p) \Rightarrow B = A \text{ implies } q_\mu \times \cdots \times u(p) = 0$$

So we've reduced $\Gamma 6\mu$ to two independent form factors $A(q^2)$ and $C(q^2)$. We can rewrite this in more standard notation with Gordon identities

$$\Gamma^\mu = F_{\text{electric}}(q^2) \times \frac{(p'+p)^\mu}{2M} + F_{\text{magnetic}}(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2M}$$

$$\text{where } F_{\text{e}}(q^2) = 2MA(q^2) + C(q^2) \text{ and } F_{\text{m}} = C(q^2)$$

So what's up with these names? Well F_{e} doesn't give a hoot about spins so it refelects electric current. Also for a particle with charge Q we have

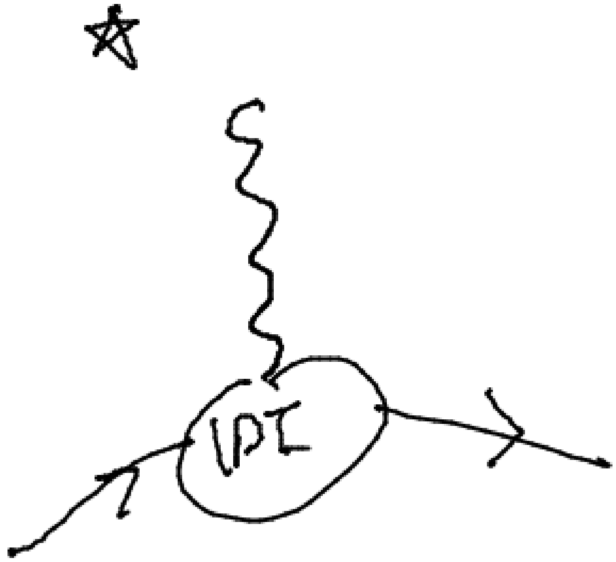
$$e \times F_{\text{e}}(q^2) \xrightarrow{q^2 \to 0} Q$$

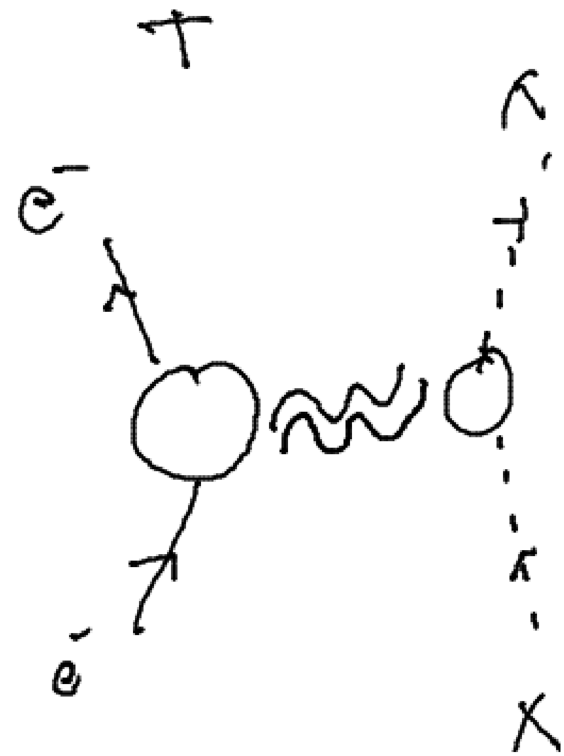
Neutrons also have $F_{\text{e}}(0) = 0$ but they have a wonky internal charge distribution. What about the electric form factor? It measures the distribution of magnetic moment due to spin. This leads us to a result that the gyromagnetic moment $g = 2F_{\text{m}}(0)$. The non-relativistic quark model actually gives a good explanaition for the experimentally measured form factors but we're not going into that here. Another basis we can expand Γ^μ in is

$$\Gamma^\mu = F_1(q^2) \times \gamma^\mu + F_2(q^2) \times \frac{i\sigma^{\mu\nu}q_\nu}{2M} \text{ where } F_1 = F_{\text{e}} \text{ and } F_2 = F_{\text{m}} - F_{\text{e}}$$

This is useful for almost pointlike particles. Higher orders of perturbation theory correspond to electrons not being point particles anymore and we can systemcatically calcualte corrections. We'll do these calculations starting now.

We want to calculate the dressed $e-e-\gamma$ vertex in QED.





We can write

$$\Gamma^\mu = F_1(q^2)\gamma^\mu + F_2(q^2)\frac{i\sigma^{\mu\nu}q_\nu}{2M}$$

Now in perturbation theory we have the loop correction, counterterm, and normal vertex. So we have

$$ie\Gamma^\mu_{\text{net}}(p',p) = (ie\gamma^\mu)_{\text{tree}} + ie\Gamma^\mu_{\text{loops}}(p',p) + iee\delta_1\times\gamma^\mu$$

In termsof the form factors this means that

$$F_1^{\text{net}}(q^2) = 1^{\text{tree}} + F_1^{\text{loops}}(q^2) + \delta_1 \text{ and } F_2^{\text{net}}(q^2) = 0^{\text{tree}} + F_2^{\text{loops}}(q^2)$$

This means that the counters set by the renormalization condition becomes

$$\delta_1 = -F_1^{\text{loops}}(q^2=0) \Rightarrow F_1^{\text{net}}(q^2=0) = 1$$

Namely we don't need to know anything about counterterms to calculate teh anomolous magnetic moment or the net F_1 form factor. We're almost out of time so we're in a great place to start the algebra.

At one loop level there is only one digram. Lets use feynmann gauge for our sanity. We have

$$ie\Gamma^\mu = \int\frac{d^4k}{(2\pi)^4}\frac{-ig^{\nu\lambda}}{(k^2+i0)}\times ie\gamma_\nu\times\frac{i}{\not{p}'+\not{k}-m+i0}\times ie\gamma^\mu\times\frac{i}{\not{p}+\not{k}-m+i0}\times ie\gamma_\lambda = e^3\int\frac{d^4k}{(2\pi)^4}\frac{\mathcal{N}^\mu}{\mathcal{D}}$$

Where we have let

$$\mathcal{N}^\mu = \gamma^\nu(\not{k}+\not{p}'+m)\gamma^\mu(\not{k}+\not{p}+m)\gamma_\nu$$

$$\mathcal{D} = [k^2 + i0] \times [(p+k)^2 - m^2 + i0] \times [(p'+k)^2 - m^2 + i0]$$

And now we can again fall into the loving arms of Feynamnn parameters. We have

$$\mathcal{D} = \iiint dxdydz\ \delta(x+y+z-1)\frac{2}{[x\{(p_k)^2 - m^2\} + y\{(p'+k)^2 - m^2\} + zk^2 + i0]^3}$$

We can do a big bunch of algebra to simplify things. to get

$$[\dots] = (k + xp + yp')^2 - \Delta \text{ where } \Delta = (xp + yp')^2 - xp^2 - yp'^2 + (x+y)m^2$$

And if we go on shell we can let $p^2 = p'^2 = m^2$ meaning that

$$\Delta = (1-z)^2 \times m^2 - xy \times q^2$$

All together if we define the shifted momentum $\ell = k + xp + yp'$ we can write

$$\frac{1}{\mathcal{D}} = \iiint dxdydz\ \delta(x+y+z-1)\frac{2}{[\ell^2 - \Delta + i0]^3}$$

Plugging this into our equation we get

$$\Gamma^\mu_{1-\text{loop}} = -2ie^2 \iiint dxdydz\ \delta(x+y+z-1) \int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}^\mu}{[\ell^2 - \Delta + i0]^3}$$

We gotta regulate this somehow. Next time we'll start with dealing with $\mathcal{N}^\mu$.

Lecture 16: One-loop QED Correction

March 02, 2021

Last time we stated by calculating the one-loop correction to QED. We will now deal with $\mathcal{N}^\mu$. To dothis we need to get rid of the γ matrices using γ matrix properties. Since we're using dimensional regularization we want to go to spacetime dimension D. Note that $\gamma^\nu\gamma_\nu = D$. Let X be some product of n dirac matrices. Consider

$$\gamma^\nu X \gamma_\nu = [\gamma^\nu, X\}\gamma_\nu + (-1)^n \times X \times (\gamma^\nu\gamma_\nu = D)$$

For $D \neq 4$ we have

$$\gamma^\nu X \gamma_\nu = (\gamma^\nu X \gamma_\nu)_{4d} + (-1)^n (D-4) \times X$$

We have some useful formulas in (25)* that we use to really expand out $\mathcal{N}^\mu$

$$\mathcal{N}^\mu = -2m^2\gamma^\mu + 4m(p' + p + 2k)^\mu - 2(\not{p} + \not{k})\gamma^\mu(\not{p}' + \not{k}) + (4-D)(\not{p}' + \not{k} - m)\gamma^\mu(\not{p} + \not{k} - m)$$

The next step is to go to ℓ momentum. Due to odd terms vanishing in the integral with $\ell \to -\ell$ we can ignore odd terms when we expand out $\mathcal{N}^\mu$ in terms of ℓ but ignoring odd terms. This gives us a big fucking equation that's given in (27) in vertex.pdf. From here is a lot of simplification, asserting that we're on-shell, using gordon identites, using integral symmetry. I'm not gonna type it out, they're all in vertex.pdf[7].

[7]he knows our eyes have already glazed over!

