Information Theory

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Some notes I write while learning (classical and quantum) information theory. If you have any comments let me know at `hi@delonshen.com`.

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- Witten refers to Witten's *"A Mini-Introduction To Information Theory"*
- Wilde refers to Wilde's *"Quantum Information Theory"*

Witten: Classical Information Theory

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Disaster! You've been struck by an acute and permanent case of locked in syndrome. In a tragic turn of events, your eyes begin to randomly move either up or down with probability p and $(1-p)$ every half a second. Even worse, you're on a conveyor belt that's slowly but steadily going towards a furnace (like that scene in Toy Story 3) and you know for certain that you'll be no more in 3 days. How inconvenient! For some reason you have some machines attached to you that records when you move your eyes up(denoted by H) and when you move your eyes down(denoted by A). Using this machine you can usually send messages to your loved ones that are of the form

$$HAHAHAHAHAHAHAHAAAHAAHAHAHAHAHHAA\dots$$

But now it's just a random string generator. Given your deadline you guess that the resulting message will be N letters long. As you slowly inch towards your death you start to think about strange things. For example you think that when your loved ones look back on your final message that there will be around pN "H" characters and $(1-p)N$ "A" characters. How many messages with this combination of characters are there? Well from basic combinatorics we know that the number is

$$\frac{N!}{(pN)!((1-p)N)!} \approx \frac{N^N}{(pN)^{pN}((1-p)N)^{(1-p)N}} = \frac{1}{p^{p\times N}(1-p)^{(1-p)\times N}}$$

Now lets define a quantity called *Shannon entropy* S as

$$2^{NS} = \frac{1}{p^{p\times N}(1-p)^{(1-p)\times N}} \Rightarrow S = -p\log_2(p) - (1-p)\log_2(1-p)$$

Say we were sending the same kind of message that's N characters long but **we don't know the probabilities each letter would occur** then by combinatorics there would be 2^N possible messages. However we have more information than that fool, we know that H occurs with probability p and A occurs with probability $(1-p)$. That means we can send 2^{NS} messages. Or at least I think this is what we're saying. Just by knowing the probabilities each letter can occur has effectively (though not in reality) increased the length of our message by a factor of S! Each and every single letter in the string carries more information(or are maybe messages harder to decipher? I don't know yet). Lets extend this to the general case

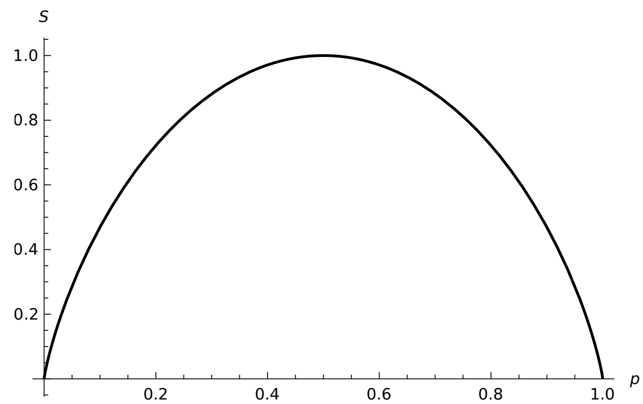
> Definition 1: (Shannon Entropy) Lets say we have a message that is composed of n different letters $\{a_1,\dots,a_n\}$ where each letter occurs with probability $\{p_1,\dots,p_n\}$. We define the Shannon Entropy S as
>
> $$2^{NS} = \frac{N^N}{(p_1N)^{p_1N}\dots(p_nN)^{p_nN}} \Rightarrow S = -\sum_{i=1}^{N} p_i \log_2(p_i)$$

Lets notice a few things about Shannon entropy. First of all since the numbere of messages we can send has to be at least 1 we know that $S \geq 0$. Now lets also ask, what's the maximum possible entropy for an alphabet of k letters.

Example 1: (Maximizing Shannon Entropy) Before we consider a general alphabet of k letters lets just consider our simple two alphabet case. We had that

$$S = -p\log_2(p) - (1-p)\log_2(1-p)$$

Now lets plot the Shannon entropy versus p and see if we can't qualitatively guess the maximum.



So right off the bat we guess that Shannon entropy is maximized when the probability is equally distributed to each letter in the alphabet. Lets see how we can prove this for a general alphabet. Witten prescribes using Lagrange multipliers with the constraint $\Sigma_i p_i = 1$. So we want to solve the system of equations

$$\frac{dS}{dp_i} = -\frac{\ln p_i + 1}{\ln 2} = \lambda = \frac{d(\text{Constraint})}{dp_i} \qquad \sum_{i=1}^{k} p_i = 1$$

The blue term gives us

$$\ln p_i = \ln p_j \Rightarrow p_i = p_j \ \forall i,j \Rightarrow \sum_{i=1}^{k} p_i = kp = 1 \Rightarrow \boxed{p_i = \frac{1}{k} \ \forall i}$$

The Shannon Entropy is maximized if each letter has the same probability of occuring!

