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Tensors are generalizations of vectors or matrices. Consider D-dimensional manifold M described in two different coordinate systems x^μ and $x^{\mu'}$. A **scalar** on M is a number assigned to each point in M. We can also call a $(0,0)$ tensor. A **vector** is a $(1,0)$ tensor. Or in other words, a vector is a set of D function v^μ which transform under coord transform

$$x \to x^{\mu'} \qquad v^\mu(x^\mu) \to v^{\mu'}(x^{\mu'}) = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu(x^\mu(x^{\mu'})) = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu$$

The **tangent space** of M at a point $p \in M$ is the set of all vectors at that point.

Example of example. Consider a world line $x^\mu(\lambda)$. We define the **tangent vector** as

$$v^\mu = \frac{\partial x^\mu}{\partial \lambda}$$

To show that this is a vector we show it transforms like we expect tensors to

$$v^{\mu'} = \frac{\partial x^{\mu'}}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda} = \frac{\partial x^{\mu'}}{\partial x^\mu} v^\mu$$

A **one-form** or a $(0,1)$ tensor or a **covariant vector** or a **dual vectors** is a set of D function which transform as follows

$$w_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} w_\mu$$

Useful since you can comine vectors and one-forms to get a scalar. One forms are arrows living in the dual tangent space. Some examples: one-forms are row vectors instead of column vectors.

$$w_\mu = (w_0, \ldots, w_3) \quad \text{and} \quad v^\mu = \begin{pmatrix} v_0 \\ \vdots \\ v_3 \end{pmatrix}$$

For any two non-negative integers k, l we define a (k,l) tesnor as a set of D^{k+l} function $T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$ which transforms as

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \ldots \nu_l'} = \left(\frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\right) \cdots \left(\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\right) T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \ldots \nu_l}$$

A antisymmmetric $(0,n)$ tensor is a **differential form**, or an **n-form**. For example f is a $(0,0)$ tensor and thus a 0-form. ω_μ is a 1-form. $F_{\mu\nu}$ is a two-form. An n-form is a collection of $\binom{D}{n}$ functions. The **wedge product** of a p-form A and q-form B denoted by $A \wedge B$ is

$$(A \wedge B)_{\mu_1 \ldots \mu_{p+q}} = \frac{(p+q)!}{p!q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{p+q}]}$$

Note that $A \wedge B$ is a $p+q$ form. Differential forms are introduced since they are easy to differentiate and integrate. Exterior derivative of a p-form A is a $p+1$ form denoted by dA

$$(dA)_{\mu_1 \dots \mu_{p+1}} = (p+1)\partial_{[\mu_1} A_{\mu_2 \dots \mu_p]}$$

This transforms as a tensor.

(a) $p = 0$. Then $A = f$ a scalar and $(df)_\mu = \partial_\mu f$.

(b) $p = 1$. Then $A = \omega_\mu dx^\mu$ is a 1-form in index free notation.

$$(d\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$$

Dogs. Review: A **differential form** is a $(0,n)$ tensor which is completely antisymmetric.

$$F_{\mu\nu} \text{ is a 2-form}$$

Differential forms makes calculus easy. A **wedge product** take a p-form and a q-form and outputs a $p+q$ form.

$$(A \wedge B)_{\mu_1 \dots \mu_{p+q}} = CA_{[\mu_1 \dots} B_{\dots \mu_{p+q}]}$$

We take the derivative of a p-form to get a $p+1$ form using exterior derivatives

$$(dA)_{\mu \dots \mu_{p+1}} = C\partial_{[\mu_1} A_{\mu_2 \dots \mu_{p+1}]}$$

And **index free** form of differential forms we use dx^μ is a basis of one-forms.

$$df = \partial_\mu f \, dx^\mu$$

$$\omega = \omega_\mu dx^\mu$$

$$d\omega = -\frac{\partial \omega_\mu}{\partial x^\nu} dx^\mu \wedge dx^\nu$$

Since partial derivatives equal we find that $d^2 = 0$.

$$(d^2 A)_{\mu_1 \dots \mu_{p+2}} = 0$$

Example in $\mathbb{R}^3$

A 1-form is a quantity with 3 components: vectors. A 2-form is a quantity with $\binom{3}{2}$ components: vectors. A 0-form is a function. A 3-form is a quantity with $\binom{3}{3}$ components: a "function."

$$\begin{aligned} \wedge &: \text{fn} \times \text{fn} \to \text{fn} \\ &: \text{fn} \times \text{vector} \to \text{vector} \\ &: \text{vector} \times \text{vector} \to \text{vector} \Rightarrow \text{the cross product} \end{aligned}$$

The "wedge product" is a generalization of cross product to higher dimensions.

$$\begin{aligned} d &\text{ on a 0-form scalar} \to \text{vector: the gradient } \nabla \\ &\text{on a 1-form vector} \to \text{vector: the curl } \nabla \times v \\ &\text{on a 2-from vector} \to \text{scalar: divergence } \nabla \cdot v \end{aligned}$$

$$\begin{aligned} d^2 &\text{ on a 0-form: } \nabla \times (\nabla f) = 0 \\ &\text{on a 1-form: } \nabla \cdot (\nabla \times v) = 0 \end{aligned}$$

To emphasize a *wedge product and exterior derivative are just higher dimensional version of each of these we've seen in calculus*. Another useful identity:

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

Where α is a p-form. Integrals need to be coordinate invariant, that's why we've turned to differential forms. The usual

$$\begin{aligned} \int d^d x f(x) &= \int dx^1 \dots dx^d f(x) \\ &= \int dx^{1'} \dots dx^{d'} \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) f(x(x')) \\ &\neq \int dx^{1'} \dots dx^{d'} f(x(x')) \end{aligned}$$

Therefore integral is not coordinate invariant. The reason this is true is because we're integrating the wrong thing. The right thing to integrate over a d-manifold is a d-form.

$$\begin{aligned} \int_{M_d} \omega_d &= \int_{M_d} \omega(x) dx^1 \wedge \cdots \wedge dx^d \\ &= \int_{M_d} \omega(x) dx^1 \dots dx^d \end{aligned}$$

If we transform coordinates

$$\omega(x(x')) = \omega(x)\left(\frac{\partial x^1}{\partial x^{\mu_1'}}\right)\dots\left(\frac{\partial x^d}{\partial x^{\mu_d}}\right) dx^{\mu_1'} \wedge \cdots \wedge dx^{\mu_d'} = \omega(x)\det\left(\frac{\partial x}{\partial x'}\right) dx^{1'} \wedge \cdots \wedge dx^{d'}$$

Thus we find that

$$\begin{aligned} \int_{M_d} \omega_d &= \int \omega(x) dx^1 \dots dx^d \\ &= \int \omega(x') dx^{1'} \dots dx^{d'} \end{aligned}$$

Stokes theorem: If we have a manifold M with boundary $\partial M = B$

$$\int_M d\omega = \int_{\partial M} \omega$$

We have learned how to take derivatives and integrate differential forms. These are general operations which didn't require a metric. We will need differentiation and integration of general tensors. So far we've only talked about the "topology" of spacetime, not geometry. To move to geometry, we need to talk about a metric.

Geometry: Metrics
We concluded that spacetime is a manifold with a metric $g_{\mu\nu}$ which defines a line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

For every point in spacetime x_0^μ there exists a coordinate system where $g_{\mu\nu} = \eta_{\mu\nu}$ plus higher order terms (locally looks like minkowski spacetime). Some things to note $g_{\mu\nu}$ is a (0,2) tensor that is symmetric, invertible ($\deg(g_{\mu\nu}) = g \neq 0$), and is diagonalizable with eigenvalues $(-,+,+,+)$ which all follow from $g_{\mu\nu} = \eta_{\mu\nu} + \dots$. Important to note that these properties are coordinate invariant. Formally we call this a metric with these properties a pseudo-reimannian metric. The inverse of a metric is a (2,0) tensor is $g^{\mu\nu}$. Some stuff about raising and lowering indices. The norm is

$$v_\mu v^\mu = g_{\mu\nu} v^\mu v^\nu = v^2$$

The metric determines the geometry of spacetime.

Geometry
Recall that spacetime is a pseudo-reimannian manifold that comes equipped with a metric $g_{\mu\nu}$ which is used to calculate intervals

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

With properties

(a) $g_{\mu\nu}$ is symmetric

(b) $g_{\mu\nu}$ is invertible with inverse $g^{\mu\nu}$

(c) $g_{\mu\nu}$ has one negative and 3 positive eigenvalues

Sometimes people use $(+,-,-,-)$ instead of the $(-,+,+,+)$ signature (which is used here.) We also use the metric to raise and lower indices. So why does the metric let us study the geometry of spacetime? Well, we can measure the interval between different points in space-time. The interval will depend on the path through spacetime connecting two points.

Given a worldline $x^\mu(\lambda)$ the distance between two points $x^\mu(\lambda_0)$ and $x^\mu(\lambda_1)$ as measured along this wordline is

$$S = \int_{\lambda_0}^{\lambda_1} \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}d\lambda$$

The propert time elapsed for an obersever travelling along this worldline is $T = iS$. This integral can be thought of as a functional. The path that extrimzes S is a geodesic. The length of the path as measured along this geodisc is the geodisc distance. Also the two points are timelike separated if the geodisic distance is imaginary, spacelike separated if the geodisic distance is real, and null if the geodisic distance is 0.

So what do geodiscs look like? Well we can use the Euler-Lagrange equations.

$$S = \int \sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}d\lambda \qquad \text{where} \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

This is extremized under $x^\mu \to x^\mu + \delta x^\mu(\lambda)$ (a small variation of the path)

$$0 = \frac{\delta s}{\delta x^\mu} = \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu}\right) \quad \text{where} \quad L = \sqrt{g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}$$

Expanding this out

$$0 = \frac{d}{d\lambda}\left(\frac{g_{\mu\alpha}\dot{x}^\alpha}{dS/\lambda} - \frac{1}{2}\frac{g_{\alpha\beta,\mu}\dot{x}^\alpha\dot{x}^\beta}{dS/d\lambda}\right)$$

$$\vdots$$

(algebra algbera)

$$\vdots$$

$$\ddot{x}^\nu + \Gamma^\nu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = \dot{x}^\nu\frac{d^2S/d\lambda^2}{ds/d\lambda} \qquad \text{where} \qquad \Gamma^\nu_{\rho\sigma} = \frac{1}{2}g^{\nu\mu}(g_{\rho\mu,\sigma} + g_{\sigma\mu,\rho} - g_{\rho\sigma,\mu})$$

Where $\Gamma^\nu_{\rho\sigma}$ is the Christoffel symbol. The differential equation that is obeyed by the geodesic is

$$\ddot{x}^\nu + \Gamma^\nu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = f(\lambda)\dot{x}^\nu$$

Note that λ is just some arbitarary parameters to albel points on the wordline. We could choose a new paramaters $\lambda'(\lambda)$ to label points on the worldline. This will change $f(\lambda)$. We can also choose λ such that $f(\lambda) = 0$. Namely choose a λ' such that $dS/d\lambda' = 0$. Physically this means we choose S as our paramters, i.e. for a wworld line $x^\mu(\lambda)$ connecting 2 poitns let

$$s(\lambda) = \int_{\lambda_0}^{\lambda} ds$$

Basically we're choosing our λ' as the arclength along the path from our starting point to the point we're at $x^\mu(\lambda)$. In this scenario is known as a affine parameter. This also gives us

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0}$$

A few comments on the chirstoffel symbol, $\Gamma^{\nu}_{\rho\sigma} = \frac{1}{2}g^{\nu\mu}(g_{\rho\mu,\sigma} + g_{\sigma\mu,\rho} - g_{\rho\sigma,\mu})$

(a) $\Gamma^{\mu}_{\nu\rho}$ is symmetric in ν and ρ.

(b) $\Gamma^{\mu}_{\nu\rho}$ is not a tensor.

(c) Instead the transformation rule is

$$\Gamma^{\mu'}_{\nu'\rho'} = \left(\frac{\partial x^{\mu'}}{\partial x^{\mu}}\right)\left(\frac{\partial x^{\nu}}{\partial x^{\nu'}}\right)\left(\frac{\partial x^{\rho}}{\partial x^{\rho'}}\right)\Gamma^{\mu}_{\nu\rho} - \frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\mu'}}{\partial x^{\rho}\partial x^{\nu}}$$

Two ways to prove this formula: left as an exercies.

Some physical interprestation of the geodesic equation. First note that differs the usual notion of a straight line $\ddot{x}^{\mu} = 0$. The $\Gamma^{\mu}_{\nu\rho}$ encodes the deviation from a straight line, either due to choice of coordinates (e.g. rotating coordinate system and coriorlis force) or due to the curvature of the metric. A geodesic is as straight as possible given a curved geometry. For example, if you think of a sphere, there is notion of a straight line but the geodesic is the arc of a great circle.

Covariant Derivatives

We still don't know how to take derivatives of arbitarry tensors...

$$\partial_{\mu} f = f_{,\mu}$$

$$\partial_{\mu} v^{\rho} = v^{\rho}{}_{,\mu} \text{ not a (1,1) tensor}$$

So how do we define a covariant tensor which transforms as a tensor? Lets start by thinking about vecotrs. We wnat a derivative where

$$\nabla_{\mu} v^{\rho} \text{ a (1,1) tensor}$$

We want this derivative to be *Linear* and that it *obeys a Leibniz rule*.

$$\text{Leibniz} : \partial_{\mu}(f v^{\rho}) = (\partial_{\mu} f)v^{\rho} + f\nabla_{\mu} v^{\rho}$$

These imply that

$$\nabla_{\mu} v^{\rho} = \partial_{\mu} v^{\rho} + G^{\rho}_{\mu\nu} v^{\nu}$$

For some $G^{\rho}_{\mu\nu}$.

$$\nabla_{\mu'} v^{\rho'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\rho'}}{\partial x^{\rho}}\nabla_{\mu} v^{\rho} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\rho'}}{\partial x^{\rho}}(\partial_{\mu} v^{\rho} + G^{\rho}_{\mu\nu} v^{\nu}) = \partial_{\mu'}\left(\frac{\partial x^{\rho'}}{\partial x^{\rho}} v^{\rho}\right) + G^{\rho'}_{\mu'\nu'} v^{\nu'}$$

Thus we see that

$$G^{\rho'}_{\mu'\nu'} v^{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\rho'}}{\partial x^{\rho}} G^{\rho}_{\mu\nu} v^{\nu} - \frac{\partial^2 x^{\rho'}}{\partial x^{\mu} x^{\nu}}\frac{\partial x^{\mu}}{\partial x^{\mu'}} v^{\nu}$$

Under coordinate trnsfomration.

$$G^{\rho'}_{\mu'\nu'} = \frac{\partial x^{\rho'}}{\partial x^{\rho}}\frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}} G^{\rho}_{\mu\nu} - \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial^2 x^{\rho'}}{\partial x^{\mu}\partial x^{\nu}}$$

Just like the Christoffel symbol. The christoffel symbold is the unique object that you can form from the metric which transforms in this way(no proof but trust me!). Thus the covariant derivative

$$\nabla_{\mu\nu}{}^{\rho} = \partial_\mu v^\rho + \Gamma^\rho_{\mu\nu} v^\nu$$

Covariant Derivatives: We started with the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = f(\lambda)\dot{x}^\mu$$

For an affine λ $ds/d\lambda$ = constant. So how do you take the derivative of a tensor to get a another tensor(with proper transformation rules)

$$\nabla_\mu f = \partial_\mu f \text{ is a (0,1) tensor}$$

$$\nabla_\mu v^\rho \text{ should be a (1,1) tensor, linear, and obey leibniz rule}$$

Thus we find that

$$\nabla_\mu v^\rho = \partial_\mu v^\rho + G^\rho_{\mu\nu} v^\nu$$

And we see that the Christoffel symbol is the only possible G t hat works the way we want. Sometimes we introduce different geometric connections (e.g. gauge fields) thus in general G might not equal Γ and Γ is called the "metric connection."

