PHY 396L: Quantum Field Theory II

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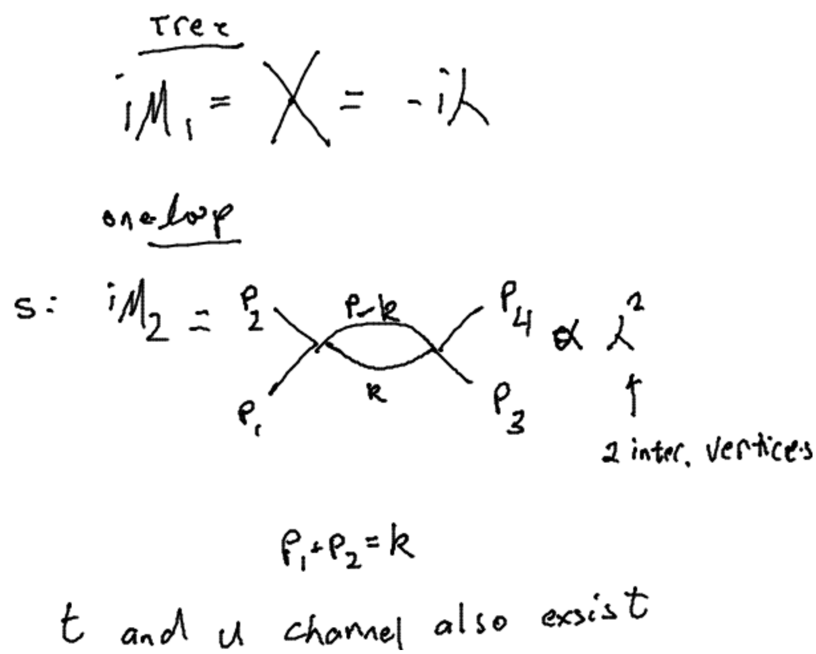
Notes for Prof. Kaplunovsky's Quantum Field Theory II course at UT Austin during Spring 2021. The official reference for the course is Peskin and Schroeder's *An Introduction to Quantum Field Theory* supplemented by Weinberg's first two volumes on QFT. However in practice we mostly follow Prof. Kaplunovsky's notes. I didn't type out any notes for QFT I and have no idea where my notebook for that course is. Also there will be some stuff in here from Prof. Maloney's QFT I and QFT II course. If you have any comments let me know at `hi@delonshen.com`.

Maloney QFT II Lecture 1: Systematics of Renormalization

Last term we focused on the leading terms in perturbation theory. If we want to understand this more deeply we have to go beyond the tree-level to loop corrections. We also saw that loop corrections often are (often unphysically) divergent if we don't regulate them somehow. Divergences only arise when we compute unphysical quantities, physical quantities are finite. To see this divergence lets consider the theory $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda}{4!}\phi^4$. We want to consider $\phi\phi \to \phi\phi$. The only vertex that contributes to this process is the seagull vertex ($i\mathcal{M}_1$). At the one-loop level there are s t and u one loop corrections. The s channel is shown in $i\mathcal{M}_2$



Now how do we compute the contribution of the $i\mathcal{M}_2$ diagram? We integrate over undetermined momenta

$$i\mathcal{M}_2 = \frac{(-i\lambda)^2}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2}\frac{i}{(p-k)^2}$$

This integral is kinda like d^4k/k^4 which diverges(TODO Maloney says "logarithmically divergent"[1]). So how do we deal with this? Well we could try cutting off $|k| < \Lambda$. We also know by lorentz invariance that the integral has to depend on the mandelstam variable $s = p^2$. Thus from a dimensional argument and from the fact that the integral is lambda divergent we can say that

$$i\mathcal{M}_2 \approx \log(s/\Lambda^2)$$

(TODO feels sketchy). Maloney tells us the answer that $i\mathcal{M}_2 = -\lambda^2\log(s/\Lambda^2)/32\pi^2$. The full matrix element is

$$\mathcal{M} = -\lambda - \frac{\lambda^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

This seems like a disaster since as the cutoff $\Lambda \to 0$ we get another divergence. To fix this we need to phrase this result in terms of physically observable quantities. The observable that we'll

[1] Okay so googling leads to this. $dU/U = d\log U$ and at large values the integral diverges logarithmically.

consider is the 4-pt function. How do we rephrase $\mathcal{M}$ in terms of an observable. We'll define the "physical" coupling constant λ_R as the matrix element for some s_0.

$$\lambda_R = -\mathcal{M}(s_0) = -\lambda - \frac{\lambda^2}{32\pi^2}\log(s_0/\Lambda^2)$$

Solving for λ we get

$$\lambda = \lambda_R - \frac{\lambda_R^2}{32\pi^2}\log(s_0/\Lambda^2) + \dots$$

Plugging this into our formula for $\mathcal{M}$ above is

$$\mathcal{M}(s) = -\lambda_R - \frac{\lambda_R^2}{32\pi^2}\log s/s_0 + \dots$$

What we can do now is relate two different scattering amplitudes. This generalizes to saying that in QFT we can only relate different observables to one another. So we could study QFT by looking at renormalized coupling from physical observables. But a simpler approach is counterterms. Consider $\mathcal{L} = -\frac{1}{2}\phi\partial^2\phi - \frac{\lambda_R}{4!}\phi^4 - \frac{\delta_\lambda}{4!}\phi^4$. The δ_λ is the counter term that asserts at each order of perturbation theory that λ_R is the matrix element for some specific s_0 for $2 \to 2$ process. Note that δ_λ when we write it out in terms of λ_R is order λ_R^2. So we have

$$\mathcal{M}(s) = -\lambda_R - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s/\Lambda^2 + \dots$$

Now again if we let $\lambda_R = -\mathcal{M}(s_0)$ and compute $\mathcal{M}(s_0)$ we get

$$\mathcal{M}(s_0) = \mathcal{M}(s_0) - \delta_\lambda - \frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \delta_\lambda = -\frac{\lambda_R^2}{32\pi^2}\log s_0/\Lambda^2 \Rightarrow \mathcal{M}(s) = -\lambda_R + \frac{\lambda_R^2}{32\pi^2}\log(s/s_0) + \dots$$

From this we can formulate a general strategy. For each coupling in $\mathcal{L}$ we introduce a counterterm to "absorb the divergence", fix the counterterm order by order in pert. theory to enforce a physical condition such as the definition of a physical coupling. Note that when we consider ϕ^4 theory the coupling constant is dimensionless.

Another example! $\mathcal{L} = -\frac{1}{2}\phi(\partial^2 + m^2)\phi$ we have

$$\langle 0|\phi|0\rangle = 0 \qquad \langle k|\phi(x)|0\rangle = e^{ikx}$$

Also k is on shell meaning that $k^2 = m^2$. In interacting QFT we can't describe the Hilbert space with a Fock space so we impose the conditions on the 0-particle and 1-particle states. (todo what?)

Explicitly lets consider ϕ^3 theory:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \frac{1}{3!}g\phi^3$$

Now if we want to shfit this theory we'll insert some terms

$$\mathcal{L} = \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_m m^2\phi^2 + Y\phi + \frac{1}{3!}Z_g g\phi^3$$

Where the Z renormalize constants and Y removes the 1-pt. function. These constants are fixed by four physical conditions.

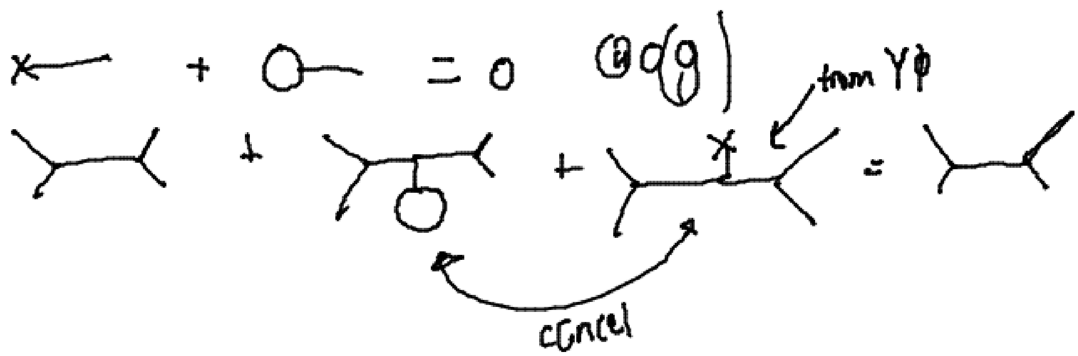
(a) Z_ϕ is fixed by the normalization of the one particle state $\langle k|\phi(x)|0\rangle = e^{ikx}$

(b) Y is fixed by $\langle 0|\phi(x)|0\rangle = 0$

(c) Z_g fixed by g = physical 3-pt funct at some energy

(d) Z_m fixed by the $(\text{mass})^2$ of a one particle state should be m^2. The physical mass is not necessarily m^2.

At tree level g^0 we have $Z_\phi = Z_m = Z_g = 1$ and $Y = 0$. At higher order in g we fix the recoupling constants order by order in perturbation theory in g. For example Y is fixed by the cancellation of the one point function. The existance of Y_ϕ implies some new "incoming vertex" and at first order Y is fixed by the fact that that new "incoming vertex" plus a loop is equal to 0. In practice we don't need to compute Y but instead just remember it cancels tadpole diagrams. This means that in any Feynman diagram expansions of a scattering amplitude we can ignore any diagram which has the property that if you cut one line in two then it will fall into two pieces one of which is not connected to any external vertex.



For other counterterms we have to do some computations.

Maloney QFT I Lecture 12 Part 1: Interactions

A free QFT where $S[\phi]$ is quadratice in ϕ has no interactions. Consider some sort of free theory with two kinds of particles a and b. Schematically lets have some Hamiltonian

$$H = a^\dagger a + b^\dagger b \Rightarrow a^\dagger|0\rangle \overset{e^{iHt}}{\rightarrow} a^\dagger|0\rangle \times \text{phase}$$

Basically for a free theory if you start with particle a, no matter how much time passes you'll still always have that particle a. For an interacting theory

$$H = a^\dagger a + b^\dagger b + \lambda(a(b^\dagger)^2 + a^\dagger b^2)$$

Where λ is small. The term in the parenthesis is the interaction term where the first term destroys the a particle and creates two b particle and the second term destroys the two b particles and gives us back the a particle(hermitian means process is reversible which is why we have both terms instead of just one term.) Solving this exactly should be a disaster but we can use approximation methods. First just by taylor expanding we can see that

$$a^\dagger|0\rangle \to e^{iHt}(a^\dagger|0\rangle) \approx a^\dagger|0\rangle + i\lambda t(b^\dagger)^2|0\rangle + \ldots$$

Basically there's some probably per unit time where a particle becomes two b particles. What we wanted to show here is that the non-quadratic terms that create interactions. There should be some things we should take away

(a) The vacuum state of $|0\rangle_{\text{free}}$ no longer works since. We defined this state as the state that's annihilated by lowering operators. However this is no longer the case. $|0\rangle_{\text{free}} \neq |0\rangle_{\text{interacting}}$. The true vacuum state $|\Omega\rangle$ is more complex in interacting theories.

(b) We shouldn't think of a and $a^\dagger$ as creating/destroying particles anymore.

To make things explicit lets consider a typical interacting theory

$$\mathcal{L} = \frac{1}{2}((\partial\phi)^2 - m^2\phi^2) - \mathcal{L}_{\text{int}} \Leftrightarrow \mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi)$$

Expanding $V(\phi)$ around $\phi = 0$ we get

$$V(\phi) = V_0 + V_1\phi + \frac{1}{2}m^2\phi^2 + \frac{g}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4$$

V_0 is ignorable. We can also let $\phi \to \phi + \text{const}$ so that $V_1 = 0$ Consider $\phi(x^\mu)$ where ϕ is slowly varying of $\mathbf{x}$.

$$\Rightarrow \mathcal{L} \approx \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Energy is minimized when ϕ sits at the minimum of $V(\phi)$. This means the ground state or "vacuum" where $V'(\phi_*) = 0$. So if $V_1 \neq 0$ then the field will realx to a value of $V(\phi)$ with $V'(\phi) = 0$ and if we want to study the properties near this vacuum we have a new field $\phi' = \phi - \phi_*$. So for now lets just choose vacuum so that in the vacuum $\phi = 0$. e.g. $\langle\Omega|\phi|\Omega\rangle = 0$.

Question: what interaction terms matter? Why should we care about ϕ^3 and not ϕ^{100}? We can answer this question with dimensional analysis

$$S = \int d^4x((\partial\phi)^2 - m^2\phi^2 + \cdots + \lambda_{ij}\partial^j\phi^i)$$

First we know in natural units $[x] = E^{-1}$ and thus $[\partial/\partial x] = E$. First action is dimensionless. Thus $[\phi] = E$ ($[d^4x\partial_x^2] = E^{-2}$). We can also see that $[m] = E$ and $[\lambda_{ij}] = E^{4-i-j}$. Now imagine we're doing some experiment which probes the theory at scale E. For example $\phi\phi \to \phi\phi$ with COM energy $\approx E$. What physical effects would come from our λ_{ij} term? well $\lambda_{ij}E^{i+j-4} = \lambda_{ij}E^{-[\lambda_{ij}]}$. What this means is that at low energies. we should only care about $[\lambda] > 0$ since if $[\lambda < 0]$ then E is raised to a positive power and thus is exponentially decreasing.

(a) $[\lambda] > 0$ is relevant

(b) $[\lambda] < 0$ is irrelevant

(c) $[\lambda] = 0$ is marginal

For a scalar theory at low energies the only relevant terms are kinetic terms and ϕ^3 and ϕ^4. Here's an idea from Landau: To study a system at low energy we can follow a recipe

(a) Guess DOF

(b) Guess symmetries

(c) Write down the most general theory (e.g. action)that satisfies the symmetries

(d) Study the relevant interaction terms.

To constrain our theory futher we should also ask what sort of symmetries we have? For example for the symmetry $\phi \to -\phi$ then there can't be a ϕ^3 term. We call this theory $\lambda\phi^4$ theory.

Maloney QFT I Lecture 13: Path Integrals

Thought I should take a look at this since QFT I at UT didn't approach Feynmann diagrams from path integrals.

We have some QM system with DOF q_i (e.g. for a field theory $\phi(\mathbf{x})$). We want to compute the transition from some inital time t_i with some configuration q_i to the final state (q_f, t_f).

$$\langle q_f, t_f | q_i, t_i \rangle$$

So we use the fact that we can always insert a complete set of basis states $|q,t\rangle$ at some intermediate time t where $t \in (t_i, t_f)$. So what happens? Let $|q_i, t_i\rangle = q_1, t_1$ and $|q_f, t_f\rangle = |q_n, t_n\rangle$. This gives us

$$\begin{aligned}
\langle q_f, t_f | q_i, t_i \rangle &= \int dq \, \langle q_f, t_f | q, t \rangle \langle q, t | q_i, t_i \rangle \\
&= \int dq_2 \ldots dq_{n-1} \, \langle q_n, t_n | q_{n-1}, t_{n-1} \rangle \ldots \langle q_2, t_2 | q_1, t_1 \rangle
\end{aligned}$$

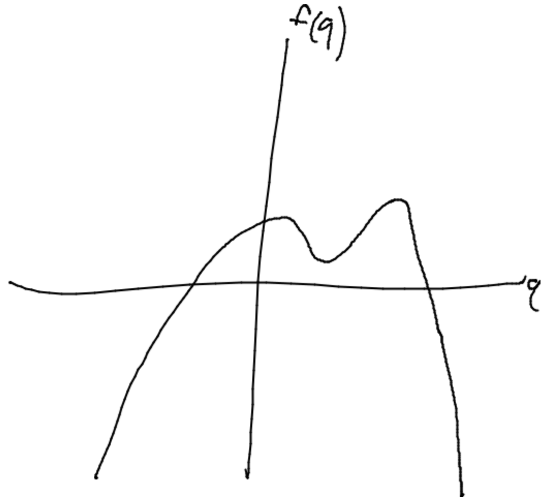
Basically we're integrating over all possible paths (including unphysical paths) from the inital state to the final state. So we can think of this transition amplitude as

$$\langle q_f, t_f | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq(t)$$

To make this idea precise we need to figure out how to integrate over a space of functions. The next thing we need to do is figure out what the fuck we're integrating. First lets just guess. In the limit $\hbar \to 0$ the integrand should be a function only of the classical solution of the EOM $\delta S/\delta q = 0$. First lets take a detour into approximating finite dimensional integrals which can be generalized to infinite dimensional integrals: the method of saddle points. Consider the one dimensional integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

And we want the integral to converge so f should look something like.



And we want to perform this integral when $\lambda \to \infty$. In this limit the integral should be dominated by the local maxima of $f(q)$. Namely by the q_* where $f'(q_*) = 0$. And what is this function Z in the limit where λ is large?

$$Z = \sum_{q_*} e^{\lambda f(q_*)}$$

It turns out there are a bunch of subleading corrections which are

$$Z = \sum_{q_*} e^{\lambda f(q_*)} \left(\sqrt{\frac{2\pi}{\lambda f''(q_*)}} + \dots \right)$$

(I proved first term in the next subsection). The higher order corrections are basically from Feynmann diagrams. Comparing this method of saddle point with the problem of path integrals we can guess that the integrand should be a function of $\frac{1}{\hbar}S$ (since the integrand does to infinity as $\hbar \to 0$ meaning that only things that solve $\delta S/\delta Q = 0$ contribute like in the method of steepest descent) and we can guess that

$$\langle q_n, t_n | q_i, t_i \rangle = \int Dq(t) \exp\left\{ \frac{i}{\hbar} S[q(t)] \right\}$$

Lets just check our work. Consider $H = \frac{1}{2}p^2 + V(q)$. Then

$$e^{-iH\delta t} = e^{-i(p^2/2+V(q))t} = e^{-i\frac{1}{2}p^2\delta t} e^{-iV(q)\delta t} e^{O(\delta t^2)}$$

Where the second equality comes from Cambell-Backer-Houstouder?? formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]+\dots}$$

Now lets insert an identity again in the following.

$$\langle q_n | e^{-iH\delta t} | q_{n-1} \rangle = \int dp_{n-1} \langle q_n | e^{-i\delta t p^2/2} | p_{n-1} \rangle \langle p_{n-1} | e^{-iV(q)\delta t} | q_{n-1} \rangle$$

Now we can use that $|p_{n-1}\rangle$ is a momentum eigenstate and $|q_{n-1}\rangle$ is a position eigenstate to replace the operators in the exponential with eigenvalues

$$\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle = \int dp_{n-1}\, \langle q_n|e^{-i\delta t p_{n-1}^2/2}|p_{n-1}\rangle \langle p_{n-1}|e^{-iV(q_{n-1})\delta t}|q_{n-1}\rangle$$

Taking out the identity we now just have

$$\begin{aligned}\langle q_n|e^{-iH\delta t}|q_{n-1}\rangle &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))}\langle q_n|p_{n-1}\rangle \langle p_{n-1}|q_{n-1}\rangle\\ &= \int dp_{n-1} e^{-i\delta t(p_{n-1}^2/2+V(q_{n-1}))} e^{ip_{n-1}(q_n-q_{n-1})}\\ &= \int dp_{n-1} e^{-i\delta t H(q_{n-1},p_{n-1})} e^{ip_{n-1}(q_n-q_{n-1})}\end{aligned}$$

So what we see is that

$$\begin{aligned}\langle q_n, t_n|q_i, t_i\rangle &= \int dq_2 \dots dq_{n-1}\, \langle q_n|e^{-iH\delta t}|q_{n-1}\rangle \dots \langle q_2|e^{-iH\delta t}|q_1\rangle\\ &= \int dq_2 \dots dq_{n-1} dp_2 \dots dp_{n-1} \prod_{i=2}^{n-1} e^{-i\delta t H(q_i,p_i)} e^{ip_i(q_{i+1}-q_i)}\\ &= \int dq_2 \dots dq_{n-1} dp_2 \dots dp_{n-1} \exp\left\{ i\sum_{i=2}^{n-1}\left(\frac{p_i}{\delta t}(q_{i+1}-q_i) - H(q_i,p_i)\right)\delta t\right\}\end{aligned}$$

Now in the limit when $n \to \infty$ then $\delta t \to 0$. This gives us

$$\langle q_n, t_n|q_i, t_i\rangle = \int DqDp \exp\left\{ i\int dt\,(p\dot{q} - H = L)\right\}$$

This is a path integral over phase space, not just position(configuration) space. We can reduce this to an integral in configuration space by noting that the integral over phase space is a gaussian integral. Note that

$$\int dp_i e^{i(-p_i^2/2-p_i(q_i+1-q_i)/\delta t)\delta t} = \text{const} \times e^{i\dot{q}_i^2/2}$$

Some comments about this derivation:

(a) $\int Dq$, the integral over a space of functions is not a well defined objectet

(b) This derivation makes manifest ideas of interference and symmetries (e.g. double slit experiment)

(c) Useful in perturbation theory.