Now lets talk about two good friend, Dan and Phil. They're sending messages to each other. Dan sends an instance of a discrete random variable x to Phil who recieves an instance of a discrete random variable y. What all this means physically is that Dan is speaking a random letter into a really bad microphone and Phil is listening on really shitty headphones and can't tell with 100% certainty what Dan is saying. Let $p(x_i, y_j)$ be the probability that Dan says x_i and Phil hears y_j. Now lets say for a second that Phil is certain that he just heard y_j on his headphone. In this case the probability that Dan said x_i is given by Bayes rule

$$p(x_i|y_j) = \frac{p(x_i, y_j)}{p(y_j)} \Rightarrow S(x|y = y_j) = -\sum_i p(x_i|y_j)\log_2(p(x_i|y_j))$$

Knowing that Phil heard y_j for certain will make us more sure Phil can guess what Dan said and thus should lower the entropy. So we can guess that the Shannon entropy corresponding to the conditional probability distribution is the entropy of the message given that we heard y_j for certain. But in reality this will never happen since y is a random variable. But we can take a weighted sum of all these entropies to see on average what the entropy of the message Phil recieves is. To take this weighted sum we first need to reduce the dimension of the probability distribtuion to get $p(y)$

$$p(y) = \sum_i p(y, x_i)$$

And from here we can take a weighted sum

$$\begin{aligned}
\sum_j p(y_j)S(x|y = y_j) &= -\sum_j\sum_i p(y_j)p(x_i|y_j)\log_2(p(x_i|y_j)) \\
&= -\sum_i\sum_j p(y_j)\frac{p(x_i, y_j)}{p(y_j)}\log_2\left(\frac{p(x_i, y_j)}{p(y_j)}\right) \\
&= -\left(\sum_i\sum_j p(x_i, y_j)\log_2(p(x_i, y_j)) - \sum_i\sum_j p(x_i, y_j)\log_2(p(y_j))\right) \\
&= -\left(\underbrace{\sum_i\sum_j p(x_i, y_j)\log_2(p(x_i, y_j))}_{S_{XY}} - \underbrace{\sum_i p(y_j)\log_2(p(y_j))}_{S_Y}\right)
\end{aligned}$$

So we find that the average entropy of a message sent between Dan and Phil is the difference between the entropy of the joint probability distribution minus the entropy of the probability distribution of y. It turns out this little guy is so important that they have a name

Definition 2: (Conditional Entropy) The entropy of some probability distribution x given that we have observed y is

$$S_{XY} - S_Y$$

Conditional Entropy is related to another concept which is called **mutual information** denoted by $I(X;Y)$. This is the information that we can get about X given that we have observed Y. This would just be the difference of the actual entropy of x (S_X) and how much we don't know about x (S_{XY}).

Definition 3: (Mutual Information) Meaesurement of how much we know about a probability distribution x once we have observed y

$$I(X;Y) = S_X - S_{XY} + S_Y$$

Lets consider an example to motivate the definition of another cool guy.

Example 2: (Your Friend is a Sketchy Loser who likes Gambling and Dice) You and your friend are playing a game with two die. If the sum of the two die is > 7 you win a buck and if they're < 7 then your friend wins a buck. Rolling a 7 is a tie. We know that you both should have an equal chance of winning. However you're suspicious of your friend. He is a big chater after all. Even more suspicious, he insits on using his special die. He says he got them from a famous die maker on etsy for free after buying a space themed DnD dice set. First lets review what you'd expect to happen. There are 36 possible outcomes to this game.

$$\begin{aligned} \text{Your Friend Win}: &(1,1)(1,2)(1,3)(1,4)(1,5)(2,1)(2,2)(2,3)(2,4)(3,1)(3,2)(3,3)(4,1)(4,2)(5,1) \\ \text{You Win}: &(2,6)(3,5)(3,6)(4,4)(4,5)(4,6)(5,3)(5,4)(5,5)(5,6)(6,2)(6,3)(6,4)(6,5)(6,6) \end{aligned}$$

Let the outcome of the i^{th} game be denoted by x_i. If your friend isn't being a dick and cheating then we'd have the probability of $G(x_i)$ occuring as

$$G(x_i) = \begin{cases} \text{You Win} & \frac{15}{36} \\ \text{You Lose} & \frac{15}{36} \\ \text{Tie} & \frac{5}{36} \end{cases}$$

You and your friend play N games. For large N we'd expect you to have won $\frac{15}{36} \times N$, your friend to have won the same number of games, and to have tied $\frac{5}{36} \times N$ times. The number of sequences of N games with this many of each ending state is

$$\frac{N!}{\left(\frac{15}{36}N\right)!\left(\frac{15}{36}N\right)!\left(\frac{5}{36}N\right)!}$$

And thus we can judge the probability of what we've seen by considering [a]

$$\mathcal{P} = \underbrace{\left(\underbrace{\left(\frac{15}{36}\right)^{N\times\frac{15}{36}}}_{\text{You Win}} \underbrace{\left(\frac{15}{36}\right)^{N\times\frac{15}{36}}}_{\text{Your Friend Wins}} \underbrace{\left(\frac{5}{36}\right)^{N\times\frac{5}{36}}}_{\text{Tie}}\right)}_{\text{Probability you expect}} \underbrace{\frac{N!}{\left(\frac{15}{36}N\right)!\left(\frac{15}{36}N\right)!\left(\frac{5}{36}N\right)!}}_{\text{What you see}}$$

The first big parenthesis corresponds to the probability of the given sequence occuring. Each time x_i occurs we multiply by a factor of $G(x_i)$. Since for large N we expect x_i to occur $N \times G(x_i)$ times the resulting probability will be the first term. The second term is the number of sequences where each x_i occurs $N \times G(x_i)$ times. But what if the die are rigged (presumably in your friends favor unles he's just a real interesting guy)? What if instead your friend can manipulate the die so that the probability of each outcome $P(x_i)$ is

$$P(x_i) = \begin{cases} \text{You Win} & \frac{10}{36} \\ \text{You Lose} & \frac{22}{36} \\ \text{Tie} & \frac{3}{36} \end{cases}$$

You'd still expect the same probability based on $G(x_i)$ if you go into this thinking your friend is honest but after large N what you see will be different and will instead be caused by $P(x_i)$.

$$\mathcal{P} = \underbrace{\left(\underbrace{\left(\frac{15}{36}\right)^{N\times\frac{10}{36}}}_{\text{You Win}} \underbrace{\left(\frac{15}{36}\right)^{N\times\frac{22}{36}}}_{\text{Your Friend Wins}} \underbrace{\left(\frac{5}{36}\right)^{N\times\frac{3}{36}}}_{\text{Tie}} \right)}_{\text{Probability you expect}} \underbrace{\frac{N!}{\left(\frac{10}{36}N\right)!\left(\frac{22}{36}N\right)!\left(\frac{3}{36}N\right)!}}_{\text{What you see}}$$

How can we quantify this discrepency? That's where **relative entropy** comes in

[a] I'm still not too certain on what Witten means by "judge." I think it has something to do with trying to see the "distance" between two probability distributions but from what I can tell it seems like he just multiplied two probability distribution dependent quantities together and called it a day. TODO intuition.

