

BLACK HOLE INFORMATION PARADOX NOTES

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Notes for Suvrat Raju's Black Hole Information Paradox course at ICTP(well really online) during Spring 2021. Course website can be found [here](#) which contains notes, assignments, and links to lecture videos. This course closely follow a review posted [here](#). If you have any comments let me know at hi@delonshen.com.

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LECTURE 1: INTRODUCTION AND TWO-POINT QFT CORRELATORS

The main organization of this course

- (a) Hawking's Original Paradox \rightarrow Thermalization and exponentially small corrections.
- (b) Paradoxes about interior of evaporating Black Holes \rightarrow holography of information, islands and page curve.
- (c) Paradoxes about large Black Holes in AdS/CFT \rightarrow Mirror operators, state-dependence, and firewalls/furballs

Lets start by talking about **Hawking Radiation**, it's the effect that underlies the information paradox. Take a black hole in asymptotically flat space. This black hole radiates with a temperature \propto surface gravity(TODO ???). We should also recall that hawking radiation relies on short distance QFT physics and on global late-time properties of the black hole geometry. The interesting thing is that the derivation for Hawking's radiation also implies the existence of the entangled modes across horizons. So what are the common derivations of hawking radiation(TODO (a) is in appendix of review paper and (b) might be in wald)?

- (a) Hawking's original derivation
- (b) Rindler \leftrightarrow Minkowski Bogolivlov transformation

In this course we'll consider a different derivation from both of these

Lets take a second to step back from black hole and look at Quantum Fields near a null surface. We'll apply what we learn here to black holes later. What we want to show is that across any null surface in a smooth state (TODO smooth state who?) we can isolate a "local" QFT (which we'll define in a bit) with universal entanglement. This is useful because we'll find that in a black hole spacetime local degrees of freedom near the horizon gives global modes in blackhole geometry.

First lets define what we mean by a smooth metric around some point. Consider a point in some $D = d + 1$ space and let this point be the origin. We have U, V , two null coordinates, and $d - 1$ transverse coordinates. A metric is smooth around some point if around some point we can locally choose some coordinates so the metric takes the following form. (think light cone variant Kruskal coordinates in arbitrary dimensions?)

$$ds^2 = -dUdV + \delta_{\alpha\beta} dy^\alpha dy^\beta + \dots$$

Where $dUdV$ are two null coordinates and α, β is over $d - 1$ indices and where the \dots terms vanish near origin.

figure

We also want to make an additional demand. Consider a scalar field ϕ and points near $U = 0$. If we're still thinking in terms of Kruskal coordinates this means we're thinking of things close to eachother on each side of the horizon? In the limit where x_1 approaches x_2 for any nonsingular state the two point correlation function (Wightman function?) becomes.

$$\langle \phi(x_1) \phi(x_2) \rangle = \frac{N}{|x_1 - x_2|^{d-1}} + \dots$$

We also impose the following scales

$$(a) |x_1 - x_2| \ll \ell_{\text{curvature}}$$

$$(b) |x_1 - x_2| \ll \frac{1}{m}$$

$$(c) |x_1 - x_2| \gg \ell_{\text{Pl}} \text{ or any } UV \text{ scale where EFT breaks down.}$$

These length scales give us the normalization if we consider a free field (e.g. $\mathcal{L} = 1/2(\partial_\mu \phi)^2$). (TODO how???)

$$N = \frac{\Gamma(d-1)}{2^d \pi^{d/2} \Gamma(d/2)} \Rightarrow \langle \phi(x_1) \phi(x_2) \rangle = \frac{\Gamma(d-1)}{2^d \pi^{d/2} \Gamma(d/2)} \frac{1}{|x_1 - x_2|^{d-1}} + \dots$$

Because of the length scales we assume we can say that the structure of the two point function is universal (TODO what in the world.) Before we continue lets look at a few things that will be useful

$$|x_1 - x_2|^2 = -\delta U \delta V + \delta y^\alpha \delta y^\beta \delta_{\alpha\beta} \quad \delta O = O_1 - O_2$$

Also if we grind through some calculations we'll find that

$$\langle \partial_{U_1} \phi(x_1) \partial_{U_2} \phi(x_2) \rangle = -\frac{d^2 - 1}{4} \frac{N(\delta V)^2}{|x_1 - x_2|^{d+3}} + \dots$$

Taking $\delta V \rightarrow 0$, e.g. we take δV to be the smallest separation, then we find that

$$\lim_{\delta V \rightarrow 0} \frac{(\delta V)^2}{(-\delta U \delta V + \delta y^\alpha \delta y^\beta \delta_{\alpha\beta})^{(d+3)/2}} \neq 0$$