For a world line $x^\mu(\lambda)$ if we have an affine paramter

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0$$

Recall that $v^\mu = \frac{\partial x^\mu}{\partial \lambda}$ is a tangent vector. This means

$$\frac{\partial}{\partial\lambda} = v^\mu\partial_\mu = v^\mu\frac{\partial}{\partial x^\mu}$$

This makes the geodesic equation

$$\frac{\partial}{\partial\lambda} v^\mu + \Gamma^\mu_{\nu\rho} v^\nu v^\rho = 0$$

Or more succienctly

$$v^\nu\nabla_\nu v^\mu = 0$$

Physically this means the derivative of the tangent vector along the curve is zero. For non affine paramtere

$$v^\nu\nabla_\nu v^\mu = f(\lambda) v^\mu$$

This is to say the derivative of the tangent vector in the direction of the worldline is zero (or proportional to v^μ).

How do we define ∇_μ for an arbitrary tensor. Consider a one-form

$$w_\mu \Rightarrow \nabla_\rho w_\mu \text{ is linear and obey leibtz}$$

$$\nabla_\alpha(w_\mu v^\mu) = \partial_\alpha(w_\mu v^\mu) = (\nabla_\alpha w_\mu)v^\mu + w_\mu(\nabla_\alpha v^\mu)$$

This gives us the following definition

$$\nabla_\alpha w_\nu = \partial_\alpha w_\nu - \Gamma^\rho_{\alpha\nu} w_\rho$$

Compare this with a vector

$$\nabla_\alpha v^\nu = \partial_\alpha v^\nu + \Gamma^\nu_{\alpha\rho} v^\rho$$

Now for an arbitrary tensor $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$.

$$\nabla_\alpha T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l} = \partial_\alpha T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l} + \Gamma^{\mu_1}_{\alpha\rho} T^{\rho\mu_2\cdots\mu_k}{}_{\nu_1\ldots\nu_l} + \cdots - \Gamma^\rho_{\alpha\nu_1} T^{\mu_1\cdots\mu_k}{}_{\rho\nu_2\ldots\nu_l} - \cdots$$

Which obeys all toe properties we want (linearity and leibnitz rule).

One thing to note is that covariant derivatives do not commute. It turns out

$$\nabla_\mu g_{\rho\sigma} = 0$$

This statement is saying that the covariant derivative is "metric compatible". Here's a proof

$$\begin{aligned}\nabla_\mu g_{\rho\sigma} &= g_{\rho\sigma,\mu} - \Gamma^\lambda_{\mu\rho} g_{\lambda\sigma} - \Gamma^\lambda_{\mu\rho} g_{\lambda\rho} \\ &\text{Asserting the definition of the Christoffel symbol} \\ &= g_{\rho\sigma,\mu} - \frac{1}{2}(g_{\sigma\mu,\rho} + g_{\sigma\rho,\mu} - g_{\mu\rho,\sigma}) - \frac{1}{2}(g_{\rho\mu,\sigma} + g_{\sigma\rho,\mu} - g_{\mu\sigma,\rho}) \\ &= 0\end{aligned}$$

We can also show $\nabla_\mu g^{\rho\sigma} = 0$ and $\nabla_\mu \delta^\rho_\sigma = 0$. Since we raise and lower indeces with $g_{\rho\sigma}$ and $g^{\rho\sigma}$ we can pass derivatives through raised and lowered indices. For example

$$\nabla_\alpha(v^\mu v_\mu) = v^\mu \nabla_\alpha v_\mu + (\nabla_\alpha v^\mu)v_\mu = 2v^\mu \nabla_\alpha v_\mu$$

Where the second equality comes from the fact that we can raise and lower indices in the second term.

$$v^\mu \nabla_\alpha w_\mu = v_\mu \nabla_\alpha w^\mu$$

Also note that

$$\nabla^\alpha = g^{\alpha\beta}\nabla_\beta = \nabla_\beta g^{\alpha\beta}$$
$$\Rightarrow \nabla_\mu v^\mu = \nabla^\mu v_\mu$$

Integration

We know how to integrate a p-form but we also can ingetrate scalars. Remember that under a coordinate tranfosrmion

$$x^\mu \to x^\mu(x^{\mu'})$$

$$\Rightarrow \int d^d x f(x) = \int d^d x' \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) f(x^\mu(x'))$$

Let $G = \det(g_{\mu\nu}) \neq 0$. $g < 0$ for a lorentzian manifold. Under $x^\mu \to x^{\mu'}$ then

$$g_{\mu'\nu'} = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) g_{\mu\nu} \left(\frac{\partial x^\nu}{\partial x^{\nu'}}\right)$$

$$G' = J^T G J$$

$$\Rightarrow \det(g_{\mu'\nu'}) = \det(g_{\mu\nu}) \det\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)^2$$

Thus G is not a scalar but a "density. Thus the integral of a scalar f on a curved geometry

$$\int d^d x \sqrt{|g|} f(x) = \int d^d x' \det\left(\frac{\partial x}{\partial x'}\right) \sqrt{|g|} f(x) = \int d^d x' \sqrt{|g'|} f(x(x'))$$

Is coordinate invariant. It is the only sensible definition of integration in general. Let's see how this is identitcal to the integration of d-forms ω which has only 1 degree of freedom $\omega = \omega(x)$. Any two d-forms are rleated by $w_1 = f(x) w_2$ (????). We want to builda canonical d-form. Recall the Levi-Civita symbol($\tilde{\epsilon}$)(it's not a tensor). A dforms

$$w = w(x) dx^1 \wedge \cdots \wedge dx^d$$

Or equivaelnty

$$w_{\mu_1 \dots \mu_d} = w(x) \tilde{\epsilon}_{\mu_1 \dots \mu_d}$$

Now under coordinate transform

$$w(x) \to w(x') = w(x) \det\left(\frac{\partial x}{\partial x'}\right)$$

Thus $w(x)\tilde{\epsilon}_{\mu_1 \dots \mu_d}$ is a tensor if $w(x') = w(x)\det(\dots)$. This transformation is exactly how the square root ofa metric tensroms.

$$\epsilon_{\mu_1 \dots \mu_d} = \sqrt{|g|} \tilde{\epsilon}_{\mu_1 \dots \mu_d}$$

Is a d-form and ϵ is the Levi-Civita tensor. And every d-form

$$w = f(x) \epsilon$$

Thus the integral of a d-form

$$\int_M w = \int f \epsilon = \int f \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^d = \int f \sqrt{|g|} dx^1 \dots dx^d$$

The ϵ tensor is called the "volume form" since it's the infitesmial volume form we use to integrate. For example if we want to comput ethe volume of manifold (or some part of M.

$$\mathrm{Vol}(M) = \int \sqrt{|g|} d^d x = \int \epsilon$$

This d-dimensional volume. For $d = 1$ it's called length, for $d = 2$ it's called the area, and for $d = 3$ it's called the volume. Lets also take a look at integration by parts:

$$\int \partial_\mu v^\mu = \text{boundary term(stokkes theorem) in flat space}$$

For curved geometry

$$\int_M \sqrt{|g|} d^d x \nabla_\mu v^\mu = \text{boundary terms}$$

Now we can use integration by arts as well

$$\int \sqrt{|g|} f \nabla_\mu v^\mu = -\int \sqrt{|g|} \nabla_\mu f v^\mu + \text{ boundary terms}$$

He has Baby! Covariant Derivatives. Lets' review what we did last time.

$$\text{Scalar} : \nabla_\mu f = \partial_\mu f$$

$$\text{Vector} : \nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu_{\mu\rho} v^\rho$$

$$\text{One} - \text{form} : \nabla_\mu v_\nu = \partial_\mu v_\nu - \Gamma^\rho_{\mu\nu} v_\rho$$

Where

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\alpha} (g_{\nu\alpha,\rho} + g_{\rho\alpha,\nu} - g_{\nu\rho,\alpha})$$

For a (1,1) tensor $T^\mu{}_\nu$ we have

$$\nabla_\rho T^\mu{}_\nu = \partial_\rho T^\mu{}_\nu + \Gamma^\mu_{\rho\alpha} T^\alpha{}_\nu - \Gamma^\alpha_{\rho\nu} T^\mu{}_\alpha$$

This definition came from the fact taht we wanted linearlity and it to obey Leibnitz's rule and it is also metric compatible

$$\nabla_\rho g_{\mu\nu} = 0$$

Also note that the covariant derivatives do not commute. This accounts for the fact that we're in curved space time.

Curvature

What do we mean when when we say that a geometry is curved?

(a) Flat Geometry: Draw a triangle, then draw a vector at some vertex. We can transport this vector so that it stays parallel to itself. OR as the man says: A vector comes back to itself when transported around a loop. This is a consequence that we're in a flat geometry

(b) Curved geometry: Now consider a sphere and a triangle that connects the north pole with two points on the equator. Do the same exercise with drawing a vector in your head.

This isn't super great so lets try tobe formal. What does it mean to transport a vector around a loop in our geometry? For a scalar f we can define v^μ a tangent vector to the curve. We can say that $\frac{\partial f}{\partial \lambda} = 0 = v^\mu \partial_\mu f = v^\mu \nabla_\mu f$. Thus we can identify

$$v^\mu \nabla_\mu = \text{ directional derivative along the path}$$

A vector w^μ is parallely tranported along a curve if

$$v^\mu \nabla_\mu w^\nu = 0$$

A geodesic is a curve with an affine paramter such that

$$v^\mu \nabla_\mu v^\nu = 0$$

Thus the tangent vector is parallely transported along the geodesic. Do how do vectors change when parllely tranposrted around a very small loop. Consider A^μ and B^μ, some infitesmial displacement vectors. How does v^μ when transported $x^\mu \to x^\mu + B^\mu$

$$v^\mu \to v^\mu + B^\nu \nabla_\nu v^\mu + \dots$$

So if we tranpsort this vector around the parallelagram the linear terms in A and B vanish thus we find the variation of the vector δv^ρ

$$\delta v^\rho = A^\mu B^\nu [\nabla_\mu, \nabla_\nu] v^\rho$$

And since the covariant derivatives do not commute we know that the covariant derivatives do not commute.

Define: the <u>curvature tensor</u> or <u>reimann tensor</u>

$$\delta v^\rho = \underline{R^\rho_{\sigma\mu\nu}} A^\mu B^\nu v^\sigma$$

This tensor is a $(1,3)$ tensor which is a precise notion of curvature at some point in space time. Lets go ahead and try to compute the reimann tensor

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu] v^\rho = R^\rho{}_{\sigma\mu\nu} v^\sigma &= \nabla_\mu \nabla_\nu v^\rho - (\mu \leftrightarrow \nu) \\
&\text{just write stuff out and use symmetry} \\
&= (\partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}) v^\sigma
\end{aligned}$$

Thus

$$\boxed{R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}}$$

This is a tensor and the proof is left as an exercise for the reader (e.g. me). Lets talk about a few features of the Reimann tensor.

(a) If the Christoffel symbol is zero (e.g. in Minkowski space) then R vanishes. Thus a geometry is flat $\Leftrightarrow R^\rho{}_{\sigma\mu\nu} = 0$. There's a theorem related to this

$$\text{Geometry is flat} \Leftrightarrow \exists \text{coord system where metric is constant}$$

(b) The reimann tensor tells us how ST deviates from being flat at a given point. At a given point x_0^μ we can always find coordinates $\hat{x}^\mu$ such that $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. How would we do this? Imagine start in a coordinate system x^μ with metric $g_{\mu\nu}$. A new metric

$$g_{\hat{\mu}\hat{\nu}} = \left(\frac{\partial x^\mu}{\partial x^{\hat{\mu}}}\right)\left(\frac{\partial x^\nu}{\partial x^{\hat{\nu}}}\right)g_{\mu\nu}$$

Taylor expand around $\hat{x}^\mu = 0$ in powers of $\hat{x}^\mu$.

$$\frac{\partial x}{\partial \hat{x}} = \frac{\partial x}{\partial \hat{x}}\bigg|_{\hat{x}=0} + \frac{\partial^2 x}{\partial \hat{x}\partial \hat{x}}\bigg|_{\hat{x}=0}\hat{x} + \partial^3\bigg|_{\hat{x}=0}\hat{x}^2 + \dots$$

The first term has 16 degrees of freedom. The second term has 40 degreees of freedom. The thrid term has 80 degrees of reedom. Now consider

$$\hat{g} = \hat{g}\bigg|_{\hat{x}=0} + \partial\hat{g}\bigg|_{\hat{x}=0}\hat{x} + \dots$$

The first term has 10 degrees of freedom. The second has 40 degrees of freedom. The third term has 100 degrees of freedom. So at $O(\hat{x}^0)$ use 16 DOF in jacobian to get rid of 10 DOF. Then at $O(\hat{x}^1)$ use 40 dof in jacboian to get rid of 40 dof. At $O(\hat{x}^2)$ we cannot set the metric to be zero (there are 20 degrees of freedom that canot be removed by choice of coordinates). So just to summarize, in these coordinates, **Reimann Normal Coordinates**, the metric $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ at $\hat{x}^\mu = 0$. First derivative is 0. Second deriative is not zero. We see that thee Reimann tensor is a packaging of the 20 DOF that we can't elimnate as a tensor.

Curvature:

A spacetime is curved a point $x_0^\mu \in M$ if a vector which is transported around a little loop at x_0^μ does not come back to itself (parallel transport). We then defined the Reimann curvature tensor with

$$\delta_\nu{}^\rho = A^\mu B^\nu R^\rho{}_{\sigma\mu\nu} v^\sigma$$

Where the Riemann curvature tensor is evaluated as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \dots$$

This (1,3) tensor which describes the deviation of spacetime from being exactly flat. Now we ask the question for a point x_0^μ in our space-time manifold, can we set $g_{\mu\nu} = \eta_{\mu\nu}$ at that point through a judicious choice of coordinates. The answer is yes! In particular we argued the metric has 10 degrees of freedom while jacobian has 16 degrees of freedom and thus we can do this. Now we define Riemann Normal coordinates. Given x_0^μ $\hat{x}^\mu$ obey $x_0^\mu \to \hat{x}^\mu = 0$

(1) $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ at $\hat{x}^\mu = 0$

(2) $\partial_{\hat{\mu}} g_{\hat{\rho}\hat{\nu}} = 0$ at $\hat{x}^\mu = 0$

(3) There isn't enought to set the second derivative to zero since there aren't enough degrees of freedom. (20 dof remain)

Claim: Riemann tensor is a packaging of these second derivaties of the metric in a tensorial way.

In Riemann Normal Coordinates (RNC) $\gamma^{\hat{\mu}}_{\hat{\nu}\hat{\sigma}} = 0$ at $\hat{x}^\mu = 0$ which means that the geodesic equation says that $\ddot{\hat{x}} = 0$ (e.g. no fictitious forces). Because of this we also refer to RNC and "locally interital coordintes." They are the coordinates of af reely falling coord system at a point in space time.