Definition 4: (Relative Entropy) Let X be a random variable which denotes the outcome of an experiment. We have some theory that predicts the probability distribution for our outcome to be Q_X. However if our theory isn't quite on the mark and the actual probability distribution if P_X how could we guess that Q_X is wrong. Let x_i denote the final state of the i^{th} expieriment. Also let there be s possible final states. We do N experiments where N is large. If we go in thinking that Q_X is correct then we will judge the probability of what we have seen with

$$\mathcal{P} = \underbrace{\left(\prod_{i=1}^{s} Q_X(x_i)^{P_X(x_i)\times N}\right)}_{\text{Probability you expect}} \times \underbrace{\frac{N!}{\prod_{j=1}^{s}(P_X(x_i)\times N)!}}_{\text{What you see}}$$

When defining shannon entropy we saw that the second term could be rewritten as

$$\underbrace{\frac{N!}{\prod_{j=1}^{s}(P_X(x_i)\times N)!}}_{\text{What you see}} \approx 2^{-N\sum_i P_X(x_i)\log_2(P_X(x_i))}$$

And the first term can be trivially rewritten as

$$\underbrace{\left(\prod_{i=1}^{s} Q_X(x_i)^{P_X(x_i)\times N}\right)}_{\text{Probability you expect}} = 2^{N\sum_i P_X(x_i)\log_2(Q_X(x_i))}$$

All together this gives us

$$\mathcal{P} \approx 2^{-N\sum_i P_X(x_i)\left(\log_2\left(\frac{P_X(x_i)}{Q_X(x_i)}\right)\right)} = 2^{-N S(P_X \| Q_X)}$$

Where the red term is what we define as **relative entropy** (or Kullback-Liebler divergence if you're in the mood for a mouthfull.)

$$S(P_X \parallel Q_X) = \sum_i P_X(x_i) \times \log_2\left(\frac{P_X(x_i)}{Q_X(x_i)}\right)$$

The relative entropy has a few properties. First we note that if $P_X = Q_X$ then relative entropy is zero, else it's positive. We'll also notice that if $\mathcal{P}$ decreases and N is fixed then our $S(P_X \parallel Q_X)$ is increasing. From this we can guess that larger relative entropy means the more sure our initial hypothesis Q_X is wrong. So what can relative entropy do for us?

Example 3: (What Relative Entropy can tell us about Mutual Information) Something that relative entropy can tell us is that mutual information is positive. Consider a joint probability distribution $P_{X,Y}(x,y)$. We can reduce the distribution to a single variable in the normal way

$$P_X(x) = \int P_{XY}(x,y)dy = \sum_j P_{XY}(x,y_j) \qquad P_Y(y) = \int P_{XY}(x,y)dx = \sum_j P_{XY}(x_j,y)$$

And we'll define a new probability distribution $Q_{XY}(x,y) = P_X(x)P_Y(y)$. So what have we done? What Q_{XY} is saying is that $P_X(x)$ and $P_Y(y)$ are statistically independent. But we don't know if this actually true or not. But recall that intuitivley, relative entropy is sort of like the distance between two probability distributions. So if we find that the relative entropy between Q_{XY} and P_{XY} is zero we can say that $P_X(x)$ and $P_Y(y)$ are statistically independent. So lets calculate the relative entropy

$$\begin{aligned}
&S(P_{XY} \parallel Q_{XY}) = \\
&= \sum_{i,j} P_{XY}(x_i,y_j) \times \log_2\left(\frac{P_{XY}(x_i,y_j)}{Q_{XY}(x_i,y_j)}\right) \\
&= \sum_{i,j} P_{XY}(x_i,y_j) \times \left(\log_2 P_{XY}(x_i,y_j) - \log_2 P_X(x_i) - \log_2 P_Y(y_j)\right) \\
&= \underbrace{\left(\sum_{i,j} P_{XY}(x_i,y_j)\log_2 P_{XY}(x_i,y_j)\right)}_{-S_{XY}} - \underbrace{\left(\underbrace{\sum_{i,j} P_{XY}(x_i,y_j)}_{\sum_i P_X(x_i)}\log_2 P_X(x_i)\right)}_{-S_X} - \underbrace{\left(\underbrace{\sum_{i,j} P_{XY}(x_i,y_j)}_{\sum_j P_Y(y_j)}\log_2 P_Y(y_j)\right)}_{S_Y} \\
&= S_X - S_{XY} + S_Y = I(x;y)
\end{aligned}$$

So we see that the mutual information is the relative entropy between a joint distribution P_{XY} and the hypothesis that P_X and P_Y are statistically independent and thus **must be positive.**

$$I(X;Y) = S_X + S_Y - S_{XY} \geq 0$$

Relative entropy has another interesting property called the **monoticity of relative entropy.**

