8.044: Statistical Physics I

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Notes and assignments for Greytak's Statistical Physics I course on MIT OCW. If you have any comments let me know at `hi@delon-shen.com`.

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Blundell and Blundell Chapter 3: Probability

January 26, 2021

Let x be a discrete random variable (e.g. can only take a finite number of values). We'll say that x can become x_i with probability P_i[1]. We can define the expectation value of x_i as

$$\langle x \rangle = \sum_i x_i P_i$$

A weighted sum. This is naturally extended to expectation values of functions of x_i

$$\langle f(x) \rangle = \sum_i f(x_i) P_i$$

We can extend these notions to continuous random variables where we have a probability $P(x)dx$ of find x as a value between x and $x + dx$. In this case we have

$$\langle x \rangle = \int x P(x) dx \qquad \langle f(x) \rangle = \int f(x) P(x) dx$$

If we relate two expectation values with a linear transformation then the expectation values transform as we'd expect as well. Let y also be a random variable and a, b be constants

$$y = ax + b \Rightarrow \langle y \rangle = a \langle x \rangle + b$$

Now what if we want to quantify the spread of values. Our first guess may be just to find the average $x - \langle x \rangle$. However note that

$$\langle x - \langle x \rangle \rangle = \langle x \rangle - \langle x \rangle = 0$$

Our next guess then could be the $(x - \langle x \rangle)^2$. We define $\left\langle (x - \langle x \rangle)^2 \right\rangle$ as the *variance*. The variance of x, which we'll denote as σ_x^2, is the *mean squared deviation* and $\sqrt{\sigma_x^2}$ to be the *standard deviation*. A useful identity to keep in mind is

$$\begin{aligned}
\sigma_x^2 &= \left\langle ((x - \langle x \rangle))^2 \right\rangle \\
&= \left\langle x^2 + \langle x \rangle^2 - 2x \langle x \rangle \right\rangle \\
&= \left\langle x^2 \right\rangle + \langle x \rangle^2 - 2 \langle x \rangle^2 \\
&= \left\langle x^2 \right\rangle - \langle x \rangle^2
\end{aligned}$$

The effect of linear transformations on the variance can be found as follows. Again let $y = ax + b$. After some algebra grinding we get

$$\sigma_y^2 = a^2 \sigma_x^2$$

Namely the b parameter doesn't affect anything. This should make sense since b is an arbitrary shift of the random variable x which should have no impact on the "spread" of the values that

[1] *edit 2021-02-03: it's probably worth noting that the lectures use a different notation. Here we denote the probability density as $P(x)$ wheras in the lectures we use $p(x)$. $P(x)$ in the lectures is reserved for proability distribution functions*

the variance measures.

Independent random variables are variables that are independent! Knowing the value of one variable gives us no information about the other variables. For these kind of variables we can just multiply them together and things work out the way we'd expect them to

$$\langle uv\rangle = \langle u\rangle\langle v\rangle$$

Lets consider a random variable Y that is the sum of n independent random variable X_i such that $\langle X_i\rangle = X$. What is the variance of Y? First the easy one

$$\langle Y\rangle^2 = n\langle X\rangle^2$$

Now for $\left\langle Y^2\right\rangle$ we just have to be a bit clever. In Y^2 there are n X_i^2 terms and then $n(n-1)$ X_iX_j terms where $i \neq j$. Thus in $\left\langle Y^2\right\rangle$ we'll have n $\left\langle X^2\right\rangle$ terms and $n(n-1)$ $\langle X\rangle^2$ terms. So

$$\left\langle Y^2\right\rangle = n\left\langle X^2\right\rangle + n(n-1)\langle X\rangle^2$$

From this we can calculate the variance

$$\sigma_y^2 = \left\langle Y^2\right\rangle - \langle Y\rangle^2 = n\left\langle X^2\right\rangle + n(n-1)\langle X\rangle^2 - n\langle X\rangle^2 = n\left\langle X^2\right\rangle - n\langle X\rangle^2 = n\sigma_x^2$$

This has applications to experiments. Lets say you independently measure the same quantity X multiple times. Let X_i be the measured value of the i^{th} run. This means that if you use the average of the measured X values $Y = \frac{1}{n}\sum_i X_i$ then the standard deviation of Y would be $\sigma_x/\sqrt{n}$. This also has applications to random walks. Lets say we have some discrete random variable X which can either be $-a$ or a with equal probability. This means that $\langle X\rangle = 0$. Thus the variance is $\left\langle X^2\right\rangle = a^2$. Now lets consider a drunk man desperately trying to get to his home on the other side of a tight rope(over the grand canyon of course.) Each step the drunk man takes is either forward or backwards a units but because he's drunk the probability that he steps forwards or backwards is equal. The standard deviation of the sum of multiple X_i in this case would correspond to the root mean squared distance that degenerate has walked. Lets now calculate the variance

$$Y = \sum_{i=1}^{n} X_i \Rightarrow \sigma_Y^2 = n\sigma_X^2 \Rightarrow \sigma_Y = \sqrt{n}\sigma_X = \sqrt{n}\left\langle X^2\right\rangle = a\sqrt{n}$$

After n steps the rms length this guy has travelled is $a\sqrt{n}$.

Lecture 1: One Random Variable

Started: January 27, 2021. Finished: January 28, 2021

A *random variable* is a quantity whose state is determined by a processes which can be analyzed and thus allow us to assign some probability to the variable taking a certain state. For our purposes randomness can be introduced from

(a) Insufficient information: e.g. where my cat is. I knew he was sleeping on the couch at 1:00PM but where could he be now?

(b) Quantum Mechanics: uncertainty principles.

We can classify random variables into three categories

(i) Discrete: the variable can only take a finite set of values. For example how many children does Ms. Smith have? I hope she doesn't have a half of a child....

(ii) Continuous: the variable can take a value from a continuum of possibilities. For example how hot will it be the next time an election takes place.

(iii) Mixed: the variable can take discrete values for some conditions and continuous variables for other conditions. For example how many grapes will Chuck eat tomorrow? Well it depends on if he blends the grapes or not. If he eats grapes noramlly then the values are discrete, but if he blends them up and and puts them in a smoothie then we have a continuous random variable. (I don't know if this example is correct.)

Lets consider a system of n similarly prepared systems. We'll let x_i denote the output of the i^{th} system. Now imagine creating a histogram that counts how many x_i lie in some bin. This is the notion of a probability density function $\rho_x(\xi)$.

$$\rho_x(\xi) = \lim_{M\to\infty\ d\xi\to 0} \frac{\text{Number of times } x \text{ landed in } [\xi, \xi + d\xi)}{M d\xi}$$

We'll drop the $p_x(\xi)$ for the much nicer $p(x)$ notation when it should be clear. From this we can also define a probability distribution function $P_x(\xi)$ where

$$P_x(\xi) = \int_{-\infty}^{\xi} \rho_x(\xi) d\xi \Rightarrow \frac{dP_x(\xi)}{d\xi} = \rho_x(\xi)$$

Where we used Leibniz integral rule implicitly assume that $\rho_x(\xi)$ vanishes at $-\infty$. The probability density completley specified the random variable. There are two questions we could ask about random variables. How do we find the probability distribution of a random variable and what can we gleam about this random variable from this probability distribution.

Example 1: (Radioactive Decay) Lets say we have some radioactive source. The probability density function to detect the first decay t seconds after beginning the experiment is

$$\rho(t) = \begin{cases} \tau^{-1} e^{-\frac{t}{\tau}} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Where τ is inverse the mean rate of events per second. We note that the probability is normalized

$$\int_{-\infty}^{\infty} \rho(t) dt = 1$$

And the probability distribution is what we would expect to see (as time increases the probability that we see an event occuring also increases)

$$P(t) = \int_0^t \rho(t) dt = 1 - e^{-\frac{t}{\tau}}$$

To model discrete or mixed random variables we can use dirac delta functions.

$$\rho(x) = \sum_i P_i \delta(x - x_i)$$

This gives us all the properties we'd expect when dealing with probability distributions and probability densities.

When measuring physical quantities usually we want to extract *ensemble averages* and the deviation of the ensemble average from the mean via *variance* and *standard deviation*. (All of these derived in B&B Ch 3).

Example 2: (Poisson Density) We're in a situation where in a interval ΔX the probability that an event happens in an interval $X + \Delta X$ is proportional to the size of the interval $P = r\Delta X$. Also the probability that an event occurs in one interval $X + \Delta X$ is independent of all other intervals. It will be derived soon that the pobability that n events will occur in an interval of length L is given be

$$p(n) = \frac{1}{n!}(rL)^n e^{-rL} \qquad n \in \mathbb{N}$$

We can use this probability to create a probability density function for a random variable y (where y is how many events occur) with delta functions

$$P(y) = \sum_i p(i)\delta(y - i)$$

Lets check normalization, mean, and variance

$$\int_{-\infty}^{\infty} P(y)dy = \sum_i p(i) \int \delta(y - i) = \sum_i p(i) = e^{-rL} \sum_i \frac{1}{i!}(rL)^i = e^{-rL} e^{rL} = 1$$

We see that the normalization is what we expect

$$\langle n \rangle = \int_{-\infty}^{\infty} yP(y)dy = \sum_{i=1}^{\infty} p(i) \int_{-\infty}^{\infty} y\delta(y - i)dy = e^{-rL} \sum_{i=1}^{\infty} \frac{1}{(i-1)!}(rL)^i$$

We can pull out a rL term and let $j = i - 1$. The bounds go from 0 to ∞ and we again have our definition of the exponential

$$\langle n \rangle = e^{-rL} \sum_{i=1}^{\infty} \frac{1}{(i-1)!}(rL)^i = e^{-rL}(rL) \times \sum_{j=0}^{\infty} \frac{1}{j!}(rL)^j = rL$$

Now the variance

$$\left\langle n^2 \right\rangle = \int_{-\infty}^{\infty} y^2 P(y)dy = \sum_{i=1}^{\infty} p(i) \int_{-\infty}^{\infty} y^2 \delta(y - i)dy = e^{-rL} \sum_{i=1}^{\infty} \frac{i}{(i-1)!}(rL)^i$$

To evaluate this lets use $\langle n \rangle$. First note that our above result suggests that

$$\sum_{i=1}^{\infty} \frac{1}{(i-1)!}(rL)^i = rLe^{rL}$$

Lets take the derivative of this with respect to r

$$\frac{d}{dr}\left\{\sum_{i=1}^{\infty}\frac{1}{(i-1)!}(rL)^i\right\} = \sum_{i=1}^{\infty}\frac{i}{(i-1)!}(rL)^{i-1}L = \frac{d}{dr}\left\{rLe^{rL}\right\} = Le^{rL} + rL^2e^{rL}$$

Now lets multiply both sides by r giving us

$$\sum_{i=1}^{\infty}\frac{i}{(i-1)!}(rL)^i = Lre^{rL} + r^2L^2e^{rL}$$

Plugging this result in gives us

$$\left\langle n^2\right\rangle = e^{-rL}\left(Lre^{rL} + r^2L^2e^{rL}\right) = rL + r^2L^2$$

Which finally allows us to compute the variance

$$\left\langle n^2\right\rangle - \langle n\rangle^2 = rL$$

It should be noted that we can write $p(n)$ in terms of its variance (or equivalently is mean) meaning that the Poisson distribution can characterized by its variance alone

$$p(n) = \frac{1}{n!}\langle n\rangle^n e^{-\langle n\rangle}$$

Lecture 2: Two Random Variables

Started: January 28, 2021. Finished: January 31, 2021

Lets introduce the concept of *joint probability distributions*. There's a lot they have in common a single random variable. For example

$$\begin{aligned}
p_{x,y}(\xi,\eta)d\xi d\eta &= \text{Probability } \xi \le x \le \xi + d\xi \text{ and } \eta \le y \le \eta + d\eta \\
P_{x,y}(\xi,\eta) &= \int_{-\infty}^{\xi}\int_{-\infty}^{\eta} p_{x,y}(\xi,\eta)d\eta d\xi \\
p_{x,y}(\xi,\eta) &= \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}P_{x,y}(\xi,\eta) \\
\langle f(x,y)\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x,y}(\xi,\eta)d\xi d\eta
\end{aligned}$$

We can also reduce the probability distribution to one varibale

$$p(x) = \int_{-\infty}^{\infty} p(x,y)dy$$

Now lets talk about conditional probability. The lecture notes approach this from a different viewpoint from what I'm going to do. I'll try to approach it from a discrete viewpoint and then extend it to the continuous case. Lets consider an example.

Example 3: (Oh Fuck I Tested Positive for Corona) Lets say that you're an unfortunate soul who has to take Senior Lab during the pandemic to graduate on time and your lab partner tested positive for corona! So you go to CVS for some rapid COVID-19 testing and, what's this? You've tested positive! Now don't fret yet. Rapid testing isn't the most accurate thing in the world. Given that you tested positive, what is the probability that you actually have corona? Lets say (these aren't real numbers) that if you have corona you have an 80% chance of testing positive from CVS. Furthermore 10% of people who don't have corona test positive for corona. We also know that 40% of the population of the world has corona. So what are the chances you have corona given that you tested positive? So first we need to figure out what's the probability that you test positive P_{pos}.

$$\begin{aligned} P_{\text{pos}} = & \text{ (\% of People who have Virus)} \times \text{(Probability of Testing Positive with Virus)} \\ & + \text{(\% of People who don't have the Virus)} \times \text{(Probability of Testing Positive w/o Virus)} \\ & = \frac{4}{10} \times \frac{8}{10} + \frac{6}{10} \times \frac{1}{10} = \frac{19}{50} \end{aligned}$$

Now this basically becomes or whole probability space. So to figure out the probability that you have the virus given that you tested positive can be found by dividing the blue term by P_{pos}

$$P(\text{have virus}|\text{tested positive}) = \frac{P\left(\text{Testing Positive with Virus}\right) \times P\left(\text{Have Virus}\right)}{P\left(\text{Testing Positive}\right)} = \frac{16}{19}$$

Nice!