(d) Not useful for calculations. Not really necessary in QM but necessary in QFT.

So how do we compute expectation values? Let $Q(t)$ operator that measure $q(t)$.

$$\begin{aligned}\langle q_n, t_n | Q(t) | q_1, t_1\rangle &= \int dq q(t) \langle q_n, t_n | q, t\rangle \langle q, t | q_1, t_1\rangle \\ &= \int_{q(t_1)=q_1}^{q(t_n)=q_n} Dq e^{iS[q]} q(t)\end{aligned}$$

Now what happens if weant to compute a two point function

$$\int Dq e^{iS[q]} q(t) q(t') = \Theta(t'-t)\langle q_n, t_n | Q(t') Q(t) | q_1, t_1\rangle + \Theta(t-t')\langle q_n, t_n | Q(t) Q(t') | q_1, t_1\rangle$$

Namely the thing that the path integral computes is time ordered expectation values. Thus note that if we have many observables

$$\int Dq e^{iS[q]} q_1 \ldots q_n = \langle | \mathbf{T} q_1 \ldots q_n | \rangle$$

So far we've considered position eigenstates. In a more general state

$$\psi = \int dq \, |q\rangle \langle q | \psi\rangle$$

To compute VEV:

$$\langle 0 | \mathbf{T}(Q_1 \ldots Q_n) | 0\rangle = \lim_{\substack{t_f \to -\infty \\ t_i \to \infty}} \int dq_i dq_f \psi_0(q_i) \psi_0^*(q_f) \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS} q_1 \ldots q_n$$

Theres' a trick to avoid this

$$|q, t\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n | q, t=0\rangle e^{iE_n t}$$

Now let $t \to (1 - i\epsilon)t$ and $t \to -\infty$ (very early times). Then $e^{iE_n t} \to e^{i(1-i\epsilon)tE_n}$. Thus we will pick out terms near the vacuum energy. Thus

$$|q, t\rangle \to \langle 0 | q, 0\rangle |0\rangle$$

This analytic continuation projects onto the goround state for ket at $t \to -\infty$ and for bra this projects onto the ground state for $t \to \infty$. So then we can compute the VEV as

$$\langle 0 | \mathbf{T} Q_1 \ldots Q_n | 0\rangle = \int Dq \exp\{iS[q]\} q_1 \ldots q_n$$

But where $S = \int d\tilde{t} L$ where $\tilde{t} = (1 - i\epsilon)t$. Sometimes we take this analytical continuation where $\tilde{t} = it$. This is a "Euclidean continuation." The reason that this is called "euclidean" comes from the following. Consider the invariant interval in special relativity. Under that analytical continuation

$$ds^2 = dt^2 - d\mathbf{x}^2 \to -(d\tilde{t}^2 + d\mathbf{x}^2)$$

It's useful in finite temperature physics.

SOME POINTS ABOUT THE METHOD OF SADDLE POINTS

Based on Problem Set 7. We're considering that integral

$$Z = \int_{-\infty}^{\infty} dq e^{\lambda f(q)}$$

To derive the first correction we'll expand $f(q)$ around the saddle points $\{q_*\}$ where $f'(q_*)$ is zero.

$$f(q) \approx f(q_*) + (f'(q_*)(q-q_*) = 0) + \frac{f''(q_*)}{2}(q-q_*)^2 + \dots$$

This gives us the integral (after we note that at the saddle point $f''(q_*)$ is negative)

$$Z \approx e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{\lambda f''(q_*)(q-q_*)^2/2} dq = e^{\lambda f(q_*)} \int_{-\infty}^{\infty} e^{-\lambda |f''(q_*)|(q-q_*)^2/2} dq$$

OwO whats this? *Notices your Gaussian integral.* We can do a change of variables

$$u = \sqrt{\frac{\lambda |f''(q_*)|}{2}}(q-q_*) = C(q-q_*) \Rightarrow du = C dq$$

Now we can evaluate the integral with the standard $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

$$\boxed{Z \approx \sqrt{\frac{2\pi}{\lambda |f''(q_*)|}} e^{\lambda f(q_*)}}$$

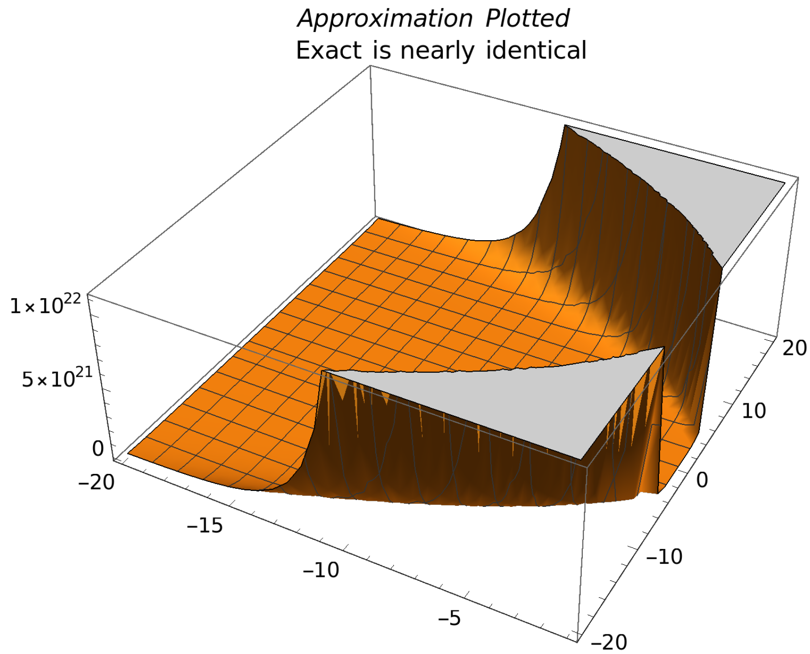
This prefactor we just defined in the context of quantum mechanics is the one-loop correction. Lets see our boy in action. Consider the Gaussian integral where $a < 0$

$$\int_{-\infty}^{\infty} dq\, e^{aq^2+bq+c} = \sqrt{\frac{\pi}{|a|}} e^{c-b^2/4a}$$

We can find the approximation with our saddle point method

$$f'(q_*) = 2aq_* + b = 0 \Rightarrow q_* = -\frac{-b}{2a} \Rightarrow \int_{-\infty}^{\infty} dq\, e^{aq^2+bq+c} \approx \sqrt{\frac{\pi}{|a|}} e^{c+bq+aq^2}$$

Lets fix $c = 42$ and plot both the approximate and the exact answer for some values of a and b just to get a feel for this approximation



It's a good approximation!

Lecture 1: Amputating Bad Legs and Feynman's Cool Trick to Evaluate Integrals

January 19, 2021

Lets start by considering $\lambda\phi^4$ theory and a $2 \to 2$ scattering. The corresponding Feynman diagrams up to 1-loop level is found in Figure 1 Why are the bad diagrams bad? Well consider one of the bad diagrams

TODO Figure

Well $q_2 = p_1$ for any q_1. So the integral becomes

TODO Integral, tl;dr divergence

Namely all of them have bad propagators. Going further we can look at the two-loop example and see that the exact same thing happens. These bad features are called *external leg bubbles.* All diagrams with external leg bubbles are bad, the propagator is frozen on shell and thus blows up.

Lets look at a practical solution. He'll give us a half-assed justification for this solution and then will talk about what's really going on next week. The solution is: *amputate all the external leg bubbles.* To do this we need to figure out how we find these external leg bubbles and the carefully figure out what do we need to amputate. If cutting 1 propagator at a connected diagram breaks it into two disconnected pieces and if one of the pieces has just one external leg

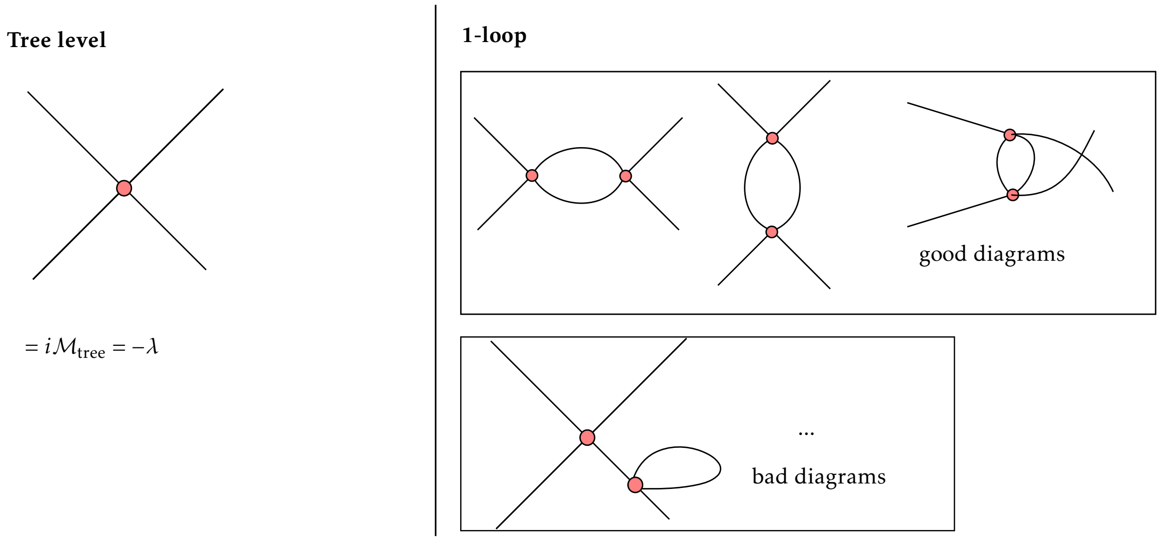
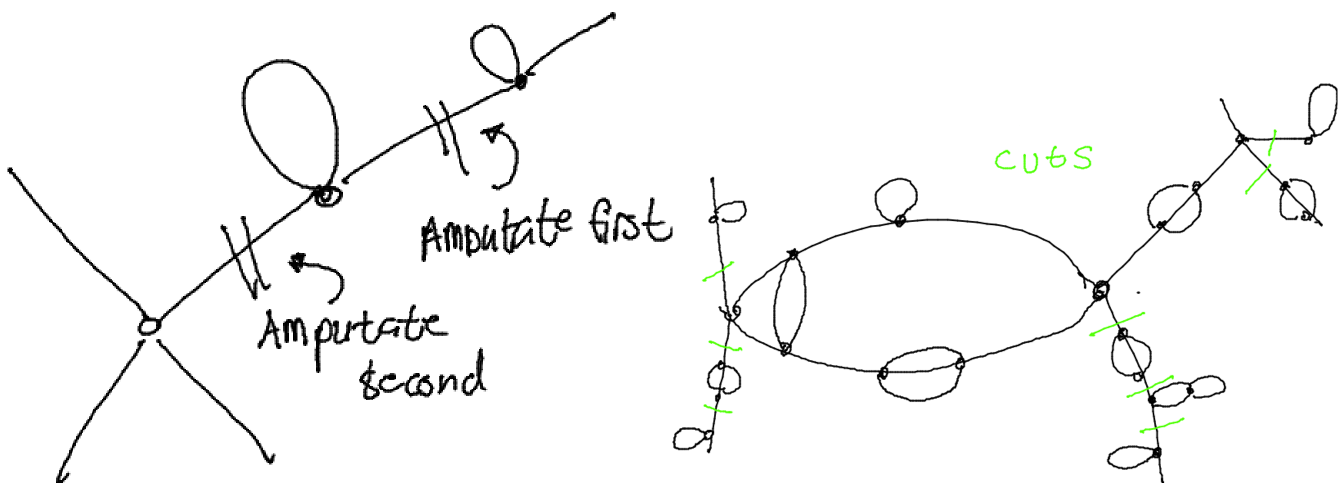


Figure 1: Illustraction of good and bad diagrams in a $2 \to 2$ proceess for $\lambda\phi^4$ theory.

then amputate that piece.



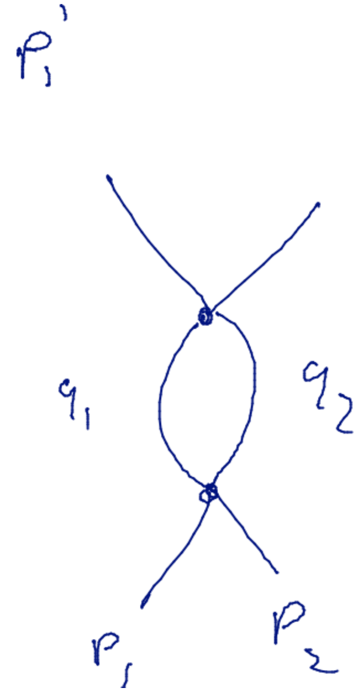
The bottom line is that

$$i\mathcal{M}_{\text{scattering}} = \sum \text{amputated connected diagrams}$$

So where does this rule come from. Well last semester when we were looking at S-matrix elements we were given a formula that was unjustified

$$\langle p_1', p_2', \ldots |\hat{S}| p_1, p_2, \ldots \rangle = \text{Limit regulazation going away} \left[C_{\text{vac}} \times \prod_{\text{ext legs}} F(\text{leg}) \times \langle 0 | a_{p_1'} a_{p_2'} \hat{S} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle \right]$$

Now note that free matrix elemnts is the sum over all Feynmann diagrams. The C_{vac} cancels all vacuum bubbles. What we want to show soon is that $F(\text{ext leg})$ cancels all leg bubbles in that leg. So in the end we're left with diagrams that are unproblematic. This is the "justification" of the no external leg bubble rule. Now the right way to derive this rule is to calculate the correlation function of multiple field, then get (some name) action formula, and then we'll get out result. It turns out that off shell the external leg bubbles are sometimes useful. So now lets go back to the very first diagrams we wrote down and figure out how to calculate loop diagrams.



$$\Rightarrow q_1 + q_2 = p_1 + p_2 = p_1' + p_2'$$

Depends only on the net $p_1 + p_2$ meaning that the matrix element $iF(s)$ only depends on $(p_1 + p_2)^2 = s$. Thus

$$i\mathcal{M}_{1-\text{loop}} = iF(s) + iF(t) + iF(u)$$

By crossing symmetry if we find one of them we find all of them. For example

$$iF(t) = \frac{1}{2}(-i\lambda)^2 \int \frac{d^4 q_1}{(2\pi)^4} \frac{i}{q_1^2 - m^2 + i\epsilon} \times \frac{i}{(q_2 = q_{\text{net}} - q_1)^2 - m^2 + i\epsilon}$$

Where $q_{\text{net}} = p_1' - p_1 = p_2 - p_2'$ and $q_{\text{net}}^2 = t$. So how do we perform integrals like this? We'll use what's called the *Feynman Parameter Trick.*

Using Feynman's Parameter Trick

Consider the following integral

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A + xB]^2}$$

This is true because $(1-x)A + xB$ interpolates between A and B. This integral was know ages before Feynman. However Feynman made good use of it.

$$\frac{1}{q_1^2 - m^2 + i\epsilon} \times \frac{1}{q_1^2 - m^2 + i\epsilon} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i\epsilon) + x(q_2^2 - m^2 + i\epsilon)]^2}$$

So looking inside the square bracket we can see that

$$\begin{aligned}
[(1-x)(q_1^2-m^2+i\epsilon)+x(q_2^2-m^2+i\epsilon)] &= (1-x)q_1^2+xq_2^2-m^2+i\epsilon \\
&= (1-x)q_1^2+xq_1^2-2x(q_{\text{net}}q_1)+xq_{\text{net}}^2-m^2+i0 \\
&\text{complete the square in the } 2x \text{ term} \\
&= (q_1-xq_{\text{net}})^2-x^2q_{\text{net}}^2+xq_{\text{net}}^2+m^2+i0 \\
&= (q_1-xq_{\text{net}})^2-\Delta(x)+i0
\end{aligned}$$

Where we define $\Delta(x)=m^2+(x^2-x)q_{\text{net}}^2=m^2+(x^2-x)t$. This means that

$$\frac{1}{q_1^2-m^2+i0}\times\frac{1}{q_2^2-m^2+i0}=\int_0^1\frac{dx}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2}$$

So now we can write $iF(t)$ more cleanly

$$\begin{aligned}
iF(t) &= \frac{1}{2}\lambda^2\int\frac{d^4q_1}{(2\pi)^4}\int_0^1\frac{dx}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2} \\
&= \frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4q_1}{(2\pi)^4}\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x)+i0]^2}
\end{aligned}$$

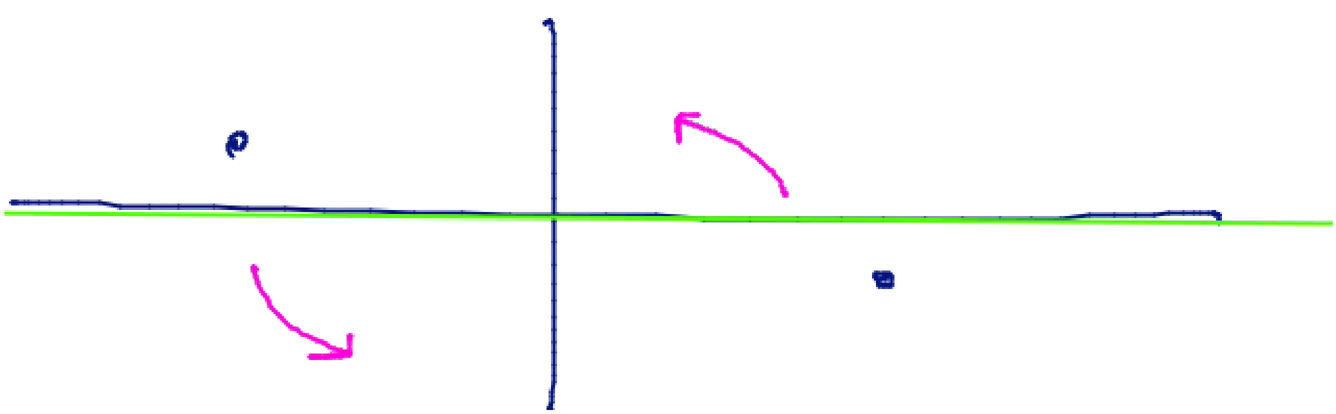
Changing the order of integration is a bit suspect here but at this level it turns out this actually works. Now once we've changed the order of integration for each x change momentum integration variable form q_1 to $k=q_1-xq_{\text{net}}$. This gives us

$$\boxed{iF(t)=\frac{\lambda^2}{2}\int_0^1 dx\int\frac{d^4k}{(2\pi)^2}\frac{1}{[k^2-\Delta(x)+i0]^2}\qquad\text{where}\qquad\Delta(x)=m^2-tx(1-x)>0}$$

Now how do we perform this integral? Me thinks residual. Lets focus on the d^4k integral

$$\int\frac{d^4k}{(2\pi)^2}\frac{1}{[k^2-\Delta(x)+i0]^2}=\int d^3\mathbf{k}\int dk_0\frac{1}{[k_0^2-\mathbf{k}^2-\Delta(x)+i0]^2}$$

The $\int dk_0$ integral has double poles at $k_0=\pm[\sqrt{\mathbf{k}^2+\Delta}-i\epsilon]$. So lets deform the contour in the complex k_0 plane. Also note that as $k_0\to\infty$ the term is negligible since the integral is about proportional to k_0^{-4}. So we may change something? can't read hand writing. Now we can use Wick rotations.