Definition 5: (Monoticity of Relative Entropy) Lets say we have two joint probability distribution P_{XY} and Q_{XY} where P_{XY} is the "true" probability distribution and Q_{XY} is the hypothesized probability distribution. After N measurements we can quantify our certainty or uncertainty that our hypotheszied distribtuion Q_{XY} is correct with the relative entropy $S(P_{XY} \parallel Q_{XY})$. But lets say we're down bad. We can only measure X. So reducing to one variable gives us P_X and Q_X and again we can asses our hypothesis with the relative entropy $S(P_X \parallel Q_X)$. But lets think very hard for a second. Would $S(P_X \parallel Q_X)$ be greater than or less than $S(P_{XY} \parallel Q_{XY})$. Or physically we want to know, is our assesment of our hypothesis going to be more or less optimistic given that we've reduced the number of random variables[a].

$$S(P_{XY} \parallel Q_{XY}) \geq S(P_X \parallel Q_X)$$

We call this the monotonicity of relative entropy. To get a feel for why this is true. Consider observing a sequence of outcomes $\{x_{i_1}, \ldots, x_{i_n}\}$. For this sequence of outcomes there should also exists some $\{y_{i_1}, \ldots, y_{i_n}\}$ that minimizes the relative entropy. But any sequence of y we view will at best be equal to the relative entropy of only viewing one dimension of the joint probability distribution and will probably be increase the relative entropy. So the relative entropy of being able to measure one degree of freedom of the joint probability distribution must be lower than being able to measure the entire joint probability distribution.

[a]Witten just says "it is harder to disprove the inital hyptohesis if we observe only X" like it's clear but unfortunately I don't have any inspired thoughts that make this statement obvious intuitivley. My best guess is that reducing the number of random variables helps us cheat a bit sort of like looking at the first few lines of the solution to a problem. We've sort of assumed that we're good for one variable and are just trying to find the distance between two simpler distributions and thus $S(P_{XY} \parallel Q_{XY})$ is more chaotic then its single variable counterpart. Or maybe I'm overthinking it and there isn't an intuition for it. In other word's Witten is just quoting the result of the proof we're about to do without saying we should know this intuitively. Edit: nevermind he literally gives the intuition in the next paragraph.

If my ham-handed explanations and Witten's very nice intuition for the monotinicity of relative entropy don't satisfy you, don't fret! We'll also show it the brain dead way with some algebra. We can restate the monoticity of relative entropy as

$$S(P_{XY} \parallel Q_{XY}) - S(P_X \parallel Q_X) \geq 0$$

Which we can rewrite as

$$S(P_{XY} \parallel Q_{XY}) - S(P_X \parallel Q_X) = \sum_{i,j} P_{XY}(x_i, y_j) \times \left(\log_2 \left[\frac{P_{XY}(x_i, y_j)}{Q_{XY}(x_i, y_j)} \right] - \log_2 \left[\frac{P_X(x_i)}{Q_X(x_i)} \right] \right)$$

$$= \sum_{i,j} \underbrace{\frac{P_{XY}(x_i, y_j)}{P_X(x_i)}}_{\substack{P_X(x_i)P(y_j|x_i) \\ \text{by Bayes rule}}} \times P_X(x_i) \times \log_2 \left[\underbrace{\frac{P_{XY}(x_i, y_j)}{P_X(x_i)}}_{\substack{P(y_j|x_i) \\ \text{by Bayes rule}}} \times \underbrace{\frac{Q_X(x_i)}{Q_{XY}(x_i, y_j)}}_{\substack{Q(y_j|x_i)^{-1} \\ \text{by Bayes rule}}} \right]$$

$$= \sum_i P_X(x_i) \underbrace{\sum_j P_X(y_j|x_i) \log_2 \left[\frac{P(y_j|x_i)}{Q(y_j|x_i)} \right]}_{\substack{S(P_{Y|X=x_i} \| Q_{Y|X=x_i}) \\ \geq 0 \text{ since it's a} \\ \text{relative entropy}}}$$

$$= \sum_i P_X(x_i) S(P_{Y|X=x_i} \parallel Q_{Y|X=x_i}) \geq 0$$

There we go! Monoticity of relative entropy can show us something very interesting. Consider some probability distributions $P_{XYZ}(x_i, y_j, z_k)$ and $Q_{XYZ} = P_X(x_i)P_{YZ}(y_j, z_k)$ where we integrate out variables in the normal way. From the monoticity of relative entropy we know that

$$S(P_{XYZ} \parallel Q_{XYZ}) \geq S(P_{XY} | Q_{XY})$$

When we were proving that mutal information is positive we showed that

$$S(P_{XY} \parallel Q_{XY}) = S_X - S_{XY} + S_Y$$

We can apply this result here

$$\cancel{S_X} - S_{XYZ} + S_{YZ} \geq \cancel{S_X} - S_{XY} + S_Y$$

Giving us the result

$$S_{XY} + S_{YZ} \geq S_{XYZ} + S_Y$$

We call this strong subadditivity. From our definition of mutual information this is equivalent to saying that

$$I(X; YZ) \geq I(X; Y)$$

Which makes sense because being able to view the whole distribution will give you more information about X than only viewing a slice of the probability distribution.

