PHY 387M: Relativity Theory Notes

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Notes for Prof. Matzner's Relativity Theory(PHY 387M) course at UT Austin during Spring 2021. The course follows Misner, Thorne, and Wheeler's "Gravitation" as well as Prof. Matzner's own notes. This will also contain my reading notes from some parts of Wald's *General Relativity* and some material from Prof. Hirata's lecture notes. If you have any comments let me know at `hi@delonshen.com`.

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Lecture 1a: Historical Background and Special Relativity

Historical background I'm leaving out goes here

Let $\mathcal{E}$ be an event in a $D = 4$ spacetime $\mathcal{M}$. $\mathcal{E}$ could be a camera flash going off at position $x^\mu = \{t, x, y, z\}$. Lets say we have two such events $\mathcal{E}_1$ and $\mathcal{E}_2$. The interval between these two events is

$$x^\mu x_\mu = x^\mu g_{\mu\nu} x^\nu = -c^2 t^2 + x^2 + y^2 + z^2$$

This interval is the same in any reference frame. Lets try to derive the Lorentz transform. Consider some fast guy walking towards you with speed v. You're standing still in your reference frame. Your position in the fast guy's reference frame x' if we assume Galilean relativity is $x = x' - vt$. However this clearly doesn't keep the speed of light c the same in every reference frame. Thus lets introduce an undetermined function $\gamma(|v|)$ where we use $|v|$ to impose isotropy

$$x' = \gamma(|v|)\left(x - \frac{v}{c} ct\right) \tag{1}$$

In Galilean relativity $ct' = ct$ but this also doesn't work. However, if we're somehow inspired to, we can also guess for special relativity

$$ct' = \gamma(|v|)\left(ct - \frac{v}{c} x\right) \tag{2}$$

Now we use the invariant interval to get γ

$$-c^2 t^2 + x^2 = -\gamma^2 \left(ct - \frac{v}{c} x\right)^2 + \gamma^2 \left(x - \frac{v}{c} ct\right)^2$$

Solving for γ with Mathematica gives us the following

$$\boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}}$$

We should also know that the invariant interval can become infinitesimal giving us an infinitesimal arc length in flat space time

$$-ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

This leads us to define the four velocity

$$\frac{dx^\mu}{ds} = \left\{c\frac{dt}{ds}, \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right\}$$

We define spacelike as $ds^2 > 0$, timelike as $ds^2 < 0$, and null (e.g. light ray) as $ds^2 = 0$. Note that if $ds^2 = 0$ we can't define the 4-velocity as above. We introduce some parameter (affine parameter?) λ and have

$$0 = -c^2 \left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2 + \ldots$$

From now on we will let $c = 1$.

Lecture 1b: Some examples in special relativity

Lightcone stuff here

We'll now introduce the metric $\eta_{\mu\nu}$ (for some reason he uses opposite signature as what he did last lecture?)

$$\eta_{\mu\nu} = \text{diag}(-1,1,1,1) \Rightarrow ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + dx^2 + dy^2 + dz^2$$

Lets look at time dilation. Say we're standing still in our reference frame K. Now consider a moving frame K' with velocity v. In the K' frame we know that $d\tau' = dt'$. From (1) (2) we know that

$$dx' = 0 = \gamma(dx - vdt) \Rightarrow dx = vdt \Rightarrow dt' = \gamma(dt - v^2dt) = \gamma dt/\gamma^2 = dt/\gamma$$

The last equality is time dilation. Now for length contraction. Consider a meter stick sitting at rest in reference frame K. Now consider an observer moving with velocity v with resepct to K. The moving observer's rest frame is K'. Now we measure the lenght of the meter stick in the K' frame as l', (in this case $dt' = 0$). Now again with what we found in lecture 1a

$$dt' = 0 = \gamma(dt - vl) \Rightarrow dt = vl \Rightarrow l' = \gamma(l - (vdt = v^2l)) = \gamma(1 - v^2) = l\gamma/\gamma^2 \Rightarrow l' = l/\gamma$$

Here we have length contraction.