Note that we rotate CCW since if we rotated CW then we'd hit the poles and they don't like being hit. So let $k_0 = ik_4$ for real k_4 running from $-\infty$ to ∞. So what does this give us?

$$k_\mu k^\mu - \Delta = k_0^2 - \mathbf{k}^2 - \Delta = \underline{-k_4^2 - \mathbf{k}^2} - \Delta$$

Notice the sign is the same in the underlined section. It becomes the "euclidean continuation?." So we can combine the 3-vector $\mathbf{k}$ and k_4 into a euclidean 4-vector $k_E = (k_1, k_2, k_3, k_4)$.So $k^2 = g_{\mu\nu} k^\mu k^\nu = k_0^2 - \mathbf{k}^2$ becomes $-k_E^2 = -k_4^2 - \mathbf{k}^2$. At the same time $dk_0 = idk_4$ so Minkowski $d^4k = id^4k_E$. And with this setup we get the integral

$$\begin{aligned}\int \frac{d^4 k_{\text{minkowski}}}{[k^2 - \Delta + i0]^2} &= \int d^3\mathbf{k} \int \frac{dk_0}{\dots} \\ &= \int d^3\mathbf{k} \int \frac{idk_4}{[-k_4^2 - \mathbf{k}^2 - \Delta]} \\ &= i \int \frac{d^4 k_E}{[k_E^2 + \Delta]^2}\end{aligned}$$

Where in the second equality since everything is negative we can get rid of the $i\epsilon$ perscription. The bottom line is

$$\boxed{F(t) = \frac{\lambda^2}{2} \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2}}$$

So the $SO^+(3,1)$ Lorentz symmetry of the Minkowski space analytically continues to $SO(4)$ rotation symmetry in $D = 4$ euclidean space under a wick rotation by $\pi/2$. We can go even futher and change variables from $d^4k_E = dk_E^{\text{real}} \times (k_E^{\text{rot?}})^3 \times d^3\Omega_k$ where Ω is solid angle which means $\int_{D=4} d^3\Omega_k = 2\pi^2$ and so $\int d^4k_E \to \int 2\pi^2 k_E^3 dk_E$ meaning that our integral is

$$\int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \frac{2\pi^2}{16\pi^4} \int_0^\infty \frac{k_E^3 dk_E}{[k_E^2 + \Delta]^2}$$

There is one subtlety however. AT $k_E \to \infty$ we know

$$\frac{k_E^3}{[k_E^2 + \Delta]^2} \approx \frac{1}{k_E} \Rightarrow \frac{dk_E}{k_E} = d \log k_E \text{ is logarithmically divergent}$$

Aside: Generalizations for Integral Used Feynman Parameter Trick

We proved some generalization of Feynman's Parameter Trick on the problem set. They're not too hard to prove for anyone with some free time. I'll put them here for reference.

$$\frac{1}{A^nB} = \int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} \tag{F.a}$$

$$\frac{1}{A^nB^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!} \times \int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}} \tag{F.b}$$

$$\begin{aligned}\frac{1}{ABC} &= \int_0^1 dx \int_0^{1-x} \frac{2dy}{[xA+yB+(1-x-y)C]^3} \\ &= \int_{x,y,z\geq 0} dxdydz\ \delta(x+y+z-1) \times \frac{2}{[xA+yB+zC]^3}\end{aligned} \tag{F.c}$$

$$\begin{aligned}\frac{1}{A_1 \dots A_k} &= \int\limits_{x_1 \dots x_k \geq 0} \dots \int d^kx\ \delta(x_1+\cdots+x_k-1) \times \frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^k} \\ &= \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \dots \int\limits_0^{1-x_2-\dots-x_{k-2}} dx_{k-1} \frac{(k-1)!}{[x_1A_1+\cdots+(1-x_2-\dots-x_{k-1}A_k]^k}\end{aligned} \tag{F.d}$$

$$\frac{1}{A_1^{n_1}A_2^{n_2}\dots A_k^{n_k}} = \frac{n_1+\cdots+n_k-1)!}{(n_1-1)!\dots(n_k-1)!} \times \tag{F.e}$$

$$\times \int\limits_{x_1,\dots,x_k\geq 0} \dots \int d^kx\ \delta(x_1+\cdots+x_k-1) \times \frac{x_1^{n_1-1}\dots x_k^{n_k-1}}{[x_1A_1+\cdots+x_kA_k]^{(n_1+\cdots+n_k)}} \tag{F.f}$$

(F.a) We wish to verify

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[xA+(1-x)B]^2}$$

Let $u = [xA+B-Bx]$ which gives us $du = (A-B)dx$. The bounds go from B to A. The integral then becomes

$$\boxed{\frac{1}{A-B}\int_B^A \frac{du}{u^2} = -\frac{1}{A-B}\left(\frac{B-A}{AB}\right) = \frac{1}{AB}}$$

(F.b) We wish to verify

$$\frac{1}{A^nB} = \int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}}$$

Lets do the same change of variables we did in (F.a). Also note that $x = (u-B)/(A-B)$.

$$\begin{aligned}
\int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} &= \frac{n}{(A-B)^n}\int_B^A \frac{(u-B)^{n-1}du}{u^{n+1}} \\
&= \frac{n}{(A-B)^n}\int_B^A \frac{du}{u^2}\left(1-\frac{B}{u}\right)^{n-1} \\
&\text{Let } w = \frac{1}{u} \Rightarrow dw = -u^{-2}du \\
&= -\frac{n}{(A-B)^n}\int_{1/B}^{1/A} dw\,(1-Bw)^{n-1} \\
&= \frac{1}{B(A-B)^n}(1-Bw)^n\Big|_{1/B}^{1/A} \\
&= \frac{1}{B(A-B)^n}(1-B/A)^n \\
&\boxed{\int_0^1 \frac{nx^{n-1}dx}{[xA+(1-x)B]^{n+1}} = \frac{1}{A^nB}}
\end{aligned}$$

(F.c) We wish to verify

$$\frac{1}{A^nB^m} = \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\int_0^1 \frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}} = Z$$

Lets do the same change of variables again

$$\begin{aligned}
Z &= \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_B^A \frac{(u-B)^{n-1}(A-u)^{m-1}du}{u^{n+m}} \\
&= \frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_B^A \frac{1}{u^2}(1-B/u)^{n-1}(A/u-1)^{m-1}du \\
&= -\frac{(n+m-1)!}{(n-1)!(m-1)!}\times\frac{1}{(A-B)^{n+m-1}}\int_{1/B}^{1/A}(1-Bw)^{n-1}(Aw-1)^{m-1}dw
\end{aligned}$$

Now lets integrate by parts (using the tabular method)

$$\begin{array}{c|c}
(1-Bw)^{n-1} & (Aw-1)^{m-1} \\
-B(n-1)(1-Bw)^{n-2} & \frac{1}{Am}(Aw-1)^m \\
\vdots & \vdots \\
(-1)^{n-1}B^{n-1}(n-1)! & \frac{1}{A^{n-1}m(m+1)\ldots(m+n-2)}(Aw-1)^{m+n-2}
\end{array}$$

Because of our bounds each non-integral term vanishes. Also note the sign of each term from integration by parts is determined by $(-1)^{n+1}$. Thus the final integral is positive.

This leaves us with

$$\begin{aligned}\int_{1/B}^{1/A}(1-Bw)^{n-1}(Aw-1)^{m-1}dw &= \frac{B^{n-1}(n-1)!}{A^{n-1}m(m+1)\dots(m+n-2)}\int_{1/B}^{1/A}(Aw-1)^{m+n-2}dw \\ &= -\frac{B^{n-1}(n-1)!}{A^{n}m(m+1)\dots(m+n-1)}(A/B-1)^{m+n-1} \\ &= -\frac{B^{n-1}(n-1)!}{A^{n}B^{m+n-1}m(m+1)\dots(m+n-1)}(A-B)^{m+n-1} \\ &= -\frac{(n-1)!(m-1)!}{A^{n}B^{m}(m+n-1)!}(A-B)^{m+n-1}\end{aligned}$$

Plugging this integral back in gives us

$$\boxed{\frac{(n+m-1)!}{(n-1)!(m-1)!}\times\int_0^1\frac{x^{n-1}(1-x)^{m-1}dx}{[xA+(1-x)B]^{n+m}}=\frac{1}{A^nB^m}}$$

(F.d) We can evaluate this in a straightforward manner

$$\begin{aligned}Z &= -\frac{1}{B-C}\int_0^1 dx\Big\{[xA+(1-x)B]^{-2}-[xA+(1-x)C]^{-2}\Big\} \\ &= \frac{1}{B-C}\left\{\frac{1}{A-B}[xA+(1-x)B]^{-1}-\frac{1}{A-C}[xA+(1-x)C]^{-1}\right\}\Bigg|_0^1 \\ &= \frac{1}{B-C}\left\{\frac{1}{A(A-B)}-\frac{1}{A(A-C)}-\frac{1}{B(A-B)}+\frac{1}{C(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{BC(A-C)-BC(A-B)-AC(A-C)+AB(A-B)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{(A-C)(BC-AC)+(A-B)(AB-BC)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{C(A-C)(B-A)+B(A-B)(A-C)}{ABC(A-B)(A-C)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{-C(A-B)+B(A-B)}{ABC(A-B)}\right\} \\ &= \frac{1}{B-C}\left\{\frac{-C+B}{ABC}\right\} \\ &= \frac{1}{ABC}\end{aligned}$$

(F.e) We know the given integral is equivalent to

$$Z=\int_0^1 dx_1\int_0^{1-x_1}dx_2\dots\int_0^{1-x_1-\dots-x_{n-2}}dx_{n-1}\frac{(n-1)!}{[x_1A_1+\dots+(1-x_1-\dots-x_{n-1})A_n]^n}$$

From our result above we know that evaluating all the integrals will result in a factor of $1/(n-1)!$. This is because of the exponent in the dnominator. From examing (F.d) we also note that after each integral something of the following form

$$\begin{aligned}\int_0^{1-x_1-\ldots-x_{n-2}} dx_{n-1}\frac{(n-1)!}{[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-1})A_n]^n} =& \\ -\frac{(n-2)!}{(A_{n-1}-A_n)[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-2})A_{n-1}]^{n-1}}& \\ +\frac{(n-2)!}{(A_{n-1}-A_n)[x_1A_1+\cdots+(1-x_1-\cdots-x_{n-2})A_n]^{n-1}}&\end{aligned} \tag{1}$$

I think we can analyze this with recursion. Lets define the following

$$\begin{aligned}Z(\{A_1,\ldots,A_{n-2},A_{n-1},A_n\},n) =& \\ \int_0^1 dx_1\int_0^{1-x_1}dx_2\ldots\int_0^{1-x_1-\ldots-x_{n-2}}dx_{n-1}\frac{(n-1)!}{[x_1A_1+\ldots x_{n-1}A_{n-1}+(1-x_1-\ldots-x_{n-1})A_n]^n}&\end{aligned} \tag{2}$$

We see from (1)

$$Z(\{A_1,\ldots,A_{n-1},A_n\},n) = \frac{-1}{A_{n-1}-A_n}\{Z(\{A_1,\ldots,A_{n-2},A_{n-1}\},n-1)-Z(\{A_1,\ldots,A_{n-2},A_n\},n-1)\}$$

Also from (F.d) we see that

$$Z(\{A_1,A_2\},2) = -\frac{1}{A_1-A_2}\left(\frac{1}{A_1}-\frac{1}{A_2}\right)$$

Now instead of being rigorous (I'm just trying to convince myself afterall) we'll just check this recursion relation with mathematica. Consider the following code which implements this recursive definition.

```
In[105]:= Clear[f]
          f[A_, 2] := -1 / (A[[1]] - A[[2]]) (1 / A[[1]] - 1 / A[[2]])
          f[A_, n_] := -1 / (A[[n - 1]] - A[[n]]) (f[A[[1 ;; n - 1]], n - 1] - f[Join[A[[1 ;; n - 2]], {A[[n]]}], n - 1])
          Table[f[Table["A" <> ToString[i], {i, 1, maxn}], maxn] // FullSimplify, {maxn, 2, 10}]
```

```
Out[108]= {1/(A1 A2), 1/(A1 A2 A3), 1/(A1 A2 A3 A4), 1/(A1 A2 A3 A4 A5), 1/(A1 A2 A3 A4 A5 A6), 1/(A1 A2 A3 A4 A5 A6 A7),
           1/(A1 A2 A3 A4 A5 A6 A7 A8), 1/(A1 A2 A3 A4 A5 A6 A7 A8 A9), 1/(A1 A10 A2 A3 A4 A5 A6 A7 A8 A9)}
```

In fact we can do better than just check it with mathematica. We're given

$$\frac{1}{A_1\ldots A_k} = \int\ldots\int_{x_1,\ldots,x_k\geq 0} d^kx\ \delta(x_1+\cdots+x_k-1)\times\frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^k}$$

And we've proven a base case of $k=3$. From here we just need to prove that if the relationship is satisfied for k then it also satisfies $k+1$. Consider

$$\frac{1}{A_1\ldots A_{k+1}} = \int\ldots\int_{x_1,\ldots,x_k\geq 0} d^kx\ \delta(x_1+\cdots+x_k-1)\times\frac{(k-1)!}{[x_1A_1+\cdots+x_kA_k]^kA_{k+1}}$$

We use (F.b) to get

$$\frac{1}{A_1 \dots A_{k+1}} = \int_{x_1,\dots,x_k \geq 0} \dots \int d^k x \int_0^1 dy\ \delta(x_1 + \dots + x_k - 1) \times \frac{k! y^{k-1}}{[y(x_1 A_1 + \dots + x_k A_k) + (1-y)A_{k+1}]^{k+1}}$$

Let $yx_i = z_i$ and $(1-y) = z_{k+1}$ meaning that $dx_i = dz_i/y$ and $dy = -dz_{k+1}$ and the limits in the innermost integral go from 1 to 0 allowing us to change sign in order to keep the prefactor positive.

$$\begin{aligned}\frac{1}{A_1 \dots A_{k+1}} = &\int_{z_1,\dots,z_k \geq 0} \dots \int d^k z \int_0^1 dz_{k+1}\ \delta(y^{-1}(z_1 + \dots + z_k + z_{k+1} - 1)) \\ &\times \frac{k! y^{-1}}{[z_1 A_1 + \dots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}}\end{aligned}$$

Now we use $\delta(ax) = \delta(x)/|a|$ to get

$$\begin{aligned}\frac{1}{A_1 \dots A_{k+1}} = &\int_{z_1,\dots,z_k \geq 0} \dots \int d^k z \int_0^1 dz_{k+1}\ \delta(z_1 + \dots + z_k + z_{k+1} - 1) \\ &\times \frac{k!}{[z_1 A_1 + \dots + z_k A_k + z_{k+1} A_{k+1}]^{k+1}}\end{aligned}$$

The last thing we want to show is that the bounds can equivalently be stated as z_{k+1} is positive. First we note that the delta function asserts

$$z_1 + \dots + z_{k+1} = 1$$

And the other bounds of the integral assert $z_1, \dots z_k \geq 0$. Now considere the situation when $z_{k+1} > 1$. Then we have

$$z_1 + \dots + z_{k+1} > z_1 + \dots + z_k + 1$$

From here we see that no combination of positive $z_1 \dots z_k$ will satisfy the condition that $z_1 + \dots + z_{k+1} = 1$ meaning that there is no contribution for $z_{k+1} > 1$. Thus the bounds of the innermost integral can be restated as $z_{k+1} \geq 0$ since any contribution from $z_{k+1} > 1$ is killed by the delta function. This gives us

$$\frac{1}{A_1 \dots A_{k+1}} = \int_{z_1,\dots,z_{k+1} \geq 0} \dots \int d^{k+1} z \delta(z_1 + \dots + z_{k+1} - 1) \frac{k!}{[z_1 A_1 + \dots + z_{k+1} A_{k+1}]^{k+1}}$$

Therefore we've proven the inductive hypothesis thus completing the proof for $(F.e)$.

(F.f) You can just get this result by taking the partial derivative of $(F.e)$ a million times. Typing that out seem wasteful.