Witten: Quantum Information Theory

Started: February 07, 2021. Finished: Probably Never :pensive:

Lets say we're studying a system A with Hilbert space $\mathcal{H}_A$ and let B be anything else relevant (or even the entire rest of the universe if you want to be spicy) with hilbert space $\mathcal{H}_B$. We can describe the joint Hilbert space with a tensor product $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and vectors in this Hilbert space as $\psi_{AB} = \psi_A \otimes \psi_B$ where $\psi_I \in \mathcal{H}_I$. If ψ_{AB} is a unit vector then we can choose ψ_A and ψ_B to also be unit vectors. And thus from here we can ignore B completley when finding observables for the system we're studying, A. Consider some observable $\mathcal{O}_A$ which is an operator on $\mathcal{H}_A$. Then

$$\langle \psi_{AB} | \mathcal{O}_A \otimes 1_B | \psi_{AB} \rangle = \langle \psi_A | \mathcal{O}_A | \psi_A \rangle \langle \psi_B | \psi_B \rangle = \langle \psi_A | \mathcal{O}_A | \psi_A \rangle$$

However most days aren't good days. It's not usually the case where ψ_{AB} is the product state of two vectors ψ_A and ψ_B. Lets say $\dim(\mathcal{H}_A) = 2$ and $\dim(\mathcal{H}_B) = 3$. We then find that any pure state in $\mathcal{H}_{AB}$ can be written as a 2×3 matrix. Furthermore through some unitary transformations we can diagonalize the pure state so that it has the form

$$\psi_{AB} = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \end{bmatrix}$$

More generally we can write this as a Schmidt decomposition

Definition 6: (Schmidt decomposition) A pure state ψ_{AB} in the hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written in terms of a orthonormal basis ψ_A^i and ψ_B^i as

$$\psi_{AB} = \sum_i \sqrt{p_i} \psi_A^i \otimes \psi_B^i$$

Where $p_i > 0$ which we can interpret as probilities. We see that ψ_{AB} is a unit vector if $\sum_i p_i = 1$ (just consider $\psi_{AB}^* \psi_{AB}$).

Lets try using the Schmidt decomposition to find observables. Consider the same observable $\mathcal{O}_a \otimes 1_B$

$$\langle \psi_{AB} | \mathcal{O}_a \otimes 1_B | \psi_{AB} \rangle = \sum_i \sum_j \sqrt{p_i p_j} \langle \psi_A^i | \mathcal{O}_a | \psi_A^j \rangle \langle \psi_B^i | 1_B | \psi_B^j \rangle = \sum_i p_i \langle \psi_A^i | \mathcal{O}_a | \psi_A^i \rangle = \mathrm{Tr} \rho_A \mathcal{O}_A$$

Where in the last equality we defined the **density matrix**

Definition 7: (Density Matrix) The density matrix of a system A can be written as

$$\rho_A = \sum_i p_i | \psi_A^i \rangle \langle \psi_A^i |$$

We can see the matrix is hermitian. Also apprently it's obvious that this matrix is positive semi-definite. Furthermore we've asserted that $\sum_i p_i = 1$ (the probabilites have to sum to 1) meaning that $\mathrm{Tr} \rho_A = 1$.

Wilde Chapter 2: Classical Shannon Theory

Started: February 08, 2021. Finished:

Lets start with an example in coding.

Example 4: (Efficient Coding) Suppose we have two friends, Dan and Phil. Dan wants to transmit messages to Phil through a noiseless bit channel. Dan also want to use this channel as little as possible since Dan is lazy. Now suppose Dan wants to transmit a message where each letter of the message is drawn from a probability distribution where

$$P(\{a,b,c,d\}) = \left\{\frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right\}$$

How efficiently can we encode a message if the probability distribution is the one we have above? The simplest way would be something like

$$\{a,b,c,d\} \xrightarrow{\text{encoding}} \{00, 01, 10, 00\}$$

We can evaluate the efficiency of this coding by calculating the expected length of each letter transmitted. But if we do this then we realize that the encoding isn't as efficient as it could be. We're not exploiting the skewed probability distribution we have. Another encoding we could consider is

$$\{a,b,c,d\} \xrightarrow{\text{encoding}} \{0, 110, 10, 111\}$$

With this encoding the message should still be readable (I guess you could make a state machine to confirm this) and we should have decreased the expected length of the message. This is because the expected value of each letter transmitted should have decreased. To see this we can look at

$$\frac{1}{2} \times 1 + \frac{1}{8} \times 3 + \frac{1}{8} \times 3 + \frac{1}{4} \times 2 = \frac{7}{4} < 2$$

The example above leads us to consider the questions: how do we measure information? One way we could try to do it is consider at which symbol we would be most surprised to see. This leads us to the definition of information content

Definition 8: (Information Content) If the probability of some event happening is $p(x)$ then the information content of this event is

$$i(x) = \log_2\left(\frac{1}{p(x)}\right) = -\log_2(p(x))$$

Now consider the scenario where we measure two events x_1 and x_2 independently. Then the probability that we measure both these events is

$$p(x_1, x_2) = p(x_1)p(x_2)$$

Then the information content of these two events happening is

$$i(x_1, x_2) = -\log_2(p(x_1, x_2)) = -\log_2(p(x_1)p(x_2)) = -\log_2(p(x_1)) - \log_2(p(x_2)) = i(x_1) + i(x_2)$$

We call this property the *additivity* of information content. We can also calculate the expectation value of the information content of some information source. This quantity is important enough to have a name: **entropy**.

