Starting an iterative process

Search Bracket

- Another relationship that might be useful for determining the search intervals that contain the real roots of a polynomial is

$$|x^*| \le \sqrt{\left(\frac{a_{n-1}}{a_n}\right)^2 - 2\left(\frac{a_{n-2}}{a_n}\right)}$$

where x is the root of the polynomial. This will be the maximum absolute value of the roots

- That means that no roots exceed x_{max} in absolute magnitude and thus, all real roots lie within the interval

$$(-|x^*_{max}|, |x^*_{max}|)$$

Starting an iterative process (continued)

- There is another relationship that suggests an interval for roots.
- All roots x satisfy the inequality

$$|x^*| \le 1 + \frac{1}{|a_n|} \max\{|a_0|, |a_1|, |a_2|, \ldots\ldots |a_{n-1}|\}$$

where the 'max' denotes the maximum of the absolute values of $|a_0|$, $|a_1|$, $|a_2|$,...... $|a_{n-2}|$, $|a_{n-1}|$

Bisection Method: Example 1

Find the root of the equation $x^3 + 4x^2 - 1 = 0$.

Solution

Let, $a = 0$ and $b = 1$.

Now, $f(0) = (0)^3 + 4(0)^2 - 1 = -1 < 0$ and
$f(1) = (1)^3 + 4(1)^2 - 1 = 4 > 0$.
i.e., $f(a)$ and $f(b)$ has opposite signs.

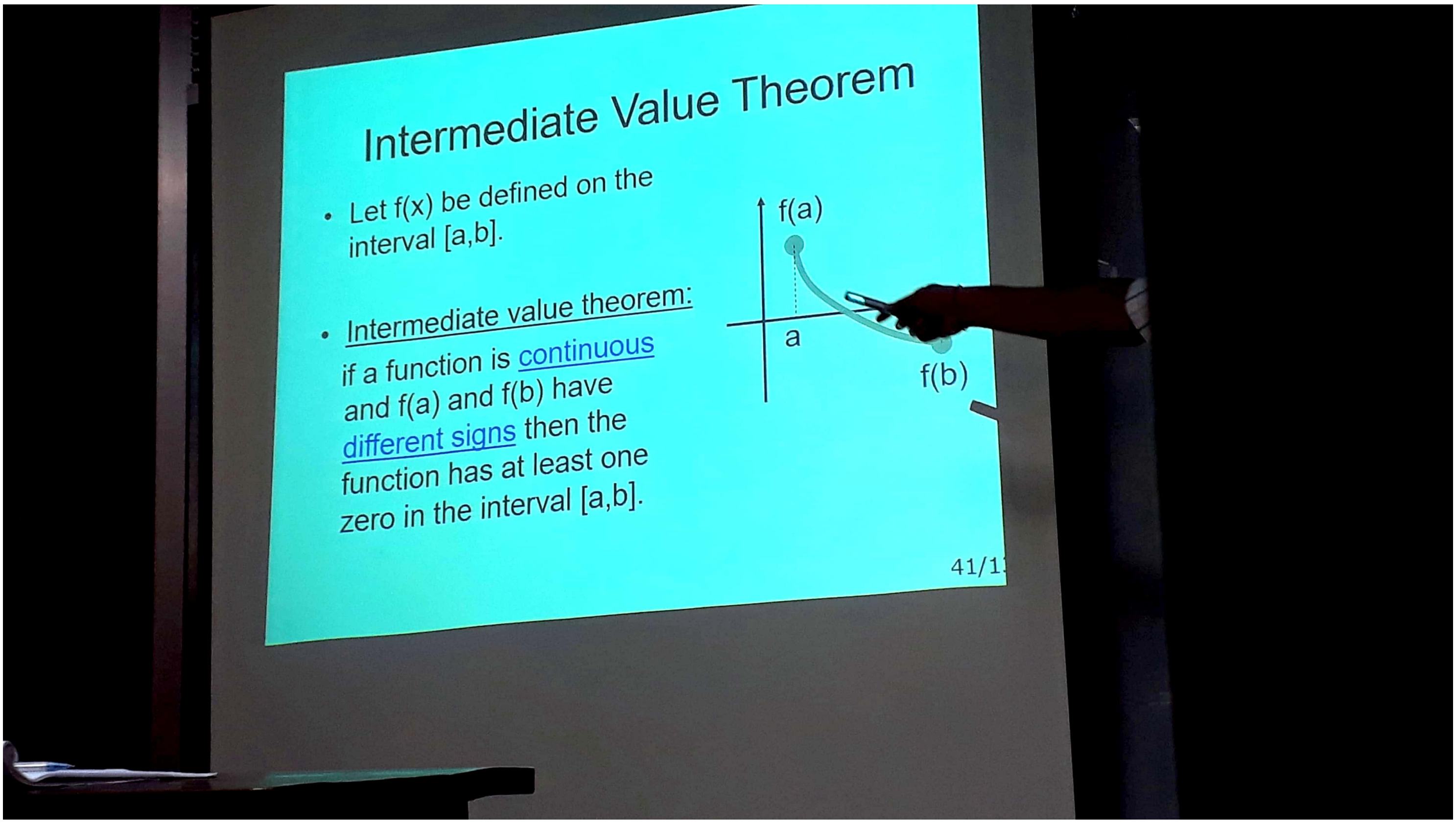
Therefore, $f(x)$ has a root in the interval $[a, b] = [0, 1]$