Maloney QFT I Lecture 14: Feynman Diagrams I

In this lecture we finally become men? To recap in QM we can compute observables with the following

$$\langle q_f, t_f | T Q(t_q) \dots Q(t_n) | q_i, t_i \rangle = \int_{q(t_i)=q_i}^{q(t_f)=q_f} Dq e^{iS/\hbar} \, q(t_i) \dots q(t_n)$$

So for example if we're doing a double slit experiment and want to measure which slit the the particle goes through we'll insert an operator q in the integral. We can also rephrase this integral as a integral through phase space

$$= \int DqDp \, \exp\left\{ i \int (p\dot{q} - H(p,q))dt \right\} q(t_1) \dots q(t_n)$$

In the end of class we introduced a trick to compute correlcation function instead of transition amplitudes. Since

$$e^{it(1-i\epsilon)H} |\psi\rangle = |0\rangle\langle 0|\psi\rangle \text{ at } t \to -\infty$$

Similarly we could apply this to a bra-vector

$$\langle \psi | e^{it(1-i\epsilon)H} = \langle \psi | 0 \rangle \langle 0 | \text{ at } t \to \infty$$

This means if you want to calucate VEV

$$\langle 0 | T Q(t_1) \dots Q(t_n) | 0 \rangle \propto \int DqDp \, \exp\left\{ i \int (\dot{q}p - (1 - i\epsilon)H)dt \right\} q(t_1) \dots q(t_n)$$

Effectivley what we're doing is adding a small imaginary part to the energies and basically projects us onto the ground state. At the tree level this $i\epsilon$ factor isn't very important but when we do loop calculation this $i\epsilon$ factor becomes important. Something to emphasize the equation above is that we have propto instead of equality since we ignored some normalizations. To get equality we could do

$$\langle 0 | \dots | 0 \rangle = \frac{\int Dq \; e^{iS/\hbar} q(t_1) \dots q(t_n)}{\int Dq \; e^{iS/\hbar}}$$

Where the denominator is there to keep $\langle 0|0\rangle = 0$.

To generalize this to many DOF $q_i(t) \to \int Dq_i(t)$. In a QFT where the DOf is a local $\phi(t, \mathbf{x}) \to \int D\phi(\mathbf{x}, t)$ (integrate over all possible field configurations ϕ). Today our goal is to compute

$$\langle 0 | T\phi(x_1) \dots \phi(x_n) | 0 \rangle = \frac{\int D\phi \; e^{iS} \phi(x_1) \dots \phi(x_n)}{\int D\phi \; e^{-S}}$$

To do this lets define the partition function or generating function of a QFT. We call J the source

$$Z[J(x)] = \int D\phi \; e^{iS + i\int d^4x \; J(x)\phi(x)}$$

It should be clear that if we computed the EOM, $J(x)$ would be a source term on the RHS of the EOM. An if we wanted to we can expand the exponential

$$Z[J(x)] = \int D\phi\ e^{iX}\left(1 + i\int d^4xJ(x)\phi(x) - \int d^4x\ d^4y\ J(x)J(y)\phi(x)\phi(y) + \dots\right)$$

This partition function includes all possible expectation values that we care about. First lets consider how we could get expectation values from this partition function. Well lets remeber how we extract single terms from a power series. We take the derivative wrt the variable and then set that variable to zero. For example $e^x = c_0 + c_1x + c_2x^2 + \dots$, $de^x/dx = c_1 + 2c_2x + \cdots \Rightarrow (de^x/dx)(x =) = c_1$. This generalizes to what we have here. However here we need to take the functional derivative

$$\boxed{\text{Define } \frac{\delta}{\delta J(x)} \text{ as } \frac{\delta J(y)}{\delta J(x)} = \delta^{(4)}(x-y)}$$

More compliacted derivatives are defined using this formula and the chain rule

$$\frac{\delta V(\phi(y))}{\delta\phi(x)} = \frac{\partial V(\phi(y))}{\partial\phi(x)}\frac{\delta\phi(y)}{\delta\phi(x)} = V'(\phi(y))\delta^{(4)}(x-y)$$

Now we just need to put these two things together

$$Z[J] = \int D\phi\ e^{iS+i\int J(x)\phi(x)d^4x}$$

So if we want the VEV we compute

$$\boxed{\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \frac{(-i)^n}{Z}\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0}}$$

Lets look at an example, free QFT $S = -\frac{1}{2}\int d^4x\ \phi(\partial^2 + m^2)\phi$. This means that

$$Z[J] = \int D\phi\exp\left\{-\frac{i}{2}\int d^4\ \phi(x)(\partial^2 + m^2)\phi(x) + i\int d^4x\ J(x)\phi(x)\right\}$$

Lets consider a simpler but illuminating problem

$$\int d\phi\exp(\phi A\phi + j\phi) \approx \exp\left(jA^{-1}j\right)$$

Where the second $\approx$ comes from shifting ϕ to complete the square. For multidimensional integral we get (TODO PSET)

$$Z[J]\int d\phi_i\exp\left(\phi_iA^{ij}\phi_j + j^i\phi_i\right) \approx \exp(j^i(A^{-1})_{ij}j^j)$$

Lets think of a discrete model of QFT (e.g. space is lattice). We can identify A^{ij} with $(\partial^2 + m^2)$. We also met that $A^{-1} = (\partial^2 + m^2)^{-1} = D_F$, the feynman propagator. To compute the integral we can use fourier transform. One thing we can note is that the kinetic operator $(\partial^2 + m^2)$ is matrix that is diagonal in momentum space. What does that mean? Lets take our $\phi(x)$ and think about it in momentum space

$$\phi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\phi}(k)$$

And lets change variables so that our path integral isn't over all configurations of $\phi(x)$ but all possible fourier components of $\tilde{\phi}(k)$.

$$\int D\phi(x) \to \int D\tilde{\phi}(k)$$

To think about this more clearly we can consider the integral over all field configurations of $\phi(x)$ as the integral over all possible values of x (π_x) in some one dimensional integral (? TODO huh)

$$\int \pi_x d\phi(x) \to \int \pi_k d\tilde{\phi}(k)$$

Whenever we do a change of variables we have a jacobian determinent. Here it's not too hard to compute since we're going from ϕ to a linear combination of ϕ. But we're just gonna ignore it since it's some overall constant for now. So what does the integrand look like in fourier space.

$$\begin{aligned}\int d^4x\ \phi(x)(\partial^2 + m^2)\phi(x) &= \int d^4x \int d^4k d^4k'\ e^{-ikx}\tilde{\phi}(k)(\partial^2 + m^2 = -k'^2 + m^2)e^{-ik'x}\tilde{\phi}(k') \\ &= \int d^4k\ \tilde{\phi}(k)(-k^2 + m^2)\tilde{\phi}(-k)\end{aligned}$$

This is what we mean when we say taht the kinetic operator is diagonal in mometnum space. We also have

$$\int d^4x\ J(x)\phi(x) = \int d^4k \tilde{J}(k)\tilde{\phi}(-k) = \frac{1}{2}\int d^4k \left(\tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right)$$

This means we have

$$Z[J] = \int D\tilde{\phi}(k)\ \exp\left\{\frac{i}{2}\int d^4k(\tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\tilde{\phi}(k)\right\}$$

We can separate out the integrand into the product of a bunch of one dimensional integrals over each independent momentum mode

$$Z[J] = \prod_k \int d\phi(k)\ \exp\left\{\frac{i}{2}\left(\tilde{\phi}(k)(k^2 - m^2)\tilde{\phi}(-k) + \tilde{J}(k)\tilde{\phi}(-k) + \tilde{J}(-k)\phi(k)\right)\right\}$$

That's just a one dimensional gaussian integral. Let $\chi(k) = \tilde{\phi}(k) + \tilde{J}(k)/(k^2 - m^2)$. this means

$$Z[J] = \prod_k \int d\chi(k)\ \exp\left\{\frac{i}{2}\chi(k)\chi(-k)(k^2 - m^2) + \frac{i}{2}\tilde{J}(k)\tilde{J}(-k)/k^2 - m^2)\right\}$$

So we can then write

$$Z[J] = Z[0]\exp\left\{\frac{i}{2}\int d^4k \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2}\right\}$$

If we then do the inverse fourier transform we now have

$$Z[J] = Z[0]\exp\left\{\frac{1}{2}\int d^4x d^4y\ J(x)D(x-y)J(y)\right\}$$

Where $D(x-y) = \int d^4k\ e^{ik(x-y)}\frac{i}{k^2-m^2+i\epsilon}$ is the feynman propagator. Now lets try to compute a two-point function

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = (-i)^2\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\left(1-\frac{1}{2}\int d^4xd^4y\ J(x)J(y)D(x-y)+\dots\right)\Bigg|_{J=0}$$

Applying chain rule we just get

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = D(x-y)$$

In fact this is a more general result for any free QFT which is

$$\langle 0|T\phi(x_1)\dots\phi(x_n)|0\rangle = \sum_{\text{pairings}}(\text{product of free propagators})$$

So for example

$$\langle 0|\phi(x_1)\dots\phi(x_4)|0\rangle = D(x_1-x_2)D(x_3-x_4)+D(x_1-x_3)D(x_2-x_4)+D(x_1-x_4)D(x_2-x_3)$$

This is Wick's theorem. Lets introduce a graphical notation to keep track of things. First recall

$$Z[J] = \exp\left(-\frac{1}{2}\int d^4xd^4y\ J(x)D(x-y)J(y)\right)$$

When we introduce interactions what we'll end up with is a more complex diagramatic set of

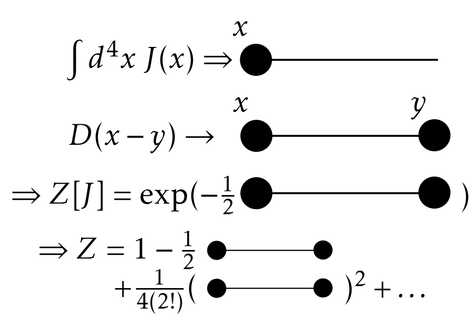


Figure 2: Graphical represntation of partition function

rules. e.g. for a ϕ^3 theory we ahve new sets of ingridients.

Lecture 2: Example of Renormalization

January 21, 2021

At the very end of the last lecture for the two scalar scaterring we found out that the momentum integral suffers from ultraviolet divergence. Divergences like that are all over the place in QFT. So people started to develop a renormalization procedure which deals with diffent physics couplings. All this machinary however has a nasty flavor of sweeping infinities under the carpet. It wasn't very clear what the hell was going on. Even though it worked perturbtivley well for $\lambda\phi^4$ and QED it didn't work for some other theories and doesn't work non-perturbativley. So in the 1950s there was a backlash which lead to alternates like the bootstrap program (aparently this name has something to do with lifting yourself up by your bootstraps?) What really comes to understanding renormalization is by undersatnding effective field theories. This paradigm came from two sides. First there were some people studying critical phenomena in condensed matter. When you're working near a critical point you have long distance correlations. Pretty soon those people realized the behavior at long distances doesn't really care about short distance details. It doesn't matter what kind of lattice we're dealing with (except for maybe some lattice contant). The knowledge of short distance could be summarized in a few parameters. So what Ken Wilson and friends figured out is that you want to worry about effective long distance theories and ignore short distance behavior. The other approach came from a completley different end: current algebra. People studying low-energy interaction between π-mesons and nucleons noticed that there was partial conservation of axial current(fancy spontaneously broken symmetry). They then wrote currents of this broken symmetry and then Weinberg and company figured out that the best way to understand what goes on with current algebra is use effective field theory for low energy(cut all meson zoo except pion.) And what came the big understanding: what we're doing in real life is an effective field theory. We know particles up to some GeV and we don't know anything above that scale. We don't know what kind of particles might occur at 10 TeV. If we had a full QFT then we'd need to include those particles. Even if we say something silly like "there is nothing beyond the standard model" we still have to still consider quantum gravity which becomes relevant at 10^{-19} GeV. So the only thing we can do is effective field theory, a theory which is only known up to some cutoff. So the best we can do is parametrize our ignorance of whats beyond what we know. And in practice this translates to is cutting off ultraviolet modes by saying "we don't know what happens before this" and just say there is literally nothing. And you emulate this nothing with a UV cutoff. And then we paramterize our ignorance by introducing bear(?) couplings that aren't actually measurable experimentailly but whatever gives us the right answer. Fortunatley there is only a few parameters we need to adjust. Before we do any examples lets revisit something we should all know. Lets consider the Debye model of solids.

Aside: Debye Model of Solids

What is relevant to low energies?

(a) For $|\mathbf{k}| \ll 1/(\text{lattice spacing})$ we know that $\omega(k) = c_s|\mathbf{k}|$ where c_s is the speed of sound.

(b) Finite volume of $\mathbf{k}$ space. In 3D we have $\mathbf{k}$ modulo inverse lattice times 2π and k belongs to a 3-torous.

So what did Debye do? He approximated $\omega(\mathbf{k}) = c_s|\mathbf{k}|$ exactly and k-space ball of radius θ/c_s (θ is the Debye temperature?) meaning that

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d^3k}{(2\pi)^3}$$

Up to $|\mathbf{k}| = \theta/c_s$. So θ acts like kind of a cutoff for the condensed matter theory. In QFT we introudce a UV cutoff λ that will shove all the gory details of the UV physics in some redefinition.

Back to 1-loop correction in $\lambda\phi^4$ theory

We found that for some diagram

$$F(t) = \frac{\lambda^2}{2}\int_0^1 dx \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \qquad \Delta(x) = m^2 - tx(1-x) > 0$$

We want to evaluate

$$\begin{aligned} I &= \int \frac{d^4k_E}{(2\pi)^4}\frac{1}{[k_E^2+\Delta]^2} \\ &= \frac{2\pi^2}{(2\pi)^4}\int_0^\infty \frac{dk_E\ k_E^3}{[k_E^2+\Delta]^2} \end{aligned}$$

Now we set UV cutoff: cut the integral at $|k_E| = \Lambda$ for $\Lambda \gg$ what?.

$$I_{\text{regulated}} = \frac{1}{2\pi^2}\int_0^\Lambda \frac{dk_E k_E^3}{[k_E^2+\Delta]^2}$$

Let $\nu = k_E^2 + \Delta$ meaning that

$$I_{\text{regulated}} = \frac{1}{8\pi^2}\times\frac{1}{2}\int_\Delta^{\Delta+\Lambda^2}\frac{d\nu(\nu-\Delta)}{\nu^2} = \frac{1}{16\pi^2}\left(\log\nu + \frac{\Delta}{\nu}\right)\Bigg|_\Delta^{\Delta+\Lambda^2} = \frac{1}{16\pi^2}\Big(\log[(\Delta+\Lambda^2)/\Delta] + \Delta/(\Delta+\Lambda^2) - 1\Big)$$

Now we take an assumption that $\Lambda^2 \gg m^2$ or q_{net}^2 and therefoer we can neglect all negative powers of Λ. So we can say

$$\log[(\Lambda^2+\Delta)/\Delta] = \log(\Lambda^2/\Delta) + \frac{\Delta}{\Lambda^2} - \frac{\Delta^2}{2\Lambda^2} + \cdots \approx \log(\Lambda^2/\Delta)$$

$$\frac{\Delta}{\Delta+\Lambda^2} - 1 \approx -1$$

This all gives us

$$I_{\text{regulated}} = \frac{1}{16\pi^2}\Big(\log(\Lambda^2/\Delta) - 1\Big) \Rightarrow F(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\ (\log(\Lambda^2/\Delta(x)) - 1)$$

To do this integral we just do $\log(\Lambda^2/\Delta(x)) = \log(\Lambda^2/m^2) - \log(\Delta^2/m^2 = 1 - tx(1-x)/^2)$. This means

$$F(t) = \frac{\lambda^2}{32\pi^2}\Big[\log(\lambda^2/m^2) - 1 - J(\lambda/m^2)\Big] \qquad J(t/m^2) = \int_0^1 dx\ \log(1 - tx(1-x)/m^2)$$

There are three diagrams at the 1-loop level (s,t,u) shown in Figure 1 meaning that

$$\mathcal{M}_{1-\text{loop}} = F(t) + F(s) + F(u)$$

Which can be calculated with crossing symmetry.

$$\mathcal{M}(s,t,u) = -\lambda + \frac{\lambda^2}{32\pi^2?}\left[3\log(\lambda^2/m^2) - 3 - J(t/m^2) - J(u/m^2) - J(s/m^2)\right] + O(\lambda^3)$$

So how do we deal with infinite parameters. Here's an overview of the renormalization procedure. λ in the above formula is the *bare* coupling in the bare lagrangian of the theory. This λ_{bare} is not directly measured by any experiment. We adjust it as needed so tah the perturbative amplitudes we calulate fit experimental data. To renormalize our procdure goes like this

(1) Start by definiting physcia coupling λ_{phys} in terms of some scattering amptliude. For example $\lambda_{\text{phys}} = -\mathcal{M}_{\text{elastic}}$ at the threshold.

(2) Use perturbation theory (feynmann graphs?) to calculate $\lambda_{\text{physical}}$ as a power series in λ_{bare}.

$$\lambda_{\text{phys}} = \lambda_{\text{bare}} + A_1\lambda_b^2 + A_2\lambda_b^3 + A_3\lambda_b^4 + \dots$$

Note that $A_1, A_2, A_3, \dots$ depend on $\log\Lambda_{\text{UV}}$. Now formally assume that not only λ_{bare} is small but alos $\lambda_{\text{bare}} \times \log\Lambda_{\text{UV}}$ is small so this perturbative theory makes sense. Do formal perturbation theory in λ_{bare}.

(3) Reverse the power series for λ_{phys}

$$\Rightarrow \lambda_{\text{bare}} = \lambda_{\text{phys}} + B_1\lambda_{\text{phys}}^1 + B_2\lambda_{\text{p}}^3 + \dots$$

This tells us $B_1 = -A_1$ and $B_2 = 2A_1^2 - A^2$ and so on.

$$\Rightarrow \lambda_{\text{bare}} = \lambda_p - A_1\lambda_p^2 + (2A_1^2 - A_2)\lambda_p^3 + \dots$$

(4) For any interesting ampltiude $\mathcal{M}$(kinematical params), use feynman graphs to calculate $\mathcal{M}$

$$\mathcal{M}(\text{kine. para.}) = \lambda_b^n\mathcal{M}_0 + \lambda_b^{n+1}\mathcal{M}_1 + \lambda_b^{n+2}\mathcal{M}_2 + \dots$$

(5) Re-expand in terms of $\lambda_b = \lambda_{\text{phys}} + B_1\lambda_{\text{phys}}^2 + B_2\lambda_{\text{phys}}^3 + \dots$. This means that

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_0 \times \lambda_{\text{phys}}^n(1 + nB_1\lambda_{\text{phys}} + n(n-1)/2B_1^2\lambda_{\text{phys}}^2 + nB_2\lambda_{\text{phys}}^2 \dots) \\
&\quad + \lambda_{\text{phys}}^{n+1}(1 + (n+1)B_1\lambda_{\text{phys}} + \dots) \\
&\quad + \lambda_{\text{phys}}^{n+2}(1 + \dots) + \dots \\
&= \lambda_{\text{phys}}^n\mathcal{M}_0 + \lambda_{\text{phys}}^{n+1}[nB_1\mathcal{M}_0 + \mathcal{M}_1] \\
&\quad + \lambda_{\text{phys}}^2[(n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0 + (n+1)B_1\mathcal{M}_1 + \mathcal{M}_2] + \dots
\end{aligned}$$

The point of all this is to see that we get a power series in λ_{phys}.