Lets notice a few things aboout the above result. We can restate the final statement as

$$\boxed{(\text{Probability that } \xi = x \text{ given that } \eta = y) = p(\xi = x|\eta = y) = \frac{p(\xi = x, \eta = y)}{p(\eta = y)}}$$

This is *Bayes' Theorem* and is true in the continuous case as well as the discrete case. We can look at this result in two ways.

(a) Using the conditional probability to construct the joint probability distribution

(b) Using the joint probability distribution to guess the conditional probability

Bayes's Rule also gives as a good notion of statistical independence. We know that statistically independent means that having information on one variable gives no information on the other varaible. What this means in the context of Bayes' rule is that knowing $\eta = y$ gives us no information on whether $\xi = x$ which means

$$\text{If } p(\xi = x|\eta = y) = p(\xi = x) \text{ then } x, y \text{ are statistically independent}$$

Probability is god awful to think about without examples so lets take a look at some

Example 4: (People keep getting blown up on my friend's farm!) You're a young man living in the fictional war torn country of Bratlastinavia. You know that there's a perfectly circular area of radius 1km in the corn field by your friend's house that's just full of land mines. In fact the probability of stepping a land mine (assume they're infinitesimally small land mines) is uniform in the circle.

$$p(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \leq 1 \\ 0 & x^2+y^2 > 1 \end{cases}$$

One night you have some particularly incurable insomnia so you try to figure out some things. If you just walk in a straight line parallel to one of the axes through the danger zone, at any given point what's the probability that you'll step on a land mine(find $p(x)$ and $p(y)$). Also you want to know whether x and y are statistically independent[a].

$$p(x) = \int p(x,y)dy = \frac{1}{\pi}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi}\sqrt{1-x^2} \qquad x^2+y^2 \leq 1$$

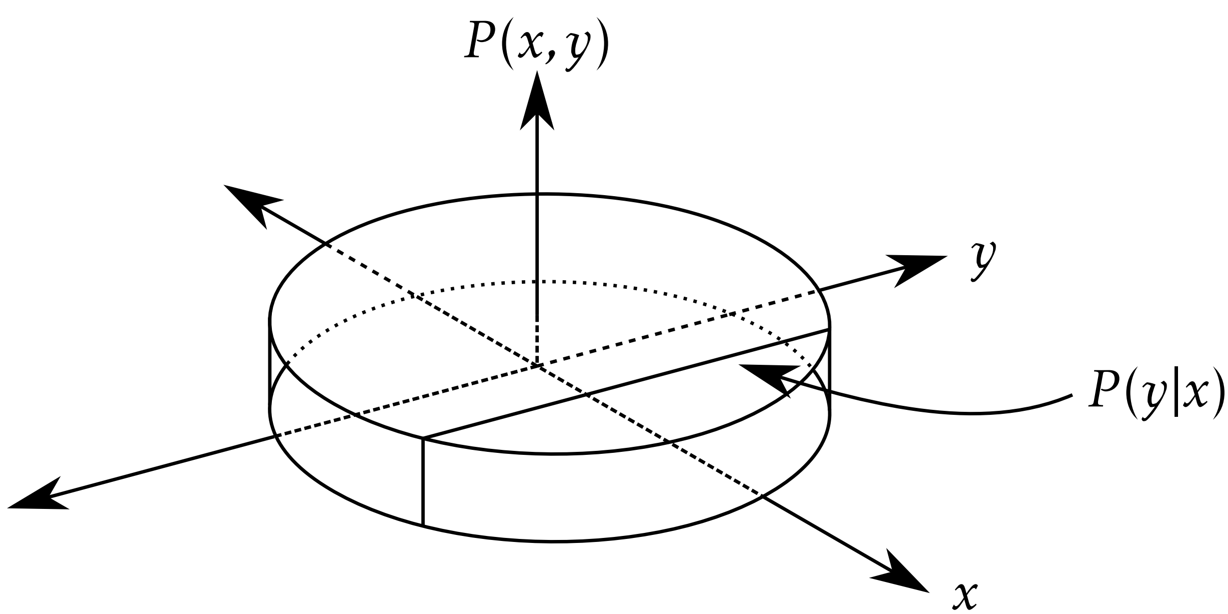
By symmetry we can also say that

$$p(y) = \frac{2}{\pi}\sqrt{1-y^2} \qquad x^2+y^2 \leq 1$$

Now applying our favorite new rule for S.I. we have

$$p(x|y) = \frac{p(x,y)}{p(y)} = \frac{1/\pi}{\frac{2}{\pi}\sqrt{1-y^2}} = \frac{1}{2\sqrt{1-y^2}} \neq p(x) \qquad x^2+y^2 \leq 1$$

The fact that the conditional probability is constant might be suprising. However if we consider a visualization of the conditional probability below the reason should become clearer.



[a] I couldn't think of what this physically means :(

Example 5: (Poisson Density Again...) Recall the defining properties that given the Poisson distribution

- In the limit $\Delta x \to 0$ the probability that one and only one event occurs between X and $X + \Delta X$ is given by $r\Delta X$ where r is a given constnat independent of X
- The probability of an event occuring in some interval ΔX is statistically independent of events in all other portion of the line

We now want to show a few thing

(a) Find $p(n = 0, L)$, the probability that no events occur in a region of length L. Then divide L into infinite number of S.I. intervals and calculate the joint probability that none of the intervals contain an event

(b) Obtain $\frac{d}{dL}p(n;L) + rp(n;L) = rp(n-1;L)$

(c) Show that $p(n;L) = \frac{1}{n!}(rL)^n e^{-rL}$. Is the solution unique?

Lets get started then

(a) Our first guess is to do something like

$$p(n = 0, L) = 1 - \int_0^L r\,dx = 1 - rL$$

However this might not be the entire solution. We were already given $p(n;L)$ and expanding that for small L we have

$$p(0;L) \approx 1 - rL + \ldots$$

So really what we found above is only true for small L like an infinitesmial dL. I think what went wrong is the fact that rdx is the probability that one event happens in the region dx, not the probability that *at least* one event happens in the region dx. Lets try again. We'll only use $p(n = 0; dL) = 1 - rdL$ and then multiply these together for all dL.

$$p(n = 0, L) = (1 - rdL)^{\frac{L}{dL}}$$

This almost looks like the definition of e. Actually let $-rdL = \frac{1}{n}$ giving us

$$p(n = 0; L) = \left(1 + \frac{1}{n}\right)^{-nrL}$$

Now in the limit $dL \to 0$ we have $n \to \infty$ so this is exactly e

$$p(n = 0; L) = e^{-rL}$$

(b) I think we can do this one by induction. But to do that we need to prove the base case. And to do that we need to find $p(n = 1; L)$. What the probability that only one event occurs in the region dL? I think it might be something like

$$p(n = 1; L) = rdL(1 - rdL)^{\frac{L}{dL}-1} \times \frac{L}{dL}$$

In one of the dL an event takes place and the rest of the intervals have no events. We multiply by an $\frac{L}{dL}$ factor since we could pick any interval. Let $-rdL = \frac{1}{n}$ again.

$$p(n=1;L) = rL \times \left(1 + \frac{1}{n}\right)^{-rnL} \times (1 - rdL)^{-1}$$

Now taking the limit $dL \to 0 \Leftrightarrow n \to \infty$ we get

$$p(n=1;L) = rL \times e^{-rL}$$

Actually do we have to use the recursive formula. We could just derive a combinatioirail expression right? We can imagine for $p(n;L)$ we're trying to divide n balls into $\frac{L}{dL}$ bins. Lets say we put all the balls into one bin. Then the contribution to the probability would be

$$(rdL)^n (1 - rdL)^{\frac{L}{dL}-1} \times \frac{L}{dL}$$

Now lets say we put all the balls into two bins. Then the contribution would be

$$(rdL)^n (1 - rdL)^{\frac{L}{dL}-2} \times \binom{\frac{L}{dL}}{2}$$

and so on. Since we'll eventually take $dL \to 0$ we'll ignore the case when $n > \frac{L}{dL}$. So then we have

$$p(n;L) = (rdL)^n \sum_{i=1} (1 - rdL)^{\frac{L}{dL}-i} \times \binom{\frac{L}{dL}}{i} = (rdL)^n \sum_{i=1} (1 - rdL)^{\frac{L}{dL}-i} \times \frac{\left(\frac{L}{dL}\right)!}{i!\left(\frac{L}{dL} - i\right)!}$$

Now lets consider

$$(rdL)^n \times \left(\frac{L}{dL}\right)! = (r)^n dL^n \left(\frac{L}{dL}\right)\left(\frac{L}{dL} - 1\right)\ldots$$

Hmmm, I don't think this is the way to go. Now that I re-read the defining properties I realize there's a "one and only one event" statement in there meaning that there should only be one term that contributes to the probability.

$$(rdL)^n (1 - rdL)^{\frac{L}{dL}-n} \times \binom{\frac{L}{dL}}{n}$$

The above should be $p(n;L)$ instead of that whole summing business going on before. A single bin can't have multiple events. Actually I think it's because the chance of multiple events in one bin times no events happening in other bins (due to the $(1-rdL)^{\cdots}$ term) is so small compared to $p(1,dL)$ that we can neglect it in our summation since they'll be higher order terms in dL. Anyway I've hand-waved my way to

$$p(n;L) = (rdL)^n (1 - rdL)^{\frac{L}{dL}-n} \times \frac{\left(\frac{L}{dL}\right)!}{n!\left(\frac{L}{dL} - n\right)!}$$

From here I think the argument I was doing before works. Consider

$$(rdL)^n \frac{\left(\frac{L}{dL}\right)!}{\left(\frac{L}{dL}-n\right)!} = r^n(L)(L-dL)(L-2dL)(\dots)(L-(n-1)dL)$$

Where the equality comes from the fact that the division creates n terms each of which we shove a dL into. Now taking the limit $dL \to 0$ we get

$$\boxed{p(n;L) = \frac{1}{n!}(rL)^n e^{-rL}}$$

No differential equations needed! I'm sure there's some glaring mistake I'm making and things just happened to work out.

Lecture 3: Functions of a Random Variable

Started: February 01, 2021. Finished: February 02, 2021

After some experimenting we found the probability density function for the velocity of atoms, $p_v(\xi)$. Given this how would we find the probability density function for a function of the velocities like the kinetic energy $T = \frac{1}{2}mv^2$. Well I don't know but the lecture notes gave a formula without proof. Lets say we have $x, p_x(\xi)$ and we want the probability distribution for $f(x)$ denoted by $p_f(\eta)$.

(a) Sketch $f(x)$ and find region of x where $f(x) < \eta$

(b) Integrate p_x over the indicated region in order to find the cumulative distribution function for f, $P_f(\eta)$

(c) Differentiate the distribution function to get the density function

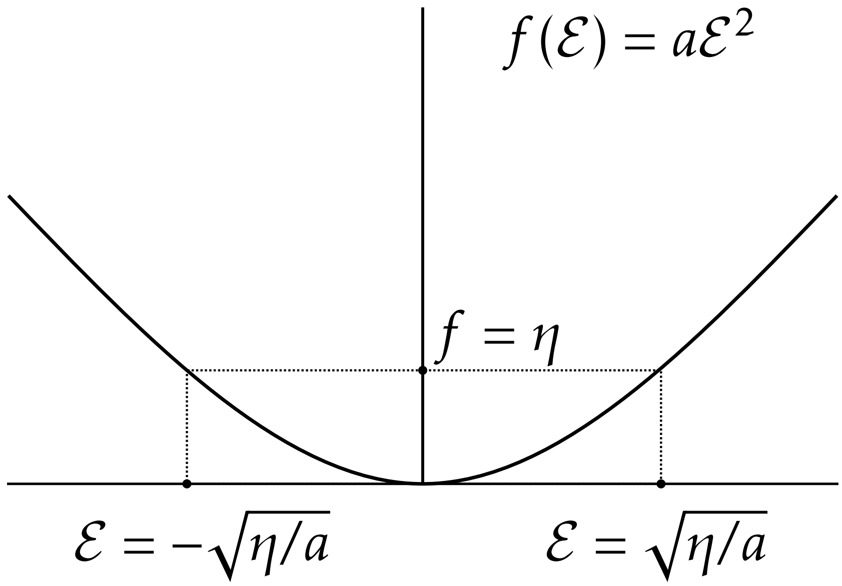
Example 6: (Classical Intensity of Polarized Thermal Light) The intensity of a linearly polarized electromagnetic wave is

$$I = a\mathcal{E}^2$$

Where $\mathcal{E}$ is the amplitude of the wave. The probability density function of the amplitude is Gaussian.

$$p(\mathcal{E}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathcal{E}^2}{2\sigma^2}}$$

Lets try to find $p(I)$. We'll start by drawing a graph.