$x_c = (0 + 1)/2 = 0.5$,
$f(0.5) = 0.125$. Now $f(a)$ and $f(x_c)$ has opposite signs

So, the next interval is $[0, 0.5]$

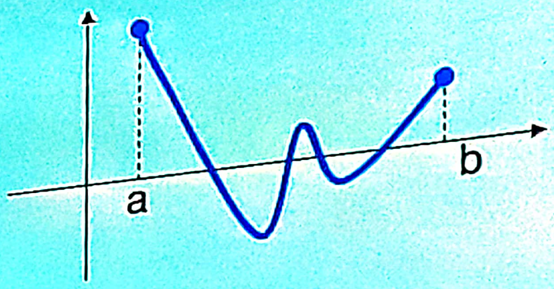
Bisection Method

- The **Bisection method** is one of the simplest methods to find a zero of a nonlinear function.
- It is also called **interval halving** method.
- To use the Bisection method, one needs an initial interval that is known to contain a zero of the function.
- The method systematically reduces the interval. It does this by dividing the interval into two equal parts, performs a simple test and based on the result of the test, half of the interval is thrown away.
- The procedure is repeated until the desired interval size is obtained.


Intermediate Value Theorem
• Let f(x) be defined on the interval [a,b].
• Intermediate value theorem:
if a function is continuous and f(a) and f(b) have different signs then the function has at least one zero in the interval [a,b].
f(a)
a
f(b)

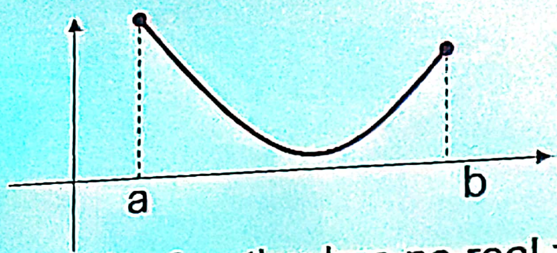
Examples

- If f(a) and f(b) have the same sign, the function may have an even number of real zeros or no real zeros in the interval [a, b].



The function has four real zeros

- Bisection method can not be used in these cases.



The function has no real zeros

Bisection Method

- If the function is continuous on [a,b] and f(a) and f(b) have different signs, Bisection method obtains a new interval that is half of the current interval and the sign of the function at the end points of the interval are different.

- This allows us to repeat the Bisection procedure to further reduce the size of the interval.