Skipping einstein summation stuff since QFT has beaten that into me

Twins Paradox

I got the material in this subsection from here as well as Matzner's lecture on this stuff. A and B are a couple who happen to be born at exactly the same time. B is going on a space mission. He will get on a rocket ship and travel away from earth at a velocity V for some time T and then will travel back to earth with velocity $-V$ for the same time T. Thus A will have aged $2T$ in the time that B has been gone but from time dilation he expects B to be younger than he is when B gets back. However by symmetry B would expect A to be younger when B gets back since from B's perspective, A is travelling away from him. Lets resolve this paradox. First we'll formalize what we said above by defining a few events. Let a_1 be the event when B leaves earth, a_2 be the event when B turns around, and a_3 be the event when B returns to earth. The proper time elapsed for A from a_1 to a_2 is $\tau_A(a_1 \to a_2) = T$ and similarly $\tau_A(a_2 \to a_3) = T$. This gives us $\tau_A(a_1 \to a_3) = 2T$. From our result on time dilation above we get that $\tau_B(a_1 \to a_3) = 2T\sqrt{1 - V^2}$. Now to second order in V (we could go to higher order but the first non-vanishing term in the Taylor expansion illustrates what we'll want to get across)

$$\tau_A - \tau_B = 2T(1 - \sqrt{1 - V^2}) \approx TV^2 + O(V^3)$$

A is older than B. But in B's reference frame we'd expect B to be older than A by symmetry. There a $2TV^2$ term missing somewhere that points to an asymmetry. So where does the asymmetry come in? Lets look at a_2 more closely. Lets assume B accelerates backwards with acceleration g for some $\delta t'$ where $\delta t' \ll T$. We know that

$$g\delta t' = 2V$$

Now note that to first order in V from (1) (2) we have

$$x' = \gamma(x - Vt) \approx x - tV + O(V^2) \Rightarrow x \approx x' + tV$$
$$t' = \gamma(t - Vx) \approx \gamma(t - V(x' + tV)) \approx t - x'V \Rightarrow t = t' + x'V$$

If we want to assert acceleration we'll let $V = g\delta t'$ meaning that

$$\delta t = \delta t'(1 + gx') \Rightarrow \frac{\delta t}{\delta t'} = 1 + gx'$$

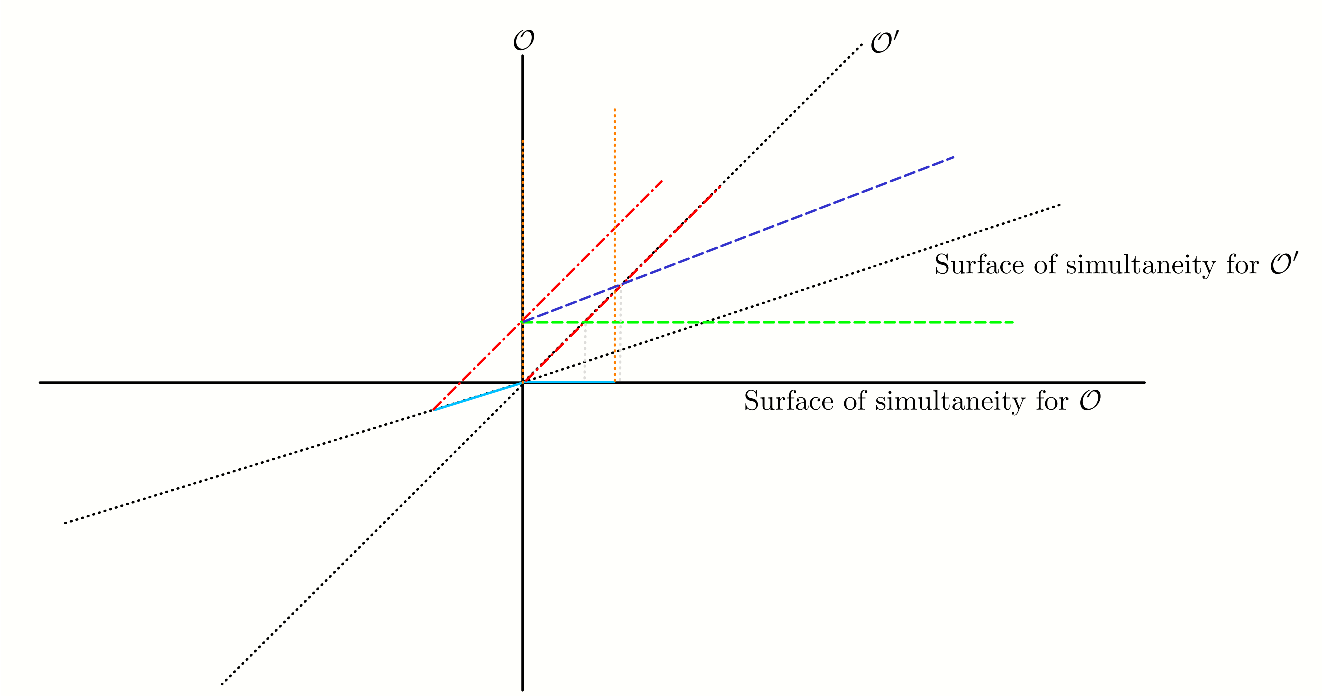
What this equation is saying is that in an accelerating frame at different "height" (e.g. x' which is TV in this case), A is aging at a different rate than B. This will resolve our paradox. Since we're accelerating to the left the acceleration is $-g$ and the position of A in B's frame is $-TV$ meaning that