He said wake up! It's time for the punchline. After about a billion simplifications we can write

$$\mathcal{N}^\mu = \mathcal{N}_1 \times \gamma^\mu - \mathcal{N}_2 \times \frac{i\sigma^{\mu\nu} q - \nu}{2m}$$

Where

$$\begin{aligned}
\mathcal{N}_1 &= \frac{(D-2)^2}{D} \times \ell^2 - (D-2) \times \Delta + 2z \times (2m^2 - q^2) \\
\mathcal{N}_2 &= 2(1-z)(2z + (4-D)(1-z)) \times m^2
\end{aligned}$$

ANd the whole point of decomposing things is that F_1 corresponds to the $\mathcal{N}_1$ and the F_2 corresponds to the $\mathcal{N}_2$. To evaluate thiese integrals, specifically the anamolous magnetic moment. After our normal tricks our momentum integral reduces to.

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\mathcal{N}_2}{[\ell^2 - \Delta + i0]^3} = \mathcal{N}_2 \times \int \frac{i\ d^4\ell_E}{2(\pi)^4} \frac{1}{-(\ell_E^2 + \Delta)^3} = \frac{-i}{32\pi^3} \times \frac{4z}{1-z}$$

After some grinding we get that the gyromagnetic moment is

$$g = 2 + \frac{\alpha}{\pi} + O(\alpha^2)$$

Sparse notes today

Lecture 17: Infared Divergence

March 03, 2021

Infared divergences are things that diverge as we go to lower dimensions. What happens when we go to the infared regime? Consider the electron propagator when we go on shell

$$(p+k)^2 - m^2 = k^2 + 2kp + p^2 - m^2 = k^2 + 2kp = O(|k|) \text{ as } k \to 0$$

The same thing happens for the outgoing electron. So we have $O(|k|^4)$ in the denominator (ingoing+outgoing electron+photon). Thus we have

$$\int d^D k \frac{\mathcal{N}^\mu}{\mathcal{D}} \propto \int \frac{d^D k}{|k|^4}$$

So as $k \to 0$ we get a divergence. In dimensions $D < 4$ we get greater divergences $O((1/k)^{4-D})$. This shows up when we're calculating one loop correction to the form factors(which came up when we're integrating feynmann parameters.) This infared divergence comes up in QED and comes from the masslessness of photons. We can't help it. So what do we regulate this divergence? In condensed matter people analytically continue the dimension to high enough dimension to avoid the divergence. This is because ultraviolet divergences don't exisint in CMP (?). However in QFT we have to worry a lot about ultraviolet divergences and so we need to find something that works for us. Lets be naughty and whip out a dirty trick. First we analytically to $D < 4$ and sfhit the loop momentum. After that evaluate the shifted momentum and then analytically continue to $D > 4$ and integrate the Feynman parameters. The problem with this *dirty* trick is that we get poles from both UV and IR poles that are hard to disentangle. So we won't talk about this *naughty, dirty* trick in this notes. We could also do lattice QFT. The way we'll regulate the UV and IR divergence is to actually us DR for the UV divergence and assume that photons have a SUPER TINY mass $m_\gamma \ll m_e$. In the tiny mass propagator regime we have the photon propagator

$$\frac{-i}{k^2 - m_\gamma^2 + i0} \times \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m_\gamma^2} \right)$$

This will kill us. But if we assume that the tiny mass of of photon is due to the Higgs mechanism and use the R_ξ gauge we can have the propagator

$$\frac{-ig^{\mu\nu}}{k^2 - m_\gamma^2 + i0}$$

The price we pay for doing this (also naughty) trick is that we have a survivng goldstone scalar with mass$^2 = \xi m_\gamma^2$. However this is okay since we like waving our hands very fast. We will use this to work with the IR divergence. When we let photons have mass nothing changes from our old calculation except for our denominator. In particular

$$\tilde{\Delta} = \Delta + z \times m_\gamma^2$$

Plugging this in and spelling out the gory details leads us to (87) in vertex.pdf.

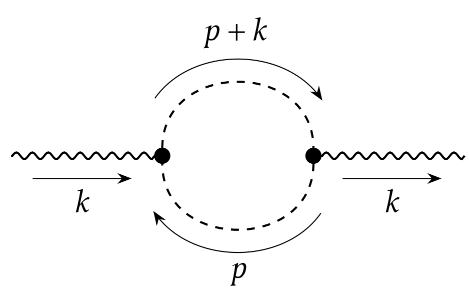
lots of calculations here.

Example: Scalar QED Charge Renormalization

March 03, 2021

Problem 2 of problem set 17

(a) At one loop there are two diagrams, the first diagram is s



$$= \int \frac{d^4p}{(2\pi)^4} ie(2p+k)^\mu \times \frac{i}{p^2 - M^2 + i0} \times ie(2p+k)^\nu \times \frac{i}{(p+k)^2 - M^2 + i0}$$

Lets start doing our regular tricks to evaluate this diagram. First is our feynmann parameter trick to simplify the denominator. We have

$$\frac{1}{p^2-M^2+i0}\times\frac{1}{(p+k)^2-M^2+i0}=\int_0^1\frac{dx}{[(1-x)(p^2-M^2+i0)+(x)((p+k)^2-M^2+i0)]^2}$$

Expanding out the denominator we see that we want something of the form

$$\ell^2-\Delta=-M^2+p^2+k^2x+2kpx$$

From this we can define $\ell=(p+xk)$ and $\Delta=M^2+k^2(x-1)x$ thus letting us write

$$=\int_0^1\frac{dx}{[\ell^2-\Delta+i0]^2}$$

So our whole integral becomes

$$\text{diagram}=-ie^2\int\frac{id^4\ell}{(2\pi)^4}\int_0^1 dx\times\frac{(2p+k)^\mu(2p+k)^\nu}{[\ell^2-\Delta+i0]^2}$$

We need to change the numerator to be in terms of ℓ. Solving for p in the definition of ℓ and plugging this in gives us (as well as ignoring terms first order in ℓ since they vanish in the $\ell\to-\ell$ symmetry)

$$\begin{aligned}(2p+k)^\mu(2p+k)^\nu&=(2\ell-2xk+k)^\mu(2\ell-2xk+k)^\nu\\&=4\ell^\mu\ell^\nu+4x^2k^\mu k^\nu-4xk^\mu k^\nu+k^\mu k^\nu+\text{vanishing}\\&=4\ell^\mu\ell^\nu+(1-2x)^2k^\mu k^\nu+\text{vanishing}\end{aligned}$$

TODO[8] $\ell^\mu\ell^\nu=g^{\mu\nu}\times\frac{\ell^2}{D}$. Asserting this gives us

$$=\frac{4}{D}\ell^2g^{\mu\nu}+(1-2x)^2k^\mu k^\nu$$

We want to verify that to 1-loop

$$\Sigma^{\mu\nu}_{1-\text{loop}}=(k^2g^{\mu\nu}-k^\mu k^\nu)\times\Pi_{1-\text{loop}}(k^2)$$

Motivated by this we can write the numerator as

$$(2p+k)^\mu(2p+k)^\nu=(k^\mu k^\nu-k^2g^{\mu\nu})\times(1-2x)^2+g^{\mu\nu}\left((1-2x)^2k^2+\frac{4}{D}\ell^2\right)$$

The red term is what we want. So from here on out we want to verify that the lavendar term vanishes when we consider both 1-loop diagrams. Lets do this by first integrating this term. First we'll consider the momentum integral.

$$\begin{aligned}\int\frac{id^4\ell}{(2\pi)^4}\frac{(1-2x)^2k^2+\frac{4}{D}\ell^2}{[\ell^2-\Delta+i0]^2}&=\int\frac{d^4\ell_E}{(2\pi)^4}\frac{(1-2x)^2k^2-\frac{4}{D}\ell_E^2}{[\ell_E^2+\Delta]^2}\\(\text{dim reg})&=\int\frac{\mu^{4-D}d^D\ell}{(2\pi)^D}\frac{(1-2x)^2k^2-\frac{4}{D}\ell^2}{[\ell^2+\Delta]^2}\\&=I_1-I_2\end{aligned}$$

[8] asked in lecture 17. march 03. It comes from lorentz symmetry. the prefactor comes from contracting a $g_{\mu\nu}$ on both sides.

Where we have

$$\begin{aligned}
I_1 &= \mu^{4-D}(1-2x)^2k^2\int\frac{d^D\ell}{(2\pi)^D}\frac{1}{[\ell+\Delta]^2}\\
\text{(gamma integral)} &= (1-2x)^2\mu^{4-D}k^2\int_0^\infty dt\int\frac{d^D\ell}{(2\pi)^D}t\exp\{-t(\ell+\Delta)\}\\
\text{(gaussian integral)} &= (1-2x)^2\mu^{4-D}k^2\int_0^\infty dt\ te^{-t\Delta}(4\pi t)^{-D/2}\\
\text{(mathematica)} &= \frac{\mu^{4-D}}{\Delta^2}\times k^2(1-2x)^2\times\left(\frac{\Delta}{4\pi}\right)^{D/2}\Gamma\left(2-\frac{D}{2}\right)
\end{aligned}$$

And similarly we have

$$\begin{aligned}
I_2 &= \frac{4}{D}\times\mu^{4-D}\times\int\frac{d^D\ell}{(2\pi)^D}\frac{\ell^2}{[\ell^2+\Delta]^2}\\
\text{(gamma integral)} &= \frac{4}{D}\times\mu^{4-D}\times\int\frac{d^D\ell}{(2\pi)^D}\int_0^\infty dt\ \ell^2t\times\exp\left\{-t[\ell^2+\Delta]\right\}\\
&= -\frac{4}{D}\times\mu^{4-D}\times\int_0^\infty dt\ t\exp\{-t\Delta\}\times\frac{\partial}{\partial t}\int\frac{d^D\ell}{(2\pi)^D}\exp\left\{-t\ell^2\right\}\\
\text{(gaussian integral)} &= -\frac{4\mu^{4-D}}{D}\int_0^\infty dt\times te^{-t\Delta}\frac{\partial}{\partial t}(4\pi t)^{-D/2}\\
\text{(mathematica)} &= 2\mu^{4-D}\Delta^{\frac{D}{2}-1}(4\pi)^{-\frac{D}{2}}\Gamma(1-D/2)
\end{aligned}$$