Bisection Method

Assumptions:

Given an interval [a,b]

f(x) is continuous on [a,b]

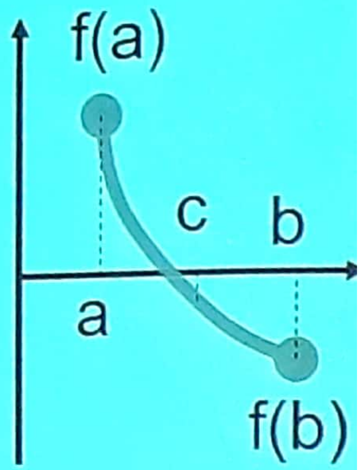
f(a) and f(b) have opposite signs.

These assumptions ensure the existence of at least one zero in the interval [a,b] and the bisection method can be used to obtain a smaller interval that contains the zero.

Bisection Algorithm

Assumptions:

- f(x) is continuous on [a,b]
- f(a) f(b) < 0



Algorithm:

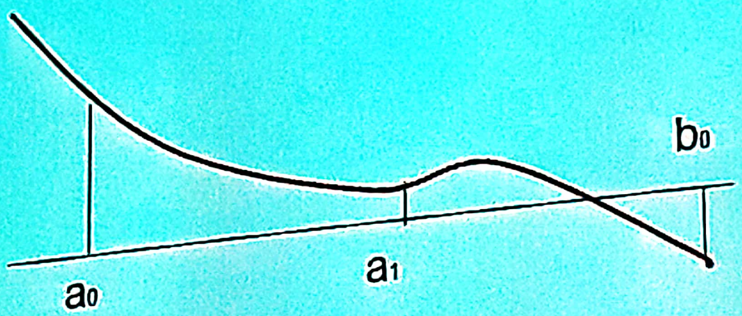
Loop

1. Compute the mid point c=(a+b)/2
2. Find $xc = (a+b)/2$.
3. If $f(xc) = 0$, then xc is the root of the equation.
3. Otherwise, If f(a) f(c) < 0 then new interval [a, c]
   If f(a) f(c) > 0 then new interval [c, b]

End loop

We continue this process until we find the root (i.e., $f(x_c)=0$), or the latest interval is smaller than some specified tolerance.