It's not zero since it does receive a contribution when $y^\alpha = 0$. To see this we can do an integral over all the transverse separations.

$$\int \frac{(\delta V)^2}{(\delta U \delta V + \delta y^\alpha \delta y^\beta \delta_{\alpha\beta})^{(d+3)/2}} d^{d-1} \delta y^\alpha$$

We'll also take the $|x_1 - x_2|$ is positive. Now with the substitution $\delta \tilde{y}^\alpha = \delta y^\alpha / \sqrt{-\delta U \delta V}$ we get

$$\frac{1}{(\delta V)^2} \int \frac{d\delta \tilde{y}^\alpha}{[1 + \delta \tilde{y}^\alpha \delta \tilde{y}^\alpha]^{(d-3)/2}}$$

What happens in the end is that all factors of δV cancel. In the notes Suvrat says that for $\delta y^\alpha \neq 0$ the integral vanishes. I think this might be due to symmetry, there's a ring of δy^α with appropriate sign that cancels out when we do the integral. But since $\delta y^\alpha = 0$ doesn't have this cancellation it remains finite. In the end we get

$$\boxed{\lim_{\delta V \rightarrow 0} \langle \partial_{U_1} \phi(x_1) \partial_{U_2} \phi(x_2) \rangle = -\frac{1}{4\pi} \frac{\delta^{d-1}(\delta y^\alpha)}{(U_1 - U_2 - i\epsilon)^2}}$$

Something to note: the reason we did an integral was to pick up the coefficient of the delta function. How do we sniff out the presence of a delta function. We use the property $\int \delta(x) dx = 1$ with the fact that the integral vanishes for $x \neq 0$. Thus if the integral we considered above gave a finite answer then we know the two point correlation function of the derivatives of the states would be proportional to a delta function.

The next step which will happen in the next lecture would be to define modes as approximately

$$\int \partial_\mu \phi(-U)^{i\omega}$$

What is this doing? It's picking up the right moving modes with constant V . (TODO huh?)

LECTURE 2: ENTANGLED MODES ACCROSS NULL SURFACES

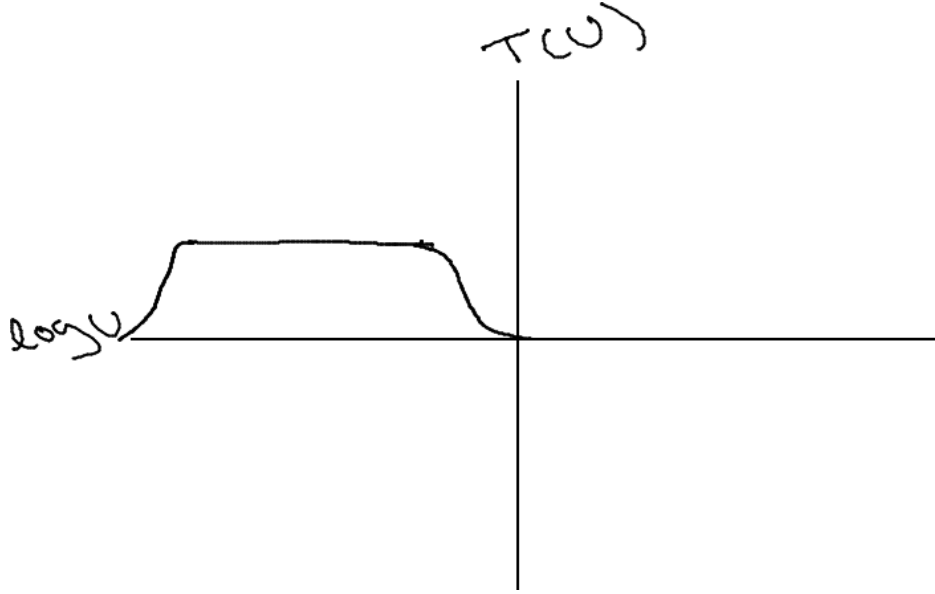
Last time we mentioned that we can extract right moving modes accross the null surface $U = 0$ with an integral (this is an approximate expression, we'll get a more precise integral in a bit)

$$\int \partial_U \phi(-U)^{i\omega} dU \quad \int \partial_U \phi U^{i\omega} dU$$

Note that we can't integrate over a large region in U because this would violate our limit of x_1 approaching x_2 which we derived the form of the two point function for in the last lecture. So then we know that we should integrate over a small region of U instead (with respect to the length scales defined in the last lecture).