Now all together we have

$$\text{diagram} = -ie^2\int_0^1 dx\times(I_1-I_2)$$

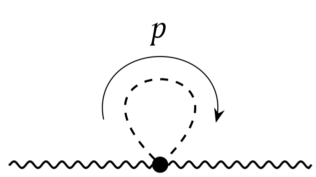
Now notice that we can do a few simplications

$$\begin{aligned}
I_1-I_2 &= \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}\left(k^2(1-2x)^2\Delta^{\frac{D}{2}-2}\Gamma\left(2-\frac{D}{2}\right)-2\Delta^{\frac{D}{2}-1}\Gamma(1-D/2)\right)\\
\text{(using gamma property)} &= \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}\Gamma(1-D/2)\left(k^2(1-2x)^2\Delta^{\frac{D}{2}-2}(1-D/2)-2\Delta^{D/2-1}\right)\\
\text{(lil big of algebra)} &= \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\times\frac{\partial}{\partial x}\left(\Delta^{\frac{D}{2}-1}(1-2x)\right)
\end{aligned}$$

So now we have the integral for the lavendar term

$$\text{integral} = -ie^2\frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}}\Gamma\left(1-\frac{D}{2}\right)\int_0^1\frac{\partial}{\partial x}\left(\Delta^{D/2-1}(1-2x)\right)dx = 2(M^2)^{\frac{D}{2}-1}ie^2\frac{\mu^{4-D}}{(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)$$

For our one loop correction to have the desired form we need this to cancel with something in the second diagram which is



$$= \int\frac{d^4p}{(2\pi)^4}2ie^2g^{\mu\nu}\times\frac{i}{p^2-M^2+i0} = 2ie^2g^{\mu\nu}\int\frac{id^4p}{(2\pi)^4}\times\frac{1}{p^2-M^2+i0}$$

This is simple enoughto evaluate

$$\begin{aligned}
\text{integral} &= \int \frac{id^4p}{(2\pi)^4} \times \frac{1}{p^2 - M^2 + i0} \\
&= -\int \frac{d^4p_E}{(2\pi)^4} \times \frac{1}{p_E^2 + M^2} \\
(\text{gamma integral}) &= -\int_0^\infty dt \times \exp\left\{-tM^2\right\} \int \frac{d^4p_E}{(2\pi)^4} \times \exp\left\{-tp_E^2\right\} \\
(\text{dim reg}) &= -\mu^{4-D} \int_0^\infty dt \times \exp\left\{-tM^2\right\} \int \frac{d^Dp}{(2\pi)^D} \times \exp\left\{-tp^2\right\} \\
(\text{gaussian}) &= -\mu^{4-D} \int_0^\infty dt \times \exp\left\{-tM^2\right\} \times (4\pi t)^{-D/2} \\
(\text{mathematica}) &= -\frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}} (M^2)^{\frac{D}{2}-1} \Gamma\left(1 - \frac{D}{2}\right)
\end{aligned}$$

Thus we see that the diagram evaluates to

$$\text{diagram} = -2ie^2 g^{\mu\nu} \frac{\mu^{4-D}}{(4\pi)^{\frac{D}{2}}} (M^2)^{\frac{D}{2}-1} \Gamma\left(1 - \frac{D}{2}\right)$$

Now compare this with the lavendar term we had before. These two cancel exactly so the one loop contribution is of the form

$$\boxed{\Sigma^{\mu\nu}_{1-\text{loop}} = (k^\mu k^\nu - k^2 g^{\mu\nu}) \times \Pi_{1-\text{loop}}(k^2)}$$

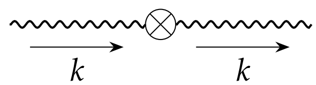
(b) The integral we want to evaluate is the red term we had in part (a)

$$\begin{aligned}
-i\Pi_{1-\text{loop}} &= ie^2 \int \frac{id^4\ell}{(2\pi)^4} \int_0^1 dx \times \frac{(1-2x)^2}{[\ell^2 - \Delta + i0]^2} \\
(\text{dim reg}) &= \mu^{4-D} ie^2 \int \frac{id^D\ell}{(2\pi)^D} \int_0^1 dx \times \frac{(1-2x)^2}{[\ell^2 - \Delta + i0]^2} \\
(\text{wick rotate}) &= \mu^{4-D} ie^2 \int \frac{d^D\ell}{(2\pi)^D} \int_0^1 dx \times \frac{(1-2x)^2}{[\ell^2 + \Delta]^2} \\
(\text{gamma integral}) &= \mu^{4-D} ie^2 \int_0^1 dx \times (1-2x)^2 \times \int_0^\infty dt\ t \times \exp\{-t(\Delta)\} \int \frac{d^D\ell}{(2\pi)^D} \exp\left\{-t\ell^2\right\} \\
(\text{gaussian integral}) &= \mu^{4-D} ie^2 \int_0^1 dx \times (1-2x)^2 \times \int_0^\infty dt\ t \times \exp\{-t\Delta\} (4\pi t)^{-D/2} \\
(\text{mathematica}) &= \frac{\mu^{4-D} ie^2}{(4\pi)^{D/2}} \times \Gamma\left(2 - \frac{D}{2}\right) \times \int_0^1 dx \times (1-2x)^2 \times \Delta^{D/2-2} \\
(D \to 4 - 2\epsilon \text{ and expand}) &= \frac{ie^2}{16\pi^2} \int_0^1 dx \times \left(\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{\Delta}\right)\right) \\
&= \frac{ie^2}{3 \times 16\pi^2} \left(\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 3\int_0^1 dx \times (1-2x)^2 \log\left(\frac{M^2}{\Delta}\right)\right)
\end{aligned}$$

Thus we find that

$$\boxed{\Pi_{1-\text{loop}} = -\frac{\alpha}{12\pi}\left(\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2}\right) + 3\int_0^1 dx \times (1-2x)^2 \log\left(\frac{M^2}{\Delta}\right)\right)}$$

Using this we can evaluate δ_3 which is respresented by the diagram



With our boxed result we can read off δ_3 as

$$\delta_3 = \Pi_{1-\text{loop}}(k^2 = 0) + \text{finite}$$

If $k^2 = 0$ the only things that is affected is the remaining integral in $\Pi_{1-\text{loop}}$. However on examination that integral vanishes for $k^2 = 0$ and thus

$$\boxed{\delta_3 = -\frac{\alpha}{12\pi}\left(\frac{1}{\epsilon} + \log\left(\frac{\mu^2}{M^2}\right)\right) + \text{finite}}$$

And the net amplitude is

$$\boxed{\Pi^{\text{net}} = -\frac{\alpha}{4\pi}\int_0^1 dx \times (1-2x)^2 \log\left(\frac{M^2}{\Delta}\right)}$$

(c) From out notes we have

$$e_{\text{eff}}^2 = \frac{e_0^2}{1-\Pi}$$

Lets us consider the limit $k^2 \gg M^2$ giving us

$$\Pi^{\text{net}} \approx \frac{-\alpha}{4\pi}\int_0^1 dx \times (1-2x)^2\left(\log\left(-\frac{k^2}{M^2}\right) - \log\left(x(1-x)\right)\right) = -\frac{\alpha}{12\pi}\left(-\log\left(-\frac{k^2}{M^2}\right) + \frac{8}{3}\right)$$

This means that

$$\frac{1}{e_{\text{eff}}^2} = \frac{1-\Pi}{e_0^2} = \frac{1}{e_0^2} - \frac{1}{48\pi}\left(\log\left(-\frac{k^2}{M^2}\right) - \frac{8}{3}\right)$$

Or in terms of $\alpha = \frac{e^2}{4\pi}$ we have

$$\boxed{\frac{1}{\alpha_{\text{eff}}} = \frac{1}{\alpha_0} - \frac{1}{12}\left(\log\left(-\frac{k^2}{M^2}\right) - \frac{8}{3}\right)}$$

LECTURE 18: MORE INFARED DIVERGENCE

March 04, 2021

We're doing something about one-loop corrections for the scattering crossection. However I'm sitting here in class right now and I don't know what is going on. It seems that I've fallen off the wagon[9].

Lets dicuss tree level scattering for Brehmstrallang(is that how you spell it?) readiation(electron scattering with some particle X and releasing a photon.) We have two diagrams that contribute. Evaluating these diagrams leads to (126) in vertex.pdf. We'll start with some algebra exercsies

$$(p-k)^2 - m^2 = k^2 - 2pk + p^2 - m^2 = -2pk$$

And this result leads us to (after some algebra)

$$\frac{i}{\not{p} - \not{k} - m}(ie\not{\epsilon}^*)u(p) \approx \frac{e(p\epsilon^*)}{pk}u(p)$$

And after some more algebra we get

$$\overline{u}(p')(ie\not{\epsilon}^*)\frac{i}{\not{p}' + \not{k} - m} \approx -e\overline{u}(p') \times \frac{(p'\epsilon)}{(p'k)}$$

And so the result we get is

$$\mathcal{M}_{\text{tree}}(e^-X \to e^-X\gamma) = /\mathcal{M}_{\text{tree}}(e^-X \to e^-X) \times \frac{e}{\omega}\left(\epsilon^*_\mu\left(\frac{p'^\mu}{(np')} - \frac{p^\mu}{(np)}\right)\right)$$

Where we defined $\omega = k^0$ and n^μ is the 4-vector in some direction(which direction? I don't know....). Now with this we can get the spin averaged matrix element

$$\overline{|\mathcal{M}|^2_{\text{tree}}}(e^-X \to e^-X\gamma) = \overline{|\mathcal{M}|^2_{\text{tree}}}(e^-X \to e^-X) \times \frac{e^2}{\omega^2}\left[-\left(\frac{p'}{(np')} - \frac{p}{(np)}\right)^2\right]$$