Bisection Method



Example #1

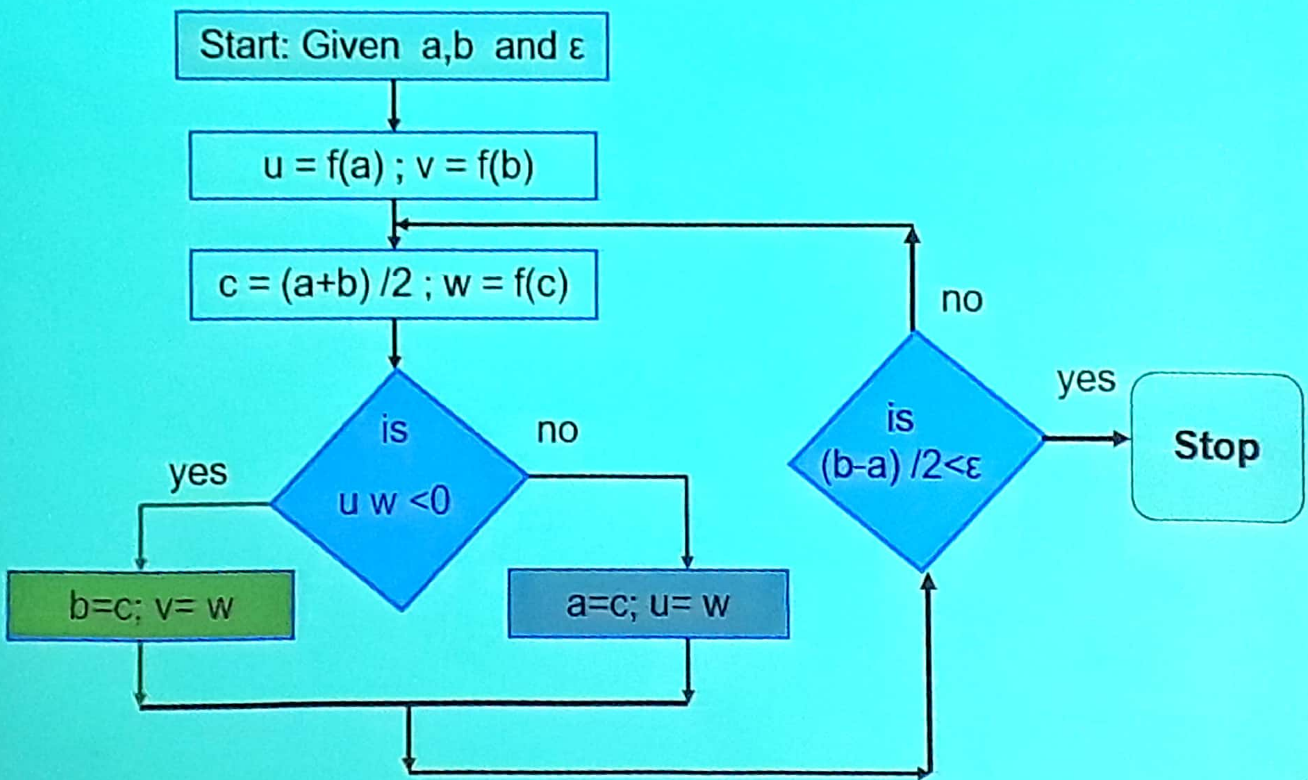
- Consider the polynomial equation $2x^3 - 8x^2 + 2x + 12 = 0$
- Estimate the possible initial guess value

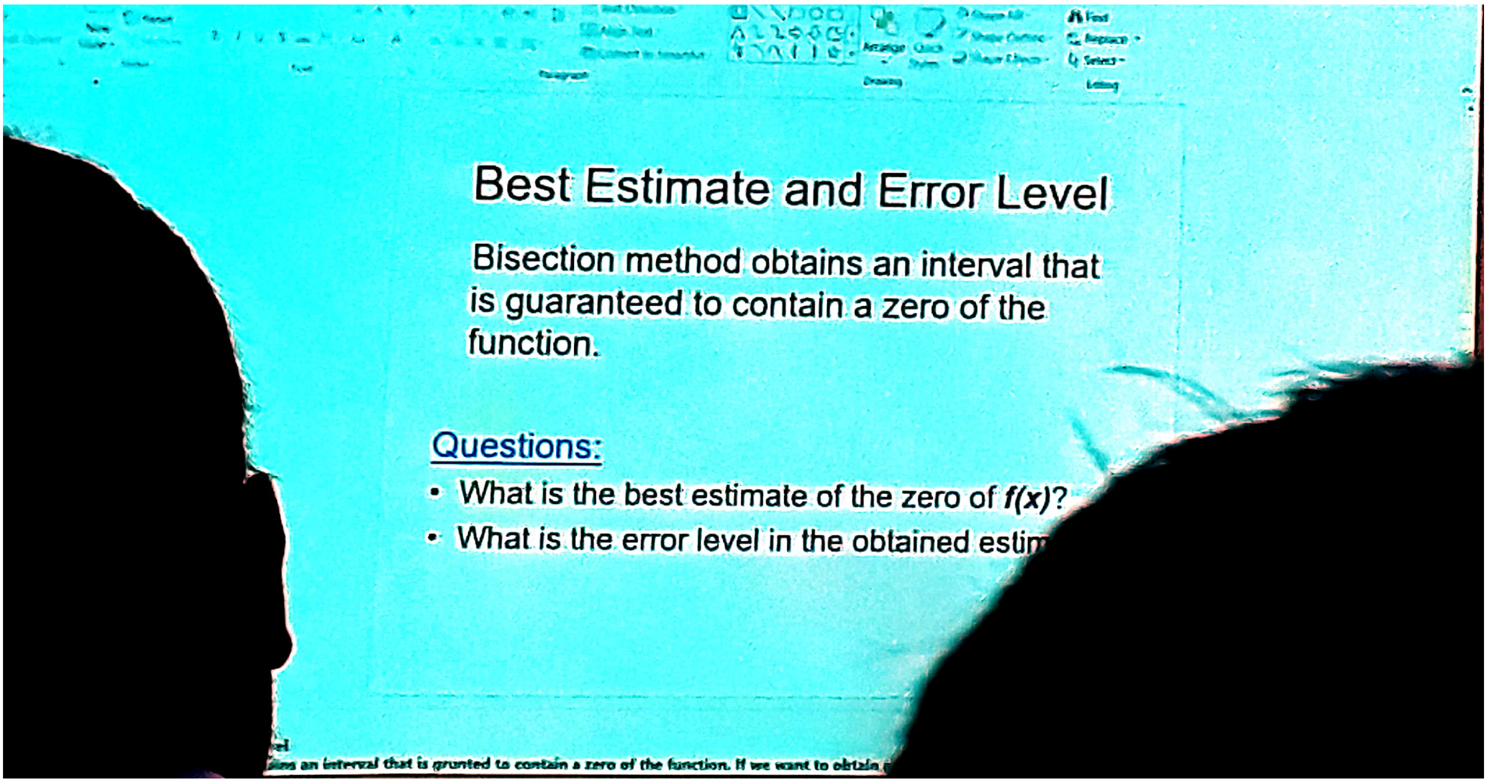
- The largest possible root is $x_1^* = -\dfrac{a_{n-1}}{a_n}$