We can write the Riemann tensor very simply in RNC.

$$R_{\mu\nu\rho\sigma} = g_{\mu\alpha} R^{\alpha}{}_{\nu\rho\sigma} \qquad \text{at } \hat{x}^\mu = 0$$

(from here we're just gonna leave the hats off)

$$R_{\mu\nu\rho\sigma} = g_{\mu\alpha}\left(\partial_\rho \Gamma^\alpha_{\nu\sigma} - \partial_\sigma \Gamma^\alpha_{\nu\rho}\right)$$

Since we're at $\hat{x} = 0$ we can assert that the metric is the minkowski metric and thus

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\rho \partial_\nu g_{\mu\sigma} + \dots)$$

This has a lot of symmetries

(a) Antisymmetric under interchage of (1,2) and (3,4)

(b) Symmetric under integrange of first pair with second pair

(c) $R_{[\rho\sigma\mu\nu]} = 0 \Leftrightarrow R_{\rho[\sigma\mu\nu]} = 0$.

The interesting thing is that this is true in general. Now an interesting question to ask is how many degrees of freedom are there in a Riemann tensor. First each of the pair of the indices is a anti-symmetric $n \times n$ matrix thus $n(n-1)/2$. The (b) condition implies that the two pairs of indices together have the number of degrees of freedom as a symmetric $n(n-1)/2 \times n(n-1)/2$ matrix e.g.

$$\frac{n(n-1)}{2}\left(\frac{n(n-1)}{2} + 1\right)/2$$

The final condition imposes $\binom{n}{4}$ which me must subtract. The final result is

$$\text{DOF} = \frac{1}{12} n^2 (n^2 - 1)$$

This is the number of independent components of the Riemann tensor. And in $D = 4$ there are 20 degrees of freedom. This should be familiar, this is the number of degrees of freedom in the second derivative of the metric. Thus $R^\rho{}_{\mu\nu\sigma}$ has all the information on second derivative of the metric. For higher derivatives we use covariatn derivties. The Riemann tensor also obeys an identity in it's derivaties, the Bianchi idneitty

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$$

Proof: use RNC. Some other tensors:

(a) The Ricci Tensor: (0,2) tensor

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$$

Up to a sign the Ricci tensor found by tracing over indices of the Riemann tensor. Thus the Ricci tensor encodes all of the trace information in the Riemann tensor. Also note that the Ricci tensor is symmetric and thus the number of degrees of freedom is the same as symmetric matrix $n(n+1)/2$ (e.g. in $D = 4$ DOF is 10)

(b) The Ricci Scalar:

$$R = R_{\mu\nu} g^{\mu\nu}$$

UP to a sign this is the unique scalar that can be found from contracing of Riemann curvature tensor. This scalar is the intuitive notiong of curvature

The bianchi identity of the Ricci tensor

$$\nabla^{\mu} R_{\mu\nu} = \frac{1}{2} \nabla_{\nu} R$$

This motivates the definition of the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

This is a symmetric (0,2) tensor which has some nice properties

$$\nabla^{\mu} G_{\mu\nu} = \nabla^{\mu} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^{\mu} R = 0$$

In other words, the Einstein tensor is conserved. The Einstein tensor is, up to a sign, the unique (0,2) tensor which can be formed from Riemann tensor and metric which is conserved.

Problem sets where passed back. So lets get started and do some physics. We can now escape the mathematics and get into physics.

1 General Relativity

Space time curved. So how do we formulate laws of physics in curved spacetime. More specifically: if we have some law that works in flat space time, how do we generalize to into curved geometry? There is a technique known as the principle of minimum coupling

1.1 Principle of Minimal Coupling

This isn't a law but a guess that needs to be checked with experiment. The einstein equivlance principle says taht physics in a very small region of Spacetime is equivalent to physics in flat spacetime. Indeed we saw last lecture, near a given point in space time, we can find inertial coordinates (RNC) where the metric is approximately flat

$$g_{\mu\nu} = \eta_{\mu\nu} + \dots$$

So lets look into this procedure

(1) Take a law of physics in S.R. (e.g. maxwell equations)

(2) Interpret this as a tensor equation in curvered space written in RNC system at a given point.

(3) This tensor equation is the law of phsycis in curved space.

From a practical point of view all this means is that we covariantize laws of physics in the obvious way.

$$\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$$
$$\partial_\mu \longrightarrow \nabla_\mu$$
$$\int d^4x \longrightarrow \int d^4x\sqrt{|g|}$$

These replacement are what we call the principle of minimum coupling. For example maxwell equations

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$
$$dF = 0 \longrightarrow dF = 0$$
$$\partial^\mu F_{\mu\nu} = 0 \longrightarrow \nabla^\mu F_{\mu\nu} = 0$$

The exterior derivative doesn't change since the exterior derivative operator is defined on any manifold. A second and more relevant example is a particle in special relativity which is not subject to any external forces. That means the particle will move on a staraight line. So if we wrote down a world line $\ddot{x}^\mu = 0$. More formally $v^\mu = \dot{x}^\mu$ and $(v^\mu\partial_\mu = \partial/\partial\lambda)v^\nu = 0$. What is the equation of this particle in curved spacetime. By the principle couline we get

$$v^\mu \nabla_\mu v^\nu = 0$$

To conlude: a freely falling particle in GR will follow a geodesic. A normal (finite mass) object will follow a time-like geodesic, $ds^2 < 0$. A light ray which corresponds with a zero mass particle $ds^2 = 0$. Gravity in general relativity is not htought of as an external force but rater as due to the curvature of space time . What that means is that the statement that particels freely falling will folow a geodesic must reproduce the motion of objects subject to a gravitational field. For example how can geodesic motion reproduce for example Newton's law? Consider a particle $x^\mu = (x^0, x^i)$ moving very slowly. $\dot{x}^0 \approx 1$. and $\dot{x}^i \ll 1$. The geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0$$

In this limit this becomes

$$\ddot{x}^i = -\Gamma^i_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -\Gamma^i_{00} + \dots$$

Compare this to Newton's law for a particle in a gravitiaonal field Φ:

$$\ddot{\mathbf{x}} = -\boldsymbol{\nabla}\Phi \Leftrightarrow \ddot{x}^i = -\partial_i \Phi$$

Comparing the two we see that

$$\phi \approx \text{metric} \qquad \nabla\phi \approx \text{christoffel}$$

Imagine that the metric is very close to $\mathbb{R}^{3,1}$.

$$g_{\mu\nu} = \eta\mu\nu + h_{\mu\nu}$$

Where $h_{\mu\nu}$ is very small.

$$g_{\mu\nu} = \eta^{\mu\nu} + O(h)$$

Computing $\Gamma^{\mu}_{\nu\rho}$ to $O(h)$.

$$\Gamma^{\mu}_{\nu\rho} = \frac{1}{2}\eta^{\mu\rho}\left(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\nu\rho\sigma}\right)$$

Lets in addition take the metric to be time independent. This means that $h_{\mu\nu}$ is independent of t

$$\Gamma^{i}_{00} = \frac{1}{2}\left(h_{00,i}\right)$$

$$\Rightarrow \ddot{x}^i = -\Gamma^{i}_{00} = \frac{1}{2}h_{00,i}$$

Newtons equation become the geodesic equation if we set the metirc to $\eta_{\mu\nu}+h_{\mu\nu}$ with $h_{00} = -2\Phi$. Newtonian gravity can be thought of as a theory of objects in a wekkly curved space time. The geometry that gives us the equation of motion corresponding to a newtonian potential $\Phi(\mathbf{x})$ is

$$ds^2 = -(1+2\Phi(x))dt^2 + d\mathbf{x}^2$$

We expect non-newtonian corrections to arise when some approximations are violated. For example

(a) Φ depends on time

(b) $\mathbf{v} \approx 1$

(c) if Φ is not very small

(d) any other modes of metric are non-zero

So what we've seen here is that it's easy to reproduce the force law

$$\ddot{\mathbf{x}} = -\boldsymbol{\nabla}\phi$$

The next thing we want to repoduce is law

$$\nabla^2\Phi = 4\pi G\rho$$

Lets look at this equation. This should be replaced in the general theory of relativity by a second order differential euaiotn for the metric. It should be a tensor equation. Lets consider first teh case where $\rho = 0$. this would describe the gravitiaonal field away from sources. This would not be true if we were in the center of the earth. For example the orbit of a stattielite around the erath or the orbit of the earth around the sun we can use $\rho = 0$. e.g. we have no source terms and we'rel ooking for a tesnor equation where the second derivative in $g_{\mu\nu}$. the only tensor which is second order in deritiave of $g_{\mu\nu}$ is the riemann tensor. So what possible equations could we write down? First possibilty

$$R^{\rho}{}_{\mu\nu\sigma} = 0$$

This doesn't work since that means that spacetime is flat. Then the metirc of spacetime is flat and then there would be no gravity. Next possiblity

$$\boxed{R_{\mu\nu} = 0}$$

This looks okay, there are 10 degrees of freedom. And the metric has 10 degrees of freedom. In fact this is the only possible equations. In fact this is **Einstein's equation in vaccum**. It is a set of 10 second order PDE's for the metric $g_{\mu\nu}$. This equation describes the curvature of spacetime away from sources. For much of this class we'll be content to study this in vacuum. This equaion is equivalent ot $G_{\mu\nu} = 0$. Recall

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Why is this so?

$$G_{\mu\nu} = 0 \Rightarrow R - \frac{1}{2}(4)(R) = 0 \Rightarrow R = 0$$

The term after right arrow comes from taking the trace. What if we have sources?

$$\nabla^2 \Phi = 4\pi G \rho$$

so what is the correst relativist notion of energy density which could sit on the right hand side of einstien's equations. For example we already know the LHS should be $R_{\mu\nu}$ of $G_{\mu\nu}$. First we know that it should be a gymmetric 2 tensor. What is energy? It's a conserved quantityt that is associated with time translation. So what is the equivlanet symmetry in minkowski space? It's infact the Poincare symmetries (4 transoaltioans and 6 lorentz symmetries). Associated with these 10 symmetries we should be able to construct 10 quantities which are analog of energy. Note that 10 is the same number of componets of a symmetry 2-index tensor. Just like the Ricci tensor or eintsetin tensor. So maybe there's a symmetric tensor that correpsonds to relativistic energy. In fact the 10 quantites are typically packaged into $T_{\mu\nu}$, the stress tensor or energy-momentum tensor. This is a symmetry 2 index tensor. First we start with a definiotn

$$T_{\mu\nu} = \text{Flux of } p^{\mu} \text{ accross a surface of const } x^{\nu}$$

For example

$$T_{00} = \text{Energy density}$$

$$T_{i0} = \text{Momentum density}$$
$$T_{0i} = \text{Shear of E accrossa surface of const } x^i$$
$$T_{ij} = \text{Shear of p accross a surface of const } x^i$$

There's a new problem set up there! To get a law of physics in cruved space

(a) Write S.R.

(b) Tensor equation in curved by covaritizing

$$\eta_{\mu\nu} \to g_{\mu\nu}$$
$$\partial_\mu \to \nabla_\mu$$
$$\int d^4x \to \int d^4x\sqrt{-g}$$

The most basic example is free partciles. Particles not acted on by external forces follow geodesics and in particular they follow time-like geodesics.

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu\dot{x}^\rho = 0$$

this can reproduce newtonian gravity for particles which are moving slowly through a weakly curved geometry ($\dot{x}^0 = 0$ and $\dot{x}^i \ll 1$). The next question is can we reproduce the field equation of newotnian gravity

$$\nabla^2\Phi = 4\pi G\rho$$

We decided that we needed to write down a equation that is second order in the derivative of the metric. We found that the only tensor that works is

$$\nabla^2 G_{\mu\nu} = \text{something}$$

In a vaccum the RHS is zero. The goal for today is to write Einstein's equation in matter. In special relativity there is arelativitsic generalizaiton of the notion of energy: the stress tensor

$$T_{\mu\nu} = \text{Flux of } p^\mu \text{ accross surface of const } x^\nu$$

This is something we've encountered before. Although it's not obvious, the $T_{\mu\nu}$ have a lot of special properties

(a) (0,2) symmetric tensor

(b) $\nabla^\mu T_{\mu\nu} = 0$. This basically says that $T_{\mu\nu}$ is "conserved"

We're gonna be given these without proof. In fact, these properties are easy to prove with the Lagrangian formulation of General Relavity (next class). And indeed $T_{\mu\nu}$ is easy to commute with this formulation. For now we'll just go over some examples.

2 Perfect Fluid

A perfect fluid is a fluid that is parameterized by two independent functions

$$\rho = \text{energy density}$$

$$p = \text{pressure density}$$

There is no viscosity or other higher order effects. What is the stress tensor for a perfect lfuid. For a fluid at rest?

$$T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}$$

"that was easy enough" he says. what if the fluid is moving in some way. Specifically lets says the fluid has a velocity $\mathbf{v}$. It's typical to define teh 4-velocity of the fluid as

$$v^\mu = (\gamma, \gamma\mathbf{v}) \qquad \gamma = \frac{1}{1 - v^2}$$

This is defined so that $v^\mu v_\mu = 1$. Now lets try to ask what the stress tensor is ? We guess that

$$T_{\mu\nu} = C v_\mu v_\nu + D\eta_{\mu\nu}$$

Since these are theo nly symmetric t ensors in two dimensions. So what are the coefficients? Well if the fluid is not moving then it has to reduce to the equation we write above. Thus we see that

$$T_{\mu\nu} = (p + \rho) v_\mu v_\nu + p\eta_{\mu\nu}$$

This is the only possiblitiy that is consistint with symmetry. Now we covariantize to get stress tensor of a perfect fluid in GR

$$T_{\mu\nu} = (p + \rho) v_\mu v_\nu + p g_{\mu\nu}$$

It turns out that for many fluids we have a equation of state which relates the energy density and momentum as

$$p = w\rho$$

Where w is the equation of state paramater. So lets look at some more equations of states for perfect fluids

(i) **Dust:** is a pressureless fluid and it is the fluid with no pressure $p = 0$ e.g. $w = 0$. This just describes some mass distribtuion $\rho(x^\mu)$

(ii) **Radiation:** (i.e. a gas of amsselss particles)

$$p = \frac{1}{3}\rho \qquad \text{i.e.} \qquad w = \frac{1}{3}$$

(iii) **Vacuum Energy:** is a fluid with $p = -\rho$. The stress tensor of this fluid is very simple

$$T_{\mu\nu} = pg_{\mu\nu} = -\rho g_{\mu\nu}$$

"Cosmological Constant".