Definition 9: (Entropy) The entropy of an information source is the expected value of the information content of that source

$$\sum_x p(x)i(x) = -\sum_x p(x)\log_2(p(x))$$

We now have some tools to ask a very interesting question: is there a more efficient coding scheme than the one we came up with example 4? To consider this question we'll consider the general case. We can model a general information source as a random variable X with probability density $p_X(x)$ whose instances x we'll call *letters* of some alphabet χ. The entropy of this random variable is

$$H(X) = -\sum_{x\in\chi} p(x)\log(p_X(x))$$

And the information content of the random variable X is

$$i(X) = -\log(p_X(X))$$

Which might look strange. What we have defined is another random variable $i(X)$ that is a function of a random variable plugged into its own probability distribution. We'll see how this is useful in a bit. Now lets go into a quick aside

Example 5: (Entropy of a Uniform Random Variable) Lets consider a uniform random variable. This means that $p(x) = \frac{1}{|\chi|}$ for all $x \in \chi$. This means that

$$H(X) = -\sum_{x\in\chi} p(x)\log(p(x)) = -\frac{1}{|\chi|}\times|\chi|\times(-\log(|\chi|)) = \log|\chi|$$

Anyways back to the plot

One of Shannon's cool ideas for efficient coding was to let the information source emit a block of information, code this entire block (as opposed to letter by letter as we did in the example) with some possible error, and then show that this error vanishes for large block size. Lets try to make this concrete. Suppose the information source emits a sequence of letters

$$x^n = x_1 \dots x_n$$

We'll introduce the notation that X^n that denotes the random variable that is associated with the entire sequence x^n and X_i to denote the random variable associated with the i^{th} letter spit out by the information source. We'll assert that the information source is independent and identically distributed (IID) meaning that $X_i = X_j = X$ for all i, j and

$$p_{X^n}(x^n) = p_{X_1\dots X_n}(x_1,\dots,x_n) = p_{X_1}(x_1)\times\cdots\times p_{X_n}(x_n) = \prod_i p_X(x_i)$$

But we can go even further! First we'll introduce the notation that $N(a_i|x^n)$ is the number of times the letter $a_i \in \chi$ appears in the sequence x^n. For large n we can assume that

$$N(a_i|x^n) = n\times p_X(a_i)$$

And thus $p(x^n)$ can be simplified even more into

$$p(x^n) = \prod_{x \in \mathcal{X}} p_X(x)^{N(x|x^n)}$$

What this formula is also saying is that we can (if we squint our eyes a little) permute the block x^n into a nicer sequence

$$x^n \to \underbrace{a_1 \ldots a_1}_{N(a_1|x^n)} \ldots \underbrace{a_{|\mathcal{X}|} \ldots a_{|\mathcal{X}|}}_{N(a_{|\mathcal{X}|}|x^n)}$$

So far what we've done is derive properties of a instance of X^n. From here we'll try to use the result we have so far to derive some nice result about the more general X^n. The first thing we'll consider is the (currently mysteriously named) *sample entropy*

Definition 10: (Sample Entropy) For a random variable X^n that corresponds to a IID information source that generates a sequence $x^n = x_1 \ldots x_n$ we define the sample entropy as the average information content of each letter in the random sequence. Namely

$$\frac{1}{n} \times i(X^n) = -\frac{1}{n} \log(P_{X^n}(X^n))$$

We can use the natural generalization of the result above to derive a more suggestive form of sample entropy

$$\begin{aligned} -\frac{1}{n} \log(P_{X^n}(X^n)) &= -\frac{1}{n} \log\left(\prod_{x \in \mathcal{X}} p_X(x)^{N(x|X^n)} \right) \\ &= -\frac{1}{n} \sum_{x \in \mathcal{X}} \log\left(p_X(x)^{N(x|X^n)} \right) \\ &= -\sum_{x \in \mathcal{X}} \frac{N(x|X^n)}{n} \log(p_X(x)) \end{aligned}$$

Now notice that for large n we can approximate $N(x|X^n) = n \times p_X(x)$. Which means that we can write for large n that

$$-\frac{1}{n} \log(P_{X^n}(X^n)) \approx -\sum_{x \in \mathcal{X}} p_X(x) \log(p_X(x)) = H(X)$$

But what's this? Isn't the RHS just the entropy of the random variable X? So what we have shown is that for large n, the information source emits a sequence whose sample entropy is close to true entropy. Wilde started saying some weird things about empircal distributions and starting whipping out our favorite $\epsilon - \delta$ arguments but I think the above is the core of what he's saying. Just for future reference the punchline of his argument was that

$$\lim_{n \to \infty} \Pr\{|(\text{Sample Entropy}) - (\text{True entropy of } X)| \leq \delta\} = 1 \quad \forall \delta > 0$$

Now this leads us to a definition

Definition 11: (Typical Sequence, Typical Set, and their properties) We define a typical sequence as a instance of X^n which we'll denote as x^n whose sample entropy is near the true entropy (it's not given in the book but I assume for some $\delta > 0$ we want the difference to be less than δ.) The typical set is the set of all typical sequences. For now we'll assert without proof that the size of the typical set is $\approx 2^{nH(X)}$. The typical set has a few properties

(a) The probability that a emitted sequence by the information source is in the typical set approaches 1 for large n

(b) The size of a typical set is exponentially smaller the the size of the set containing all possible sequences.

(c) The probablity of a particular typical sequence is roughly uniform and is $\approx 2^{-nH(X)}$.

(a), (b), and (c) make up what is called the asymptotic equipartition theorem.

Lets return to our original motivating problem, trying to come up with efficient coding schemes. Given the notion of a typical set we could try a coding scheme that's a bijective function from the typical set whose cardinality is $2^{nH(s)}$ to some binary string of length $nH(X)$. If we encounter a sequence that is not part of the typical set (the probability of this is vanishingly small for large n) then we throw an error! Lets try to evaluate the efficiency of this coding scheme. We can do this with a *compression rate*

Definition 12: (Compression Rate) The compression rate is defined as

$$\text{(compression rate)} = \frac{\text{(\# of noiseless channel bits needed)}}{\text{(\# of source symbols)}}$$

The compression rate of this typical set block coding scheme is

$$\frac{nH(X)}{n} = H(X)$$

This also gives us a operational definition of the shannon entropy as the compression ratio of our coding scheme. We won't prove this yet but this is as good as we can do.