(6) In the power series in λ_{phys} the dependence on $\log\Lambda_{\text{UV}}$ cancels out from each term. That is $B_1, B_2, B_3 \dots$ depend on $\log\Lambda$. $\mathcal{M}_1, \mathcal{M}_2, \dots$ depend onf $\log\Lambda$. But $\mathcal{M} - 1 + nB_1\mathcal{M}_0$ is independent of $\log\Lambda_{\text{UV}}$ and likewise $\mathcal{M}_2 + (n+1)B_1\mathcal{M}_1 + (n(n-1)/2 \times B_1^2 + nB_2)\mathcal{M}_0$ is indpendent of $\log\Lambda$.

And that's the renormalization procedure. Lets look at an example for elastic scattering

1-loop calculation for Elastic Scattering

First we define

$$\lambda_{\text{phys}} = -\mathcal{M}@\text{threshold } t = u = 0 \text{ and } s = 4m^2$$

And thus from EQREFHERE we get

$$\lambda_{\text{b}} = \lambda_{\text{phys}} + \frac{\lambda_b^2}{32\pi^2}\left(3\log\Lambda^2/m^2 - 3 - 2(J(0) = 0) - J(4)\right)$$

And when we rewrite $\mathcal{M}(s,t,u)$ in terms of λ_{phys} we get

$$\mathcal{M}(s,t,u) = -\lambda_{\text{phys}} + \frac{\lambda_{\text{phys}}^2}{32\pi^2}\left(J(4) - J(s/m^2) - J(t/m^2) - J(u/m^2)\right)$$

And that's how the renormalization theoyr works. Now he focused on λ but in reality there is also a renormalization of mass $m_b \neq m_{\text{phys}}$. Also nowaways they do perturbation theory with counterturm which directly reorganizes perturbation theory in terms of $\lambda_{\text{phys}}, m^2_{\text{phys}}, \ldots$ directly at the level of feynman graphs.

Lecture 3: UV Regularization Schemes

January 22, 2021

Here are some ways to regularize

(a) Wilson's hard edge cutoff: Used in condensed matter but not commonly used in particle physics

(b) Pauli Villars (1949): Commonly used today and one of the older ones

(c) Covariant Higher Derivatives: Used in supersymmetry.

(d) Dimensional Regularization: Most commonly used today. This basically is taking the dimension of ST $D = 4 - 2\epsilon$. This will be the main subject of lecture 4 if we don't get to it today.

(e) Lattice (discrete spacetime): the hardest and most physical cutoff. This is the only cutoff that works nonperturbativley

Wilson's Hard Edge Cutoff

Limit all euclidan momenta to $|k_E| \leq \Lambda$. Example

$$q_2 = q_{\text{net}} - q_1 \Rightarrow |q_{1E} - q_E^{\text{net}}| < \Lambda$$

This means that q_1 is the intersection of two balls in eucledian momentum space. In terms of $k = q_1 + xq_{\text{net}}$. We have

$$|k_E - xq_{\text{net}}| \leq \Lambda \qquad \text{and} \qquad |k_E - (x-1)q_E^{\text{net}}| \leq \Lambda$$

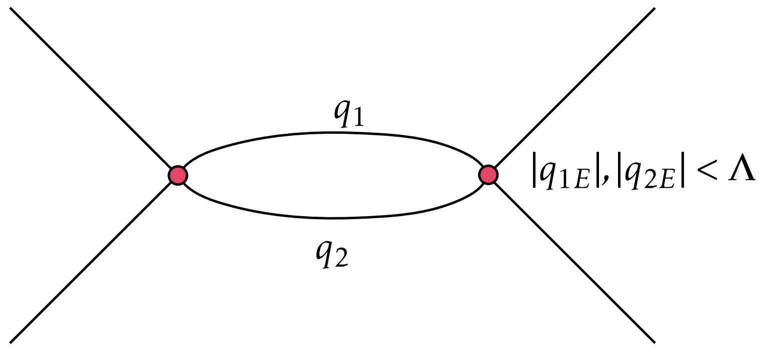


Figure 3: Setup for 1-loop diagram in Wilson's Hard Edge Cutoff

For $|q^{\text{net}}| \ll \Lambda$ we then have

$$|k_E| \le \Lambda + (\text{direction-dependent } O(q^{\text{net}}))$$

Now for lograithmically divergent integrals

$$\Lambda + O(q_{\text{net}}) \text{ is as good as } \Lambda$$

$$\int_0^{\Lambda + O(q^{\text{net}})} \frac{2k_E^3 dk_E}{(k_E^2 + \Delta)^2} = \log\left(\frac{(\Lambda + O(k_{\text{net}}))^2}{\Delta}\right) - 1 \approx \log \Lambda^2/\Delta - 1 + O(q_{\text{net}}/\Lambda)$$

But for worse divergences this is no good. For example if

$$\int = (\text{upper limit})^2 = \Lambda^2 + \underline{O(\Lambda q_{\text{net}})}$$

The underlined term is definitely trouble. That's not the only problem with the hard edge cutoff. Here are some other theories

(a) In gauge theories like QED or QCD hard edge cutoff breaks gauge invariance. Arbitrary phase change can only go up to momentum. In practice breaking gauge invariance means broken ward identites which is trouble.

(b) Hard edge cutoff changes analytical properties of the amplitudes. Thinking about this kind of stuff was all the rage in the 60's.

The bottom line is that the hard edge cutoff mostly works in perturbation theory but makes no physical sense as a complete non-perturbative theory. That's all we can say about the hard edge cutoff.

Pauli Villars regularization

In this case loop momenta q^μ are unlimited but the effects of high $q \ge O(\Lambda)$ is **cancelled** by similar-loops of very heavy particles(fields). So we have

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - \Lambda^2 + i0} \right\} \qquad q_2 = q_{\text{net}} - q_1$$

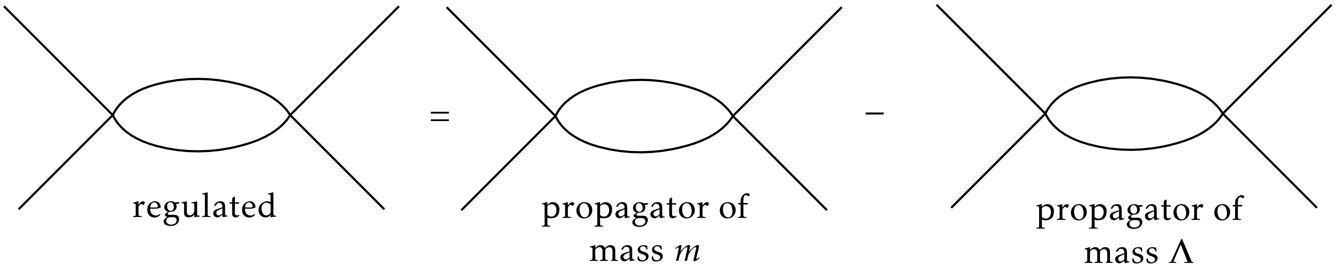


Figure 4: Visual representation of the Pauli Villars regularzation scheme

And notice that we subtract before integration. So for $q_1, q_2 \ll \Lambda$

$$Firstterm \approx \frac{1}{q_2^2},\ Secondterm = O(1/\Lambda^4)$$

Note that the second term is much less than the first term so physical first term. However for $q_1, q_2 \gg \Lambda$ the integrand I

$$I \approx \frac{1}{q_2^2} - \frac{1}{(q^2 - \Lambda^2)^2} \approx O(\Lambda^2/q^2)$$

And so the integral converges. Evaluating the integral is basically introducing feynman paramters and thus finding that it behavies like $\log(\Lambda/\Delta)$ plus some constant.

So to summarize Pauli Villars is good for pertubation theory. when $qp_{\text{external}} \ll \Lambda$. But this cannot be extened to a physical theory at all energies. This is because of the propagators in the compensating loops. On one side we have scalar propagators meaning that we need a scalar field of mass Λ. But on the other hand the minus sign of the loop means fermi statistics. Basically we have a scalar fermion which breaks spin-statistics. Spin-statics assuem relativity, positive energy of all particles, and positive norm on hilbert space. What we break is wrong sign of Hilbert space norm for the compensating scalar which is very unphysical. And this is why we cannot just incorporate that scalar in the theory and then just say "it's a physcial theorywith one fat boy." You cannot extend the theory to the energy where that particle can be produced. That would be unphyscial.

Covariant Higher Derivatives

In this case we have softer propagators at $q > O(\Lambda)$ by adding higher derivatives terms to the lagrangian. For example for a scalar field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{24}\phi^4 - \frac{1}{2\Lambda^2}(\partial^2\phi)^2$$

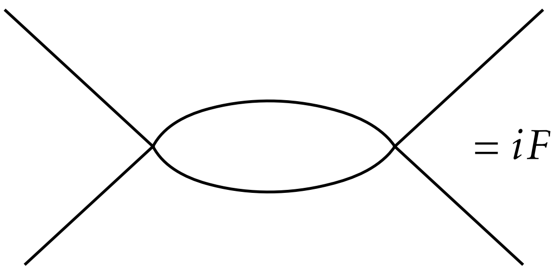
So free ϕ obeys $(\partial^2 + m^2 + \lambda^4/\Lambda^2)\phi = 0$ and that makes the greens function for a propagator

$$G = \frac{i}{q^2 - m^2 - q^4/\Lambda^2 + i\epsilon}$$

And that means for $q^2 \ll \Lambda^2$ the propagator is the usual thing and for large $q \gg \Lambda$ the propagator becomes

$$G \approx -\frac{i\Lambda^2}{q^4} \ll \frac{1}{q^2}$$

Basically this is supressed But for $q^2 \gg \Lambda^2$ we get $\int \frac{\Lambda^4}{q^8} d^4 q$ which is convergent. Bottom line



$$= iF = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 - m^2 - q_1^4/\Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 - q_2^4/\Lambda^2 + i0}$$

Figure 5: Visual represntation of Higher Order Derivative regulariztion scheme

is that higher derivative regular is good for perturbation theory but does not work as a complete theory for energies about equal to Λ. Trouble with higer derivativve regulator: higher derivatives means that ϕ encodes several particles.

$$\frac{i}{q^2 - m^2 - q^4/\Lambda + i0} \approx \frac{i}{q^2 - m^2 + i0} - \frac{i}{q^2 - \Lambda^2 + i0}$$

We will learn soon that a pole in the propagator coresponds to a physical mass of the particle. So 2 scalar particles are encoded in this scalar field. Also for $q^2 = \Lambda^2$ residue of the pole has the wrong sign which causes with particles $m = \Lambda$ has negative Hilbert space norm. A unphysical ghost particle has appeared. This regulation scheme works for some things.

Aside: Using Pauli-Villars and Higher Derivative Reularization Scheme

We used hard edge to evaluate the loop diagram in lecture. In this section we use PV and HD.

(a) **Pauli-Villars:** In this regularization scheme we deal with the UV divergence by introducing a super heavy particle. Lets start by examinig the t-channel contribution. From out notes we have

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} - \frac{1}{q_1^2 - \Lambda^2 + i0} \times \frac{1}{q_2^2 - \Lambda^2 + i0} \right\}$$

As perscribed in the lecture we can evaluate this integral by introducing Feynman parameters. So first we will consider

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(1-x)(q_1^2 - m^2 + i0) + x(q_2^2 - m^2 + i0)]^2}$$

The term in the square brackets we can expand out and simplify using $q_2 = q_{\text{net}} - q_1$ and mathematica

$$\begin{aligned}
(1-x)(q_1^2 - m^2 + i0) + x(q_2^2 - m^2 + i0) &= i0 - m^2 + q1^2 - q1^2 x + q_2^2 x \\
&= i0 - m^2 + q_1^2(1-x) + xq_{\text{net}}^2 + xq_1^2 - 2xq_{\text{net}}q_1 \\
&= i0 - m^2 + xq_{\text{net}}^2 + q_1^2 - 2xq_{\text{net}}q_1 \\
\text{Complete the square} &= i0 - m^2 + xq_{\text{net}}^2 + (q_1 - xq_{\text{net}})^2 - x^2 q_{\text{net}}^2 \\
&= (q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0
\end{aligned}$$

Where $\Delta(x)$ is defined implicitly above. So now we have

$$\frac{1}{q_1^2 - m^2 + i0} \times \frac{1}{q_2^2 - m^2 + i0} = \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2}$$

Meaning that

$$iF(t) = \frac{\lambda^2}{2} \int \frac{d^4 q_1}{(2\pi)^4} \left\{ \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} - \int_0^1 \frac{dy}{[(q_1 - yq_{\text{net}})^2 - \Delta(y,\Lambda) + i0]^2} \right\}$$

First let us consider the red term. We can change the order of integration

$$\begin{aligned}
\int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} &= \int_0^1 dx \int \frac{d^4 q_1}{(2\pi)^4} \frac{1}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} \\
\text{Let } k = q_1 - xq_{\text{net}} &= \int_0^1 dx \int \frac{d^4 k}{(2\pi)^4} \frac{1}{[k^2 - \Delta(x,m) + i0]^2}
\end{aligned}$$

Now focusing on the $d^4 k$ integral. Note in the k_0 integral we have poles when $k_0 = \pm[\sqrt{\mathbf{k}^2 + \Delta(x,y} - i0]$. Lets perform a wick rotation CCW by letting $k_0 = ik_4$ meaning that $k_\mu k^\mu = -k_4^2 - \mathbf{k}^2$. Namely the metric becomes euclidean. So we can define a $k_E = (k_1, k_2, k_3, k_4)$ and get the integral

$$\int \frac{d^4 q_1}{(2\pi)^4} \int_0^1 \frac{dx}{[(q_1 - xq_{\text{net}})^2 - \Delta(x,m) + i0]^2} = i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2}$$

However we can go even further. The integral only depends on k_E^2 so we have a $SO(4)$ symmetry. This means we can change $d^4 k_E = k_E^3 dk_E d\Omega^3$. This is in analogy to the spherical integral we're used to $d^3 x = r^2 dr d\Omega$. In 4 dimensions $\int d\Omega^3 = 2\pi^2$. So now we have

$$i \int_0^1 dx \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2} = i \int_0^1 dx \int \frac{k_E^3 dk_E \, 2\pi^2}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta(x,m)]^2}$$

We can get a similar result for the blue term. So in all we have

$$F(t) = \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \int_0^1 dx \int_0^\infty dk_E \times k_E^3 \left\{ \frac{1}{[k_E^2 + \Delta(x,m)]^2} - \frac{1}{[k_E^2 + \Delta(x,\Lambda)]^2} \right\}$$

Lets evaluate these integrals. First let $u = k_E^2 + \Delta(x,m)$ meaning that $du = 2k_E dk_E$ meaning that $k_E^3 dk_E = du(u - \Delta(x,m))/2$

$$\begin{aligned}\int_0^\infty \frac{k_E^3}{[k_E^2 + \Delta(x,m)]^2} dk_E &= \int_\Delta^\infty \frac{du(u-\Delta(x,m))}{2u^2} \\ &= \frac{1}{2}\left(\ln(u) + \Delta(x,m)u^{-1}\right)\Big|_\Delta^\infty \\ &= \frac{1}{2}\left(\ln(\infty) - \ln(\Delta(x,m)) - 1\right)\end{aligned}$$

Now adding the two contributaions gives us

$$\begin{aligned}F(t) &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\,\{\ln(\infty) - \ln(\Delta(x,m)) - 1 - \ln(\infty) + \ln(\Delta(x,\Lambda)) + 1\} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Delta(x,\Lambda)}{\Delta(x,m)}\right)\right\} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}\right)\right\}\end{aligned}$$

Take $\Lambda^2 \gg m^2, q_{\text{net}}^2$ giving us

$$\begin{aligned}&= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2}{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}\right)\right\} \\ &= \frac{\lambda^2}{2} \times \frac{2\pi^2}{(2\pi)^4} \times \frac{1}{2}\int_0^1 dx\left\{\ln\left(\frac{\Lambda^2}{m^2}\right) - \ln\left(\frac{m^2 + x^2 q_{\text{net}}^2 - x q_{\text{net}}^2}{m^2}\right)\right\}\end{aligned}$$

$$\boxed{F(t) = \frac{\lambda^2}{32\pi^2} \times \left\{\ln(\Lambda^2/m^2) - J(t/m^2)\right\}}$$

(b) **Higher Derivatives:** In this regularization scheme we deal with UV divergence by introducing higher derivative terms to the lagrangian. This gives us the propagator

$$iF = \frac{\lambda^2}{2}\int \frac{d^4 q_1}{(2\pi)^4}\left\{\frac{1}{q_1^2 - m^2 - q_1^4/\Lambda^2 + i0} \times \frac{1}{q_2^2 - m^2 - q_2^4/\Lambda^2 + i0}\right\}$$

In the problem set it is suggested we use

$$\frac{i}{q^2 - m^2 - (q^4/\Lambda^2) + i0} \approx \frac{i}{q^2 - m^2 + i0} \times \frac{-\Lambda^2}{q^2 - \Lambda^2 + i0}$$

To see where this approximation comes fromes lets multiply the things together. The RHS becomes

$$\text{RHS} = \frac{i(-\Lambda^2)}{-0^2 - 0i\Lambda^2 - 0im^2 + L^2 m^2 + 2iq^2 0 - L^2 q^2 - m^2 q^2 + q^4}$$

Now we'll ignore higher order $i0$ terms and assume that $\Lambda^2 \gg m^2, q^2$ to get

$$\text{RHS} \approx \frac{i}{i0 - m^2 + q^2 - q^4/\Lambda^2} = LHS$$

Lets plug this prescribed approximation in

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\left\{\frac{1}{q_1^2-m^2+i0}\times\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\times\frac{1}{q_2^2-m^2+i0}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\right\}$$

The red terms are out of control so lets apply Feynman's very poggers trick to deal with them. From part (a) of this problem we have

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\times\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}$$

The problem again prescribes that we use

$$\frac{-\Lambda^2}{q_1^2-\Lambda^2+i0}\approx\frac{-\Lambda^2}{q_2^2-\Lambda^2+i0}\approx\frac{-\Lambda^2}{(q_1-xq_{\text{net}})^2-\Lambda^2+i0}$$