One can also check that the stress tensor obeys the conservation law $\nabla^\mu T_{\mu\nu} = 0$. In $\mathcal{R}^{3,1}$ this gives the usual conservation equations for a relativistic fluid. For example if $\mathbf{v} \ll 1$ and $p \ll \rho$ then the conservation equation says that

$$\nabla^\mu T_{\mu 0} = 0 \Rightarrow \dot{\rho} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0$$

$$\nabla^\mu T_{\mu i} \Rightarrow \text{Euler equation}$$

3 Einstein's Equation Redux

Now lets ask what is the appropriate equation of GR in the presenece of matter $T_{\mu\nu}$. We would like to reproduce the newtonian equation

$$\nabla^2 \Phi = 4\pi G\rho \to R_{\mu\nu} = ()T_{\mu\nu}$$

What if we wrote down

$$R_{\mu\nu} = kT_{\mu\nu}$$

This can't be conserved because $\nabla^\mu T_{\mu\nu} = 0$ but $\nabla^\mu R_{\mu\nu} \neq 0$. But recall that there is a unique tensor which can be constructed out of the $R_{\mu\nu}$ which obeys the conservation law

$$G_{\mu\nu} = kT_{\mu\nu}$$

This describes the curvature of spacetime due to some sources. Does this equation of motion reproduce newtons law in the approximation when spacetime is weakly curved. We saw that the metric weakly curved space

$$ds^2 = -(1 + 2\phi)dt^2 + dx^2 + O(\phi^2)$$

For dust at rest.

$$T_{\mu\nu} = \text{Diag}(\rho, 0, 0, 0)$$

Now consider

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = kT_{\mu\nu}$$

We want to compute G to order ϕ and the RHS to order ϕ^0. Take the trace of the both sides

$$R - 2R = -R = kT = kg^{\mu\nu}T_{\mu\nu} = -k\rho + O(\phi)$$

Note that

$$R_{00} = \frac{1}{2} g_{00} R + kT_{00} = \frac{1}{2} k\rho$$

Computer $R_{00} = R^\lambda{}_{0\lambda 0}$ to $O(\phi)$ for weakly curved ds^2.

$$\vdots$$

We have shown that Einstein'sequation reproduces Newtonian gravity.

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu} \text{ "Einstein's equation"}}$$

3.1 Some comments

Taking the trace we also get

$$R = -8\pi G T$$

Which also gives us

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu})$$

This is called the trace subtracted einstine's equations. Some more comments, these are tensor equations meaning that they are ture in any coordinate system. Another note is that the einstein equation is a set of 10 second order differential equation for the 10 componenets of $g_{\mu\nu}$. These are nonlinear equations

$$R_{...} \approx \partial^2 g + (\partial g)^2$$

If g_1 and g_2 are solutions then $g_1 + g_2$ is not a soltuions. This is because gravity has energy which are sources for GR. This is unlike EM where you could just add two solutions but the reason we can do this in EM is that the fields (and photons) have no electric charge. The EM fields themselves are sourceless. Because of this Einstine's equations are very difficult to solve. Just a preview of what we'll see, solutions of einstine's equation are known exactly only if there is some sort of symmetry. For example spherically symmetric (sun, earth). A lot of other times we use approxiamte solutions. And finally we have a lots of numerical relativity.

Reading week is called spring break??

4 Lagrangian Formulation

We concluded last time

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}$$

Is there any other way to get to Einstein's equaiton? Namely can this come from an actio principle?

For most classical systems the EOM are the Euler-Lagrang equatio that arise from minimizing the action $S[q_i(t)]$. For example if we have one degree of freedom then the action is dependent on the path

$$S[q(t)] = \int dt L(q, \dot{q}, t)$$

When $\delta S/\delta q = 0 = \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)$.

A field theory isa theory with 1 or more DOF at each point in space. For example

$$\Phi(t, \mathbf{x}) \qquad \text{1 DOF per point in space}$$

$$A_\mu(t, \mathbf{x}) \qquad \text{4 DOF} \ldots$$

$$g_{\mu\nu}(t,\mathbf{x}) \qquad 10 \text{ DOF} \ldots$$

So how do we get EOM in a field theory? Well the field equations. We seek an action the depends on our fields

$$S[\phi(x^\mu)]$$

This is a functional of our fields. For most field theories the correction action has the fllowing form

$$S = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi)$$

Where $\mathcal{L}$ is the larange density. If we can formulate a theory in this matter then we call the theory a local field thory. Now how would we minimize this action?

$$\phi \to \phi + \delta\phi$$

This means the action varies as such

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu(\delta\phi) \right)$$

Now we integrate by parts

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \partial_\mu(\ldots) \right)$$

If ϕ vanishes at infinity then we just ignore boundary term. Then the variation of the action leads us to

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

This was an example of a scalar field theory since ϕ was a scalar function and had no indices.

Now we can ask if there is a action that is a functional of $g_{\mu\nu}$. We can figure this out by thinking **very** hard. We note that

(a) The action should have the form of

$$S = \int d^4x \sqrt{-g} \mathcal{L}$$

Such that $\mathcal{L}$ to be scalar. This is the only way that the action is a scalar.

(b) The Lagrangian should involve two derivatives of the field since we know that the EOM have a quadratic derivative and if we apply the EOM to a lagrangian that is quadratic in the derivatives then we get EOM that are quadratic.

Recall that there is only a single scalar that is second derivative in the metric.

$$\mathcal{L} = cR$$

The action must take the form

$$S = c \int d^4x \sqrt{-g} R$$

And we call this the Einstien-Hilbert action. Lets try to use this action to derive the EOM for general relativity.

$$g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$$

Lets recast the action a bit

$$S = c \int d^4x \sqrt{-g} g^{\mu\nu} R^{\mu\nu}$$

How do these components vary?

(a) $g^{\mu\lambda} g_{\lambda\sigma} = \delta^\mu{}_\sigma$. We want this to be true to first order in varation.

$$(g^{\mu\lambda} + \delta g^{\mu\lambda})(g_{\lambda\sigma} + \delta g_{\lambda\sigma}) = \delta^\mu{}_\sigma + O(\delta g)$$

After some work we see that there is just a minus sign in front of the variaton of the inverse of the metric.

(b) To derive $\delta\sqrt{-g}$ we use $\det g = e^{\mathrm{Tr}(\log g)}$. Log and exp of matrix are found with the tyalor expansion. This allows us to calculate variation

$$\delta g = g \frac{\partial \mathrm{Tr} \log g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = g g^{\mu\nu} \delta g_{\mu\nu} = -g g_{\mu\nu} \delta g^{\mu\nu}$$

Thus we get that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}$$

(c) The final part we need to consider is the Ricci tensor. He's not gonna do this for us is a total derivative and is really messy so we're not gonna look at it

With all this we can now consider the equation of motion.

$$\delta S = c \int d^4 \sqrt{-g} \Big(R_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta R_{\mu\nu} \Big)$$

This gives us

$$R_{\mu\nu} = \frac{1}{2} R g_{\mu\nu}$$

The einstein equation in the vacuum. This now gives us a very easy way to couple Einstein equation to matter. We have

$$S = S_{EH} + S_{\text{Matter}}$$

Then this gives us (let partial derivates be variations)

$$c G_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\partial S}{\partial \delta g^{\mu\nu}}$$

Now we can claim that

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\partial S}{\partial g^{\mu\nu}}$$

with $c = \frac{1}{16\pi} G$. this gives us a extremley easy way to compute $T_{\mu\nu}$. Remarkably this definition usually agrees with our previous definition.

4.1 Some examples

$$S_{\text{matter}} = -\int \sqrt{-g}\left(\frac{d\Lambda}{d8\pi G}\right)$$

This gives us the following EOM

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\Lambda g_{\mu\nu}$$

This is a perfect fluid where the pressure is equal the negative energy. This kind of matter is very important. We have that Λ is the cosmological constant and sometimes also reffered to vacuum energy because $S_\Lambda = \int \text{const}$ and $L = T + V$ so adding a constant to V this is equivalent to adding some energy to the empty space. (note I think he had some sign error here).

It should be bizzare that this appears in the EOM since in classicla mechanics the potential is defind up to a constant, we shouldn't have to worry about this in our EOM. However here it does have some important implications in GR. It is likely that the cosmological constant (or something like it) is in the Lagrangian of nature.

Just a quick note: GR + Cosmological Contant we have (after taking the trace of both sides

$$R = 4\Lambda$$

Such geometries with $R_{\mu\nu} = \text{const} g_{\mu\nu}$ are known as "Einstein spaces."

Hard problems teach wowie! They're talking about the final exam. For them it's handed out 4/22 and due at 4/23 at noon.

So here's the plan for the rest of the semester: we have the EOM for GR. We will study other general features of the geometry of spacetime. Next we will explore the schwarzfield solution (spherically symmetric static object.) Next we'll talk about tests for GR. Then we'll have one lecture on black holes and then talk about cosmology and gravitational waves and then a varietey of things.

5 Equation of Motion

Recall that the EOM for GR is

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Another way to think about these EOM is to think of them from following from an action principle

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int \sqrt{-g}R$$

If we wish to include coupling to matter then

$$S[g_{\mu\nu}] = \frac{1}{16\pi G}\int \sqrt{-g}R + S_{\text{matter}}[g_{\mu\nu}, \text{matter}]$$

Extremizing the action leads to

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial S_{\mathrm{M}}}{\partial g^{\mu\nu}}$$

This also gives a simple way to compute $T_{\mu\nu}$. For example

(a) cosmological constant: In SR we can let $\mathcal{L}$ = cosnst but in general relativity $S_{\mathrm{m}} = -\int d^4x\sqrt{-g}\Lambda/8\pi G$, we have an extra $\sqrt{-g}$ that leads to the constant shift to affect the EOM.

(b) next simplest matter field theory is where the dynamicla variable is simply a scalar $\phi(x^\mu)$. What possible action could we write down? First we expect that the action should be quadractic in derivatives of ϕ (e.g. kinetic term) and also the action should be an integral of a scalar.

$$S[\phi] = \int d^4x\sqrt{-g}\mathcal{L}$$

My guess: $\mathcal{L} = \partial_\mu\phi\partial^\mu\phi$. I was wrong it's actually

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi) - V(\phi)$$

The ϕ EOM in special relativity is $\partial^\mu\partial_\mu\phi = V'(\phi)$.

Now in General relavitiy we have

$$S[\phi, g] = \int d^4x\sqrt{-g}\Big(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\Big)$$

The ϕ EOM would then become

$$\nabla^\mu\nabla_\mu\phi = V'(\phi) = \nabla^2\phi$$

We sometimes call ∇^2 is called laplacian or d'lambertian. What's more interesting is the stress tensor

$$T^{\text{scalar}}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}((\nabla\phi)^2 + V)$$

there's a factor of 1/2 missing somewhere. This equation is very hard to solve. However there is one case where his is very easy. This is where ϕ = const. This means that

$$T^{\text{const}}_{\mu\nu} = -\frac{1}{2}g_{\mu\nu}V(\phi)$$

Namely we look just like eqn with a cosmological constant.

(c) Electromagnetism: The basic DOf we have the EM potential $\mathbf{A}$ and V so we have

$$A_\mu = (V, \mathbf{A})$$

From this we can write maxwell's equation very simply. From this one-form $A = A_\mu dx^\mu$ we can define a two form $F = dA = F_{\mu\nu} dx^\mu \wedge dx^\nu$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

the field strength tensor is a 2-form. From here it is very easy to write down maxwell's equations

$$dF = 0 \qquad \nabla^\mu F_{\mu\nu} = j_\nu$$

Where $j^\mu = (\rho, \mathbf{j})$. Finally we can write down the action of EM in special relativity

$$S = \int d^4x(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Which we get by thinking about what $\mathcal{L}$ can we write down that is second derivative in A^μ and is a scalar. Now with EOM we can get maxwell equations. Now we covariantize this.

$$S[g, A_\mu] = -\frac{1}{4}\int d^4x\sqrt{-g}\left(g^{\mu\nu}g^{\rho\sigma}F_{\mu\rho}F_{\nu\sigma}\right)$$

From this we get

$$T_{\mu\nu} = F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^2$$

General relavity coupled to EM is very special: we call it einstein-maxwell theory. This is so important because gravity and EM are the only two long range forces that we know of.

$$G_{\mu\nu} = 8\pi G(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^2)$$

$$dF = 0 \qquad \nabla^\mu F_{\mu\nu} = 0$$

these EOM describe EM+GR in abscence of sources. So the equation above describe basically 99% of the dynamics of the universe.

6 Solving EOM and Symmetries

these are hard to solve (10 coupled 2nd order diff-eq). So lets instead think about spacetimes with symmetries. Indeed many of the EOM willy simply in this case.

What does it mean for a geometry to have a symmetry? What's he simplest pacetime?

$$\mathbb{R}^{3,1} : ds^2 = -dt^2 - d\mathbf{x}^2$$

This has translation symmetry $x^\mu \to x^\mu + a^\mu$ and lorentz transofmration $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. Thus the symmetires of $\mathbb{R}^{3,1}$ are coordinate transformation which left metric invariant. So the correct way to think of a symmetry in SR is a coordinate transfomration which leaves the metric invaraint. An isometry is a coordinate transformation $x^\mu \to x^{\mu'}$ which leaves $g_{\mu\nu}$ unchanged.

The basic point is metric with lots of isometries are easy to study. Some pedantics: the set of isometries of a spacetime is the "isometry group" of the ST.

Isometries can either be discrete (more difficult e.g. $t \to -t$) or continous (e.g. $t \to t+s$). It is the cotinous isometries that are useful for us? Why are these useful? Well continous isometries are useufl because they lead to conserved quantites as they do in classicla mechanics. Here we can now derive a noether charge similar to how we derived conserved quantities in classicla mechanics. Lets consider a continous family of coordinate transformations. In particular

$$x^\mu \to x^{\mu'}(x^\mu, s) \text{ s.t. } x^{\mu'}(x^\mu, 0) = x^\mu$$

Technically called "continous somehting conneted to identity." Now we expand around the trivial coordinate transofrmation

$$x^{\mu'}(x^\mu, s) = x^\mu + s\frac{dx^{\mu'}}{ds}\bigg|_{s=0}$$

The v^μ are a vector which describe a coordinate tranformation. In particular the vector v^μ generate this family ofcoordinate transfomration. for example in $\mathbb{R}^{3,1}$ when we have $t \to t+s$ is generated by $v^\mu\partial_\mu = \partial_t$. How does $g_{\mu\nu}$ transofrms under a coordinate transform generated by v^μ.

Quick questions: homework psets...

7 Symmetries

An isometry of an ST w/ metric $g_{\mu\nu}$ is a coord transofmr which leaves the metric unchacnged

$$g_{\mu'\nu'}(x') = g_{\mu\nu}(x)$$

A continous isometry is a family of coordinate transformations (is a one-parameter family) of isometries

$$x^\mu \to x^{\mu'}(x^\mu, s) \text{ s.t. } x^{\mu'}(x^\mu, 0) = x^\mu$$

Lets call $x^{\mu'} = y^\mu$. Expanding the transformd coord around the identity

$$y^\mu(x^\mu, s) = x^\mu + s\frac{dy^\mu}{ds}\bigg|_{s=0} + \dots$$

Where v^μ is the vector which generates this coordinate transform. so how do tensors ranform under coordinate tranfomrs gneerated by the vector v^μ.

7.1 Scalar

So how doe a scalar $f(x^\mu)$ transform when $x^\mu \to y^\mu$.

$$\begin{aligned} f'(y^\mu) &= f(x^\mu) \\ &= f(y^\mu - sv^\mu + \dots) \\ &= f(y^\mu) - sv^\mu\partial_\mu f + \dots \end{aligned}$$

This means that

$$f'(y^\mu) - f(y^\mu) = -s(v^\mu \partial_\mu f) + \dots$$

Note that $v = v^\mu \partial_\mu$ takes a scalar to a scalar.