- That is, no root can be larger than the value 4
- All roots must satisfy the relation $|x^*| \le \sqrt{\left(\dfrac{-8}{2}\right)^2 - 2\left(\dfrac{2}{2}\right)} = \sqrt{14}$

- Therefore, all real roots lie in the interval $\left(-\sqrt{14}, \sqrt{14}\right)$.
- We can use these two points as initial guesses for the bracketing methods and one of them for the open end methods

Flow Chart of Bisection Method




Best Estimate and Error Level
Bisection method obtains an interval that is guaranteed to contain a zero of the function.
Questions:
• What is the best estimate of the zero of f(x)?
• What is the error level in the obtained estim

Example

$a = 6,\ b = 7,\ \varepsilon = 0.0005$

How many iterations are needed such that: $|x - r| \le \varepsilon$?

$$n \ge \frac{\log(b-a) - \log(\varepsilon)}{\log(2)} = \frac{\log(1) - \log(0.0005)}{\log(2)} = 10.9658$$

$\Rightarrow n \ge 11$

Example

- Use Bisection method to find a root of the equation $x = \cos(x)$ with absolute error <0.02 (assume the initial interval $[0.5, 0.9]$)

(assume the initial interval [[illegible], [illegible]])

Question 1: What is $f(x)$?

Question 2: Are the assumptions satisfied ?

Question 3: How many iterations are needed ?

Question 4: How to compute the new estimate ?

Example

- Use Bisection method to find a root of the equation x = cos (x) with absolute error <0.02 (assume the initial interval [0.5, 0.9])

Question 1: What is *f (x)* ?
Question 2: Are the assumptions satisfied ?
Question 3: How many iterations are needed ?
Question 4: How to compute the new estimate ?

Bisection Method

Advantages

- Simple and easy to implement
- One function evaluation per iteration
- The size of the interval containing the zero is reduced by 50% after each iteration
- The number of iterations can be determined a priori
- No knowledge of the derivative is needed
- The function does not have to be differentiable

Disadvantage

- Slow to converge
- Good intermediate approximations may be discarded
- We need two initial guesses *a* and *b* which bracket the root.
- It is among the *slowest* methods to find the root.
- When an interval contains more than one root, the bisection method can find *only* one of them.