$$\delta t - \delta t' = \delta t' g x' = 2TV^2$$

The missing $2TV^2$ term that resolves the idea that A should be older than B by TV^2 if we're following along with B. This difference in passage of time at different heights in an accelerating frame can also be measured by GPS's (I think Matzner mentioned this in the lecture.) One thing to note is that we only resolved the paradox to order V^2 but it gets the point across and is valid for higher orders according to Hirata (I'll just take his word on this here.)

Wald Problem 1.1: Car and Garage Paradox

Taking inspiration from figure 1.3 of Wald we get



Where O is the observer at rest in the garage and O' is the moving vehicle observer frame. From the dashed green line we see that for $\mathcal{O}$ when the back of the car enters the garage the front of the car is still in the garage thus the doorman is correct in his frame. From the dashed purple line we see that for $\mathcal{O}'$ when the back of the car enters the garage the front of the car has already gone through the back of the garage and thus $\mathcal{O}'$ is also correct in his assumption.

Wald Chapter 2: Manifolds and Tensor Fields

Lets start by motivating the idea of manifolds. Before general relativity we could assume that globally space time was flat, $\mathbb{R}^{3,1}$. However with the entrance of general relativity we'll be solving for the global structure of spacetime. Locally however we can still say things looks flat. Trying to solve for the structure of ST is similar to trying to determine the shape of the Earth as a sailor. Locally we know that the surface of Earth looks like $\mathbb{R}^2$ but globally it wouldn't be safe to assume that the surface of the Earth is $\mathbb{R}^2$. We do know however that the surface of the Earth is some surface embedded in $\mathbb{R}^3$ so this could motivate us to study ST as an embedding in some higher dimensional space. However ST doesn't have a natural higher dimensional space which we can embed it into.

So lets try to formalize this notion of a manifold. First we'll define a open ball int $\mathbb{R}^n$ of radius r centered at point $y = (y^1, \ldots, y^n)$ as all point $x = (x^1, \ldots, x^n)$ where

$$|x - y| = \left\{ \sum_{i=1}^{n} (x_1 - y_1)^2 \right\}^{1/2} < r$$

A open set is a set which is the union of a set of open balls. Now a manifold is sort of like a patchwork of open subsets of $\mathbb{R}^n$. Or more formally a real, smooth manifold $\mathcal{M}$ is a set and a set of subsets $\{O_\alpha\}$ that satisfy the following property

(a) $\{O_\alpha\}$ cover $\mathcal{M}$

(b) For every α there exists a bijective function ψ_α between O_α and U_α where U_α is some subset of $\mathbb{R}^n$

(c) If there exists α and β such that $O_\alpha \cap O_\beta \neq \emptyset$ then there exists a smooth function $\psi_\beta \circ \psi_\alpha^{-1}$ which takes points from $\psi_\alpha(O_\alpha \cap O_\beta) \subset U_\alpha \subset \mathbb{R}^n$ to points $\psi_\beta(O_\alpha \cap O_\beta) \subset U_\beta \subset \mathbb{R}^n$. We'll also require that these subsets U_α be open subsets.