7.2 Vector

How does w^μ transform?

$$\begin{aligned} w'^\mu(y^\mu) &= \frac{\partial y^\mu}{\partial x^\nu} w^\nu(x^\mu) \\ &= (\delta^\mu_\nu + s v^\mu{}_{,\nu} + \dots)(\omega^\nu(y^\mu) - s v^\alpha \partial_\alpha w^\nu + \dots) \\ &= w^\mu(y^\mu) - s(v^\alpha \partial_\alpha w^\mu - w^\nu \partial_\nu v^\mu) + \dots \\ \Rightarrow w'^\mu - w^\mu &= -s(v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu) \end{aligned}$$

The quantity in the parenthesis is very special, it's the lie derivative of w w.r.t. v

$$\mathcal{L}_v w^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$$

And similarly for a scalar

$$\mathcal{L}_v f = v^\mu \partial_\mu f$$

In a earlier probelm set we called the Lie derivate of a vector wrt v^μ is also called the lie bracket, orcommutaor, of the two vectors

$$\mathcal{L}_v w^\mu = [v, w]^\mu = v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu$$

The reason that is called a commutator is becaused this is bliear in v and w and is anti-symmetric.

Using the same logic we can unerstand how a tensor transforms under the infitesmail coordinate transformion generated by v^μ

$$T'^{\mu_1 \cdots \mu_k}{}_{\nu_1 \dots \nu_l}(y^\mu) - T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \dots \nu_l}(y^\mu) = -s\mathcal{L}_v T^{\cdots}{}_{\dots}$$

Where

$$\begin{aligned} \mathcal{L}_v T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \dots \nu_l} &= v^\alpha \partial_\alpha T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \dots \nu_l} \\ &- T^{\alpha \mu_2 \dots \mu_k}{}_{\nu_1 \dots \nu_l} \partial_\alpha v^{\mu_1} - \dots \\ &+ T^{\mu_1 \cdots \mu_k}{}_{\alpha \nu_2 \dots \nu_l} \partial_{\nu_1} v^\alpha \end{aligned}$$

This has some interesting properties

(a) $\mathcal{L}_v T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \dots \nu_l}$ is a (k, l) tensor

(b) A new way to taking a derivative of tensors! We've seen exterior derivative, covariant derivative, and now we have a lie derivative. It coresponds to a derivatie of a tensor of a vector v^μ. It does not require a metric.

(c) The Lie derivitve is linear and obeys a Lebnitz rule(I'm not gonna write it out, we all know what it is). There is anothe rem

Fact: In the fomrulation of $\mathcal{L}_v T^{\cdots}{}_{\dots}$ I can repalce $\partial_\mu \to \nabla_\mu$ and the result is unchanged! The proof is left for us on the on the next problem set

7.3 Metric

How does $g_{\mu\nu}$ transform under coordinate transform generated by v^μ? We get that from the lie derivative

$$\begin{aligned}\mathcal{L}_v g_{\mu\nu} &= v^\alpha \partial_\alpha g_{\mu\nu} \\ &+ g_{\alpha\nu}\partial_\mu v^\alpha \\ &+ g_{\mu\alpha}\partial_\nu v^\alpha \\ &= v^\alpha \nabla_\alpha g_{\mu\nu} \\ &+ g_{\alpha\nu}\nabla_\mu v^\alpha \\ &+ g_{\mu\alpha}\nabla_\nu v^\alpha\end{aligned}$$

Now since the covariant derivative is metric compatible we know that the first term is zero and that we can move g into the covariant derivative giving us

$$\mathcal{L}_v g_{\mu\nu} = \nabla_\mu v_\nu + \nabla_\nu v_\mu$$

This measn that the metric is invarint under the coord. transform given by v^μ iff

$$\nabla_\mu v_\nu = -\nabla_\nu v_\mu$$

This is the **killing equation** and a solution to this equaion is known as a **killing vector**

To generalize a continous isometry is generated by a killing vector.

For a given geotetry the number of contious isometries is the number of independt solution to the kiilling equation. Here are some xample

(a) $\mathbb{R}^{3,1}$ we have translation generated by ∂_μ and loretnz ransfomration $x_\mu\partial_\nu - x_\nu\partial_\mu$.

(b) In $\mathbb{R}^3$ we have rotations generated by $x_i\partial_j - x_j\partial_i$ e.g. $SU(2)$ or $SO(3)$.

So to flesh out our knowledge of killing vectors lets note that thecomutator (lie bracket) of two killing vectors i a killing vector. This means that they form a vector space or algebra. This is sometimes called an isometr algebra.

The real reason that killing vectors are useful: if ds^2 has an isometry generated by k_μ this means that we can choose coords x^μ such that the compoentnes of hte metric $g_{\mu\nu}$ are independent of one of the coordinates. This makes solving EOM easier. Here's a proof: consider

$$k^\mu\partial_\mu = \frac{\partial}{\partial y}$$

Where we choose a coordinate system y where the above is true. Namely we choose a coord system such that

$$k^\mu = (1,0,0,0)$$

This means that the lie derivative is

$$\mathcal{L}_k g_{\mu\nu} = k^\alpha\partial_\alpha g_{\mu\nu} = \frac{d}{dy} g_{\mu\nu} = 0$$

QED

If the geometry has n commuting killing vectors then one can find a coordinate system such that $g_{\mu\nu}$ is indepndent of n of the four coordinates.

(a) $\mathbb{R}^{3,1}$ we know that killing vector ∂_μ commute meaning that we can choose cartesian coordinate $\eta_{\mu\nu}$

(b) S^2 has 3 killing vectors which do not commute (rotations do not commute). At best we can find a metric that is independt of one of the coordinates but not all three. e.g. $k^\mu \partial_\mu = \partial_\phi$ then metric indep of ϕ thus $ds^2 = d\theta^2 + \cos^2\theta d\phi^2$.

Claim: Up to a change of coordinates this is he unique metric in 2-dims which has $SU(2)$ or $SO(3)$ isometries

This is all well and good but why do we care? Symmetires are useful because they lead to conserved quanties from Noether's theorem. Take a metric $g_{\mu\nu}$ with a killing vector k^μ. Take an observer moving ona geodesic in this geometry

$$x^\mu(\lambda)$$

Where we take λ to be a affine parameter. The tangent vector is

$$v^\mu = \dot{x}^\mu$$

And from here we have

$$I(\lambda) = k_\mu v^\mu$$

And from this we can prove that I is independent of λ. To this we just use the fact that the traj is a geoedisc

$$v^\mu \nabla_\mu v^\nu = 0$$

And the killing equation

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0$$

And to compute

$$\frac{dI}{d\lambda} = v^\mu \partial_\mu I = v^\mu \nabla_\mu I = 0$$

In $\mathbb{R}^{3,1}$ this gives us the usual conserved enegy and momentum in SR.

This fact gives a definition of energy in any spacetime with a time-translation symmetry nad likewise gives a notion of linear(angular) momentum in a ST with a tranlsation or rotation symmetry. something to note is that not all geometries have this symmetry. This means that sometimes thereis no unique notion of conserved energy (e.g. expanding energy and cosmology).

It turns out he sent a hint for 4b) that I spent too long on.

8 Schwarzschild Solution

From this point forward we're gonna shit gears and study application of GR.

Question: What is the gravitaional field (aka metric) around a static, spherically symmetric object. (for most observational effects we can treat most things as static spherically symmetric.)

In newtonian gravity: Outside object $\nabla^2\Phi = 0$. We can show that the only solution is

$$\phi = GM/r$$

In General Relativity: we seeks a metric that is static and sypherically symmetric and slve Einstein's equaiton in vacuum oustide the interior of the object. Static means there is a time-translation symmetry (time-like killing vector $k^\mu\partial_\mu = \partial_t$) and it will have a rotation symmetry (3 spacelike killing vector which generate the $SU(2)$ algebra). **Claim:** the most general such metric is

$$ds^2 = e^{2\alpha}dt^2 + e^{2\beta}dr^2 + e^{2\gamma}d\Omega^2$$

Note that because of the killing vectors there is no t, ϕ, θ dependence. Thus α, β, γ are functions of r. To recapitualte spherical symmetry $\Rightarrow d\Omega^2$ while rest of ds^2 is indept of θ and ϕ. Static $\Rightarrow ds^2$ is indept of t. There are no off diagonal terms: these diagonal terms would describe rotating solutions. So indeed if we write metric for earth and sun and we take into account rotation of the earth then there is rotation. Let write $\hat{t}$ instead of t for a second. First lets choose a new radial coordinate so that $r = e^\gamma$

$$\Rightarrow ds^2 = -e^{2\alpha}d\hat{t}^2 + e^{2\beta}dr^2 + r^2 d\Omega^2$$

This means that we are choosing a radial coordinate r to measure the size of S^2 just as the radial coordinate of $\mathbb{R}^3$ or $\mathbb{R}^{3,1}$ in plar coordinates

$$ds^2_{\mathbb{R}^3} = dr^2 + r^2 d\Omega^2$$

So what is the behavior of this geometry outside sources we need to solve

$$R_{\mu\nu} = 0$$

It's straightforward to show that

$$ds^2 = -e^{2\alpha}d\hat{t}^2 + e^{2\beta}dt^2 + r^2 d\Omega^2$$

Leads to

$$R_{\mu\nu} = \dots$$

Lets consider the following combination

$$e^{-2(\alpha-\beta)}R_{\hat{t}\hat{t}} + R_{rr} = \frac{2}{r}(\dot{\alpha} + \dot{\beta}) = 0 \to \alpha = -\beta + c$$

This gives us

$$ds^2 = -e^{-2\beta}(e^{2c}d\hat{t}^2) + e^{2\beta}dr^2 + r^2 d\Omega^2$$

Let us let $t = e^c \hat{t}$ giving us

$$ds^2 = -e^{-2\beta} dt^2 + e^{2\beta} er^2 + r^2 d\Omega^2$$

Consider the other parts of the Ricci tensor

$$R_{\theta\theta} = 0 = e^{-2\beta}(2\dot{\beta}r - 1) = -1 \Rightarrow -r\frac{\partial}{\partial r}e^{-2\beta} - e^{-2\beta} = -\frac{\partial}{\partial r}\left(re^{-2\beta}\right) = -1$$

$$\Rightarrow e^{-2\beta} = 1 - \frac{R}{r}$$

This leads to the following metric

$$\boxed{ds^2 = -(1 - R/r)dt^2 + (1 - R/r)^{-1} dr^2 + r^2 d\Omega^2}$$

Note that R is a const of integration with units of length and is usually called the schwarzschild raidus. The boxed metric is important, it's called the Schwarzschild Metric.

We claim that this describes the geometry around the static spherically symmetric object. Lets compare this to the Newtonian limit $\phi = GM/r$. Recall that we also showed that a metric that leads to this is $g_{tt} = -(1 - 2\phi)$+higher order. Now note

$$g_{tt} = -(1 - R/r) \stackrel{?}{=} -(1 - 2\phi) = -(1 - 2GM/r)$$

We see then that $R = 2GM$. We can think of R as the gravitational length scale of a massive object. As $r \to \infty$ the metric just becomes $\mathbb{R}^{3,1}$. Also this is the unique solution with spherical symmetry. Note that when r comes close to R then funny things happen. Namely when $r = R$ the metric becomes

$$ds^2 = 0 \cdot dt^2 + \infty dr^2 + R^2 d\Omega^2$$

This is a signature that the metric is breaking down. However the geometry is completley smooth at $r = R$. We call $r = R$ is the "horizion" of the solution.

Something funny also happens when $r = 0$. The metric diverges a lot. There is a singularity in the geometry. This is kinda like the delta function singularity in the $1/r$ potential.

For "normal" objects the schwarzschield raidus is tiny. For example for the earth

$$R_s(\text{Earth}) = 1\text{cm} \ll R_{\text{earth}}$$

$$R_s(\text{Sun}) = 3\text{Km} \ll R_{\text{sun}}$$

What does this mean? Well for a normal object, the horizion for $r = R_s$ is not physical since the schwarzschield metric isa good description of an object only outside the object. If we wanted to, we could try to write down the metric insde the eart or the sun and would depend on how we measure the source.

There are other sorts of objects where their radius is $< R_s$ is a black hole... But first lets consider more normal objects

Consider the geodesic motion of an observer in the schwarschield background. So a massive object follows a time-like geodesic

$$x^\mu(\lambda) = (t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda)) \qquad \left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu < 0$$

Where the proper time measured along this geodesic is

$$\frac{d\tau}{d\lambda} = \sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} = 1$$

We will let λ be affine. We can write down $\ddot{x}^\mu + \Gamma^\mu_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0$. By spherical symmetry we can consider a geodesic in the plane where $\theta = 0$. In this case the geodesic is quite easy to analayze. Also note that the metric has 2 killing vectors ∂_t and ∂_ϕ which leads to two conserved quantities E and L (energy and angular momentum). What is the formula for a conserved quantity we find that

$$I = k_\mu v^\mu = k_\mu \dot{x}^\mu \text{ proved on pset6 is indep of } \lambda$$

So lets find the conserved quantities

$$E = k_\mu \dot{x}^\mu \qquad k^\mu = (1,0,0,0) \Leftrightarrow k_\mu = (-(1-R/r),0,0,0) \Rightarrow = (1-R/r)\dot{t} \Rightarrow \dot{t} = \frac{E}{1-R/r}$$

$$L = K_\mu \dot{x}^\mu \qquad K^\mu = (0,0,0,1) \Leftrightarrow K_\mu = (0,0,0,1/r^2\cos^2\theta) = (0,0,0,1/r^2) \Rightarrow L = \dot{r}^2\dot{\phi} \Rightarrow \dot{\phi} = L/r^2$$

This is one of his favorite lectures of this course!!!!

9 Geodesics for Schwarzschield Metric

The schwarzschield metric describes ST outside a spherically symmetric static object.

$$ds^2 = -(1-R/r)dt^2 + (1-R/r)^{-1}dr^2 + r^2 d\Omega^2$$

Where $R = 2GM$ is the schwarzschild radius. It's the typical length scale of this geometry. Right now we're worried about "normal" objects where radius $\gg R$.

A massive object will travel along a timelike geodesic so we need to write down the geodesic equation. Lets see a trick to derive the geodesic very quickly

$$x^\mu(\lambda) = (t(\lambda), r(\lambda), \theta(\lambda), \phi(\lambda))$$

Where λ is an affine parameter meaning that $(ds/d\lambda)^2 = -1$. We also choose to study geodesic $\theta = 0$. This means that (note that $\dot{x} = \partial x/\partial\lambda$.)

$$\left(\frac{ds}{d\lambda}\right)^2 = -1 = -(1-R/r)\dot{t}^2 + (1-R/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2$$

In addition we have 2 conserved quantites $\partial_t \Rightarrow E$ and $\partial_\phi \Rightarrow L$. We computed these last time

$$E = (1 - R/r)\dot{t} \qquad L = r^2\dot{\phi}$$

Plugging these in gives us

$$\left(\frac{ds}{d\lambda}\right)^2 = -1 = (1 - R/r)^{-1}E^2 + (1 - R/r)^{-1}\dot{r}^2 + r^{-2}L^2$$

After a little bit of algbera

$$\frac{1}{2}\dot{r}^2 + (V_{\text{eff}} = -R/2r + L^2/2r^2 - RL^2/2r^3 + 1/2)L^2r^{-2} + \frac{1}{2}(1 - R/r)) = \frac{1}{2}E^2$$

This is the EOM for a particle in a 1-D potential. So if we want to descibe the motion of aprticle moving in a geodesic we just need to solve this central force problem. Comparing this to the analogous formula for newtonian gravity

$$V_{\text{eff,newtonian}} = -R/2r + L^2/2r^2$$

Lets remember the story in newtonian gravity. The effective potential is the sum of two terms. This leads to various properties

(a) for all L there eixsts a stable circular orbit $r_c = 2L^2/R$

(b) Orbits are conic sections (this is a special feature of $1/r$ potential like newotnian gravity

(c) angular momentum barrier which prevents us from getting to $r = 0$.

