

PHY 396L: QUANTUM FIELD THEORY II

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Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. Also there will be some stuff in here from Prof. Maloney's QFT I and QFT II course. If you have any comments let me know at hi@delonshen.com.

MALONEY QFT II LECTURE 1: SYSTEMATICS OF RENORMALIZATION	1
MALONEY QFT I LECTURE 12 PART 1: INTERACTIONS	3
MALONEY QFT I LECTURE 13: PATH INTEGRALS	5
Some Points About the Method of Saddle Points	9
LECTURE 1: AMPUTATING BAD LEGS AND FEYNMAN'S COOL TRICK TO EVALUATE INTEGRALS	10
Using Feynman's Parameter Trick	12
Aside: Generalizations for Integral Used Feynman Parameter Trick	15
MALONEY QFT I LECTURE 14: FEYNMAN DIAGRAMS I	20
LECTURE 2: EXAMPLE OF RENORMALIZATION	23
Aside: Debye Model of Solids	24
Back to 1-loop correction in $\lambda\phi^4$ theory	25
1-loop calculation for Elastic Scattering	27
LECTURE 3: UV REGULARIZATION SCHEMES	27
Wilson's Hard Edge Cutoff	27
Pauli Villars regularization	28
Covariant Higher Derivatives	29
Aside: Using Pauli-Villars and Higher Derivative Regularization Scheme	30
OPTICAL THEOREM	34
In Quantum Mechanics	34
In Quantum Field Theory	40

LECTURE 4: MORE REGULARIZATION AND OPTICAL THEOREM	40
Dimensional Regularization	41
Optical Theorem	43
LECTURE 5: APPLYING OPTICAL THEOREM AND BEGINNING CORRELATION FUNCTIONS	45
Applying Optical Theorem to $\lambda\phi^4$ Theory	45
Introduction to Correlation Functions	47
LECTURE 6: THE 2-POINT CORRELATION FUNCTION	49
LECTURE 7: COMPUTING TWO-POINT FUNCTIONS AND YUKAWA THEORY	53
Calculating Field Strength Renormalization to 2-loop level for $\lambda\phi^4$ theory	60
LECTURE 8: FINISHING UP YUKAWA AND STARTING COUNTERTERMS	65
Counterterms	67
Confirming Optical Theorem for Decay Rates in Yukawa Theory	72
LECTURE 9: COUNTERTERMS AND SUBGRAPHS	75
LECTURE 10: NESTED/OVERLAPPING DIVERGENCES AND BEGINNING OF RENORMALIZABILITY (PHYSICAL ANALYSIS!)	81
PESKIN AND SCHROEDER 10.5: A TWO-LOOP EXAMPLE	88
LECTURE 11: RENORMALIZING QED AND WARD IDENTITIES	88

MALONEY QFT II LECTURE 1: SYSTEMATICS OF RENORMALIZATION

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence let's consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \rightarrow \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$

$$\begin{aligned} \text{tree} \\ i\mathcal{M}_1 &= X = -i\lambda \\ \\ \text{one-loop} \\ s: i\mathcal{M}_2 &= \text{diagram} \propto \lambda^2 \\ &\quad \uparrow \\ &\quad 2 \text{ inter. vertices} \\ &\quad p_1 + p_2 = k \\ &\quad t \text{ and } u \text{ channel also exist} \end{aligned}$$

Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} \frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges (TODO Maloney says "logarithmically divergent"¹). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2 \log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2} \log s/\Lambda^2 + \dots$$

This seems like a disaster since as the cutoff $\Lambda \rightarrow 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

¹Okay so googling leads to this. $dU/U = d \log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase \mathcal{M} in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0 .

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2} \log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2} \log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for \mathcal{M} above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2} \log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \rightarrow 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2 . So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2} \log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2} \log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2} \log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2} \log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in \mathcal{L} we introduce a counterterm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \quad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

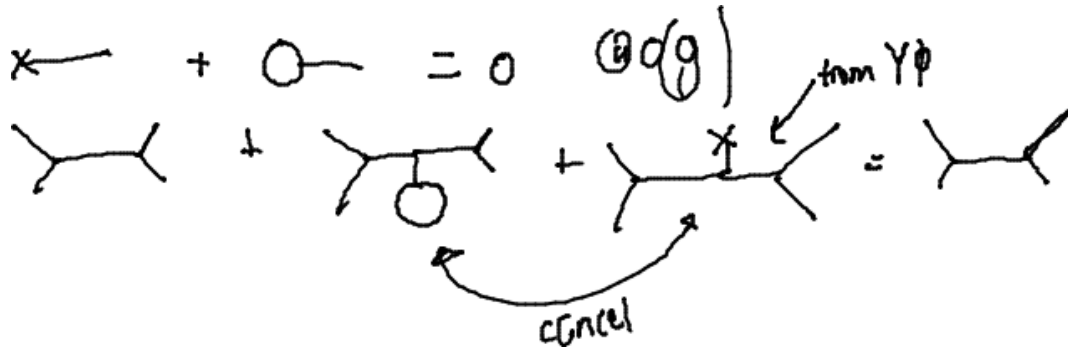
Now if we want to shift this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

- (a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$
- (b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$
- (c) Z_g fixed by $g =$ physical 3-pt funct at some energy
- (d) Z_m fixed by the (mass)² of a one particle state should be m^2 . The physical mass is not necessarily m^2 .

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g . For example Y is fixed by the cancellation of the one point function. The existence of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.



For other counterterms we have to do some computations.

MALONEY QFT I LECTURE 12 PART 1: INTERACTIONS

A free QFT where $S[\phi]$ is quadratic in ϕ has no interactions. Consider some sort of free theory with two kinds of particles a and b . Schematically let's have some Hamiltonian

$$H = a^\dagger a + b^\dagger b \Rightarrow a^\dagger |0\rangle \xrightarrow{e^{iHt}} a^\dagger |0\rangle \times \text{phase}$$

Basically for a free theory if you start with particle a , no matter how much time passes you'll still always have that particle a . For an interacting theory

$$H = a^\dagger a + b^\dagger b + \lambda(a(b^\dagger)^2 + a^\dagger b^2)$$

Where λ is small. The term in the parenthesis is the interaction term where the first term destroys the a particle and creates two b particles and the second term destroys the two b particles and gives us back the a particle (hermitian means process is reversible which is why we have both terms instead of just one term.) Solving this exactly should be a disaster but we can use approximation methods. First just by Taylor expanding we can see that

$$a^\dagger |0\rangle \rightarrow e^{iHt}(a^\dagger |0\rangle) \approx a^\dagger |0\rangle + i\lambda t(b^\dagger)^2 |0\rangle + \dots$$

Basically there's some probability per unit time where a particle becomes two b particles. What we wanted to show here is that the non-quadratic terms that create interactions. There should be some things we should take away

- (a) The vacuum state of $|0\rangle_{\text{free}}$ no longer works since. We defined this state as the state that's annihilated by lowering operators. However this is no longer the case. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$. The true vacuum state $|\Omega\rangle$ is more complex in interacting theories.
- (b) We shouldn't think of a and a^\dagger as creating/destroying particles anymore.

To make things explicit let's consider a typical interacting theory

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \mathcal{L}_{\text{int}} \Leftrightarrow \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Expanding $V(\phi)$ around $\phi = 0$ we get

$$V(\phi) = V_0 + V_1\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

V_0 is ignorable. We can also let $\phi \rightarrow \phi + \text{const}$ so that $V_1 = 0$. Consider $\phi(x^\mu)$ where ϕ is slowly varying of \mathbf{x} .

$$\Rightarrow \mathcal{L} \approx \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Energy is minimized when ϕ sits at the minimum of $V(\phi)$. This means the ground state or "vacuum" where $V'(\phi_*) = 0$. So if $V_1 \neq 0$ then the field will relax to a value of $V(\phi)$ with $V'(\phi) = 0$ and if we want to study the properties near this vacuum we have a new field $\phi' = \phi - \phi_*$. So for now let's just choose vacuum so that in the vacuum $\phi = 0$. e.g. $\langle\Omega|\phi|\Omega\rangle = 0$.

Question: what interaction terms matter? Why should we care about ϕ^3 and not ϕ^{100} ? We can answer this question with dimensional analysis

$$S = \int d^4x ((\partial\phi)^2 - m^2\phi^2 + \dots + \lambda_{ij}\partial^j\phi^i)$$

First we know in natural units $[x] = E^{-1}$ and thus $[\partial/\partial x] = E$. First action is dimensionless. Thus $[\phi] = E$ ($[d^4x\partial_x^2] = E^{-2}$). We can also see that $[m] = E$ and $[\lambda_{ij}] = E^{4-i-j}$. Now imagine we're doing some experiment which probes the theory at scale E . For example $\phi\phi \rightarrow \phi\phi$ with COM energy $\approx E$. What physical effects would come from our λ_{ij} term? well $\lambda_{ij}E^{i+j-4} = \lambda_{ij}E^{-[\lambda_{ij}]}$. What this means is that at low energies, we should only care about $[\lambda] > 0$ since if $[\lambda] < 0$ then E is raised to a positive power and thus is exponentially decreasing.

- (a) $[\lambda] > 0$ is relevant
- (b) $[\lambda] < 0$ is irrelevant
- (c) $[\lambda] = 0$ is marginal

For a scalar theory at low energies the only relevant terms are kinetic terms and ϕ^3 and ϕ^4 . Here's an idea from Landau: To study a system at low energy we can follow a recipe

- (a) Guess DOF

- (b) Guess symmetries
- (c) Write down the most general theory (e.g. action) that satisfies the symmetries
- (d) Study the relevant interaction terms.

To constrain our theory further we should also ask what sort of symmetries we have? For example for the symmetry $\phi \rightarrow -\phi$ then there can't be a ϕ^3 term. We call this theory $\lambda\phi^4$ theory.

MALONEY QFT I LECTURE 13: PATH INTEGRALS

Thought I should take a look at this since QFT I at UT didn't approach Feynmann diagrams from path integrals.

We have some QM system with DOF q_i (e.g. for a field theory $\phi(\mathbf{x})$). We want to compute the transition from some initial time t_i with some configuration q_i to the final state (q_f, t_f) .

$$\langle q_f, t_f | q_i, t_i \rangle$$

So we use the fact that we can always insert a complete set of basis states $|q, t\rangle$ at some intermediate time t where $t \in (t_i, t_f)$. So what happens? Let $|q_i, t_i\rangle = |q_1, t_1\rangle$ and $|q_f, t_f\rangle = |q_n, t_n\rangle$. This gives us

$$\begin{aligned} \langle q_f, t_f | q_i, t_i \rangle &= \int dq \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \\ &= \int dq_2 \dots dq_{n-1} \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \dots \langle q_2, t_2 | q_1, t_1 \rangle \end{aligned}$$

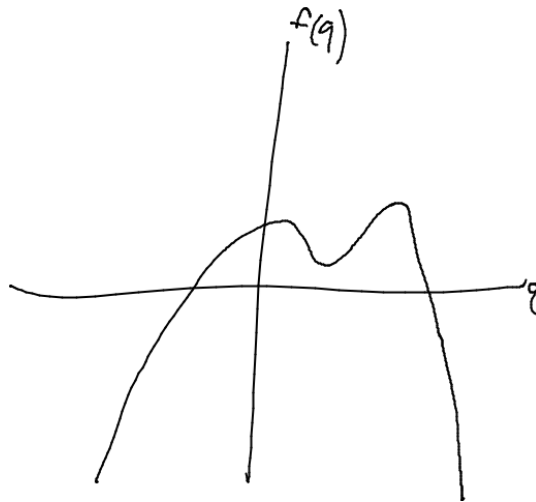
Basically we're integrating over all possible paths (including unphysical paths) from the initial state to the final state. So we can think of this transition amplitude as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq(t)$$

To make this idea precise we need to figure out how to integrate over a space of functions. The next thing we need to do is figure out what the fuck we're integrating. First lets just guess. In the limit $\hbar \rightarrow 0$ the integrand should be a function only of the classical solution of the EOM $\delta S / \delta q = 0$. First lets take a detour into approximating finite dimensional integrals which can be generalized to infinite dimensional integrals: the method of saddle points. Consider the one dimensional integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

And we want the integral to converge so f should look something like.



And we want to perform this integral when $\lambda \rightarrow \infty$. In this limit the integral should be dominated by the local maxima of $f(q)$. Namely by the q_* where $f'(q_*) = 0$. And what is this function Z in the limit where λ is large?

$$Z = \sum_{q_*} e^{\lambda f(q_*)}$$

It turns out there are a bunch of subleading corrections which are

$$Z = \sum_{q_*} e^{\lambda f(q_*)} \left(\sqrt{\frac{2\pi}{\lambda f''(q_*)}} + \dots \right)$$

(I proved first term in the next subsection). The higher order corrections are basically from Feynmann diagrams. Comparing this method of saddle point with the problem of path integrals we can guess that the integrand should be a function of $\frac{1}{\hbar} S$ (since the integrand goes to infinity as $\hbar \rightarrow 0$ meaning that only things that solve $\delta S / \delta Q = 0$ contribute like in the method of steepest descent) and we can guess that

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq(t) \exp \left\{ \frac{i}{\hbar} S[q(t)] \right\}$$

Lets just check our work. Consider $H = \frac{1}{2}p^2 + V(q)$. Then

$$e^{-iH\delta t} = e^{-i(p^2/2 + V(q))t} = e^{-i\frac{1}{2}p^2\delta t} e^{-iV(q)\delta t} e^{O(\delta t^2)}$$

Where the second equality comes from Cambell-Backer-Houstouder?? formula

$$e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A}, \hat{B}] + \dots}$$

Now lets insert an identity again in the following.

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q)\delta t} | q_{n-1} \rangle$$

Now we can use that $|p_{n-1}\rangle$ is a momentum eigenstate and $|q_{n-1}\rangle$ is a position eigenstate to replace the operators in the exponential with eigenvalues

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p_{n-1}^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q_{n-1})\delta t} | q_{n-1} \rangle$$

Taking out the identity we now just have

$$\begin{aligned} \langle q_n | e^{-iH\delta t} | q_{n-1} \rangle &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2 + V(q_{n-1}))} \langle q_n | p_{n-1} \rangle \langle p_{n-1} | q_{n-1} \rangle \\ &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2 + V(q_{n-1}))} e^{ip_{n-1}(q_n - q_{n-1})} \\ &= \int dp_{n-1} e^{-i\delta t H(q_{n-1}, p_{n-1})} e^{ip_{n-1}(q_n - q_{n-1})} \end{aligned}$$

So what we see is that

$$\begin{aligned} \langle q_n, t_n | q_i, t_i \rangle &= \int dq_2 \dots dq_{n-1} \langle q_n | e^{-iH\delta t} | q_{n-1} \rangle \dots \langle q_2 | e^{-iH\delta t} | q_1 \rangle \\ &= \int dq_2 \dots dq_{n-1} dp_2 \dots dp_{n-1} \prod_{i=2}^{n-1} e^{-i\delta t H(q_i, p_i)} e^{ip_i(q_{i+1} - q_i)} \\ &= \int dq_2 \dots dq_{n-1} dp_2 \dots dp_{n-1} \exp \left\{ i \sum_{i=2}^{n-1} \left(\frac{p_i}{\delta t} (q_{i+1} - q_i) - H(q_i, p_i) \right) \delta t \right\} \end{aligned}$$

Now in the limit when $n \rightarrow \infty$ then $\delta t \rightarrow 0$. This gives us

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq Dp \exp \left\{ i \int dt (p\dot{q} - H = L) \right\}$$

This is a path integral over phase space, not just position(configuration) space. We can reduce this to an integral in configuration space by noting that the integral over phase space is a gaussian integral. Note that

$$\int dp_i e^{i(-p_i^2/2 - p_i(q_{i+1} - q_i)/\delta t)\delta t} = \text{const} \times e^{i\dot{q}_i^2/2}$$

Some comments about this derivation:

- (a) $\int Dq$, the integral over a space of functions is not a well defined object
- (b) This derivation makes manifest ideas of interference and symmetries (e.g. double slit experiment)
- (c) Useful in perturbation theory.
- (d) Not useful for calculations. Not really necessary in QM but necessary in QFT.

So how do we compute expectation values? Let $Q(t)$ operator that measure $q(t)$.

$$\begin{aligned}\langle q_n, t_n | Q(t) | q_1, t_1 \rangle &= \int dq q(t) \langle q_n, t_n | q, t \rangle \langle q, t | q_1, t_1 \rangle \\ &= \int_{q(t_1)=q_1}^{q(t_n)=q_n} Dq e^{iS[q]} q(t)\end{aligned}$$

Now what happens if we want to compute a two point function

$$\int Dq e^{iS[q]} q(t) q(t') = \Theta(t' - t) \langle q_n, t_n | Q(t') Q(t) | q_1, t_1 \rangle + \Theta(t - t') \langle q_n, t_n | Q(t) Q(t') | q_1, t_1 \rangle$$

Namely the thing that the path integral computes is time ordered expectation values. Thus note that if we have many observables

$$\int Dq e^{iS[q]} q_1 \dots q_n = \langle \mathbf{T} q_1 \dots q_n \rangle$$

So far we've considered position eigenstates. In a more general state

$$\psi = \int dq |q\rangle \langle q | \psi \rangle$$

To compute VEV:

$$\langle 0 | \mathbf{T}(Q_1 \dots Q_n) | 0 \rangle = \lim_{t_i \rightarrow \infty}^{t_f \rightarrow -\infty} \int dq_i dq_f \psi_0(q_i) \psi_0^*(q_f) \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS} q_1 \dots q_n$$

There's a trick to avoid this

$$|q, t\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n | q, t=0 \rangle e^{iE_n t}$$

Now let $t \rightarrow (1 - i\epsilon)t$ and $t \rightarrow -\infty$ (very early times). Then $e^{iE_n t} \rightarrow e^{i(1-i\epsilon)tE_n}$. Thus we will pick out terms near the vacuum energy. Thus

$$|q, t\rangle \rightarrow \langle 0 | q, 0 \rangle | 0 \rangle$$

This analytic continuation projects onto the ground state for ket at $t \rightarrow -\infty$ and for bra this projects onto the ground state for $t \rightarrow \infty$. So then we can compute the VEV as

$$\langle 0 | \mathbf{T} Q_1 \dots Q_n | 0 \rangle = \int Dq \exp \{iS[q]\} q_1 \dots q_n$$

But where $S = \int d\tilde{t} L$ where $\tilde{t} = (1 - i\epsilon)t$. Sometimes we take this analytical continuation where $\tilde{t} = it$. This is a "Euclidean continuation." The reason that this is called "euclidean" comes from the following. Consider the invariant interval in special relativity. Under that analytical continuation

$$ds^2 = dt^2 - d\mathbf{x}^2 \rightarrow -(d\tilde{t}^2 + d\mathbf{x}^2)$$

It's useful in finite temperature physics.

SOME POINTS ABOUT THE METHOD OF SADDLE POINTS

Based on Problem Set 7. We're considering that integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

To derive the first correction we'll expand $f(q)$ around the saddle points $\{q_*\}$ where $f'(q_*)$ is zero.

$$f(q) \approx f(q_*) + (f'(q_*)(q - q_*) = 0) + \frac{f''(q_*)}{2}(q - q_*)^2 + \dots$$

This gives us the integral (after we note that at the saddle point $f''(q_*)$ is negative)

$$Z \approx e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{\lambda f''(q_*)(q - q_*)^2/2} dq = e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{-\lambda |f''(q_*)|(q - q_*)^2/2} dq$$

OwO whats this? *Notices your Gaussian integral.* We can do a change of variables

$$u = \sqrt{\frac{\lambda |f''(q_*)|}{2}}(q - q_*) = C(q - q_*) \Rightarrow du = C dq$$

Now we can evaluate the integral with the standard $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$Z \approx \sqrt{\frac{2\pi}{\lambda |f''(q_*)|}} e^{\lambda f(q_*)}$$

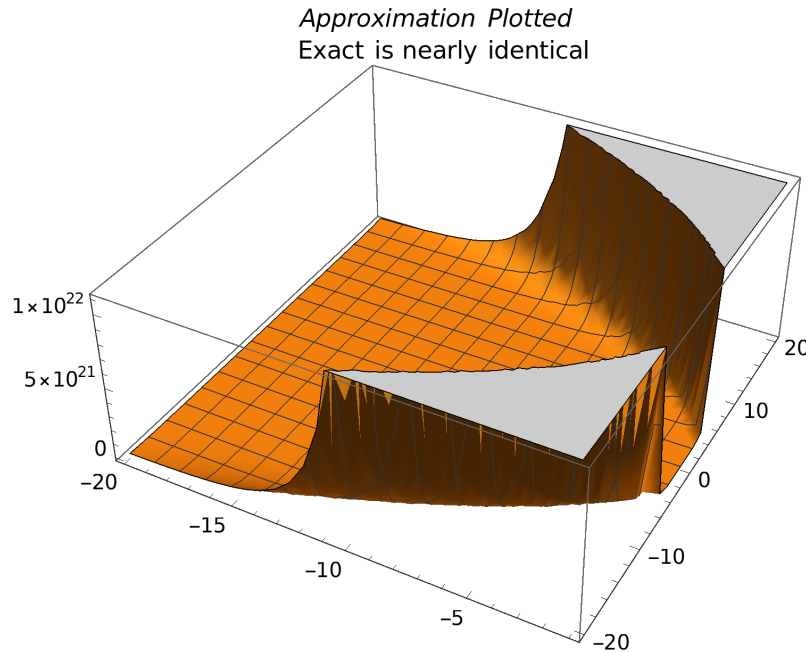
This prefactor we just defined in the context of quantum mechanics is the one-loop correction. Lets see our boy in action. Consider the Gaussian integral where $a < 0$

$$\int_{-\infty}^{\infty} dq e^{aq^2 + bq + c} = \sqrt{\frac{\pi}{|a|}} e^{c - b^2/4a}$$

We can find the approximation with our saddle point method

$$f'(q_*) = 2aq_* + b = 0 \Rightarrow q_* = -\frac{b}{2a} \Rightarrow \int_{-\infty}^{\infty} dq e^{aq^2 + bq + c} \approx \sqrt{\frac{\pi}{|a|}} e^{c + bq + aq^2}$$

Lets fix $c = 42$ and plot both the approximate and the exact answer for some values of a and b just to get a feel for this approximation



It's a good approximation!

LECTURE 1: AMPUTATING BAD LEGS AND FEYNMAN'S COOL TRICK TO EVALUATE INTEGRALS

January 19, 2021

Lets start by considering $\lambda\phi^4$ theory and a $2 \rightarrow 2$ scattering. The corresponding Feynman diagrams up to 1-loop level is found in Figure 1 Why are the bad diagrams bad? Well consider one of the bad diagrams

TODO Figure

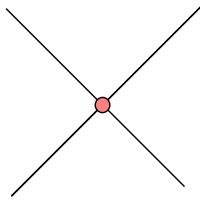
Well $q_2 = p_1$ for any q_1 . So the integral becomes

TODO Integral, tl;dr divergence

Namely all of them have bad propagators. Going further we can look at the two-loop example and see that the exact same thing happens. These bad features are called *external leg bubbles*. All diagrams with external leg bubbles are bad, the propagator is frozen on shell and thus blows up.

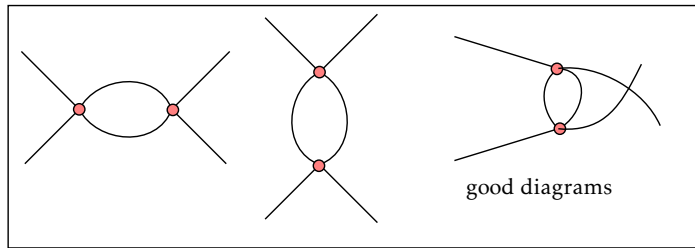
Lets look at a practical solution. He'll give us a half-assed justification for this solution and then will talk about what's really going on next week. The solution is: *amputate all the external leg bubbles*. To do this we need to figure out how we find these external leg bubbles and the carefully figure out what do we need to amputate. If cutting 1 propagator at a connected diagram breaks it into two disconnected pieces and if one of the pieces has just one external leg

Tree level

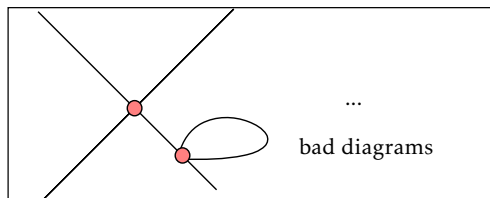


$$= i\mathcal{M}_{\text{tree}} = -\lambda$$

1-loop



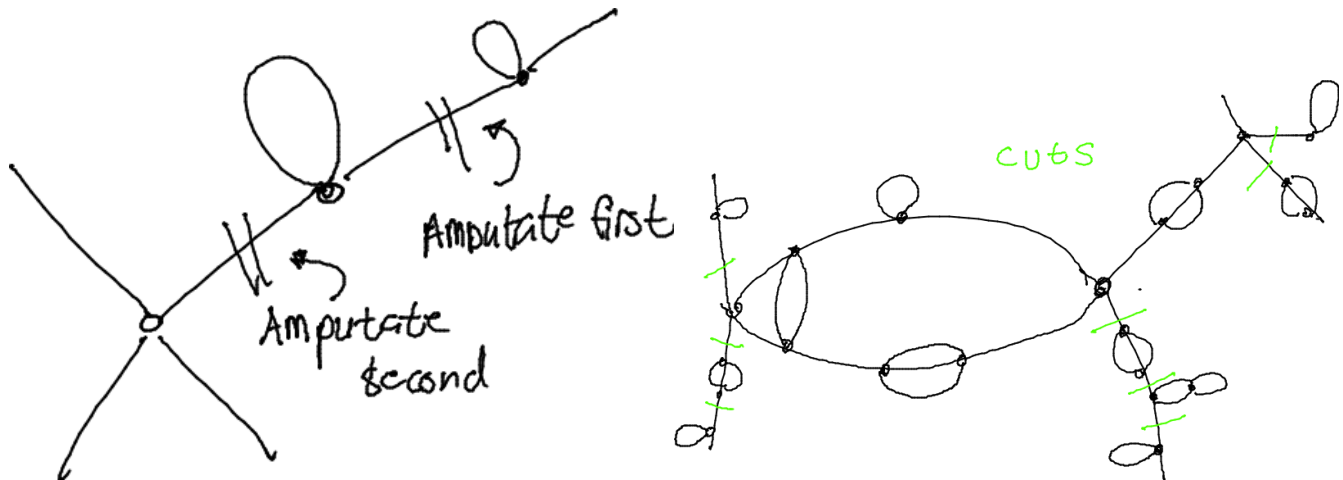
good diagrams



bad diagrams

 Figure 1: Illustration of good and bad diagrams in a $2 \rightarrow 2$ process for $\lambda\phi^4$ theory.

then amputate that piece.



The bottom line is that

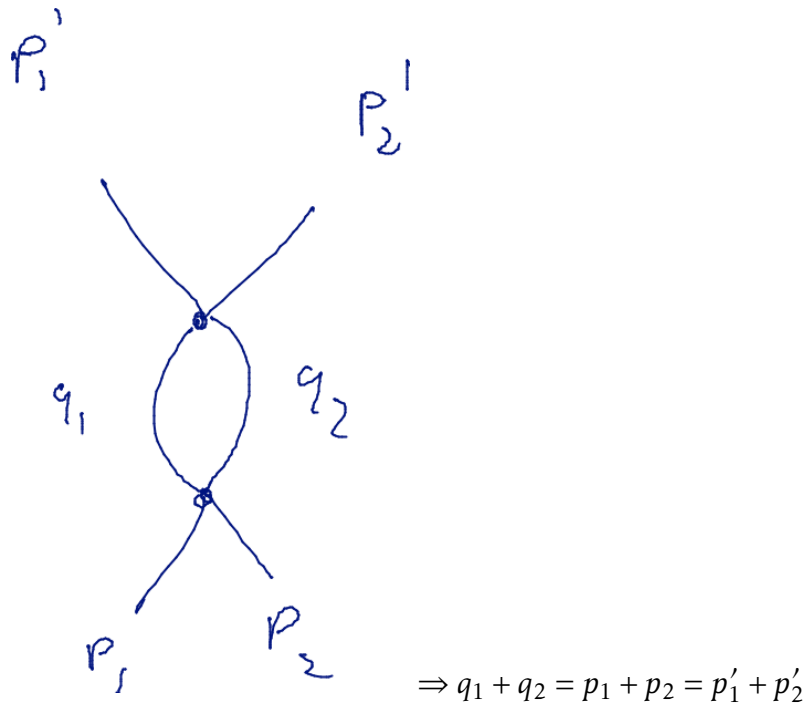
$$i\mathcal{M}_{\text{scattering}} = \sum \text{amputated connected diagrams}$$

So where does this rule come from. Well last semester when we were looking at S-matrix elements we were given a formula that was unjustified

$$\langle p'_1, p'_2, \dots | \hat{S} | p_1, p_2, \dots \rangle = \text{Limit regularization going away} \left[C_{\text{vac}} \times \prod_{\text{ext legs}} F(\text{leg}) \times \langle 0 | a_{p'_1} a_{p'_2} \hat{S} a_{p_1}^\dagger a_{p_2}^\dagger | 0 \rangle \right]$$

Now note that free matrix elements is the sum over all Feynmann diagrams. The C_{vac} cancels all vacuum bubbles. What we want to show soon is that $F(\text{ext leg})$ cancels all leg bubbles in that leg. So in the end we're left with diagrams that are unproblematic. This is the "justification" of the no external leg bubble rule. Now the right way to derive this rule is to calculate the

correlation function of multiple field, then get (some name) action formula, and then we'll get out result. It turns out that off shell the external leg bubbles are sometimes useful. So now lets go back to the very first diagrams we wrote down and figure out how to calculate loop diagrams.



Depends only on the net $p_1 + p_2$ meaning that the matrix element $iF(s)$ only depends on $(p_1 + p_2)^2 = s$. Thus

$$i\mathcal{M}_{1\text{-loop}} = iF(s) + iF(t) + iF(u)$$

By crossing symmetry if we find one of them we find all of them. For example

$$iF(t) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \frac{i}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i\epsilon}$$

Where $q_{\text{net}} = p_1' - p_1 = p_2 - p_2'$ and $q_{\text{net}}^2 = t$. So how do we perform integrals like this? We'll use what's called the *Feynman Parameter Trick*.