The ψ_α are what we call coordinate systems. To make sure we can't create new manifolds by removing or adding coordinate systems we could also require $\{\psi_\alpha\}$ to contain all functions that satisfy (b) and (c) (e.g. maximal).

Wald Appendix A.1: Some Definitions for Topological Spaces

The reason we care about topological spaces is because general relativistic space time has the structure of a topological space (and more!) So lets define a topological space . Let X be a set and $\mathcal{J}$ be a collection of subsets of X. We define $(X, \mathcal{J})$ to be a topological space that satisfies the following properties

(a) Let $\{O_\alpha\} \subset \mathcal{J}$. We require $\bigcup_\alpha O_\alpha \subset \mathcal{J}$

(b) Let $\{O_\alpha\}$ be a finite subset of $\mathcal{J}$. We require $\bigcap_\alpha O_\alpha \subset \mathcal{J}$.

(c) We require $\emptyset$ and X to be members of $\mathcal{J}$.

$\mathcal{J}$ is a topology on X and contains only open sets. Lets look at some fun things we can do with topological spaces *(a)* We can make a topology out of any set pretty easily. For example $\{X, \{X, \emptyset\}\}$ works pretty well. That's a fun party trick I guess.

(b) Let $X = \mathbb{R}$. We can let $\mathcal{J}$ to be the set of all sets which can be formed by the union of some open interval (a, b). This generalizes to $\mathbb{R}^n$ with open balls of $\mathbb{R}^n$.

(c) For a given topological space $\{X, \mathcal{J}\}$ any subset $A \subset X$ can also be made into a topology by definition of a topology of A as $\mathcal{I} = \{U | U = A \cap V \text{ s.t. } V \in \mathcal{J}\}$. This is called an *induced topology.*

(d) Let $\{X_1, \mathcal{J}_1\}$ and $\{X_2, \mathcal{J}_2\}$ be topological spaces. We can define a topology $\mathcal{J}$ for the set $X = \{(x_1, x_2) \text{ s.t. } x_1 \in X_1,\ x_2 \in X_2\}$ as all sets that are the unions of sets of the form $O_1 \times O_2$ where $O_i \in \mathcal{J}_i$. In this way we can build up to a topological space for $\mathbb{R}^n$

(e) Let $\{X, \mathcal{J}\}$ and $\{Y, \mathcal{J}\}$ be topological spaces. Consider a map $f : X \to Y$. If for any open subset $O \in J$, $f^{-1}(O)$ is open as well we call this map continuous. Now if f is continuous and bijective and f^{-1} is continuous we say that f is a *homeomorphism* and the two topological spaces are *homeomorphic*, they have the same topological properties.

(f) From here on out assume $\{X, \mathcal{J}\}$ is a topological space

(g) A set $C \subset X$ is closed if $X - C$ is open. In the topology we described in (a) we see that all members of $\mathcal{J}$ are both open and closed. We define topologies where the only subsets that are both open and closed are the set itself and $\emptyset$ as *connected* .

(h) Let $A \subset X$. The *closure* of A denoted by $\overline{A}$ is the intersection of all closed sets containing A. $\overline{A}$ contains A, is closed, and equal A iff A is closed as well. Similarly the *interior* of A denoted by $\tilde{A}$ is the union of all open sets inside A. $\tilde{A}$ is a subset of A, is open, and equal A iff A is open as well. The *boundary* of A is the set of point in $\overline{A}$ that are not in $\tilde{A}$.

(i) If $\forall p, q \in X$ we can find $O_p, O_q \in \mathcal{J}$ such that $p \in O_p$ and $q \in O_q$ then we call that topological space *Hausdorff*

(j) A set A is said to be compact if for any open cover of A there exists a finite subcover of A.