(d) For all values of r there exists a stable circular orbit. $L^2 = r_c R/2$.

In einstein gravity we have (if we put the $c = 1$ terms back in)

$$V_{\text{eff}} = \frac{1}{2} - \frac{R}{2r} + \frac{L^2}{r^2} - \frac{RL^2}{2r^3c^2}$$

We note that the last term is supprsed by inverse c term. Drawing the V_{eff} we see a different landscape. So what the dynamics here? Again we can solve for the stable circular orbit r_c and unstable r_{crit}

$$r_c = \frac{L^2 + \sqrt{L^4 - 3R^2L^2}}{R} \qquad r_{\text{crit}} = \frac{L^2 - \sqrt{L^4 - 3R^2L^2}}{R}$$

When $L = \sqrt{3}R$ then these two coincide. What happens now? When $\sqrt{3}R < L < \infty$ we have $3R < r_c < \infty$ and $3R < r_{\text{crit}} < 3R/2$. So unlike the newotinan case stable ciruclar orbits exist only for $r > 3R$ but there are unstable circular orbots for $3/2R < r < 3R$. Now $V_{\text{eff}}(r_h) = 0$ gives us

$$r_h = \frac{L^2 - \sqrt{L^4 - 4R^2L^2}}{2R}$$

You shouldn' trust this since we're beyond the schwarzschield radius where the geometry became kinda fucky. For $r < R$ it is impossible for a timelike geodesic to $r \to \infty$ no matter how much inital velocity or angular momentum we give it.

For the rest of this class we'll only consider normal objects with $r \gg R$. Here the discrepancies from Newtonian gravity are small but are measurable. These lead to the three classic tests to GR

(a) Precession of Perihelion of Mercury

(b) Deflection of light by Sun (gravitaional lensing)

(c) Gravitational Redshift

These are really tests of the schwarzschield metric. Experimental tests of GR are few and far between and so a lot of aspects of GR aren't tested as rigorously as say particle physics or qunautm mechanics.

10 Precession of Perehelion of Mercury

Recall the effective potential

$$V_{\text{eff}} = -\frac{R}{2r} + \frac{L^2}{2r^2} - \frac{RL^2}{2r^3}$$

We'll treat the last term as a small perturbation. This correction will lead to non-periodic orbits... Lets remember how we compute the periodicity

$$E = \frac{1}{2}\dot{r}^2 + V_{\text{eff}}$$

Lets consider $r(\phi)$ instead of $r(\lambda)$ and how do we do that? Well $\dot{\phi} = L/r^2$ this means that $r' = \frac{\partial r}{\partial \phi} = \dot{r}/\dot{\phi} = r^2\dot{r}/L$. This gives us

$$E = \frac{1}{2}r'^2(L^2/r^4) - R/2r + L^2/2r^2 + RL^2/2r^3$$

Multiplying everybing by $2r^2/L^2$

$$E2r^2/L^2 = r'^2/r^2 + Rr/2L^2 + 1 - R/r$$

Now let $u = 2L^2/Rr$ meaning that $u' = -2L^2/Rr^2r'$ giving us that $u'/u = -r'/r$ rewritting the above as

$$u'^2/u^2 - 2/u + 1 - R^2u/2L^2 = 8L^2E/R^2 1/u^2 \Rightarrow u'^2 - 2u + u^2 = \text{const} + R^2u^3/2L^2$$

Taking the derivative wrt ϕ then gives us

$$\underline{u'' + u = 1} + \frac{3}{2}\frac{R^2}{L^2}u^2$$

The underlined term is an EOM of a SHO w a constant driving force. In newtonain gravity $u_0 = 1 + e\cos\phi$. If we wish to include the GR correction lets work in the limit where that correction is small. Using perturbation theory...

$$u = u_0 + u_1 + \dots$$

How is u_1 obtained? Well it's obtained by pluggin in u_0 into our differential equation

$$u_1'' + u_1 = \frac{3}{2}\frac{R^2}{L^2}$$

It might be a $\frac{3}{4}$ instead of $\frac{3}{2}$. There was a numerica eroor somehwere. We'l fix it from here

$$u_1'' + u_1 = \frac{3}{4}\frac{R^2}{L^2}u_0^2 = \frac{3}{4}\frac{R^2}{L^2}(1 + e\cos\phi)^2$$

EOM of a SHO a driving force . We can solve this with a green's function

$$u_1(\phi) = \frac{3}{4}\frac{R^2}{L^2}\int (1 + e\cos\phi')^2 \sin(\phi - \phi')d\phi = \frac{3R^2}{4L^2}e\phi\sin\phi + \text{periodic in } \phi$$

The terms that are periodic in ϕ the peridoci terms don't grow in ϕ. This means that

$$u = 1 + e\cos\phi + e\frac{3R^2}{4L^2}\phi\sin\phi + \dots \approx 1 + e\cos(1 - \alpha)\phi + \dots \qquad \alpha = \frac{3R^2}{4L^2}$$

What this means is that the period changes slightly for each orbit. The angle that is added per orbit is $\Delta\phi = 2\pi\alpha$ =advance in perhilon of mercury in orbit. We can compute this numericlaly as

$$\Delta\phi = 43 \text{ arcseconds/century}$$

This was measured in mid-19^{th} century was the first success of GR. There are other planets that also cause $\Delta\phi$ which gives us

$$\Delta\phi_{3-\text{body}} \gg \Delta\phi_{GR}$$

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11 Graivtional Lensing

Consider the motion of light moving near a massive object... Recall the schwazschield metric

$$ds^2 = -(1 - R/r)dt^2 + (1 - R/r)^{-1}dt^2 + r^2 d\Omega^2$$

And use the fact that light travels along null geodesics. This means that "light has mass." this is because light is afffect by mass??? Recall a null geodesic in the plane $\theta = 0$ and simiarly the two killing vectors leads to conserved energy and angular momentum. Let $\dot{a} = \frac{\partial a}{\partial \lambda}$ where λ is an affine parameter

$$\dot{t} = E/(1-R/r) \qquad \dot{\phi} = L/r^2$$

We can use the fact that ds^2 meaning that $(ds/d\lambda)^2 = 0$ and thus

$$-(1-R/r)\dot{t}^2 = (1-R/r)^{-1}\dot{r}^2 + r^2\dot{\phi}^2 = 0$$

Now we can plug in $\dot{t}$ and $\dot{\phi}$ to give us

$$0 = -(1-R/r)^{-1}E^2 + (1-R/r)^{-1}\dot{r}^2 + L^2/r^2$$

Or if we wanted to solve for $\dot{r}^2$

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \frac{1}{2}E^2 = \hat{E} \qquad V_{\text{eff}}(r) = \frac{L^2}{2r^2} - \frac{L^2 R}{2r^3}$$

So what do we see here? Well wee see that light behaves like a particle in a 1-d effective potential. Before we discuss the details lets draw the effecive potneal vs r diagram

insert diagram here

We find that the unstable circular orbit $r_{\text{crit}} = \frac{3}{2}R$. So light is either "repelled" out to infinity or falls into the interior of the geometry and there is a circular orbit for light that is unstable at radius $r = 3R/2$. We can think of this as some limit of the time-like solution we found last lecture.

To understand the effects for normal objects consider the case $r \gg R$ which means that we can treat the additional GR term as a small peturbation. So the equation of motion

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}} = \text{const}$$

Now we can use the exact same procedure. Using $r(t) \to r(\phi)$ and using $\dot{\phi} = L/r^2$ we get $u = 2L^2/Rr$ thus giving us

$$u'^2 + u^2 = \text{cons} + \frac{R^2}{2L^2}u^3$$

Taking the derivative wrt ϕ

$$u'' + u = \frac{3}{4}\frac{R^2}{L^2}u^2$$

This again is the SHO with a driving force. Unlke the timelike geodesic there is no constant term. What we then want to do is expand in a perturbation series $u = u_0 + u_1 + \dots$. First lets talk about the newtonian case. This occurs when $r \gg R$ meaning that we have

$$u_0'' + u_0 = 0 \Rightarrow u_0(\phi) = c\cos\phi$$

Where we set inital conditions so that the solution is pretty like this. This describes a trajectory with

$$r = \frac{2L^2}{ru} = \frac{2L^2}{Rc\cos\phi} = \frac{b}{\cos\phi} \Rightarrow r\cos\phi = x = b$$

This is just a light ray travelling on a straight line with impact parameter b. The first correction is then found with

$$u_1'' + u_1 = \frac{3}{4}\frac{R^2}{L^2}u_0 = \frac{3}{4}\frac{R^2 c}{L^2}\cos\phi = \frac{3L^2}{b^2}\cos^2\phi$$

The easiest way to find a solution is with green's functions

$$u_1(\phi) = \frac{3L^2}{b^2}\int d\phi' \cos^2\phi' \sin(\phi - \phi')$$

So what are the limits on integration? AT late times we want to integrate $u_1 \approx \frac{3L^2}{b^2}\int_{-\pi/2}^{\pi/2} = 4L^2\sin\phi/b^2$. So at early times $u \approx 2L^2\cos\phi/Rb$ and at late times

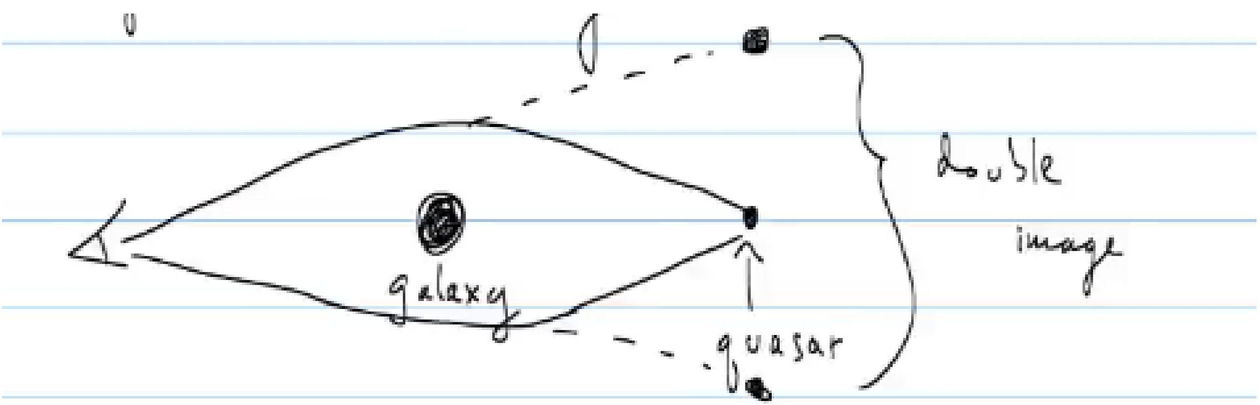
$$u \approx 2L^2/Rb(\cos\phi + 2R/b\sin\phi) = 2L^2/Rb\cos(\phi - \phi_0)$$

These both describe straight lines but we're deflected some direction. We can compute ϕ_0 with

$$\phi_0 = \frac{2R}{b} = 4GM/b$$

This angle describes the deflection angle of the light ray due to general relativity. So what we conclude is that light is deflected by massive object by an angle $4GM/b$. For a light ray near the sun this can be as large asa few arcseconds. The best way to observe gravitaonail lensing is to use a total solar ecplipse. This was observed by Eddinton... since then a lot of people think he fudged the data a bit.

In astronomy this effect is very commonly observed and is known as gravitational lensing. To explain this a bit further. Suppose we have some quasar and galaxy and us all in a straight line.



12 Gravitational Redshift

Lets consider two observers which are sitting at different radii in the schwarzschield geometry. e.g. I'm sitting on theearth surface and my friend is sitting at the top of a tall buidling. For an observer at r_i = const the proper time elapsed per coordainte time t

$$d\tau^2 = -ds^2 = (1 - R/r)dt^2 \Rightarrow \tau = (1 - R/r)^{1/2}t$$

If I send a signal once per proper time λ_1 and the r_2 observer will recieve a signal once per λ_2 which can be found with

$$\lambda_2 = \left(\frac{1-R/r_2}{1-R/r_1}\right)^{1/2} > \lambda_1 \text{ if } r_2 > r_1$$

Now instead we shine a light with wavelength λ_1 then the wavelength of the light recieved by my friend is λ_2 where

$$\frac{\lambda_2}{\lambda_1} = \left(\frac{1-R/r_2}{1-R/r_1}\right)^{1/2} = \frac{\omega_1}{\omega_2}$$

This effect is known as gravitaional red-shift. Why is this called redshift? The light recieved is "redder" since $\lambda_1 < \lambda_2$ and thus the light reived from deep inside a gravitiaonl potential is red shifted. This is important and is measurable for a GPS. It's also important in cosmology. Light in the early universe is red-shifted. This is also important for black holes. Lets consider the case when I'm very close to the event horizion of a black hole. $r_1 \to R$. Close to the event horiziong is **very** redshifted.

Gravitional redhisft is relaly due to equivalence principle

$$\frac{\lambda_2}{\lambda_1} = 1 + \frac{R}{2}(1/r_1 - 1/r_2) + \dots = 1 + \Delta\phi + \dots$$

This can be derived from the equivalence principle. It's basically just the doppler effect and folows that we identify gravity with acceleration.

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13 Schwarzschield Black Hole

$$ds^2 = -(1-R/r)dt^2 + (1-R/r)dr^2 + r^2 d\Omega^2$$

What happens for $r \approx R$. We saw that many funny things happen. At $r \to R$ this metric becomes singular. This singularity is coordinate artifact just like the singularity in the $\mathbb{R}^{3,1}$ metric in spherical coordinates

$$ds^2 = -t^2 + dr^2 + r^2 d\Omega^2$$

Indeed all of the scalars that can be formed from the metric and the Riemann tensor are nice and finite at the schwarzschield radius. There is no "coordinate independt" notion of curvature that blows up at the schwarzschield radius. As we take the limit as $R \to \infty$ i.e $M \to \infty$ these curvature invariants at the horizion become small. The event horizion can be weakly curved. Honestly we could be in an event horizion right now.

Imagine throwing Fergus the dog into a BH and give him a flashlight. Also lets tell him to send us a light signal for each proper time $\delta\tau$ as he approaches the event horizion. A null geodesic has

$$ds^2 = 0 = -(1-R/r)dt^2 + (1-R/r)^{-1}dr^2 \Rightarrow \frac{dr}{dt} = \pm(1-R/r)$$

As $r \to R$ we see that $\frac{dr}{dt} \to 0$. The time it takes the null geodesic to reach us is

$$t = \left| \int_r^{r_{\text{us}}} \frac{dr}{1 - R/r} \right| = |r - r_{\text{us}} + R\ln((r-R)/(r_{\text{us}} - R))|$$

So as $r \to R$ we see that $t \to \infty$. Thus from our point of view we will never observe fergus crossing the horizion. We can sit for a million years and never see him cross the black hole.