Proving a coding theorem is usually broken down into two parts. First the direct coding theorem where we construct a coding scheme with some specific compression rate. This is meant to show that

$$\begin{array}{c}\text{Rate of compression greater}\\ \text{than entropy of source}\end{array} \longrightarrow \begin{array}{c}\text{Exists a coding scheme}\\ \text{that can achieve losless compression}\end{array}$$

Where here loselss compression means probablilty of error in decoding is small. Then we have to prove the converse

$$\begin{array}{c}\text{Exists a coding scheme}\\ \text{that can achieve losless compression}\end{array} \longrightarrow \begin{array}{c}\text{Rate of compression greater}\\ \text{than entropy of source}\end{array}$$

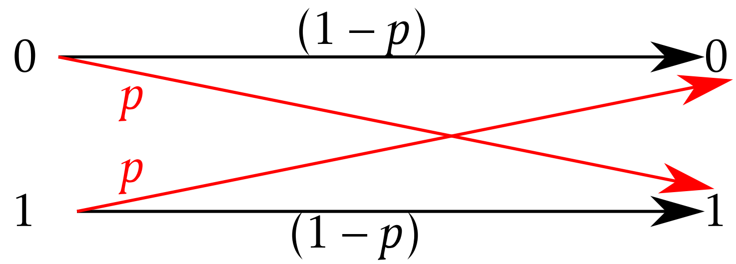


Figure 1: Illustration of a classical noisy bit-flip channel

We've had a lot of fun in noiseless channels so far so lets finally start talking about noise. In particular we'll talk about a bit-flip channel. When we send a bit, the probability that the bit is flipped is $(1-p)$. This is illustrated in Figure 1. Also assume that the output of this channel is IID. Again lets consider our two favorite people who send messages to eachother, Dan and Phil. The found this noisy transmitter that's cheaper than the noiseless one (in reality there probably aren't any noiseless ones) and they want to figure out how they can efficiently and without error send messages through this channel. If they just use the channel as is then there is a probability p that Phil misinterprets what Dan is saying. What about if we try a coding scheme like

$$0 \to 000 \quad 1 \to 111$$

And then take a majority vote out of the triplets to decode what the triplet actually is. For example 110 would decode to 1 and 001 would decode to 0. Lets analyze this coding scheme. First if we want to trasmit a 0 then we have

$$\begin{gathered}
\text{Output} : \text{Probability} \\
\{(000)\} : (1-p)^3 \\
\{(001),(010),(100)\} : 3 \times (1-p)^2 \times p \\
\{(110),(101),(011)\} : 3 \times (1-p) \times p^2 \\
\{(111)\} : p^3
\end{gathered}$$

And we have a similar table for wanting to transmit a 1. So what's the probability of error here? Well it would just be

$$p(e) = p(0)p(e|0) + p(1)p(e|1)$$

From the above table we can read off

$$p(e|0) = 3(1-p)p^2 + p^3 = 3p^2 - 2p^3 \xrightarrow{by symmetry} p(e|1) = 3p^2 - 2p^3 = p(e|\cdot) = p(e|0)$$

Thus giving us

$$p(e) = p(e|\cdot)\left(\underbrace{p(0)+p(1)}_{=1}\right) = 3p^2 - 2p^3$$

So now the most natural question to ask is, when is this naive coding scheme better than just not doing anything at all? We can formalize this as

$$p(e) < p \Rightarrow 3p - 2p^2 < 1 \to 0 < 2p^2 - 3p + 1$$

Solving this for p yields

$$p > \frac{3 \pm \sqrt{9-8}}{4} = \frac{3 \pm 1}{4}$$

We can throw out the $p > 1$ since that's unphysical leaving us with $p > 1/2$. So if the probability flipping a bit is more than $\frac{1}{2}$ the above coding scheme is better than doing nothing. However if $p < \frac{1}{2}$ the coding scheme is worse than doing nothing at all[1]. In general the rate of error decreases as we increase the redundancy of our code but do you really want to do it that way? Lets look for a more aesthetically pleasing coding scheme. To do this lets try solving a more general problems(that's a good trick.) We'll first need a more general model of a noisy channel, we'll call it the *discrete memoryless channel*

Definition 13: (Discrete Memoryless Channel) Dan wants to transmit to Phil a message $m \in M$ where M is a finite set of possible messages that Dan samples with uniform probability. To generalize the "bit-flip noise" we saw before we can introduce a conditional probability distribution $p_{Y|X}(y|x)$ for a random variable X and Y where we'll input a instance of X and the channel will poop out an instance of Y. We'll denote a sequence of messages as $x^n = x_1 \ldots x_n$ and the random variable associated with x^n and x_i to be X^n and X_i with similarl notation for y_n and Y^n. We'll assume the channel is IID. The conditional probability distribution then becomes

$$p_{Y^n|X^n} = p_{Y_1|X_1}(y_1|x_1) \times \cdots \times p_{Y_n|X_n}(y_n|x_n) = \prod_{i=1}^{n} p_{Y|X}(y_i|x_i)$$

Any message $m \in \mathcal{M}$ Dan wants to send can be encoded in $b = \log_2(|\mathcal{M}|)$ bits and the actual instantiationi of this message in bits (which may or may not be the same as b) we'll denote as $x^n(m)$. The decoding of the recieved string of bits y^n by Phil we'll put off for now. The rate of this channel is

$$(\text{rate}) = \frac{(\text{numbers of bits needed to encode messag})}{(\text{number of bits used to transmit message over channel})}$$