We'll start by introducing a smearing function (TODO what?) $T(U)$ with the following properties

- (a) $T(U)$ dies off smoothly near $U \rightarrow 0$
- (b) Support in interval $[U_l, U_r]$ where $\ell_{UV} \ll U_r, U_l \ll \ell_{\text{curv}}$ and $\frac{U_r}{U_l} \gg 1$ and $U_l, U_r > 0$
- (c) We normalize the smearing function $\int T(U)^2 dU/U = 2\pi$. Note that $dU/U = d \log U$.
- (d) $T(U)$ is flat for a large range of $\log U$.



The next thing we need to do is define what's happening in the transverse direction by integrating over a volume Vol in the transverse direction which is smaller than a cube of the curvature scale. From this we can write a more precise expression for the mode.

$$a_{\omega_0} = \int (\partial_U \phi(U, V = 0, y^\alpha)) (-U)^{-i\omega_0} T(-U) dU \frac{d^{d-1} y^\alpha}{\sqrt{\pi \omega_0 \text{Vol}}}$$

We can similarly define a similar integral for the other side of the null surface. Note we let $V = -\epsilon$ to ensure that we're considering points that are spacelike separated.

$$\tilde{a}_{\omega_0} = \int (\partial_U \phi(U, V = -\epsilon, y^\alpha)) (U)^{i\omega_0} T(U) dU \frac{d^{d-1} y^\alpha}{\sqrt{\pi \omega_0 \text{Vol}}}$$

The $T(U)$ insures we only integrate over a small region in U and we have some normalization factors put in that aren't motivated from what I can tell.

Let's now compute the two point function of a and \tilde{a} and we will find that this will only depend on the short distance field correlator that we found last lecture. First spelling things out

$$\langle a\tilde{a} \rangle = \frac{1}{\pi \text{Vol} \omega_0} \int dU_1 dU_2 \langle \partial_{U_1} \phi(U_1, V=0, y_1) \partial_{U_2} \phi(U_2, V=-\epsilon, y_2) \rangle \times \\ \times (-U_1)^{-i\omega_0} U_2^{i\omega_0} T(-U_1) T(U_2) d^{d-1} y_1 d^{d-1} y_2$$

Remember that the correlator gives a delta function

$$= -\frac{1}{4\pi^2 \omega_0} \int \frac{1}{(U_1 - U_2)^2} \left(\frac{U_2}{-U_1} \right)^{i\omega_0} T(-U_1) T(U_2) dU_1 dU_2$$

To do this integral we need the identity

$$\frac{1}{U_1 - U_2}^2 = \frac{1}{(-U_1)U_2} \int_{-\infty}^{\infty} \frac{\omega e^{-\pi\omega}}{1 - e^{-2\pi\omega}} (U_2/(-U_1))^{-i\omega} d\omega$$

When $U_1 < 0$ and $U_2 > 0$. Assume that $|U_1| > |U_2|$. This lets do a contour integral where there are poles at $\omega = in$ where $n \in \mathbb{N}$. From here we can sum the residuals. What are the residuals at the poles? Well mathematica can tell us and so can Suvrat.

$$\frac{1}{(U_1 - U_2)^2} = \frac{1}{|U_1|U_2} \sum_{n=1}^{\infty} -n(-1)^n (U_2/|U_1|)^n$$

We can also plug in the identity (before computing the residuals) into $\langle a\tilde{a} \rangle$ to get

$$\langle a\tilde{a} \rangle = \frac{1}{4\pi^2 \omega_0} \int \frac{dU_1}{U_1} \frac{dU_2}{U_2} (U_2/(-U_1))^{-i(\omega-\omega_0)} \frac{\omega e^{-\pi\omega}}{1 - e^{-2\pi\omega}} T(-U_1) T(U_2) d\omega \\ = \int T(-U_1) (1/(-U_1))^{i(\omega_0-\omega)} dU_1/U_1 \times \int T(U_2) U_2^{i(\omega_0-\omega)} dU_2/U_2 \times \int \frac{\omega e^{-\pi\omega}}{1 - e^{-2\pi\omega}} d\omega$$

Now note that we can rewrite this in terms of $\log(U)$ since $dU/U = d\log U$.

$$= \int T(-U_1) e^{-i(\log[-U_1])(\omega_0-\omega)} d\log U_1 \times \int T(U_2) e^{i(\log U_2)(\omega_0-\omega)} d\log U_2 \times \int \frac{\omega e^{-\pi\omega}}{1 - e^{-2\pi\omega}} d\omega$$

The first two integrals are fourier transforms of T . Namely $S(\gamma) = \frac{1}{2\pi} \int_0^{\infty} T(U) U^{-i\gamma} dU/U$

$$= \frac{1}{\omega_0} \int \frac{\omega e^{-\pi\omega}}{1 - e^{-2\pi\omega}} |S(\omega - \omega_0)|^2 d\omega$$

