PHY 387M: Relativity Theory Notes

Delon Shen

Notes for Prof. Matzner's Relativity Theory(PHY 387M) course at UT Austin during Spring 2021. The course follows Misner, Thorne, and Wheeler's "Gravitation" as well as Prof. Matzner's own notes. This will also contain my reading notes from some parts of Wald's *General Relativity* and some material from Prof. Hirata's lecture notes. If you have any comments let me know at `hi@delonshen.com`.

Last updated on January 18, 2021

LECTURE 1A: HISTORICAL BACKGROUND AND SPECIAL RELATIVITY

Historical background I'm leaving out goes here

Let $\mathcal{E}$ be an event in a $D = 4$ spacetime $\mathcal{M}$. $\mathcal{E}$ could be a camera flash going off at position $x^\mu = \{t, x, y, z\}$. Lets say we have two such events $\mathcal{E}_1$ and $\mathcal{E}_2$. The interval between these two events is

$$x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = -c^2 t^2 + x^2 + y^2 + z^2$$

This interval is the same in any reference frame. Lets try to derive the Lorentz transform. Consider some fast guy walking towards you with speed v. You're standing still in your reference frame. Your position in the fast guy's reference frame x' if we assume Galilean relativity is $x = x' - vt$. However this clearly doesn't keep the speed of light c the same in every reference frame. Thus lets introduce an undetermined function $\gamma(|v|)$ where we use $|v|$ to impose isotropy

$$x' = \gamma(|v|)\left(x - \frac{v}{c}ct\right) \tag{1}$$

In Galilean relativity $ct' = ct$ but this also doesn't work. However, if we're somehow inspired to, we can also guess for special relativity

$$ct' = \gamma(|v|)\left(ct - \frac{v}{c}x\right) \tag{2}$$

Now we use the invariant interval to get γ

$$-c^2 t^2 + x^2 = -\gamma^2\left(ct - \frac{v}{c}x\right)^2 + \gamma^2\left(x - \frac{v}{c}ct\right)^2$$

Solving for γ with Mathematica gives us the following

$$\boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}}$$

We should also know that the invariant interval can become infinitesimal giving us an infinitesimal arc length in flat space time

$$-ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

This leads us to define the four velocity

$$\frac{dx^\mu}{ds} = \left\{c\frac{dt}{ds}, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right\}$$

We define spacelike as $ds^2 > 0$, timelike as $ds^2 < 0$, and null (e.g. light ray) as $ds^2 = 0$. Note that if $ds^2 = 0$ we can't define the 4-velocity as above. We introduce some parameter (affine parameter?) λ and have

$$0 = -c^2\left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2 + \dots$$

From now on we will let $c = 1$.

Lecture 1b: Some examples in special relativity

Lightcone stuff here

We'll now introduce the metric $\eta_{\mu\nu}$ (for some reason he uses opposite signature as what he did last lecture?)

$$\eta_{\mu\nu} = \text{diag}(-1,1,1,1) \Rightarrow ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2$$

Lets look at time dilation. Say we're standing still in our reference frame K. Now consider a moving frame K' with velocity v. In the K' frame we know that $d\tau' = dt'$. From (1) (2) we know that

$$dx' = 0 = \gamma(dx - vdt) \Rightarrow dx = vdt \Rightarrow dt' = \gamma(dt - v^2dt) = \gamma dt/\gamma^2 = dt/\gamma$$

The last equality is time dilation. Now for length contraction. Consider a meter stick sitting at rest in reference frame K. Now consider an observer moving with velocity v with resepct to K. The moving observer's rest frame is K'. Now we measure the lenght of the meter stick in the K' frame as l', (in this case $dt' = 0$). Now again with what we found in lecture 1a

$$dt' = 0 = \gamma(dt - vl) \Rightarrow dt = vl \Rightarrow l' = \gamma(l - (vdt = v^2l)) = \gamma(1 - v^2) = l\gamma/\gamma^2 \Rightarrow l' = l/\gamma$$

Here we have length contraction.

Skipping einstein summation stuff since QFT has beaten that into me

Twins Paradox

I got the material in this subsection from here as well as Matzner's lecture on this stuff. A and B are a couple who happen to be born at exactly the same time. B is going on a space mission. He will get on a rocket ship and travel away from earth at a velocity V for some time T and then will travel back to earth with velocity $-V$ for the same time T. Thus A will have aged $2T$ in the time that B has been gone but from time dilation he expects B to be younger than he is when B gets back. However by symmetry B would expect A to be younger when B gets back since from B's perspective, A is travelling away from him. Lets resolve this paradox. First we'll formalize what we said above by defining a few events. Let a_1 be the event when B leaves earth, a_2 be the event when B turns around, and a_3 be the event when B returns to earth. The proper time elapsed for A from a_1 to a_2 is $\tau_A(a_1 \to a_2) = T$ and similarly $\tau_A(a_2 \to a_3) = T$. This gives us $\tau_A(a_1 \to a_3) = 2T$. From our result on time dilation above we get that $\tau_B(a_1 \to a_3) = 2T\sqrt{1 - V^2}$. Now to second order in V (we could go to higher order but the first non-vanishing term in the Taylor expansion illustrates what we'll want to get across)