(k) Let $\{O_\alpha\}$ be a open cover of X. We call another open cover $\{V_\alpha\}$ a *refinement* of $\{O_\alpha\}$ if for all β there exists an O_α such that $V_\beta \subset O_\alpha$.

(l) We call a cover $\{V_\alpha\}$ locally finite if for all $x \in X$ there exists a neighborhood W such that the number of sets $V \in V_\beta$ that satisfy $V \cap W = \emptyset$ is finite.

(m) A topological space is paracompact if for each open cover $\{O_\alpha\}$ of X there exists a locally finite refinement of $\{O_\alpha\}$.

We'll stop with appendix A here. For chapter two we only need the definitions.

Back to Manifolds

If we take the topological route to define manifolds we would require the set of $\{\psi_\alpha\}$ to contain only homeomorphic functions. The only topological spaces considered in this book are *Hausdorff* and *paracompact.*

Lets consider an example of a manifold, a 2-sphere S^2, which we define as

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_1^2 + x_2^2 + x_2^3 = 1\}$$

To map this onto $\mathbb{R}^2$ we define our elements of the covering set as $O_i^\pm$ such that

$$O_i^\pm = \{(x_1, x_2, x_3) \in S^2 \wedge \pm x_i > 0\}$$

Namely the set of $O_i^\pm$ is the set of six hemispheres that cover S^2. Also we can use a homomorphic function to project each $O_i^\pm$ onto $U_\alpha = D \subset \mathbb{R}^2$ where D is the disk on the j,k plane and $i \neq j \neq k$. This also satisfies the condition for overlapping elements of the cover of S^2. Namely $\psi_\beta \circ \psi_\alpha^{-1}$ behaves exactly as we expect it to.

We can also define products of manifolds. Consider two manifolds M and M'. We can define $M \times M' = \{(p, p') | p \in M \;\; \wedge p' \in M'\}$. From here we can construct the covering set for $M \times M'$ by considering the covering set for M which we denote by O and M' which we denote by O'. We get $O_{\alpha\beta} = \{(O_\alpha, O'_\beta) | O_\alpha \in O \wedge O'_\beta \in O'\}$. Also for all α and β there should exists an $\psi_{\alpha\beta}$ such that $\psi_{\alpha\beta}(O_{\alpha\beta}) = U_{\alpha\beta}$. We can construct this as you would expect. By letting $\psi_{\alpha\beta}(\{O_\alpha, O'_\beta\}) = \{\psi_\alpha(O_\alpha), \psi'_\beta(O'_\beta)\}$ where ψ_α is the corresponding function for O_α in M and ψ'_β is the corresponding function for O'_β in M'.

We can now describe differentiability and smoothness. Consider manifolds M and M' with coordinate systems $\{\psi_\alpha\}$ and $\{\psi'_\alpha\}$ respectively. We call a map f smooth if for all α and β we have that $\psi'_\beta \circ f \circ \psi_\alpha^{-1}$ is a smooth function between U_α and U_β. Furthermore if this f is one-to-one, onto, and has a smooth inverse map then we call this function a *diffeomorphism* and the two manifolds *diffeomorphic*.

Vectors

We all know about vector spaces. You know about them, I know about them. However our intuitive notion of vector spaces start to break down in curved manifolds. For example, how do we define a vector space on a 2-sphere so that the vector space is still closed under vector addition? We'll find that we can recover our intuitive notion of vector spaces by considering *infinitesimal* vectors which stems from the fact that in general relativity we can assume that locally space looks flat (think about flat earthers.) However it turns out that our intuition for infinitesimal vectors breaks a little in curved geometry as well. For a sphere we have an intuitive picture of a tangent vector to the sphere since it's embedded in $\mathbb{R}^n$. However when we no longer have ourselves embedded in $\mathbb{R}^n$ our intuition for tangent spaces becomes a little shaky. So we'll start by trying to construct tangent spaces from only the properties of the manifold and keep $\mathbb{R}^n$ as our favorite special case for checking our work.