USING FEYNMAN'S PARAMETER TRICK

Consider the following integral

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

This is true because $(1-x)A + xB$ interpolates between A and B . This integral was known ages before Feynman. However Feynman made good use of it.

$$\frac{1}{q_1^2 - m^2 + i\epsilon} \times \frac{1}{q_2^2 - m^2 + i\epsilon} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)]^2}$$

So looking inside the square bracket we can see that

$$\begin{aligned}
 [(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)] &= (1-x)q_1^2 + xq_2^2 - m^2 + i\epsilon \\
 &= (1-x)q_1^2 + xq_1^2 - 2x(q_{\text{net}}q_1) + xq_{\text{net}}^2 - m^2 + i0 \\
 &\text{complete the square in the } 2x \text{ term} \\
 &= (q_1 - xq_{\text{net}})^2 - x^2q_{\text{net}}^2 + xq_{\text{net}}^2 + m^2 + i0 \\
 &= (q_1 - xq_{\text{net}})^2 - \Delta(x) + i0
 \end{aligned}$$

Where we define $\Delta(x) = m^2 + (x^2 - x)q_{\text{net}}^2 = m^2 + (x^2 - x)t$. This means that

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2}$$

So now we can write $iF(t)$ more cleanly

$$\begin{aligned}
 iF(t) &= \frac{1}{2} \lambda^2 \int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2} \\
 &= \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x) + i0]^2}
 \end{aligned}$$

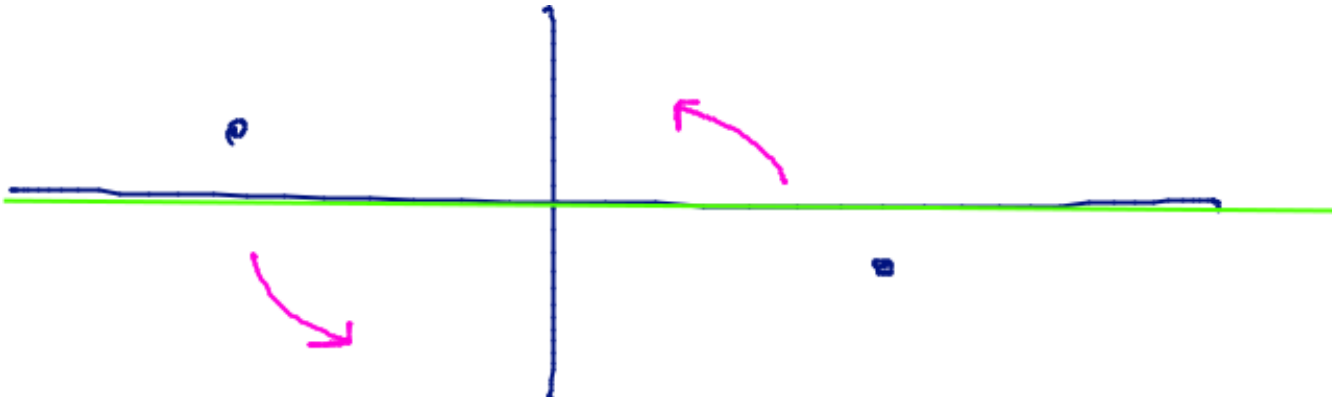
Changing the order of integration is a bit suspect here but at this level it turns out this actually works. Now once we've changed the order of integration for each x change momentum integration variable from q_1 to $k = q_1 - xq_{\text{net}}$. This gives us

$$iF(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4 k}{(2\pi)^2} \frac{1}{[k^2 - \Delta(x) + i0]^2} \quad \text{where} \quad \Delta(x) = m^2 - tx(1-x) > 0$$

Now how do we perform this integral? Me thinks residual. Lets focus on the $d^4 k$ integral

$$\int \frac{d^4 k}{(2\pi)^2} \frac{1}{[k^2 - \Delta(x) + i0]^2} = \int d^3 \mathbf{k} \int dk_0 \frac{1}{[k_0^2 - \mathbf{k}^2 - \Delta(x) + i0]^2}$$

The $\int dk_0$ integral has double poles at $k_0 = \pm[\sqrt{\mathbf{k}^2 + \Delta} - i\epsilon]$. So lets deform the contour in the complex k_0 plane. Also note that as $k_0 \rightarrow \infty$ the term is negligible since the integral is about proportional to k_0^{-4} . So we may change something? can't read hand writing. Now we can use Wick rotations.



Note that we rotate CCW since if we rotated CW then we'd hit the poles and they don't like being hit. So let $k_0 = ik_4$ for real k_4 running from $-\infty$ to ∞ . So what does this give us?

$$k_\mu k^\mu - \Delta = k_0^2 - \mathbf{k}^2 - \Delta = \underline{-k_4^2 - \mathbf{k}^2 - \Delta}$$

Notice the sign is the same in the underlined section. It becomes the "euclidean continuation?". So we can combine the 3-vector \mathbf{k} and k_4 into a euclidean 4-vector $k_E = (k_1, k_2, k_3, k_4)$. So $k^2 = g_{\mu\nu} k^\mu k^\nu = k_0^2 - \mathbf{k}^2$ becomes $-k_E^2 = -k_4^2 - \mathbf{k}^2$. At the same time $dk_0 = idk_4$ so Minkowski $d^4k = id^4k_E$. And with this setup we get the integral

$$\begin{aligned} \int \frac{d^4 k_{\text{minkowski}}}{[k^2 - \Delta + i0]^2} &= \int d^3 \mathbf{k} \int \frac{dk_0}{\dots} \\ &= \int d^3 \mathbf{k} \int \frac{idk_4}{[-k_4^2 - \mathbf{k}^2 - \Delta]} \\ &= i \int \frac{d^4 k_E}{[k_E^2 + \Delta]^2} \end{aligned}$$

Where in the second equality since everything is negative we can get rid of the $i\epsilon$ perscription. The bottom line is

$$F(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2}$$

So the $SO^+(3,1)$ Lorentz symmetry of the Minkowski space analytically continues to $SO(4)$ rotation symmetry in $D = 4$ euclidean space under a wick rotation by $\pi/2$. We can go even futher and change variables from $d^4 k_E = dk_E^{\text{real}} \times (k_E^{\text{rot?}})^3 \times d^3 \Omega_k$ where Ω is solid angle which means $\int_{D=4} d^3 \Omega_k = 2\pi^2$ and so $\int d^4 k_E \rightarrow \int 2\pi^2 k_E^3 dk_E$ meaning that our integral is

$$\int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \frac{2\pi^2}{16\pi^4} \int_0^\infty \frac{k_E^3 dk_E}{[k_E^2 + \Delta]^2}$$

There is one subtlety however. AT $k_E \rightarrow \infty$ we know

$$\frac{k_E^3}{[k_E^2 + \Delta]^2} \approx \frac{1}{k_E} \Rightarrow \frac{dk_E}{k_E} = d \log k_E \text{ is logarithmically divergent}$$

ASIDE: GENERALIZATIONS FOR INTEGRAL USED FEYNMAN PARAMETER TRICK

We proved some generalization of Feynman's Parameter Trick on the problem set. They're not too hard to prove for anyone with some free time. I'll put them here for reference.

$$\frac{1}{A^n B} = \int_0^1 \frac{nx^{n-1} dx}{[xA + (1-x)B]^{n+1}} \quad (\text{F.a})$$

$$\frac{1}{A^n B^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{x^{n-1}(1-x)^{m-1} dx}{[xA + (1-x)B]^{n+m}} \quad (\text{F.b})$$

$$\begin{aligned} \frac{1}{ABC} &= \int_0^1 dx \int_0^{1-x} \frac{2dy}{[xA + yB + (1-x-y)C]^3} \\ &= \int_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \frac{2}{[xA + yB + zC]^3} \end{aligned} \quad (\text{F.c})$$

$$\begin{aligned} \frac{1}{A_1 \dots A_k} &= \int_{x_1, \dots, x_k \geq 0} \dots \int d^k x \delta(x_1 + \dots + x_k - 1) \times \frac{(k-1)!}{[x_1 A_1 + \dots + x_k A_k]^k} \\ &= \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \int_0^{1-x_2-\dots-x_{k-2}} dx_{k-1} \frac{(k-1)!}{[x_1 A_1 + \dots + (1-x_2-\dots-x_{k-1}) A_k]^k} \end{aligned} \quad (\text{F.d})$$

$$\frac{1}{A_1^{n_1} A_2^{n_2} \dots A_k^{n_k}} = \frac{(n_1 + \dots + n_k - 1)!}{(n_1 - 1)! \dots (n_k - 1)!} \times \quad (\text{F.e})$$

$$\times \int_{x_1, \dots, x_k \geq 0} \dots \int d^k x \delta(x_1 + \dots + x_k - 1) \times \frac{x_1^{n_1-1} \dots x_k^{n_k-1}}{[x_1 A_1 + \dots + x_k A_k]^{(n_1 + \dots + n_k)}} \quad (\text{F.f})$$

(F.a) We wish to verify

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA + (1-x)B]^2}$$

Let $u = [xA + B - Bx]$ which gives us $du = (A - B)dx$. The bounds go from B to A . The integral then becomes

$$\boxed{\frac{1}{A-B} \int_B^A \frac{du}{u^2} = -\frac{1}{A-B} \left(\frac{B-A}{AB} \right) = \frac{1}{AB}}$$

(F.b) We wish to verify

$$\frac{1}{A^n B} = \int_0^1 \frac{nx^{n-1} dx}{[xA + (1-x)B]^{n+1}}$$

Lets do the same change of variables we did in (F.a). Also note that $x = (u - B)/(A - B)$.

$$\begin{aligned}
 \int_0^1 \frac{nx^{n-1}dx}{[xA + (1-x)B]^{n+1}} &= \frac{n}{(A-B)^n} \int_B^A \frac{(u-B)^{n-1}du}{u^{n+1}} \\
 &= \frac{n}{(A-B)^n} \int_B^A \frac{du}{u^2} \left(1 - \frac{B}{u}\right)^{n-1} \\
 \text{Let } w &= \frac{1}{u} \Rightarrow dw = -u^{-2}du \\
 &= -\frac{n}{(A-B)^n} \int_{1/B}^{1/A} dw (1-Bw)^{n-1} \\
 &= \frac{1}{B(A-B)^n} (1-Bw)^n \Big|_{1/B}^{1/A} \\
 &= \frac{1}{B(A-B)^n} (1-B/A)^n \\
 \boxed{\int_0^1 \frac{nx^{n-1}dx}{[xA + (1-x)B]^{n+1}} &= \frac{1}{A^n B}}
 \end{aligned}$$

(F.c) We wish to verify

$$\frac{1}{A^n B^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xA + (1-x)B]^{n+m}} = Z$$

Lets do the same change of variables again

$$\begin{aligned}
 Z &= \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \frac{1}{(A-B)^{n+m-1}} \int_B^A \frac{(u-B)^{n-1}(A-u)^{m-1}du}{u^{n+m}} \\
 &= \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \frac{1}{(A-B)^{n+m-1}} \int_B^A \frac{1}{u^2} (1-B/u)^{n-1} (A/u-1)^{m-1} du \\
 &= -\frac{(n+m-1)!}{(n-1)!(m-1)!} \times \frac{1}{(A-B)^{n+m-1}} \int_{1/B}^{1/A} (1-Bw)^{n-1} (Aw-1)^{m-1} dw
 \end{aligned}$$

Now lets integrate by parts (using the tabular method)

$$\begin{array}{c|c}
 (1-Bw)^{n-1} & (Aw-1)^{m-1} \\
 -B(n-1)(1-Bw)^{n-2} & \frac{1}{Am}(Aw-1)^m \\
 \vdots & \vdots \\
 (-1)^{n-1}B^{n-1}(n-1)! & \frac{1}{A^{n-1}m(m+1)\dots(m+n-2)}(Aw-1)^{m+n-2}
 \end{array}$$

Because of our bounds each non-integral term vanishes. Also note the sign of each term from integration by parts is determined by $(-1)^{n+1}$. Thus the final integral is positive.

This leaves us with

$$\begin{aligned}
 \int_{1/B}^{1/A} (1-Bw)^{n-1} (Aw-1)^{m-1} dw &= \frac{B^{n-1}(n-1)!}{A^{n-1}m(m+1)\dots(m+n-2)} \int_{1/B}^{1/A} (Aw-1)^{m+n-2} dw \\
 &= -\frac{B^{n-1}(n-1)!}{A^n m(m+1)\dots(m+n-1)} (A/B-1)^{m+n-1} \\
 &= -\frac{B^{n-1}(n-1)!}{A^n B^{m+n-1} m(m+1)\dots(m+n-1)} (A-B)^{m+n-1} \\
 &= -\frac{(n-1)!(m-1)!}{A^n B^m (m+n-1)!} (A-B)^{m+n-1}
 \end{aligned}$$

Plugging this integral back in gives us

$$\boxed{\frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{x^{n-1}(1-x)^{m-1} dx}{[xA + (1-x)B]^{n+m}} = \frac{1}{A^n B^m}}$$

(F.d) We can evaluate this in a straightforward manner

$$\begin{aligned}
 Z &= -\frac{1}{B-C} \int_0^1 dx \{ [xA + (1-x)B]^{-2} - [xA + (1-x)C]^{-2} \} \\
 &= \frac{1}{B-C} \left\{ \frac{1}{A-B} [xA + (1-x)B]^{-1} - \frac{1}{A-C} [xA + (1-x)C]^{-1} \right\} \Big|_0^1 \\
 &= \frac{1}{B-C} \left\{ \frac{1}{A(A-B)} - \frac{1}{A(A-C)} - \frac{1}{B(A-B)} + \frac{1}{C(A-C)} \right\} \\
 &= \frac{1}{B-C} \left\{ \frac{BC(A-C) - BC(A-B) - AC(A-C) + AB(A-B)}{ABC(A-B)(A-C)} \right\} \\
 &= \frac{1}{B-C} \left\{ \frac{(A-C)(BC-AC) + (A-B)(AB-BC)}{ABC(A-B)(A-C)} \right\} \\
 &= \frac{1}{B-C} \left\{ \frac{C(A-C)(B-A) + B(A-B)(A-C)}{ABC(A-B)(A-C)} \right\} \\
 &= \frac{1}{B-C} \left\{ \frac{-C(A-B) + B(A-B)}{ABC(A-B)} \right\} \\
 &= \frac{1}{B-C} \left\{ \frac{-C+B}{ABC} \right\} \\
 &= \frac{1}{ABC}
 \end{aligned}$$

(F.e) We know the given integral is equivalent to

$$Z = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \int_0^{1-x_1-\dots-x_{n-2}} dx_{n-1} \frac{(n-1)!}{[x_1 A_1 + \dots + (1-x_1-\dots-x_{n-1}) A_n]^n}$$

From our result above we know that evaluating all the integrals will result in a factor of $1/(n-1)!$. This is because of the exponent in the denominator. From examining (F.d) we also

note that after each integral something of the following form

$$\int_0^{1-x_1-\dots-x_{n-2}} dx_{n-1} \frac{(n-1)!}{[x_1 A_1 + \dots + (1-x_1-\dots-x_{n-1}) A_n]^n} =$$

$$- \frac{(n-2)!}{(A_{n-1} - A_n)[x_1 A_1 + \dots + (1-x_1-\dots-x_{n-2}) A_{n-1}]^{n-1}}$$

$$+ \frac{(n-2)!}{(A_{n-1} - A_n)[x_1 A_1 + \dots + (1-x_1-\dots-x_{n-2}) A_n]^{n-1}} \quad (1)$$

I think we can analyze this with recursion. Lets define the following

$$Z(\{A_1, \dots, A_{n-2}, A_{n-1}, A_n\}, n) =$$

$$\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \int_0^{1-x_1-\dots-x_{n-2}} dx_{n-1} \frac{(n-1)!}{[x_1 A_1 + \dots x_{n-1} A_{n-1} + (1-x_1-\dots-x_{n-1}) A_n]^n} \quad (2)$$

We see from (1)

$$Z(\{A_1, \dots, A_{n-1}, A_n\}, n) = \frac{-1}{A_{n-1} - A_n} \{Z(\{A_1, \dots, A_{n-2}, A_{n-1}\}, n-1) - Z(\{A_1, \dots, A_{n-2}, A_n\}, n-1)\}$$

Also from (F.d) we see that

$$Z(\{A_1, A_2\}, 2) = -\frac{1}{A_1 - A_2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right)$$

Now instead of being rigorous (I'm just trying to convince myself afterall) we'll just check this recursion relation with mathematica. Consider the following code which implements this recursive definition.

```
In[105]:= Clear[f]
f[A_, 2] := -1 / (A[[1]] - A[[2]]) (1 / A[[1]] - 1 / A[[2]])
f[A_, n_] := -1 / (A[[n-1]] - A[[n]]) (f[A[[1];; n-1], n-1] - f[Join[A[[1];; n-2], {A[[n]]}], n-1])
Table[f[Table["A" <> ToString[i], {i, 1, maxn}], maxn] // FullSimplify, {maxn, 2, 10}]
```

```
Out[108]= { 1 / (A1 A2), 1 / (A1 A2 A3), 1 / (A1 A2 A3 A4), 1 / (A1 A2 A3 A4 A5), 1 / (A1 A2 A3 A4 A5 A6), 1 / (A1 A2 A3 A4 A5 A6 A7),
1 / (A1 A2 A3 A4 A5 A6 A7 A8), 1 / (A1 A2 A3 A4 A5 A6 A7 A8 A9), 1 / (A1 A10 A2 A3 A4 A5 A6 A7 A8 A9) }
```

In fact we can do better than just check it with mathematica. We're given

$$\frac{1}{A_1 \dots A_k} = \int \dots \int_{x_1, \dots, x_k \geq 0} d^k x \delta(x_1 + \dots + x_k - 1) \times \frac{(k-1)!}{[x_1 A_1 + \dots + x_k A_k]^k}$$

And we've proven a base case of $k = 3$. From here we just need to prove that if the relationship is satisfied for k then it also satisfies $k + 1$. Consider

$$\frac{1}{A_1 \dots A_{k+1}} = \int \dots \int_{x_1, \dots, x_k \geq 0} d^k x \delta(x_1 + \dots + x_k - 1) \times \frac{(k-1)!}{[x_1 A_1 + \dots + x_k A_k]^k A_{k+1}}$$

We use (F.b) to get

$$\frac{1}{A_1 \dots A_{k+1}} = \int \dots \int_{x_1, \dots, x_k \geq 0} d^k x \int_0^1 dy \delta(x_1 + \dots + x_k - 1) \times \frac{k! y^{k-1}}{[y(x_1 A_1 + \dots + x_k A_k) + (1-y)A_{k+1}]^{k+1}}$$

Let $yx_i = z_i$ and $(1-y) = z_{k+1}$ meaning that $dx_i = dz_i/y$ and $dy = -dz_{k+1}$ and the limits in the innermost integral go from 1 to 0 allowing us to change sign in order to keep the prefactor positive.

$$\begin{aligned} \frac{1}{A_1 \dots A_{k+1}} &= \int \dots \int_{z_1, \dots, z_k \geq 0} d^k z \int_0^1 dz_{k+1} \delta(y^{-1}(z_1 + \dots + z_k + z_{k+1} - 1)) \\ &\quad \times \frac{k! y^{-1}}{[z_1 A_1 + \dots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}} \end{aligned}$$

Now we use $\delta(ax) = \delta(x)/|a|$ to get

$$\begin{aligned} \frac{1}{A_1 \dots A_{k+1}} &= \int \dots \int_{z_1, \dots, z_k \geq 0} d^k z \int_0^1 dz_{k+1} \delta(z_1 + \dots + z_k + z_{k+1} - 1) \\ &\quad \times \frac{k!}{[z_1 A_1 + \dots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}} \end{aligned}$$

The last thing we want to show is that the bounds can equivalently be stated as z_{k+1} is positive. First we note that the delta function asserts

$$z_1 + \dots + z_{k+1} = 1$$

And the other bounds of the integral assert $z_1, \dots, z_k \geq 0$. Now consider the situation when $z_{k+1} > 1$. Then we have

$$z_1 + \dots + z_{k+1} > z_1 + \dots + z_k + 1$$

From here we see that no combination of positive $z_1 \dots z_k$ will satisfy the condition that $z_1 + \dots + z_{k+1} = 1$ meaning that there is no contribution for $z_{k+1} > 1$. Thus the bounds of the innermost integral can be restated as $z_{k+1} \geq 0$ since any contribution from $z_{k+1} > 1$ is killed by the delta function. This gives us

$$\frac{1}{A_1 \dots A_{k+1}} = \int \dots \int_{z_1, \dots, z_{k+1} \geq 0} d^{k+1} z \delta(z_1 + \dots + z_{k+1} - 1) \frac{k!}{[z_1 A_1 + \dots + z_{k+1} A_{k+1}]^{k+1}}$$

Therefore we've proven the inductive hypothesis thus completing the proof for (F.e).

(F.f) You can just get this result by taking the partial derivative of (F.e) a million times. Typing that out seem wasteful.

MALONEY QFT I LECTURE 14: FEYNMAN DIAGRAMS I

In this lecture we finally become men? To recap in QM we can compute observables with the following

$$\langle q_f, t_f | T Q(t_q) \dots Q(t_n) | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS/\hbar} q(t_i) \dots q(t_n)$$

So for example if we're doing a double slit experiment and want to measure which slit the particle goes through we'll insert an operator q in the integral. We can also rephrase this integral as a integral through phase space

$$= \int Dq Dp \exp \left\{ i \int (p \dot{q} - H(p, q)) dt \right\} q(t_1) \dots q(t_n)$$

In the end of class we introduced a trick to compute correlation function instead of transition amplitudes. Since

$$e^{it(1-i\epsilon)H} |\psi\rangle = |0\rangle \langle 0 | \psi \rangle \text{ at } t \rightarrow -\infty$$

Similarly we could apply this to a bra-vector

$$\langle \psi | e^{it(1-i\epsilon)H} = \langle \psi | 0 \rangle \langle 0 | \text{ at } t \rightarrow \infty$$

This means if you want to calculate VEV

$$\langle 0 | T Q(t_1) \dots Q(t_n) | 0 \rangle \propto \int Dq Dp \exp \left\{ i \int (\dot{q} p - (1-i\epsilon)H) dt \right\} q(t_1) \dots q(t_n)$$

Effectively what we're doing is adding a small imaginary part to the energies and basically projects us onto the ground state. At the tree level this $i\epsilon$ factor isn't very important but when we do loop calculation this $i\epsilon$ factor becomes important. Something to emphasize the equation above is that we have \propto instead of equality since we ignored some normalizations. To get equality we could do

$$\langle 0 | \dots | 0 \rangle = \frac{\int Dq e^{iS/\hbar} q(t_1) \dots q(t_n)}{\int Dq e^{iS/\hbar}}$$

Where the denominator is there to keep $\langle 0 | 0 \rangle = 1$.

To generalize this to many DOF $q_i(t) \rightarrow \int Dq_i(t)$. In a QFT where the DOF is a local $\phi(t, \mathbf{x}) \rightarrow \int D\phi(\mathbf{x}, t)$ (integrate over all possible field configurations ϕ). Today our goal is to compute

$$\langle 0 | T \phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{\int D\phi e^{iS} \phi(x_1) \dots \phi(x_n)}{\int D\phi e^{-S}}$$

To do this let's define the partition function or generating function of a QFT. We call J the source

$$Z[J(x)] = \int D\phi e^{iS + i \int d^4x J(x) \phi(x)}$$

It should be clear that if we computed the EOM, $J(x)$ would be a source term on the RHS of the EOM. And if we wanted to we can expand the exponential

$$Z[J(x)] = \int D\phi e^{iX} \left(1 + i \int d^4x J(x)\phi(x) - \int d^4x d^4y J(x)J(y)\phi(x)\phi(y) + \dots \right)$$

This partition function includes all possible expectation values that we care about. First let's consider how we could get expectation values from this partition function. Well let's remember how we extract single terms from a power series. We take the derivative wrt the variable and then set that variable to zero. For example $e^x = c_0 + c_1x + c_2x^2 + \dots$, $de^x/dx = c_1 + 2c_2x + \dots \Rightarrow (de^x/dx)(x=0) = c_1$. This generalizes to what we have here. However here we need to take the functional derivative

Define $\frac{\delta}{\delta J(x)}$ as $\frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)$

More complicated derivatives are defined using this formula and the chain rule

$$\frac{\delta V(\phi(y))}{\delta \phi(x)} = \frac{\partial V(\phi(y))}{\partial \phi(x)} \frac{\delta \phi(y)}{\delta \phi(x)} = V'(\phi(y)) \delta^{(4)}(x-y)$$

Now we just need to put these two things together

$$Z[J] = \int D\phi e^{iS + i \int J(x)\phi(x) d^4x}$$

So if we want the VEV we compute

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{(-i)^n}{Z} \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} Z[J] \Big|_{J=0}$$

Let's look at an example, free QFT $S = -\frac{1}{2} \int d^4x \phi(\partial^2 + m^2)\phi$. This means that

$$Z[J] = \int D\phi \exp \left\{ -\frac{i}{2} \int d^4x \phi(x)(\partial^2 + m^2)\phi(x) + i \int d^4x J(x)\phi(x) \right\}$$

Let's consider a simpler but illuminating problem

$$\int d\phi \exp(\phi A \phi + j \phi) \approx \exp(j A^{-1} j)$$

Where the second \approx comes from shifting ϕ to complete the square. For multidimensional integral we get (TODO PSET)

$$Z[J] \int d\phi_i \exp(\phi_i A^{ij} \phi_j + j^i \phi_i) \approx \exp(j^i (A^{-1})_{ij} j^j)$$

Let's think of a discrete model of QFT (e.g. space is lattice). We can identify A^{ij} with $(\partial^2 + m^2)$. We also met that $A^{-1} = (\partial^2 + m^2)^{-1} = D_F$, the feynman propagator. To compute the integral we can use fourier transform. One thing we can note is that the kinetic operator $(\partial^2 + m^2)$ is matrix

that is diagonal in momentum space. What does that mean? Lets take our $\phi(x)$ and think about it in momentum space

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\phi}(k)$$

And lets change variables so that our path integral isn't over all configurations of $\phi(x)$ but all possible fourier components of $\tilde{\phi}(k)$.

$$\int D\phi(x) \rightarrow \int D\tilde{\phi}(k)$$

To think about this more clearly we can consider the integral over all field configurations of $\phi(x)$ as the integral over all possible values of x (π_x) in some one dimensional integral (? TODO huh)

$$\int \pi_x d\phi(x) \rightarrow \int \pi_k d\tilde{\phi}(k)$$

Whenever we do a change of variables we have a jacobian determinant. Here it's not too hard to compute since we're going from ϕ to a linear combination of ϕ . But we're just gonna ignore it since it's some overall constant for now. So what does the integrand look like in fourier space.

$$\begin{aligned} \int d^4x \phi(x)(\partial^2 + m^2)\phi(x) &= \int d^4x \int d^4k d^4k' e^{-ikx} \tilde{\phi}(k)(\partial^2 + m^2 = -k'^2 + m^2)e^{-ik'x} \tilde{\phi}(k') \\ &= \int d^4k \tilde{\phi}(k)(-k^2 + m^2)\tilde{\phi}(-k) \end{aligned}$$

This is what we mean when we say taht the kinetic operator is diagonal in mometnum space. We also have

$$\int d^4x J(x)\phi(x) = \int d^4k \tilde{J}(k)\tilde{\phi}(-k) = \frac{1}{2} \int d^4k (\tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k))$$

This means we have

$$Z[J] = \int D\tilde{\phi}(k) \exp \left\{ \frac{i}{2} \int d^4k (\tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)) \right\}$$

We can separate out the integrand into the product of a bunch of one dimensional integrals over each independent momentum mode

$$Z[J] = \prod_k \int d\phi(k) \exp \left\{ \frac{i}{2} (\tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\phi(k)) \right\}$$

That's just a one dimensional gaussian integral. Let $\chi(k) = \tilde{\phi}(k) + \tilde{J}(k)/(k^2 - m^2)$. this means

$$Z[J] = \prod_k \int d\chi(k) \exp \left\{ \frac{i}{2} \chi(k)\chi(-k)(k^2 - m^2) + \frac{i}{2} \tilde{J}(k)\tilde{J}(-k)/k^2 - m^2 \right\}$$

So we can then write

$$Z[J] = Z[0] \exp \left\{ \frac{i}{2} \int d^4k \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2} \right\}$$

If we then do the inverse fourier transform we now have

$$Z[J] = Z[0] \exp \left\{ \frac{1}{2} \int d^4x d^4y J(x) D(x-y) J(y) \right\}$$

Where $D(x-y) = \int d^4k e^{ik(x-y)} \frac{i}{k^2 - m^2 + i\epsilon}$ is the feynman propagator. Now lets try to compute a two-point function

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = (-i)^2 \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} \left(1 - \frac{1}{2} \int d^4x d^4y J(x) J(y) D(x-y) + \dots \right) \Big|_{J=0}$$

Applying chain rule we just get

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D(x-y)$$

In fact this is a more general result for any free QFT which is

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \sum_{\text{pairings}} (\text{product of free propagators})$$

So for example

$$\langle 0|\phi(x_1)\dots\phi(x_4)|0\rangle = D(x_1-x_2)D(x_3-x_4) + D(x_1-x_3)D(x_2-x_4) + D(x_1-x_4)D(x_2-x_3)$$

This is Wick's theorem. Lets introduce a graphical notation to keep track of things. First recall

$$Z[J] = \exp \left(-\frac{1}{2} \int d^4x d^4y J(x) D(x-y) J(y) \right)$$

When we introduce interactions what we'll end up with is a more complex diagrammatic set of

$$\begin{aligned} \int d^4x J(x) &\Rightarrow \text{---} \bullet^x \text{---} \\ D(x-y) &\rightarrow \text{---} \bullet^x \text{---} \bullet^y \text{---} \\ \Rightarrow Z[J] &= \exp \left(-\frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} \right) \\ \Rightarrow Z &= 1 - \frac{1}{2} \text{---} \bullet \text{---} \bullet \text{---} + \frac{1}{4(2!)} (\text{---} \bullet \text{---} \bullet \text{---})^2 + \dots \end{aligned}$$

Figure 2: Graphical represntation of partition function

rules. e.g. for a ϕ^3 theory we ahve new sets of ingridients.

LECTURE 2: EXAMPLE OF RENORMALIZATION

January 21, 2021

At the very end of the last lecture for the two scalar scattering we found out that the momentum integral suffers from ultraviolet divergence. Divergences like that are all over the place in QFT. So people started to develop a renormalization procedure which deals with different physics couplings. All this machinery however has a nasty flavor of sweeping infinities under the carpet. It wasn't very clear what the hell was going on. Even though it worked perturbatively well for $\lambda\phi^4$ and QED it didn't work for some other theories and doesn't work non-perturbatively. So in the 1950s there was a backlash which led to alternatives like the bootstrap program (apparently this name has something to do with lifting yourself up by your bootstraps?) What really comes to understanding renormalization is by understanding effective field theories. This paradigm came from two sides. First there were some people studying critical phenomena in condensed matter. When you're working near a critical point you have long distance correlations. Pretty soon those people realized the behavior at long distances doesn't really care about short distance details. It doesn't matter what kind of lattice we're dealing with (except for maybe some lattice constant). The knowledge of short distance could be summarized in a few parameters. So what Ken Wilson and friends figured out is that you want to worry about effective long distance theories and ignore short distance behavior. The other approach came from a completely different end: current algebra. People studying low-energy interaction between π -mesons and nucleons noticed that there was partial conservation of axial current (fancy spontaneously broken symmetry). They then wrote currents of this broken symmetry and then Weinberg and company figured out that the best way to understand what goes on with current algebra is use effective field theory for low energy (cut all meson zoo except pion.) And what came the big understanding: what we're doing in real life is an effective field theory. We know particles up to some GeV and we don't know anything above that scale. We don't know what kind of particles might occur at 10 TeV. If we had a full QFT then we'd need to include those particles. Even if we say something silly like "there is nothing beyond the standard model" we still have to still consider quantum gravity which becomes relevant at 10^{-19} GeV. So the only thing we can do is effective field theory, a theory which is only known up to some cutoff. So the best we can do is parametrize our ignorance of what's beyond what we know. And in practice this translates to is cutting off ultraviolet modes by saying "we don't know what happens before this" and just say there is literally nothing. And you emulate this nothing with a UV cutoff. And then we parametrize our ignorance by introducing bare(?) couplings that aren't actually measurable experimentally but whatever gives us the right answer. Fortunately there is only a few parameters we need to adjust. Before we do any examples let's revisit something we should all know. Let's consider the Debye model of solids.