Next we'll find the probability distribution of $\mathcal{E}$

$$P_I(\eta) = \int_{-\sqrt{\frac{\eta}{a}}}^{\sqrt{\frac{\eta}{a}}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\mathcal{E}^2}{2\sigma^2}} d\mathcal{E}$$

Now we need to take the derivative wrt. η to get $p_I(\eta)$. By Leibnitz's integral rule we know that we can ignore the integral and we're left with

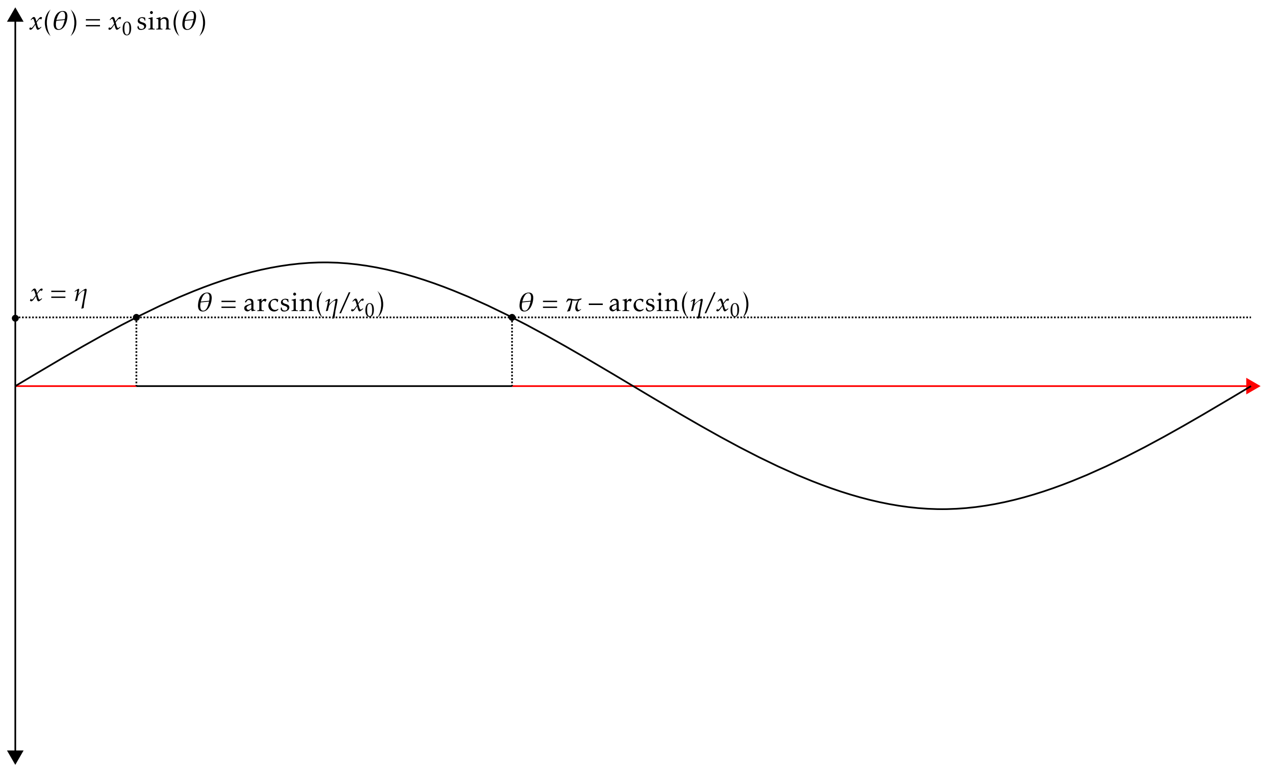
$$p_I(\eta) = \frac{dP_I(\eta)}{d\eta} = \frac{e^{-\frac{\eta}{2a\sigma^2}}}{\sqrt{2\pi\sigma^2}}\sqrt{\frac{1}{a\eta}} = \frac{1}{\sqrt{2\pi\sigma^2 a\eta}} \times \exp\left\{-\frac{\eta}{2a\sigma^2}\right\}$$

Not so bad I think. Lets try another one.

Example 7: (Harmonic Motion) Today, you decide, is the day you'll finally get into magic. The first trick you come up with is guessing the position of a harmonic oscillator with fixed total energy without ever looking at the harmonic oscillator. Your nephew is in the audience. He's also a big nerd so he knows that

$$p(\theta) = \frac{1}{2\pi} \qquad 0 \le \theta < 2\pi$$

Now he want to find the probability that you guess correctly given that you're truly randomly guessing (I guess he's setting up a null hypothesis?) Lets try to help him along.



Instead of integrating over the two red regions what we can do is integrate over the complement

$$P_x(\eta) = 1 - \int_{\arcsin(\eta/x_0)}^{\pi-\arcsin(\eta/x_0)} p(\theta)d\theta$$

From here we take the derivative wrt. η to get

$$p_x(\eta) = -\frac{d(\pi - \arcsin(\eta/x_0))}{d\eta} p(\pi - \arcsin(\eta/x_0)) + \frac{d(\arcsin(\eta/x_0))}{d\eta} p(\arcsin(\eta/x_0))$$

Now what in the world is the derivative of arcsin? First lets let $y = \arcsin(x)$ meaning that $\sin(y) = x$. Then we know that

$$1 = \frac{d(\sin(y))}{dx} = \frac{d\sin(y)}{dy}\frac{dy}{dx} = \cos(y)\frac{dy}{dx} = \cos(y)\frac{d(\arcsin(x))}{dx}$$

We can find $\cos(y)$ from basic trig

$$\cos(y) = \cos(\arcsin(x)) = \sqrt{1-x^2}$$

And plugging this all in gives us

$$\frac{d(\arcsin(x))}{dx} = \frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{d(\arcsin(\eta/x_0))}{d\eta} = \frac{1}{\sqrt{1-(\eta/x_0)^2}}\frac{1}{x_0}$$

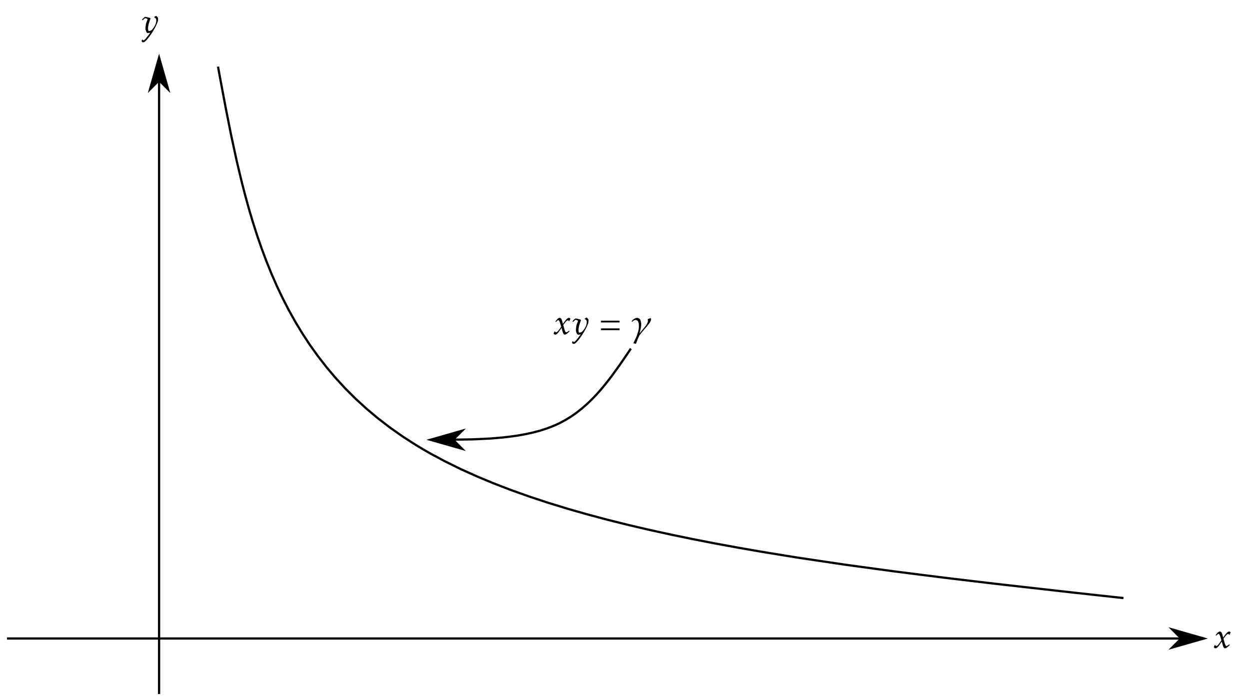
Plugging this all in gives us

$$p_x(\eta) = \frac{1}{\pi\sqrt{x_0^2 - \eta^2}} \qquad x > 0$$

The case when $x < 0$ is pretty much the same.

That was fun and all but in the previous few examples everything was done with a single random variable. What happens when we functions of multiple random variables

Example 8: (A Gambling Game like Pachinko except it's noting like Pachinko) You're in a casino playing with some slot machines. Howevere the machine you're playing with is kinda strange. The game goes as follows: the machine generates two random numbers x and y and multiplies them together. You know the probability $p(x,y)$. It is your job to bet some money. If you bet more than $x \times y$ then you get $x \times y$ and your money back. If not you lose the money you bet. Lets assume that the casino has unlimited money so that they can go as high as they like and you have the regular amount of money a person in a casino would have. To bet most efficiently you decide to figure out $p(z)$ where $z = x \times y$. How do you do this? We'll restrict ourselves to $x, y, z > 0$.



The region we want to integrate over is

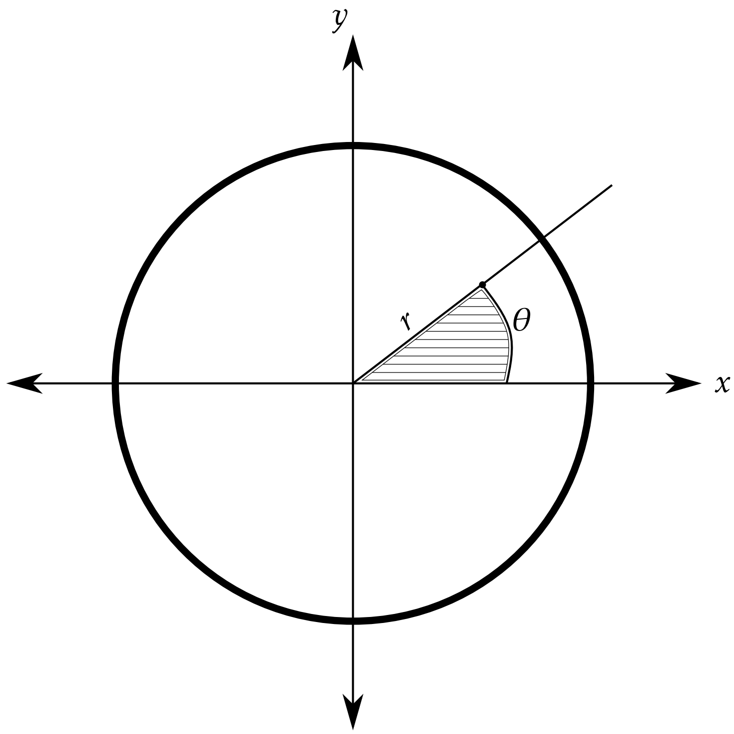
$$P_z(\gamma) = \int_0^\infty \int_0^{\gamma/y} p(x,y)dxdy \Rightarrow p_z(\gamma) = \frac{dP_z}{d\gamma} = \int_0^\infty dy \left(\frac{1}{y}p(\gamma/y, y)\right)$$

One useful application of functions of random variables is that they can be used to chnage of variables.

Example 9: (People keep getting blown up on my friend's farm!(But this time it's polar)) Same idea as last time

$$p(x,y) = \begin{cases} \frac{1}{\pi} & x^2+y^2 \leq 1 \\ 0 & x^2+y^2 > 1 \end{cases}$$

But this time we want to use this to find the joint probability distribtuion $p(r,\theta)$.



From the picture we can compute the probabilty

$$P(r,\theta) = \underbrace{\left(\pi r^2\right)}_{\text{total area}} \underbrace{\left(\frac{\theta}{2\pi}\right)}_{\text{percent filled}} \underbrace{\left(\frac{1}{\pi}\right)}_{p(x,y)}$$

From this we can compute some probability densities

$$p(r,\theta) = \frac{\partial}{\partial r}\frac{\partial}{\partial \theta}P(r,\theta) = \frac{r}{\pi}$$

$$p(r) = 2r \qquad p(\theta) = \frac{1}{2\pi}$$

Using our rule for statistical independence we also note that $p(r)$ and $p(\theta)$ are statistically independent.