First lets try to understand the $(q_1-xq_{\text{net}})^2$ term. In the limit where $\Lambda^2 \gg q_1^2, q_2^2$ we know that the difference between q_1^2/Λ^2 and q_2/Λ^2 is negligible and so interpolating between q_1^2 and q_2^2 using $q_1-xq_{\text{net}})^2$ makes sense due to the fact that it's basically a constant as well. So for the low energy limit I kinda see what this trick is doing. Now for the $\Lambda^2 \ll q_1^2, q_2^2$ limit the whole term is $O(1/q^2)$ so I guess q_{net} is negligible so we can interpolate between the q_1 and q_2 values as above without too much trouble. This feels sketchy but we'll go along with it for now. If we plug this interpolation into our above iF we get

$$iF = \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}$$

Lets try to do the same wick rotation trick we did in the last part of this problem.

$$\begin{aligned}iF &= \frac{\lambda^2}{2}\int \frac{d^4q_1}{(2\pi)^4}\int_0^1 dx\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}\\ &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4q_1\left\{\frac{\Lambda^4}{[(q_1-xq_{\text{net}})^2-\Lambda^2+i0]^2}\times\frac{1}{[(q_1-xq_{\text{net}})^2-\Delta(x,m)+i0]^2}\right\}\\ &\text{Let } k = q_1 - xq_{\text{net}} \text{ giving us}\\ &= \frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4k\left\{\frac{\Lambda^4}{[k^2-\Lambda^2+i0]^2}\times\frac{1}{[k^2-\Delta(x,m)+i0]^2}\right\}\end{aligned}$$

Focusing on the d^4k integral. We have four poles but we can avoid them in the same way with a Wick rotation $k_0 = ik_4$ which allows us to wrtie

$$iF = i\frac{\lambda^2}{2(2\pi)^4}\int_0^1 dx\int d^4k_E\left\{\frac{\Lambda^4}{[k_E^2+\Lambda^2]^2}\times\frac{1}{[k^2+\Delta(x,m)]^2}\right\}$$

Using the $SO(4)$ symmetry to write

$$iF = i\frac{\lambda^2}{2(2\pi)^4}\times 2\pi^2\int_0^1 dx\int_0^\infty dk_E\times k_E^3\left\{\frac{\Lambda^4}{[k_E^2+\Lambda^2]^2}\times\frac{1}{[k_E^2+\Delta(x,m)]^2}\right\}$$

Now how do we evaluate this? Excellent question! I have no idea. But mathematica does!

$$\int_0^\infty dk_E \times \ldots = \frac{\Lambda^4}{2\left(\Lambda^2 - \Delta(x,m)\right)^3}\left(\frac{\Delta(x,m)\left(\Lambda^2 - \Delta(x,m)\right)}{\Delta(x,m) + k_{\mathrm{E}}^2} - \log\left(k_{\mathrm{E}}^2 + \Lambda^2\right)\left(\Delta(x,m) + \Lambda^2\right)\right.$$
$$\left. + \left(\Delta(x,m) + \Lambda^2\right)\log\left(\Delta(x,m) + k_{\mathrm{E}}^2\right) + \frac{\Lambda^4 - \Lambda^2\Delta(x,m)}{k_{\mathrm{E}}^2 + \Lambda^2}\right)\Bigg|_0^\infty$$
$$= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x,m)^3)}\Big(0 + (\log(\infty) - \log(\infty))(\Delta(x,m) + \Lambda^2) + 0$$
$$-\Lambda^2 + \Delta(x,m) + (\Delta(x,m) + \Lambda^2)\log(\Delta(x,m)) - \log(\Lambda^2)) - \Lambda^2 + \Delta(x,m)\Big)$$
$$= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x,m))^3}\Big(-2(\Lambda^2 - \Delta(x,m)) + (\Lambda^2 - \Delta(x,m))\log(\Lambda^2/\Delta(x,m))\Big)$$
$$= \frac{\Lambda^4}{2(\Lambda^2 - \Delta(x,m)^2}\Big(-2 + \log(\Lambda^2/\Delta(x,m))\Big)$$

When we assert $\Lambda^2 \gg m^2, q_{\mathrm{net}}^2$ we get that

$$\frac{\Lambda^4}{2(\Lambda^2 - \Delta(x,m))^2} \approx \frac{\Lambda^4}{2\Lambda^4} = \frac{1}{2} \qquad \log\left(\frac{\Lambda^2}{\Delta(x,m)}\right) = \log\left(\frac{\Lambda^2}{m^2}\right) - \log\left(\frac{\Delta(x,m)}{m^2}\right)$$

$$\Rightarrow \int_0^1 dk_E \times \cdots = \frac{1}{2}\Big(\log(\Lambda^2/m^2) - 2 - \log(\Delta(x,m)/m^2)\Big)$$

Putting this all together gives us

$$\boxed{F = \frac{\lambda^2}{32\pi^2}\left\{\log\left(\frac{\Lambda^2}{m^2}\right) - J\left(\frac{t}{m^2}\right) - 2\right\}}$$

OPTICAL THEOREM

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IN QUANTUM MECHANICS

Mostly from L19 and L20 of Zwiebach's course on OCW. First lets review scattering in quantum mechanics. Lets say we want to study a potential. One way we can do this is sending particles in this potential and seeing how they scatter. In general we have a beam of particles going towards a target that get scattered to detectors. Lots of weird things can happen in these scattering experiments. For example

$$p + p \to p + p + \pi^0$$

Two protons turns into two protons and a pion. There is no notion of a conservation of particles. Recall this was one of our motivations for quantum field theory. However for right now we won't consider changes of particles in the process. Namely our processes will consider processes of the flavor

$$a + b \to a + b$$

We'll also restrict our discussion to elastic scattering(internal states do not changes.) An example of inelastic scattering is the Frank-Hertz experiment. Here will also assume no spin, non-relativistic, and interactions are of the form $V(\mathbf{r}_1 - \mathbf{r}_2)$ which allows us to analyze this problem like a central force scattering.

So what do we need to solve? First we have the Hamiltonian

$$H = \frac{\mathbf{p}^2}{2m} + V(\mathbf{r})$$

and assume we have a energy eigenstate $\psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-\frac{iEt}{\hbar}} \Rightarrow$ We want to solve the time-independent Schrodinger equation

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\right]\psi(\mathbf{r}) = E\psi(\mathbf{r})$$

First we'll consider the potential to have finite volume. We'll assume that outgoing states are plane waves which have energy $E = \frac{\hbar^2 k^2}{2m}$. Putting this onto the RHS of our equation gives us

$$\left[-\frac{\hbar^2}{2m}(\nabla^2 + k^2) + V(\mathbf{r})\right]\psi(\mathbf{r}) = 0$$

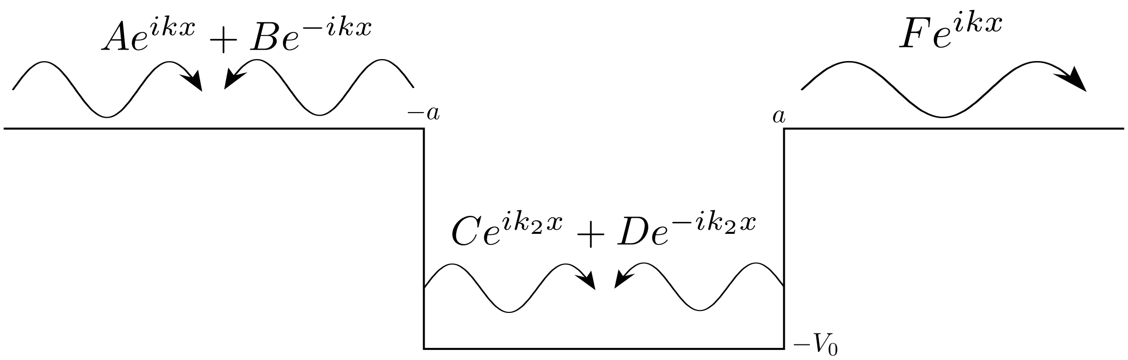

Figure 6: Illustration of a one dimensional potential where we decompose the solution into ingoing and outgoing waves for various regions. Intuition behind scattering

Before we continue lets remember how we dealt with one-dimensional potentials (e.g. Figure 6). We'd be throwing some wave into the potential and then use our intuition to write out a form of the solution in terms of ingoing and outgoing waves. This is what we'll be doing here in scattering. Lets consider the simplest case $V = 0$. Then $\psi = e^{i\mathbf{k}\cdot\mathbf{x}}$ are solutions. This is the same as saying that Ae^{ikx} in Figure 6 is a solution before we ever see the potential. From this lets define an incident wave function $\phi(\mathbf{x}) = e^{ikz}$. We say ϕ instead of ψ since ψ is usually used to denote the full solution whereas ϕ indicates that it might not be the full solution. We know that ϕ is a solution of the S.E. away from $V(r)$. To complete our solution we should expect a spherical outgoing waves. To write this spherical wave we write e^{ikr}. However there is a physical reason that this doesn't work very well. The solution doesn't fall off meaning that we're just accumulating probability as we go radially outwards. To see this from equations just note that $(\nabla^2 + k^2)e^{ikr} \neq 0$. It turns out the one we want is

$$(\nabla^2 + k^2)e^{ikr}/r = 0$$

So lets try to write the scattering solution as

$$\psi_{\text{scattering}}(\mathbf{x}) = \frac{e^{ikr}}{r}$$

But there is no reason that the solution doesn't also depend on θ and ϕ. So lets add a function to our solution

$$\psi_{\text{scattering}}(\mathbf{x}) = f(\theta,\phi)\frac{e^{ikr}}{r}$$

We'll see later that this is the leading term of a solution. So our full solution outside of the potential can be written as

$$\psi(\mathbf{r}) \approx \phi(\mathbf{r}) + \psi_{\text{scattering}}(\mathbf{r},\theta,\phi) = e^{ikz} + f(\theta,\phi)\frac{e^{ikr}}{r}$$

What we need to figure out is $f(\theta,\phi)$, the scattering amplitude. So here we have finished setting up the problem! We'll build to a computation of $f(\theta,\phi)$ using partial waves and phase shifts. But before we do that we need to talk about cross-sections. Lets say in some solid angle $d\Omega$ that corresponds to the detector we see n particles. we can associate a $d\sigma$ on the volume of the potential that captures n particles. What this means is that we get a idea of how the particle sees the particles.

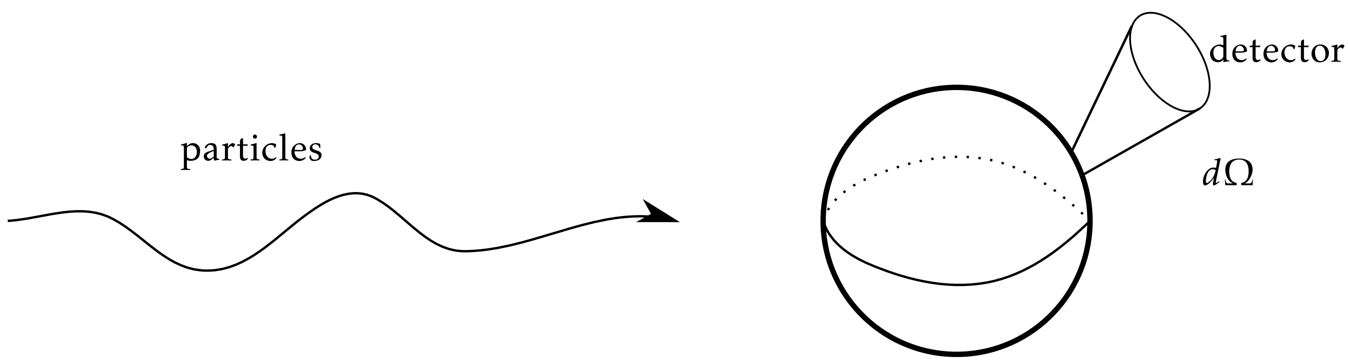

Figure 7: Visualization of scattering cross section.

$$d\sigma = \frac{\text{number of particles scattered per unit time into solid angle } d\Omega \text{ at } (\theta,\phi)}{\text{flux of incidnt particles}}$$

We can think of incdient flux IF (really the probability current) as

$$IF = \frac{\hbar}{m}\text{Im}\left(\phi(\mathbf{r})^*\nabla\phi(\mathbf{r})\right) = \frac{\hbar k}{m}\hat{\mathbf{z}}$$

Now lets try to find the number of particles in the small volume dr in Figure 8

$$dn = \text{number of particles in } dr$$

To find this we need to find the square of the wave function in the scattered region $\psi_{\text{scattering}}$. This gives us

$$dn = \left|f(\theta,\phi)\frac{e^{ikr}}{r}\right|^2 r^2 d\Omega dr = |f(\theta,\phi)|^2 d\Omega dr$$

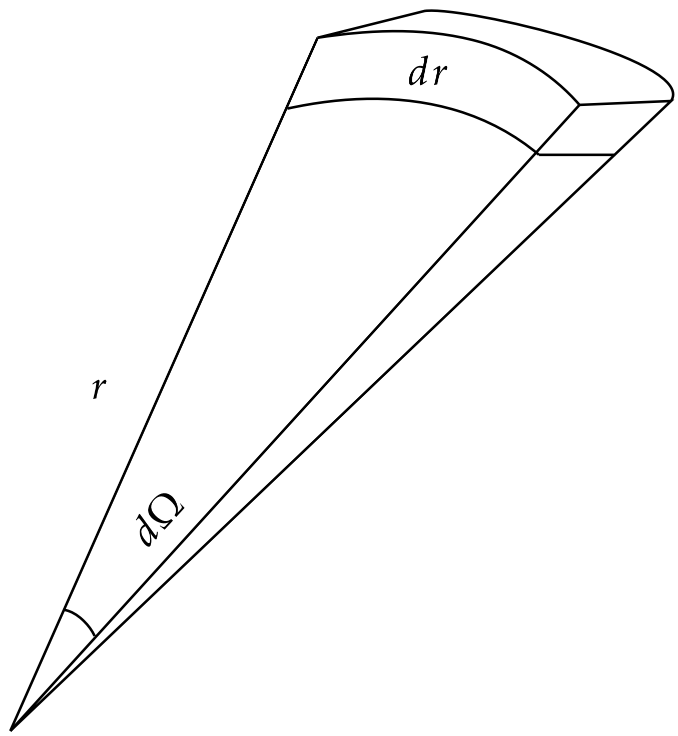

Figure 8: Illustration used in derivation of scattered particle density

Now all these particles will go through the box a infinitesmial $dt = \frac{dr}{v}$ which is

$$dt = \frac{mdr}{\hbar k}$$

Thus we have

$$\frac{dn}{dt} = \frac{\hbar k}{m}\left|f(\theta,\phi)\right|^2 d\Omega$$

And finally we get

$$d\sigma = \frac{\frac{\hbar k}{m}\left|f(\theta,\phi)\right|^2 d\Omega}{\frac{\hbar k}{m}} = |f(\theta,\phi)|^2 d\Omega$$

Which can be written as

$$\boxed{\frac{d\sigma}{d\Omega} = |f(\theta,\phi)|^2}$$

And integrating this we can get a total cross section

$$\boxed{\sigma = \int d\Omega\ \left|f(\theta,\phi)\right|^2}$$

Now lets reduce to a central force. First will we assume that

$$V(\mathbf{r}) = V(r)$$

So we should expect only $f(\theta)$. Now lets consider the solution of the free particle solution in a spherically symmetric potential

$$\psi(\mathbf{r}) = \frac{\mathcal{U}_{E\ell}(r)}{r} Y_{\ell m}(\Omega)$$

And from this we can get the schordinger equation for $\mathcal{U}_{E\ell}$ as

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m}\frac{\ell(\ell+1)}{r^2}\right)\mathcal{U}_{E\ell}(r) = \frac{\hbar^2 k^2}{2m}\mathcal{U}_{E\ell} \Rightarrow \left(-\frac{d^2}{dr^2} + \frac{\ell(\ell+1)}{r^2}\right)\mathcal{U}_{E\ell} = k^2\mathcal{U}_{E\ell}$$

Now we'll define a new variable $\rho = kr$ which makes the quation simpler

$$\left(-\frac{2}{\rho^2} + \frac{\ell(\ell+1)}{\rho^2}\right)\mathcal{U}_{E\ell} = \mathcal{U}_{E\ell}$$

These are solved by Bessel functions

$$\mathcal{U}_{E\ell} = A_\ell \rho j_\ell(\rho) + B_\ell \rho n_\ell(\rho)$$

Both of these behave nicely at infinity. Now we sould be able to write the plane wave solution e^{ikz} as a superposition of our solutions above. To do this first note that the plane wave isn't singular at the origin so we can ignore the spherical Neumann function. Furthermore we have azimuthal symmetry thus we can also ingore the m quantum number. All of this together we get

$$e^{ikz} = \sum_\ell a_\ell j_\ell Y_{\ell,0}$$

Now by nature of the spherical bessel function and the spherical harmonics we can find that

$$e^{ikz} = e^{ikr\cos\theta} = \sqrt{4\pi}\sum \ell\sqrt{2\ell+1}\, i^\ell Y_{\ell,0}(\theta) j_\ell(kr)$$

Lets take a closer look at the spherical bessel function. As we go towards infinitiy we have

$$j_\ell(kr) \to \frac{1}{kr}\sin\left(kr - \frac{\ell\pi}{2}\right)\frac{1}{2ik}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

The terms in the brackets are just ingoing and outgoing waves. This fact will be useful when we start trying to solve this problem. Lets plug this ins

$$e^{ikz} \approx \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\, i^\ell Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2}}{r}\right\}$$

The "partial wave" refers to the fact that each ℓ term are independent. To understand how this works better lets consider the simplest case. $D = 1$. Lets say we have some hard wall with some potential $V(x)$ before we hit the hard wall. If $V = 0$ we have the incoming $\phi = \sin kx = \frac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$. The full solution then

$$\psi(x) = \frac{1}{2i}\left(\cdots - e^{-ikx}\right)$$

So we have the same incoming wave but some undetermined outgoing wave that we must to solve for. What do we know about the outgoing wave? Well the probabilty current must be conserved and must have the same energy. So what can we write? Well we could write in a phase shift

$$\psi(x) = \frac{1}{2i}\left(e^{-ikx+2i\delta_k} - e^{-ikx}\right)$$

Which is valid for $x > a$. We can also assert that

$$\psi(x) = \phi(x) + \psi_{\text{scattered}}(x)$$

Solving for the scattered scattered wave

$$\psi_{\text{scattered}} = \frac{1}{2i}e^{ikx}\left(e^{2i\delta_k} - 1\right) = e^{ikx}e^{i\delta_k}\sin\delta_k$$

How do we extend this to three dimensions? Well consider

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\} + f_\ell(\theta)\frac{e^{ikr}}{r}$$

Where the hell is the incoming wave in this mess? There's only one and it's the red term. We can also write

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{??? - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

Namely we have some undetermined outgoing wave and the same incoming wave. Now we need some inspiration(whisky?) to figure out what this ??? term is. First we know that the probability must be conserved. Futhermore we know each ℓ works independently. So that means that the amplitude of the ??? term has to be the same as the incoming wave for each ℓ or else the probabiliy would not be conserved which would be a disaster. So using our intuition from the 1-d case we will write an outgoing wave with some phase shift

$$\psi(\mathbf{r}) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell,0}(\theta)\frac{1}{2i}\left\{\frac{e^{i(kr-\ell\pi/2)+2i\delta_\ell}}{r} - \frac{e^{-i(kr-\ell\pi/2)}}{r}\right\}$$