We can then find the phase space factor t o get the scattering cross section

$$d\sigma_{\text{tree}}(e^-X \to e^-X\gamma = d\sigma_{\text{tree}}(e^-X \to e^-X) \times \frac{\omega^2 d\omega d\Omega_n}{(2\pi)^3 2\omega} \times \frac{e^2}{\omega^2}\left[-\left(\frac{p'}{(np')} - \frac{p}{(np)}\right)^2\right]$$

We can plot this to find that this corss section is peaked when phooton is emitted in direction of incoming or outgoing electron. We can average over photon directions **n** by integrating (which we define as $\mathcal{I}(p',p)$) and all together we get

$$\frac{d\sigma^{\text{tree}}(eX \to eX\gamma)}{d\Omega_e d\omega_\gamma} = \frac{d\sigma^{\text{tree}}(eX \to eX)}{d\Omega} \times \left(\frac{\alpha}{\pi}\right) \times \frac{\mathcal{I}(p',p)}{\omega_\gamma}$$

After some algebra we find that $\mathcal{I}(p',p) = 2f_{\text{IR}}(q^2)$ which was defined earlier in vertex.pdf. What follows is some algebra to show this explicitly. It's just kinematics so I'll skpi writing it down. Now if we integrate over photon energy we immediatley we run into a infared divergence :(. We regulate this by assuming that the photon has a tiny non-zero mass m_γ which leads to

$$\int \frac{d\omega_\gamma}{\omega_\gamma} = \log\left(\frac{E_e}{m_\gamma}\right) + \text{finite}$$

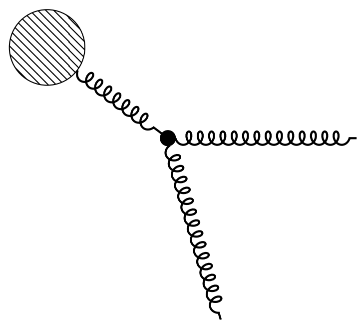
[9] and I can't get up!

However a miracle occurs (114 and 119 in vertex.pdf): the combined cross-section between tree level and 1-loop level has no infared divergence.

Lets move on to something else: in QED IR divergences leads to the idea that there is no clear separation between quantum states $|e^-\rangle$ and $|e^- + \text{soft } \gamma\rangle$ and so on(this has something to due because of poles being at the physical mass and branch cuts becoming kinda wonky.) What happens is that state mix and we get no clear eigen space (similar to $|ee\rangle$ and $|ee\gamma\rangle$ and so on.) What this means is that

$$\boxed{\text{QED has ill defined Fock space}}$$

Physically this is ok. Theoretically this doesn't stop us from using feynmann diagrams on fock space but what happens is that we have infared divergent intermediate steps. This problem comes up in QCD as well (and in general an un-higgsed gauge theory.) This is a big theoretical problem. A similar problem is colinear gluons e.g. $|\text{gluon}\rangle$ versus $|2 \text{ colinear gluons}\rangle$. Why is this a problem?[10]



This gives us

$$q^2 = (k_1 + k_2)^2 = 2k_1k_2 = 2\omega_1\omega_2(1 - \cos\theta_{12})$$

We see that as $\theta_{12} \to 0$ we have $1/q^2$ blowing up.

Likewise $|\text{quarks + colinear quarks}\rangle$ vs $|\text{just quarks}\rangle$ for $m_q \to 0$.

In QCD it is hard to distinusih $|\text{quarks}\rangle$ or $|\text{gluons}\rangle$ from $|\text{quarks + soft or collinear quarks}\rangle$ and $|\text{gluons+soft or collinear gluons}\rangle$. So in practice what do we do? We calculate

$$\sigma(\cdots \to \text{jets})$$

Where a jet is a bunch of quakrs or antiquarks or gluons flying within a small angle cone. This also allows for extra soft gluons.

Start with a hard collision that gives us a few $q, \bar{q}, g$ (quarks, antiquarks, and gluons) and this gives us a few jets which fragment into more jets (perturbation theory) and the final quarks, antiquarks, and gluons turn into hadrons. So experimentally a jet is a stream of hadrons flying in a narrow cone.

LECTURE 19: RENORMALIZING QED GAUGE DEPENDENCE AND A GRAB BAG OF OTHER THINGS

March 05, 2021

[10] I'm unreasonably proud of the fact that I managed to live-tex a feynmann diagram.

So we're almost done with QED. We're almost done with infared issues. Lets finish QED today by talking about gauge dependene. But before we get started lets talk about the dependence of the photon propagator on gauge choices. It turns out that physical quantities like cross seections and amplitudes are gauge invariant but that only works on shell. The moment we start calculating off shell quantities they become gauge dependent. By the time we et to physical amplitudes but in the intermediate stages things might be gauge dependent so this means that we need to be sure when we're doing some yucky computation we need to make sure we're using the same gauage at each step. We will explore this today by seeing how δ_1 and δ_2 counterterms depend (or don't depend) on gauge conditions. Lets consider the following vertex

I give up, see kaplunovsky's notes

This evaluates to

$$ie\Gamma^{\mu}_{1-\text{loop}}(p',p) = \int \frac{d^4k}{(2\pi)^4} ie\gamma_\nu \times \frac{i}{\not{p}' + \not{k} - m + i0} \times ie\gamma^\mu \times \frac{i}{\not{p} + \not{k} - m + i0} \times ie\gamma_\lambda$$
$$\times \frac{-i}{k^2 + i0}\left[g^{\lambda\nu} + (\xi - 1)\frac{k^\lambda k^\nu}{k^2 + i0}\right]$$

After regularizing stuff we get

$$ie\Gamma^\mu = ie\gamma^\mu_F + (\xi - 1) \times ie\Delta\Gamma^\mu(p',p)$$

we can make some simplicifacitons so that

$$\Delta\Gamma^\mu = e^2\gamma^\mu \times \int \frac{d^4k}{(2\pi)^4}\frac{-i}{(k^2 + i0)^2}$$

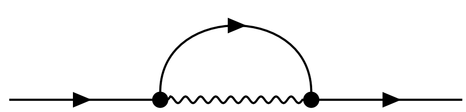
And from here we can get

$$\delta_1(\xi) = \delta_1^{\text{feynman}} - (\xi - 1) \times e^2 \int \frac{d^4k}{(2\pi)^4}\frac{-i}{(k^2 + i0)^2}$$

Now lets consider the δ_2 term which is set by

$$\Sigma^e_{\text{net}} = \Sigma^e_{\text{loops}} + \delta_m - \Delta_2\not{p}$$

We have the one loop diagram



After evaluating this integral we get (in hw 18)

$$\Delta\Sigma = -ie^2 \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 + i0)^2} \times \not{k} \times \frac{1}{\not{k} + \not{p} - m_e + i0} \times \not{k}$$

And spelling out the algebra for our renormalization conditions we have

$$\delta_2 = \delta_2^{\text{feynman}} + (\xi - 1)\Delta\delta_2 \text{ and } \delta_m = \delta_m^{\text{feynman}} + (\xi - 1)\Delta\delta_m$$

Where we have defined

$$\Delta\delta_2 = \frac{d\Delta\Sigma}{d\not{p}}\Big|\not{p} = m \text{ and } \Delta\delta_m - m\Delta\delta_2 = -\Delta\Sigma(\not{p} = m)$$

And after some inplace derivatives we can find that

$$\Delta\delta_2 = -e^2 \int \frac{d^4k}{(2\pi)^4} \times \frac{-i}{(k^2 + i0)^2}$$

Thus we can see that

$$\Delta\delta_1 = \Delta\delta_2$$

So once we verify ward identities $\delta_1 = \delta_2$ in the feynman gauge (which we will do in the hw) we will find that

$$\delta_1(\xi) = \delta_2(\xi) \text{ in any gauge}$$

Notice so far that some of our computations have relied and thus are only true for $\not{p} = m$. If twe don't assert this then things become messy. this song and dance was to show us that if we go off shell we should worry about gauge dependence.

At this point we're done with QED. It will be on the midterm and wehave one problem on it in the HW but we're not talking about it anymore. We are now talking about something completley different:

Consider any renormalizable QFT.

$$\mathcal{L}_\text{b} = \mathcal{L}_\text{phys} + \mathcal{L}_\text{counter terms}$$

In general this contains all opertaors of dimensions ≤ 4 (field products and their derivatives) which respect the symmetires of the physical lagrangiann $\mathcal{L}_\text{phys}$. However if we get an operator that breaks the symmetry of the physcial lagrangian then the counterterm is not generated. Lets look at a specific example: yukawa theory

$$\mathcal{L}_\text{phys} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{M^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 + \overline{\psi}(i\not{\partial} - m)\psi + ig\phi\overline{\psi}\gamma^5\psi$$

Now suppose $M_\text{phys} = 0$ and/or $\lambda_\text{phys} = 0$. The symmetires of the theory will be the exact same. So we sill can get $\delta_\phi^M \neq 0$ and $\delta^\lambda \neq 0$ from the diagrams we evaluated on the homework. On the other hand what if we tkae $m_\text{phys} = 0$. Then we now have an extra symmetry $\mathbb{Z}_2$.

$$\psi \to \gamma^5\psi,\ \overline{\psi} \to -\overline{\psi}\gamma^5,\ \phi \to -\phi$$

What happens when we let $m_\text{phys} = 0$ and get a extra symmetry is that $\delta_\psi^m = 0$.

Another example: QED with massless electron $m_e = 0$ giving an axial symmetry

$$\psi \to e^{i\theta\gamma^5},\ \overline{\psi} \to \overline{\psi}e^{i\theta\gamma^5} \Rightarrow \delta^m = 0$$

Bottom line: lagrangain has symmetry then no counterterms will break that theoyr.