Assignment 1

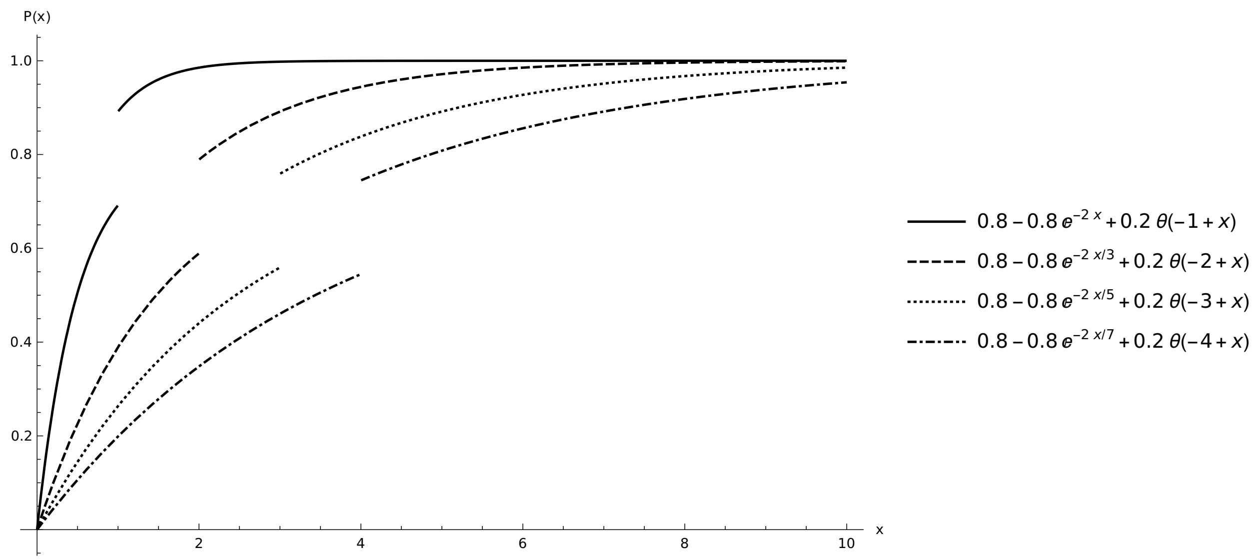
Started: February 03, 2021. Finished: February 04, 2021

Doping a Semiconductor

(a) They say we don't need to give an analytic expression for $P(x)$ but we'll do it anyways since we have mathematica. We have

$$P(x) = 0.2 \times \theta(x-d) - 0.8e^{-\frac{x}{l}} + 0.8$$

Where θ is the Heaviside function. Plotting this in mathematica as well



(b) To find $\langle x\rangle$ we just take an integral

$$\boxed{\langle x\rangle = \int_0^\infty p(x)x\,dx = \int_0^\infty x\left(\left(\frac{0.8}{l}\right)\exp\left\{-\frac{x}{l}\right\} + 0.2\delta(x-d)\right)dx = 0.2d + 0.8l}$$

(c) The variance we found in lecture is

$$\sigma_x^2 = \left\langle x^2\right\rangle - \langle x\rangle^2$$

We find that

$$\left\langle x^2\right\rangle = \int_0^\infty p(x)x^2dx = 0.2d^2 + 1.6l^2$$

Putting this together gives us

$$\boxed{\sigma_x^2 = .16d^2 - .32dl + .96l^2}$$

(d) Taking another integral

$$\boxed{\left\langle \exp\left\{-\frac{x}{s}\right\}\right\rangle = 0.2\times\exp\left\{-\frac{d}{s}\right\} + \frac{0.8s}{l+s}}$$

Peculiar Probability Distribution

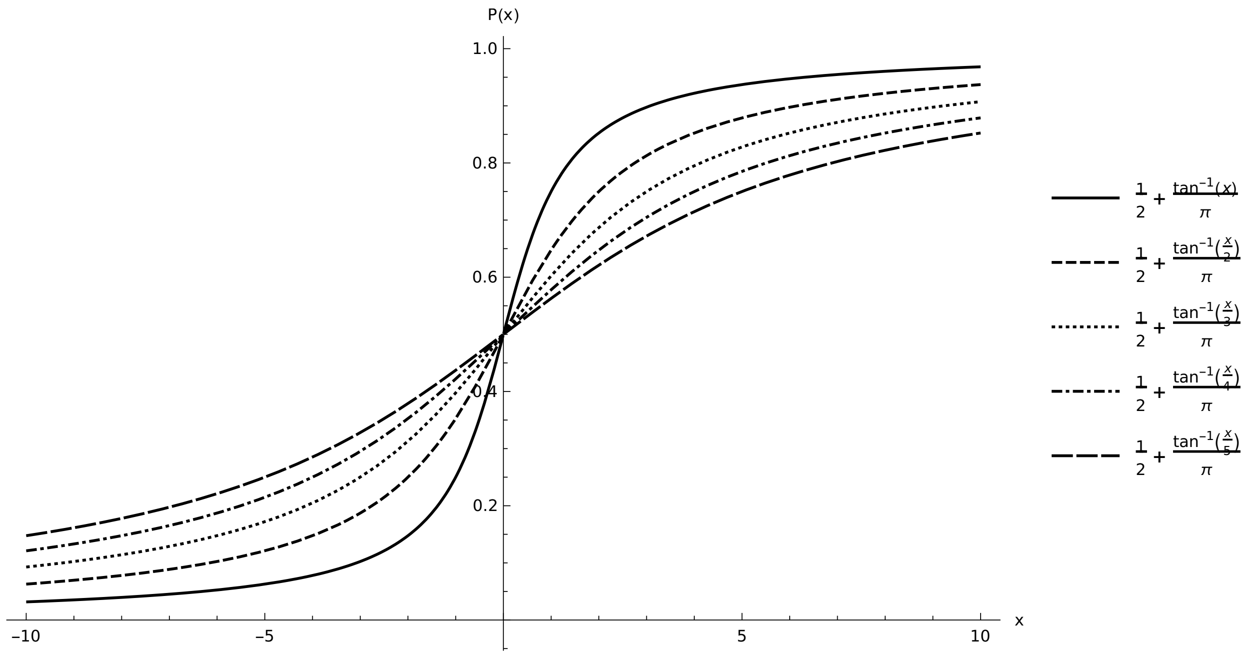
(a) Lets integrate

$$1 = \int_{-\infty}^{\infty} p(x)dx = a\sqrt{\frac{1}{b^2}}\pi \Rightarrow \boxed{a = \frac{\sqrt{b^2}}{\pi}}$$

(b) Taking another integral

$$P(x) = \int_{-\infty}^{x} p(x)dx = \frac{1}{2} + \frac{\arctan\left(\frac{x}{b}\right)}{\pi}$$

We can plot this



(c) I don't think we need to take an integral here. $p(x)$ is an even function so

$$\boxed{\langle x \rangle = 0}$$

(d) We want to solve

$$p(x) = \frac{a}{2b^2} = \frac{1}{2\pi b} \Rightarrow x = \pm b$$

(e) Well lets just look at the integral real quick

$$\left\langle x^2 \right\rangle = \int_{-\infty}^{\infty} \frac{bx^2}{(b^2 + x^2)\pi}$$

As $x \to \pm\infty$ we have the integrand converging to a constant. So it can't be convergent. This means that

$$\text{Var}(x) \text{ diverges}$$

Probability Density for a Clasical Harmonic Oscillator

(a) We can take the derivative of $x(t)$ wrt t to find the velocity

$$\frac{dx}{dt} = \omega x_0 \cos(\omega t + \phi)$$

To find the speed we can square things and then take the square root.

$$v^2 = (\omega x_0)^2 \underbrace{(1 - \sin^2(\omega t + \phi))}_{\cos^2(\omega t+\phi)} = \omega^2(x_0^2 - x^2) \Rightarrow \boxed{\text{Speed} = \omega\sqrt{x_0^2 - x^2}}$$

(b) We know the probability of finding a particle in $[x, x+dx]$ is inverseley proportional to the speed at x. Thus we have

$$p(x) = \frac{A}{\omega\sqrt{x_0^2 - x^2}}$$

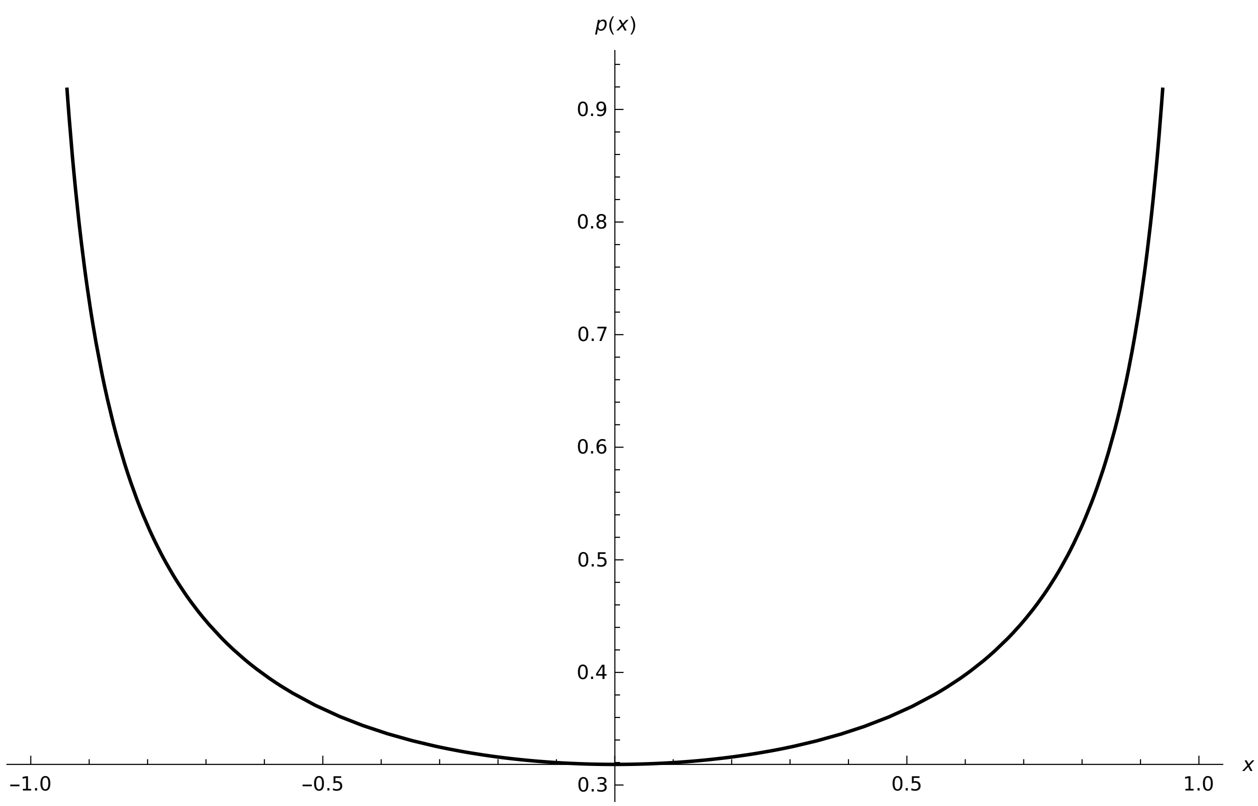
Where A is a normalization constant. We can find A by integrating between the bounds of x

$$1 = \int_{-x_0}^{x_0} p(x)dx = \frac{A\pi}{\omega} \Rightarrow A = \frac{\omega}{\pi}$$

This gives us

$$\boxed{p(x) = \frac{1}{\pi\sqrt{x_0^2 - x^2}}}$$

(c) Using mathematica we can plot



We can see that the most probable values of x are when the particle is moving the slowest (at the max displacements) and the least probable is when x is moving the fastest ($x = 0$). The mean is $\langle x \rangle = 0$ by symmetry.

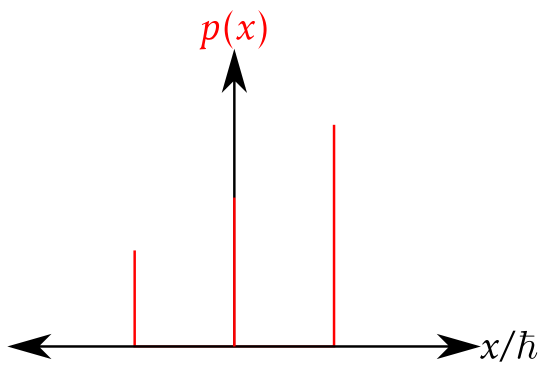
QUANTIZED ANGULAR MOMENTUM

(a) All we have is $\langle L_x \rangle = \frac{1}{3}\hbar$ and $\left\langle L_x^2 \right\rangle = \frac{2}{3}\hbar$. Since we have only two unknowns(the third is determined by $1 - p_1 - p_0$) this should be enough. Lets setup our system of equations

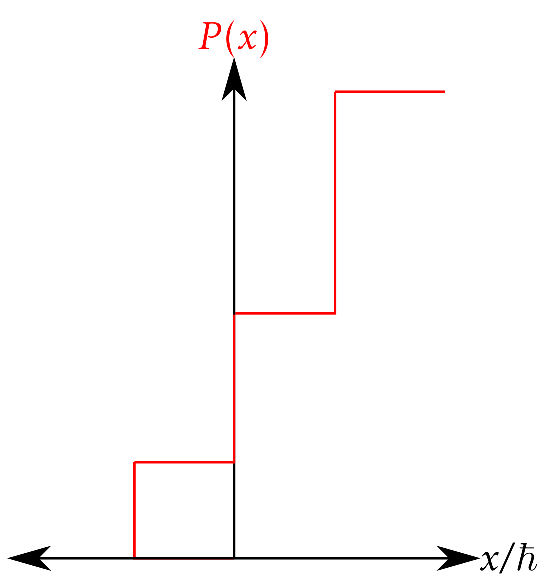
$$-p_{-1} + p_1 = \frac{1}{3}$$
$$p_{-1} + p_1 = \frac{2}{3}$$
$$\Rightarrow p_1 = \frac{1}{2} \text{ and } p_{-1} = \frac{1}{6} \Rightarrow p_0 = \frac{1}{3}$$

We can then write $p(L_x)$ as

$$\boxed{p(L_x) = \frac{\delta(x+\hbar)}{6} + \frac{\delta(x)}{3} + \frac{\delta(x-\hbar)}{2}}$$



(b) We can sketch the cumulative function as



Coherent State of a Quantum Harmonic Oscillator

(a) The terms with i in the exponential are just gonna vanish so we're left with

$$\Psi^*\Psi = \frac{1}{\sqrt{2\pi x_0^2}}\exp\left\{-\frac{(x-2\alpha x_0\cos\omega t)^2}{2x_0^2}\right\}$$

(b) Looks kinda like a Gaussian wave packet huh?