Insider the horizion the metric

$$ds^2 = (R/r - 1)dt^2 - (R/r-1)^{-1}dr^2 - r^2 d\Omega^2$$

We see that t is a space-lke coordinate and r is a time-like coordinate.

The other think we should point out is the horizion is the point where t is a null coordinate $|\partial_t|^2 = 0$. Not only does Fergus appear to slow down at the horizion, the light he emits is redshifted. As $r \to R$ the light is infinitley redshifted.

What does fergus see? Lets define $r^* = |r + R\ln(r/R - 1)|$ so r^* is the amount of time the null geodesic took. Lets then rewrite the metric with this. This is sometimes called the tortorise coordinate

$$dr^* = (1 - R/r)^{-1} dr \Rightarrow ds^2 = (1 - R/r)(-dt^2 + dr_*^2) + r^2 d\Omega^2$$

Why is this useful? If we forget about the angular compoent then we have some "conformal factor" times two dimensional minkowski space. So a null geodesic in the (r^*, t) plane is the same as minkowsi metric thus null geodesic travel on 45 degrees lin in (r_*, t) plane. This stiill breka downs at the schwarzschield radius.

Whe nstudying null geodesics it's useful to use coordinates to study ??? Basiclaly

$$u = t + r^* \qquad v = t - r^* \Rightarrow ds^2 = -(1 - R/r)du dv + r^2 d\Omega^2$$

An outgoing null gedesics $v =$ const and ingoing null geodesic $u =$ const. these coordinate are usually called null coordinates. If instead of taking u and v to be coordinates we could take v and r to be our coordinates "Eddington-Finkelstien" coordinates

$$ds^2 = -(1 - R/r)dv^2 + 2dvdr + r^2 d\Omega$$

These are perfect smooth accroos the schwarzschield radius. at $r \to R$ there are some cross terms.

Finally let: $U = e^{u/2R}$ and $V = -e^{-v/2R}$. This is really just some resalcing. $du = 2RdU/u$ and $dv = 2RdV/v$. The metric then becomes

$$ds^2 = -(1 - R/r)dudv + r^2 d\Omega^2 = -(1 - R/r)4R^2/UVdUdV = r^2 d\Omega^2$$

Now recaling that

$$r^* = r + R\ln(r/R - 1) = (u - v)/2$$

Gives us

$$\frac{1 - R/r}{UV} = \frac{1 - R/r}{e^{(u-v)/2R}} = \frac{1 - R/r}{}\exp\left\{-\frac{1}{R}(r + R)\ln(r/R - 1)\right\} = e^{-r/R}R/r$$

$$\Rightarrow ds^2 = 4R^3 \frac{e^{-r/R}}{r} dUdV + r^2 d\Omega^2$$

These are "Kruskul coordinates." The compoents of this metric are smooth for $r \neq 0$. So the horizion doesn't look like some particuluarly importnat place. When $r = R$ when $UV = 0$ meaning that U or V is zero. The singularity is at $R = 0$. Something sketchy is going on here and I ddidn't get enough sleep last night to follow.

Let $U = T + \rho$ and $V = T - \rho$ then the metric is

$$ds^2 = \frac{4R^3}{r} e^{-r/R}(-d\tau^2 + d\rho^2) + r^2 d\Omega^2$$

14 Relativistic Doppler Shift (From Guth's MIT course)

Some definitions. We're talking about the case where the observer is staionary and the source is moving with velocity v. We're talking about soundwaves with veloctiy u with respect to some medium. Δt_s (source) is the time interval between wave crests as measured by the source. Δt_o is the time interval between wave crets as measured by the observer.

Lets try to figure this out. If there is no motion $\Delta t_0 = \Delta t_s$. If there is some movement then there is some change $\Delta t_o = \Delta t_s + v\Delta t_s/u$ and this gives us the ratio

$$\frac{\Delta t_o}{\Delta t_s} = 1 + \frac{v}{u} = \frac{\lambda_0}{\lambda_s} = 1 + z \Rightarrow \boxed{z = \frac{v}{u} = \text{Nonrelativistic moving source}}$$

Now lets do the other simple case where the observer is moving and the source is stationary. This gives us $\Delta t_0 = \Delta t_s + \frac{v\Delta t_0}{u}$. From this we get

$$\frac{\Delta t_o}{\Delta t_s} = (1 - v/u)^{-1} \Rightarrow z = \frac{\Delta t_o}{\Delta t_s} - 1 = \text{something}$$

14.1 The Relativistic Case

$$\Delta t_o = \gamma \Delta t_s + \frac{v\gamma\Delta t_s}{u} = \gamma(1 + v/c)\Delta t_s = \sqrt{\frac{1+\beta}{1-\beta}}\Delta t_s$$

David Gross is coming to McGill!

15 More Black Holes

The schwarzschild black hole: the schwarzschild geometry has some funny things that happen. To find what happens to the person accross the event horizion we need to find a new coordinate system. In particular we started looking at Kruska's coordinates $(t, r) \to (U, V)$ such that

$$ds^2 = -\frac{4R^3}{r} e^{-r/R} dUdV + r^2 d\Omega^2$$

Where $r(U, V)$. What is the important point of this schwarzschild metric? Well this coordinate system is smooth except at $r = 0$ i.e. $UV = -1$ and the horizion is at $UV = 0$. We ended last class by drawing a diagram in these coordinates.

15.1 Blackhole Formation

Quick overview. It is very difficult for matter to collapse into a blackhole because a dense collection of matter typically hashigh pressure which makes the object want to expand. We're really just playing two forces against eachother: gravity and forces that make matter want to expand.

For a typical star: fusion pressure will keep radius bigger than schwarzschild radius. Eventaully as the star evolves, higher and higher mass elemts are formed until fusion is exhausted.

There are 3 possible outcomes once fusion is exhausted..Fermions: normal matter. Fermions don't like to be in the same space. The pressure that comes from fermions not wanting to be in the same state we have degeneracy pressure.

(a) $M \leq 1.4M_{\odot}$: "White dwarf" $R > R_s$ and the electron degeneracy pressure (e.g. white dwarf is a giant atom)

(b) $M \leq 4M_{\odot}$: "Neutron star": kinda like a giant nucleus with $R > R_s$. There is a neutron degeneracy pressure that keeps things from collpasing to black hole

(c) $M \geq 4M_{\odot}$: We get black hole.

The $4M_{\odot}$ should be thought of as estimate. There's very good evidence that black holes exsist (this is 2010 we know better now in 2020 I think).

16 Rotating Black Holes

A typical massive object is rotating and not spherically symmetric and hense is not exactly described by schwarzschild geometry.

We seek a solution with two killing vector ∂_t and ∂_ϕ. For example the earth is axially symmetric. Einstein's equation are much more complicated in this case. Our coordinate (r, t, θ, ϕ) and our solution will depend on r, θ

No hair theorem: The solution is uniquely parameterized by M and J.

16.1 Kerr Metric

Not very simple. Deriving this metric is "quite beautiful" and is given in Wald's book but in the interest of time we'll just get theresult here

$$ds^2 = -(1-2GMr/\rho)dt^2 - \frac{4GMar\sin^2\theta}{\rho^2}dtd\phi + \frac{\rho^2}{\Delta^2}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)^2 + a^2\Delta\sin^2\theta)d\phi^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2GMr + a^2 \qquad a = \frac{J}{M}$$

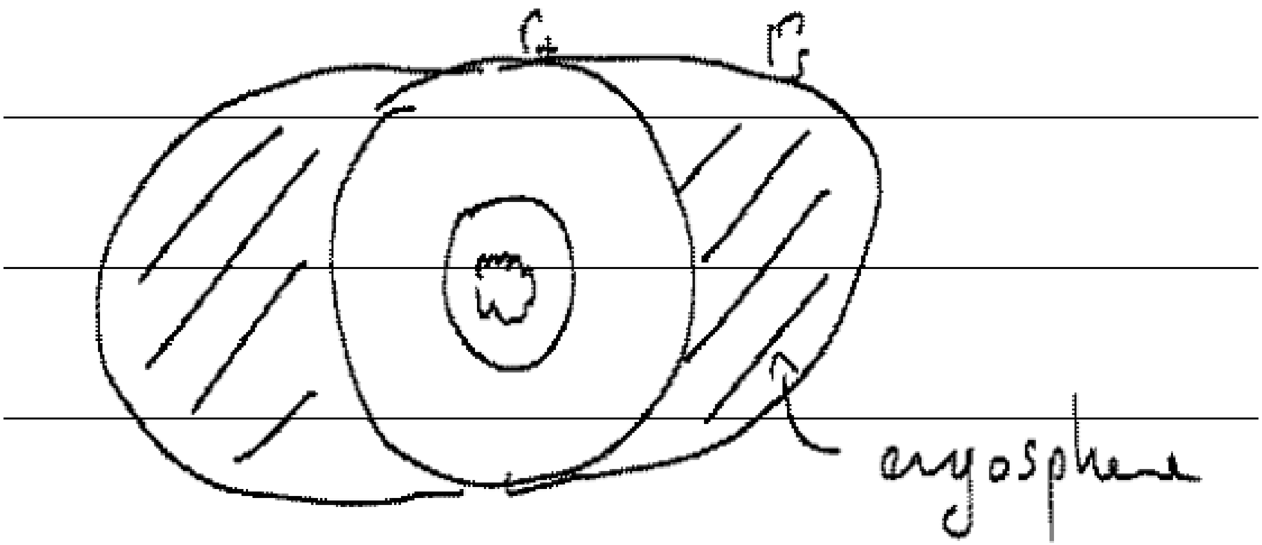
Here the metric has two killing vector ∂_t and ∂_ϕ. Note that when $J = 0$ we note that ds^2 becomes schwarzschild. Also note that as $r \to \infty$ the metric becomes minkowski.

There is a very important point he wants to point out. In schwarzschild the coeff of dt^2 and dr^2 become singular at the same point. However here this is not true. We now have two surfaces where the coeffs diverge at different points. When the coeff dr^2 diverges we have a horizion. This happens when $\rho = 0$ which gives $r = r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$. Namely we have an "outer" horizion and "inner" horizion. A timelke worldline cannot escape to ∞ from behind the outer horizion. The coeff of dt^2 vanishes when $1 - 2Gmr/(r^2 + a^2\cos\theta) =$ 0and solving this gives $r = GM + \sqrt{G^2M^2 - a^2\cos^2\theta}$. Note that $r_s > r_t$ except at $\theta = 0$ or $\theta = \pi$. This surfaces is known as the stationary limit surface. Why is this interesting? Lets say we're inside the stationry limit surface. The coef of dt^2 is positive. This means that a wordline with constant r, θ, ϕ is spacelike. This means that no matter how poweful your rocket ship is you can't stay at fixed r. Basically this meanas you have to rate along with the black hole. Namely every timelike geodesic rotates. This is a consequence of what's known as "frame dragging." Basically ST itself is being "dragged around".

17 Even More Blackholes

A review of no hair theorem: A stationary axially symmetric BH of vacuum GR is parameterized by mass and angular momentum.

Also recall that there are two horizions $r = r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$. It was stated (but not proved) that no timelike wordline can escape from behind the outer horizion $r = r_+$. In addition we had the stationary limit surface $|\partial_t|^2 = 0$ where $r = r_s = GM + \sqrt{G^2M^2 - a^2\cos^2\theta} \geq r_+$. Inside the starionary limit surface the trajcectory of constant r, θ, ϕ i impossible(spacelike), we're basically forced to rotate with the black hole. This effect is usually called frame dragging. If we wanted to draw a picture it would look something like

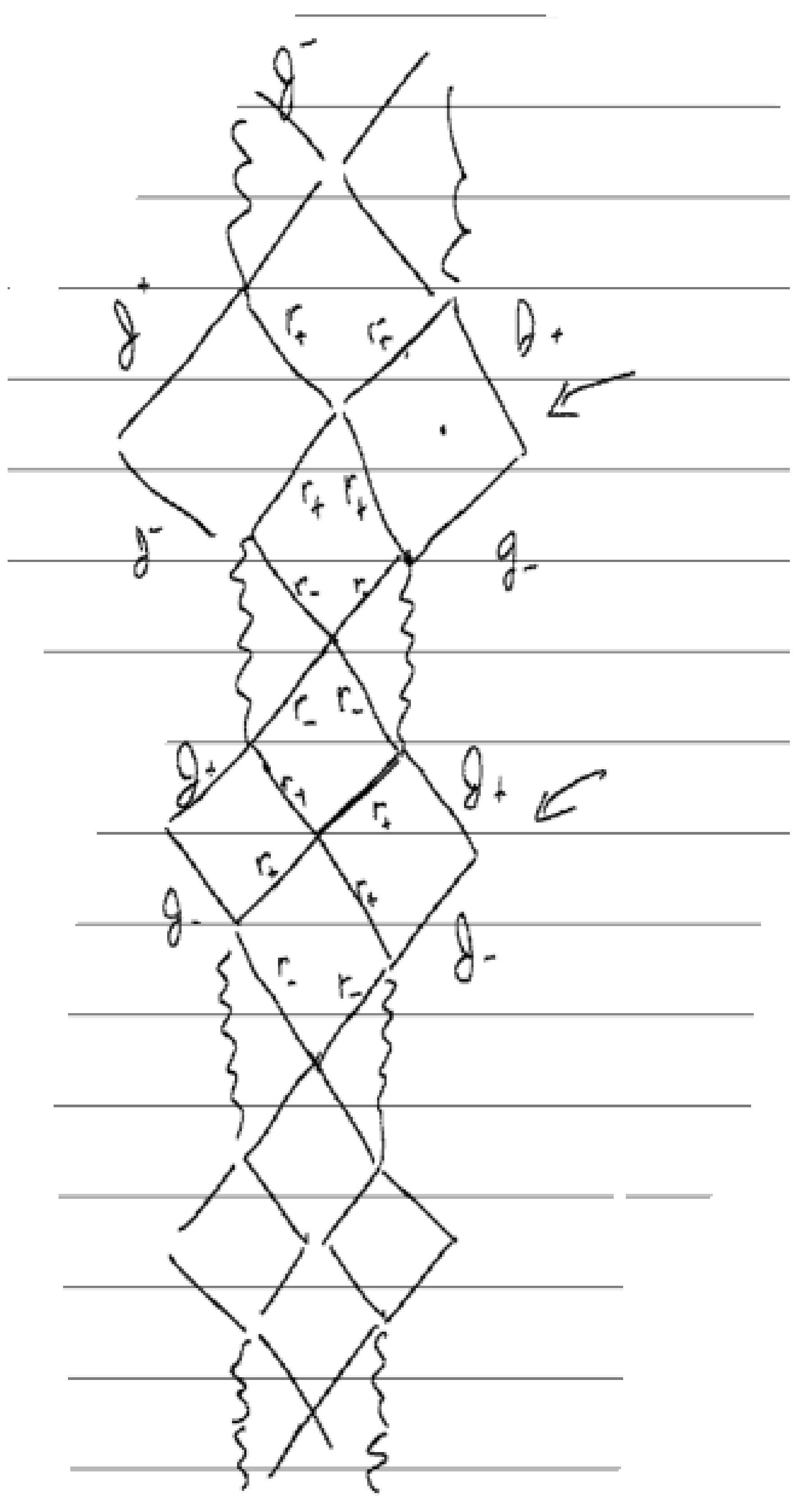


Mechanics inside the ergosphere is kinda wonky. It honestly could take up it's own course.

$$ds^2 = \frac{\rho^2}{\Delta^2} dr^2 + \dots \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

So for in order for ρ to vanish we need $r = 0$ AND $\theta = \pi/2$. We have a "ring" singuarity inside the black hole. In the schwarzschild metric there was an "alternate" universe. However in kerr black hles there is are infinite "alternate universes" OwO.