ASIDE: DEBYE MODEL OF SOLIDS

What is relevant to low energies?

- (a) For $|\mathbf{k}| \ll 1/(\text{lattice spacing})$ we know that $\omega(k) = c_s|\mathbf{k}|$ where c_s is the speed of sound.
- (b) Finite volume of \mathbf{k} space. In 3D we have \mathbf{k} modulo inverse lattice times 2π and k belongs to a 3-torus.

So what did Debye do? He approximated $\omega(\mathbf{k}) = c_s|\mathbf{k}|$ exactly and \mathbf{k} -space ball of radius θ/c_s (θ is the Debye temperature?) meaning that

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d^3k}{(2\pi)^3}$$

Up to $|\mathbf{k}| = \theta/c_s$. So θ acts like kind of a cutoff for the condensed matter theory. In QFT we introduce a UV cutoff λ that will shove all the gory details of the UV physics in some redefinition.

BACK TO 1-LOOP CORRECTION IN $\lambda\phi^4$ THEORY

We found that for some diagram

$$F(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} \quad \Delta(x) = m^2 - tx(1-x) > 0$$

We want to evaluate

$$\begin{aligned} I &= \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} \\ &= \frac{2\pi^2}{(2\pi)^4} \int_0^\infty \frac{dk_E k_E^3}{[k_E^2 + \Delta]^2} \end{aligned}$$

Now we set UV cutoff: cut the integral at $|k_E| = \Lambda$ for $\Lambda \gg$ what?.

$$I_{\text{regulated}} = \frac{1}{2\pi^2} \int_0^\Lambda \frac{dk_E k_E^3}{[k_E^2 + \Delta]^2}$$

Let $\nu = k_E^2 + \Delta$ meaning that

$$I_{\text{regulated}} = \frac{1}{8\pi^2} \times \frac{1}{2} \int_\Delta^{\Delta+\Lambda^2} \frac{d\nu(\nu - \Delta)}{\nu^2} = \frac{1}{16\pi^2} \left(\log \nu + \frac{\Delta}{\nu} \right) \Big|_\Delta^{\Delta+\Lambda^2} = \frac{1}{16\pi^2} \left(\log[(\Delta + \Lambda^2)/\Delta] + \Delta/(\Delta + \Lambda^2) - 1 \right)$$

Now we take an assumption that $\Lambda^2 \gg m^2$ or q_{net}^2 and therefoer we can neglect all negative powers of Λ . So we can say

$$\begin{aligned} \log[(\Delta + \Lambda^2)/\Delta] &= \log(\Lambda^2/\Delta) + \frac{\Delta}{\Lambda^2} - \frac{\Delta^2}{2\Lambda^2} + \dots \approx \log(\Lambda^2/\Delta) \\ \frac{\Delta}{\Delta + \Lambda^2} - 1 &\approx -1 \end{aligned}$$

This all gives us

$$I_{\text{regulated}} = \frac{1}{16\pi^2} \left(\log(\Lambda^2/\Delta) - 1 \right) \Rightarrow F(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left(\log(\Lambda^2/\Delta(x)) - 1 \right)$$

To do this integral we just do $\log(\Lambda^2/\Delta(x)) = \log(\Lambda^2/m^2) - \log(\Delta^2/m^2 = 1 - tx(1-x)/^2)$. This means

$$F(t) = \frac{\lambda^2}{32\pi^2} \left[\log(\Lambda^2/m^2) - 1 - J(\lambda/m^2) \right] \quad J(t/m^2) = \int_0^1 dx \log(1 - tx(1-x)/m^2)$$

There are three diagrams at the 1-loop level (s,t,u) shown in Figure 1 meaning that

$$\mathcal{M}_{1\text{-loop}} = F(t) + F(s) + F(u)$$

Which can be calculated with crossing symmetry.

$$\mathcal{M}(s, t, u) = -\lambda + \frac{\lambda^2}{32\pi^2} \left[3\log(\lambda^2/m^2) - 3 - J(t/m^2) - J(u/m^2) - J(s/m^2) \right] + O(\lambda^3)$$

So how do we deal with infinite parameters. Here's an overview of the renormalization procedure. λ in the above formula is the *bare* coupling in the bare lagrangian of the theory. This λ_{bare} is not directly measured by any experiment. We adjust it as needed so that the perturbative amplitudes we calculate fit experimental data. To renormalize our procedure goes like this

- (1) Start by defining physical coupling λ_{phys} in terms of some scattering amplitude. For example $\lambda_{\text{phys}} = -\mathcal{M}_{\text{elastic}}$ at the threshold.
- (2) Use perturbation theory (feynmann graphs?) to calculate $\lambda_{\text{physical}}$ as a power series in λ_{bare} .

$$\lambda_{\text{phys}} = \lambda_{\text{bare}} + A_1 \lambda_b^2 + A_2 \lambda_b^3 + A_3 \lambda_b^4 + \dots$$

Note that A_1, A_2, A_3, \dots depend on $\log \Lambda_{\text{UV}}$. Now formally assume that not only λ_{bare} is small but also $\lambda_{\text{bare}} \times \log \Lambda_{\text{UV}}$ is small so this perturbative theory makes sense. Do formal perturbation theory in λ_{bare} .

- (3) Reverse the power series for λ_{phys}

$$\Rightarrow \lambda_{\text{bare}} = \lambda_{\text{phys}} + B_1 \lambda_{\text{phys}}^1 + B_2 \lambda_{\text{phys}}^3 + \dots$$

This tells us $B_1 = -A_1$ and $B_2 = 2A_1^2 - A^2$ and so on.

$$\Rightarrow \lambda_{\text{bare}} = \lambda_p - A_1 \lambda_p^2 + (2A_1^2 - A_2) \lambda_p^3 + \dots$$

- (4) For any interesting amplitude \mathcal{M} (kinematical params), use feynman graphs to calculate \mathcal{M}

$$\mathcal{M}(\text{kine. para.}) = \lambda_b^n \mathcal{M}_0 + \lambda_b^{n+1} \mathcal{M}_1 + \lambda_b^{n+2} \mathcal{M}_2 + \dots$$

- (5) Re-expand in terms of $\lambda_b = \lambda_{\text{phys}} + B_1 \lambda_{\text{phys}}^2 + B_2 \lambda_{\text{phys}}^3 + \dots$. This means that

$$\begin{aligned} \mathcal{M} &= \mathcal{M}_0 \times \lambda_{\text{phys}}^n (1 + nB_1 \lambda_{\text{phys}} + n(n-1)/2 B_1^2 \lambda_{\text{phys}}^2 + nB_2 \lambda_{\text{phys}}^2 \dots) \\ &\quad + \lambda_{\text{phys}}^{n+1} (1 + (n+1)B_1 \lambda_{\text{phys}} + \dots) \\ &\quad + \lambda_{\text{phys}}^{n+2} (1 + \dots) + \dots \\ &= \lambda_{\text{phys}}^n \mathcal{M}_0 + \lambda_{\text{phys}}^{n+1} [nB_1 \mathcal{M}_0 + \mathcal{M}_1] \\ &\quad + \lambda_{\text{phys}}^2 [(n(n-1)/2 \times B_1^2 + nB_2) \mathcal{M}_0 + (n+1)B_1 \mathcal{M}_1 + \mathcal{M}_2] + \dots \end{aligned}$$

The point of all this is to see that we get a power series in λ_{phys} .

- (6) In the power series in λ_{phys} the dependence on $\log \Lambda_{\text{UV}}$ cancels out from each term. That is B_1, B_2, B_3, \dots depend on $\log \Lambda$. $\mathcal{M}_1, \mathcal{M}_2, \dots$ depend on $\log \Lambda$. But $\mathcal{M} - 1 + nB_1 \mathcal{M}_0$ is independent of $\log \Lambda_{\text{UV}}$ and likewise $\mathcal{M}_2 + (n+1)B_1 \mathcal{M}_1 + (n(n-1)/2 \times B_1^2 + nB_2) \mathcal{M}_0$ is independent of $\log \Lambda$.

And that's the renormalization procedure. Lets look at an example for elastic scattering

1-LOOP CALCULATION FOR ELASTIC SCATTERING

First we define

$$\lambda_{\text{phys}} = -\mathcal{M} @ \text{threshold } t = u = 0 \text{ and } s = 4m^2$$

And thus from EQREFHERE we get

$$\lambda_b = \lambda_{\text{phys}} + \frac{\lambda_b^2}{32\pi^2} \left(3 \log \Lambda^2/m^2 - 3 - 2(J(0) = 0) - J(4) \right)$$

And when we rewrite $\mathcal{M}(s, t, u)$ in terms of λ_{phys} we get

$$\mathcal{M}(s, t, u) = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2} \left(J(4) - J(s/m^2) - J(t/m^2) - J(u/m^2) \right)$$

And that's how the renormalization theory works. Now he focused on λ but in reality there is also a renormalization of mass $m_b \neq m_{\text{phys}}$. Also nowadays they do perturbation theory with counterterm which directly reorganizes perturbation theory in terms of $\lambda_{\text{phys}}, m_{\text{phys}}^2, \dots$ directly at the level of Feynman graphs.

LECTURE 3: UV REGULARIZATION SCHEMES

January 22, 2021

Here are some ways to regularize

- (a) Wilson's hard edge cutoff: Used in condensed matter but not commonly used in particle physics
- (b) Pauli Villars (1949): Commonly used today and one of the older ones
- (c) Covariant Higher Derivatives: Used in supersymmetry.
- (d) Dimensional Regularization: Most commonly used today. This basically is taking the dimension of ST $D = 4 - 2\epsilon$. This will be the main subject of lecture 4 if we don't get to it today.
- (e) Lattice (discrete spacetime): the hardest and most physical cutoff. This is the only cutoff that works nonperturbatively

WILSON'S HARD EDGE CUTOFF

Limit all euclidian momenta to $|k_E| \leq \Lambda$. Example

$$q_2 = q_{\text{net}} - q_1 \Rightarrow |q_{1E} - q_E^{\text{net}}| < \Lambda$$

This means that q_1 is the intersection of two balls in euclidian momentum space. In terms of $k = q_1 + xq_{\text{net}}$. We have

$$|k_E - xq_{\text{net}}| \leq \Lambda \quad \text{and} \quad |k_E - (x-1)q_E^{\text{net}}| \leq \Lambda$$

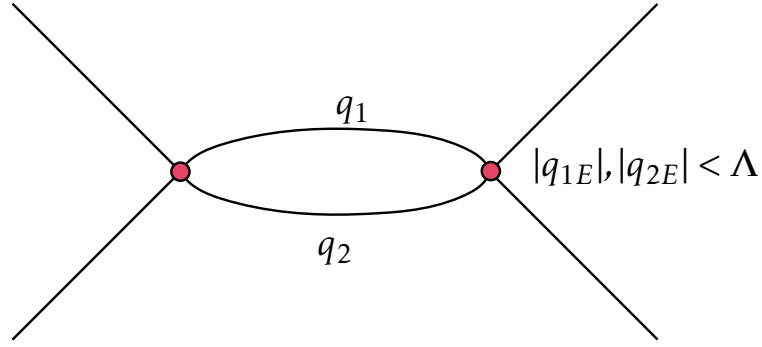


Figure 3: Setup for 1-loop diagram in Wilson's Hard Edge Cutoff

For $|q^{\text{net}}| \ll \Lambda$ we then have

$$|k_E| \leq \Lambda + (\text{direction-dependent } O(q^{\text{net}}))$$

Now for lograithmically divergent integrals

$$\Lambda + O(q_{\text{net}}) \text{ is as good as } \Lambda$$

$$\int_0^{\Lambda + O(q^{\text{net}})} \frac{2k_E^3 dk_E}{(k_E^2 + \Delta)^2} = \log \left(\frac{(\Lambda + O(k_{\text{net}}))^2}{\Delta} \right) - 1 \approx \log \Lambda^2 / \Delta - 1 + O(q_{\text{net}} / \Lambda)$$

But for worse divergences this is no good. For example if

$$\int = (\text{upper limit})^2 = \Lambda^2 + \underline{O(\Lambda q_{\text{net}})}$$

The underlined term is definitely trouble. That's not the only problem with the hard edge cutoff. Here are some other theories

- (a) In gauge theories like QED or QCD hard edge cutoff breaks gauge invariance. Arbitrary phase change can only go up to momentum. In practice breaking gauge invariance means broken ward identites which is trouble.
- (b) Hard edge cutoff changes analytical properties of the amplitudes. Thinking about this kind of stuff was all the rage in the 60's.

The bottom line is that the hard edge cutoff mostly works in perturbation theory but makes no physical sense as a complete non-perturbative theory. That's all we can say about the hard edge cutoff.

PAULI VILLARS REGULARIZATION

In this case loop momenta q^μ are unlimited but the effects of high $q \geq O(\Lambda)$ is **cancelled** by similar-loops of very heavy particles(fields). So we have

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - \Lambda^2 + i0} \right\} \quad q_2 = q_{\text{net}} - q_1$$

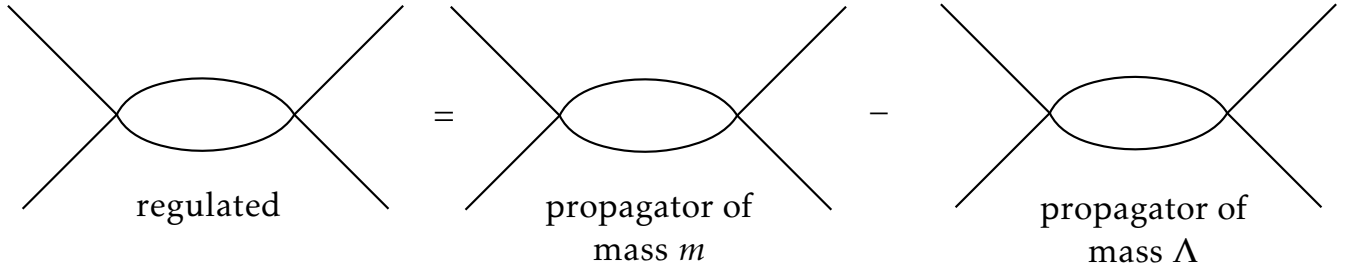


Figure 4: Visual representation of the Pauli Villars regularization scheme

And notice that we subtract before integration. So for $q_1, q_2 \ll \Lambda$

$$\text{First term} \approx \frac{1}{q_2^2}, \text{ Second term} = O(1/\Lambda^4)$$

Note that the second term is much less than the first term so physical first term. However for $q_1, q_2 \gg \Lambda$ the integrand I

$$I \approx \frac{1}{q_2^2} - \frac{1}{(q^2 - \Lambda^2)^2} \approx O(\Lambda^2/q^2)$$

And so the integral converges. Evaluating the integral is basically introducing feynman paramters and thus finding that it behaves like $\log(\Lambda/\Delta)$ plus some constant.

So to summarize Pauli Villars is good for pertubation theory. when $qp_{\text{external}} \ll \Lambda$. But this cannot be extended to a physical theory at all energies. This is because of the propagators in the compensating loops. On one side we have scalar propagators meaning that we need a scalar field of mass Λ . But on the other hand the minus sign of the loop means fermi statistics. Basically we have a scalar fermion which breaks spin-statistics. Spin-statics assuem relativity, positive energy of all particles, and positive norm on hilbert space. What we break is wrong sign of Hilbert space norm for the compensating scalar which is very unphysical. And this is why we cannot just incorporate that scalar in the theory and then just say "it's a physcial theorywith one fat boy." You cannot extend the theory to the energy where that particle can be produced. That would be unphyscial.

COVARIANT HIGHER DERIVATIVES

In this case we have softer propagators at $q > O(\Lambda)$ by adding higher derivatives terms to the lagrangian. For example for a scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 - \frac{1}{2\Lambda^2}(\partial^2 \phi)^2$$

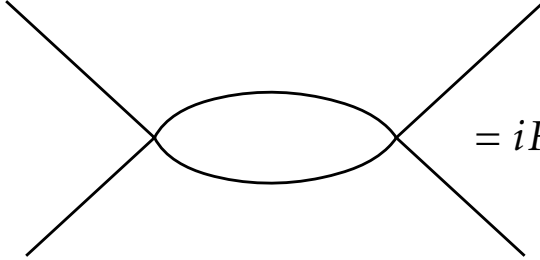
So free ϕ obeys $(\partial^2 + m^2 + \lambda^4/\Lambda^2)\phi = 0$ and that makes the greens function for a propagator

$$G = \frac{i}{q^2 - m^2 - q^4/\Lambda^2 + i\epsilon}$$

And that means for $q^2 \ll \Lambda^2$ the propagator is the usual thing and for large $q \gg \Lambda$ the propagator becomes

$$G \approx -\frac{i\Lambda^2}{q^4} \ll \frac{1}{q^2}$$

Basically this is suppressed But for $q^2 \gg \Lambda^2$ we get $\int \frac{\Lambda^4}{q^8} d^4q$ which is convergent. Bottom line



$$= iF = \frac{\lambda^2}{2} \int \frac{d^4q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 - q_1^4/\Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 - q_2^4/\Lambda^2 + i0}$$

Figure 5: Visual representation of Higher Order Derivative regularization scheme

is that higher derivative regular is good for perturbation theory but does not work as a complete theory for energies about equal to Λ . Trouble with higher derivative regulator: higher derivatives means that ϕ encodes several particles.

$$\frac{i}{q^2 - m^2 - q^4/\Lambda + i0} \approx \frac{i}{q^2 - m^2 + i0} - \frac{i}{q^2 - \Lambda^2 + i0}$$

We will learn soon that a pole in the propagator corresponds to a physical mass of the particle. So 2 scalar particles are encoded in this scalar field. Also for $q^2 = \Lambda^2$ residue of the pole has the wrong sign which causes with particles $m = \Lambda$ has negative Hilbert space norm. A unphysical ghost particle has appeared. This regulation scheme works for some things.

ASIDE: USING PAULI-VILLARS AND HIGHER DERIVATIVE REGULARIZATION SCHEME

We used hard edge to evaluate the loop diagram in lecture. In this section we use PV and HD.

- (a) **Pauli-Villars:** In this regularization scheme we deal with the UV divergence by introducing a super heavy particle. Lets start by examinig the t-channel contribution. From out notes we have

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - \Lambda^2 + i0} \right\}$$

As perscribed in the lecture we can evaluate this integral by introducing Feynman parameters. So first we will consider

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i0) + x(q_2^2 - m^2 + i0)]^2}$$

The term in the square brackets we can expand out and simplify using $q_2 = q_{\text{net}} - q_1$ and mathematica

$$\begin{aligned}
 (1-x)(q_1^2 - m^2 + i0) + x(q_2^2 - m^2 + i0) &= i0 - m^2 + q_1^2 - q_1^2 x + q_2^2 x \\
 &= i0 - m^2 + q_1^2(1-x) + xq_{\text{net}}^2 + xq_1^2 - 2xq_{\text{net}}q_1 \\
 &= i0 - m^2 + xq_{\text{net}}^2 + q_1^2 - 2xq_{\text{net}}q_1 \\
 \text{Complete the square} &= i0 - m^2 + xq_{\text{net}}^2 + (q_1 - xq_{\text{net}})^2 - x^2q_{\text{net}}^2 \\
 &= (q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0
 \end{aligned}$$

Where $\Delta(x)$ is defined implicitly above. So now we have

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2}$$

Meaning that

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} - \int_0^1 \frac{dy}{[(q_1 - yq_{\text{net}})^2 - \Delta(y, \Lambda) + i0]^2} \right\}$$

First let us consider the red term. We can change the order of integration

$$\begin{aligned}
 \int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} &= \int_0^1 dx \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} \\
 \text{Let } k &= q_1 - xq_{\text{net}} = \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta(x, m) + i0]^2}
 \end{aligned}$$

Now focusing on the $d^4 k$ integral. Note in the k_0 integral we have poles when $k_0 = \pm[\sqrt{\mathbf{k}^2 + \Delta(x, y - i0)}$. Lets perform a wick rotation CCW by letting $k_0 = ik_4$ meaning that $k_\mu k^\mu = -k_4^2 - \mathbf{k}^2$. Namely the metric becomes euclidean. So we can define a $k_E = (k_1, k_2, k_3, k_4)$ and get the integral

$$\int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} = i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x, m)]^2}$$

However we can go even further. The integral only depends on k_E^2 so we have a $SO(4)$ symmetry. This means we can change $d^4 k_E = k_E^3 dk_E d\Omega^3$. This is in analogy to the spherical integral we're used to $d^3 x = r^2 dr d\Omega$. In 4 dimensions $\int d\Omega^3 = 2\pi^2$. So now we have

$$i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x, m)]^2} = i \int_0^1 dx \int \frac{k_E^3 dk_E}{(2\pi)^4} \frac{2\pi^2}{[k_E^2 + \Delta(x, m)]^2}$$

We can get a similar result for the blue term. So in all we have

$$F(t) = \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^\infty dk_E \times k_E^3 \left\{ \frac{1}{[k_E^2 + \Delta(x, m)]^2} - \frac{1}{[k_E^2 + \Delta(x, \Lambda)]^2} \right\}$$

Lets evaluate these integrals. First let $u = k_E^2 + \Delta(x, m)$ meaning that $du = 2k_E dk_E$ meaning that $k_E^3 dk_E = du(u - \Delta(x, m))/2$

$$\begin{aligned} \int_0^\infty \frac{k_E^3}{[k_E^2 + \Delta(x, m)]^2} dk_E &= \int_\Delta^\infty \frac{du(u - \Delta(x, m))}{2u^2} \\ &= \frac{1}{2} \left(\ln(u) + \Delta(x, m)u^{-1} \right) \Big|_\Delta^\infty \\ &= \frac{1}{2} (\ln(\infty) - \ln(\Delta(x, m)) - 1) \end{aligned}$$

Now adding the two contributaions gives us

$$\begin{aligned} F(t) &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2} \int_0^1 dx \{ \ln(\infty) - \ln(\Delta(x, m)) - 1 - \ln(\infty) + \ln(\Delta(x, \Lambda)) + 1 \} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2} \int_0^1 dx \left\{ \ln \left(\frac{\Delta(x, \Lambda)}{\Delta(x, m)} \right) \right\} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2} \int_0^1 dx \left\{ \ln \left(\frac{\Lambda^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2} \right) \right\} \end{aligned}$$

Take $\Lambda^2 \gg m^2, q_{\text{net}}^2$ giving us

$$\begin{aligned} &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2} \int_0^1 dx \left\{ \ln \left(\frac{\Lambda^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2} \right) \right\} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2} \int_0^1 dx \left\{ \ln \left(\frac{\Lambda^2}{m^2} \right) - \ln \left(\frac{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2} \right) \right\} \end{aligned}$$

$$\boxed{F(t) = \frac{\lambda^2}{32\pi^2} \times \left\{ \ln(\Lambda^2/m^2) - J(t/m^2) \right\}}$$

- (b) **Higher Derivatives:** In this regularization scheme we deal with UV divergence by introducing higher derivative terms to the lagrangian. This gives us the propagator

$$iF = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 - q_1^4/\Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 - q_2^4/\Lambda^2 + i0} \right\}$$

In the problem set it is suggested we use

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i0} \approx \frac{i}{q^2 - m^2 + i0} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i0}$$

To see where this approximation comes fromes lets multiply the things together. The RHS becomes

$$\text{RHS} = \frac{i(-\Lambda^2)}{-0^2 - 0i\Lambda^2 - 0im^2 + L^2 m^2 + 2iq^2 0 - L^2 q^2 - m^2 q^2 + q^4}$$

Now we'll ignore higher order $i0$ terms and assume that $\Lambda^2 \gg m^2, q^2$ to get

$$\text{RHS} \approx \frac{i}{i0 - m^2 + q^2 - q^4/\Lambda^2} = \text{LHS}$$

Lets plug this prescribed approximation in

$$iF = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} \times \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i0} \right\}$$

The red terms are out of control so lets apply Feynman's very poggers trick to deal with them. From part (a) of this problem we have

$$iF = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 dx \left\{ \frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i0} \times \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i0} \times \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} \right\}$$

The problem again prescribes that we use

$$\frac{-\Lambda^2}{q_1^2 - \Lambda^2 + i0} \approx \frac{-\Lambda^2}{q_2^2 - \Lambda^2 + i0} \approx \frac{-\Lambda^2}{(q_1 - xq_{\text{net}})^2 - \Lambda^2 + i0}$$

First lets try to understand the $(q_1 - xq_{\text{net}})^2$ term. In the limit where $\Lambda^2 \gg q_1^2, q_2^2$ we know that the difference between q_1^2/Λ^2 and q_2^2/Λ^2 is negligible and so interpolating between q_1^2 and q_2^2 using $(q_1 - xq_{\text{net}})^2$ makes sense due to the fact that it's basically a constant as well. So for the low energy limit I kinda see what this trick is doing. Now for the $\Lambda^2 \ll q_1^2, q_2^2$ limit the whole term is $O(1/q^2)$ so I guess q_{net} is negligible so we can interpolate between the q_1 and q_2 values as above without too much trouble. This feels sketchy but we'll go along with it for now. If we plug this interpolation into our above iF we get

$$iF = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 dx \left\{ \frac{\Lambda^4}{[(q_1 - xq_{\text{net}})^2 - \Lambda^2 + i0]^2} \times \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} \right\}$$

Lets try to do the same wick rotation trick we did in the last part of this problem.

$$\begin{aligned} iF &= \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 dx \left\{ \frac{\Lambda^4}{[(q_1 - xq_{\text{net}})^2 - \Lambda^2 + i0]^2} \times \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} \right\} \\ &= \frac{\lambda^2}{2(2\pi)^4} \int_0^1 dx \int d^4 q_1 \left\{ \frac{\Lambda^4}{[(q_1 - xq_{\text{net}})^2 - \Lambda^2 + i0]^2} \times \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x, m) + i0]^2} \right\} \end{aligned}$$

Let $k = q_1 - xq_{\text{net}}$ giving us

$$= \frac{\lambda^2}{2(2\pi)^4} \int_0^1 dx \int d^4 k \left\{ \frac{\Lambda^4}{[k^2 - \Lambda^2 + i0]^2} \times \frac{1}{[k^2 - \Delta(x, m) + i0]^2} \right\}$$

Focusing on the $d^4 k$ integral. We have four poles but we can avoid them in the same way with a Wick rotation $k_0 = ik_4$ which allows us to wrtie

$$iF = i \frac{\lambda^2}{2(2\pi)^4} \int_0^1 dx \int d^4 k_E \left\{ \frac{\Lambda^4}{[k_E^2 + \Lambda^2]^2} \times \frac{1}{[k^2 + \Delta(x, m)]^2} \right\}$$

Using the $SO(4)$ symmetry to write

$$iF = i \frac{\lambda^2}{2(2\pi)^4} \times 2\pi^2 \int_0^1 dx \int_0^\infty dk_E \times k_E^3 \left\{ \frac{\Lambda^4}{[k_E^2 + \Lambda^2]^2} \times \frac{1}{[k_E^2 + \Delta(x, m)]^2} \right\}$$

Now how do we evaluate this? Excellent question! I have no idea. But mathematica does!

$$\begin{aligned}
 \int_0^\infty dk_E \times \dots &= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x, m))^3} \left(\frac{\Delta(x, m)(\Lambda^2 - \Delta(x, m))}{\Delta(x, m) + k_E^2} - \log(k_E^2 + \Lambda^2)(\Delta(x, m) + \Lambda^2) \right. \\
 &\quad \left. + (\Delta(x, m) + \Lambda^2) \log(\Delta(x, m) + k_E^2) + \frac{\Lambda^4 - \Lambda^2 \Delta(x, m)}{k_E^2 + \Lambda^2} \right) \Big|_0^\infty \\
 &= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x, m))^3} \left(0 + (\log(\infty) - \log(\infty))(\Delta(x, m) + \Lambda^2) + 0 \right. \\
 &\quad \left. - \Lambda^2 + \Delta(x, m) + (\Delta(x, m) + \Lambda^2) \log(\Delta(x, m)) - \log(\Lambda^2) - \Lambda^2 + \Delta(x, m) \right) \\
 &= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x, m))^3} \left(-2(\Lambda^2 - \Delta(x, m)) + (\Lambda^2 - \Delta(x, m)) \log(\Lambda^2 / \Delta(x, m)) \right) \\
 &= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x, m))^2} \left(-2 + \log(\Lambda^2 / \Delta(x, m)) \right)
 \end{aligned}$$

When we assert $\Lambda^2 \gg m^2, q_{\text{net}}^2$ we get that

$$\begin{aligned}
 \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x, m))^2} &\approx \frac{\Lambda^4}{2\Lambda^4} = \frac{1}{2} \quad \log\left(\frac{\Lambda^2}{\Delta(x, m)}\right) = \log\left(\frac{\Lambda^2}{m^2}\right) - \log\left(\frac{\Delta(x, m)}{m^2}\right) \\
 \Rightarrow \int_0^1 dk_E \times \dots &= \frac{1}{2} \left(\log(\Lambda^2 / m^2) - 2 - \log(\Delta(x, m) / m^2) \right)
 \end{aligned}$$

Putting this all together gives us

$$F = \frac{\lambda^2}{32\pi^2} \left\{ \log\left(\frac{\Lambda^2}{m^2}\right) - J\left(\frac{t}{m^2}\right) - 2 \right\}$$

OPTICAL THEOREM

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IN QUANTUM MECHANICS

Mostly from L19 and L20 of Zwiebach's course on OCW. First lets review scattering in quantum mechanics. Lets say we want to study a potential. One way we can do this is sending particles in this potential and seeing how they scatter. In general we have a beam of particles going towards a target that get scattered to detectors. Lots of weird things can happen in these scattering experiments. For example

$$p + p \rightarrow p + p + \pi^0$$

Two protons turns into two protons and a pion. There is no notion of a conservation of particles. Recall this was one of our motivations for quantum field theory. However for right now we won't consider changes of particles in the process. Namely our processes will consider processes of the flavor

$$a + b \rightarrow a + b$$

We'll also restrict our discussion to elastic scattering(internal states do not changes.) An example of inelastic scattering is the Frank-Hertz experiment. Here will also assume no spin, non-relativistic, and interactions are of the form $V(\mathbf{r}_1 - \mathbf{r}_2)$ which allows us to analyze this problem like a central force scattering.