Now lets ask the questions: what happens if the coupling or mass is a non-zero but small number. The answer depends on what happens when we take that couling/mass to zero. Does an extra symmetry occur or not.

(a) If $\lambda = 0$ and $M = 0$ does not increase symmetry then counter terms don't vanish. Example: small λ or M_ϕ in Yukawa theory.

(b) If something does create an extra symmetry then the counterterms should start to vanish. For example in QED if we let m_e approach 0 then

$$\delta^m = O(\alpha m_e \log \Lambda^2) \text{ instead of } O(\alpha\Lambda)$$

Likewise in yukawa theory we et

$$\delta^m_\psi = O(g^2 m \log \Lambda^2) \text{ instead of } O(g^2\Lambda)$$

A point of terminology: if a small coupling or mass enhances the symmetry of the physical lagrangian then we call it naturally small.

Now the last subject of today. Lets consider $\lambda\phi^4$ theory and consider some scattering at very high energies but at fixed angles.

$$= -\lambda + \frac{\lambda^2}{32\pi^2}\left(J\left(\frac{s}{m^2}\right) + J\left(\frac{t}{m^2}\right) + J\left(\frac{u}{m^2}\right) - \dots\right) + O\left(\frac{\lambda^3}{(4\pi)^4}\right)$$

Where $J(t/m^2)$ is defiend in the notes a million years ago. All together we get

$$\mathcal{M} = -\lambda + \frac{\lambda^2}{32\pi^2}\left(3\log\left(\frac{E^2}{m^2}\right) + \text{finite}\right) + \text{higher loops}$$

And for higher loops (see homework) we find that

$$\mathcal{M}_{2-\text{loops}} \propto O\left(\frac{\lambda^3}{(16\pi^2)^2} \times \log^2\left(\frac{E^2}{M^2}\right)\right)$$

And generally for $E \to \infty$

$$\mathcal{M}_{\text{L}-\text{loops}} \propto \frac{\lambda^{L+1}}{(16\pi^2)^2} \times \left(\log\frac{E^2}{m^2}\right)^L$$

This means that

$$\mathcal{M}_{\text{net}} = \sum_{L=0} M_{L-\text{loops}} = \text{power seires in } \left(\frac{\lambda}{16\pi^2} \times \log\frac{E^2}{m^2}\right)$$

This is a problem since if the log becomes large and $\lambda/16\pi^2$ isn't small enough to ocunteract this then we get that the $\lambda \times \log = O(1)$ which fucks up perturbation theory. How do we hand problems like this? Well we can use partial resummation of the leading logs which explicitly means reorganize the perturbation theory so that

$$\mathcal{M} = \sum_{\text{Loops}} \frac{\lambda_{\text{eff}}^{L+1}(E)}{(16\pi^2)^L} \times O(1)$$

So that all the large logs are hiding inside the energy dependent efffective coupling. This is also called running coupling. We'll talk about this more on tuesday. Also the renormalization group isn't a group.

Lecture 20: Beginning of Renormalization Group

March 09, 2021

Lets start by considering $\lambda\phi^4$ theory. We have the treelevel term givein in (3) of RG.pdf. In general we have that

$$O\left(\lambda \times \left(\frac{\lambda_0}{16\pi^2} \times \log\frac{E^2}{M^2}\right)^{\#\text{loops}}\right)$$

What the renormalization group does is reorganize perturbation theory in terms of effective coupling. This means that the energy scale becomes a power series with $O(1)$ coefficents $F_L(\text{momenta}/E)$

$$\mathcal{M} = \lambda^{(E-2)/2}\sum_{L=0}^{\infty}\left(\frac{\lambda(E)}{16\pi^2}\right)^L \times F_L(\text{momenta}/E)$$

Where we have the large logs burried inside the effeictive coupling $\lambda(E)$. Furthermore the expansion of $\lambda(E)$ to get a differential equation that we call the renormalization group equation

$$\frac{d\lambda(E)}{d(\log E)} = \beta(\gamma(E)) = \sum_{n=1}^{\infty} b_n \frac{\lambda^{n+1}(E)}{(16\pi^2)^n}$$

Where b_n shows up at n loop level. If we approximate the sum with the one loop term then we immediately get $\lambda(E)$ as we have it up to one-loop level. What we're going to do this week is to learn how we can use renormalization group and some implications of this renormalization. Now lets get to the basics

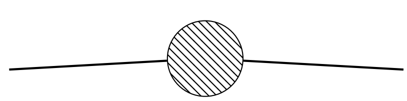
The first thing we need to do is define $\lambda(E)$ which is done in terms of a physcial ampltiude at some energy scale E. Consider a 1PI 4-scalar amplitude which we will identify with $-\lambda(E)$. First we let the external mometna $p_i^2 = -E^2$ and $s = t = u = -\frac{4}{3}E^2$. We choose $\lambda(E)$ to be negative so to avoid obtrusive complex analysis. If we redefine coupling with $-\lambda(E)$ then we should also modify the counterterm

$$V^{\text{net}}(\{p_i\}) = (V^{\text{tree}} = -\lambda(E) + V^{\text{loops}} - \delta^\lambda(E) \Rightarrow \delta^\lambda(E) = V^{\text{loops}}(\{p_i\})$$

Where $p_i^2 = -E^2$. What's going on here is that we're reorganizing the perturbation theory. Another implication of introducing an effective coupling is that a lot of things become energy dependent like $M^2(E)$ and $\delta^Z(E)$. Because of this we have the propagator

$$\frac{i}{p^2 - M^2(E) + i0}$$

And the seagull vertices $-i\lambda(E)$. How are we defining $M(E)$. Consider the self energy correction

 $= \Sigma^{\text{net}}$

The above gives us the dressed propagator

$$\frac{i}{p^2 - M^2(E) - \Sigma^{\text{net}} + i0}$$

We're considering $p^2 = -E^2$ which we can use to determine

$$\delta^Z(E) = \frac{\partial \Sigma^{\text{loops}}}{\partial p^2}\bigg|_{p^2=-E^2} \quad \cdots$$

What we've done above is choose a renormalization scheme. In reality there are a lot of renormaizlation schees we could've chosen. But all of them has something in common. Consider λ and λ' that are different running couplings according to different renormailzation schemes then we have

$$\lambda(E) - \lambda'(E) = O(\lambda^2(E))$$

Calssically we have that scalar fields scale with energy as E but in quantum theory there is a scaling dimension $E^{1+O(\lambda^2)}$. We call this $O(\lambda^2)$ as the anomalous dimension. LEts look at this anomlous dimension a little more. Consider energy dependence of the $\Delta^Z(E)$ counterterm. We have

$$\gamma(E) = \frac{1}{2}\cdots \Rightarrow Z(E) = \text{const} \times E^{2\gamma}$$

Now consider a time ordered two-point correlation function of two bare fields. We get that

$$\mathcal{F}_2^{\text{bare}}(x-y) = Z(e) \times \mathcal{F}_2(x-y) \Rightarrow \mathcal{F}_2^{\text{b}}(p) = Z(E) \times \mathcal{F}_2(p)$$

We know that $\mathcal{F}_2(p)$ is the dressed propagator of the renormalized scalar field which we know so we have

$$\mathcal{F}_2^{\text{b}}(p) = \frac{iZ(E)}{p^2 - M^2(E) - \Sigma_{\text{tot}}(p^2; E)}$$

Ins the limit $-p^2 \gg M^2$ and $E^2 = -p^2$ we get that the denominator becomes $\approx p^2$. So we can say that

$$\mathcal{F}_2^{\text{b}}(p) \approx -i \times (\text{const}) \times \frac{(-p^2)^\gamma}{(-p^2)}$$

How do we interpret this. consider a correlation function of local operators of known scaling dimensions. Any local operator which scales with energy as E^Δ then the correlation function of the operator with its hermitan conjugate scales as

$$\langle \Omega | \mathbf{T} \hat{\mathcal{O}}(x) \hat{\mathcal{O}}^\dagger(y) | \Omega \rangle \propto |x-y|^{-2\Delta} \text{ for } x - y \to 0$$

We can fourier transfrom this into momentum we can get that we get the momentum space correlation function is $\propto (-p^2)^{\Delta-2}$. On comparing this with our previous result we see that

$$\Delta - 2 = \gamma - 1 \Rightarrow \Delta = \gamma + 1$$

This means that

$$\gamma = \frac{1}{2}\frac{d\ \log Z}{d\ \log E}$$

To calculate γ we need the two loop results for $\lambda\phi^4$ theory (remember δ^Z vanishes for one-loop order. From our results and taking $-p^2 \approx E^2 \gg m^2$ we get

$$\frac{d\Sigma^{2-\text{loops}}}{dp^2} = \frac{-\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} + 2\log\frac{E^2}{-p^2} + \text{const} + O(m^2/E^2)\right)$$

Which in our off-shell renormalization gives us

$$\delta^Z_{2-\text{loops}}(E) = -\frac{\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} + \text{const}\right)$$

If we switch to energy dependent $Z(E) = 1 + \delta^Z(E)$. If we expand $\log Z$ in terms of λ we have

$$\log(Z = 1 + \delta^Z) = \delta^z - \frac{1}{2}(\delta^Z)^2 + \cdots \Rightarrow \log Z(E) = -\frac{\lambda^2}{24(4\pi)^4}\left(\frac{1}{\epsilon} + 2\log\frac{\mu^2}{E^2} + \text{const}\right) + O(\lambda^3)$$

the energy dependence only comes from the energy log. This means we can find $\gamma(E)$, the anomlous dimension as

$$\gamma(E) = \frac{\lambda^2(E)}{12(4\pi)^4} + O(\lambda^3)$$

Now what about higher orders of perturbation theory? In general we have

$$\delta^Z(E) = \lambda^2(E) \times A_2(E,\epsilon) + \lambda^3(E) \times A^3(E,\epsilon) + \dots$$