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-a)^2}{2\sigma^2}\right\} \Rightarrow \sigma = x_0 \text{ and } a = 2\alpha x_0\cos\omega t$$

From the properties of Gaussians' we have

$$\langle x\rangle = 2\alpha x_0\cos\omega t \qquad \mathrm{Var}(x) = \sigma^2 = x_0^2$$

(c) It's just a oscillating Gaussian wave packet.

Bose-Einstein Statistics

(a) Lets first normalize

$$1 = (1-a)\sum_n a^n$$

Remember those slick derivative tricks we did in lecture 1 to calculate properties of the poisson distribution? Lets try that here

$$\frac{d}{da}\frac{1}{1-a} = \frac{1}{(1-a)^2} = \frac{d}{da}\sum_{n=0}^{\infty}a^n = \sum_{n=0}^{\infty}a^{n-1}\times\underbrace{(n-1+1)}_{=n} = \sum_{n=0}^{\infty}a^{n-1}\times(n-1) + \sum_{n=0}^{\infty}a^{n-1}$$

In the LHS lets let $m = n-1$ giving us

$$\frac{d}{da}\sum_{n=0}^{\infty}a^n = \underbrace{\cancel{-a^{-1}} + \sum_{m=0}^{\infty}a^m\times(m)}_{\sum_{m=-1}^{\infty}a^m\times m} + \underbrace{\cancel{a^{-1}} + \sum_{m=0}^{\infty}a^m}_{\sum_{m=-1}^{\infty}a^m}$$

Now that the red term is $\langle m\rangle/(1-a)$. Furthermore by normalization we know that the blue term is $1/(1-a)$. We'll set what we have above equal to the green term.

$$\frac{1}{(1-a)^2} = (1-a)^{-1}(\langle m\rangle + 1) \Rightarrow \frac{1}{1-a} - 1 = \langle m\rangle \Rightarrow \langle m\rangle = \boxed{\langle n\rangle = \frac{a}{1-a}}$$

(b) For the variance we need $\left\langle n^2\right\rangle$. We're given the hint to take the derivative of $\langle n\rangle$ wrt a so lets try that

$$\frac{d\langle n\rangle}{da} = \underbrace{\frac{1}{1-a}}_{\langle n\rangle+1} + \underbrace{\frac{a}{(1-a)^2}}_{\langle n\rangle^2} = \frac{d}{da}\left((1-a)\sum_{n=0}^{\infty}a^n\times n\right) = (1-a)\times\underbrace{\left\langle n^2\right\rangle/a}_{\sum_{n=0}^{\infty}a^{n-1}\times n^2} - \underbrace{\frac{\langle n\rangle}{(1-a)}}_{\sum_{n=0}^{\infty}a^n\times n}$$

Setting the LHS equal to the green term ant noting $(1-a)/a = \frac{1}{\langle n \rangle}$ we get

$$\frac{1}{1-a} + \frac{a}{(1-a)^2} = \frac{\left\langle n^2 \right\rangle}{a} - \frac{a}{(1-a)^2} \Rightarrow \left\langle n^2 \right\rangle = \frac{a}{1-a} + \frac{2a^2}{(1-a)^2} = \langle n \rangle + 2\langle n \rangle^2$$

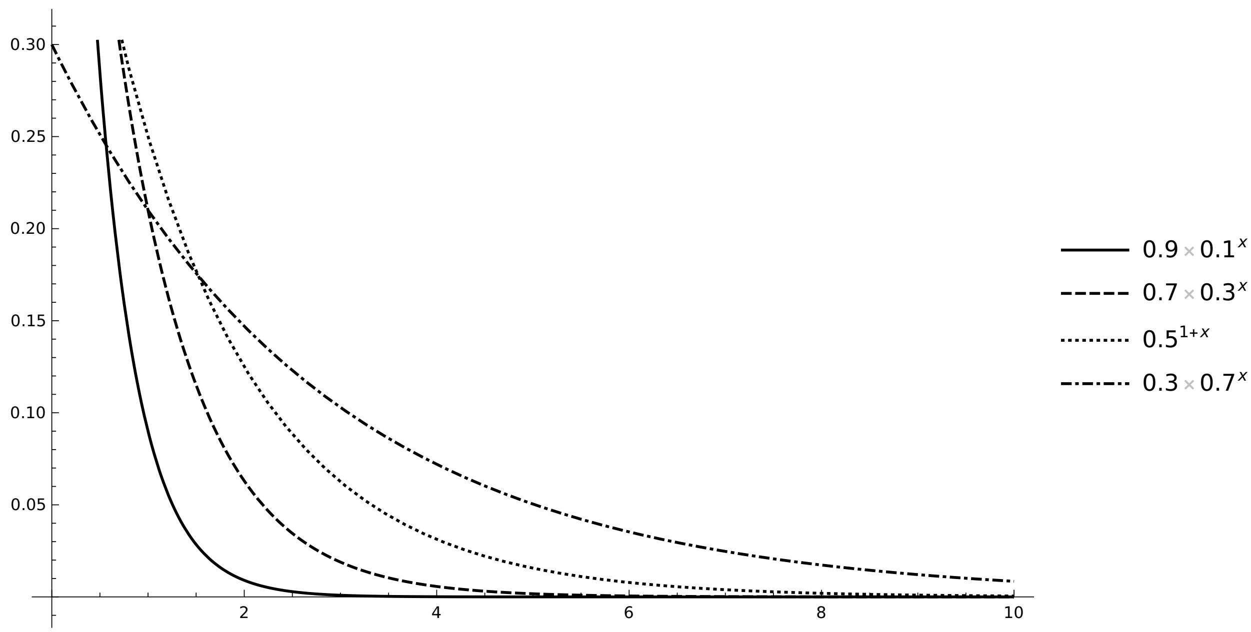
Putting everything together gives us

$$\boxed{\mathrm{Var}(n) = \left\langle n^2 \right\rangle - \langle n \rangle^2 = \langle n \rangle (1 + \langle n \rangle)}$$

(c) We are told to write the probability density as the product of an envelope function $F(x)$ times a train of δ functions.

$$p(x) = F(x) \times \left(\sum_{n=0}^{\infty} \delta(x-n) \right) = \underbrace{\sum_{n=0}^{\infty} \delta(x-n)(1-a)a^n}_{\substack{\text{original} \\ \text{probabilty}}}$$

Lets for a second imagine that the probabilty distribution is continuous. Would would it look like then? Using mathematica we plot the continous limit of the original probability for a few values of a



That does look pretty exponentially decreasing. Since the sum of δ takes care of discretizing everything we're trying to find

$$F(x) = \lambda e^{-x/\phi} = (1-a)a^x \Rightarrow \lambda = (1-a) \text{ and } \underbrace{e^{\ln(a)x}}_{a^x} = e^{-x/\phi} \Rightarrow \phi = -\frac{1}{\ln(a)}$$

The red terms are the parameters of our exponential. Lets try to get $\langle n \rangle$ in here somehow. First lets try to solve a in terms of $\langle n \rangle$. From our result in part (a) we have

$$\langle n \rangle = \frac{a}{1-a} \Rightarrow a = \frac{\langle n \rangle}{1 + \langle n \rangle}$$

Plugging this into our formula for ϕ we get

$$\phi = -\left(\ln\left\{\frac{\langle n\rangle}{1+\langle n\rangle}\right\}\right)^{-1}$$

Lets rewrite the argument of ln as

$$\frac{\langle n\rangle}{1+\langle n\rangle} = \left(1-\frac{1}{\langle n\rangle}\right)^{-1} = (1+\omega)^{-1}$$

Note as $\langle n\rangle \to \infty$ we have $\omega \to 0$ so we can do a binomial expansion

$$\frac{\langle n\rangle}{1+\langle n\rangle} = 1+\omega+\dots$$

This for large $\langle n\rangle$ we have

$$\ln\left\{\frac{\langle n\rangle}{1+\langle n\rangle}\right\} \approx \ln\{1+\omega\}$$

Now lets expand ln for $\omega \to 0$ again. This gives us

$$\ln\left\{\frac{\langle n\rangle}{1+\langle n\rangle}\right\} \approx \omega = -\frac{1}{\langle n\rangle}$$

Thus as $\langle n\rangle \to \infty$ we get

$$\boxed{\phi \to \langle n\rangle}$$

Lecture 4: Sums of Random Variables

Started: February 06, 2021. Finished: February 10, 2021

Lets start by proving some things about means! Consider a bunch of statistically independent random varaibles $\{x_i\}$ where the probability density of each x_i can be different. The sum of these random variables is $S_n = \sum_{i=1}^n x_i$ which is a new random variable. Lets consider the properties of S_n

$$\begin{aligned}
\langle S_n\rangle &= \int dx_1 \dots \int dx_i \dots \int dx_n \left(\sum_{i=1}^n x_i\right) \underbrace{p(x_1,\dots,x_i,\dots,x_n)}_{\substack{p(x_1)\times\cdots\times p(x_i)\times\cdots\times p(x_n) \\ \text{due to S.I.}}} \\
&= \sum_{i=1}^n \left(\underbrace{\int x_i p(x_i) dx_i}_{\langle x_i\rangle} \times \prod_{i\neq j} \underbrace{\int p(x_j) dx_j}_{1} \right) \\
&= \sum_{i=1}^n \langle x_i\rangle
\end{aligned}$$

$$\langle S_n^2\rangle = \int dx_1 \dots \int dx_i \dots \int dx_n \underbrace{\left(\sum_{i=1}^n x_i^2 + \sum_{i\neq j} 2x_i x_j\right)}_{\left(\sum_{i=1}^n x_i\right)^2} p(x_1)\times\dots\times p(x_i)\times\dots p(x_n)$$

$$= \sum_{i=1}^n \underbrace{\left(\underbrace{\int x_i^2 p(x_i)dx_i}_{\langle x_i^2\rangle} \times \prod_{i\neq j}\underbrace{\int p(x_j)dx_j}_{1}\right)}_{\langle x_i^2\rangle} + 2\sum_{i\neq j}\underbrace{\int x_i dx_i}_{\langle x_i\rangle}\times\underbrace{\int x_j dx_j}_{\langle x_j\rangle}\times\prod_{i\neq j\neq k}\underbrace{\int p(x_k)dx_k}_{1}$$

$$= \sum_{i=1}^n \langle x_i^2\rangle + \sum_{i\neq j} 2\langle x_i\rangle\langle x_j\rangle$$

From this we can find the variance

$$\mathrm{Var}(S_n) = \langle S_n^2\rangle - \langle S_n\rangle^2 = \sum_{i=1}^n \langle x_i^2\rangle + \cancel{\sum_{i\neq j} 2\langle x_i\rangle\langle x_j\rangle} - \left(\sum_{i=1}^n \langle x_i\rangle^2 + \cancel{\sum_{i\neq j} 2\langle x_i\rangle\langle x_j\rangle}\right) = \sum_{i=1}^n \underbrace{\langle x_i^2\rangle - \langle x_i\rangle^2}_{\sigma_i^2}$$

So from all this mess we find that

$$S_n = \sum_{i=1}^n x_n \Rightarrow \langle S_n\rangle = \sum_{i=1}^n \langle x_i\rangle \text{ and } \mathrm{Var}(S_n) = \sum_{i=1}^n \mathrm{Var}(x_i)$$

In the special case where we're summing the same variable we have

$$\langle S_n\rangle = n\langle x\rangle \text{ and } (S_n) = n\times\mathrm{Var}(x)$$

The probability density of S_n trends to spiking at $\langle S_n\rangle$ for large n since the mean grows as n while the width of the peak grows as $\sqrt{n}$.