So what do we need to solve? First we have the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

and assume we have a energy eigenstate $\psi(\mathbf{r}, t) = \psi(\mathbf{r})e^{-\frac{iEt}{\hbar}} \Rightarrow$ We want to solve the time-independent Schrodinger equation

$$\left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) \right] \psi(\mathbf{r}) = E\psi(\mathbf{r})$$

First we'll consider the potential to have finite volume. We'll assume that outgoing states are plane waves which have energy $E = \frac{\hbar^2 k^2}{2m}$. Putting this onto the RHS of our equation gives us

$$\left[-\frac{\hbar^2}{2m} (\nabla^2 + k^2) + V(\mathbf{r}) \right] \psi(\mathbf{r}) = 0$$

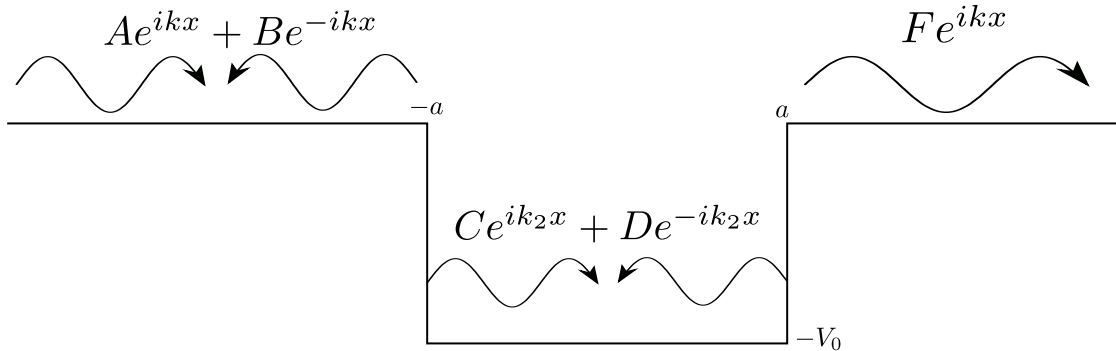


Figure 6: Illustration of a one dimensional potential where we decompose the solution into ingoing and outgoing waves for various regions. Intuition behind scattering

Before we continue lets remember how we dealt with one-dimensional potentials (e.g. Figure 6). We'd be throwing some wave into the potential and then use our intuition to write out a form of the solution in terms of ingoing and outgoing waves. This is what we'll be doing here in scattering. Lets consider the simplest case $V = 0$. Then $\psi = e^{i\mathbf{k}\cdot\mathbf{x}}$ are solutions. This is the same as saying that Ae^{ikx} in Figure 6 is a solution before we ever see the potential. From this lets define an incident wave function $\phi(\mathbf{x}) = e^{ikz}$. We say ϕ instead of ψ since ψ is usually used to denote the full solution whereas ϕ indicates that it might not be the full solution. We know that ϕ is a solution of the S.E. away from $V(r)$. To complete our solution we should expect a spherical outgoing waves. To write this spherical wave we write e^{ikr} . However there is a physical reason that this doesn't work very well. The solution doesn't fall off meaning that we're just accumulating probability as we go radially outwards. To see this from equations just note that $(\nabla^2 + k^2)e^{ikr} \neq 0$. It turns out the one we want is

$$(\nabla^2 + k^2)e^{ikr}/r = 0$$

So lets try to write the scattering solution as

$$\psi_{\text{scattering}}(\mathbf{x}) = \frac{e^{ikr}}{r}$$

But there is no reason that the solution doesn't also depend on θ and ϕ . So lets add a function to our solution

$$\psi_{\text{scattering}}(\mathbf{x}) = f(\theta, \phi) \frac{e^{ikr}}{r}$$

We'll see later that this is the leading term of a solution. So our full solution outside of the potential can be written as

$$\psi(\mathbf{r}) \approx \phi(\mathbf{r}) + \psi_{\text{scattering}}(\mathbf{r}, \theta, \phi) = e^{ikz} + f(\theta, \phi) \frac{e^{ikr}}{r}$$

What we need to figure out is $f(\theta, \phi)$, the scattering amplitude. So here we have finished setting up the problem! We'll build to a computation of $f(\theta, \phi)$ using partial waves and phase shifts. But before we do that we need to talk about cross-sections. Lets say in some solid angle $d\Omega$ that corresponds to the detector we see n particles. we can associate a $d\sigma$ on the volume of the potential that captures n particles. What this means is that we get a idea of how the particle sees the particles.

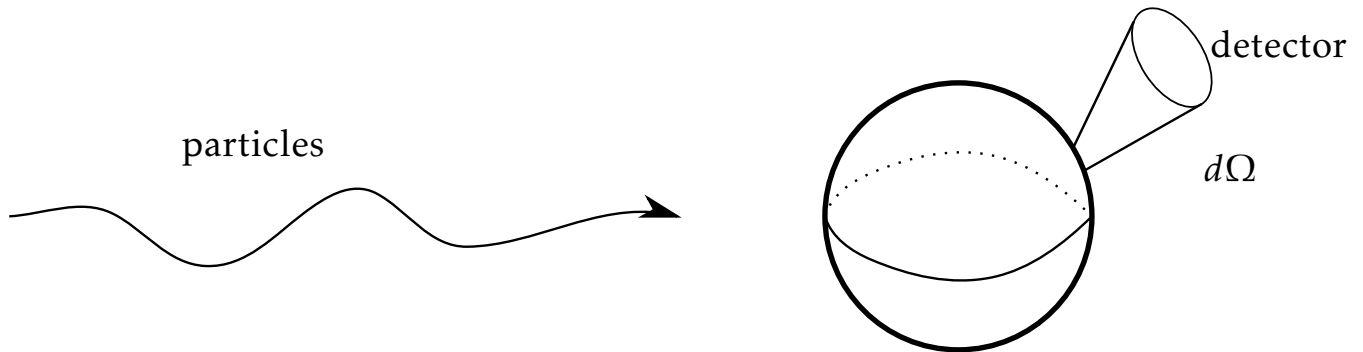


Figure 7: Visualization of scattering cross section.

$$d\sigma = \frac{\text{number of particles scattered per unit time into solid angle } d\Omega \text{ at } (\theta, \phi)}{\text{flux of incident particles}}$$

We can think of incident flux IF (really the probability current) as

$$IF = \frac{\hbar}{m} \text{Im}(\phi(\mathbf{r})^* \nabla \phi(\mathbf{r})) = \frac{\hbar k}{m} \hat{\mathbf{z}}$$

Now lets try to find the number of particles in the small volume dr in Figure 8

$$dn = \text{number of particles in } dr$$

To find this we need to find the square of the wave function in the scattered region $\psi_{\text{scattering}}$. This gives us

$$dn = \left| f(\theta, \phi) \frac{e^{ikr}}{r} \right|^2 r^2 d\Omega dr = |f(\theta, \phi)|^2 d\Omega dr$$

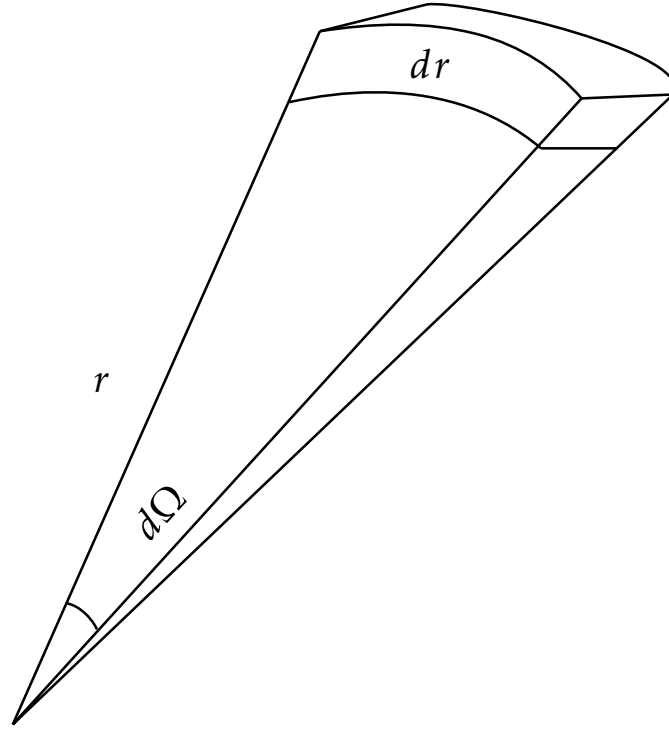


Figure 8: Illustration used in derivation of scattered particle density

Now all these particles will go through the box a infinitesimal $dt = \frac{dr}{v}$ which is

$$dt = \frac{m dr}{\hbar k}$$

Thus we have

$$\frac{dn}{dt} = \frac{\hbar k}{m} |f(\theta, \phi)|^2 d\Omega$$

And finally we get

$$d\sigma = \frac{\frac{\hbar k}{m} |f(\theta, \phi)|^2 d\Omega}{\frac{\hbar k}{m}} = |f(\theta, \phi)|^2 d\Omega$$

Which can be written as

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2}$$

And integrating this we can get a total cross section

$$\boxed{\sigma = \int d\Omega |f(\theta, \phi)|^2}$$

Now lets reduce to a central force. First will we assume that

$$V(\mathbf{r}) = V(r)$$

So we should expect only $f(\theta)$. Now lets consider the solution of the free particle solution in a spherically symmetric potential

$$\psi(\mathbf{r}) = \frac{\mathcal{U}_{E\ell}(r)}{r} Y_{\ell m}(\Omega)$$

And from this we can get the schordinger equation for $\mathcal{U}_{E\ell}$ as

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{\ell(\ell+1)}{r^2} \right) \mathcal{U}_{E\ell}(r) = \frac{\hbar^2 k^2}{2m} \mathcal{U}_{E\ell} \Rightarrow \left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2} \right) \mathcal{U}_{E\ell} = k^2 \mathcal{U}_{E\ell}$$

Now we'll define a new variable $\rho = kr$ which makes the quation simpler

$$\left(-\frac{d^2}{d\rho^2} + \frac{\ell(\ell+1)}{\rho^2} \right) \mathcal{U}_{E\ell} = \mathcal{U}_{E\ell}$$

These are solved by Bessel functions

$$\mathcal{U}_{E\ell} = A_\ell \rho j_\ell(\rho) + B_\ell \rho n_\ell(\rho)$$

Both of these behave nicely at infinity. Now we could be able to write the plane wave solution e^{ikz} as a superposition of our solutions above. To do this first note that the plane wave isn't singular at the origin so we can ignore the spherical Neumann function. Furthermore we have azimuthal symmetry thus we can also ignore the m quantum number. All of this together we get

$$e^{ikz} = \sum_{\ell} a_{\ell} j_{\ell} Y_{\ell,0}$$

Now by nature of the spherical bessel function and the spherical harmonics we can find that

$$e^{ikz} = e^{ikr \cos \theta} = \sqrt{4\pi} \sum_{\ell} \ell \sqrt{2\ell+1} i^{\ell} Y_{\ell,0}(\theta) j_{\ell}(kr)$$

Lets take a closer look at the spherical bessel function. As we go towards infinity we have

$$j_{\ell}(kr) \rightarrow \frac{1}{kr} \sin\left(kr - \frac{\ell\pi}{2}\right) \frac{1}{2ik} \left\{ \frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r} \right\}$$

The terms in the brackets are just ingoing and outgoing waves. This fact will be useful when we start trying to solve this problem. Lets plug this ins

$$e^{ikz} \approx \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} i^{\ell} Y_{\ell,0}(\theta) \frac{1}{2i} \left\{ \frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r} \right\}$$

The "partial wave" refers to the fact that each ℓ term are independent. To understand how this works better lets consider the simplest case. $D = 1$. Lets say we have some hard wall with some potential $V(x)$ before we hit the hard wall. If $V = 0$ we have the incoming $\phi = \sin kx = \frac{1}{2i} (e^{ikx} - e^{-ikx})$. The full solution then

$$\psi(x) = \frac{1}{2i} (\dots - e^{-ikx})$$

So we have the same incoming wave but some undetermined outgoing wave that we must to solve for. What do we know about the outgoing wave? Well the probabiltiy current must be

conserved and must have the same energy. So what can we write? Well we could write in a phase shift

$$\psi(x) = \frac{1}{2i} \left(e^{-ikx+2i\delta_k} - e^{-ikx} \right)$$

Which is valid for $x > a$. We can also assert that

$$\psi(x) = \phi(x) + \psi_{\text{scattered}}(x)$$

Solving for the scattered wave

$$\psi_{\text{scattered}} = \frac{1}{2i} e^{ikx} (e^{2i\delta_k} - 1) = e^{ikx} e^{i\delta_k} \sin \delta_k$$

How do we extend this to three dimensions? Well consider

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} i^{\ell} Y_{\ell,0}(\theta) \frac{1}{2i} \left\{ \frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r} \right\} + f_{\ell}(\theta) \frac{e^{ikr}}{r}$$

Where the hell is the incoming wave in this mess? There's only one and it's the **red** term. We can also write

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} i^{\ell} Y_{\ell,0}(\theta) \frac{1}{2i} \left\{ ??? - \frac{e^{-i(kr-\ell\pi/2)}}{r} \right\}$$

Namely we have some undetermined outgoing wave and the same **incoming wave**. Now we need some inspiration(whisky?) to figure out what this ??? term is. First we know that the probability must be conserved. Furthermore we know each ℓ works independently. So that means that the amplitude of the ??? term has to be the same as the **incoming wave** for each ℓ or else the probability would not be conserved which would be a disaster. So using our intuition from the 1-d case we will write an outgoing wave with some phase shift

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} i^{\ell} Y_{\ell,0}(\theta) \frac{1}{2i} \left\{ \frac{e^{i(kr-\ell\pi/2)+2i\delta_{\ell}}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r} \right\}$$

The δ_{ℓ} parameterizes our ignorance. We can now solve for $f_{\ell}(\theta)$

$$f_{\ell}(\theta) \frac{e^{ikr}}{r} = \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} i^{\ell} Y_{\ell,0} \underbrace{\frac{1}{2i} (e^{2i\delta_{\ell}} - 1)}_{e^{i\delta} \sin \delta} \frac{e^{ikr}}{r} \underbrace{e^{-i\pi\ell/2}}_{(-i)^{\ell}}$$

And after a lot of happy cancellations we get

$$f_{\ell}(\theta) = \frac{\sqrt{4\pi}}{k} \sum_{\ell=0}^{\infty} \sqrt{2\ell+1} Y_{\ell,0}(\theta) e^{i\delta_{\ell}} \sin \delta_{\ell}$$

A lot of simple things come out of this. Recall

$$\sigma = \int |f_{\ell}(\theta)|^2 d\Omega = \int f_{\ell}(\theta)^* f_{\ell}(\theta) d\Omega$$

Now plugging in our formula for $f_\ell(\theta)$ give us

$$\sigma = \frac{4\pi}{k^2} \sum_{\ell, \ell'} \sqrt{2\ell+1} \sqrt{2\ell'+1} e^{-i\delta_\ell} \sin \delta_\ell e^{i\delta_{\ell'}} \sin \delta_{\ell'} \int d\Omega Y_{\ell 0}^*(\Omega) Y_{\ell' 0}(\Omega) = \frac{4\pi}{k^2} \sum (2\ell+1) \sin^2 \delta_\ell \quad (\text{OT}^*)$$

There is one more important result that we should talk about which was the whole point of this song and dance. **Optical Theorem.** Lets say you have some detector surrounding some scattering experiment. The "shadow" of the scattering object equal to the scattered particles that you can detect. In EM the intuition for this shadow should be clear. You shine some light on a object and the "shadow" relates to the photons that are scattered off the surface. However here the intuition of the shadow is not so clear. We have to parts of the solution $\psi(\mathbf{r})$ that contribute to the shadow. So lets see how this works in quantum mechanics. First we need the result that

$$Y_{\ell 0}(\theta = 0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

We can then write

$$f_k(\theta = 0) = \frac{1}{k} \sum_{\ell=0}^{\infty} (2\ell+1) e^{i\delta_\ell} \sin \delta_\ell \Rightarrow \text{Im } f_k(\theta = 0) = \frac{1}{k} \sum_{\ell} (2\ell+1) \sin^2 \delta_\ell$$

But compare this to (OT*). What we see is that

$$\boxed{\sigma = 4\pi \text{Im } f_k(\theta = 0)}$$

IN QUANTUM FIELD THEORY

See Lecture 4.

LECTURE 4: MORE REGULARIZATION AND OPTICAL THEOREM

January 26, 2021

One thing he did not finish is that in gauge theories the higher derivative regularization scheme becomes the **covariant higher derivative regularization scheme**. For example in QED

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \bar{\psi}(i\mathcal{D} + e\mathcal{A} - m)\psi + \frac{1}{4\Lambda^2} (\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu}) + \frac{1}{2\Lambda^2} \bar{\psi}(i\mathcal{D} + e\mathcal{A})^3 \psi$$

If you look at the final term you find that we get a softer propagator

$$\text{three point vertex} = ie\gamma^\nu \left(1 + \mathcal{O}\left(\frac{p^2}{\Lambda^2}\right) \right)$$

For all multi loop graphs the propagator allows for UV regularization. But for one loop graphs and especially the graphs that look like two, three, and four legged spiders we get no regularization with the above lagrangian. So we need extra regulators like Pauli-Villars. So in practice this is used for proving all sorts of theorems. That's it's for higher derivatives.

There is a renormalization scheme that is a perfectly good non-perturbative quantum theory: lattice renormalization. We need to go through the path integrals to really talk about this so Prof. Kaplunovsky will come back to it after we discuss path integrals (after spring break.) For now let's put this aside and talk about dimensional regularization as the UV regular most commonly used (in high energy physics.)

DIMENSIONAL REGULARIZATION

What we want to do is generalize loop integrals to fractional (or even not rational) space time dimensions and then analytically continue back to $D = 4$.

$$\int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} f(k_E^\nu) = \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} f(k_E^\nu)$$

Where μ is the reference energy scale at which the spherical momentum space shell dk_D^{rad} has the same volume in D dimensions as in 4-dimensions. Now why is this regulated? Let's consider the radial integral. We start with

$$d^4 k_E \propto (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

Now in D dimensions we get

$$(k_E^{\text{rad}})^3 dk_E^{\text{rad}} \rightarrow \mu^{4-D} \times (k_E^{\text{rad}})^{D-1} dk_E^{\text{rad}} = \left(\frac{\mu}{k_E^{\text{rad}}} \right)^{4-D} \times (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

The red term is what regularizes things. Let's take a closer look at the red factor. Let's consider $D = 4 - 2\epsilon$

$$\left(\frac{\mu}{k_E^{\text{rad}}} \right)^{4-D=2\epsilon} = \left(\frac{k_E^2}{\mu^2} \right)^{-\epsilon} = \exp\left(-\epsilon \times \log \frac{k_E^2}{\mu^2}\right)$$

The exponential becomes small when the log term becomes proportional to $\frac{1}{\epsilon}$.

Let's consider a generic logarithmically divergent momentum integral. From our other regularization schemes we saw that

$$\text{regulated integral} = C \times \log \frac{\Lambda^2}{m^2} + \dots$$

So for dimensional regularization we expect

$$\text{regulated integral} = C \times \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{m^2} \right) + \dots$$

TODO why do we expect this??? Before we continue let's learn how to take integrals in arbitrary dimensions. Consider

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2)$$

For an integer dimension D we have $k_E^2 = k_1^2 + \dots + k_D^2$. From this we get

$$\exp(-t k_E^2) = \prod_i \exp(-t k_i^2)$$

Using this allows us to rewrite the integral

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2) = \prod_i \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \exp(-t k_i^2) = (4\pi t)^{-\frac{D}{2}}$$

It's just a gaussian integral! Now lets just analytically continue this formula for non-integer D byt saying

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2) = (4\pi t)^{-\frac{D}{2}}$$

So what will we do for non-Gaussian momentum integrals? I would cry.

$$I = \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \int \dots$$

Well first lets recall the gamma function integral

$$\int_0^{\infty} dt t^{n-1} \times \exp(-t(k_E^2 + \Delta)) = \frac{\Gamma(n)}{[k_E^2 + \Delta]^n}$$

For $n = 2$ we have

$$\frac{1}{[k_E^2 + \Delta]^2} = \frac{1}{\Gamma(2) = 1} \times \int_0^{\infty} dt t \times \exp(-t(k_E^2 + \Delta))$$

This gives us

$$\begin{aligned} \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} &= \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} \int_0^{\infty} dt t \times \exp(-t(k_E^2 + \Delta)) \\ &= \int_0^{\infty} dt t e^{-t\Delta} \times \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} e^{-t k_E^2} \end{aligned}$$

Now just using our Gaussian integral we get (μ shouldn't be there?)

$$\int \dots = \int_0^{\infty} dt t e^{-t\Delta} \times (4\pi t)^{-\frac{D}{2}} = (4\pi)^{-\frac{D}{2}} \int_0^{\infty} dt t^{1-\frac{D}{2}} \times e^{-t\Delta}$$

Now as long as the power of $t > -1$ the integral converges. This means for $D < 4$ or we have a complex D , whenever $\text{Re}(E) < 4$. What's the physcial meaning of this divergence? It's just the ultraviolet divergence of the original integral. In particular for $D = 4 - 2\epsilon$ we get

$$\mu^{4-D} \times \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} = \frac{(4\pi\mu^2)^\epsilon}{16\pi^2} \times \Gamma(\epsilon) \times \Delta^{-\epsilon}$$

From this we can extract the matrix element for the t-channel process

$$\mathcal{F}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta(x)} \right)^\epsilon$$

This is actually an exact answer for $D = 4 - 2\epsilon$. Now how do we see that this diverges as $D \rightarrow 4$? Well the gamma function blows up as $\epsilon \rightarrow 0$ due to the poles at $\epsilon = 0$. How do we calculate these poles? Using $\Gamma(x+1) = x\Gamma(x)$ we can consider

$$\begin{aligned}\Gamma(\epsilon) &= \frac{\Gamma(\epsilon+1)}{\epsilon} = \frac{1}{\epsilon} \left(\Gamma(1) + \epsilon \times \Gamma'(1) + \frac{\epsilon^2}{2} \Gamma''(1) + \dots \right) \\ &= \frac{1}{\epsilon} - \gamma_E + \frac{\pi^2 + 6\gamma_E^2}{12} \times \epsilon + O(\epsilon^2)\end{aligned}$$

Where γ_E is the Euler-Mascheroni constant. In this we can clearly see the pole at $\epsilon = 0$. Lets try expanding the term inside the parenthesis in the integral in powers of ϵ

$$\left(\frac{4\pi\mu^2}{\Delta(x)} \right)^\epsilon = \exp \left(\epsilon \times \log \left(\frac{4\pi\mu^2}{\Delta(x)} \right) \right) = 1 + \epsilon \times \log \frac{4\pi\mu^2}{\Delta(x)} + \dots$$

The point of all this is to show that we should be careful about taking $\epsilon \rightarrow 0$. What the hell do we do with this expression? For other regularization schemes the idea was that we toss out all negative powers of Λ . In this case we note that a positive power of ϵ corresponds to a negative power of Λ . This means the integrand over the feynman parameter becomes

$$\Gamma(\epsilon) \times \left(\frac{4\pi\mu^2}{\Delta(x)} \right)^\epsilon \rightarrow \frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)}$$

Giving us

$$\mathcal{F}_{\text{DR}}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx \left(\frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{\Delta(x)} \right) = \frac{\lambda^2}{32\pi^2} \left(\frac{1}{\epsilon} - \gamma_E + \log \frac{4\pi\mu^2}{m^2} - J \left(\frac{t}{m^2} \right) \right)$$

So far all our answers for regularizations have been of a similar form. They can be the exact same however if

$$\log \Lambda_{HE}^2 - 1 = \log \Lambda_{PV}^2 = \log \Lambda_{HD}^2 - 2$$

All regularization schemes are in the same ballpark and related by a numerical constant. We can extend this identification by letting

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi\mu^2) = \log \Lambda_{HE}^2 - 1 = \log \Lambda_{PV}^2 = \log \Lambda_{HD}^2 - 2$$

OPTICAL THEOREM

This theorem relates forward amplitude with total cross section. For example consider scattering forward scattering amplitude for elastic scattering $f_e(\theta = 0)$. We have

$$\text{Im } f_e(\theta = 0) = \frac{k_{\text{reduced}}}{4\pi} \times \sigma_{\text{total}}$$

Now in the partial wave analysis we decompose the the scattering function into eigenstates of angular momentum

$$f(\theta) = \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) P_l(\cos \theta) \times \left(\frac{e^{2i\delta_l} - 1}{2i} = e^{i\delta_l} \sin \delta_l \right)$$

Using this we can do all our gymnastics to get

$$\sigma_{\text{total}} = \frac{4\pi}{k^2} \times \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

Taking the imaginary part of forward scattering gives us

$$\text{Im } f(\theta = 0) = \frac{1}{k} \times \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

Now let's talk about this in a relativistic notation. We rewrite our first statement as

$$\text{Im } \mathcal{M}_{\text{elastic, forward}} = 2E_1 E_2 |\mathbf{v}_1 - \mathbf{v}_2| \times \sigma_{\text{total}}$$

The prefactor is invariant under the boosts along the axis of collision. If we go to the center of mass frame the prefactor evaluates to $4E_{\text{cm}}^{\text{net}} |\mathbf{p}_{\text{cm}}|$. Let's prove this formula today. On Thursday we'll apply this formula to $\lambda\phi^4$ theory. This theorem follows from the unitarity of the S-matrix (or the scattering operator $S^\dagger S = 1$). Let's write S as $S = 1 + iT$. Thus

$$1 = S^\dagger S = 1 + iT - iT + T^\dagger T \Rightarrow iT^\dagger - iT = T^\dagger T$$

Now let's sandwich the above with diagonal matrix elements $\langle i | \dots | i \rangle$. First the LHS

$$\langle i | iT^\dagger - iT | i \rangle = i \langle i | T | i \rangle^* - i \langle i | T | i \rangle = 2\text{Im} \langle i | T | i \rangle$$

Whereas on the RHS we'll resolve the identity

$$\langle i | T^\dagger T | i \rangle = \sum_{|f\rangle} |\langle f | T | i \rangle|^2$$

Now we have the result

$$2\text{Im} \langle i | T | i \rangle = \sum_{|f\rangle} |\langle f | T | i \rangle|^2$$

There's something wrong here though. Let's factor out the energy-momentum conservation which gives us

$$\langle f | T | i \rangle = (2\pi)^4 \delta^{(4)}(p_f - p_i) \times \langle f | M | i \rangle$$

There will be troublesome δ functions of both sides. To resolve this consider a state $|i\rangle$ that is similar to $|i\rangle$ except for the momentum. We'll take $|i'\rangle$ so that we can approximate

$$\langle i' | M | i \rangle = \langle i | M | i \rangle$$

Now we can sandwich our things between $|i'\rangle$ and i . This gives us

$$\langle i' | iT^\dagger - iT | i \rangle = (2\pi)^4 \delta^{(4)}(p_{i'} - p_i) \times 2\text{Im} \langle i | M | i \rangle$$

And on the RHS

$$\langle i' | T^\dagger T | i \rangle = \sum_{|f\rangle} \langle f | T | i' \rangle^* \langle f | T | i \rangle = (2\pi)^4 \delta^{(4)}(p_{i'} - p_i) \times \sum_{|f\rangle} |\langle f | M | i \rangle| \times (2\pi)^4 \delta^{(4)}(p_f - p_i)$$

Which finally gives us the result

$$2\text{Im} \langle i|M|i \rangle = \sum_{\langle f|} |\langle f|M|i \rangle|^2 \times (2\pi)^4 \delta^{(4)}(p_f - p_i)$$

Now let $|i\rangle$ be an initial state with two particles and $|f\rangle$ be all possible final states. the sum sums overall possible reactions

$$\sum_{\langle f|} = \sum_{\text{channels}} \prod_{a=1}^n \left(\sum_{s'_a} \int \frac{d^3 p'_a}{(2\pi)^3 2E'_a} \right)$$

Plugging this into our above result gives us a big boy that is just the phase space integral (with spin sums). This gives us

$$\begin{aligned} 2\text{Im} \langle i|M|i \rangle &= \sum_{\text{channels}} 4E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{net}}(1+2 \rightarrow 1' + \dots + n') \\ &= 4E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1+2 \rightarrow \text{anything}) \end{aligned}$$

And from this we have our optical theorem

$$\text{Im} \mathcal{M}_{\text{elastic,forward}} = 2E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1+2 \rightarrow \text{anything})$$

LECTURE 5: APPLYING OPTICAL THEOREM AND BEGINNING CORRELATION FUNCTIONS

January 28, 2021

Lets see how optical theorem work in $\lambda\phi^4$ theory.

APPLYING OPTICAL THEOREM TO $\lambda\phi^4$ THEORY

At the tree level

$$\frac{d\sigma_{\text{elastic}}}{d\Omega_{cm}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s}$$

Integrating we can get the net elastic scattering cross section

$$\sigma_{\text{net}}^{\text{elastic}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s} \times \frac{4\pi}{2} = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

For inelastic processes $2 \rightarrow n$ where $n \geq 4$ we find that the tree amplitude is $O(\lambda^{\frac{n}{2}})$ and thus $\mathcal{M} = O(\lambda^{\frac{n}{2}})$. This means that the total scattering cross section

$$\sigma_{\text{total}} = \sigma_{\text{elastic}}^{\text{net}} + O(\lambda^4) = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

Now applying optical theorme to this we get

$$\text{Im } \mathcal{M}(\text{elastic,forward}) = 2E_1 E_2 |v_{12}^{\text{rel}}| \times \frac{\lambda^2 + O(\lambda^3)}{32\pi s} = \frac{\lambda^2 v}{32\pi} + O(\lambda^3)$$

From this we can see that

$$\text{Im}\mathcal{M}_{\text{tree}}(\dots) = 0 \quad \text{Im}\mathcal{M}_{1\text{-loop}}(\dots) = \frac{\lambda^2 v}{32\pi} > 0$$

Aside: You can apply exactly the same analysis to QED. Take some tree level process. Amplitude is of order E^2 and so the cross section is of order α^2 and by optical theorem if we take an amplitude in the forward direction with unchanged spin then the tree level amplitude must be real but the 1-loop amplitude has an imaginary part.

Lets verify the optical theorem. At tree level

$$\mathcal{M}_{\text{tree}}^{\text{elastic}} = -\lambda, \quad \text{Im}\mathcal{M}_{\text{tree}}^{\text{elastic}} = 0$$

We'll see soon that

$$\forall \theta: \quad \text{Im}\mathcal{M}_{1\text{-loop}}^{\text{elastic}} = \frac{\lambda^2 v}{32\pi} \quad v = \frac{|p|}{E} = \sqrt{1 - \frac{m^2}{E^2}} = \text{sqrt}1 - \frac{4m^2}{s}$$

In an earlier lecture we hsoed that

$$\mathcal{M}_{1\text{-loop}}^{\text{elastic}} = \frac{\lambda^2}{32\pi^2} \left(J(4) - J(t/m^2) - J(u/m^2) - J(s/m^2) \right)$$

Recalling the form of the J function we note that the **green** term is imaginary. Or more specifically for $s > 4m^2$ the argument of the logarithm becomes negative at some values of x . Now recall from complex analysis

$$\log(z) = \log(|z|) + i \arg(z)$$

some complex analysis here

We then find that

$$\text{Im}J\left(\frac{s}{m^2}\right) = \int_0^1 dx \text{Im} \log \frac{m^2 - sx(1-x)}{m^2} = \pm\pi \times (x_2 - x_1) = \pm\pi v$$

The sign convention comes from the branch cut and how we choose to analytically continue. In the end we get

$$\text{Im}J\left((s + i\epsilon/m^2)\right) = \mp\pi v$$

Plugging this into our $\mathcal{M}_{1\text{-loop}}^{\text{elastic}}$ we get

$$\text{Im}\mathcal{M}_{1\text{-loop}}^{\text{elastic}}(s \pm i\epsilon, t) = \pm \frac{\lambda^2 v}{32\pi}$$

Now comparing with our result from optical theorem we need to choose $s + i\epsilon$. It turns out that there is another way to extract the imaginary part of \mathcal{M} . A guy named Kolkovol(?) created a diagrammatic way to extract these imaginary parts.