The δ_ℓ parameterizes our ignorance. We can know solve for $f_\ell(\theta)$

$$f_\ell(\theta)\frac{e^{ikr}}{r} = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,i^{\ell}Y_{\ell 0}\underbrace{\frac{1}{2i}\left(e^{2i\delta_\ell} - 1\right)}_{e^{i\delta}\sin\delta}\frac{e^{ikr}}{r}\underbrace{e^{-i\pi\ell/2}}_{(-i)^{\ell}}$$

And after a lot of happy cancellations we get

$$\boxed{f_\ell(\theta) = \frac{\sqrt{4\pi}}{k}\sum_{\ell=0}^{\infty}\sqrt{2\ell+1}\,Y_{\ell 0}(\theta)e^{i\delta_\ell}\sin\delta_\ell}$$

A lot of simple things come out of this. Recall

$$\sigma = \int |f_\ell(\theta)|^2 d\Omega = \int f_\ell(\theta)^* f_\ell(\theta) d\Omega$$

Now plugging in our formula for $f_\ell(\theta)$ give us

$$\sigma = \frac{4\pi}{k^2}\sum_{\ell,\ell'}\sqrt{2\ell+1}\sqrt{2\ell'+1}e^{-i\delta_\ell}\sin\delta_\ell e^{i\delta_{\ell'}}\sin\delta_{\ell'}\int d\Omega\ Y^*_{\ell 0}(\Omega)Y_{\ell' 0}(\Omega) = \frac{4\pi}{k^2}\sum(2\ell+1)\sin^2\delta_\ell \tag{OT*}$$

There is one more important result that we should talk about which was the whole point of this song and dance. **Optical Theorem.** Lets say you have some detector surrounding some scattering experiment. The "shadow" of the scattering object equal to the scattered particles that you can detect. In EM the intuition for this shadow should be clear. You shine some light on a object and the "shadow" relates to the photons that are scattered off the surface. However here the intuition of the shadow is not so clear. We have to parts of the solution $\psi(\mathbf{r})$ that contribute to the shadow. So lets see how this works in quantum mechanics. First we need the result that

$$Y_{\ell 0}(\theta=0) = \sqrt{\frac{2\ell+1}{4\pi}}$$

We can then write

$$f_k(\theta=0) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1)e^{i\delta_\ell}\sin\delta_\ell \Rightarrow \text{Im } f_k(\theta=0) = \frac{1}{k}\sum_\ell(2\ell+1)\sin^2\delta_\ell$$

But compare this to (OT*). What we see is that

$$\boxed{\sigma = 4\pi\text{Im } f_k(\theta=0)}$$

In Quantum Field Theory

See Lecture 4.

Lecture 4: More Regularization and Optical Theorem

January 26, 2021

One thing he did not finish is that in gauge theories the higher derivative regularization scheme becomes the **covariant higher derivative regularization scheme.** For example in QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\overline{\psi}(i\not{D}+e\not{A}-m)\psi) + \frac{1}{4\Lambda^2}(\partial_\alpha F_{\mu\nu})(\partial^\alpha F^{\mu\nu}) + \frac{1}{2\Lambda^2}\overline{\psi}(i\not{\partial}+e\not{A})^3\psi$$

If you look at the final term you find that we get a softer propagator

$$\text{three point vertex } = ie\gamma^\nu\left(1+O\left(\frac{p^2}{\Lambda^2}\right)\right)$$

For all multi loop graphs the propagator allows for UV regularization. But for one loop graphs and especially the graphs that look like two, three, and four legged spiders we get no regularization with the above lagrangian. So we need extra regulators like Pauli-Villars. So in practice this is used for proving all sorts of theorems. That's it's for higher derivatives.

There is a renormalization scheme that is a perfectly good non-perturbative equantum theory: lattice renormalization. We need to go through the path integrals to really talk about this so Prof. Kaplunovsky will come back to it after we discuss path integrals (after spring break.) For now lets put this aside and talk about dimensional regularization as the UV regular most commonly used (in high energy physics.)

Dimensional Regularization

What we want to do is generalize loop integrals to fractional (or even not rational) space time dimensions and then analytically continue back to $D = 4$.

$$\int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} f(k_E^\nu) = \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} f(k_E^\nu)$$

Where μ is the reference energy scale at which the spherical momentum space shell dk_D^{rad} has the same volume in D dimensions as in 4-dimensions. Now why is this regulated? Lets consider the radial integral. We start with

$$d^4 k_E \propto (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

Now in D dimensions we get

$$(k_E^{\text{rad}})^3 dk_E^{\text{rad}} \to \mu^{4-D} \times (k_E^{\text{rad}})^{D-1} dk_E^{\text{rad}} = \left(\frac{\mu}{k_E^{\text{rad}}}\right)^{4-D} \times (k_E^{\text{rad}})^3 dk_E^{\text{rad}}$$

The red term is what regularizes things. Lets take a closer look at the red factor. Lets consider $D = 4 - 2\epsilon$

$$\left(\frac{\mu}{k_E^{\text{rad}}}\right)^{4-D=2\epsilon} = \left(\frac{k_E^2}{\mu^2}\right)^{-\epsilon} = \exp\left(-\epsilon \times \log \frac{k_E^2}{\mu^2}\right)$$

The exponential becomes small when the log term becomes proportional to $\frac{1}{\epsilon}$.

Lets consider a generic logarithmically divergent momentum integral. From our other regularization schemes we saw that

$$\text{regulated integral} = C \times \log \frac{\Lambda^2}{m^2} + \dots$$

So for dimensional regularization we expect

$$\text{regulated integral } = C \times \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{m^2}\right) + \dots$$

TODO why do expect this??? Before we continue lets learn how to take integrals in arbitrary dimensions. Consider

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-t k_E^2)$$

For an integer dimension D we have $k_E^2 = k_1^2 + \cdots + k_D^2$. From this we get

$$\exp(-t k_E^2) = \prod_i \exp(-t k_i^2)$$

Using this allows us to rewrite the integral

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-tk_E^2) = \prod_i \int_{-\infty}^{\infty} \frac{dk_i}{2\pi} \exp(-tk_i^2) = (4\pi t)^{-\frac{D}{2}}$$

It's just a gaussian integral! Now lets just analytically continue this formula for non-integer D byt saying

$$\int \frac{d^D k_E}{(2\pi)^D} \exp(-tk_E^2) = (4\pi t)^{-\frac{D}{2}}$$

So what will we do for non-Gaussian momentum integrals? I would cry.

$$I = \int_{\text{reg}} \frac{d^4 k_E}{(2\pi)^4} \frac{1}{[k_E^2 + \Delta]^2} = \int \dots$$

Well first lets recall the gamma function integral

$$\int_0^{\infty} dt\ t^{n-1} \times \exp(-t(k_E^2 + \Delta)) = \frac{\Gamma(n)}{[k_E^2 + \Delta]^n}$$

For $n = 2$ we have

$$\frac{1}{[k_E^2 + \Delta]^2} = \frac{1}{\Gamma(2) = 1} \times \int_0^{\infty} dt\ t \times \exp(-t(k_E^2 + \Delta))$$

This gives us

$$\begin{aligned} \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} &= \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} \int_0^{\infty} dt\ t \times \exp(-t(k_E^2 + \Delta)) \\ &= \int_0^{\infty} dt\ te^{-t\Delta} \times \int \frac{\mu^{4-D} d^D k_E}{(2\pi)^D} e^{-tk_E^2} \end{aligned}$$

Now just using our Gaussian integral we get (μ shouldn't be there?)

$$\int \dots = \int_0^{\infty} dt\ te^{-t\Delta} \times (4\pi t)^{-\frac{D}{2}} = (4\pi)^{-\frac{D}{2}} \int_0^{\infty} dt\ t^{1-\frac{D}{2}} \times e^{-t\Delta}$$

Now as long as the power of $t > -1$ the integral converges. This means for $D < 4$ or we have a complex D, whenever $\text{Re}(E) < 4$. What's the physcial meaning of this divergence? It's just the ultraviolet divergence of the original integral. In particular for $D = 4 - 2\epsilon$ we get

$$\mu^{4-D} \times \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{[k_E^2 + \Delta]^2} = \frac{(4\pi\mu^2)^{\epsilon}}{16\pi^2} \times \Gamma(\epsilon) \times \Delta^{-\epsilon}$$

From this we can extract the matrix element for the t-channel process

$$\mathcal{F}(t) = \frac{\lambda^2}{32\pi^2} \int_0^1 dx\ \Gamma(\epsilon) \left(\frac{4\pi\mu^2}{\Delta(x)} \right)^{\epsilon}$$

This is actually an exact answer for $D = 4-2\epsilon$. Now how do we see that this diverges as $D \to 4$? Well the gamma function blows up as $\epsilon \to 0$ due to the poles at $\epsilon = 0$. How do we calculate these poles? Using $\Gamma(x+1) = x\Gamma(x)$ we can consider

$$\begin{aligned}\Gamma(\epsilon) &= \frac{\Gamma(\epsilon+1)}{\epsilon} = \frac{1}{\epsilon}\left(\Gamma(1) + \epsilon \times \Gamma'(1) + \frac{\epsilon^2}{2}\Gamma''(1) + \dots\right) \\ &= \frac{1}{\epsilon} - \gamma_E + \frac{\pi^2 + 6\gamma_E^2}{12} \times \epsilon + O(\epsilon^2)\end{aligned}$$

Where γ_E is the Euler-Mascheroni constant. In this we can clearly see the pole at $\epsilon = 0$. Lets try expanding the term inside the parenthesis in the integral in powers of ϵ

$$\left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon = \exp\left(\epsilon \times \log\left(\frac{4\pi\mu^2}{\Delta(x)}\right)\right) = 1 + \epsilon \times \log\frac{4\pi\mu^2}{\Delta(x)} + \dots$$

The point of all this is to show that we should be careful about taking $\epsilon \to 0$. What the hell do we do with this expression? For other regularization schemes the idea was that we toss out all negative powers of Λ. In this case we note that a positive power of ϵ corresponds to a negative power of Λ. This means the integrand over the feynman parameter becomes

$$\Gamma(\epsilon) \times \left(\frac{4\pi\mu^2}{\Delta(x)}\right)^\epsilon \to \frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta(x)}$$

Giving us

$$\mathcal{F}_{\mathrm{DR}}(t) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{\Delta(x)}\right) = \frac{\lambda^2}{32\pi^2}\left(\frac{1}{\epsilon} - \gamma_E + \log\frac{4\pi\mu^2}{m^2} - J\left(\frac{t}{m^2}\right)\right)$$

So far all our answers for regularizations have been of a similar form. They can be the exact same however if

$$\log\Lambda_{HE}^2 - 1 = \log\Lambda_{PV}^2 = \log\Lambda_{HD}^2 - 2$$

All regularization schemes are in the same ballpark and related by a numerical constant. We can extend this identification by letting

$$\frac{1}{\epsilon} - \gamma_E + \log(4\pi\mu^2) = \log\Lambda_{HE}^2 - 1 = \log\Lambda_{PV}^2 = \log\Lambda_{HD}^2 - 2$$

OPTICAL THEOREM

This theorem relates forward amplitude with total cross section. For example consider scattering forward scattering ampltiude for elastic scattering $f_{\mathrm{e}}(\theta = 0)$. We have

$$\mathrm{Im}\ f_{\mathrm{e}}(\theta = 0) = \frac{k_{\mathrm{reduced}}}{4\pi} \times \sigma_{\mathrm{total}}$$

Now in the partial wave analysis we decompose the the scattering function into eigenstates of angular momentum

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta) \times \left(\frac{e^{2i\delta_l - 1}}{2i} = e^{i\delta_l}\sin\delta_l\right)$$

Using this we can do all our gymanstics to get

$$\sigma_{\text{tota}} = \frac{4\pi}{k^2} \times \sum_{l=0}^{\infty} (2l+1)\sin^2\delta_l$$

Taking the imaginary part of forward scattering gives us

$$\text{Im } f(\theta = 0) = \frac{1}{k} \times \sum_{l=0}^{\infty} (2l+1)\sin^2\delta_l$$

Now lets talk about this in a relativistic notation. We rewrite our first statement as

$$\text{Im } \mathcal{M}_{\text{elastic,forward}} = 2E_1E_2|\mathbf{v}_1 - \mathbf{v}_2| \times \sigma_{\text{total}}$$

The prefactor is invariant under the boosts along the axis of collision. If we go to the center of masss frame the prefactor evaluates to $4E_{\text{cm}}^{\text{net}}|\mathbf{p}_{\text{cm}}|$. Lets prove this formula today. On thursday we'll apply this formula to $\lambda\phi^4$ theory. This theorem follows from the unitarity of the S-matrix (or the scattering operator $S^\dagger S = 1$). Lets write S as $S = 1 + iT$. Thus

$$1 = S^\dagger S = 1 + iT - iT + T^\dagger T \Rightarrow iT^\dagger - iT = T^\dagger T$$

Now lets sandwith the above with diagonal matrix elements $\langle i|\ldots|i\rangle$. First ths LHS

$$\left\langle i|iT^\dagger - iT|i\right\rangle = i\langle i|T|i\rangle^* - i\langle i|T|i\rangle = 2\text{Im}\langle i|T|i\rangle$$

Wheras on the RHS we'll resolve the idnetity

$$\left\langle i|T^\dagger T|i\right\rangle = \sum_{|f\rangle} |\langle f|T|i\rangle|^2$$

Now we have the result

$$2\text{Im}\langle i|T|i\rangle = \sum_{|f\rangle} |\langle f|T|i\rangle|^2$$

Ther's something wrong here though. Lets factor out the energy-momentum conservatio nwhich gives us

$$\langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_f - p_i) \times \langle f|M|i\rangle$$

There will be troublesome δ functions of both sides. To resolve this consider a state $|i\rangle$ that is similar to $|i\rangle$ except for the momentum. We'll take $|i\rangle$ so that we can approximaite

$$\langle i'|M|i\rangle = \langle i|M|i\rangle$$

Now we can sandhiwhc our things between $|i\rangle'$ and i. This gives us

$$\left\langle i'|iT^\dagger - iT|i\right\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \times 2\text{Im}\langle i|M|i\rangle$$

And on the RHS

$$\left\langle i'|T^\dagger T|i\right\rangle = \sum_{|f\rangle} \langle f|T|i'\rangle^* \langle f|T|i\rangle = (2\pi)^4\delta^{(4)}(p_{i'} - p_i) \times \sum_{|f\rangle} |\langle f|M|i\rangle| \times (2\pi)^4\delta^{(4)}(p_f - p_i)$$

Which finally gives us the result

$$2\mathrm{Im}\,\langle i|M|i\rangle = \sum_{\langle f|} |\langle f|M|i\rangle|^2 \times (2\pi)^4 \delta^{(4)}(p_f - p_i)$$

Now let $|i\rangle$ be an inital state with two particles and $|f\rangle$ be all possible final states. the sum sums overall possible reactions

$$\sum_{\langle f|} = \sum_{\text{channels}} \prod_{a=1}^{n} \left(\sum_{s'_a} \int \frac{d^3 p'_a}{(2\pi)^3 2E'_a} \right)$$

Plugging this into our above result gives us a big boy that is just the phase space integral (with spin sums). This gives us

$$\begin{aligned} 2\mathrm{Im}\,\langle i|M|i\rangle &= \sum_{\text{channels}} 4E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{net}}(1+2 \to 1' + \cdots + n') \\ &= 4E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1+2 \to \text{ anything} \end{aligned}$$

And from this we have our optical theorem

$$\boxed{\mathrm{Im}\mathcal{M}_{\text{elastic,forward}} = 2E_1 E_2 |v_{12}^{\text{rel}}| \times \sigma_{\text{total}}(1+2 \to \text{ anything})}$$

Lecture 5: Applying Optical Theorem and Beginning Correlation Functions

January 28, 2021

Lets see how optical theorem work in $\lambda\phi^4$ theory.

Applying Optical Theorem to $\lambda\phi^4$ Theory

At the tree level

$$\frac{d\sigma_{\text{elastic}}}{d\Omega_{cm}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s}$$

Integrating we can get the net elastic scattering cross section

$$\sigma_{\text{net}}^{\text{elastic}} = \frac{\lambda^2 + O(\lambda^3)}{64\pi^2 s} \times \frac{4\pi}{2} = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

For inelastic processes $2 \to n$ where $n \geq 4$ we find that the tree amplitude is $O(\lambda^{\frac{n}{2}})$ and thus $\mathcal{M} = O(\lambda^{\frac{n}{2}})$. This means that the total scattering cross section

$$\sigma_{\text{total}} = \sigma_{\text{elastic}}^{\text{net}} + O(\lambda^4) = \frac{\lambda^2 + O(\lambda^3)}{32\pi s}$$

Now applying optical theorme to this we get

$$\mathrm{Im}\ \mathcal{M}(\text{elastic,forward}) = 2E_1 E_2 |\mathbf{v}_{12}^{\text{rel}}| \times \frac{\lambda^2 + O(\lambda^3)}{32\pi s} = \frac{\lambda^2 v}{32\pi} + O(\lambda^3)$$

From this we can see that

$$\mathrm{Im}\mathcal{M}_{\text{tree}}(\dots) = 0 \qquad \mathrm{Im}\mathcal{M}_{1-\text{loop}}(\dots) = \frac{\lambda^2 v}{32\pi} > 0$$

Aside: You can apply exactly the same analysis to QED. Take some tree level process. Amplitude is of order E^2 and so the cross section is of order α^2 and by optical theorem if we take an amptliude in the forward direction with unchanged spin then the tree level amplitude must be real but the 1-loop amplitude has an imaginary part.