The way we'll construct this notion of tangent vectors is through direction derivatives along those tangent vectors. For a quick refresher on directional derivatives lets consider $\mathbb{R}^2$ say we want to find the change of a function $f(x,y)$ along a vector $\mathbf{v} = a\hat{x} + b\hat{x}$. Then we can write the direction derivative of f along $\mathbf{v}$ as $D_{\mathbf{v}}f$ with the definition of derivatives we learned in our first calculus class

$$D_{\mathbf{v}}f = \lim_{h\to 0}\frac{f(x+ah, y+bh) - f(x,y)}{h}$$

This so far isn't very illuminating but we can bring this into a cleaner form. Consider the function $g(z) = f(x_0 + az, y_0 + ba)$ where everything except for z is fixed. The derivative of this with respect to z can be again found with our regular derivative definition

$$g'(z) = \lim_{h\to 0}\frac{g(z+h) - g(z)}{h} \Rightarrow g'(0) = \lim_{h\to 0}\frac{g(h) - g(0)}{h} = \lim_{h\to 0}\frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\mathbf{v}}f(x_0, y_0)$$

Still not very illuminating. But now lets consider $g(z) = f(x = x_0 + az, y = y_0 + bz)$, the same function but dressed differently. Using the chain rule we get

$$g'(z) = \frac{\partial g}{\partial x}\frac{dx}{dz} + \frac{\partial g}{\partial y}\frac{dy}{dz} = \frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b \Rightarrow g'(0) = \left(\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b\right)\Bigg|_{x=x_0,\ y=y_0} = D_{\mathbf{v}}f(x_0, y_0)$$

Now since we fixed x_0, y_0 arbitrarily we now have a much more useful definition for directional derivatives

$$D_{\mathbf{v}}f(x,y) = v_x\frac{\partial f}{\partial x} + v_y\frac{\partial f}{\partial y}$$

This inspires the definition of direction derivatives which we'll use to implicitly define vectors and thus tangent vectors on our manifold.

$$v = v^\mu \partial_\mu$$

Let F be the set of smooth functions from M to $\mathbb{R}$. We define a tangent vector at a point $p \in M$ as the map $v : \mathsf{F} \to \mathbb{R}$. Note that this map is linear and obeys its own Leibnitz rule.

Theorem: Consider a n–dimensional manifold called M and some $p \in M$. Also let V_p denote the tangent space at p. We will show that $\dim V_p = n$.

Proof: We can do this by explicitly constructing an orthogonal basis for V_p with n elements. To do this first consider some coordinate system ψ and some function $f : U \to \mathbb{R}$. (I think) a concrete example of what f could be is the temperature at each point on the manifold. With these two function we can define a function

$$F = f \circ \psi^{-1} : \mathbb{R}^n \to \mathbb{R}$$

This is a function on the coordinates that label a manifold instead of the manifold itself. E.g. we can ask what's the temperature at $(r = R, \theta = 0, \phi = 0)$ with F wheras with f we could only ask what's the temperature at the pole. From here we will define X_μ which we'll show is our orthogonal basis for V_p

$$X_\mu = \frac{\partial}{\partial x^\mu}F\bigg|_{\psi(p)}$$

This function is determing the rate of change of f along the basis coordinates defined by coordinate system ψ(or something like that?). These are tangent vectors. Now we'll use a result

(TODO: it'll be proven in the end of chapter problems) that basically says for smooth functions g shift the origin.

$$g(x) = g(a) + (x_\mu - a_\mu)H^\mu(x)$$

I tried my best to illustrate this in Figure 1 Asserting that $g = F$ and $a = \psi(p)$ we get for a point