INTRODUCTION TO CORRELATION FUNCTIONS

We define the n -point correlation function

$$\mathcal{F}_n(x_1, \dots, x_n) = \langle \Omega | T \Phi_H(x_1) \dots \Phi_H(x_n) | \Omega \rangle$$

Or for vector or tensor fields or something

$$\mathcal{F}_n^{a_1 \dots a_n}(\dots) = \langle \Omega | T \Phi_H^{a_1}(x_1) \dots \Phi_H^{a_n}(x_n) | \Omega \rangle$$

Last semester we learned how to calculate correlation functions of interacting theory by using perturbation theory and free correlation function. To figure out how we relate the correlation function of an interacting theory with the correlation function we defined above we first consider

$$\Phi_H(\mathbf{x}, t) = e^{iHt} \Phi_S(\mathbf{x}) e^{-iHt} = e^{iHt} e^{-iH_0 t} \Phi_I e^{iH_0 t} e^{-iHt}$$

We can define a unitary operator

$$U_I(t, t_0) = e^{iH_0 t} e^{-iH(t-t_0)} e^{-iH_0 t_0}$$

Using this we can get

$$\begin{aligned} U_I(t, 0) &= e^{iH_0 t} e^{-iHt} \quad \text{and} \quad U_I(0, t) = e^{iHt} e^{-iH_0 t} \\ \Rightarrow \Phi_H(x) &= U_I(0, x^0) \Phi_I U_I(x^0, 0) \end{aligned}$$

Now let's consider two fields

$$\Phi_H(x) \Phi_H(y) = U_I(0, x^0) \Phi_I(x) U_I(x^0, y^0) \Phi_I(y) U_I(y^0, 0)$$

The above expression is true because $U_I(x^0, 0) U_I(0, y^0) = U_I(x^0, y^0)$. From here we can generalize this to

$$\Phi_H(x_1) \Phi_H(x_2) \dots \Phi_H(x_n) = U_I(0, x_1^0) \Phi_I(x_1) U_I(x_1^0, x_2^0) \Phi_I(x_2) \dots$$

Now we need to relate the free vacuum to a true vacuum $|\Omega\rangle$. To do this we consider the state $U_I(0, -T)|0\rangle$ for a complete T . Taking the limit $T \rightarrow (1 - i\epsilon) \times \infty$. Basically the real part grows to infinity faster than the imaginary part grows to $-\infty$. Let's now consider (where $H_0|0\rangle = 0$)

$$U_I(0, -T)|0\rangle = e^{-iHT} e^{iH_0 T} |0\rangle = e^{-iHT} |0\rangle$$

We can expand the vacuum state in terms of eigenstates of H denoted by $|Q\rangle$

$$|0\rangle = \sum_Q |Q\rangle \times \langle Q|0\rangle \Rightarrow e^{-iHT} |0\rangle = \sum_Q |Q\rangle \times e^{-iTE_Q} \langle Q|0\rangle$$

What happens is that we can pick out the lowest energy state because of how we defined T . What this means is that $U_I(0, -T)|0\rangle$ becomes $|\Omega\rangle \times e^{-iTE_\Omega} \langle \Omega|0\rangle$. And thus

$$|\Omega\rangle = \lim_{T \rightarrow (1-i\epsilon)\infty} U_I(0, -T)|0\rangle \times e^{iTE_\Omega} / \langle \Omega|0\rangle$$

In a similar way we can find with the same $T \rightarrow \dots$ trick

$$\langle \Omega | = \lim_{T \rightarrow (1-i\epsilon)\infty} \frac{e^{iTE_\Omega}}{\langle 0 | \Omega \rangle} \times \langle 0 | U_I(+T, 0)$$

Now using these two results we can find that big thing inside the big correlation function becomes

$$\begin{aligned} \langle \Omega | \dots | \Omega \rangle &= C(T) \langle 0 | \dots | 0 \rangle \\ \dots &= U_I(T, 0) U_I(0, x^0) \Phi_I(x) U_I(x^0, y^0) \dots \end{aligned}$$

Time time ordering this gives us

$$= T \left(\Phi_I(x) \Phi_I(y) \times \exp \left(-\frac{i\lambda}{24} \int_{-T}^T dt \int d^3\mathbf{z} \phi_I^4(t, \mathbf{z}) \right) \right)$$

Which follows from a Dyson series expansion of the evolution operator. This gives

$$\begin{aligned} \mathcal{F}_2(x, y) &= \langle \Omega | T \Phi_H(x) \Phi_H(y) | \Omega \rangle \\ &= \lim_{T \rightarrow (1-i\epsilon)\infty} C(T) \langle 0 | \dots | 0 \rangle \end{aligned}$$

So far we have x is later than y . But what if x is earlier than y ? Well we reverse some things around.

Now what about the n -point function

$$\begin{aligned} \mathcal{F}_n(x_1, \dots, x_n) &= \langle \Omega | T \Phi \dots | \Omega \rangle \\ &= \lim_{T \rightarrow (1-i\epsilon)\infty} C(T) \times \left\langle 0 | T \left(\Phi_I(x_1) \dots \Phi_I(x_n) \times \exp \left\{ \frac{-i\lambda}{24} \int d^4z \Phi_I^4(z) \right\} \right) | 0 \right\rangle \end{aligned}$$

The coefficient $C(T)$ is the same for all correlation function for the same n . What happens if $n = 0$? Then $\mathcal{F}_0 = \langle \Omega | \Omega \rangle = 1$ meaning that

$$\lim_{T \rightarrow \dots} C(T) \times \langle 0 | T(\exp \dots) | 0 \rangle = 1$$

Then we can eliminate the $C(T)$ factors

$$\mathcal{F}_n(x_1 \dots) = \lim_T \frac{\langle 0 | \dots | 0 \rangle}{\langle 0 | T(\exp \dots) | 0 \rangle}$$

We can use this mess for perturbation theory!

(26)

Using this perturbation theory we can write Feynman diagram for n -point correlation functions.

Feynman rules for correlation function here

Just like for scattering amplitudes we can factorize things into vacuum bubble diagrams And other. The ratio we took above is what cancels out the vacuum bubbles.

$$\langle 0 | T(\exp \dots) \rangle = \sum (\text{vacuum bubbles w/o external vertices})$$

What's good about killing these vacuum bubbles? Well the leading divergence cancels out when we do this. So we can set $T \rightarrow (1 + i0)\infty$ diagram by diagram. In practice this means when we integrate over a vertex instead of integrating from $-T$ to T we integrate from $-\infty$ to ∞ . This is important since then we can go to momentum space with fourier transform and then get momentum space feynman rules.

Momentum Space Feynman Rules here

When we did scattering amplitudes we learned that we only needed to consider connected diagrams (disconnected meant basically no scattering.) For correlation functions like define **connected correlation functions**

$$\mathcal{F}_n^{\text{connected}} = \sum \dots$$

We can use this to get the original \mathcal{F}_n through (block expansions(?)) the following

$$\begin{aligned}\mathcal{F}_2(x, y) &= \mathcal{F}_2^{\text{conn}}(x, y) \\ \mathcal{F}_4(x, y) &= \mathcal{F}_4^{\text{conn}}(x, y, z, w) + \mathcal{F}_2^{\text{conn}}(x, y) \times \mathcal{F}_2^{\text{conn}}(z, w) \\ &\quad + \mathcal{F}_2^{\text{conn}}(x, z) \times \mathcal{F}_2^{\text{conn}}(y, w) + \mathcal{F}_2^{\text{conn}}(x, w) \times \mathcal{F}_2^{\text{conn}}(y, z) \\ &\dots\end{aligned}$$

Connected 4-point, 6-point, and so on correlation functions are related to the scattering amplitude with the **LSZ reduction formula**. Next time we'll go more in depth into 2-point correlation functions which are important for renormalization and knowing what the hell is going on.

LECTURE 6: THE 2-POINT CORRELATION FUNCTION

Last lecture we learned about correlation functions for quantum fields and ended up with connected correlation function that can be defined recursively. Today we're going to focus on the 2-point function. To do this we're going to ignore Feynman diagrams and worry about the actual physics. Consider $\mathcal{F}_2(x - y)$ where $x^0 > y^0$. We have

$$\mathcal{F}_2(x - y) = \langle \Omega | T \Phi_H(x) \Phi_H(y) | \Omega \rangle = \sum_{|\Psi\rangle} \langle \Omega | \Phi_H(x) | \Psi \rangle \times \langle \Psi | \Phi_H(y) | \Omega \rangle$$

The Ψ is all quantum states of the theory which can occur from the action of the field $\phi(y)$. For example for QED you could have a photon, or three photon, or electron positron pairs. these pairs could be free and flying away or they could be bound like in hydrogen. Different types of states will have different parameters. But all states will have a net momentum. In addition we might have some other quantum states where we could just consider relative momentum. So for convenience we'll denote $|\Psi\rangle = |\psi, p^\mu\rangle$. With this we can define

$$\mathcal{F}_2(x - y) = \sum_{\psi} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{1}{2E(\mathbf{p}, M(\psi))} \times \langle \Omega | \Phi_H(x) | \psi, p \rangle \times \langle \psi, p | \Phi_H(y) | \Omega \rangle$$

Lets try to take out the x dependence. To do this we use translational symmetry in all 4-dimensions which means that

$$\Phi_H(x) = \exp(ix_\mu P^\mu) \Phi_H \exp(-ix_\mu P^\mu)$$

If we take the vacuum to have energy 0 then

$$\langle \Omega | \exp(ix_\mu P^\mu) = \langle \Omega | \exp(-ix_\mu P^\mu) | \psi, p \rangle = e^{-ix_\mu p^\mu} \times | \psi, p \rangle$$

Asserting all of this we can get

$$\langle \Omega | \Phi_H(x) | \psi, p \rangle \times \langle \psi, p | \Omega_H(y) | \Omega \rangle = e^{-ip(x-y)} \times |\langle \psi, p | \Phi_H(0) | \Omega \rangle|^2$$

Now since the state $\Phi_H(0) | \Omega \rangle$ is invariant under orthochronous Lorentz symmetry we have that the matrix element is the same for all \mathbf{p} on the mass shell. From all this we can get

$$\mathcal{F}_2(x-y) = \sum_{\psi} |\langle \psi | \Phi_H(0) | \Omega \rangle|^2 \times D(x-y; M(\psi))$$

What happens if $x^0 < y^0$? If we work through this carefully we just see that x and y are exchanged. To combine everything we just use the Feynman propagator.

$$\mathcal{F}_2(x-y) = \sum_{\psi} |\langle \psi | \Phi_H(0) | \Omega \rangle|^2 \times G_F(x-y, M(\psi))$$

The above equation can be rewritten more generally by redefining the one-point function as a measure. This equation is usually called the **Kallen-Lehmann spectral representation** (TODO Kallen has some fancy dots in their name)

$$\mathcal{F}_2(x-y) = \int_0^\infty \frac{dm^2}{2\pi} \rho(m^2) \times \int \frac{d^4 p}{(2\pi)^4} \frac{i e^{-ip(x-y)}}{p^2 - m^2 + i0}$$

$$\rho(m^2) = \sum_{\psi} |\langle \psi | \Phi_H(0) | \Omega \rangle|^2 \times (2\pi \delta(M^2(\psi) - m^2))$$

The second function is the *spectral density function*. Don't get distracted by the fancy math though. Lets talk about some features about this spectral density.

(a) Real and non-negative.

(b) In free field theory $\rho(m^2) = 2\pi \delta(m^2 - M^2)$. This is because for a free theory the only thing you can make out of a vacuum with a field is a single particle.

The spectral density becomes more interesting when we consider an interacting theory. For example for $\lambda\phi^4$ theory we can make three particles and then can make a continuous spectrum. A more general kind of theory can have even funkier spectral densities (see Kaplunovsky's lecture notes page 13 for some examples.)

Lets move on from this spectral representation to its consequences on the two point function. The two point function only depends on x and y so lets do a fourier transform

$$\mathcal{F}_2(p) = \int_0^\infty \frac{dm^2}{2\pi} \rho(m^2) \times \frac{i}{p^2 - m^2 + i0}$$

How do features of the spectral density function transform into analytically features of $\mathcal{F}_2(p)$? In general one particle states will give delta spike contribution while multiparticle states give a continuum.

$$\rho(m^2) = Z \times 2\pi\delta(m^2 - M_{\text{particle}}^2) + \text{smooth continuum}$$

The amplitude of the delta function will have amplitude

$$Z = |\langle 1\text{particle} | \Phi_H(0) | \Omega \rangle|^2 > 0$$

\sqrt{Z} is the strength with which that the quantum field creates the single particle state. With all this we have

$$\mathcal{F}_2(p^2) = \frac{iZ}{p^2 - M_{\text{particle}}^2 + i\epsilon} + \text{smooth}(p^2)$$

An implication of this we'll find is that a pole in the two point function is at physical particle masses. Why do we care about all of this? We know that the physical and bare coupling are different by perturbation theory. This implies that the physical mass and the bare mass are different in perturbation theory. We can find the physical mass in perturbation theory by looking at pole masses. We shall see Tuesday that

$$M_{\text{pole}}^2 = m_{\text{bare}}^2 + \text{loop correction} = f(m_{\text{bare}}^2, \lambda_b, \Lambda_{UV})$$

We can then identify $M_{\text{pole}}^2 = M_{\text{particle}}^2$ to solve equations. Something also to note is that the continuum creates a branch cut running from the threshold to ∞ . We can see this from

$$\mathcal{F}_2(p^2 + i0) - \mathcal{F}_2(p^2 - i0) = \frac{1}{2\pi i} \int_0^\infty \frac{dm^2 \rho(m^2)}{m^2 - p^2 - i0} - \dots = \frac{1}{2\pi i} \oint_{\text{around } p^2} \frac{dm^2 \rho(m^2)}{m^2 - p^2} = \rho(p^2)$$

This means that there is a discontinuity across the real axis meaning that there is a branch cut. First let's look at the real axis of this Riemann surface. For spacelike p^2 , the two point function becomes

$$i\mathcal{F}_2(p^2) = \int_0^\infty \frac{dm^2}{2\pi} \frac{\rho(m^2)}{m^2 - p^2}$$

This is well behaved. Now let's go timelike p^2 and stay below the threshold of the branch cut our $i\mathcal{F}_2$ is still fine and good. However once we reach the threshold we get a singularity which leads to a branch cut

$$(\text{above the branchcut}) \quad i\mathcal{F}_2(p^2 \pm i0) = R \pm i \frac{\rho(p^2)}{2}$$

So the question is if we want to evaluate the two point function above the branch cut threshold which \pm sign do we take? The answer is we take $+i0$. This is because we want the thing to look like the Feynman propagator (oh?). The physical value is on the top side of the branch cut. This is something to remember: **the physical side of the branch cut is the top side.**

The Riemann surface of the 2point function has an infinite number of Riemann sheets but only one physical sheet. So which one is the physical sheet? According to the notes the physical sheet begins on the upper side of the branch cut, extends CCW to negative real axis and then back to positive axis. TODO HUH? On the physical sheet the integral can be taken literally and all real

poles correspond to physical particle states. We can have poles off-axis on the unphysical sheet. These poles off-axis actually correspond to resonances. Lets be careful in how we define these unphysical poles. First define on the upper side of the branch cut

$$i\mathcal{F}_2(p^2 + i0)$$

Once we have this lets analytically continue to complex p^2 . If we go up in the complex plane $\text{Im}(p^2) > 0$. If we go below the branch cut we're on the unphysical sheet. On this unphysical sheet we might hit a pole. For an example lets say we have a pole at $p^2 = M^2 - iM\Gamma$. Suppose this pole is close the real axis meaning that $\Gamma \ll 1$. Then when we stay on the real axis, this pole dominates over the known pole contributions. So for real p^2 in the vicinity of M^2 we have

$$\mathcal{F}(p^2) = \frac{iZ}{p^2 - M^2 + iM\Gamma} + \text{smooth}(p^2)$$

This is the **Breit-Wigner resonance**. By optical theorem we know that Γ is the total decay rate of the unstable particle (resonances are unstable particles.) $\frac{1}{\Gamma}$ is the average lifetime of the particle.

Now lets turn back to perturbation theory. At the tree level we have the free feynman propagator. AT one loop we start multiplying by other factors. And it just keeps going on. So when you naively calculate \mathcal{F}_2 with diagrams, you just get a lot of poles

$$\left(\frac{i}{p^2 - m_b^2 + i0} \right)^{\text{power} \geq 2} \times \text{others}$$

This is unphysical, we should only have simple poles. The perturbation theory, instead of shifting tree-level poles to physical mass, just creates a bunch of higher order poles. To resolve this we can re-sum the perturbation expansion so higher order poles add up to a simple shifted pole. E.g.

$$\sum_{n=0}^{\infty} \left(\frac{i}{p^2 - m_b^2 + i0} \right)^{n+1} \times (-i\Delta)^n = \frac{i}{p^2 - m_b^2 + i0} \times \frac{1}{1 - \frac{\Delta}{p^2 - m_b^2 + i0}} = \frac{i}{p^2 - (m_b^2 + \Delta) + i0}$$

To see this graphically consider an arbitrary multi-loop feynman graph for two-point correlation function. We have a line with a bunch of (irreducible one-particle) subgraphs. Irreducible means that if you cut any one propagator then it stays connected. Technically is one-propagator irreducible but people call it one-particle. We'll skip the graph theory proof but all contributing feynmann graph that contribute to the 2-point correlator look like that line picture we should have in our head. So the formal sum over all such diagrams with N bubbles (then we'll have $N + 1$ propagators) looks like

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0} \right)^{N+1} \times \prod_{i=1}^N [1PI \text{ bubble } \#i]$$

We can reorganize this so that

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0} \right)^{N+1} \times \{\text{sum over bubbles}\}^N$$

Now we reframe our perturbation theory as summing over bubble diagrams to some order to calculate the physical mass in perturbation theory

$$\mathcal{F}_2(p^2) = \frac{i}{p^2 - m_b^2 + i[\text{sum over bubbles}] + i0}$$

LECTURE 7: COMPUTING TWO-POINT FUNCTIONS AND YUKAWA THEORY

February 02, 2021

On friday we spent most of our time on analytical properties of the two point function, discussing the meaning of its poles and branch cuts. IN particular we learned that the position of the pole is the physical mass of the particles. Furthermore the residue of the poles is the strength with which the quantum field is created from the vacuum. Today we'll talk more about two point functions

The first thing we notice is that if we add things order by order in perturbation theory you notice that we get higher order poles exactly at the bare mass. However higher order poles are unphysical. The only way to shift the pole is to resum a infinite number of diagrams. Ideally all of them but in practice a well defined subset of diagrams. So lets see what's going on. We did this on friday really quickly so lets do it slowly today. The basic idea is we look at the most general diagram contributing to the one-point function. They all have this form (equation (66) in notes). A bunch of propagators connecting one-particle irreducible (1PI) subgraph (if you cut a propagator in the subgraph the graph remains connected.) So we're summing over a bunch of bubbles

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0} \right)^{N+1} \times \prod_{i=1}^N [1PI \text{ bubble } \#i]$$

What all of this gives us is

$$\mathcal{F}_2(p^2) = \frac{i}{p^2 - m_b^2 - \sigma(p^2) + i0}$$

The thing that we forgot to address is what do we do in practice. The idea is we sum to a certain loop order. Once we have down that we go back to that arrangement (equation (69)) and we say instead of summing all such diagrams up to L loops altogether we sum up to L loops for each bubbles. Then we sum over arbitrary number of bubbles. We are summing over any number of bubbles and for each number of bubbles we go up to L loops. That's the partial resummation and how we'll reorganize our summation. That's the right way to do perturbation theory for the two point function.

Ok we get the red expression, what do we do with it? We know that the physical mass is where the pole is. This means that

$$p^2 - m_b^2 - \Sigma(p^2) = 0 \Rightarrow m_b^2 = M^2 - \Sigma(p^2 = M^2)$$

We need to solve for this bare mass parameter of the perturbation theory in order to get physically correct result. That was the bare mass versus the physical mass. What about the field

strengths factor Z . We can get that from the residue of the pole as well. Let $p^2 = M^2 + \delta p^2$. Expand $\Sigma(p^2)$

$$\Sigma(p^2) = \Sigma(M^2) + \delta p^2 \times \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2} + \dots$$

All of this gives us a new form of the denominator

$$p^2 - m_b^2 - \Sigma(p^2) = M^2 + \delta p^2 - m_b^2 - \Sigma(M^2) - \delta p^2 \times \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2} + \dots$$

We can make some cancellations if we're considering a pole giving us

$$p^2 - m_b^2 - \Sigma(p^2) = \delta p^2 \left(1 - \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2} \dots \right)$$

What happens when we invert this? Well we get

$$\mathcal{F}_2(p^2 = M^2 + \delta p^2) = \frac{i}{p^2} \times \left(1 - \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2} + \dots \right)^{-1} = \frac{i}{\delta p^2} \times \frac{1}{1 - (d\Sigma/dp^2)A + \text{finite}}$$

All together we get that near the pole

$$mcF_2(p^2 \text{ near } M^2) = \frac{iZ}{p^2 - M^2 + i0} + \text{finite} \Rightarrow Z = \frac{1}{1 - \left. \frac{d\Sigma}{dp^2} \right|_{p^2=M^2}}$$

And that is how we calculate the field strength renormalization in perturbation theory

EXAMPLE 1: (1-LOOP EXAMPLE FOR FIELD STRENGTH RENORMALIZATION) $-i\Sigma(p^2)$ is the sum of all 1PI graphs with two external legs. Lets go up to L loops. We have

$$\Sigma(p^2) = \lambda \times f_1(p^2) + \lambda^2 \times f_2(p^2) + \dots + \lambda^L f_L(p^2) + \dots$$

Lets start with the one loop order. We only have one graph.

$$(86) \text{ TODO}$$

Evaluating the above graph gives us

$$-i\Sigma_{1\text{-loop}}(p^2) = \text{TODO}$$

And thus to one-loop order we have a mass shift of

$$m_b^2 = M^2 - \Sigma$$

However we have no strength renormalization

$$\frac{d\Sigma_{1\text{-loop}}}{dp^2} = 0 \Rightarrow Z_{1\text{-loop}} = 1$$

Quick aside: The two loop diagrams are given in (91) of the lecture notes. The calculation of these graphs is on the next homework. Some hints. First we use the Feynmann

parameter trick to get everything to a signal denominator. Then we have a denominator that depends on a linear combination of terms quadratic in the propagators. We can then diagonalize this polynomial for some constants. Something to note here is that we need to keep unit jacobian. The next step is to take the derivative before we take an integral and express it in a specific term. This way will make things less divergent. And then we do what we learned in class. Wick rotate momenta, dimensional regularization. At the end of the day we get a huge bunch of shit(it's very long.) After all this we verify some Feynman parameter integrals aren't don't give divergences. And as an optional exercise we actually calculate the relevant integrals for $p^2 = M^2$ using mathematica. Finally we assemble to field strength renormalization.

Enough about the HW lets go back to the 1-loop case. We have the momenta integral

$$\int \frac{d^4 q}{(2\pi)^4} \frac{i}{q^2 - m_b^2 + i0}$$

It diverges quadratically. Lets wick rotate to Euclidean momentum space $d^4 q \rightarrow i d^4 q_E$. This means

$$\frac{i}{q^2 - m_b^2 + i0} = \frac{i}{-q_E^2 - m_b^2} = \frac{-i}{q_E^2 + m_b^2}$$

Once we're in this form we no longer need $i0$. All of this gives us

$$\Sigma_{1\text{-loop}} = \frac{\lambda}{2} \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2 + m_b^2}$$

This integral diverges quadratically. Lets use Wilson's hard edge cutoff instead of dimensional regularization. We now have

$$\int_{\text{reg}} \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2 + m_b^2} = \int_0^\Lambda d q_E \times q_E^2 \frac{1}{q_E^2 + m_b^2} \times \int \frac{d^3 \Omega(q_E^\mu)}{(2\pi)^4}$$

After doing our usual tricks we get

$$\Sigma_{1\text{-loop}} = \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m_b^2 \times \log \left(\frac{\Lambda^2}{m_b^2} \right) \right)$$

Anyhow what does this equation give us? Well according to (77) in Kaplunovsky's notes that's how the mass shifts

$$M_{\text{phys}}^2 = m_b^2 + \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m_b^2 \times \log \frac{\Lambda^2}{m_b^2} \right) + O(\lambda^2)$$

That is an example of mass shift calculation.

And now we have arrived at the fine-tuning problem. Lets say we want to put UV cutoff way above mass. Formally the prefactor $\frac{\lambda}{32\pi^2}$ is infinitesimal but in reality it's still significant. And

if we put Λ^2 100 times the mass, we have a decent chance of overshooting. Take the Higgs particle. It has $M = 125$ GeV and self-coupling $\lambda \approx 0.25$. So the UV cutoff just above the LHC reach at 10 TeV we end up with a one-loop correction to the Higgs field of > 200 GeV. If we continue the standard model to the GUT scale of 10^{16} GeV and $\lambda_b = 0.3$ then the quantum correction to the particle mass would be about 10^{29} GeV². So we've gotta fine tune m_b to 25 significant digits. Moreover if we go to two loops we get $(\lambda/32\pi^2)^{\text{loops}}\Lambda^2$ so we would need to adjust the fine tuned m_b^2 order by order up to about ten-loop order. All of this is to say that fine tuning the bare mass feels kinda yucky. Lots of people have tried all kinds of ways to avoid this problem. Supersymmetry is one of them. Unfortunately supersymmetric standard model is on life support because of the LHC. Composite higgs also doesn't seem to work. By the way people in condensed matter people also have fine tuning problem. Ok lets move on.

Lets address a technical issue. A regulator dependce on divergences. For logarithmic divergences we saw in HW1 that all wehave to do is rescale λ a little bit for each regularization so that they give same cross section . Unfortunately this doesn't work for quadratic divergences. Recall we got

$$\Sigma = \frac{\lambda}{32\pi^2} \left(\Lambda_{\text{HE}}^2 - m_b^2 \times \log \frac{\Lambda_{\text{HE}}^2}{m_b^2} \right)$$

If we plug in some constant C we get

$$\Sigma = \frac{\lambda}{32\pi^2} \left(C \times \Lambda_{\text{HE}}^2 - m_b^2 \times \log \frac{\Lambda_{\text{HE}}^2}{m_b^2} - m_b^2 \log(C) \right)$$

But if we use another cutoff we'd have

$$\Sigma = \frac{\lambda}{32\pi^2} \left(a\Lambda^2 - m_b^2 \times \log \frac{\Lambda^2}{m_b^2} - b \times m_b^2 \right)$$

So not only do we have to adjust things order by order but we also we need to be very careful in specyng the details of the regulator. Now that's the general problem. Dimensional regularization has a more particular problem. The **green** term is just zero in dimensional regularization terms. You might think is very fortunate, we don't need to fine tune. But all of this is BS. All of this regulator is just a cover-up for the unknown UV behavior. Regulators are just intermediate step. But generally we'd expect **the green** term to be non-zero and we'd have to worry about fine-tuning. Well, lets take a closer look at dimensional regularization.

EXAMPLE 2: (DIMENSIONAL REGULARIZATION AND MASS SHIFT) Consider

$$\Sigma(D) = \frac{\lambda}{2} \times \int \frac{\mu^{4-d} d^D q_D}{(2\pi)^D} \frac{1}{q_E^2 + m_b^2}$$

This diverges for $D \geq 2$ so we need low dimension $D < 2$. We can relate the numerator to a gaussian integral

$$\frac{1}{q_E^2 + m_b^2} = \int_0^\infty dt \exp\{-t(q_E^2 + m_b^2)\}$$

All of this gives us

$$\Sigma(D) = \frac{\lambda}{2} \mu^{4-D} \int \frac{d^D q_E}{(2\pi)^D} \int_0^\infty (\text{gaussian integral}) = \frac{\lambda}{2} \mu^{4-D} \int_0^\infty dt \exp\{-tm_b^2\} \times (4\pi t)^{-\frac{D}{2}}$$

If we analytically continue the Σ to $D < 2$ dimensions then the t integral evaluates to a gamma function (equation (105) in notes.) *missed some stuff here but what we get is that UV divergences is worse than logarithmic.*

Now let's analytically continue the mass shift (105) to $D = 4 - 2\epsilon$. This gives us

$$\Sigma = \frac{\lambda}{32\pi^2} (4\pi\mu^2)^\epsilon (m_b^2)^{1-\epsilon} \Gamma(\epsilon - 1)$$

We can rewrite the gamma function as

$$\Gamma(\epsilon - 1) = \frac{\Gamma(\epsilon)}{\epsilon - 1} = \frac{-1}{1 - \epsilon} \times \left(\frac{1}{\epsilon} - \gamma_E + O(\epsilon) \right) = - \left(\frac{1}{\epsilon} - \gamma_E + 1 + O(\epsilon) \right)$$

The other term in Σ also becomes

$$\left(\frac{4\pi\mu^2}{m_b^2} \right)^\epsilon = \exp \left\{ \epsilon \times \log \frac{4\pi\mu^2}{m_b^2} \right\} = 1 + \epsilon \times \log \frac{4\pi\mu^2}{m_b^2} + \dots$$

So all of this gives us

$$\Gamma(\epsilon - 1) \times (\dots)^\epsilon = - \left(\frac{1}{\epsilon} - \gamma_E + 1 + \log \frac{4\pi\mu^2}{m_b^2} + \dots \right)$$

Which altogether gives us

$$\Sigma = - \frac{\lambda m_b^2}{32\pi^2} \times \left(\frac{1}{\epsilon} - \gamma_E + 1 + \log \frac{4\pi\mu^2}{m_b^2} + \dots \right)$$

Which can be restated in terms of an effective Λ_{eff}

$$\Sigma = - \frac{\lambda m_b^2}{32\pi^2} \times \left(\log \frac{\Lambda_{\text{eff}}^2}{m_b^2} + \text{const} \right)$$

Note that what we completely lose here is the leading Λ^2 term. The dimensional regularization gave us a hint that there is a pole at $D = 2$ that we should watch out for, that log divergence isn't the whole story. However dimensional regularization didn't tell us shit about what's actually going on. what a dick :(. Basically when we use dimensional regularization we should be careful about quadratic divergences.

Now let's talk about something else **Yukawa Theory**.