And this gives us

$$\log(z(E) = 1 + \delta^Z) = \delta^Z(E) - \frac{1}{2}(\delta^Z(E))^2 + \cdots = \lambda^2 \times A_2 + \lambda^3 \times A_3 + \lambda^4 \times (A_4 - \frac{1}{2}A_2^2)$$

some stuff about higher loops here. I'm banking on not doing 3+ loop calculations in this class

In general we have that the anomalous dimension $\gamma(E)$ obtains as a power series in the in the running coupling

$$\gamma(E) = \sum_{n=2}^{\infty} C_n \lambda^n(E)$$

Lets move on to another subject. The renormalizagion group equations can be written as

$$\frac{d\lambda(E)}{d\log E} = \beta(\lambda(E)) = b_! \times \lambda^2(E) + b_2 \times \lambda^3(E) + \dots$$

In this secion we'll derive th above equation and learn how we can calcuate the β function. The key to the baove equation is the relation between the runing and bare coupling.

$$\lambda(E) + \delta^\lambda(E,\text{cutoff}) = Z^2(E,\text{cutoff}) \times \lambda_\text{b}(\text{cutoff}) = (1 + \delta^Z(E,\text{cutoff}))^2 \times \lambda_\text{bare}(cutoff)$$

Taking the derivative of both sides of the equation wrt $\log E$ which gives us

$$\frac{d\lambda}{d\log E} + \frac{\partial\delta^\lambda}{\partial\log E} = \frac{\partial Z^2\lambda_\text{bare}}{\partial\log E} = \frac{\partial Z^2}{\partial\log E} \times \lambda_\text{b} = 4\gamma \times (\lambda + \delta^\lambda)$$

By counting the dependence of the variables in terms of power of λ we can note that in leading order

$$\frac{d\lambda(E)}{d\log E} = -\frac{\partial\delta^\lambda}{\partial\log E} + O(\lambda^2)$$

We can calcualte δ^λ to one loop order which we did a loooooong time ago. We got

$$\delta^\lambda_{1-\text{loop}} = \frac{3\lambda^2}{32\pi^2}\times\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{E^2}+\text{const}\right)\Rightarrow\frac{\partial\delta^\lambda_{1-\text{loop}}}{\partial\log E} = -\frac{3\lambda^2}{16\pi^2}\Rightarrow\frac{d\lambda(E)}{d\log E} = \frac{3\lambda^2}{16\pi^2}+O(\lambda^3)$$

Where we can read off b_1 explicitly. At $O(\lambda^3$ term we can also get

$$\beta^{\text{order }\lambda^3} = 4_{\gamma,2-\text{loop}}\times\lambda-\frac{\partial}{\partial\log E}\delta^\lambda_{2-\text{loop}}$$

At next order $O(\lambda^4)$ we get some more stuff which includes a UV divergent term. But the UV divergent terms cancel. We can extract a general pattern here

$$\beta(\lambda) = \frac{d\lambda(E)}{d\log E} = \sum b_n\times\lambda^{n+1}(E)\text{ for finite }b_n$$

Now lets talk about actually solving the RGE. If we have the beta function $\beta(\lambda)$ we have

$$\frac{d\lambda(E)}{d\log E} = \beta(\lambda)$$

This is easy to solve. For example at one loop level we approximated above $\beta(\lambda)\approx(3/16\pi^2)\lambda^2$ and this gives us

$$\frac{d\lambda}{d\beta(\lambda)} = d\left(-\frac{16\pi^2}{3\lambda}\right)\Rightarrow\frac{16\pi^2}{3\lambda(E_1)}-\frac{16\pi^2}{3\lambda(E_2)} = \log\frac{E_2}{E_1}$$

To find the constant term we need to find λ for some speicic energy. The equation we found to one loop for λ is only true for $E^@\gg m^2$. In the opposite limit things become different. Namely we have

$$\delta^\lambda_{1-\text{loop}} = V_{1-\text{loop}}(s=t=u=-4/3E^2) = \frac{3\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon}+\log\frac{\mu^2}{M^2}-\left(J\left(-\frac{4E^2}{3M^2}\approx\frac{2E^2}{9M^2}\right)\right)\right)$$

And thus

$$\beta_{1-\text{loop}} = \frac{\lambda^2}{24\pi^2}\times\frac{E^2}{M^2}\ll\frac{3\lambda^2}{16\pi^2}$$

And thus below the treshold when we solve the equation we find that we can neglect the running coupling and treat it as a constant low-energy coupling. And when we match the low-energy and high-energy condition $E_0 = M\times O(1)$ we get

$$\text{for }E\gg M,\ \frac{16\pi^2}{\lambda(E)} = \frac{1}{16\pi^2\lambda_0}-3\log\frac{E}{M}+O(1)$$

Lecture 21: Renromalization Group Equation for QED

We have

$$\gamma_\gamma = \frac{1}{2}\frac{d\log Z_e}{d\log E}\quad\gamma_e = \frac{1}{2}\frac{d\log Z_2}{d\log E}$$

A while ago we calculated the electric charge renormalization which I won't repeat here. On the off shell renormalization we get

$$\Pi_{\text{net}}(k^2) = \Pi_{\text{loops}}(k^2) - \delta_3(E) = 0 \text{ for } k^2 = -E^2$$

And thus when we approximate $E \gg m_e$ we can write that

$$\delta_e = -\frac{\alpha}{3\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + O(1)\right) \Rightarrow \frac{\partial \delta_3}{\partial \log E} = \frac{2\alpha}{3\pi} \Rightarrow \gamma_\gamma = \frac{\alpha}{3\pi} + O(\alpha^2)$$

If we also did the two loop calculation we'd get

$$\gamma_\gamma = \frac{\alpha}{3\pi} + \frac{\alpha^2}{4\pi^2} + O(\alpha^3)$$

In our next homework we'll calculate δ_2 in QED and get the result

$$\delta_2 = -\frac{\xi\alpha}{4\pi}\left(\frac{1}{\epsilon} + \log\frac{\mu^2}{E^2} + O(1)\right)$$

Where ξ is a gauge fixing parameter. This means that the anomalous dimension is gauge dependent. This leads us to an important rule

(a) Neutral gauge invariant fields have gauge invariant anomalous dimensions

(b) Charged field and other gauge-dependent operators have gauge dpeendent anomalous dimensions.

Lets calcualte the beta function of the running ocupling $e(E)$. After a little bit of algebra and using the ward identity result $Z_1(E) = Z_2(E)$ we have

$$\sqrt{Z_3(E)} \times e_{\text{b}} = e(E) \Rightarrow \frac{de(E)}{d\log E} = e(E) \times \gamma_e(E) \Rightarrow \beta_e(e) = e \times \gamma_\gamma(e)$$

A little bit of algebra can also let us write thing in terms of $\alpha(E)$ to get

$$\beta_\alpha = \frac{2\alpha^2}{3\pi} + \frac{\alpha^3}{2\pi^2} + O(\alpha^4)$$

From here we can solve the RGE. At one loop order we can get the result

$$\frac{1}{\alpha(E)} = \frac{1}{\alpha(E_{\text{ref}})} - \frac{2}{3\pi}\log\frac{E}{E_{\text{ref}}}$$

And accounting for low energy we get

$$\frac{1}{\alpha(E)} = \frac{1}{\alpha_0} - \frac{2}{3\pi}\left(\log\frac{E}{m_e} + O(1)\text{constant}\right)$$

We can also do the above calculation to two-loop order and it is owrked out explicitly in RG.pdf.

Lets go beyond QED to theories with multiple fields and multiple couplings for example the yukawa theory. Furthermore in the standard model we have gauge couplings, gauge self coupling, and a shit ton of Yukawa couplings. Each coupling has its own beta function that is dependent on all the couplings of the theory

$$\frac{dg_a(E)}{d\log E} = \beta_a(\text{all of the } g_i)$$

This makes things much messier to solve. LEts see how we might solve this. Consider a general coupling $g(E)$ that couples n scalar fields or their derivatives. For this we have

$$g(E) + \delta^g(E) = g_b \times \prod_{i=1}^{n} \sqrt{Z_i(E)}$$

Taking the derivative of both sides wrt $\log E$ gives us

$$\frac{dg}{d\log E} + \frac{d\delta^g}{d\log E} = (g + \delta^g) \times \sum_{i=1}^{n} \gamma_i$$

And with this we can define

$$\beta_g = (\gamma_1 + \cdots + \gamma_n) \times (g + \delta^g) - \frac{d\delta^g}{d\log E}$$

For example in $\lambda\phi^4$ theory we know that

$$\beta_\lambda = 4\gamma(\lambda + \delta^\lambda) - \frac{d\delta^\lambda}{d\log E}$$

Similarly for Yukawa coupling $g \times i\Phi\overline{\psi}\gamma^5\psi$. In this case we have

$$\beta_g = (2\gamma_\psi + \gamma_\Phi) \times (g + \delta^g) - \frac{d\delta^g}{d\log E}$$

In homework 17 we calculated the inifte part of all counterterms. Using the fact that (shown in the notes)

$$\frac{d\delta}{d\log E} = (-2) \times \text{coefficient of the } \frac{1}{\epsilon} \text{ pole in } \delta$$

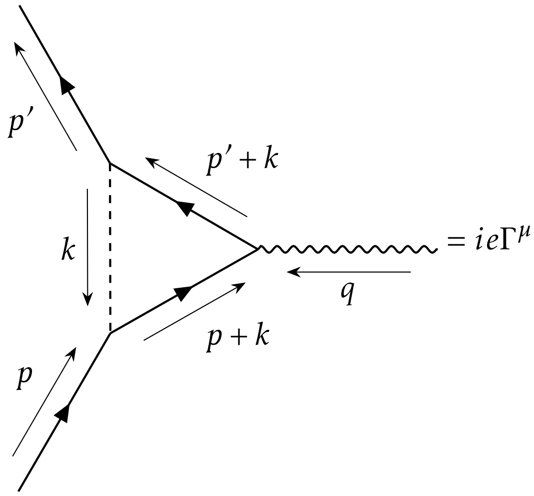
We can calculate the beta functions.