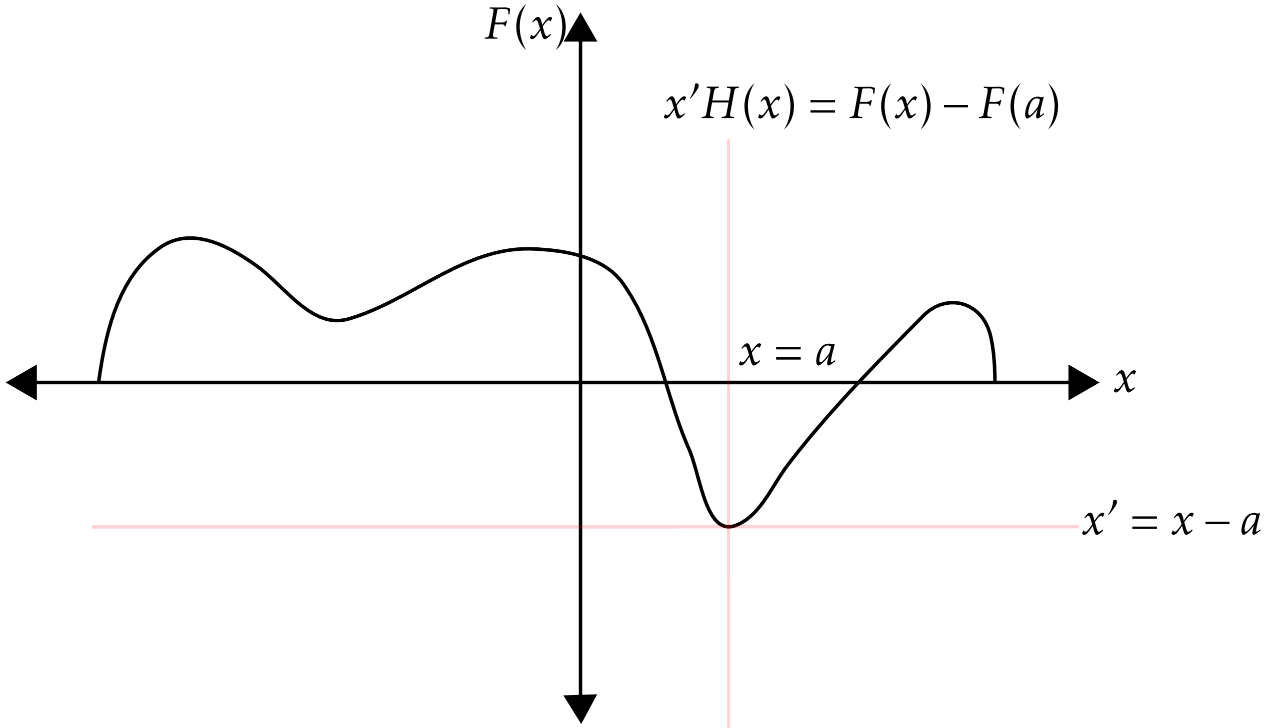


Figure 1: Visual depiction of the result of problem 2.2 for a one dimensional manifold. This was used to prove that the dimension of a tangent space V_p is n

q on the manifold

$$f(q) = f(p) + (x^\mu \circ \psi(q) - x^\mu \circ \psi(p))H_\mu(\psi(q)) \tag{3}$$

From considering an infinitesimal displacement $q = p + \delta p$ we can also intuit that

$$H_\mu(\psi(p)) = \left.\frac{\partial F}{\partial x^\mu}\right|_{q=p} \tag{4}$$

Now lets consider a arbitrary tangent vector $v \in V_p$. What we want to do is find the directional derivative of this function $f(q)$ along this v at the point on the manifold p. To do this we apply the definition of $v = v^\mu \partial_\mu$ to Equation 3. We use the linearity and leibnitz rule for v as well as the fact that the directional derivative of a constant is zero.

$$v(f) = H_\mu(\psi(q))\Big|_{q=p} v[x^\mu \circ \psi - \underline{x^\mu \circ \psi(p)}] + \underline{[x^\mu \circ \psi - x^\mu \circ \psi(p)]}\Big|_p v[H_\mu \circ \psi]$$

Now applying Equation 4 to the first term and noticing that the underlined terms vanish we're left with

$$\boxed{v(f) = \frac{\partial F}{\partial x^{\mu}}\bigg|_{q=p} v[x^{\mu} \circ \psi] = X_{\mu} v[x^{\mu} \circ \psi]} \tag{5}$$

Now we can see that X_{μ} is a basis for V_p and the components of an arbitrary tangent vector $v \in V_p$ are determined by $v[x^{\mu} \circ \psi]$. What this is saying is that any tangent vector can be constructed by the directional derivative in the direction of x_{μ} of some function f times the component vector x_{μ}. Since we have proven that X_{μ} is a basis for V_p we have shown that dim $V_p = n$.