EXAMPLE 3: (YUKAWA THEORY FOR NONUNITY FIELD STRENGTH REGULARIZATION) We have the lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial - m_f)\Psi + \frac{1}{2}(\partial_\mu\Phi)^2 - \frac{1}{2}m_s^2\Phi^2 + g\Phi\bar{\Psi}\Psi$$

Lets focus on the scalar two point correlation function and calculate it to one loop order.

$$\mathcal{F}_\phi(p^2) = \frac{i}{p^2 - m_s^2 - \Sigma_\phi(p^2) + i0}$$

At the one loop level there is only one 1PI graph contirbuting to Σ_ϕ . (114) which evlautes to

$$-i\Sigma_\phi^{1\text{-loop}}(p^2) = - \int \frac{d^4q_1}{(2\pi)^4} \text{Tr} \left(\frac{i}{q_1 - m_f + i0} (-ig) \frac{i}{q_2 - m_f + i0} (-ig) \right)$$

To calculate this we'll start by alcualting the trace

$$\text{Trace} = g^2 \frac{\text{tr}((q_1 + m_f)(q_2 + m_f))}{(q_1^2 - m_f^2 + i0)(q_2^2 - m_f^2 + i0)} = g^2 \frac{4q_1q_1 + 4m_f^2}{\dots}$$

Now we use feynman paramter trick to bring the deonomitaor here to the form

$$\frac{1}{\text{denomiantor}} = \int_0^1 \frac{d\xi}{\mathcal{D}^2}$$

We have

$$\mathcal{D} = (1 - \xi) \times (q_1^2 - m_f^2 + i0) + \xi \times (q_2^2 - m_f^2 + i0)$$

After some lots of algebra we can get this down to

$$\mathcal{D} = k^2 - \Delta(\xi) + i0 \quad k = q_1 + \xi p \text{ and } \Delta(\xi) = m_f^2 - \xi(1 - \xi)p^2$$

this also allows us to write

$$q_1 = k - \xi p \quad q_2 = k + (1 - \xi)p \Rightarrow 4(q_1q_2) + 4m_f^2 = 4k^2 + 4\Delta(\xi) + 4(1 - 2\xi)(kp)$$

Plugging everything back in we get

$$\Sigma_\phi(p^2) = -4ig^2 \int \frac{d^4q_1}{(2\pi)^4} \int_0^1 d\xi \frac{k^2 + \Delta + (1 - \xi)(kp)}{[k^2 - \Delta + i0]^2}$$

Next we're going to switch order of integration. For a convergent integral this is not a problem. For a divergent we can only do this if we proprly implement the regulator. E.g. for hard edge we have a complicated lens shaped domain of integration. What we're going to do is use the dimensional regularization.

$$\Sigma_\phi(p^2) = -4ig^2 \int_0^1 d\xi \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + \Delta + (1 - 2\xi)(kp)}{[k^2 - \Delta + i0]^2}$$

Now let's send $k \rightarrow -k$. What happens? Well the **blue** term vanishes since it's an odd function and if we integrate over a symmetric region the integral vanishes. So we're left with

$$\Sigma_\Phi = -4ig^2 \int_0^1 d\xi \int \frac{d^4k}{(2\pi)^4} \frac{k^2 + \Delta}{[k^2 - \Delta + i0]^2}$$

The trick that we learned today is to **separate even and odd parts of integral to get rid of the odd ones**. Now we do our usual wick rotation to euclidean momentum space

$$d^4k \rightarrow id^4k_E \quad (k^2 + \Delta) \rightarrow (\Delta - k_E^2) \quad (k^2 - \Delta + i0) \rightarrow -(\Delta - k_E^2)$$

This gives us

$$\Sigma_\Phi = 4g^2 \int_0^1 d\xi \int \frac{d^4k_E}{(2\pi)^4} \frac{\Delta - k_E^2}{(\Delta - k_E^2)^2}$$

This integral diverges. Let's try to make this less divergent by taking the derivative

$$\frac{d\Sigma_\Phi}{dp^2}$$

The integral depends on the scalar momentum only through $\Delta = m_f^2 - \xi(1 - \xi)p^2$. This means that

$$\frac{\partial(\text{integrand})}{\partial p^2} = -\xi(1 - \xi) \partial(\text{integrand})/\partial \Delta$$

So we get

$$\frac{d\Sigma_\Phi}{dp^2} = (130) \quad \frac{d^2\Sigma_\Phi}{d(p^2)^2} = (131)$$

And after doing some derivative magic in (133) we get

$$\frac{d\Sigma_\Phi}{dp^2} = -4g^2 \int_0^1 d\xi \xi(1 - \xi) \times \int \frac{d^4k_E}{(2\pi)^4} \frac{3k_E^2 - \Delta}{(\Delta - k_E^2)^3}$$

This is only logarithmically divergent instead of quadratically divergent like it's daddy without a derivative. Now let's take another derivative which gives

$$\frac{d^2\Sigma_\Phi}{d(p^2)^2} = 4g^2 \int_0^1 d\xi \xi^2(1 - \xi)^2 \times \int \frac{d^4k_E}{(2\pi)^4} \frac{2\Delta - 10k_E^2}{(\Delta - k_E^2)^4}$$

This is a finite function of p^2 . What this tells us is that the first derivative is the sum of some divergent constant plus a finite function of p^2 . This in turn tells us that

$$\Sigma_\Phi(p^2) = \text{divergent constant 1} + \text{divergent constant 2} \times p^2 + \text{finite function}(p^2)$$

We also know that

$$\text{divergent constant 1} = O(\Lambda^2) \text{ and divergent constant 2} = O(\log \Lambda^2)$$

It turns out that is a completely general behavior for any scalar field in any renormalizable theory to any loop order (that's a really general behavior OWO.) We'll see this again in $\lambda\phi^4$ theory when we calculate things to two loop order.

We'll finish this example next time.

CALCULATING FIELD STRENGTH RENORMALIZATION TO 2-LOOP LEVEL FOR $\lambda\phi^4$ THEORY

Based on problemset

(a) For this problem we have

$$\frac{(-i\lambda)^2}{3!} \iint \frac{dq_1^\mu}{(2\pi)^4} \frac{dq_2^\mu}{(2\pi)^4} \frac{i}{q_1^2 - m_b^2 + i0} \frac{i}{q_2^2 - m_b^2 + i0} \frac{i}{(p - q_1 - q_2)^2 - m_b^2 + i0}$$

Where the $(-i\lambda)^2$ terms comes from the coupling constant and the $3!$ comes from the interchange of the three propagator. Recall the Feynman parameter trick we proved a while ago

$$\frac{1}{ABC} = \int_0^1 dx \int_0^{1-x} \frac{2dy}{[xA + yB + (1-x-y)C]^3} = \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \frac{2}{[xA + yB + zC]^3}$$

Using this we can write

$$\begin{aligned} & \text{(Product of three propagators)} = \\ & = \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \frac{2}{[x(q_1^2 - m_b^2 + i0) + y(q_2^2 - m_b^2 + i0) + z\{(p - q_1 - q_2)^2 - m_b^2 + i0\}]^3} \\ & = \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \frac{2}{[q_1^2 x + q_2^2 y + (q_1 + q_2 - p)^2 z + (i0 - m_b^2)(x + y + z)]^3} \end{aligned}$$

From the dirac delta function we know $x + y + z = 1$ allowing up to simplify a little bit

$$\left(\begin{array}{c} \text{Product of} \\ \text{Three Propagators} \end{array} \right) = \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \frac{2}{[q_1^2 x + q_2^2 y + (q_1 + q_2 - p)^2 z + (i0 - m_b^2)]^3}$$

(b) Now we want to change variables so that the cubed term becomes

$$\alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m^2 + i0$$

$$k_1 = q_1 + c_{11} \times q_2 + c_{12} \times p \quad k_2 = q_2 + c_2 \times p$$

To achieve this lets expand everything explicitly

$$i0 - m_b^2 + p^2 z - 2pq_1 z - 2pq_2 z + q_1^2 x + q_2^2 y + q_1^2 z + q_2^2 z + 2q_1 q_2 z$$

To try to get things to meet in the middle lets try explicitly expanding out the red term.

$$\begin{aligned} \left(\begin{array}{c} \text{Red} \\ \text{Term} \end{array} \right) &= \alpha c_{12}^2 p^2 + \beta c_2^2 p^2 + 2\alpha c_{12} p q_1 + 2\alpha c_{11} c_{12} p q_2 \\ &+ 2\beta c_2 p q_2 + \alpha c_{11}^2 q_2^2 + 2\alpha c_{11} q_1 q_2 + i0 - m_b^2 + \gamma p^2 + \alpha q_1^2 + \beta q_2^2 \end{aligned}$$

On comparing the red term with the blue term we see that

$$\begin{aligned}\alpha c_{12}^2 + \beta c_2^2 + \gamma &= z \\ 2\alpha c_{12} &= -2z \\ 2\alpha c_{11}c_{12} + 2\beta c_2 &= -2z \\ \alpha c_{11}^2 + \beta &= y + z \\ 2\alpha c_{11} &= 2z \\ \alpha &= x + z\end{aligned}$$

Lets let mathematica do algebra. Hading this system to mathematica gives us

$$\begin{aligned}\alpha &= x + z & \beta &= y + \frac{xz}{x+z} & \gamma &= \frac{xyz}{yz+x(y+z)} \\ c_{11} &= \frac{z}{x+z} & c_{12} &= -\frac{z}{x+z} & c_2 &= -\frac{xz}{yz+x(y+z)}\end{aligned}$$

So we have

$$k_1 = q_1 + \frac{z}{x+z} \times q_2 - \frac{z}{x+z} \times p \quad k_2 = q_2 - \frac{xz}{yz+xy+xz} \times p$$

Which in turn gives us

$$(x+z) \times k_1^2 + \left(y + \frac{xz}{x+z}\right) \times k_2^2 + \frac{xyz}{yz+xy+xz} \times p^2 - m^2 + i0$$

Now notice that the jacobian

$$\frac{\partial k_i}{\partial q_j} = \begin{bmatrix} 1 & \frac{z}{x+z} \\ 0 & 1 \end{bmatrix} \Rightarrow \det\left(\frac{\partial k_i}{\partial q_j}\right) = 1$$

Therefore we don't need to multiply by some constant in the integral when we change variables. Our integral in total then becomes

$$\Sigma(p^2) = -\frac{\lambda^2}{3} \times \frac{1}{(2\pi)^8} \iint dk_1 dk_2 \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \frac{1}{[\alpha \times k_1^2 + \beta \times k_2^2 + \gamma \times p^2 - m_b^2 + i0]^3}$$

Lets let \mathcal{D} equal the term we are cubing in the denominator

(c) Lets take the derivative

$$\frac{d}{d(p^2)} \frac{1}{[\mathcal{D}]^3} = -\frac{3\gamma}{[\mathcal{D}]^4}$$

This gives us

$$\frac{d\Sigma(p^2)}{d(p^2)} = \frac{\lambda^2}{(2\pi)^8} \iint dk_1 dk_2 \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \gamma \times \frac{1}{[\mathcal{D}]^4}$$

- (d) To evaluate the momentum integral lets wick rotate to the euclidean momentum space $k_i^0 = ik_i^4$. This gives us

$$\frac{d\Sigma(p^2)}{d(p^2)} = -\lambda^2 \iint \frac{dk_{1E}}{(2\pi)^4} \frac{dk_{2E}}{(2\pi)^4} \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \gamma \times \frac{1}{[\alpha \times k_{1E}^2 + \beta \times k_{2E}^2 - \gamma \times p^2 + m_b^2]^4}$$

Now we'll use dimensional regularization to take care of the momentum integral

$$\text{Red Stuff} = \iint \frac{\mu_1^{4-D} \mu_2^{4-D} d^D k_{1E} d^D k_{2E}}{(2\pi)^{2D}} \frac{1}{[\alpha \times k_{1E}^2 + \beta \times k_{2E}^2 - \gamma \times p^2 + m_b^2]^4}$$

Now recall the Gamma function integral

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$$

We can use this to our advantage. Consider some function f which doesn't depend on t . We can write another integral

$$\int_0^\infty t^{n-1} \times \exp\{-tf\} dt$$

We can do a change of variables $u = tf \Rightarrow du = f dt$ giving us

$$\frac{1}{f^n} \int_0^\infty u^{n-1} \times \exp(-u) du = \frac{\Gamma(n)}{f^n} = \int_0^\infty t^{n-1} \times \exp\{-tf\} dt$$

This means we can write when $n = 4$

$$\frac{1}{\Gamma(4) = 3! = 6} \times \int_0^\infty dt t^3 \times \exp\left\{-t(\alpha \times k_{1E}^2 + \beta \times k_{2E}^2 - \gamma \times p^2 + m_b^2)\right\} = \frac{1}{[\alpha \times k_{1E}^2 + \beta \times k_{2E}^2 - \gamma \times p^2 + m_b^2]^4}$$

Which means that

$$\begin{aligned} \text{Red Stuff} &= \iint \frac{\mu_1^{4-D} \mu_2^{4-D} d^D k_{1E} d^D k_{2E}}{(2\pi)^{2D}} \times \frac{1}{6} \int_0^\infty dt t^3 \times \exp\left\{-t(\alpha \times k_{1E}^2 + \beta \times k_{2E}^2 - \gamma \times p^2 + m_b^2)\right\} \\ &= \frac{\mu_1^{4-D} \mu_2^{4-D}}{6} \int_0^\infty dt t^3 \exp\left\{-t(m_b^2 - \gamma p^2)\right\} \times \int \frac{d^D k_{1E}}{(2\pi)^D} \exp\left\{-t\alpha k_{1E}^2\right\} \times \int \frac{d^D k_{2E}}{(2\pi)^D} \exp\left\{-t\beta k_{2E}^2\right\} \end{aligned}$$

the blue and green terms are just gaussian integrals that are analytically continued to arbitrary dimensions. We can see that

$$\text{Blue Stuff} = (4\pi\alpha t)^{-\frac{D}{2}} \quad \text{Green Stuff} = (4\pi\beta t)^{-\frac{D}{2}}$$

All together we get

$$\text{Red Stuff} = \frac{(4\pi)^{-D} (\alpha\beta)^{-D/2} \mu_1^{4-D} \mu_2^{4-D}}{6} \int_0^\infty dt t^{3-D} \exp\left\{-t(m_b^2 - \gamma p^2)\right\}$$

Now let $D = 4 - \epsilon$ giving us

$$\text{Red Stuff} = \frac{(4\pi)^{-4+\epsilon}(\alpha\beta)^{-2+\epsilon/2}\mu_1^\epsilon\mu_2^\epsilon}{6} \int_0^\infty dt t^{-1+\epsilon} \exp\{-t(m_b^2 - \gamma p^2 = f)\}$$

Inside the integral we let $u = tf \Rightarrow du = f dt$ giving us

$$\text{Red Stuff} = \frac{(4\pi)^{-4+\epsilon}(\alpha\beta)^{-2+\epsilon/2}\mu_1^\epsilon\mu_2^\epsilon}{6(m_b^2 - \gamma p^2)^\epsilon} \int_0^\infty dt u^{-1+\epsilon} \exp\{-u\} = \left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2} \right)^\epsilon \times \frac{(\alpha\beta)^{-2}}{6(4\pi)^4} \times \Gamma(\epsilon)$$

Now lets expand the gamma function and the parenthesized term

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \dots \quad \underbrace{\exp\left\{\epsilon \times \log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right)\right\}}_{\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right)^\epsilon} = 1 + \epsilon \times \log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right) + \dots$$

Throwing out positive powers of ϵ leaves us with

$$\text{Purple Stuff} = \frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu_1\mu_2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right)$$

Note that μ is some reference energy scale to maintain momentum in phase space. Thus $\mu_1 = \mu_2 = \mu$. We have finally finished evaluating the momentum integral after taking $\epsilon \rightarrow 0$

$$\text{Momentum Integral} = \frac{1}{6(4\pi)^4(\alpha\beta)^2} \times \left(\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{4\pi\mu^2\sqrt{\alpha\beta}}{m_b^2 - \gamma p^2}\right) \right)$$

(e) It's just pluggin in (d) to (c)

(f) Lets make sure that the Feynmann parameter integrals converge. The δ function asserts that $x + y + z = 1$ meaning that we're just looking at the section of a plane in a octant of \mathbb{R}^3 . So just a triangle. On examining the given form of the integral

$$\iiint_{x,y,z \geq 0} dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3}$$

We see that everything seems kosher except for the vertices. However using mathematica we find that in the limit $x \rightarrow 0$ and $y \rightarrow 0$ and $z \rightarrow 1$ the integrand just vanishes so we won't have a divergence there. By symmetry the same thing happens for the other two corners. Similarly for

$$\iiint_{x,y,z \geq 0} dx dy dz \delta(x + y + z - 1) \times \frac{xyz}{(xy + xz + yz)^3} \times \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz(p^2/m^2))^2}$$

We have the same sketchy points at the vertices. We'll evaluate things on-shell meaning $p^2/m^2 = 1$. Doing a similar analysis with mathematica I think we're fine. Pardon the hand-wavyness of this part of the problem

(g) From our notes we know that the field strength renormalization is calculated as

$$Z = \left(1 - \frac{d\Sigma}{dp^2} \Big|_{p^2=m^2} \right)^{-1}$$

So at last we need to calculate the whole integral. Plugging the **red stuff** back in we get

$$\frac{d\Sigma}{dp^2} = -\lambda^2 \times \frac{1}{6(4\pi)^4} \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \frac{\gamma}{(\alpha\beta)^2} \times \left(\frac{1}{\epsilon} - \gamma_E + \log \left(\frac{4\pi\mu^2 \sqrt{\alpha\beta}}{m_b^2 - \gamma p^2} \right) \right)$$

First lets rewrite the log term to be nicer

$$\log \left(\frac{4\pi\mu^2 \sqrt{\alpha\beta}}{m_b^2(1 - p^2/m_b^2\gamma)} \right) = \log \left(\frac{4\pi\mu^2}{m_b^2} \right) + \log \left(\frac{\sqrt{\alpha\beta}}{(1 - p^2/m_b^2\gamma)} \right)$$

Now we can note that

$$\frac{\alpha\beta}{(1 - p^2/m_b^2\gamma)^2} = \frac{(yz + xy + xz)^3}{(yz + xy + xz - xyzp^2/m^2)^2}$$

Giving us

$$= \log \left(\frac{4\pi\mu^2}{m_b^2} \right) + \frac{1}{2} \log \left(\frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz(p^2/m^2))^2} \right)$$

Something else to note is that

$$\frac{\gamma}{(\alpha\beta)^2} = \frac{xyz}{(yz + xy + xz)^3}$$

Plugging stuff back in gives us (and also solves part (e))

$$\begin{aligned} \frac{d\Sigma}{dp^2} = & -\lambda^2 \times \frac{1}{6(4\pi)^4} \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \\ & \times \frac{xyz}{(xy + xz + yz)^3} \times \left(\frac{1}{\epsilon} - \gamma_E + \log \left(\frac{4\pi\mu^2}{m_b^2} \right) + \frac{1}{2} \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz(p^2/m^2))^2} \right) \end{aligned}$$

Now how in the fucking world do we evaluate this? I have no clue. But we're given in the problem set that

$$\begin{aligned} & \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \frac{xyz}{(xy + xz + yz)^3} = \frac{1}{2} \\ & \iiint_{x,y,z \geq 0} dx dy dz \delta(x+y+z-1) \times \frac{xyz}{(xy + xz + yz)^3} \times \log \frac{(xy + xz + yz)^3}{(xy + xz + yz - xyz)^2} = -\frac{3}{4} \end{aligned}$$

Note that the second integral is evaluated on shell. So now we get

$$\frac{d\Sigma}{dp^2} \Big|_{p^2=m_b^2} = -\frac{\lambda^2}{3072\pi^4} \left(\frac{1}{\epsilon} - \gamma_E + \log \left(\frac{4\pi\mu^2}{m_b^2} \right) - \frac{3}{4} \right)$$

Now plugging this into Z and expanding for $\lambda \ll 1$ we get

$$Z \approx 1 + \frac{\lambda^2}{6144\pi^4} \left(\frac{1}{\epsilon} - \gamma_E + \log \left(\frac{4\pi\mu^2}{m_b^2} \right) - \frac{3}{4} \right)$$

LECTURE 8: FINISHING UP YUKAWA AND STARTING COUNTERTERMS

February 04, 2021

Whoever is late is late >:3c. Last time we talked about what goes in perturbation theory for the two-point function. Then he showed us how in $\lambda\phi^4$ theory there is a mass shift for 1-loop but there is no field strength renormalization until we get to two loop. Lets get back to the example we were doing last time

EXAMPLE 4: (YUKAWA THEORY CONTINUED) Lets get back to the computations. We'll use dimensional regularization to evaluate $\Sigma_\Phi(p^2)$. The integrand

$$\frac{\Delta - k_E^2}{(k_E^2 + \Delta)^2} = \frac{2\Delta}{(k_E^2 + \Delta)^2} - \frac{1}{(k_E^2 + \Delta)} = \int_0^\infty dt (2\Delta \times t - 1) \times e^{-t(k_E^2 + \Delta)}$$

This means that the dimensionally regulated integral becoemes

$$\mu^{4-D} \int \frac{d^D k_E}{(2\pi)^D} \frac{\Delta - k_E^2}{(k_E^2 + \Delta)^2} = \mu^{4-D} (4\pi)^{-\frac{D}{2}} \int_0^\infty dt e^{-t\Delta} (2\Delta t^{-1-(D/2)} - t^{-(D/2)})$$

As $t \rightarrow 0$ then the integral diverges. So to evaluate this integral we analytically this integral some dimension $D < 2$ and then once we evalaute things we analytically continue back. Well for $D < 2$

$$\text{Integral} = 2\Delta \times \Gamma(2 - D/2) \times \Delta^{(D/2)-2} - \Gamma(1 - D/2) \times \Delta^{(D/2)-1}$$

All of this simplifies to

$$\text{Integral} = \Delta^{(D/2)-1} \times \frac{2D-2}{D-2}$$

This has poles at at $D = 2$. Getting everyone back together gives us

$$\Sigma_\Phi = 4g^2 \mu^{4-D} (4\pi)^{-D/2} \Gamma\left(2 - \frac{D}{2}\right) \frac{2D-2}{D-2} \times \int_0^1 d\xi [\Delta(\xi)]^{(D/2)-1}$$

Analytically continuing this to $D = 4 - 2\epsilon$ we get

$$\Sigma_\Phi = \frac{g^2}{4\pi^2} \Gamma(\epsilon) \frac{6-4\epsilon}{2-2\epsilon} \times \int_0^1 d\xi \Delta \times \left(\frac{4\pi\mu^2}{\Delta(\xi)}\right)^\epsilon$$

As we take $\epsilon \rightarrow 0$ we get

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon), \quad \frac{6-4\epsilon}{2-2\epsilon} = 3\epsilon + O(\epsilon^2), \dots$$

Thus giving us

$$\Sigma_\Phi(p^2) = \frac{3g^2}{4\pi^2} \int_0^1 d\xi \Delta(\xi) \times \left(\frac{1}{\epsilon} - \gamma_E + \frac{1}{3} + \log \frac{4\pi\mu^2}{\Delta}\right)$$

Lets reflect a bit on the example we just did. The divergent term $\frac{1}{\epsilon} \times \text{stuff}$ has exactly the form $a + b \times p^2$ which we expected from our analysis last time. Now what about the finite part of the integral $\Delta \times \log$? Can it be imaginary? Well as long as Δ is positive then the

log is real and thus Σ is real. The imaginary part happens when Δ becomes negative and then the log gets an imaginary part. The actual calculation is part of our homework. In the same homework problem we will find that

$$\text{Im}\Sigma_{\Phi}^{1\text{-loop}} = -m_s \Gamma^{\text{tree}}(\Phi \rightarrow \bar{\Psi} + \Psi)$$

Where does this formula come from? The optical theorem for decay rate

DEFINITION 1: (OPTICAL THEOREM FOR DECAY RATES)

$$\text{Im}\langle 1|\mathcal{M}|1\rangle = M \times \Gamma_{\text{total}}(1 \rightarrow \text{anything})$$

In the language of Feynman digrams the particle going to itself is just a 1PI bubble with two external legs. This is exactly $-i\Sigma_{\Phi}(p^2)$. So we can also say

$$\text{Im}\Sigma_{\Phi}(p^2 = M_s^2) = -M_s \Gamma_{\text{total}}(\Phi \rightarrow \text{anything})$$

Now at the 1-loop level just as he argued earlier that tree level scattering attitude translates to imaginary part of 1-loop. Finally lets take a derivate of our result to get teh field strength renormalization. It's actually better to take the derivative before we analatycailly continue to $D \rightarrow 4$. So we have

$$\frac{d\Sigma_{\Phi}}{dp^2} = (156)$$

Going to the $D = 4 - 2\epsilon$

$$4g^2\mu^{4-D}(D-1)\Gamma(2-D/2) \times \Delta^{(D/2)-2} \rightarrow \frac{3g^2}{4\pi^2} \left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log \frac{4\pi\mu^2}{\Delta} + O(\epsilon) \right)$$

Plugging everything to (156) and evlaulating the integral gives us

$$\frac{d\Sigma_{\Phi}^{1\text{-loop}}}{dp^2} = \frac{g^2}{8\pi^2} \left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log \frac{4\pi\mu^2}{m_f^2} - I(p^2/m_f^2) \right)$$

What is this derivative good for? Well evaluating at $p^2 = M_s^2$ and this determines the field sterngh factor

$$Z_{\phi} = |\langle 1|\text{scalarparticle}|\mathcal{M}|\Omega\rangle|^2$$

And thus

$$Z_{\Phi} = 1 - \frac{g^2}{8\pi^2} \left(\frac{1}{\epsilon} - \gamma_E - \frac{2}{3} + \log \frac{4\pi\mu^2}{m_f^2} - I(M_s^2/m_f^2) \right) + O(g^4)$$

And this is the end of this calculation.

Okay! The next subject is counterterm and perturbation theory. The way he talked about perturbation theory earlier it looked like we start with bare mass and bare coupling and we have to express everything in terms of those parametrs and then relate those paramters to the physical mass and physical coupling. It looks like a bunch of steps there. there's a way to rphrase perturbation theory directly without all this nonsense.

COUNTERTERMS

Well lets start with a $\lambda\phi^4$ theory. we have the bare lagrangian

$$\mathcal{L}_{\text{bare}} = \frac{1}{2}(\partial_\mu \phi_b)^2 - \frac{1}{2}m_b^2 \phi_b^2 - \frac{1}{24}\lambda_b \phi_b^4$$

We also know that

$$\langle 1 \text{ particle} | \phi | \text{vacuum} \rangle = \sqrt{z} e^{ipx}$$

Lets define the renormalized field

$$\phi_r(x) = \frac{1}{\sqrt{z}} \phi_b$$

So then of course

$$\langle 1\text{-particle} | \phi_r | \text{vacuum} \rangle = e^{ipx}$$

To simplify our notation

$$\phi_r = \phi(x) \quad \phi_b = \sqrt{z} \phi(x)$$

Then the lagrangian is

$$\mathcal{L}_b = \frac{1}{2}z(\partial_\mu \phi)^2 - \frac{1}{2}zm_b^2 \phi^2 - \frac{1}{24}z^2\lambda_b \phi^4$$

Which we can compare to the physical lagrangian

$$\mathcal{L}_{\text{physical}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_{\text{physical}}^2 \phi^2 - \frac{1}{24}\lambda_{\text{physical}} \phi^4$$

So consider the difference.

$$\mathcal{L}_b = \mathcal{L}_{\text{phy}} + \text{counterterms}$$

And this leads us to

$$\mathcal{L}_{\text{counterterms}} = \frac{1}{2}\delta_z(\partial_\mu \phi)^2 - \frac{1}{2}\delta_m \phi^2 - \frac{1}{24}\delta_\lambda \phi^4$$


Where


$$\delta_z = z - 1, \quad \delta_m = zm_b^2 - m_{\text{phys}}^2, \quad \delta_\lambda = z^2\lambda_b - \lambda_{\text{phys}}$$


So far all we've done is algebra. The fun part comes in the next step in defining the Feynman rules. We are going to treat all counterterms as perturbations. In other words we take


$$\mathcal{L}_{\text{free}} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m_b^2 \phi^2 \quad \mathcal{L}_{\text{pert}} = -\frac{1}{24}\lambda_{\text{phys}} \phi^4 + \text{corrections}$$


So far we've treated everything quadratic as free and everything else as perturbations. Here we treat some quadratic terms as perturbations as well. So here are the Feynmann rules

Propagator  = $\frac{i}{q^2 - m_{\text{phys}}^2 + i0}$

physical vertex  = $-i\mathcal{L}_{\text{phys}}$

counterterm vertices  = $-i\delta_\lambda$

 = $i\delta_2 p^2 - i\delta_m$



So what are counterterms good for? Well the counterterm couplings δ_i are adjusted order by order in perturbation theory so as to keep all amplitudes finite. In other words we cancel all divergences. Well let's look at some examples. We'll learn to generalize after these examples.

EXAMPLE 5: (AN EXAMPLE USING COUNTERTERMS) Take an example

$$\begin{aligned}
 -i \Sigma(p^2) &= \text{diagram: a horizontal line with a shaded circle on it} \\
 &= \text{diagram: a horizontal line with a loop on it} + \text{higher loops} \\
 &\quad + \text{diagram: a horizontal line with a box on it and a loop on the box} \quad \text{counterterm} \\
 \Rightarrow \quad \Sigma_{\text{net}}(p^2) &= \Sigma_{\text{loop}}(p^2) - \delta_z \cdot p^2 + \delta_m
 \end{aligned}$$

We know to 1-loop order

$$\Sigma_{1\text{-loop}} = \frac{\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2} \right)$$

And therefore we're going to set δ^m to cancel the first loop contribution.

$$\delta^m = -\frac{\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \log \frac{\Lambda^2}{m^2} \right) + O(\lambda^2)$$

We also know that $\Sigma_{1\text{-loop}}$ is p^2 independent so

$$\delta_z = 0 + \underbrace{O(\lambda^2)}_{\text{homework}}$$

Thus we see

$$\Sigma_{\text{net}}(p^2) \text{ is finite.}$$

More generally, finite parts of δ_m and δ_z are set so that

$$\Sigma_{\text{net}}(p^2 = m_{\text{phys}}^2) = 0 \quad \left. \frac{d\Sigma_{\text{net}}}{dp^2} \right|_{p^2 = m_{\text{phys}}^2} = 0$$

This is because the two point function

$$\mathcal{F}_2(p) = \frac{i}{p^2 - m_{\text{phys}}^2 - \Sigma_{\text{net}}(p^2) + i0}$$

HAs unshifted pole at $p^2 = m_{\text{phys}}^2$ meaning that $\Sigma_{\text{net}}(p^2) = 0$. Also we don't want to modify the residue. So we also don't want to modify the derivative of the denominator meaning that the derivative of Σ_{net} has to be zero. And what does all this mean for the counteterms?