$$\tau_A - \tau_B = 2T(1 - \sqrt{1 - V^2}) \approx TV^2 + O(V^3)$$

A is older than B. But in B's reference frame we'd expect B to be older than A by symmetry. There a $2TV^2$ term missing somewhere that points to an asymmetry. So where does the asymmetry come in? Lets look at a_2 more closely. Lets assume B accelerates backwards with acceleration g for some $\delta t'$ where $\delta t' \ll T$. We know that

$$g\delta t' = 2V$$

Now note that to first order in V from (1) (2) we have

$$x' = \gamma(x - Vt) \approx x - tV + O(V^2) \Rightarrow x \approx x' + tV$$
$$t' = \gamma(t - Vx) \approx \gamma(t - V(x' + tV)) \approx t - x'V \Rightarrow t = t' + x'V$$

If we want to assert acceleration we'll let $V = g\delta t'$ meaning that

$$\delta t = \delta t'(1 + gx') \Rightarrow \frac{\delta t}{\delta t'} = 1 + gx'$$

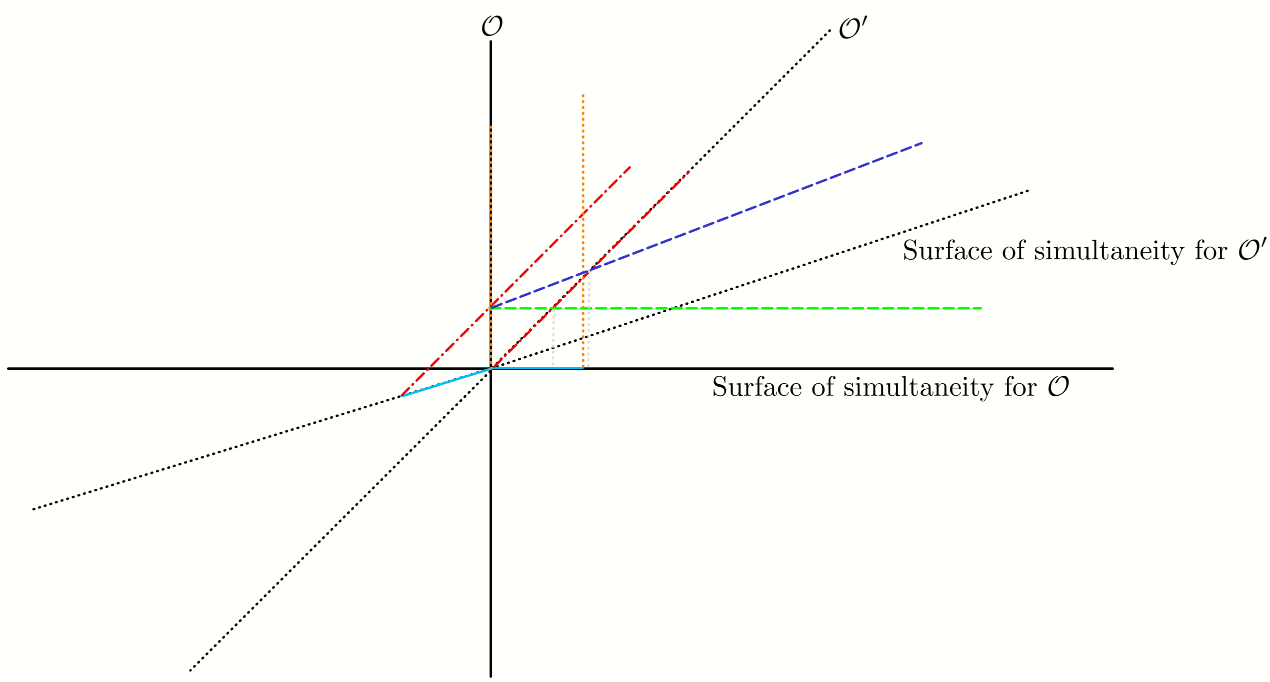
What this equation is saying is that in an accelerating frame at different "height" (e.g. x' which is TV in this case), A is aging at a different rate than B. This will resolve our paradox. Since we're accelerating to the left the acceleration is $-g$ and the position of A in B's frame is $-TV$ meaning that

$$\delta t - \delta t' = \delta t' g x' = 2TV^2$$

The missing $2TV^2$ term that resolves the idea that A should be older than B by TV^2 if we're following along with B. This difference in passage of time at different heights in an accelerating frame can also be measured by GPS's (I think Matzner mentioned this in the lecture.) One thing to note is that we only resolved the paradox to order V^2 but it gets the point across and is valid for higher orders according to Hirata (I'll just take his word on this here.)

Wald Problem 1.1: Car and Garage Paradox

Taking inspiration from figure 1.3 of Wald we get



Where O is the observer at rest in the garage and O' is the moving vehicle observer frame. From the dashed green line we see that for $\mathcal{O}$ when the back of the car enters the garage the front of the car is still in the garage thus the doorman is correct in his frame. From the dashed purple line we see that for $\mathcal{O}'$ when the back of the car enters the garage the front of the car has already gone through the back of the garage and thus $\mathcal{O}'$ is also correct in his assumption.