Now note that since we said T is very flat for a large range of U then we know that the fourier transform of T has to be very big at $T = 0$ and thus the fourier transform of T becomes basically a delta function. This gives us finally

$$\boxed{\langle a\tilde{a} \rangle = \frac{e^{-\pi\omega_0}}{1 - e^{-2\pi\omega_0}} + \dots} \quad (1)$$

There are similar calculations we can do to find

$$\langle aa^\dagger \rangle = \frac{1}{1 - e^{-2\pi\omega_0}} \quad \langle \tilde{a}\tilde{a}^\dagger \rangle = \frac{1}{1 - e^{-2\pi\omega_0}} \quad [a, \tilde{a}] = 0 \quad [a, a^\dagger] = [\tilde{a}, \tilde{a}^\dagger] = 1 \quad \langle a^\dagger \tilde{a} \rangle = \langle a^\dagger a^\dagger \rangle = 0 \quad (2)$$

Something to note is that in some quantum field theories $\langle i_\omega \tilde{i}_{\omega'} \rangle = e^{-\pi\omega} 1 - e^{-2\pi\omega} \delta(\omega - \omega')$. However in this case that is not true. Here we have (TODO how?) $\langle \tilde{a}_{\omega_0} c_{\omega'_0} \rangle \approx 0$ where $\omega_0 \neq \omega'_0$.

In the special case where we have a spacetime with spherical symmetry

$$ds^2 = -dUdV + r_0^2 d\Omega_{d-1}^2 + \dots$$

In this we can derive a analogous form of the modes where

$$a = \frac{r_0^{d-1}}{\sqrt{\pi\omega_0}} \int \partial_U \phi(U, V=0, \Omega) (-U)^{-i\omega_0} T(-U) dU Y_l^*(\Omega) d\Omega \quad \tilde{a} = \dots$$

Where Y_l are our spherical harmonic functions. These satisfy all the same correlator and commutation relation as we found before.

Now moving to a different topic. We've been writing $\langle \dots \rangle$ for correlators but we need to describe what state we're calculating these correlators for. Say we are in a state $|\psi\rangle$. What we want to show is that $\tilde{a}|\psi\rangle \propto a^\dagger|\psi\rangle$. Thinking about this geometrically this means that \tilde{a} and a^\dagger are parallel to each other. To prove this consider the decomposition of $\tilde{a}|\psi\rangle$ (TODO: is that actually a complete set?)

$$\tilde{a}|\psi\rangle = c_1 a|\psi\rangle + c_2 a^\dagger|\psi\rangle + |\chi\rangle$$

Where $|\chi\rangle$ is orthogonal to $a|\psi\rangle$ and $a^\dagger|\psi\rangle$. From here we can use the correlators we have in (2) to get $c_1 = 0$. Similarly we can use (1) to get

$$\frac{e^{-\pi\omega_0}}{1 - e^{-2\pi\omega_0}} = \langle \psi | a \tilde{a} | \psi \rangle = c_2 \langle \psi | a a^\dagger | \psi \rangle + \langle \psi | a | \chi \rangle = \frac{c_2}{1 - e^{-2\pi\omega_0}} \Rightarrow c_2 = e^{-\pi\omega_0}$$

Finally we can find $|\chi\rangle$ through the following. From (2) we have $[\tilde{a}, \tilde{a}^\dagger] = 1$. This means

$$\langle \tilde{a} \tilde{a}^\dagger \rangle - 1 = \langle \tilde{a}^\dagger \tilde{a} \rangle = \frac{e^{-2\pi\omega_0}}{1 - e^{-2\pi\omega_0}}$$

Now with this we have (after $|\dots|^2$ both sides)

$$\frac{e^{-2\pi\omega_0}}{1 - e^{-2\pi\omega_0}} = |c_2|^2 \langle \psi | a a^\dagger | \psi \rangle + \langle \chi | \chi \rangle + 0 \Rightarrow \langle \chi | \chi \rangle = 0 \Rightarrow |\chi\rangle = 0$$

Where the last term vanishes because we assume χ is orthogonal to $a^\dagger|\psi\rangle$ and $a|\psi\rangle$. After all this we get

$$\boxed{\tilde{a}|\psi\rangle = e^{-\pi\omega_0} a^\dagger|\psi\rangle} \quad (3)$$

Similarly we can also show that

$$\boxed{\tilde{a}^\dagger|\psi\rangle = e^{\pi\omega_0} a|\psi\rangle} \quad (4)$$

Next lecture we'll apply these results to black holes