$$\beta_g = \frac{5g^3}{16\pi^2} \qquad \beta_\lambda = \frac{3\lambda^2 + 8\lambda g^2 - 48g^4}{16\pi^2}$$

THE MUON'S ANOMALOUS MAGNETIC MOMENT

STARTED: March 06, 2021. FINISHED: March 07, 2021

In this section we consider what would happen if we coupled the muon to a new super heavy scalar field with a Yukawa coupling g. We'll calculate the effect of this new heavy particle on the anomalous magnetic moment of the muon. The one-loop contribution involving the yukawa coupling is



Using Feynmann rules

$$\begin{aligned}
ie\Gamma^\mu &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i0}\times(-ig)\times\frac{i(\not{p}'+\not{k}+m)}{((p'+k)^2-m^2+i0}\times(ie\gamma^\mu)\times\frac{i(\not{p}+\not{k}+m)}{((p+k)^2-m^2+i0)}\times(-ig) \\
&= -eg^2\int\frac{d^4k}{(2\pi)^4}\frac{(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)}{(k^2-M^2+i0)((p'+k)^2-m^2+i0)((p+k)^2-m^2+i0)} \\
&= -eg^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathcal{N}^\mu}{\mathcal{D}}
\end{aligned}$$

Using Feynman parameters we rewrite the numerator as

$$\frac{1}{\mathcal{D}} = \iiint d(FP)\frac{2}{[x((p+k)^2-m^2)+y((p'+k)^2-m^2)+z(k^2-M^2)]^3}$$

Let $\ell = k+xp+yp'$ and let fermions be on shell. We then can use mathematica

```
In[∘]:= FullSimplify[(x ((p + k)^2 - m^2) + y ((p ' + k)^2 - m^2) + z (k^2 - M^2) - (k + x p + y p ')^2 // ExpandAll) /. {(p ')^2 → m^2, p^2 → m^2}, Assumptions → x + y + z == 1]
Out[∘]= -m² (x² + y²) - M² z - 2 p x y p′
```

We can then write

$$[\dots] = \ell^2 - m^2(x^2+y^2) - M^2z - 2pp'xy = \ell^2 - \Delta$$

To summarize we defined

$$\ell = k+xp+yp' \quad \Delta = M^2z + m^2(x^2+y^2) + 2pp'xy$$

Since $2pp' = (p^2+p'^2-(p-p')^2)$ we have

$$\Delta = M^2z + m^2(x^2+y^2+2xy = (x+y)^2) - xyq^2 = M^2z + m^2(1-z)^2 - xyq^2$$

So all together we now have

$$\Gamma^\mu = 2ig^2\iiint d(FP)\int\frac{d^4\ell}{(2\pi)^4}\frac{\mathcal{N}^\mu}{[\ell^2-\Delta+i0]^3}$$

Next we will make use of the fact that linear in ℓ terms cancel and that we are considering on-shell muons which means that $\overline{u}(p')\not{p}' = m\overline{u}$ and $\not{u}(p) = mu(p)$ and that the integral is in the context of $\overline{u}(p')\Gamma^\mu u(p)$. The first thing we will do is insert ℓ in place of k and disregard linear terms. This gives us

$$\begin{aligned}\mathcal{N}^\mu &= (\not{p}' + \not{\ell} - x\not{p} - y\not{p}' + m)\gamma^\mu(\not{p} + \not{\ell} - x\not{p} - y\not{p}' + m)\\ &= \not{\ell}\gamma^\mu\not{\ell} + (\not{p}' - x\not{p} - y\not{p}' + m)\gamma^\mu(\not{p} - x\not{p} - y\not{p}' + m) + (\text{linear in } \ell)\end{aligned}$$

From now on we will neglect terms linear in ℓ. From here we can use $x+y+z=1$ that is asserted in the $d(FP)$ to simplify the above

$$\mathcal{N}^\mu = \not{\ell}\gamma^\mu\not{\ell} + (x(\not{p}' - \not{p} = \not{q}) + z\not{p}' + m)\gamma^\mu(y(\not{p} - \not{p}' = -\not{q}) + z\not{p} + m)$$

From here we will make our favorite dirac sandwhich (mines a reuben) to use the on shell muon conditions. this gives us

$$\mathcal{N}^\mu = \not{\ell}\gamma^\mu\not{\ell} + (x\not{q} + m(1+z))\gamma^\mu(-y\not{q} + m(1+z))$$

The next step is opening up the brakcets to get

$$\mathcal{N}^\mu = \not{\ell}\gamma^\mu\not{\ell} - xy\not{q}\gamma^\mu\not{q} + xm(1+z)\not{q}\gamma^\mu - ym(1+z)\gamma^\mu\not{q} + m^2(1+z)^2\gamma^\mu$$

Lets consider this term specifically

$$\begin{aligned}xm(1+z)\not{q}\gamma^\mu - ym(1+z)\gamma^\mu\not{q} &= m(1+z)(x\not{q}\gamma^\mu - y\gamma^\mu\not{q})\\ &= m(1+z)\left(\frac{1}{2}x\not{q}\gamma^\mu + \frac{1}{2}x\not{q}\gamma^\mu - \frac{1}{2}y\gamma^\mu\not{q} - \frac{1}{2}y\gamma^\mu\not{q}\right)\\ &= m(1+z)\left(\frac{1}{2}x\not{q}\gamma^\mu + \frac{1}{2}(1-y-z)\not{q}\gamma^\mu - \frac{1}{2}y\gamma^\mu\not{q} - \frac{1}{2}(1-x-z)\gamma^\mu\not{q}\right)\\ &= m(1+z)\left(x\left(\frac{1}{2}\{\not{q}\gamma^\mu, \gamma^\mu\not{q}\} = q^\mu\right) + \left(\frac{1}{2}[\not{q}, \gamma^\mu] = i\sigma^{\mu\nu}q_\nu\right)\right.\\ &\left.-y\left(\frac{1}{2}\{\not{q}, \gamma^\mu\} = q^\mu\right) + z\left(\frac{1}{2}[\gamma^\mu, \not{q}] = -i\sigma^{\mu\nu}q_\nu\right)\right)\\ &= m(1+z)\left((x-y)q^\mu + (1-z = x+y)i\sigma^{\mu\nu}q_\nu\right)\end{aligned}$$

Putting everything together gives us

$$\mathcal{N}^\mu = \not{\ell}\gamma^\mu\not{\ell} + xyq^2\gamma^\mu + m^2(1+z)^2\gamma^\mu + m(1+z)(x-y)q^\mu + m(1+z)(1-z)i\sigma^{\mu\nu}q_\nu$$

Now notice that due to the $x \leftrightarrow y$ symmetry of the feynmann parameter integral we can neglect the red term so we're left with

$$\mathcal{N}^\mu = \not{\ell}\gamma^\mu\not{\ell} + xyq^2\gamma^\mu + m^2(1+z)^2\gamma^\mu + m(1-z^2)i\sigma^{\mu\nu}q_\nu$$

Now notice that due to Lorentz symmetry in the ocntext of the $\int d^4\ell$ we have

$$\not{\ell}\gamma^\mu\not{\ell} \to \gamma^\lambda\gamma^\mu\gamma^\nu g_{\lambda\nu}\frac{\ell^2}{D} = \frac{\ell^2}{D}(2-D)\gamma^\mu$$

And all together we have

$$\mathcal{N}^\mu = (2-D)\frac{\ell^2}{D}\gamma^\mu + xyq^2\gamma^\mu + m^2(1+z)^2\gamma^\mu + m(1-z^2)i\sigma^{\mu\nu}q_\nu$$

The structure of F_1 and F_2 leads us to write

$$\mathcal{N}^\mu = \mathcal{N}_1\gamma^\mu + \mathcal{N}_2\frac{i\sigma^{\mu\nu}q_\nu}{2m}$$

Where we can read off the definition of $\mathcal{N}_1$ and $\mathcal{N}_2$. In the context of understanding the anomoulous magnetic moment we only need $\mathcal{N}_2$ which is

$$\mathcal{N}_2 = 2m^2(1-z^2)$$

The integral we want to evaluate to understand F_2 is

$$F_2^{1-\text{loop}} = 2ig^2\iiint d(FP)\Big(2m^2(1-z^2)\Big)\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta+i0]^3}$$

Which looks like a fun integral. First we'll go to euclidean momentum with a wick rotation

$$F_2 = -2g^2\iiint d(FP)\Big(2m^2\Big(1-z^2\Big)\Big)\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+\Delta]^3}$$

Consider only the momentum integral. First we dim-reg and then we use the gamma integral to get

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2+\Delta]^3} = \frac{\mu^{4-D}}{\Gamma(3)=2}\int_0^\infty t^2\times\exp\{-t\Delta\}\int\frac{d^D\ell}{(2\pi)^D}\exp\Big\{-t\ell^2\Big\}$$

Now we use a gaussian integral to get

$$= \frac{\mu^{4-D}}{2}\int_0^\infty t^2\times\exp\{-t\Delta\}(4\pi t)^{-D/2} = \frac{\mu^{4-D}}{2(4\pi)^{D/2}}\times\Delta^{\frac{D}{2}-3}\Gamma\left(3-\frac{D}{2}\right)$$

Plugging this back into F_2 we get that

$$F_2 = -\frac{g^2\mu^{4-D}\Gamma\left(3-\frac{D}{2}\right)}{(4\pi)^{D/2}}\iiint d(FP)(2m^2(1-z^2))\times\Delta^{\frac{D}{2}-3}$$