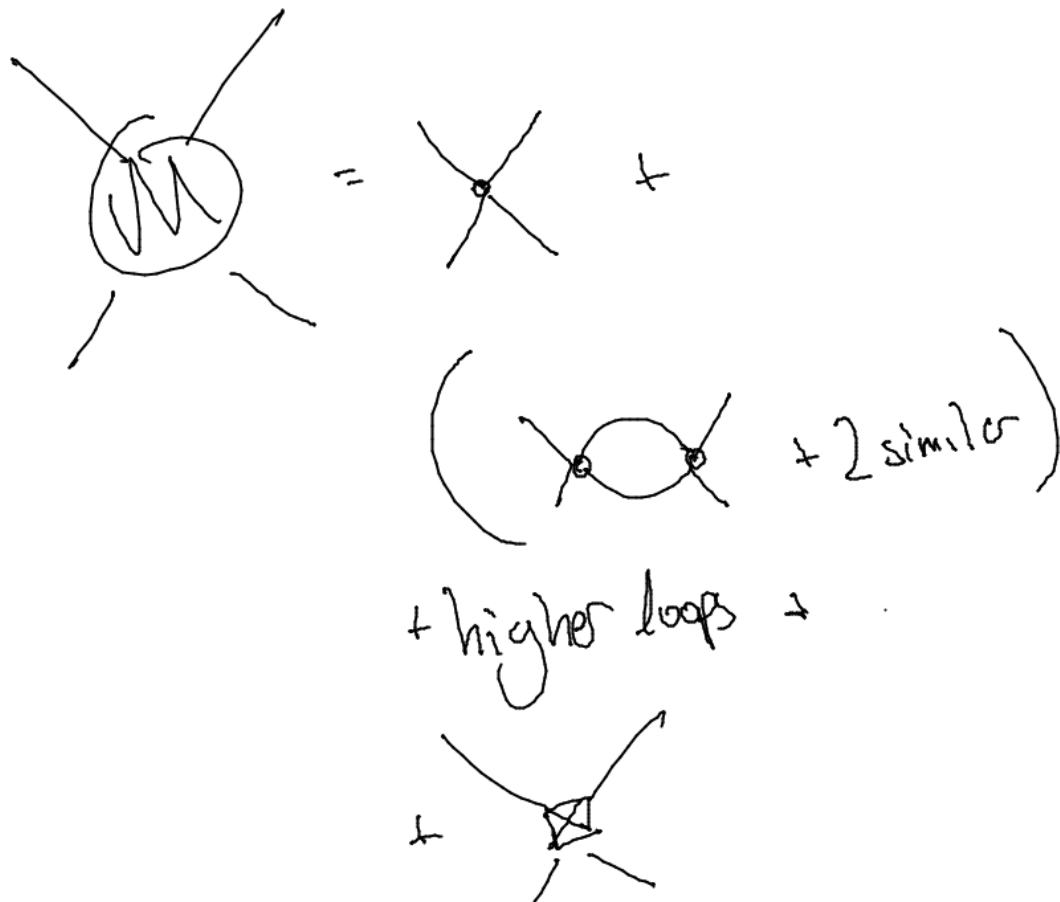
$$\Sigma_{\text{net}} = \Sigma_{1\text{-loop}} - \delta_z \times p^2 + \delta_m$$

And that means that at $p^2 = m^2$

$$\Sigma_{\text{loop}} - \delta_z \times m^2 + \delta_m = 0 \quad \frac{d\Sigma_{\text{loops}}}{dp^2} - \delta_z = 0$$

And this is what gives precise values of δ_z and δ_m . Now lets take another example.

EXAMPLE 6: (FOUR-POINT AMPTLIUDE) consider



$$M = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2} \left(3 \log \frac{\Lambda^2}{m^2} - 3 - J(4/m^2) - J(u/m^2) - J(s/m^2) \right) + O(\lambda^3) - \delta_\lambda$$

First of all δ_λ cancel the UV divergence. Thus

$$\delta_\lambda = \frac{\lambda_p^2}{32\pi^2} (3 \log \frac{\Lambda^2}{m^2} + \text{constants}) + O(\lambda^3)$$

And how do we obtain the finite constant here? Well we need to define what we mean by physical λ . Well if we define $-\lambda_{\text{phys}} = m$ at the threshold where $s = (4)m^2$ $t = u = 0$. We get

$$-\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2} \left(3 \log \frac{\Lambda^2}{m^2} - 3 - J(4) - 2J(0) \right) - \delta_\lambda = \lambda_{\text{phys}}$$

So we find that

$$\delta_\lambda = \frac{\lambda_{\text{phys}}}{32\pi^2} \left(3 \log \frac{\Lambda^2}{m^2} - 3 - J(4) - 2J(0) \right) + O(\lambda_{\text{phys}}^3)$$

And when we evaluate thing

$$J(0) = 0 \quad J(4) = -2$$

And if we plug all this into a formula for \mathcal{M} we have

$$\mathcal{M} = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2} (J(r) - J(s/m^2) - J(u/m^2) - J(t/m^2))$$

So we have seen how counterterms counter some divergences. The question is are they good enough to cancel all divergences. The answer is yes. The 3 counterterms δ_z , δ_m and δ_λ are adjusted order by order in perturbation theory can cancel all the divergences of the $\lambda\phi^4$ theory. He'll outline for us this proof for the few minutes left today. Here's a clue as to what's going on. We need to classify divergences and divergent amplitudes. Let's start with the superficial degree of divergence. Take a generic connected Feynman diagram. It will have l loops, \mathcal{P} propagators and v vertices and e external legs. You remember the drill from last semester. So we get

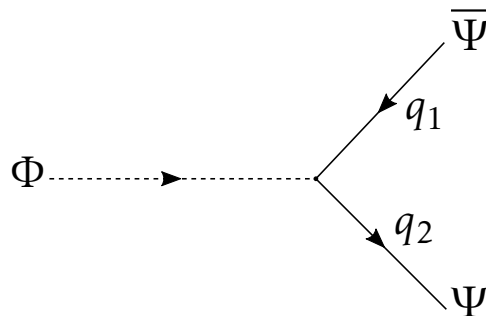
$$I = \int d^4q_1 \dots \int d^4q_l (-i\lambda)^v \prod_{j=1}^{\mathcal{P}} \frac{i}{(\text{linear combination of } q\text{'s and } p_{\text{external}}^2) - m^2 + i0}$$

If we have some superficial degree of divergence $\mathcal{D} = 4L - 2P$ the naively for $\mathcal{D} < 0$ the integral I converges. For $\mathcal{D} = 0$ I diverges as $\log \Lambda_{\text{UV}}$. And for $\mathcal{D} > 0$ the integral diverges as $\Lambda_{\text{UV}}^{\mathcal{D}}$. Why did we say naively? This is because \mathcal{D} . We must check for divergent subgraphs because we may look at the graph and say we have $\mathcal{D} < 0$ but we have a negative subgraph. The man is now showing us some graphs in his notes

CONFIRMING OPTICAL THEOREM FOR DECAY RATES IN YUKAWA THEORY

(based on problemset from course)

(a) At tree level we only have one process that contributes



We know that the decay rate is just the matrix element times a phase space factor.

$$\Gamma = \mathcal{P} \times |\mathcal{M}|^2$$

Where $|\mathcal{M}|^2 = \sum_{s_1, s_2} |\mathcal{M}|^2$ is the spin-summed matrix element. First we can read off the matrix element using Feynman rules for Yukawa Theory (P&S pg 118)

$$i\mathcal{M} = \bar{u}^{s_2}(q_2) \times (-ig) \times v^{s_1}(q_1)$$

We want to evaluate the spin sum²

$$\begin{aligned}
 \sum_{s_1, s_2} |\mathcal{M}|^2 &= g^2 \text{Tr} \left(\sum_{s_1, s_2} \bar{u}^{s_2} v^{s_1} \times \bar{v}^{s_1} u^{s_2} \right) \\
 &= g^2 \text{Tr} \left(\sum_{s_1} \bar{v}^{s_1}(q_1) v^{s_1}(q_1) \times \sum_{s_2} \bar{u}^{s_2}(q_2) u^{s_2}(q_2) \right) \\
 &= g^2 \text{Tr} \left((\not{q}_1 - m_f)(\not{q}_2 + m_f) \right) \\
 &\stackrel{\text{Kill odd number of slashed momentum}}{=} g^2 \text{Tr} (\not{q}_1 \not{q}_2 - m_f^2) \\
 &= g^2 (4q_1 q_2 - 4m_f^2)
 \end{aligned}$$

Thus we get the spin sum

$$|\mathcal{M}|^2 = 4g^2(q_1 q_2 - m_f^2)$$

We can simplify this a bit with

$$2q_1 q_2 = \underbrace{(q_1 + q_2)^2}_{M_s^2} - \underbrace{(q_1^2 + q_2^2)}_{2m_f^2} \Rightarrow |\mathcal{M}|^2 = 2g^2(M_s^2 - 4m_f^2)$$

We need to calculate the phase space factor. From Kaplunovksy's notes on phase sapce factor we know that there is a pre-integral factor of $1/2M_s$

$$\begin{aligned}
 2M_s \mathcal{P} &= \iint \frac{d^3 q_1}{(2\pi)^3 2E_1} \frac{d^3 q_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(3)}(p - q_1 - q_2) \delta(E - E_1 - E_2) \\
 &= \int \frac{d^3 q_1}{(2\pi)^3 2E_1} \frac{1}{(2\pi)^3 2E_2} (2\pi)^4 \delta(E - E_1 - E_2)
 \end{aligned}$$

The angular integral just gives a factor of 4π leaving us with

$$2M_s \mathcal{P} = \int_0^\infty \frac{4\pi (2\pi)^4 |q_1|^2 dq_1}{(2\pi)^3 2E_1 2E_2 (2\pi)^3} \delta(M_s - E_1 - E_2)$$

By (17) and (19) of Kaplunovsky's notes on phase space we're left with

$$2M_s \mathcal{P} = \left(\frac{|q_1|^2}{4E_1 E_2 \pi} \right) \times \underbrace{\left(\frac{d(E_1 + E_2)}{dq_1} \right)^{-1}}_{\frac{E_1 E_2}{q_1 M_s}} \Rightarrow \mathcal{P} = \frac{|q_1|}{8\pi M_s^2}$$

All together we have

$$\Gamma = \frac{|q_1| \times g^2 (M_s^2 - 4m_f^2)}{4\pi M_s^2}$$

²No prefactor since incoming scalar particle has no spin

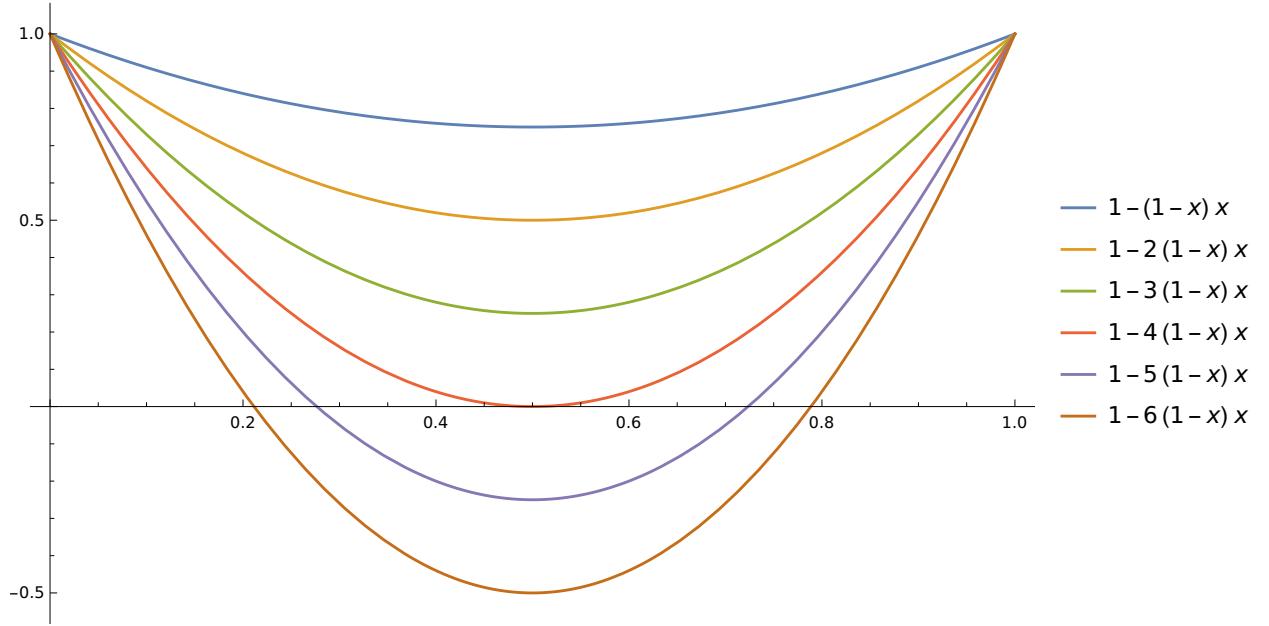
Now we can assert that $|q_1| = \sqrt{M_s^2/4 - m_f^2} = \beta M_s/2$ where $\beta = \sqrt{1 - 4m_f^2/M_s^2}$. This gives us

$$\Gamma = \frac{g^2 M_s \beta^3}{8\pi}$$

(b) If $\Delta(\xi)$ becomes negative then we will gain an imaginary part. Recall the definition of Δ .

$$\Delta(x) = m_f^2 - \xi(1 - \xi)p^2$$

Lets plot $\Delta(x)/m_f^2$ for various values of p



We can see then that once $p^2 > 4m_f^2$, the logarithm dips below zero for a certain range of x . We now want to evaluate the integral. First note that $\log(-a) = \log(a) \pm i\pi$ if a is positive and real. Now lets consider

$$\text{Im}[\log(4\pi\mu^2) - \log \Delta] = -\text{Im}\left[\underbrace{\log(\Delta)}_{\text{real} \pm i\pi}\right] = \mp\pi$$

From this we can extract the imaginary part of $\Sigma_\Phi^{1\text{-loop}}$

$$\text{Im}\left[\Sigma_\Phi^{1\text{-loop}}\right] = \mp \frac{12g^2}{16\pi} \int_a^b dx \Delta(x) =$$

Now we need the bounds which occur when $\Delta < 0$. This means

$$m_f^2 = x(1-x)M_s^2 \Rightarrow x = \frac{1 \pm \sqrt{1 - 4m_f^2/M_s^2}}{2} = \frac{1 \pm \beta}{2}$$

So we integrate

$$\frac{12g^2}{16\pi} \int_{(1-\beta)/2}^{(1+\beta)/2} dx \Delta(x) = \frac{12g^2}{16\pi} \left(\beta(m_f^2 - M_s^2/4) + \frac{1}{12}\beta^3 M_s^2 \right)$$

We note that

$$m_f^2 - M_s^2/4 = -\beta^2 M_s^2/4$$

Thus getting

$$\text{Im} \left[\Sigma_{\Phi}^{1\text{-loop}} \right] = \pm \frac{g^2 M_s^2 \beta^3}{8\pi}$$

All that is left to do is find sign arg of Δ . If we bump up $p^2 = \text{real} + i\epsilon$ then will we rotate CW or CCW to reach Δ . This will give us the arg.

$$\text{Im} \left[\Delta(p^2 = \text{real} + i\epsilon) \right] = \epsilon(x^2 - x) < 0 \text{ since } 0 < x < 1$$

This means that we go through $-\pi$ to get to Δ meaning that

$$\boxed{\text{Im} \left[\Sigma_{\Phi}^{1\text{-loop}} \right] = -\frac{g^2 M_s^2 \beta^3}{8\pi}}$$

(c) We see that

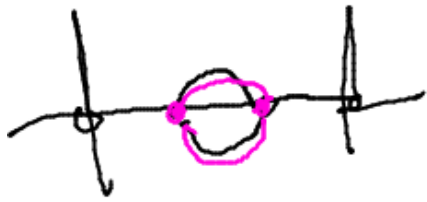
$$\text{Im} \Sigma_{\Phi}^{1\text{-loop}} = \frac{-g^2 M_s^2 \beta^3}{8\pi} = -M_s \times \frac{g^2 M_s \beta^3}{8\pi} = -M_s \times \Gamma(S \rightarrow f + \bar{f})$$

LECTURE 9: COUNTERTERMS AND SUBGRAPHS

February 05, 2021

We'll start with a notice. The last semester's lecture are posted on the class website. Now lets continue on what we started last lecture. At the very end we went through pretty fast on superficial divergences. Take ϕ^4 theory. When we write an amplitude we'll have L loops and \mathcal{P} propagotrs. So the superficial degree of divergence is $\mathcal{D} = 4L - 2\mathcal{P}$ If this superficial degree of divergence is negtaive then it'll converge. $\mathcal{D} = 0$ is logarithmically divergence and for $\mathcal{D} > 0$ we have $\Lambda^{\mathcal{D}}$ divergence. But as we said, we start with this supeficial degree of divergence but

weshould also cekck for subgraphs.



whole graph

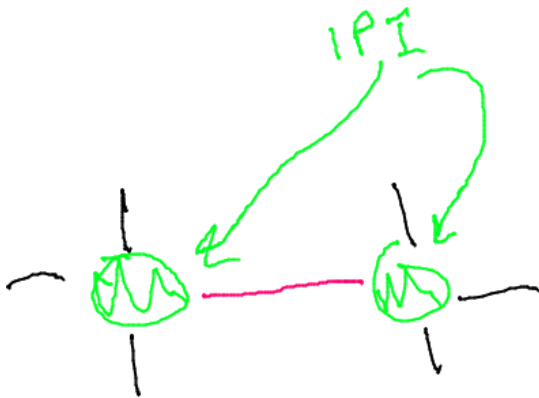
$$L=2 \quad P=5 \quad \mathcal{D} < 0$$

subgraphs

$$L=2 \quad P=3 \quad \mathcal{D}=2 > 0$$

diverges as Λ^2

So the general rule is to check \mathcal{D} for all the 1PI subgraphs. The rason is as follows



qur depend
only on P_{ext}
not q_{ext}

$$\Rightarrow \Pi(\text{propagator}) \sim (q_{loop})^{-2(p-1)}$$

$$\Rightarrow \mathcal{D}_{eff} = 4L - 2(p-1) = \mathcal{D} + 2$$

$\Rightarrow \mathcal{D}$ doesn't tell us anything
because of \mathcal{D}_{eff}

So now that we know what to check let's see what kinds of graphs are divergent and which aren't. Can we relate superficial degree of freedom to things like the number of external legs.

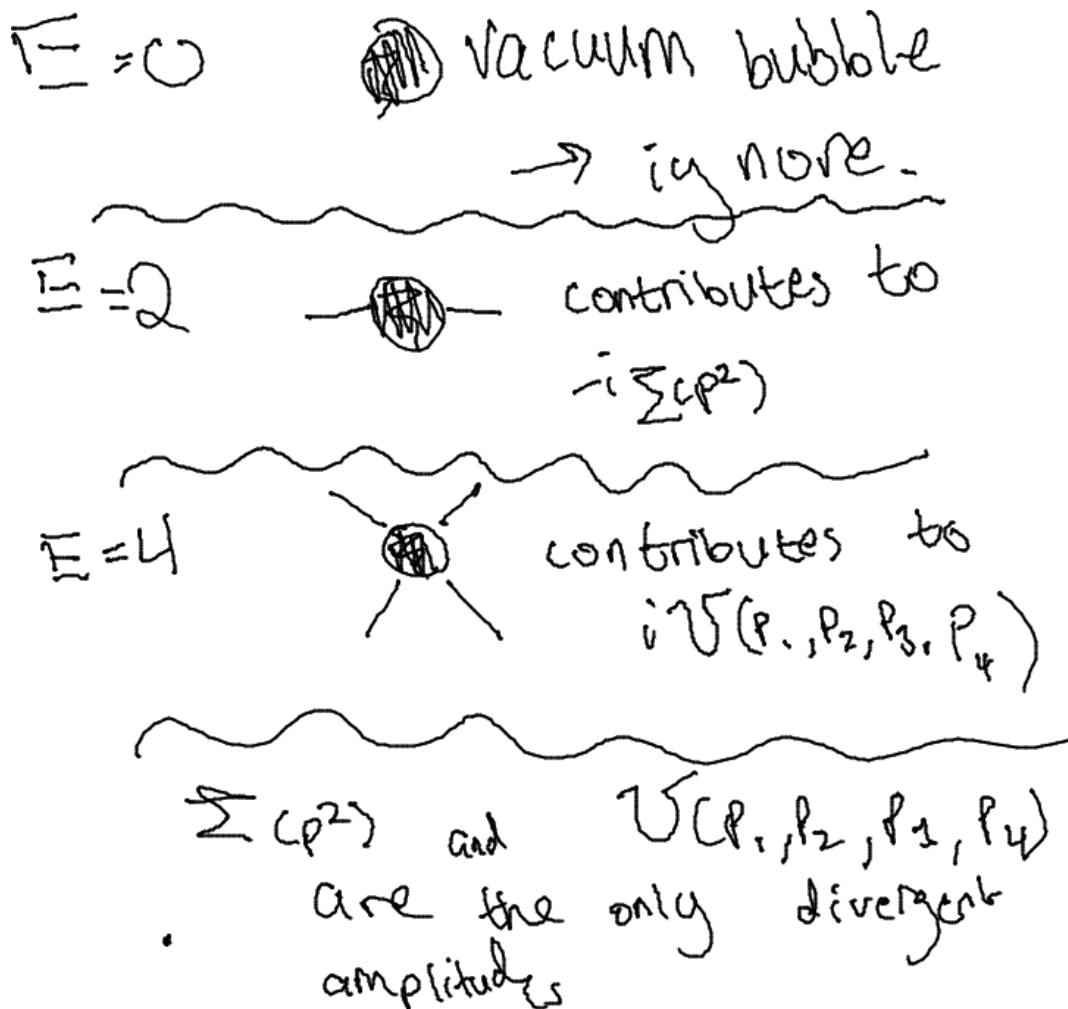
So let's take a one particle irreducible graph or subgraph. It will have v vertices, l -loops and P propagators and E external to the subgraph. So a propagator to the big graph may be an external line to the subgraph. So we're treating a subgraph as a Feynman diagram of its own. And then what do I know? I know the Euler theorem that says that $L - P + V = 1$. And I also know that if I count line ends we get $2P + E = 4V$. So if we count the superficial degree of divergence

$$D = 4L - 2P = 4(1 + P - V) - 2P = (4 - 2P + 4V - 2P - E)$$


And now for we can do some cancellations to end up with

$$D = 4 - E$$

So the subgraphs can be determined to be divergent or not just by counting how many lines are going into it. Okay! Let's move on. All divergent graphs or subgraphs have $E \leq 4$. That really limits then. Moreover we have $\phi \rightarrow -\phi$ symmetry which means that E must be even because any subgraph with odd number of external legs will have odd number of ϕ fields with is just 0 by symmetry. Or in graph theory if all vertices have even valence all graphs have even number of external legs. And that means that divergent graphs or subgraphs have $E = 0, 2, 4$. In other words, what do we have?



If we UV-regulate these divergent subgraphs and cancel their UV divergences then all amplitudes become finite. This is because any other amplitude with bigger external legs will come from $\mathcal{D} < 0$. So divergence can only come from subgraphs. And if we somehow manage to cancel the divergences of these subgraphs then everything is kosher. So let's start with $-\mathrm{i}\Sigma(p^2)$.

$-i \Sigma(p^2) =$  $D = 4 - E = +2$
 $\Rightarrow O(\Lambda^2)$ divergence

To reduce divergence lets take derivatives

$$\frac{\partial \Sigma}{\partial p^2} : O(\Lambda^1) \text{ divergence}$$

How do we see that? Well schematically

$$-i\Sigma \propto \int d^{4L}q \prod_{i=1}^{\mathcal{P}} \frac{1}{q_i^2 - m^2 + i0}$$

where $q_1 \dots q_l$ are loop momenta and $q_{l+1} \dots q_{\mathcal{P}}$ are linear combinations of $q_1 \dots q_l$ and p_{ext} . So when we write

$$\frac{d\Sigma}{dp^\mu} = \int d^4L_q \sum_{i=[+]}^{\mathcal{P}} \frac{-2(\partial q_j/\partial p)q_j^\mu}{[q_j^2 - m^2 + i0]^2} \times \prod (\text{other})$$

Now if we look at the numerator $-q_j(\partial q_j/\partial p)$ this has degree 1 in $q_1 \dots q_l$. The denominator will have degree $4 + 2(\mathcal{P} - 1) = 2\mathcal{P} + 2$. So o

$$\mathcal{D}[\partial\Sigma p] = 4L + 1 - (2P + 2) = 4L - 2P - 1 = \mathcal{D}[\Sigma] - 1 = +1$$

From this we get that

$$\frac{\partial \Sigma}{\partial p^\mu} = O(\Lambda)$$

By the same token $\partial^2 \Sigma / \partial p^\mu \partial p^\nu$ has

$$\mathcal{D}[\Sigma] - 2 = 0$$

This diverges as $O(\log \Lambda)$. And finally

$$\frac{\partial^3 \Sigma}{\partial p^\mu \partial p^\nu \partial p^\sigma} \text{ has } \mathcal{D} = -1 \Rightarrow \text{converges}$$

The third derivative of Σ is finite. This means that the infinite part of $\Sigma(p^2)$ is quadratic polynomial in p^μ .

$$A^{\mu\nu} p_\mu p_\nu + B^\mu p_\mu + C$$

And by lorentz symmetry $A^\nu = A \times y^{\mu\nu}$ and $B^\mu = 0$ (I think he just means isotropy so we can't have odd power terms). This means that

$$\Sigma(p^2) = C + A \times p^2 + \text{finite } f(p^2)$$

This gives us

$$C = O(\Lambda^2), A = O(\log \Lambda)$$

Now why is that important? because in a counterterm perturbation theory $\Sigma^{\text{net}}(p^2) = \Sigma^{\text{loops}}(p^2) + \delta_m - \delta_z p^2$. So comparing everything to each other we see that

$$\delta_z = A \quad \delta_m = -C \Rightarrow \Sigma^{\text{net}}(p^2) \text{ is finite}$$

What about the other divergent amplitude.



$$iV(p_1 \dots p_4)$$

$$D = 4 - E = 0$$

$$\Rightarrow O(\log \Lambda) \text{ divergent}$$

From the same logic above we have all derivatives of the above subgraph as finite. This means that

$$V = \text{divergent constant} + \text{finite } f(p_1, \dots, p_4)$$

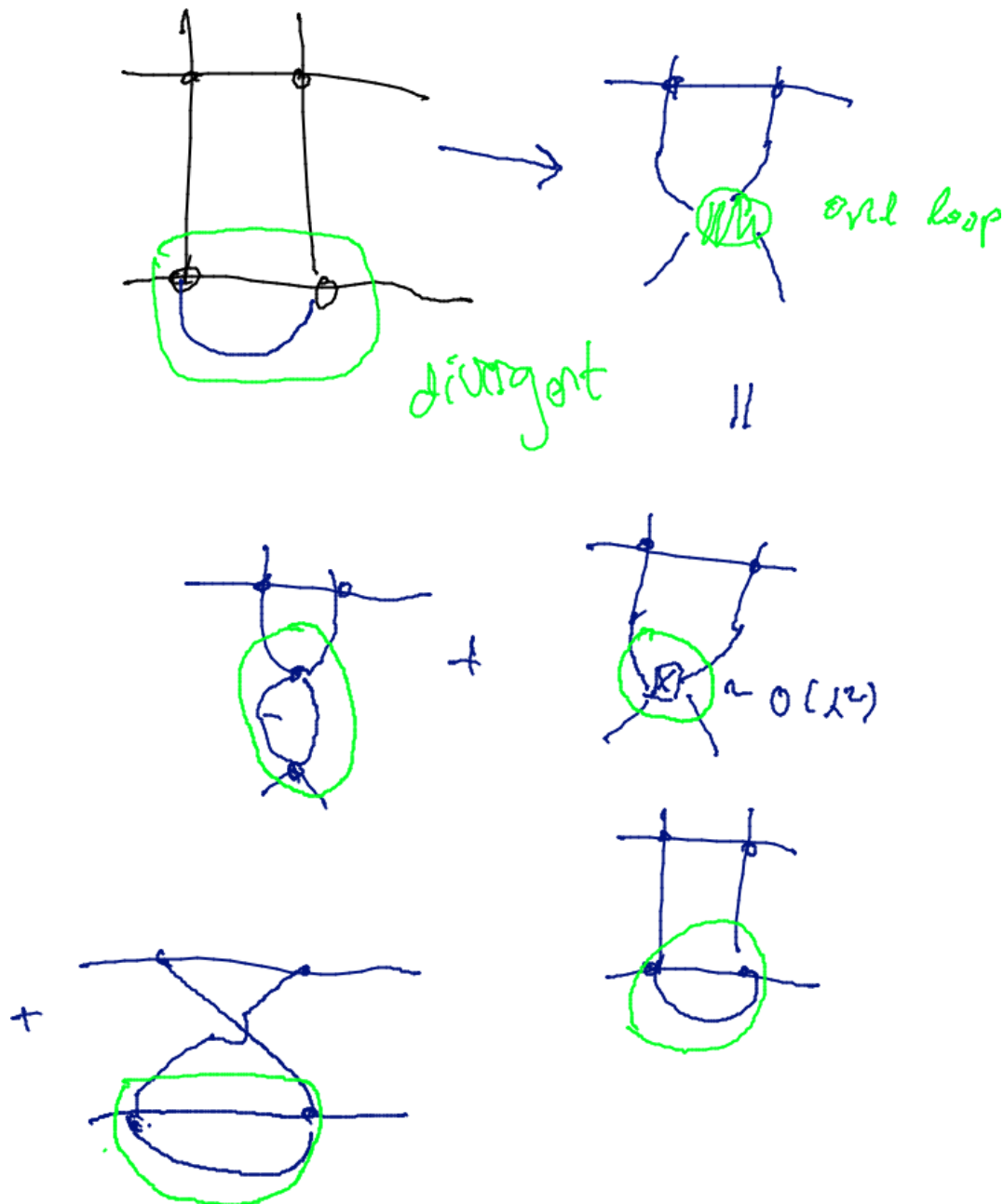
And we have

$$\Sigma V^{\text{net}}(p_1, \dots, p_4) = V^{\text{loops}}(p_1 \dots p_4) - \delta_\lambda$$

Where δ_λ conceals the divergent constant of V_{loops} . this makes $V_{\text{net}}(p_1 \dots p_4)$ finite. All other subgraphs are finite.

Here's the bottom line. The 3 counter terms $\delta_m, \delta_z, \delta_\lambda$ are adjusted order by order in perturbation theory together cancel all the UV divergences of all amplitudes. Well! We have started a little bit late so let's let him go for another five minutes. So far we've only talked about graphs

that diverge by themselves. What about subgraph divergences?



What all this is saying is that at large q the **combined one loop circle** is finite and of the order $O(\log q^2)$ and then the whole graph is going to be proportional to

$$\propto \int \frac{d^4 q}{(q^2)^3} \times O(\log(q^2)) \text{ which is finite}$$

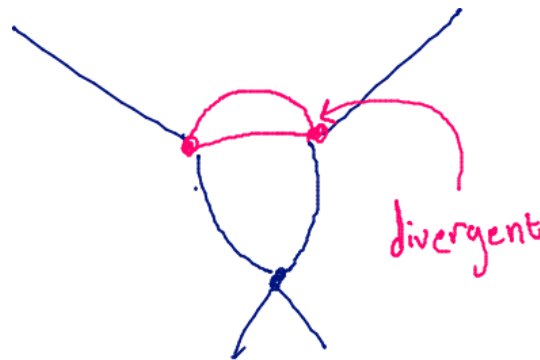
So the bottom line here is that sure we can have a divergent subgraph but if we combine that divergent subgraph that could replace it in the same order in λ then all together the net green block becomes finite and when we plug things into the whole integral we get a finite answer. To summarize the counterterms cancel UV divergences of subgraphs *in situ* (right in their place). Okay we're out of time. We'll make a break at this point. The one thing we did not finish today

is that we did not discuss nested or overlapping divergences. We'll talk about such divergences a bit on Tuesday and we'll assign the rest of the story as a reading assignment for the next homework and switch onto the general subject of renormalizability and non-renormalizability.

LECTURE 10: NESTED/OVERLAPPING DIVERGENCES AND BEGINNING OF RENORMALIZABILITY (PLUS SOME DIMENSIONAL ANALYSIS!)