This X_{μ} is called the coordinate basis. If we had chosen another coordinate system ψ' then we could find the coordinate basis in the new coordinate system. First let $x^{\nu'}$ the ν component of $\psi' \circ \psi^{-1}$. We can then assert

$$X_{\mu} = \frac{\partial}{\partial x^{\mu}}\bigg|_{\psi(p)} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} \frac{\partial}{\partial x^{\nu'}}\bigg|_{\psi(p)} = \frac{\partial x^{\nu'}}{\partial x^{\mu}} X_{\nu'}\bigg|_{\psi(p)}$$

Lets try plugging this into (5). We know geometrically v stays the same in both coorinate systems so we need to equate

$$X_{\mu} v^{\mu} = X_{\nu'} v^{\nu'} \Rightarrow \frac{\partial x^{\nu'}}{\partial x^{\mu}} X_{\nu'} v^{\mu} = X_{\nu'} v^{\nu'} \Rightarrow \boxed{\frac{\partial x^{\nu'}}{\partial x^{\mu}} v^{\mu} = v^{\nu'}} \tag{6}$$

The boxed equation is the *vector transformation law.*

Now lets consider an arbitary curve C on the manifold. We can think of C as a smooth map from $\mathbb{R}$ to the manifold. At each point p on this curve we can associate with it a tangent vector $T \in V_p$. Let $f \in \digamma$ be a function from the manifold to $\mathbb{R}$. By our definition of tangent vectors we have.

$$T(f) = \frac{\partial (f \circ C)}{\partial t}$$

Also note for an arbitrary coordinate system ψ we can write $\psi \circ C \Leftrightarrow x^{\mu}(t) \Rightarrow C = \psi^{-1}(x^{\mu}(t))$. Thus we can write

$$T(f) = \frac{\partial (f \circ \psi^{-1})}{\partial x^{\mu}} \frac{\partial x^{\mu}}{\partial t} = X_{\mu} \frac{\partial x^{\mu}}{\partial t}$$

From this we can see that the components of the tangent vector is given by

$$\boxed{T^{\mu} = \frac{\partial x^{\mu}}{\partial t}}$$

Definition $[v,w](f) = v[w(f)] - w[v(f)]$ skip formalism for fields for now. this part feels more intuitive than last part.

Tensors

First we will define a *dual vector* as a linear map that takes some number of spatial displacement vectors and maps them to a number. Or more formally consider a vector space V. Let V^* be the collection of linear maps $f : V \to \mathbb{R}$. Since V is a vector space it is easy to see that due to f being linear that V^* is also a vector space. Now let us define a basis $\{v_{\mu*}\}$ with the defining property that that the action of this basis on the basis of V is

$$v_{\mu*}v^{\nu} = \delta_{\mu}^{\nu}$$

We will show in the problems that this makes $\{v_{\mu*}\}$ a basis for V^*. From here we call V^* the dual space to V. Right now V doesn't have enough strucutre to uniquely define its dual space V^*. This extra strucutre will come from a metric. Now lets prove that the dual of the dual space V^* denoted by V^{**} can be identified to V. This part is based on these set of notes since Wald was too slick for me here. Let consider a map $g : V \to V^{**}$ defined as $g(v)(f) = f(v)$ where $v \in V$ and $f \in V^*$. To show that the two spaces can be identified with eachother we'll need an isomorphic function between the two spaces. Our canidate for this function is g. To show that g is isomorphic we'll need the following components.

(a) We know that since V and V^{**} have the same dimension we only need to prove that g is one-to-one to prove that g is isomorphic.

(b) To prove that g is one to one we need to prove that null g is the zero element.

Let $v \in \text{null}(g)$. Then $g(v)$ is the zero element of V^{**}. This means that for any $v^* \in V^*$ we have $(g(v))(v^*) = 0$. Thus by the definition of our function for any linear function $v^* \in v$ we have $v^*(v) = 0$. This property is only satisfied if $v = 0$. Thus null $g = \{0\}$ and thus g is one-to-one. Therefore V we'll from now on identify V with V^{**}.