Iteration Method

- Suppose we have an equation in the form $g(x) = 0$
- Rewrite the equation in the form $x = f(x)$.
- Start with an initial guess x_0, which is an ***approximation*** of the root.
- Calculate $x_1, \ldots, x_n, \ldots$ such that
  - $x_1 = f(x_0)$
  - $x_2 = f(x_1)$
  - $x_3 = f(x_2) \ldots$

Iteration Method

- Suppose we have an equation in the form $g(x) = 0$
- Rewrite the equation in the form $x = f(x)$.
- Start with an initial guess x_0, which is an ***approximation*** of the root.
- Calculate $x_1, \ldots, x_n, \ldots$ such that
  - $x_1 = f(x_0)$
  - $x_2 = f(x_1)$
  - $x_3 = f(x_2) \ldots$
- Iterate the same process until $(x_n - x_{n-1})$ smaller than some specified tolerance.

Iteration Method: Convergence Conditions

- Any arbitrary approximation x_0, x_1, x_2 does not assure that it will converge to the actual root x of the equation.
  - E.g. $x = 10^x + 1$,
  - if $x_0 = 0$, $x_1 = 2$, $x_2 = 101$, …. that does not converge to the actual root x
  - As n increase, x_n increases without limit!

Iteration Method: Convergence Conditions

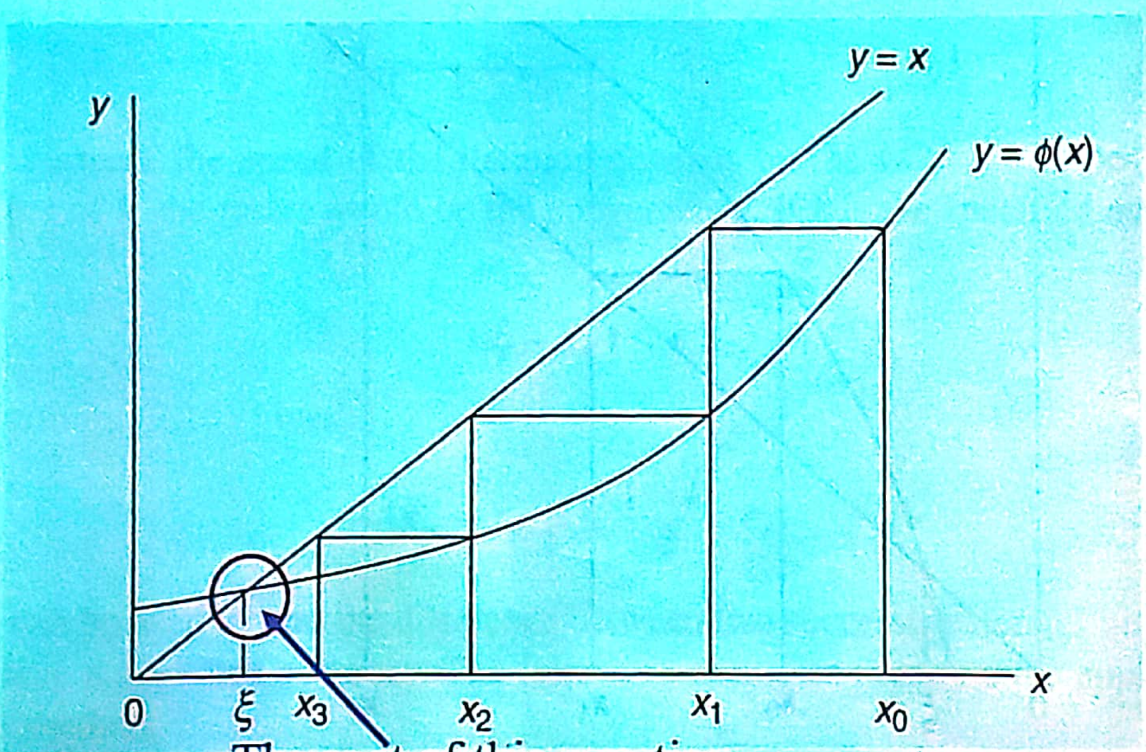
- Any arbitrary approximation x_0, x_1, x_2 does not assure that it will converge to the actual root x of the equation.
  - E.g. $x = 10^x + 1$,
  - if $x_0 = 0, x_1 = 2, x_2 = 101, \ldots$. that does not converge to the actual root x
  - As n increase, x_n increases without limit!
- The equation $x = f(x)$ converges to the real root x,
  - if $f(x)$ is continuous
  - If $|f'(x)| < 1$
- The equation $x = f(x)$ does not converges to the real root x if $|f'(x)| > 1$