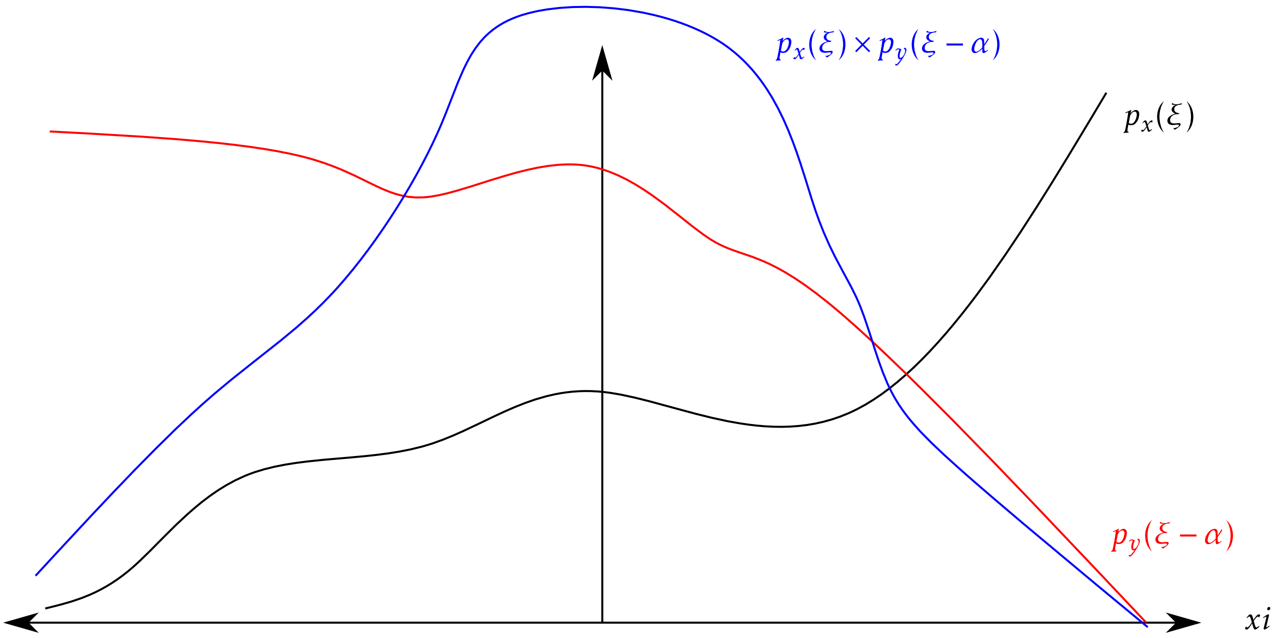
Now lets turn to finding the probability density of the sum of two random variable x and y. Let $S = x+y$. We want $P_S(\alpha)$ from $p_{x,y}(\xi,\eta)$. We can do this from our normal method

$$P_S(\alpha) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\alpha-\xi} d\eta\; p_{x,y}(\xi,\eta) \Rightarrow p_S(\alpha) = \frac{d}{d\alpha}P_S(\alpha) = \int_{-\infty}^{\infty} d\xi\; p_{x,y}(\xi,\alpha-\xi)$$

Note that if x and y are S.I then we have a convolution $p_x \otimes p_y$

$$p_S(\alpha) = \int_{-\infty}^{\infty} d\xi p_x(\xi)p_y(\alpha-\xi)$$

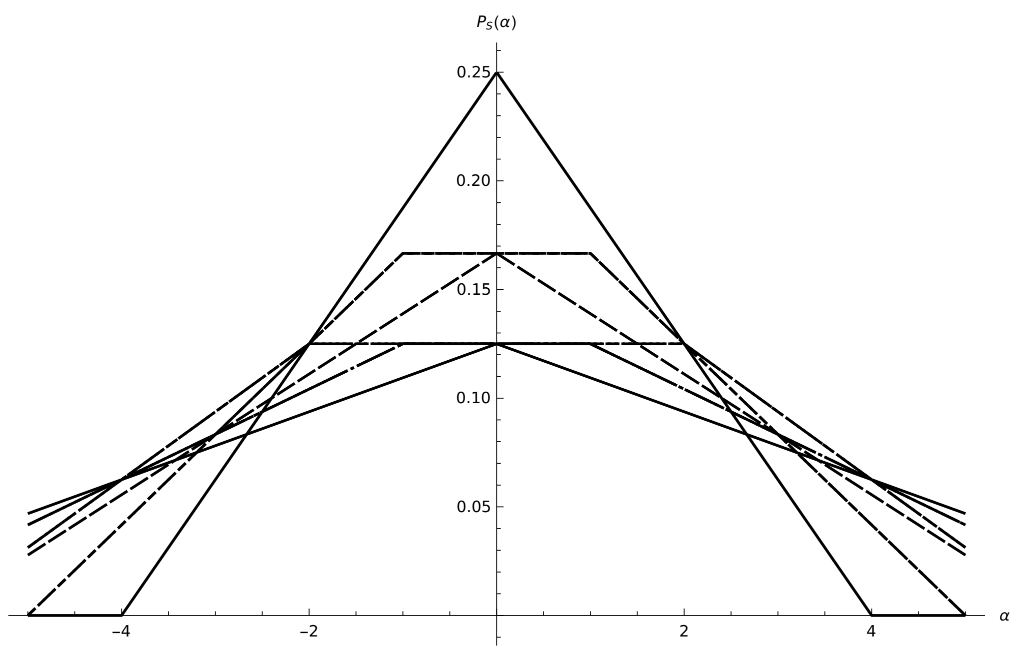
The picture you should have in your head is putting two probability distributionn top of eachother and then integrating over the intersecion.



Example 10: (Sum of two Uniformly Distributed Variables) We have the two following statistically independnet random variable with the following distributions

$$p(x) = \begin{cases} \frac{1}{2a} & |x| < a \\ 0 & \text{otherwise} \end{cases} \qquad p(y) = \begin{cases} \frac{1}{2b} & |y| < b \\ 0 & \text{otherwise} \end{cases}$$

We want to find the probability density $p(S)$ for the sum $S = x + y$. To do this we just convolve the two distributions. Setting the bounds of this problem could be tricky. However mathematica can handle piecewise functions and so mathematica will have to worry about that :3c. Plotting the result for a few values of a and b



Okay... interesting. You know what, lets look at this analytically. First lets make our lives easier by saying $b < a$. Now by the parity of the distributions we only need to consider three cases.

(a) The most trrivial one is $a + b < \alpha$ where $P_X(\alpha) = 0$

(b) p_X and p_Y intersect but neither contains the other. In this case the integral becomes

$$\int_{\alpha-b}^{a} \frac{1}{2a} \times \frac{1}{2b} dx = \frac{a+b-S}{(2a)\times(2b)}$$

(c) p_X contains p_Y completley. In this case the integral becomes

$$\int_{\alpha-b}^{\alpha+b} \frac{1}{2a} \times \frac{1}{2b} dx = \frac{1}{2a}$$

And another example!

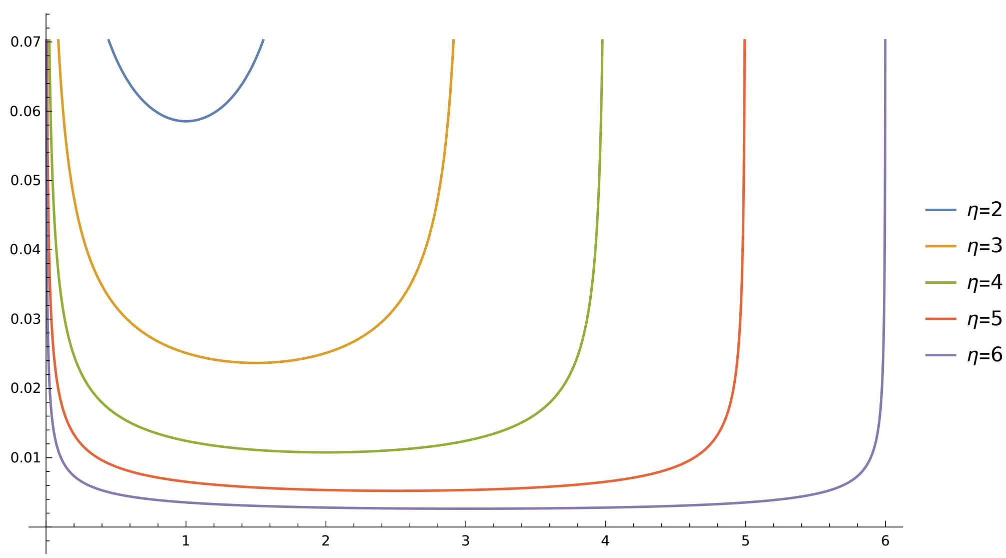
Example 11: (Classical Intensity of Unpolarized Light) We can think of an unpolarized beam of light as the sum of two statistically independent orthogonal polarized beams of equal average intensity $\langle i \rangle$. Using our result from example 6 lets find the probability density of the intensity this unpolarized beam of light $p(I_T = 2I)$. First we have from example 6

$$p(I) = \frac{1}{\sqrt{2\pi\sigma^2 a I}} \times \exp\left\{-\frac{I}{2a\sigma^2}\right\}$$

Note that we can let $\sigma^2 a = \alpha = \langle I \rangle$

$$p(I) = \frac{1}{\sqrt{2\pi\alpha I}} \times \exp\left\{-\frac{I}{2\alpha}\right\}$$

Lets plot $p_I(\xi) \times p_I(\eta - \xi)$ to see what we need to integrate over.



So it seem like the only region we need to integrate over is $[0, I_T]$ so

$$p_{I_T}(\eta) = \int_0^\eta p_I(\xi) \times p_I(\eta - \xi) d\xi = \frac{1}{2\alpha} \times \exp\left\{-\frac{\eta}{2\alpha}\right\} \text{ when } \eta \geq 0$$

Looking at the previous two examples we see that if we find the probability density of a sum of identitcal probability densities then the result is not necessarily the same kind probability density. There are three exceptions to this

(a) Gaussians densities

(b) Poisson densities

(c) Lorentzian(or Cauchy) densities $p(x) = \frac{\Gamma}{\pi}\frac{1}{(x-m)^2+\Gamma^2}$ where Γ is the half width and m is the position of the center.

Lets confirm what we said for Gaussians

Example 12: (Sum of Gaussian Densities) Consider

$$p(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left\{-\frac{(x - E_x)^2}{2\sigma_x^2}\right\}$$

$$p(y) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left\{-\frac{(y - E_y)^2}{2\sigma_y^2}\right\}$$

We want to find the probability density of $S = x + y$. In this scenario we don't need to worry about bounds too much. We can just brute force convolve with mathematica and see what we get.

$$p_S(\eta) = \int_{-\infty}^{\infty} p_x(\xi) p_y(\eta - \xi) d\xi = \frac{1}{\sqrt{2\pi(\sigma_x^2 + \sigma_y^2)}} \times \exp\left\{-\frac{\left(\eta - E_x - E_y\right)^2}{2\left(\sigma_x^2 + \sigma_y^2\right)}\right\}$$

Where we see that the result is a Gaussian with

$$\text{Var}(\eta) = \sigma_x^2 + \sigma_y^2 \quad \langle\eta\rangle = E_x + E_y$$

Note: Greytak has some fancy tricks in his lecture notes if you want some cool integration party tricks.

Before we continue we should note that this whole mess of convolulutions can be thought of in a different way using **fourier transforms.** Let $f \leftrightarrow F$ denote a fourier transformed pair. Then we have

$$a \leftrightarrow A \quad b \leftrightarrow B \Rightarrow a \otimes b \leftrightarrow AB$$

Lets move on to a fun topic: the *central limit theorem*

DEFINITION 1: (NON-RIGOROUS BUT USEFUL FORM OF THE CENTRAL LIMIT THEOREM) If S_n is the sum of n statistically independent random variables with the same mean $\langle x \rangle$ and variance σ_x^2, for large n we can approximate $P(S_n)$ as a gaussian with mean $n\langle x \rangle$ and variance $n\sigma_x^2$.

Lets assert some extensions of this theorem as well

(a) Even for modest values of n a gaussian would be ok. The convergence is quite rapid

(b) If no term dominates the sum, even if not all the random variables have the exact same mean and variance we can still approximate it with a gaussian

(c) For some cases, the variables doesn't even need to be statistically independent.

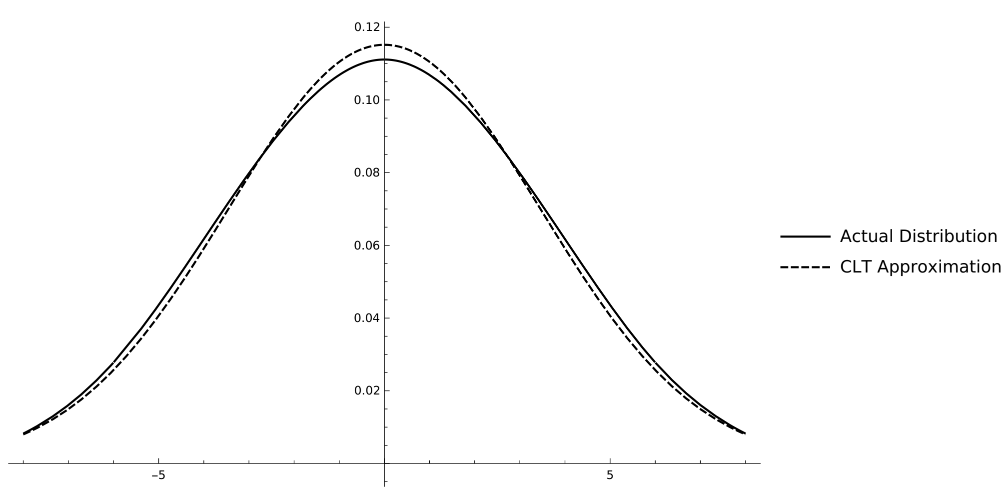
EXAMPLE 13: (SOME EXAMPLES WITH MATHEMATICA) We saw in example 10 that the sum of two uniformly distributed random variables. Now lets see what it looks like for the sum of four uniformly distributed random variables with distribution

$$p(x) = \begin{cases} 1/2a & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

Lets do our convolutions with mathematica and see what we get. We don't need to derive anything new for more than a sum of two distributions since convolutions is associative so we can just do it iteratively. By the central limit theorem we should a normal distribution with mean $n\langle x \rangle = 0$ and and variance $n\sigma_x^2 = 2/a$ to approximate the resulting distribution well.

```
$Assumptions = a ∈ PositiveReals
p[x_] := UnitBox[x / (2 a)] / (2 a)
c1[α_] := Integrate[p[x] × p[α - x], {x, -Infinity, Infinity}]
c2[β_] := Integrate[c1[α] × p[α - β], {α, -Infinity, Infinity}]
c3 = Integrate[c2[β] × p[β - γ], {β, -Infinity, Infinity}];
CLTapprox = Evaluate@PDF[NormalDistribution[0, 2 a / Sqrt[3]], γ];
toplot = {c3, CLTapprox};
Plot[Evaluate@(toplot /. a → 3), {γ, -8, 8}, PlotTheme → "Monochrome",
 PlotLegends → {"Actual Distribution", "CLT Approximation"}, Evaluated → True, ImageSize → Large]
```