So in our case we have

$$(\text{rate}) = \frac{1}{n} \times \log_2(|\mathcal{M}|)$$

How do we evaluate the performance of some coding scheme? Lets define $\mathcal{C} = \{x^n(m) | m \in \mathcal{M}\}$ and $p_e(m, \mathcal{C})$ be the probabilty that something goes wrong when using the coding scheme $\mathcal{C}$ when we transmit message m. The average and max error then is

$$\overline{p_e}(\mathcal{C}) = \frac{1}{|\mathcal{M}|} \sum p_e(m, \mathcal{C}) \quad p_e^*(\mathcal{C}) = \max_{m \in \mathcal{M}} p_e(m, \mathcal{C})$$

The first thing that should be obvious is the fact that if the max probability of error is small then the average probability of error is small. What might be less obvious is that if the average probability of error is small then max probability is small for at least half the messages. Lets

[1] just like boarding groups for flights >:(

show this. We want to prove

$$\frac{1}{|M|}\sum_m p_e(m,\mathcal{C}) \le \epsilon \to p_e(m,\mathcal{C}) \le 2\epsilon \text{ for at least half the messages } m$$

Suppose not. Then there exists $N < |M|/2$ messages such that $p_e(m,\mathcal{C}) \le 2\epsilon$. Lets set the N messages with $p_e \le 2\epsilon$ to be zero and the $|M| - N$ messages with $p_e > 2\epsilon$ equal to two epsilon. Namely

$$\frac{1}{|M|}\sum_m p_e(m,\mathcal{C}) > \frac{|M|-N}{|M|} \times 2\epsilon = 2\epsilon - \frac{N}{|M|} \times 2\epsilon$$

Now note that $N < |M|/2$ meaning that $-N/|M| > -1/2$ giving us

$$\frac{1}{|M|}\sum_m p_e(m,\mathcal{C}) > \epsilon \text{ which is not } \le \epsilon$$

Thus proving the statement by contradiction[2]

So up until now our general channel was two sepearte layers of randomness. The first layer of randomness is the choosing of the channel. The second layer of randomness is the noise in the channel when we send messages. It turns out that to prove that there exists a reliable coding scheme that is maximally efficient it's easier to introduce a third layer of randomness! This randomness will be in the actual instances of X that make up a codeword of m. Namely we'll have

$$P(X^n(m) = x^n(m)) = \prod_{i=1}^{n} P_X(x_i(m))$$

What's essentially what's happened is that the coding itself has become a random varaiable. $\mathcal{C}$ is no longer a map from $m \in M$ to some x^n but instead a probability distribution that each $m \in M$ will map to some specific x^n. A speicific instantiation of $\mathcal{C}$ which we'll now call $\mathcal{C}_0$ is what we had before, a map from $m \in M$ to some codeword. Furthermore each message will have the exact same probability distribution of x^n since there is no explicit reference to the message m, there is only a reference to what instance of X is in the slot x_i. So we now can write the probability that some coding scheme $\mathcal{C}_0$ arises as

$$P(\mathcal{C}_0 = \{x^n(m)|m \in \mathcal{M}\}) = \prod_{m}^{|\mathcal{M}|}\prod_{i=1}^{n} P_X(x_i(m))$$

So why is all this nonsense useful? Well what we can now consider is the expectation value of the mean probability of error over our space of random coding schemes. Namely

$$\mathbb{E}\left\{\frac{1}{|M|}\sum_{i=1}^{|M|} p_e(m,\mathcal{C})\right\}$$

[2]Okay I feel a bit skteched out by this proof of mine. But it did get around the need to use Markov's inequality like Wilde wanted us to. I'll leave it here for now.

We can use the linearity of expectation here to then write the above quantity as equal to

$$= \frac{1}{|M|} \sum_{i=1}^{|M|} \mathbb{E}\{p_e(m, \mathcal{C})\}$$

Now notice that our probability of error over all our possible coding schemes is independent of the actual message since the probability distribution of x^n for each m is the same(see above).. Thus we can replace m with any specific message we could transmit. This means that

$$\mathbb{E}\left\{\frac{1}{|M|} \sum_{i=1}^{|M|} p_e(m, \mathcal{C})\right\} = \mathbb{E}\{p_e(\mathcal{C})\}$$

Shannon then gives us a proof that we'll go into later[3] that says for some $\epsilon > 0$ which shrinks as block size increases we have

$$\mathbb{E}\{\overline{p}_e(\mathcal{C})\} \leq \epsilon$$

And thus by nature of expectation value we then know that there must exsist some coding scheme $\mathcal{C}_*$ such that

$$\overline{p}_e(\mathcal{C}_*) \leq \epsilon$$

We also proved by contradicition earlier that

$$\frac{1}{|M|} \sum_m p_e(m, \mathcal{C}) \leq \epsilon \rightarrow p_e(m, \mathcal{C}) \leq 2\epsilon \text{ for at least half the messages } m$$

We can now apply this result here to this special $\mathcal{C}_*$ to obtain a bound on the error of each message. First what we do is throw out all the message with $p_e(m, \mathcal{C}_*) > 2\epsilon$ which is half the messages[4]. For large n this has no effect on the rate. Recall that the number of messages $m \in \mathcal{M}$ we can send is 2^{nR}. Throwing out half the possible messages in $\mathcal{M}$ will lead to $2^{nR-1} = 2^{n(R-1/n)}$ where the red stuff is our new rate. As $n \to \infty$ we can see that the red stuff goes towards R. With this we can now put a bound on the maximum error of the coding (for our thanos snapped set of messages)

$$p_e(\mathcal{C}_0) \leq 2\epsilon$$

[3] Why is Wilde such a tease

[4] I need to think more about this but why can we do this? Is it because we're picking messages uniformly meaning that any specific message we want to send we can send by making the message we want to send be at least half of the possibilities we can choose from? Or maybe it's because we're picking uniformly that makes the message an "arbitrary message." I dunno.