Recall for the schwarzschild BH we used Kruskal coordinates to understand the BH inside the horizion. If we do a conformal transform we can draw the diaram in a finite region instead of an finite region(Penrose diagram). Maloney's drawing of Kerr BH:



17.1 Penrose process:

It's impossible for anyting to escape a black hole. For a schwarzschild black hole the mass will always increase(classically). For a kerr black hole this is no longer the case. There are classical process that lower the mass of a black hole. Lets imagine we're in our rocket ship with a massive object. We go into the ergosphere, throw the massive object in, and then leave. The sign of the timelike killing vector changes sign, thus sign of energy changes, thus mass is negative, thus energy in blackhole decreases and I gain more energy. There is a price to pay however. This process decreases mass and angular momentum of BH. This leads us to the folloing question. Is there some quantity that always increases in Kerr black hole! Yes! This increases the area of the horizion. Thus classically we have the second law of BH mechanics: $\Delta A \geq 0\ldots$ Now here's a myseterious remark: entropy is always increasing in classical processes. So morally speaking Area $\approx$ S. This has very deep ramifications. The other comment about this process, this doesn't seem to be terribly relevant to astrophysical black holes.

18 Some more questions in Blackhole physics

For a schwarzscihld BH, mass was just a constant of integration.... this means that there is no reason for us to focus on $M > 0$. But now we will say why we we restrict $M \geq 0$.

$$ds^2 = -(1 - R/r)dt^2 + (1 - R/r)^{-1}dr^2 + r^2 d\Omega^2 \qquad R = 2GM$$

What happens if $M < 0$. There is a singularity at $r = 0$ but there is no event horizion because event horizion is now at a negative radius. General relativity breaks down at the singularity. But usually this is hidden behind the event horiizon but if event horizion isn't there it becomes a disaster. It turns out that is important and not pedantic. It (which we call "naked singualrity") can occur even if $M > 0$. What happens if $J > M$ in kerr black holes we have a naked singularity. For eample there exists astrophysics blackholes but $J/M \leq .99$. As far as we can tell there are no naked singularities.

Cosmic Censorship Conjecture: every singularity which arises from the evolution of smooth initial data is behind an event horizion. This conjecture can be proven for certain types of matter.

Singularity theorems: the formation of singualrities is inevibitile in certain circumtances. Proof is difficult but understanding hsouldbt be too hard. What does this mean? Lets say we have a sphere and imagine we have a lightbulb at every point on the sphere and turn on the lightbulbs. If this is a normal sphere in a normal geometry then we know what happens. But what happens if geometry is curved. Then we could have all the light rays either go into the pshere our outside the sphere "Trapped surfaces" and "anti-trapped surfaces" respectively. A geometry with trapped surface will always form a signualrity. A geometry with anti-trapped surface always has a singuality in the past. what this means is that singuarlities are a general feature of einstein's equations. The hope then is that all singularities are all behind horizions. Another thing we could hope is that we cna make sense of the dynamics close to naked singularities. An exapmle of the second singualrrity theorm is our universe (big bang). We'll talk about this in our discussion of cosmology.

19 Cosmology

What are the symmetries of our universe?

Copernican Principle: The laws of physics are the same everywhere in the universe. What does this mean practically?

(a) Homogeneity: ST metric is the same at every point. E.g. for any two points p, q we can find an isometry (coordinate transform) that takes $p \to q$.

(b) Isotropy: spacetime has no preferred direction i.e. no preffered vector. This means that for any 2 vectors v^μ and w^μ at p in s.t. there exists an iisometry which takes $v \to w$.

A fun exercise might be to come up with metrics that have these properties (homogeneous and not isotropic, isotropic and not homogenous, etc.)

A universe that is both homogeneous and isotropic are very symmetric (e.g. $\mathbb{R}^{3,1}$ which has translation symmetry (homogeneous) and loretnz symmetry (isotropic) which has 10 isometries). Its possible to show that a $D = 4$ space has at most 10 isometries. It turns out that there are two other ST that have 10 isometreies(my guess AdS and dS). Both AdS and dS are maximally symmetric spaces. These spaces are just as symmetric as minkowski but both of these have curvature. We have now that $\mathbb{R}^{3,1}$, dS, and AdS are the maximally symmetry spaces in $D = 4$ with $R < 0, = 0, > 0$. We can think of these as equivaent to flat plane, sphere, and hyperboloid. Cosmology in these three symmetry spaces is **very** boring. Observationally the unverise is evolving. Thus our universe does not appear to be homogeneous or isotropic. Then what is the appropriate symmetry? However our universe does appear to spacially homogeneous and isotropic on very large scales. At ong distance scales we can approximate our universe with

$$ds^2 = -dt^2 + a^2(t)d\sigma_k^2$$

Where $d\sigma_k^2$ is the spatial part of the metric $D = 3$ which is maximally symmetric and a^2 is the scale factor. One question is why aren't there any off-diagonal terms? Well it's because that would indicate a preffered dirction.

There are 3 possible euclidian symmetric spaces labeled by index k in σ_k

(a) $k = 0$: Flat Euclidian space $\mathbb{R}^3$ (3 translations and 3 rotations: ISO(3))

(b) $k = 1$: Sphere $\mathbb{S}^3$ where

$$d\sigma_1^2 = d\theta^2 + \sin^2\theta(d\phi^2 + \sin^2\phi d\psi^2)$$

Where there is a noth pole at $\theta = 0$ and south pole at $\theta = \pi$. This is also a symetric space with positive curvature $R = 6$. This space also has 6 symmetries $SO(4) \Leftrightarrow SU(3)$.

(c) $k = -1$: Hyperboloid $\mathbb{H}^3$ has a metric

$$d\sigma_2^2 = d\theta^2 + \sinh^2\theta d\Omega^2$$

This is also a maximally symmetric space with constant negative curvature. What's the two dimensional hyperoloid?

$$\mathbb{H}^2 : d\theta^2 + \sinh^2\theta d\phi^2$$

As $\theta \to 0$ things look like a plane. This also has the isometry group $SO(3,1)$ which happens to be the same as the loretnz group in $D = 4$.

So we have these three different geomtires. Lets use a different coordinate system where the metric on the three different coordinate systems takes the following form

$$d\sigma_0^2 = dr^2 + r^2 d\Omega^2$$

$$d\sigma_1^2 = d\theta^2 + \sin^2\theta d\Omega^2$$

$$d\sigma_{-1}^2 = d\theta^2 + \sinh^2 d\Omega^2$$

For $k = 1$ we can let $r = \sin\theta \Rightarrow dr = -\cos\theta d\theta \Rightarrow d\theta^2 = dr^2/(1-r^2)$. We can do something analgorus for $k = -1$. Thus we get

$$d\sigma_k^2 = \frac{dr^2}{1-kr^2} + r^2 d\Omega^2$$

Remember that these spaces are symmetirc spaces thus $R = 6K$ and $R_{ij} = 2Kg_{ij}$ (these are in three dimensional geometry)

Spatial isotropy and homogenaity is enough to imply

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right\} \qquad k = 0, \pm 1$$

This is the FRW metric and the universe that descriebd by this metric is the FRW universe. When $k = 1$ the spatial slices are spheres and thus "closed." When $k = -1$ the spatial slices are open (you can see this by seeing the range of r implied by sin and sinh. When $k = 0$ spatial slices are flat.

This metric escribes our universe only at very long scales. Now all we need to do is figure out what the fuck $a(t)$ is. But to do that we need the equations of motion (where there is matter... :(.) we need to decide on the matter content of the universe. What we'll use is the Friedman Model: perfect fluid. Recall that

$$T_{\mu\nu} = (p+\rho)v_\mu v_\nu + pg_{\mu\nu}$$

Where p is pressure and ρ is mass density. What is the four velocity of the FRW model? Well because of spatial isotropy we know that

$$v^\mu = (1,0,0,0) \Rightarrow T_{00} = \rho \qquad T_{ij} = pg_{ij}$$

Plug the metric and the stress tensor into einstien's equation to get the equations of motion and these are called Friedman's equations.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{k}{a^2} \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

Lets stare at these for a few seconds. First we note that p and ρ are functions of time. This is because if we hvae some dust and the universe is expanding and since energy is conserved then energy density decreases. So how do we find p and ρ? The conservation equation

$$\nabla_\mu T^{\mu\nu}0 \Rightarrow 0 = \nabla_\mu T^\mu{}_0 = -\dot{\rho} - 3\frac{\dot{a}}{a}(p+\rho) \Rightarrow \frac{\dot{\rho}}{p\rho} = -3\frac{\dot{a}}{a}$$

At this point the asiest way to proceed we should assume that matter obeys som equation of state

$$p = w\rho \Rightarrow \frac{\dot{\rho}}{\rho}\frac{1}{1+w} = -3\frac{\dot{a}}{a} \Rightarrow \rho = \rho_0 a^{-3(1+w)}$$

So if we have a perfect fluid with an equaiton of state parameter we can just plug the above result into the friedman equations and solve for a. This tells us how the matter density evolves in time. In particular different types of matter redshift in different ways. For example, lets say we have some pressureless dust: $\omega = 0 \Rightarrow \rho = \rho_0 a^-3$. On the other hand if we have radiaiton $\omega = 1/3 \Rightarrow \rho = \rho_0 a^{-4}$. And finally if we have vacuum energy $\omega = -1$ then $\rho = \rho_0$. the curature term (e.g. curvature of spatial slices) falls off as a^{-2}.

In our universe everything we talked about is present. These all contribute to Friedman's equations.

20 More Cosmology

We're currently studying FRW cosmology where we approximat the universe with a ametric that is spatially homogenous and isotropic. Given this assumption the metric of space time will take the form

$$ds^2 = -dt^2 + a^2(t)d\sigma_k^2$$

Where $d\sigma_k^2$ is a maxmally symmetric $D = 3$ Euclidiant goemetry. We need to specifty k and matter content (collection of perfect fluid) to describe the metric fully. One can then go ahead and write a stress tensor and thus the metric

$$(\dot{a}/a)^2 = 8\pi G\rho/3 - k/a^2$$

This is often reffered to as firedman's equations. we can also derive form conservation of the stress tensor

$$\ddot{a}/a = -4\pi G(\rho + 3p)/3$$

We can also write

$$\left(\frac{\dot{a}}{a}\right)^2 == \sum \frac{8\pi G}{3}\sum_i \rho_i$$

Where we define $\rho_{\text{curvature}} = \rho_k = -3k/8\pi G a^2$ (e.g. we let curvature be another kind of matter)

For a perfect fluid, the conservation equation lets us to understand how our densites evolve. We found that

$$\rho = \rho_0 a^{-3(1+w)}$$

From this equation we see that perfect fluids with larger values of w deermine at early time and smaller w dominate at late time. So we can imagine that different forms of energy dominate at different epochs. so for example when a is small radiation dominates and then dust dominates and then when dust redshifts away the curvature dominates and then when curvature redshifts away we have vacuum energy dominate. In the transition periods it's very hard to find analytic solutions. for now lets just assume that our very simple model works great.

For a single type of perfect fluid $p = w\rho$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 a^{-3(1+w)} \Rightarrow \dot{a}^2 a^{3w+1} = \frac{8\pi G}{3}\rho_0 = \text{const}$$

Taking the square root

$$\dot{a}a^{(3w+1)/2} = \sqrt{\text{const}} = \frac{2}{3w+3}\partial_t(a^{(3w+3)/2})$$

Solving this gives us

$$a(t) = C(t-t_0)^{2/(3(w+1))}$$

So the FRW metric becomes

$$ds^2 = -dt^2 + t^{4/(3(w+1))}d\sigma_k^2$$

So we have a "polynomial expansion" in the sense that the spatial slices expand in time as a polynomial in t which depends on the matter. There are two special cases we should want to focus on.

(a) Curvature dominated: If curvature dominated then the RHS of friemdna's equation is negative thus k must be -1 and thus $w = -1/3$ and $ds^2 = -dt^2 + t^2 d\sigma_{-1}^2$. This equation is a solution of the source free Einstein equation. We call this geometry the "Milne University". We should note that this is ismilar to some geoemtry we've seen before $dr^2 + r^2 d\Omega^2$. Maybe we shouldn't be suprised that milne universe is just minkowski space just written in a "cosmological" coordinate system. In this cosomological coordinate system the isotropy becomes apparent.

(b) Maloney's lying to us. What happens when $w = -1$? Then $\dot{a}/a = \text{const}$. We usually call $\dot{a}/a = H$ which is the hubble parameter of the universe. For every universe we have H. Note that H has units 1/[Time] and thus H is the time for the universe to expand by a factor of e. Morally speaking the inverse of H should be the time that it takes for the universe to double in time. H^{-1} is the typical length scale of the universe. However when $w = -1$ and the curvature term can be neglected ($k = 0$) then H is a constant and thus $a = e^{Ht}$ and thus $ds^2 = -dt^2 + e^{2Ht}d\sigma_0^2$. Well the spacial slices expand exponetilaly. In cosmology there is a name for this, "inflation" hehe. This geometry is dS_4 namely four dimensional deSitter space.

What does our universe look like? Today in our universe radiation is negligible and thus our universe can be approximated with dust, cosmological consant, and curature. The

FRW equations can then be rewritten

$$H^2 = \frac{8\pi G}{3}\sum_i \rho_i \Leftrightarrow \Omega_{\text{matter}} = \frac{8\pi G}{3H^2}\rho_{\text{matter}} \qquad \Omega_c = -\frac{k}{H^2 a^2} \qquad \Omega_\Lambda = \frac{8\pi G}{3H^2}\rho_\Lambda \qquad \Omega_m + \Omega_c + \Omega_\Lambda = 1$$

From observation we can nail these parameters down as

$$\Omega_m \approx .25 \qquad \Omega_\Lambda \approx 0.75 \qquad \Omega_c \approx 0$$

The first two have different depence on time. However we happen to live at a time where these are comparable and in the same order of magintude. "Why now problem."

We can now understand the late time behavior of the universe. So lets make a little graph.

graph in notes

The universe appears to have that spatial slices which will expand forever. Here are a few other constnats. Of the $\Omega_m \approx 0.25$, only five percent comes from "normal matter." the rest comes from "dark matter." The exact content is not known.

Another comment: we're not entirely sure that thecurrent accleeratied expansion is due to a cosmological constant. So Ω_Λ isn't always called or said to be due to cosmological constant and sometime people call it "dark energy."

Here are some numbers

$$H_{\text{today}} \approx 70\text{km/s/MpC} \Leftrightarrow H^{-1} \approx 10^{10}\text{years} \approx 10^{17}\ s \Leftrightarrow H^{-1} \approx 10^3\text{MpC} \approx 10^{28}\text{cm}$$

20.1 Early Universe

We've been talking about time when there are no radiation or other forms of matter were important. Universe at early times was a hot dense plasma and as universe expands the universe cools and went through a lot of phase transitions. We started as a quark/gluon plasma then became nucleons and then finally became atoms. A great success of standard model was that we could study these dense plasmas. When nucleons went to atoms "recombinatoin" was when universe became translucent (photons can't traverse through charged media because they couple to charges.) We hope that we can do astronomy with gravity waves.