February 09, 2021

Today we're going to continue talking about the issue of renormalization and renormalizability. Last lecture we saw that in $\lambda\phi^4$ theory the counterterms δ_m , δ_z and δ_λ cancel overall UV divergences and divergent subgraphs. There is one potential problem with subgraph divergences which is the nested and overlapping divergences. For example



Also whole graph divergent



So we have for large q

$$\text{subgraph: } \log \Lambda^2 + \text{finite } f(\text{momenta})$$

where $f \propto \log q^2$

$$\text{Whole graph: } \int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^4} (\log \Lambda^2 + \text{finite}(\text{something} \times q^2))$$

Which eventually turns into

$$\text{Whole graph: } \int \dots = (\log \Lambda^2)^2 + (\log \Lambda^2) \times \text{momentumdependent} + \text{finite}$$

The **red term** is a potential problem. Now if you look at the counterterm we get

$$\delta_\lambda = \log \Lambda^2 + \text{const}$$

Which means we have

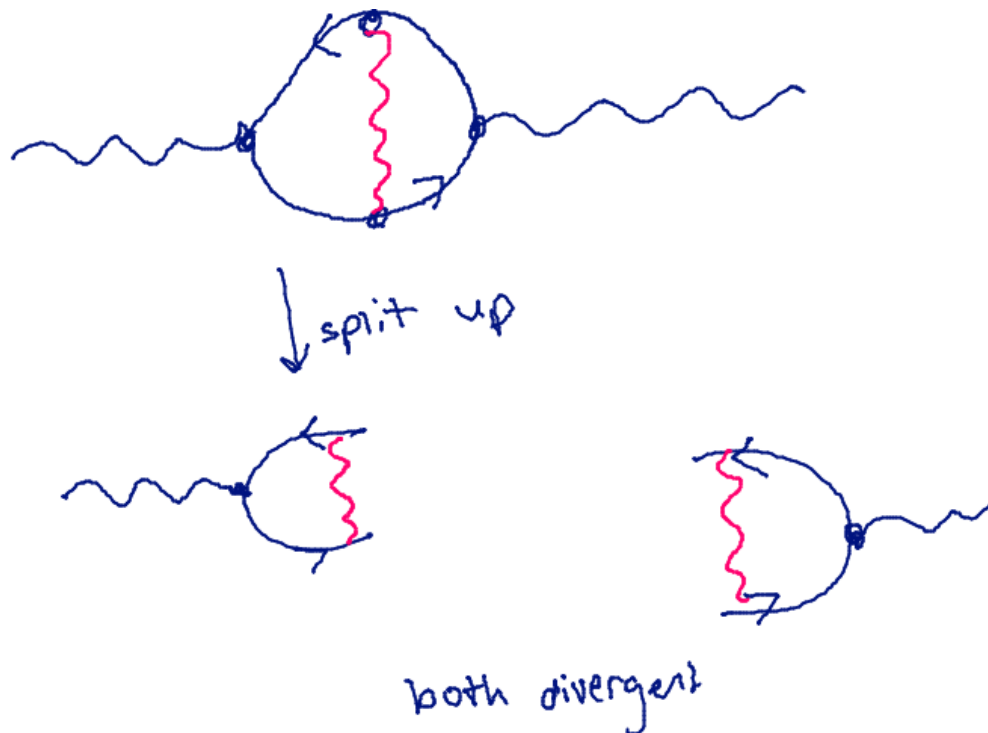
$$\text{whole graph} = (\log \Lambda^2 + \text{const}) \times (\log \Lambda^2 + \text{finite(momenta)})$$

So the whole term will have the eform

$$\text{whole graph} = (\log \Lambda^2)^2 + \log \lambda^2 + \text{momentum dependent} + \text{finite}$$

And the **red term** will be trouble. However, we have a miracle of miracles. Net graphs of the type we had above (the upper loop stands for net oneloop + δ_λ), the whole shmeer $\log \Lambda^2 \times (\text{mom. dep.})$ terms cancel out. And so the net amplitude is a divergent constant plus finite where the divergent term is cancelled by $\delta_\lambda^{2-\text{loop}}$. He's not going to do this kind of analysis in class. Instead it's for homework (Peskin and Schroeder 10.5). So that's our example of a nested divergence.

Now lets take a look at a overlapping divergence example.



However there is a theorem that says if you can handle overall divergence then all the subgraph divergences will take care of themselves (BPHZ(Bogolyubov, Parasyuk, Hess, Zimmerman) Theorem). To restate, the theorem roughly says if you can cancel all the overall divergences then the subgraph divergences including nested and overlapping divergences are okay. This is a very rough formulation. Ok. Let us happily put aside the specifics of $\lambda\phi^4$ theory and just tell us that this theory is an example of a renormalizable theory. Now lets consider a general renormalizable QFT.

DEFINITION 2: (RENORMALIZABLE QFT) In a renormalizable QFT all UV divergences are cancelled by a finite number of counterterms but their values need to be adjusted order by order to all orders of perturbation theory. A key feature of these theories is the fact that there is an infinite number of divergent graphs (because divergence happens to all orders of loop amplitude) but only a finite number of divergent amplitudes. Furthermore divergent part of amplitudes are polynomial in external momenta. $\lambda\phi^4$ theory is a nice example of a renormalizable theory.

We'll learn next lecture that QED is a renormalizable theory but QED is more complicated. However not all theories are renormalizable. Some of them are super-renormalizable and some of them are non-renormalizable. Let's get some examples here. (We'll only get one example of super-renormalizable but an *infinite* number of non-renormalizable theories.)

DEFINITION 3: (SUPER-RENORMALIZABLE QFT) A super-renormalizable QFT has only a finite number of divergent diagrams. Divergences stop after so many loops. And that means that all divergences are cancelled by a few counterterms determined at a finite order of perturbation theory. So the difference between this and a normal renormalizable QFT is that here you don't need to recalculate counterterms for infinity, you can stop after a finite number of orders.

DEFINITION 4: (NONRENORMALIZABLE QFT) A nonrenormalizable QFT needs an infinite set of counterterms. There are an infinite number of divergent amplitudes and in fact, all amplitudes diverge at high enough order of perturbation theory. So these are okay as effective theories but doing renormalization is pointless because you'll need a counterterm for basically everything.

We'll now start to look at more general examples. Let's look at a scalar theory with $c\phi^k$ interactions.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{m^2}{2}\phi^2 - \sum_{k=3}^{\infty} \frac{c_k}{k!}\phi^k$$

Now we don't have to have all the k 's. But we're trying to consider a completely general setup. What this Lagrangian tells us is that we have vertices of valence $k = 3, 4, \dots$. And now let's do loop counting or degree of divergence counting. So recall that the superficial degree of divergence is

$$\mathcal{D} = 4L - 2P$$

Let's consider the fact that

$$L - P + V_{\text{net}} = 1 \rightarrow L = 1 + P - \sum_k V_k$$

On the other hand if you look at the number of line ends, that's just counting lines meaning that

$$\text{number of line ends} = 2P + E = \sum_k k V_k$$

So the superficial degree of divergence is

$$\mathcal{D} = 4(1 + P - \sum_k V_k) - 2P + \underbrace{\sum_k k V_k - 2P - E}_{=0}$$

After cancelling some thing we get

$$\boxed{\mathcal{D} = 4 - E + \sum_k (k - 4) V_k}$$

Now lets get specific. If we take $\kappa\phi^3$ interactions only. That is $k = 3$ only. In this case we get





$$\mathcal{D} = 4 - E - V$$

And ealier we learned that for a cubic t heory $V = E - 2 + 2L$ which means that

$$\mathcal{D} = 6 - 2E - 2L$$

So we can see that the more loops the less divergence. And the only divergent amplitudes will be $E + L \leq 3$. So if $E = 0$ that's just a vacuum bubble which diverges for $L \leq 3$ only but we don't really care. Now if we take $E = 1$ we get a tadpole. Also $L \leq 2$ means 2 divergnt graphs. A baloon and a Baymax. For $E = 2$ we get the stick figure sticking arm out from the top view. This is it! No more divergent graphs. Wow, it's super-renormalizable. What about the counterterms

needed to cancel out these divergences?


 $D=0 \Rightarrow \Sigma(p^2)$ is div. const + finite
 \Rightarrow need δ_m to cancel but no δ_z

 $= -i\delta_m$ only
 furthermore δ_m fixed at 1-loop order

 tadpoles cancelled by bare lagrangian
 $\mathcal{L}_{\text{bare}} = -\delta_g \phi$

 $= -i\delta_j$. so δ_j is fixed at 2-loop order
 so ϕ^3 theory is super-renormalizable

Next lets consider $V = \frac{\chi}{3}\phi^3 + \frac{\lambda}{4!}\phi^4$. In this case the superficial degree of divergence is

$$D = 4 - E - V_3$$

Notice that it's independent of V_4 . All this means is that finite number of divergent amplitudes ($E \leq 4$) but infinite number of divergent graphs. Or more concretely any number of loops that allows for any V^4 . This too is a renormalizable theory.

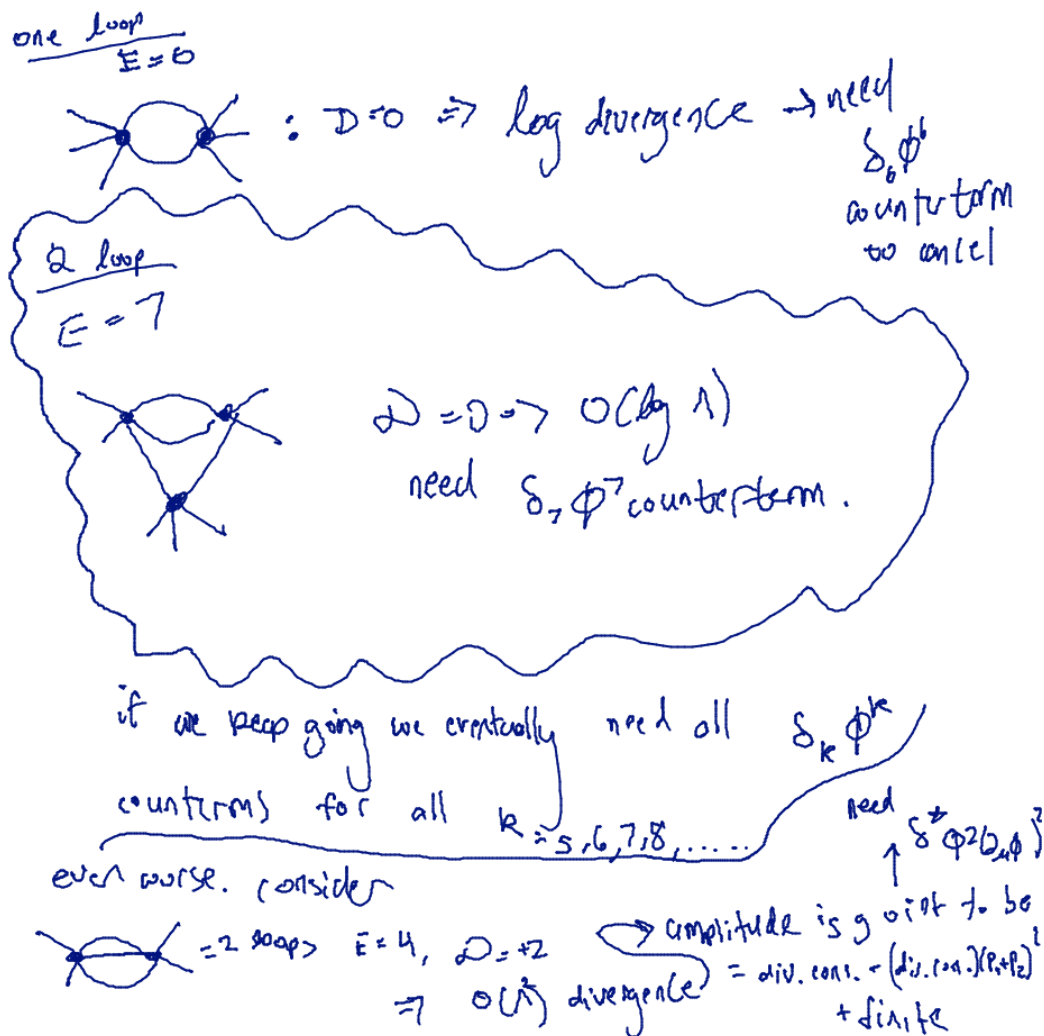
$$\mathcal{L}_{\text{bare}} = \frac{\delta_z}{2}(\partial_\mu \phi)^2 - \frac{\delta_m}{2}\phi^2 - \delta_j \phi - \frac{\delta_\chi}{6}\phi^3 - \frac{\delta_\lambda}{24}\phi^4$$

Well that was the good news. If we allow any kind of quartic or cubic coupling it looks like this theory. As long as the number of fields is finite then it is a perfectly renormalizable theory. But what if we allow $k > 4$? For example suppose we have just $k = 5$ and $V = \frac{c_5}{120}\phi^5$. In this case the superficial degree of divergence is

$$D = 4 - E + V$$

This grows with V so the higher order of perturbation theory the worse the divergence of any amplitude. So take any amplitude with any number of external legs. At high enough order of

perturbation theory $D > 0$ which means there is at least a constant divergent term. Lets look at some examples. Consider 1-loop for the ϕ^5 theory. First consider $E = 6$.



At higher loops we'll need all products of ϕ and derivatives at all powers. The point of all this is to show that we have a whole zoo of divergences. ϕ^5 needs an infinite family of counterterms more and more at each loop order. This is an example of a non-renormalizable theory. Like any ϕ^k coupling with $k > 4$ makes QFT non-renormalizable. What about theories that have more than just scalars? Back in november he gave us an idea of dimensiona analysis and the fact that negative dimensions are trouble. We can now be more technical about it and relate dimensional analysis to renormalizabliyty. Lets recap very quickly. The basic idea is that we use units where $\hbar = c = 1$ so there is just one dimensionful units, units of energy. $[m] = [p^\mu] = E^{+1}$ and $[x^\mu] = E^{-1}$. The action is dimensionless meaning that $[\mathcal{L}] = E^4$. For example for a scالر field theory $\Phi(x)$ has

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2$$

This means that $[\Phi] = E$. And in the same way all bosons have dimensions E^{+1} (mostly because of the kinetic term) for fermions the kintec terms looks like

$$\mathcal{L}_{\text{free}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi$$

So fermions have dimension *TODO*. In perturbation theory the coupling parameters has dimension

$$[C_n] = [\mathcal{L}]/[\Phi^n] = E^{4-n}$$

This turns out to be a general rule

- (a) All couplings of a renormalizable theory must have non-negative energy dimensions
- (b) coupling of theory has strictly positive energy dimensions then the theory is super-renormalizable
- (c) if any coupling has negative energy dimension then the theory is non-renormalizable.

So those are the rules. Let's let him explain us where this comes from. Take a generic interaction term. It will have n_b bosonic fields, n_f fermionic fields, n_d spacetime derivatives acting on all these fields. Thus we have

$$[\text{field product}] = E^{n_b + \frac{3}{2}n_f + n_d}$$

This means that the coupling dimension is

$$[g] = E^\Delta \text{ for } \Delta = 4 - n_b - \frac{3}{2}n_f - n_d$$

Now let's count the superficial degree of divergence. consider some L loop amplitude

$$\int d^4L q \prod (\text{propagator}) \times \prod (\text{vertices})$$

Well bosonic propagator go like $\frac{1}{q^2}$ and fermionic propagators go like $\frac{1}{q}$. But if there are derivatives we get more q terms. Altogether we get

$$\int d^4L q \frac{1}{q^{2P_b + P_f}} \times \prod_v^{\text{vertices}} q^{n_d(v)}$$

So the superficial degree of divergence is

$$\mathcal{D} = 4L - 2P_b - P_f + \sum_{v=1}^V n_d(v)$$

If we use Euler theorem then

$$L - P_{\text{net}} + V = 1 \Rightarrow L = 1 + P_b + P_f - V$$

All together this means that

$$\mathcal{D} = 4 + 2P_b + 3P_f + \sum_{v=1}^V (n_d - 4)$$

If we count the line ends we can also see that

$$2P_b + E_b = \sum_v n_b \quad 2P_f + E_f = \sum_v n_f(v)$$

Thus giving us

$$2P_b + 3P_f = \sum_v (n_b + \frac{3}{2}n_f) - E_b - \frac{3}{2}E_f$$

And consequently we get

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f + \sum_{v=1}^V \underbrace{\left(n_b + \frac{3}{2}n_f + n_d - 4 \right)}_{\text{energy dimension of field product}}$$

So we get the important relation

$$\mathcal{D} = 4 - E_b - \frac{3}{2}E_f - \sum_{v=1}^V \Delta(g_v)$$

and that is why coupling constant dimension is so important for renormalizability. If we have some dimensionless constant while the rest are positive then theory has a infinite number of divergent diagrams but all diagrams have $E_b + \frac{3}{2}E_f \leq 4$ which tells us that there is only a finite number of divergent amplitudes. So we only need a finite number of counterterms. If some term has negative dimension energy dimension then we can go to higher orders of loop order and get more and more non-renormalizable theories³.

so now we know why we care about coupling dimensions.

$$\Delta = 4 - n_b - \frac{3}{2}n_f - n_d$$

To see a zoo of what kinds of fields we can have look at diman2.pdf. I can't type as fast as he's going through these notes.

PESKIN AND SCHROEDER 10.5: A TWO-LOOP EXAMPLE

STARTED: February 10, 2021. FINISHED:

LECTURE 11: RENORMALIZING QED AND WARD IDENTITIES

February 11, 2021

Lets start! Last lecture we started talking about renormalizability of some field theories and the outlined the kinds of allowable couplings in $D = 4$. Now we're going to talk about the best know QFT that is practical, QED. And we'll take a look at renormalizing QED. So for simplicity we're going to look at the simplest version of QED, photon and electrons. We all know how to add other fields to the theory

$$\mathcal{L}_{\text{phys}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(i\not{D} - m_e)\Psi = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\Psi}(i\not{\partial} - m)\Psi + eA_\mu \bar{\Psi}\gamma^\mu\Psi$$

³this paragraph is really bad. Look at notes diman2.pdf

In the bare lagrangina we replace e with bare couling and bare mass. By convention Z_3 and Z_2 correspond to EM and electron field while Z_1 is for the electric charge. This consideration vies us

$$A_{\text{bare}}^\mu(x) = \sqrt{Z_e} \times A^\mu \quad \Psi_b = \sqrt{Z_2} \times \Psi$$

All together

$$\mathcal{L}_b = -\frac{Z_3}{4} F_{\mu\nu} F^{\mu\nu} + Z_2 \bar{\Psi}(i\not{\partial} - m_b)\Psi + Z_1 e \times A_\mu \bar{\Psi} \gamma^\mu \Psi$$

Where

$$Z_1 \times e = Z_2 \sqrt{Z_3} \times e_{\text{bare}} \quad \mathcal{L}_{\text{bare}} = \mathcal{L}_{\text{phys}} + \mathcal{L}_{\text{terms}}^{\text{counter}}$$

In counterterm perturbation theory we can also wrtie

$$\mathcal{L}_{\text{terms}}^{\text{counter}} = -\frac{\delta_3}{4} \times F_{\mu\nu} F^{\mu\nu} + \delta_2 \times \bar{\Psi} i\not{\partial} \Psi - \delta_m \bar{\Psi} \Psi + e\delta_1 A_\mu \bar{\Psi} \gamma^\mu \Psi$$

Where we can solve for δ_i . For now we need to find the effects of the gauge averaging by adding a gauge symmetry breaking term \mathcal{L}_G where

$$\mathcal{L}_G = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$$

Where ξ paramterizes a speicfic gauge (e.g. $\xi = 1$ is Feynman gauge.) So when we do counterterm perturbation theory all the counterterms are perturbations. In the notes we're given feynman rules for the rules we should remember from last semster as well as the feynmann rules for the counterterms which is the only new bunch of Feynmann rules.

Lets go to power counting, my favorite kind of counting. QED has only one physical coupling e that dimensionless. Thus we should think that QED is a renormalizable theory (we saw this last lecture from dimensional analysis.) The problem with the pwoer counting analysis is if you ask how many counterterms we need to cancel out divergences it turns out we need more counterterms than we actually have to keep gauge symmetry. Fortuantely Ward-Takahashi identities show that you don't actually need the missing counterterms, if you sum up each diagrams in each order then you have enough counterterms. Anyways that for tomorrow. Today lets do the power counting. The supericial degree of divergence is

$$\mathcal{D} = 4 - \frac{3}{2}E_e - E_\gamma$$

So we need

$$\frac{3}{2}E_e + E_\gamma \leq 4$$

And since the number of electronic lines is always even we have either $E_e = 0$ and $E_\gamma \leq 4$ or $E_e = 2$ and $E_\gamma \leq 1$. All other graphs will not contribute to a superficial divergence. So lets lit out all the possiblites

(a) $E_e = 0 = E_\gamma$, just a vacuum bubble, we ignore

(b) $E_e = 0$ and $E_\gamma = 1$, tadpole which vanishes by lorentz symmetry acting on the μ index of a zero-momentum photon ⁴

⁴what the fuck?

- (c) $E_e = 0$ and $E_\gamma = 2$. Two photonic legs. This diverges quadratically and contributes to $i\Sigma_{\mu\nu}^\gamma(k)$.
- (d) $E_e = 0$ and $E_\gamma = 3$ (looks like someone's OC Sharingan). These diagrams form pairs that cancel each other out. This is *Furry's theorem*⁵ which comes from charge conjugation.
- (e) $E_e = 0$ and $E_\gamma = 4$. This one is also non-vanishing and we need to worry about it since it contributes to 4-photon amplitude $iV_{\kappa\lambda\mu\nu}$
- (f) $E_e = 2$ and $E_\gamma = 0$ which contributes to $-i\Sigma_{\alpha\beta}^e(p)$ where $\alpha\beta$ are dirac indices. By symmetry (lorentz+parity) we have

$$\Sigma_{\alpha\beta}^e(p^\mu) = A(p^2) \times \delta_{\alpha\beta} + B(p^2) \times p_\mu \gamma_{\alpha\beta}^\mu$$

We can expand this as a power series in p^2 and use $p^2 = (\not{p})^2$ so we can also turn the term above into a power series of \not{p} slashed. Because of this we can also write $\Sigma^e(\not{p})$.

- (g) $E_e = 2$ and $E_\gamma = 1$. This is just the same vertex we had above but dressed with e- γ vertex. Apparently this is a real pain.

We can now look at the momentum dependence of the divergences. Let's keep going. First we'll do it in reverse order. We'll first consider (g). The dressed vertex has

$$\mathcal{D}[\Gamma^\mu] = 0 \Rightarrow \log \text{ divergences}$$

And the derivative of the dressed vertex wrt electron or photons momentum has $\mathcal{D} = -1$ thus we know that

$$\Gamma_{\alpha\beta}^\mu(p', p) = [O(\log \Lambda) \text{ constant}] \times \gamma_{\alpha\beta}^\mu + \text{finite}(p', p)$$

Furthermore we can see by lorentz symmetry that the constant array must be proportional to the gamma matrix and that gives the result above. By counterterm perturbation theory we also have

$$\Gamma_{\text{net}}^\mu(p', p) = (\Gamma_{\text{tree}}^\mu = \gamma^\mu) + \Gamma_{\text{loops}}^\mu + \delta_1 \times \gamma_\mu$$

So if we pick the correct γ_μ then we can counter the divergence.

Next we'll consider the electron propagator (g). The superficial degree of divergence is

$$\mathcal{D} = 1$$

So it diverges like p^2 and the taking derivatives reduces superficial degree of divergence by one. So we have

$$\Sigma^e = [O(\Lambda) \text{ constant}] + [O(\log \Lambda) \text{ constant}] \times \not{p} + \text{finite}(\not{p})$$

And in counterterm perturbation theory we have

$$\Sigma_{\text{net}}^e(\not{p}) = \Sigma_{\text{loops}}^e(\not{p}) + \delta_m - \delta_2 \times \not{p}$$

⁵0w0

So again the counterterms cancel out the divergences.

Now for some bad news. First the four-photon amplitude (e) in the list above. First superficial degree of divergence is $\mathcal{D} = 0$. So log divergence. By lorentz symmetry and bose symmetry we need

$$V_{\kappa\lambda\mu\nu}(\{k_i\}) = [O(\log \Lambda) \times (\text{a bunch of metrics } g_{\alpha\beta}) + \text{finite}]_{\kappa\lambda\mu\nu}$$

How the fuck do we deal with the the term that is proportional to the product of metrics. This is our first problem.

Now lets consider (b). For this one the third derivative is finite. This means that the divergent part of $\Sigma'_{\mu\nu}$ has to be some polynomial in k and for lorentz symmetry we need a metric

$$A \times g_{\mu\nu} + B \times k^2 \times g_{\mu\nu} + C \times k_\mu k_\nu$$

Where A is $O(\Lambda^2)$ and B, C is $O(\log \Lambda)$. Naively we could say that the counter term is

$$-i\delta_{3A} k^2 g_{\mu\nu} + i\delta_{3B} \times k_\mu k_\nu - i\delta_{m\gamma} \times g_{\mu\nu}$$

But this will create terms in the lagrangian that don't exist, namely the photon mass term.

$$\frac{\delta_{m\gamma}}{2} A_\nu A^\nu$$

All together we know that QED has a finite number of divergent amplitudes which can be cancelled by a finite number of counterterms but the gauge symmetry of QED stop some counterterms from existing. In QCD we have a similar situation. For QED there is something we can do in this class to show that we don't actually need those counterterms. In the end they cancel each other out and that miracle happens because of Ward-Takahashi identities. There's a lot of these identities and they all stem from electric current conservation and the one we care about right now is the identity relating to purely photonic amplitudes. For a diagram with n photonic lines we have

$$\text{diagram} = i\mathcal{M}^{\mu_1 \dots \mu_n}(k_1, \dots, k_n)$$

The ward identity says that for any i

$$(k_i)_{\mu_i} \times \mathcal{M}^{\mu_1 \dots \mu_n}(k_1, \dots, k_n) = 0$$

A caveat is that this doesn't work any one diagram, you have to add up all diagrams in a particular group. But once you do that the above relation holds. Lets apply this to the 4-photon amplitude. We saw a few minutes ago that

$$\mathcal{V}_{\kappa\lambda\mu\nu}(k_1, k_2, k_3, k_4) = C \times (\text{bunch of metrics}) + \text{finite}$$

Where C is $O(\log \Lambda)$ constant. By Ward identity

$$0 = k_1^\kappa \times \mathcal{V}_{\kappa\lambda\mu\nu} = C \times (\text{bunch of } k_1 \text{ times metrics}) + k_1^\kappa \times \text{finite}_{\kappa\lambda\mu\nu}$$

The finite terms cannot cancel so we can say that $C = 0$. In other words the ward identity for the 4-photon amplitude does not give an ultraviolet divergence. So that takes care of that. Lets move on to the two photon amplitude. We have

$$\Sigma'_{\mu\nu} = \Xi(k^2) \times g_{\mu\nu} + \Pi(k^2) \times k_\mu k_\nu$$

For scalar function Ξ and Π . By the ward identity we have

$$0 = k^\mu \times \Sigma_{\mu\nu}^\lambda = \Xi(k^2) \times k_\nu - \Pi(k^2) \times k^2 k_\nu$$

$$0 = k^\nu \times \Sigma_{\mu\nu}^\lambda = \Xi(k^2) \times k_\mu - \Pi(k^2) \times k^2 k_\mu$$

And thus

$$\Xi(k^2) = \Pi(k^2) \times k^2 \Rightarrow \Sigma_{\mu\nu}^\gamma = \Pi(k^2) \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

A few minutes ago we had that the divergent part of the 2-phoonto ampltiude had some form (above.). So again lets shove a ward identity into this

$$0 = A \times k_\nu + B \times k^2 k_\nu + C \times k^2 k_\nu$$

From this we can read off $A = 0$ $C = -B$ giving us

$$[\Sigma_{\mu\nu}^\gamma(k)]_{\text{divergent}} = B \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

This is exactly what we had above. This means for the entire amplitude

$$\Sigma_{\mu\nu}^\gamma(k) = \Pi(k^2)(k^2 g_{\mu\nu} - k_\mu k_\nu) \text{ where } \Pi(k^2) \text{ is a log divergent constnat plus something finite}$$

We have to be careful here. This is only true for the netamplitude or for the next combination for the whole loop order. For QED itself at the one-loop order there is only one diagram. However at higher loop orders individual diagrams may disobey the above rule but summing them all together satisfies the identity. But once you get you can ask what kind of counterterm do you need to cancel things out? Well if we add the counterterms

$$[\Sigma_{\mu\nu}^\gamma]^{\text{net}} = [\Sigma_{\mu\nu}^\gamma]^{\text{loops}} - \delta_3 \times (k^2 g_{\mu\nu} - k_\mu k_\nu)$$

So we can canel the divergence.

To summarize, the ward identites alow us to ignore the counterterms that we seem to be missing since the bad counterterms aren't needed when we add everything together.

So now we know how to use the counterterms to cancel the UV idvergent loop graphs. However how do we set the finite part of the counterterms. These follow from *renormailzation conditions* (we need pole at physicl mass with unit ampltiude). Lets try to do the electron's dressed propagator, the fourier transform of the 2-point function.

$$\mathcal{F}_2^e(p) = \int d^4(x-y) e^{ip(x-y)} \mathcal{F}_2^e(x-y) = \frac{i}{\not{p} - m + i0} + \frac{i}{\not{p} - m + i0} (-i\Sigma^3(\not{p})) \frac{i}{\not{p} - m + i0} + \dots$$

So everythign here is a 4x4 matrix with dirac indices so we need to be careful how we multiply. Thankfully the 1PI ampltiudes are powerseries in \not{p} so it commmutes with \not{p} and function of \not{p} so we can write the N -bubble term as

$$\left(\frac{i}{\not{p} - m + i0} \right)^{N+1} \times (-i\Sigma^e(\not{p}))^N \frac{i}{\not{p} - m + i0} \times \left(\frac{\Sigma^e(\not{p})}{\not{p} - m + i0} \right)^N$$

Giving us

$$\mathcal{F}_2^e(p) = \frac{i}{\not{p} - m - \Sigma^e(\not{p}) + i0}$$

Something about poles for the free propagator here. Now let's talk about the dressed propagator. It should have a pole at $\not{p} = \text{physical electron mass}$ and residue is the electron field strength factor. Namely the pole should be at the exact same position as the free propagator with unit residue. This means that

$$\mathcal{F}_2^3(p) = \frac{i}{\not{p} - m - \Sigma^3(\not{p}) + i0} = \frac{i}{\not{p} - m + i0} + \text{smooth}$$

So if we expand the denominator in powers of $(\not{p} - m)$ we get

$$\not{p} - m - \Sigma^3(\not{p}) = -\Sigma^e @(\not{p} = m) + \left(1 - \frac{d\Sigma^3}{d\not{p}} @(\not{p} = m)\right) \times (\not{p} - m) + O((\dots)^2)$$

This means the renormalization condition is

$$\text{at } \not{p} = m \text{ both } \Sigma^e = 0 \text{ and } \frac{d\Sigma^3}{d\not{p}} = 0$$

This means that

$$\Sigma_{\text{net}}^e(\not{p}) = \Sigma_{\text{loop}}^e + \delta_m - \delta_2 \times \not{p}$$

All together this means that

$$\Sigma_{\text{loops}}^e \text{ at } \not{p} = m + \delta_m - \delta_2 \times m = 0 \quad \frac{d\Sigma_{\text{loops}}^e}{d\not{p}} @(\not{p} = m) - \delta_2 = 0$$

These are the renormalization conditions that fix the finite parts of the counterterms