The above code gives us



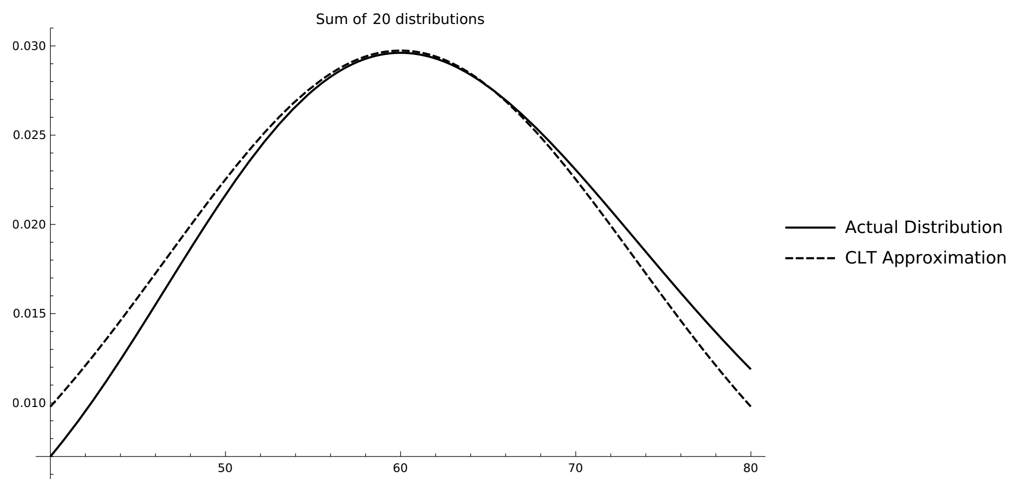
It's a good approximation! But this one is easy. What about sums of exponentially distributed variables e.g.

$$p(x) = \frac{1}{a}e^{-x/a} \text{ for } x > 0$$

We can tell on examination that $\langle x \rangle = a$ and variance is a^2 so we should expect that after n sums we can expect a gaussian with variance na^2 and mean na to approximate the sum well. Lets do some more coding!

```
$Assumptions = a ∈ PositiveReals;
p[x_] := Piecewise[{{1 / a Exp[-x / a], x > 0}}, 0]
nSum = 20;
res = p[x];
For[i = 1, i ≤ nSum, i++, res = Integrate[res p[y - x], {x, -Infinity, Infinity}] /. {y → x}]
toplot = {res, Evaluate@PDF[NormalDistribution[nSum * a, Sqrt[nSum] * a], x]};
Plot[Evaluate@(toplot /. a → 3), {x, nSum * 3 - 20, nSum * 3 + 20}, PlotTheme → "Monochrome",
 PlotLegends → {"Actual Distribution", "CLT Approximation"},
 PlotLabel → "Sum of " <> ToString[nSum] <> " distributions", ImageSize → Large]
```

The above code gives us



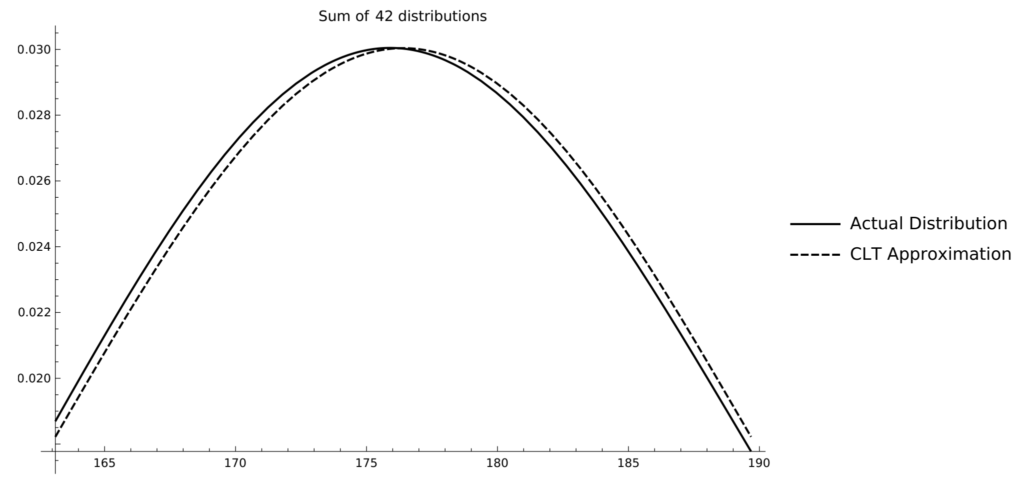
Wow, another banger! Lets also look at the poisson distribution. We actually don't even need to do any convolutions, we know that the sum of poisson distributions is just a poisson distributiom neaning that we have

$$\text{Sum of } m \text{ Poisson Distribution} = \frac{1}{n!}\left(m\langle n\rangle\right)^n e^{-m\langle n\rangle}$$

Modifying our previous mathematica code a bit more

```
$Assumptions = a ∈ PositiveReals;
mean = 4.2
nSum = 42;
res = 1 / n! (nSum mean) ^ n Exp[-nSum mean];
toplot = {res, Evaluate@PDF[NormalDistribution[nSum * mean, Sqrt[nSum mean]], n]};
Plot[Evaluate@(toplot), {n, nSum * mean - Sqrt[nSum mean], nSum * mean + Sqrt[nSum mean]},
 PlotTheme → "Monochrome", PlotLegends → {"Actual Distribution", "CLT Approximation"},
 PlotLabel → "Sum of " <> ToString[nSum] <> " distributions", ImageSize → Large]
```

The above code gives us



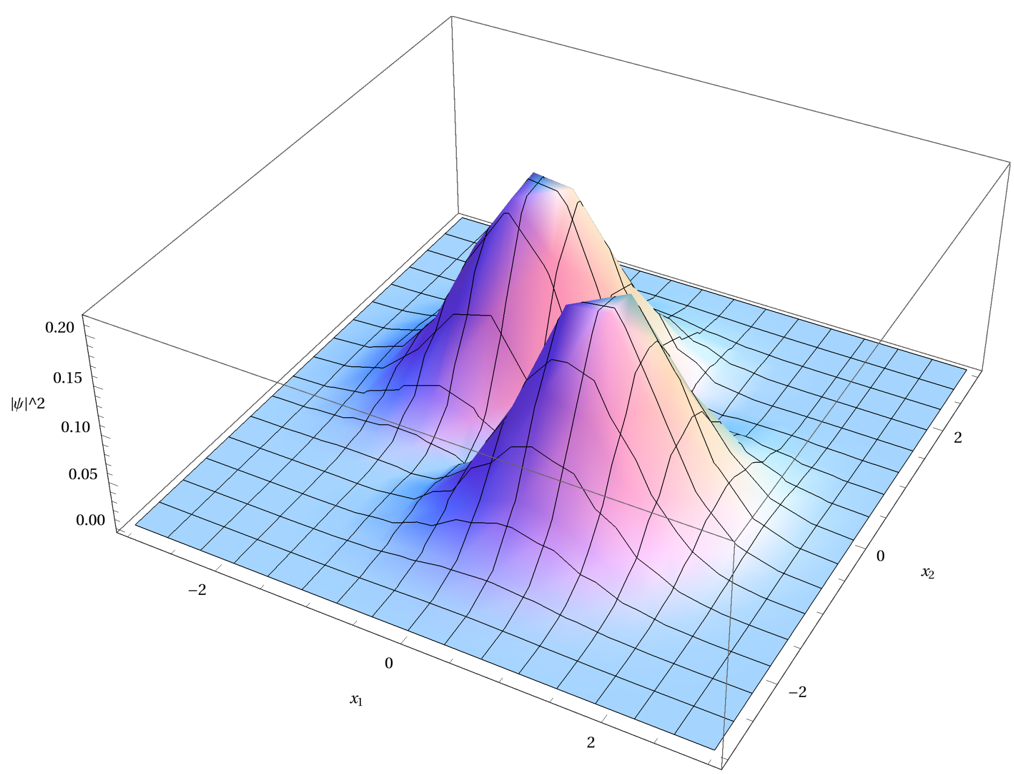
Assignment 2

Started: February 11, 2021. Finished: February 15, 2021

Two Quantum Particles

(a) The probability density is $|\psi|^2$.

$$|\psi|^2 = \frac{(x_1 - x_2)^2}{\pi x_0^4} \times e^{-\frac{x_1^2 + x_2^2}{x_0^2}}$$



We see that the joint probability is highest at $(x_1, x_2) = (1, -1) = (-1, 1)$ and lowest when $x_1 = x_2$. We can see this analytically by noting that $|\psi|^2$ vanishes when $x_1 = x_2$.

(b) We want to integrate out one of the variables which means

$$p_1(x) = \int_{-\infty}^{\infty} |\psi|^2(x, x_2)dx_2 = \frac{e^{-\frac{x^2}{x_0^2}}\left(2x^2 + x_0^2\right)}{2\sqrt{\pi}x_0^3}$$

And similarly for x_2

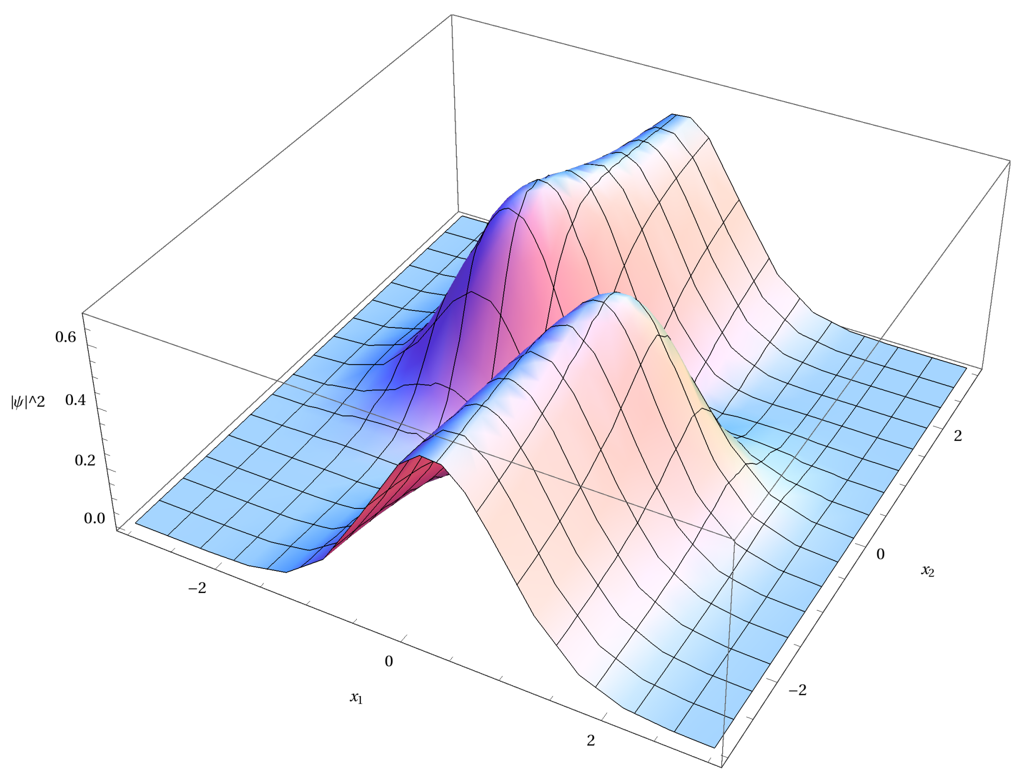
$$p_2(x) = \int_{-\infty}^{\infty} |\psi|^2(x_1, x)dx_1 = \frac{e^{-\frac{x^2}{x_0^2}}\left(2x^2 + x_0^2\right)}{2\sqrt{\pi}x_0^3}$$

Since $p_1(x_1) \times p_2(x_2) \neq |\psi|^2$ we can say that the positions of the two particles is no statistically independent

(c) We can calculate the probability density $p(x_1|x_2)$ with bayes rule

$$p(x_1|x_2) = \frac{p(x_1, x_2)}{p(x_2)} = \frac{2e^{-\frac{x_1^2}{x_0^2}}(x_1 - x_2)^2}{\sqrt{\pi}\left(x_0^3 + 2x_2^2 x_0\right)}$$

We can plot this for $x_0 = 1$ giving us.