And now I notice that dim-reg was kinda dumb since the integral wasn't ultraviolet divergent. So let $D=4$ whic gives

$$F_2 = -\frac{g^2}{16\pi^2}\times\iiint d(FP)\frac{2m^2(1-z^2)}{\Delta}$$

In the problem set we're given the hint that we should exploit $M_S^2 \gg m_{\text{muon}}^2$. Lets do that. Consider the numerator in the $q^2\to 0$ limit where we want to calculate things in

$$\Delta = M^2z + m^2(1-z)^2$$

The m^2 term only becomes signification for $z\approx 0$. When z is not closee to zero we know that the m^2 term does nothing. So we can approximate Δ as

$$\Delta \approx M^2z + m^2$$

This has the limiting behaviors we care about. Also note that the integrand does not depend on z thus we can integrate of x and y to get

$$\int_0^1 \int_0^1 dxdy \times \delta(x+y+z-1) = (1-z)$$

So our integral becomes

$$F_2 = -\frac{2g^2m^2}{16\pi^2M^2}\int_0^1 dz\,\frac{(1-z^2)(1-z)}{z+m^2/M^2} = -\frac{2g^2m^2}{16\pi^2M^2}\left(-\frac{7}{6}+\log\left(\frac{M^2+m^2}{m^2}\right)\right)$$

In the problem set we're given the discrepency between theoretical and experimental measurements of the magnetic moment of the muon

$$a_\mu^{\text{exp}} - a_\mu^{\text{theory}} \approx (26\pm 8)\times 10^{-10}$$

So to set a bound so that we're withitin two standard deviations of the mean we solve for

$$\frac{2g^2m^2}{(16\pi^2M^2)}\left(\log\left(\frac{M^2+m^2}{m^2}\right)-\frac{7}{6}\right) \le 42\times 10^{-10}$$

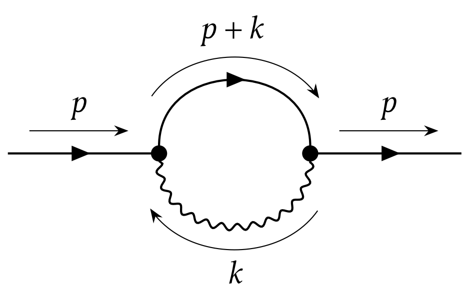
We're told that this scalar field has mass $M \ge 300\ GeV$. Solving for g^2 gives us

$$\boxed{g^2 \le 0.181}$$

δ_2 counterterm in QED

Started: March 07, 2021. Finished: March 07, 2021

In this section we'll compute the δ^2 counterterm in QED to verify manually that in QED $\delta^1 = \delta^2$ and thus in any gauge choice $\delta^1 = \delta^2$ which was proven in the notes. First to one-loop level we have the following diagram that contributes to δ_2 and δ^m



$$= -i\Sigma = \int \frac{d^4k}{(2\pi)^4}\times\frac{-ig^{\mu\nu}}{k^2+i0}\times(ie\gamma_\mu)\times\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i0}\times(ie\gamma_\nu)$$

Lets start *grinding*. First lets try to simplify. In our notes when calculating δ^1 we used dim-reg for the ultraviolet divergence and assigning the photon a small mas m_γ to fix IR divergence. This gives us

$$\Sigma = -e^2\int\frac{id^4k}{(2\pi)^4}\frac{g^{\mu\nu}\gamma_\mu(\not{p}+\not{k}+m_e)\gamma_\nu}{(k^2+i0)((p+k)^2-m_e^2+i0)} \to -e^2\mu^{4-D}\int\frac{id^Dk}{(2\pi)^D}\frac{g^{\mu\nu}\gamma_\mu(\not{p}+\not{k}+m_e)\gamma_\nu}{(k^2-m_\gamma^2+i0)((p+k)^2-m_e^2+i0)}$$

`Oh woah`[11] `what's this?` We're multiplying stuff together in the *bottom*. Smells like Feynmann paramter time.

$$\frac{1}{\mathcal{D}} = \int_0^1 \frac{dx}{[(1-x)(k^2 - m_\gamma^2 + i0) + x((p+k)^2 - m^2 + i0)]^2}$$

Define the shifted momentum $\ell = k + xp$. We can then write

$$[\dots] = \ell^2 - \Delta + i0$$

Where Δ is determined by some algebra

$$\Delta = -((1-x)(k^2 - m_\gamma^2) + x((p+k)^2 - m^2) - \ell^2) = xm_e^2 + (1-x)(m_\gamma^2 - p^2 x)$$

This gives us

$$\frac{1}{\mathcal{D}} = \int_0^1 \frac{dx}{[\ell^2 - \Delta + i0]^2}$$

The numerator can also be simplified by the identity derived in the notes

$$\gamma^\nu \not{a} \gamma_\nu = -2\not{a} + (4-D)\not{a}$$

This gives us

$$\mathcal{N} = (2-D)(\not{p} + \not{k}) + Dm_e = (2-D)(\not{p}(1-x) + \not{\ell}) + Dm_e$$

In the context of the momentum integral we know that odd powers of ℓ vanish so we're left with

$$\mathcal{N} = (2-D)(1-x)\not{p} + Dm_e$$

I think we can get to evaluating

$$\begin{aligned}
\Sigma &= -e^2\mu^{4-D}\int_0^1 dx \int \frac{id^Dk}{(2\pi)^D}\frac{(2-D)(1-x)\not{p} + Dm_e}{[\ell^2 - \Delta + i0]^2} \\
\text{(euclidean momentum)} &= e^2\mu^{4-D}\int_0^1 dx \int \frac{d^Dk_E}{(2\pi)^D}\frac{(2-D)(1-x)\not{p} + Dm_e}{[\ell^2 + \Delta]^2} \\
\text{(gamma integral)} &= e^2\mu^{4-D}\int_0^1 dx \int_0^\infty dt\ t \times \exp\{-t\Delta\}((2-D)(1-x)\not{p} + Dm_e)\int \frac{d^Dk_E}{(2\pi)^D}\exp\left\{-t\ell^2\right\} \\
\text{(gaussian integral)} &= e^2\mu^{4-D}\int_0^1 dx \int_0^\infty dt\ t \times \exp\{-t\Delta\}((2-D)(1-x)\not{p} + Dm_e) \times (4\pi t)^{-D/2} \\
\text{(mathematica)} &= \frac{e^2\mu^{4-D}}{(4\pi)^{D/2}}\int_0^1 dx((2-D)(1-x)\not{p} + Dm_e)\Delta^{\frac{D}{2}-2}\Gamma\left(2 - \frac{D}{2}\right) \\
(D \to 4 - 2\epsilon) &= \frac{\alpha}{2\pi}\left(4\pi\mu^2\right)^\epsilon \Gamma(\epsilon)\int_0^1 dx\frac{(\epsilon-1)(1-x)\not{p} + (2-\epsilon)m_e}{\Delta^\epsilon}
\end{aligned}$$

From out notes on QED renormalization we know that

$$\Sigma_{\text{net}}(\not{p}) = \Sigma_{\text{loops}}(\not{p}) + \delta_m - \delta_2\not{p} \Rightarrow \left.\frac{d\Sigma_{1-\text{loops}}}{d\not{p}}\right|_{\not{p}=m_e} = \delta_2$$

[11];o

Lets start taking derivatives. First we have

$$\frac{d\Delta}{d\not{p}} = \frac{d\Delta}{dp}\frac{dp}{d\not{p}} = \frac{d\Delta}{dp}\left(\frac{d\not{p}}{dp}\right)^{-1} = (1-x)(-2\not{p}x)$$

The derivaive of the integrand becomes

$$\frac{d}{d\not{p}}\left(\frac{(\epsilon-1)(1-x)\not{p}+(2-\epsilon)m_e}{\Delta^\epsilon}\right) = \frac{(\epsilon-1)(1-x)}{\Delta^\epsilon} - \frac{(\epsilon-1)(1-x)\not{p}+(2-\epsilon)m_e}{\Delta^{\epsilon+1}}(1-x)(-2x\not{p})\epsilon$$

Now lets go to on shell electrons. This means that

$$\Delta \to x^2 m_e^2 + (1-x)(m_\gamma^2) \approx x^2 m_e^2 + m_\gamma^2$$

Where the approximation comes from a similar idea we used in calculating the muon's anomalous magnetic moment, $m_e \gg m_\gamma$ so m_γ only matters when $x \to 0$. We also have

$$(\epsilon-1)(1-x)\not{p}+(2-\epsilon)m_e \to m_e(\epsilon - x\epsilon - 1 + x + 2 - \epsilon) = m_e(x(1-\epsilon)+1)$$

And with this we come accross a miracle. The integrand above becomes

$$-\frac{d}{dx}\left(\frac{(x(1-\epsilon)+1)(1-x)}{\Delta^\epsilon}\right) - \frac{x(1-\epsilon)+1}{\Delta^\epsilon}$$

From here we can let mathematica handle the non-trivial term (while letting $m_\gamma = 0$ since this isn't IR divergent) and evaluate the second term to get

$$\int [\dots] = m_\gamma^{-2\epsilon} - m_e^{-2\epsilon}\left(\frac{1}{1-2\epsilon}+\frac{1}{2}\right)$$

So we get

$$\boxed{\left.\frac{d\Sigma_{1-\text{loop}}}{d\not{p}}\right|_{\not{p}=m_e} = \delta_2 = \frac{\alpha}{2\pi}\left(\frac{4\pi\mu^2}{m_e^2}\right)^\epsilon\left(\left(\frac{m_e^2}{m_\gamma^2}\right)^\epsilon - \frac{1}{1-2\epsilon} - \frac{1}{2}\right) = \delta_1}$$