Lets verify the optical theorem. At tree level

$$\mathcal{M}_{\text{tree}}^{\text{elastic}} = -\lambda, \ \ \mathrm{Im}\mathcal{M}_{\text{tree}}^{\text{elastic}} = 0$$

We'll see soon that

$$\forall\theta : \quad \mathrm{Im}\mathcal{M}_{1-\text{loop}}^{\text{elastic}} = \frac{\lambda^2 v}{32\pi} \qquad v = \frac{|p|}{E} = \sqrt{1 - \frac{m^2}{E^2}} = sqrt 1 - \frac{4m^2}{s}$$

In an earlier lecture we hsoed that

$$\mathcal{M}_{1-\text{loop}}^{\text{elastic}} = \frac{\lambda^2}{32\pi^2}\Big(J(4) - J(t/m^2) - J(u/m^2) - J(s/m^2)\Big)$$

Recalling the form of the J function we note that the green term is imaginary. Or more specifically for $s > 4m^2$ the argument of the logarithm becomes negative at some values of x. Now recall from complex analysis

$$\log(z) = \log(|z|) + i\arg(z)$$

some complex analysis here

We then find that

$$\mathrm{Im}J\left(\frac{s}{m^2}\right) = \int_0^1 dx\, \mathrm{Im}\log\frac{m^2 - sx(1-x)}{m^2} = \pm\pi\times(x_2 - x_1) = \pm\pi v$$

The sign convention comes from the branch cut and how we choose to analytically continue. In the end we get

$$\mathrm{Im}J\Big(\big(s + i\epsilon/m^2\big)\Big) = \mp\pi v$$

Plugging this into our $\mathcal{M}_{1-\text{loop}}^{\text{elastic}}$ we get

$$\mathrm{Im}\mathcal{M}_{1-\text{loop}}^{\text{elastic}}(s \pm i\epsilon, t) = \pm\frac{\lambda^2 v}{32\pi}$$

Now comparing with our result from optical theorem we need to choose $s + i\epsilon$. It turns out that there is another way to extract the imaginary part of $\mathcal{M}$. A guy named Kolkovol(?) created a diagrammatic way to extract these imaginary parts.

Introduction to Correlation Functions

We define the n-point correlation function

$$\mathcal{F}_n(x_1,\dots,x_n) = \left\langle \Omega | T \hat{\Phi_H}(x_1)\dots \hat{\Phi_H}(x_n) | \Omega \right\rangle$$

Or for vector or tensor fields or something

$$\mathcal{F}_n^{a_1\dots a_n}(\dots) = \left\langle \Omega | T \hat{\Phi_H^{a_1}}(x_1)\dots \hat{\Phi_H^{a_n}}(x_n) | \Omega \right\rangle$$

Last semester we learend how to calcualte correleation functions of interacting theory by using perturabtion theory and free correlation function. To figure out how we relate the correlation function of an interacting theory with the correlation function we defined above we first consider

$$\Phi_H(\mathbf{x},t) = e^{iHt}\Phi_S(\mathbf{x})e^{-iHt} = e^{iHt}e^{-iH_0t}\Phi_I e^{iH_0t}e^{-iHt}$$

We can define a unitary operator

$$U_I(t,t_0) = e^{iH_0t}e^{-iH(t-t_0)}e^{-iH_0t_0}$$

Using this we can get

$$U_I(t,0) = e^{iH_0t}e^{-iHt} \qquad \text{and} \qquad U_I(0,t) = e^{iHt}e^{-iH_0t}$$

$$\Rightarrow \Phi_H(x) = U_I(0,x^0)\Phi_I U_I(x^0,0)$$

Now lets consider two fields

$$\Phi_H(x)\Phi_H(y) = U_i(0,x^0)\Phi_I(x)U_I(x^0,y^0)\Phi_I(y)U_I(y^0,0)$$

The above expression is true becasue $U_I(x^0,0)U_I(0,y^0) = U_I(x^0,y^0)$. From here we can generalize this to

$$\Phi_H(x_1)\Phi_H(x_2)\dots\Phi_H(x_n) = U_I(0,x^0)\Phi_I(x_1)U_I(x_1^0,x_2^0)\Phi_I(x_2)\dots$$

Now we need to relate the free vacuum to a true vacuum $|\Omega\rangle$. To do this we consider the state $U_I(0,-T)|0\rangle$ for a complet T. Taking the limit $T \to (1-i\epsilon)\times\infty$. Basically the real part grows to infinity faster than the imaginary part grows to$-\infty$. Lets now consider (where $H_0|0\rangle = 0$)

$$U_I(0,-T)|0\rangle = e^{-iHT}e^{iH_0T}|0\rangle = e^{-iHT}|0\rangle$$

We can expand the vacuum state in terms of eigenstates of H denoted by $|Q\rangle$

$$|0\rangle = \sum_Q |Q\rangle \times \langle Q|0\rangle \Rightarrow e^{-iHT}|0\rangle = \sum_Q |Q\rangle \times e^{-iTE_Q}\langle Q|0\rangle$$

What happens is that we can pick out the lowest energy state because of how we defined T. What this means is that $U_I(0,-T)|0\rangle$ becomes $|\Omega\rangle \times e^{-iTE_\Omega}\langle\Omega|0\rangle$. And thus

$$|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty} U_I(0,-T)|0\rangle \times e^{iTE_\Omega}/\langle\Omega|0\rangle$$

In a similar way we can find with the same $T \to \dots$ trick

$$\langle\Omega| = \lim_{T\to(1-i\epsilon)\infty} \frac{e^{iTE_\Omega}}{\langle 0|\Omega\rangle} \times \langle 0||U_I(+T,0)$$

Now using these two results we can find that big thing inside the big correlation function becomes

$$\langle\Omega|\dots|\Omega\rangle = C(T)\langle 0|\dots|0\rangle$$
$$\dots = U_1(T,0)U_I(0,x^0)\Phi_I(x)U_I(x^0,y^0)\dots 19 \text{ in notes}$$

Time time ordering this gives us

$$= T\left(\Phi_I(x)\Phi_I(y) \times \exp\left(-\frac{i\lambda}{24}\int_{-T}^{T} dt \int d^3\mathbf{z}\phi_I^4(t,\mathbf{z})\right)\right)$$

Which follows from a Dyson series expansion of the evolution operator. This gives

$$\begin{aligned}\mathcal{F}_2(x,y) &= \langle\Omega|T\Phi_H(x)\Phi_H(y)|\Omega\rangle \\ &= \lim_{T\to(1-i\epsilon)\infty} C(T)\langle 0|\dots|0\rangle\end{aligned}$$

So far we have x is later than y. But what if x is earlier than y? Well we reverse somet things around.

Now what about the n-point function

$$\begin{aligned}\mathcal{F}_n(x_1,\dots,x_n) &= \langle\Omega|T\Phi\dots|\Omega\rangle \\ &= \lim_{T\to(1-i0)\infty} C(T) \times \left\langle 0\left|T\left(\Phi_I(x_1)\dots\Phi_I(x_n)\times\exp\left\{\frac{-i\lambda}{24}\int d^4z\Phi_I^4(z)\right\}\right)\right|0\right\rangle\end{aligned}$$

The coeffîcent $C(T)$ is the smae for all correlation function for the same n. What happens if $n = 0$? Then $\mathcal{F}_0 = \langle\Omega|\Omega\rangle = 1$ meaning that

$$\lim_{T\to\dots} C(T)\times\langle 0|T(\exp\dots)|0\rangle = 1$$

Then we can eleminate the $C(T)$ factors

$$\mathcal{F}_n(x_1\dots) = \lim_T \frac{\langle 0|\dots|0\rangle}{\langle 0|T(\exp\dots)|0\rangle}$$

We can use this mess for perturbation theory!

$$\tag{26}$$

Using this perturbation theory we can write Feynman diagram for n-point correlation functions.

Fyenman rules for correlation function here

Just like for scattering amplitudes we can factorize things into vacuum bubble diagrams And other. The ratio we took above is what cancels out the vacuum bubbles.

$$\langle 0|T(\exp\dots)\rangle = \sum(\text{vacuum bubbles w/o external vertices})$$

What's good about killing these vacuum bubbles? Well the leading divergence cancels out when we do this. So we can set $T \to (1+i0)\infty$ diagram by digram. In practice this means when we integrate over a vertex instead of integrating from $-T$ to T we integrate from $-\infty$ to ∞. This is important since then we can go to momentum space with foureir transform and then get momentum space feynman rules.

Momentum Space Feynamn Rules here

When we did scattering amplitudes we learned that we only needed to consider connected diagrams (discconected meant basically no scattering.) For correlation functios like define connected correlation functions

$$\mathcal{F}_n^{\text{connected}} = \sum \cdots$$

We can use this to get the original $\mathcal{F}_n$ through (block expansions(?)) the following

$$\begin{aligned}
\mathcal{F}_2(x,y) &= \mathcal{F}_2^{\text{conn}}(x,y)\\
\mathcal{F}_4(x,y) &= \mathcal{F}_4^{\text{conn}}(x,y,z,w) + \mathcal{F}_2^{\text{conn}}(x,y)\times\mathcal{F}_2^{\text{conn}}(z,w)\\
&\quad + \mathcal{F}_2^{\text{conn}}(x,z)\times\mathcal{F}_2^{\text{conn}}(y,w) + \mathcal{F}_2^{\text{conn}}(x,w)\times\mathcal{F}_2^{\text{conn}}(y,z)\\
&\cdots
\end{aligned}$$

Connected 4-point, 6-point, and so on correltaion functions are related to the scattering amplitude wit the LSZ reduction formula. Next time we'll go more in depth into 2-point correlation functions which are important for renormalization and knowing what the hell is going on.

Lecture 6: The 2-Point Correlation Function

Last lecture we learned about correlation functions for quantum fields and ended up with connected correlation function that can be defined recursivley. Today we're going to focus on the 2-point function. To do this we're ignore Feynman diagrams and worry about the actual physics. Consider $\mathcal{F}_2(x-y)$ where $x^0 > y^0$. We have

$$\mathcal{F}_2(x-y) = \langle\Omega|T\Phi_H(x)\Phi_H(y)|\Omega\rangle = \sum_{|\Psi\rangle}\langle\Omega|\Phi_H(x)|\Psi\rangle\times\langle\Psi|\Phi_H(y)|\Omega\rangle$$

The Ψ is all quantum states of the theory which can occur from the action of the field $\phi(y)$. For example for QED you could have a photon, or three photon, or electron positron paris. these pairs could be free and flying away or they could be bound like in hydrogen. Different types of states will have different parametrs. But all staets will have a net momentum. In addition we might have some other quantum states where we could just consider relative momentum. So for convinience we'll denote $|\Psi\rangle = |\psi, p^\mu\rangle$. With this we can define

$$\mathcal{F}_2(x-y) = \sum_\psi\int\frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E(\mathbf{p},M(\psi))}\times\langle\Omega|\Phi_H(x)|\psi,p\rangle\times\langle\psi,p|\Phi_H|\Omega\rangle$$

Lets try to take out the x dependence. To do this we use translational symmetry in all 4-dimensions which means that

$$\Phi_H(x) = \exp(ix_\mu P^\mu)\Phi_H\exp(-ix_\mu P^\mu)$$

If we take the vacuum to have energy 0 then

$$\langle\Omega|\exp(ix_\mu P^\mu) = \langle\omega| \qquad \exp(-ix_\mu P^\mu)|\psi,p\rangle = e^{-ix_\mu p^\mu}\times|\psi,p\rangle$$

Asserting all of this we can get

$$\langle\Omega|\Phi_H(x)|\psi,p\rangle\times\langle\psi,p|\Omega_H(y)|\Omega\rangle = e^{-ip(x-y)}\times\left|\langle\psi,p|\Phi_H(0)|\Omega\rangle\right|^2$$

Now since the state $\Phi_H(0)|\Omega\rangle$ is invariant under orthocronous Lorentz symmetry we have that the matrix element is the same for all $\mathbf{p}$ on the mass shell. From all this we can get

$$\mathcal{F}_2(x-y) = \sum_\psi \left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times D(x-y;M(\psi))$$

What happens if $x^0 < y^0$? If we work through this carefully we just see that x and y are ex hagned. To combine everything we just use the Feynman propagator.

$$\mathcal{F}_2(x-y) = \sum_\psi \left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times G_F(x-y,M(\psi))$$

The above equation can be rewritten more generally by redefining the one-point function as a measure. This equation is usually called the Kallen-Lehmann spectracl represtnation(TODO Kallen has some fancy dots in their name)

$$\mathcal{F}_2(x-y) = \int_0^\infty \frac{dm^2}{2\pi}\rho(m^2)\times\int\frac{d^4p}{(2\pi)^4}\frac{ie^{-ip(x-y)}}{p^2-m^2+i0}$$

$$\rho(m^2) = \sum_\psi\left|\langle\psi|\Phi_H(0)|\Omega\rangle\right|^2\times(2\pi\delta(M^2(\psi)-m^2)$$

The second function is the *spectral density function.* Don't get distracted by the fancy math though. Lets talk about some features about this spectral density.

(a) Real and non-negative.

(b) In free field theory $\rho(m^2) = 2\pi\delta(m^2-M^2)$. This is because for a free theory the only thing you can make out of a vaccum with a field is a single particle.

The spectral density becomes more interesting when we consider a interacting theory. For example for $\lambda\phi^4$ theory we can make three particles and then can make a continuous spectrum. A more general kind of theory can have even funkier spectral densities (see Kaplunovsky's lecture notes page 13 for some examples.)

Lets move on from this spectral representation to its consequences on the two point function. The two point function only depends on x and y so lets do a fourier transform

$$\mathcal{F}_2(p) = \int_0^\infty\frac{dm^2}{2\pi}\rho(m^2)\times\frac{i}{p^2-m^2+i0}$$

How do features off the sepctral density function transform into analytically features of $\mathcal{F}_2(p)$ In general one particle states will give delta spike contribution while multparrtilce states give a continuum.

$$\rho(m^2) = Z \times 2\pi\delta(m^2 - M^2_{\text{particle}}) + \text{smooth continuum}$$

The amptliude of the delta function will have amptliude

$$Z = |\langle 1\text{particle}|\Phi_H(0)|\Omega\rangle|^2 > 0$$

$\sqrt{Z}$ is the strength with which that the quantum field creates the single particle state. With all this we have

$$\mathcal{F}_2(\rho^2) = \frac{iZ}{p^2 - M^2_{\text{particle}} + i\epsilon} + \text{smooth}(p^2)$$

An implication of this we'll find is that a pole in the two point function is at physical particle masses. Why do we care about all of this? We know that the pyscial and bare coupling are differnt by perturabtion theory. This implies that the physcial mass and the bare mass are different in perturbation theory. We can find the physcial mass in perturbation theory by looking at pole masses. We shall see tuesday that

$$M^2_{\text{pole}} = m^2_{\text{bare}} + \text{loop correction} = f(m^2_{\text{bare}}, \lambda_{\text{b}}, \Lambda_{UV})$$

We can then identify $M^2_{\text{pole}} = M^2_{\text{particle}}$ to solve equations. Something also to note is that the contiuum creates a branch cut running from the threshold to ∞. We can see this from

$$\mathcal{F}_2(p^2 + i0) - \mathcal{F}_2(p^2 - i0) = \frac{1}{2\pi i}\int_0^\infty \frac{dm^2\rho(m^2)}{m^2 - p^2 - i0} - \cdots = \frac{1}{2\pi i}\oint_{\text{around p}^2} \frac{dm^2\rho(m^2)}{m^2 - p^2} = \rho(p^2)$$

This means that there is a discontinuity accross the real axis meaning that there is a branch cut. First lets look at the real axis of this Riemann surface. For spacelike p^2, the two point function becomes

$$i\mathcal{F}_2(p^2) = \int_0^\infty \frac{dm^2}{2\pi}\frac{\rho(m^2)}{m^2 - p^2}$$

This is well behaved. Now lets go timelike p^2 and stay below the threshold of the branch cut our $i\mathcal{F}_2$ is still find and good. However once we reash the treshold we get a singularity which leads to a branch cut

$$\text{(above the branchcut)}\;\; i\mathcal{F}_2(p^2 \pm i0) = R \pm i\frac{\rho(p^2)}{2}$$

So the question is if we want to evalute the two point function above the branch cut threshold which $\pm$ sign do we take? The answer is we take $+i0$. This is because we want the thing to look like the Feynmann propagator(oh?). The physcial value is on the top side of the branch cut. This is something to remember: the physical side of the branch cut is the top side.

The RIemann surface of the 2point function has an ifnite number of Riemann sheets but only one physical sheet. So which one is the physcial sheet? According to the notes the physial sheet begin on the upper side of the branch cut, extend CCW to negatvie real axis and then back to posiive axis. TODO HUH? On the physical sheet the integral can be taken lietrally and all real poles correspond to physical particle states. We can have poles off-axis on the unphysical sheet. These poles off-axis actually correspond to resonances. Lets be careful in how we define these unphyscial poles. First define on the upper side of the branch cut

$$i\mathcal{F}_2(p^2+i0)$$

Once we have this lets analytically continue to complex p^2. If we go up i the complex plane $\text{Im}(p^2)>0$. If we go below the branch cut we 're on the unphysical sheet. On this unphysical sheet we might hit a pole. For an example lets say we have a pole at $p^2 = M^2 - iM\Gamma$. Suppose this pole is close the real axis meaning that $\Gamma \ll 1$. Then when we stay on the real axis, this pole dominates over the known pole contributions. So for real p^2 in the viscinity of M^2 we have

$$\mathcal{F}(p^2) = \frac{iZ}{p^2 - M^2 + iM\Gamma} + \text{smooth}(p^2)$$

This is the Breit-Wigner resonance. By optical theorem we know that Γ is the total decay rate of the unstbale particle(resonances are unstable particles.) $\frac{1}{\Gamma}$ is the average lifetime of the particle.

Now lets turn back to perturbation theory. At the tree level we have the free feynman propagator. AT one loop we start multiplying by other factors. And it just keeps going on. So when you naivley calcuatle $\mathcal{F}_2$ with diagrams, you just get a lot of poles

$$\left(\frac{i}{p^2 - m_b^2 + i0}\right)^{\text{power}\geq 2} \times \text{othres}$$

This is unphysical, we should only have simple poles. The perturabtion theory, instead of shifting tree-level poles to physical mass, just creates a bunch of higher order poles. To resolve this we can re-sum the perturbation exapnsion so higher order poles add upt oa simple shifted pole. E.g.

$$\sum_{n=0}^{\infty}\left(\frac{i}{pr - m_b^2 + i0}\right)^{n+1} \times (-i\Delta)^n = \frac{i}{p^2 - m_b^2 + i0} \times \frac{1}{1 - \frac{\Delta}{p^2 - m_b^2 + i0}} = \frac{i}{p^2 - (m_b^2 + \Delta) + i0}$$

To see this graphically consider an arbitrary multi-lop feynman graph for two-point correlation function. We have a line with a bunch of (irrducible one-particle) subgraphs. Irreducible means that if you cut any one propagator then it stays connected. Technically is one-propagator irreducible but people call it one-particle. We'll skip the graph theory proof but all contributing feynmann graph that csontirbute to the 2-poitn correlator look like that line picture we should have in our head. So the formal sum over all such diagrams with N bubbles (then we'll have $N+1$ propagators) looks like

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0}\right)^{N+1} \times \prod_{i=1}^{N}[1PI \text{ bubble}\#i]$$

We can reorganize this so that

$$\mathcal{F} = \left(\frac{i}{p^2 - m_b^2 + i0}\right)^{N+1} \times \{\text{sum over bubbles}\}^N$$

Now we reframe our perturbation theory as summing over bubble diagrams to some order to calculate the physical mass in perturbation theory

$$\mathcal{F}_2(p^2) = \frac{i}{p^2 - m_b^2 + i[\text{sum over bubbles}] + i0}$$