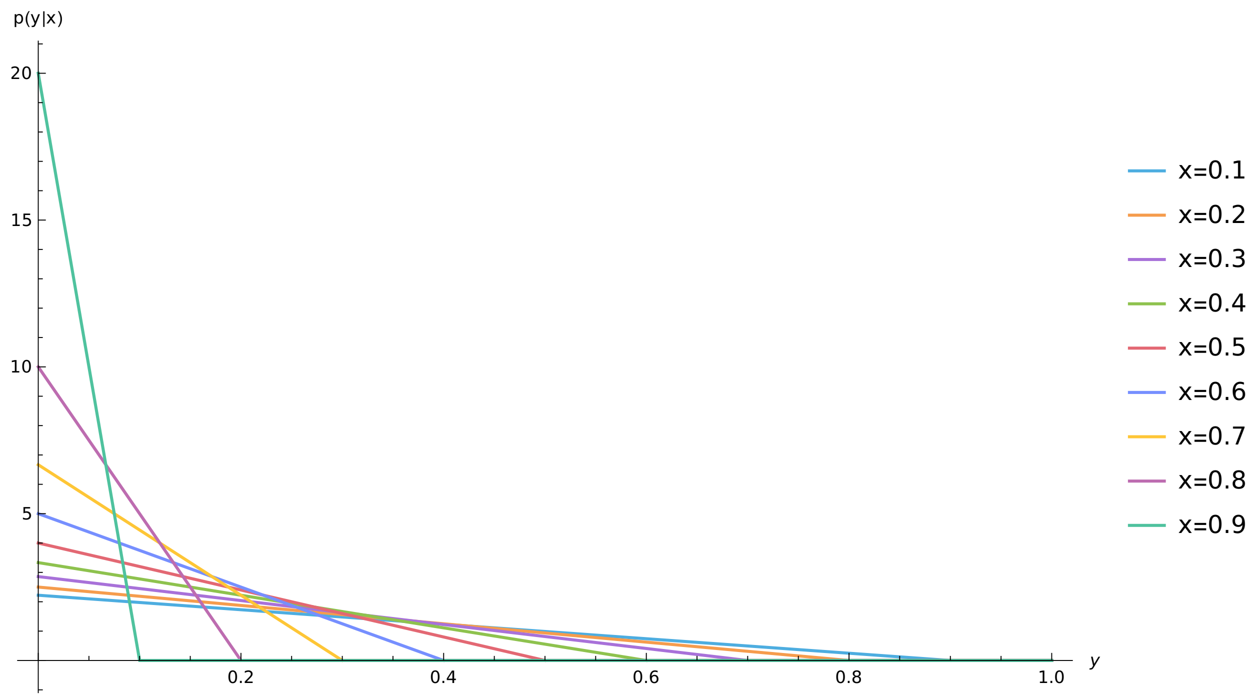


Pyramidal Density

(a) Mathematica time

$$p(x) = \begin{cases} 3\left(x^2 - 2x + 1\right) & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(b) Sometimes I feel bad for how much I rely on mathematica



Stars

(a) My first guess is that the probability density that a given star's nearest neighbor occurs at a raidal distance r away is equivalent to the complement of the probability that we find no stars with radial distance $< r$. I'm a bit concerned about this since it doesn't take into account the probability that we also won't find a star at radial distance r but lets roll with it for now. We want to evalaute

$$1 - \int d^3x \times \rho = 1 - 4\pi \int_0^r dr \times r^2 \rho = 1 - \frac{4\pi\rho r^3}{3}$$

And looking at the above result I think we can say this is absolute nonsense. If $r \to -\infty$ then we have negative (and worse infinite) probability. Actually I think what we just evaluated is 1 minus the number of stars we'd expect to find after we go a radial distance r away so what I've been doing is absolute nonsense. Maybe what we want is to find the complement for finding a star for every infinitesimal dr. Actually wait, isn't this just the copmlement of some poisson density. Like from Example 5 we can say that the probability that one event occurs in some volume element is $\rho\Delta^3 r$. So from the form of the poisson density we can read off the $n = 1$ case to get

$$p(r) = (\rho r)e^{-\rho r}$$

Lets check the normalization

$$\int d^3r \times p(r) = \frac{24\pi}{\rho^3} \neq 1 \text{ oh fuck}$$

Okay maybe we need to think a little more. So we know that the poisson distribution is characterized by $\langle n \rangle$ and I think we can find that. It'd just be

$$\langle n \rangle = \frac{4}{3}\pi r^3 \rho$$

Oh we found this already. Anyways now we have

$$p(r) = \left(\frac{4\pi\rho r^3}{3}\right) \times \exp\left\{-\frac{4\pi\rho r^3}{3}\right\} \to \int p(r) d^3r = \frac{1}{\rho} \text{ oh...}$$

Actually no this isn't right since it's the probability density of finding one star at some $r_* < r$. Okay so maybe instead we want the poisson probability distribution of finding zero stars up until distance r times $\rho \times 4\pi r^2 dr$ the probability of finding a star at r?

$$p(r) = 4\pi r^2 \rho dr \times \exp\left\{-\frac{4\pi\rho r^3}{3}\right\}$$

I think the green term above is $p(r)dr$ instead of $p(r)$ since it's the probability of finding a star at r which is the probability density function at r times dr. So instead we have

$$p(r)dr = 4\pi r^2 \rho dr \times \exp\left\{-\frac{4\pi\rho r^3}{3}\right\} \Rightarrow \boxed{p(r) = 4\pi r^2 \rho \times \exp\left\{-\frac{4\pi\rho r^3}{3}\right\}}$$

The normalization is good and the units work out as well.

(b) We know the probability distribution $p(r)$ to find the star's nearest neighbor. We know $a(r)$ blows up at $r \to 0$. We want to solve the equation $\eta = GM/r^2 \Rightarrow r = \pm\sqrt{GM/\eta}$. Now we'll integrate over the complement

$$P_a(\eta) = 1 - \int_{-\sqrt{GM/\eta}}^{\sqrt{GM/\eta}} 4\pi r^2 \rho \times \exp\left\{-\frac{4\pi\rho r^3}{3}\right\} dr$$

And now we can take a derivative wrt η to get

$$p(a) = 2\pi\rho(GM)^{3/2}/a^{5/2} \times \left(\exp\left\{-\frac{4\pi\rho}{3}\left(\frac{GM}{a}\right)^{3/2}\right\} + \exp\left\{\frac{4\pi\rho}{3}\left(\frac{GM}{a}\right)^{3/2}\right\}\right)$$

We can simplify this by shoving a cosh into $p(a)$ to get

$$p(a) = \frac{4\pi\rho(GM)^{3/2}}{a^{5/2}} \times \cosh\left\{\frac{4\pi\rho}{3}\left(\frac{GM}{a}\right)^{3/2}\right\}$$

For low a we'd expect the greatest error since that implies that the nearest neighbor is far away and thus there could be a closer neighbor.

(c) Oops, I think I misread part (b)

(d) Gravitaional lensing affecting our measurement of "nearest stars," red shift, mass.

Kinetic Energies in Ideal Gasses

(a) We know that $E < \eta$ for $|v|$ betweeen 0 and $\sqrt{2\eta/m}$. I think this would actually be easier if we did it in spherical coordinates since we're just integrating over a sphere in v_i space. So we want to evaluate

$$P_E(\eta) = 4\pi \int_0^{\sqrt{2\eta/m}} r^2 \times (2\pi\sigma^2)^{-3/2} \exp\left\{\frac{-r^2}{2\sigma^2}\right\} dr \text{ where } r^2 = r_x^2 + r_y^2 + r_z^2$$

Taking our derivative gives us

$$\boxed{p(E) = \frac{2}{m\sigma^3} \times \sqrt{\frac{E}{m\pi}} \times \exp\left\{-\frac{E}{m\sigma^2}\right\}}$$

This doesn't look gaussian :pensive:

(b) We're just doing the same integral here but in polar coordinates (and changing $(2\pi\sigma^2)^{-3/2}$ to just $(2\pi\sigma^2)^{-1}$ since one of the DOF is frozen out)

$$P_E(\eta) = 2\pi \int_0^{\sqrt{2\eta/m}} r \times (2\pi\sigma^2)^{-1} \exp\left\{\frac{-r^2}{2\sigma^2}\right\}$$

Taking our derivative gives us

$$\boxed{p(E) = \frac{1}{m\sigma^2} \times \exp\left\{-\frac{E}{m\sigma^2}\right\}}$$

Measuring an Atomic Velocity Profile

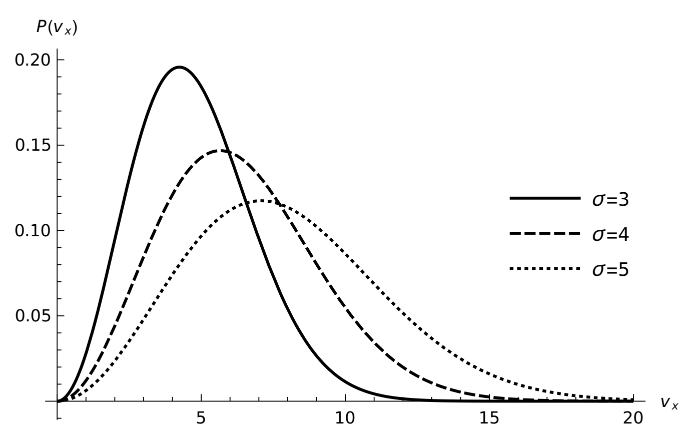
First we solve for v_x getting

$$v_x = \sqrt{\frac{A}{s}}$$

We want to integrate in the interval $[A/\eta^2, \infty]$. Doing the integral and then derivative with mathematica we get

$$p_{v_x}(\eta) = \frac{\sqrt{\frac{2}{\pi}}\eta^2 e^{-\frac{\eta^2}{2\sigma^2}}}{\sigma^3}$$

And plotting this with mathematica gives us



Planetary Nebulae

I think we can do this in two steps. First finding $p_\theta(\xi)$ and then using $r_\perp = R\sin\theta$ to get $p(r_\perp)$. So first we can see that

$$p(\theta)d\theta = \frac{2\pi R^2 \sin\theta d\theta}{4\pi R^2} = \frac{\sin\theta d\theta}{2} \Rightarrow p(\theta) = \frac{\sin\theta}{2}$$

Now we'll integrate over the complement of the region where $r_\perp < R\sin\xi$ to get

$$P(r_\perp) = 1 - \int_{\arcsin(r_\perp/R)}^{\pi-\arcsin(r_\perp/R)} p(\theta)d\theta$$

And taking the derivative gives us the final answer

$$\boxed{p(r_\perp) = \frac{r_\perp}{R\sqrt{(R-r_\perp)(R+r_\perp)}}}$$

mathematica notebook here

Zemansky Waldo Dittman Chapter 1: Temperature

Started: February 17, 2021. Finished:

First lets set down some definitions. When we want to study a system we first establish a boundary that specifies region of the system. We then also consider the surroundings of the system which may influence the physics inside the system we're studying. If there is an exchange of matter between the system and the surroundings we could call the system open while if there is no matter exchange we'd call the system closed. In stuyding this system we could take two points of view. The first point of view is a macroscopic point of view where we consider features of the system at the human scale and try to study the physics of the system there. The other point of view is the microscopic point of view where we consider the interactions of each particle in the system and then from there can study the system. Now lets consider a specific system, a car engine(I don't know how these things work so lets take ZWD's word for how things go on here.) In the inital phase we can specify the configuration of matter in the engine by describing the mass and composition of the gasses. Once the enging combusts we can do the same for the resultant. The pressure and volume are also important quantities when describing the process of combustion. And finally we can also measure the temperature of this system. The lavender things I've described above are all macroscopic properties and can be thought of as macroscopic coordinates to describe the system. The branch of thermodynamics is the study of systems with these macroscopic properties with the special condition that we also must be studying systems with temperature. We could also study this system with a microscopic point of view by considering the bunch of particles flying around inside the heat engine and interacting with each other. These particles can take a bunch of energy levels $\epsilon_1, \ldots$ and we use probability theory to determine what is the most likely state for the entire bunch of particles which we call the equilibrium. Statistical mechanics then is the study of these distributions and used to determine the population(how many particles occupy a certain energy state at equilibrium) of the states. The two point of views are reconcilable.

Lets talk a bit more about equlibrium. Say we wave two system A and B described by macroscopic coordinates $X_{A,B}, Y_{A,B}$. We define a adiabatic wall as a boundary between these two systems making it impossible for them to communicate and thus making A and B come into equlibrium independently. On the other hand a diathermic wall is a boundry between two systems that allows them to communicate and thus A and B come into equlibrium together as a single system. Physically we can think of adiabtic walls as something like a thiccc slab of conrete while a diathermic wall as something like a thin sheet of metal. We can now imagine linking two systems A and B with an adiabatic wall and each system to a third system C with a diathermic wall. This whole system will come into equlibrium and afterwards if we replace the adiabatic wall between A and C with a diathermic wall, nothing will change. This is an experimental fact and translates to thermodynamics with the zeroth law of thermdynamics: if A is in equlibrium with B and B is in equlibrium with C then A is in equlibirum with C. Now to define temperature more concretley lets go back to two systems. Say we bring A and B into thermal equilibrium and they have coordinates $X_{(A,B)1}, Y_{(A,B)1}$. It turns out if we keep B the same then there exists a whole family of X_{Ai}, Y_{Ai} that can be in thermal equlibrium with B (and by the zeroth law in thermal equilibrium with eachother.) I know we haven't gotten to isothermal curves yet but just think of an isothermal curves. By symmetry if we keep X constant then there exists a whole family of X_{Bi}, Y_{Bi} with the same characteristic. Even more so each $X_{(A,B)i}, Y(A,B)i$ is in thermodynamic equilibrium with eachother by the zeroth law. These families that are in equlibrium with eachother have a unifying property which we shall call the temperature. We can go even further to say that

$$\begin{pmatrix}\text{Two systems have}\\ \text{the same temperature}\end{pmatrix} \Leftrightarrow \begin{pmatrix}\text{Systems are in}\\ \text{thermal equilibrium}\end{pmatrix}$$