Iteration Method: Convergence Conditions

- Any arbitrary approximation x_0, x_1, x_2 does not assure that it will converge to the actual root x of the equation.
  - E.g. $x = 10^x + 1$,
  - if $x_0 = 0, x_1 = 2, x_2 = 101, \ldots$ that does not converge to the actual root x
  - As n increase, x_n increases without limit!
- The equation $x = f(x)$ converges to the real root x,
  - if $f(x)$ is continuous
  - If $|f'(x)| < 1$
- The equation $x = f(x)$ does not converges to the real root x if $|f'(x)| > 1$
- Therefore, $g(x) = 0$ has to be re-written as $x = f(x)$ in such a way that $|f'(x)| < 1$

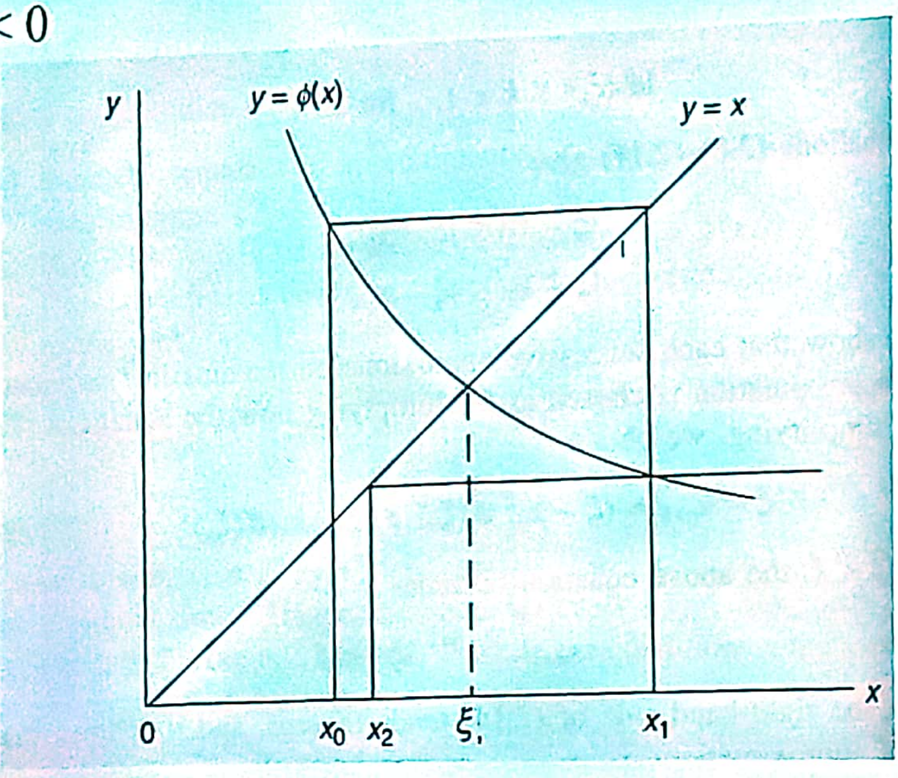
Iteration Method: Convergence Conditions

Convergence of $x_{n+1} = f(x_n)$, when $|f'(x)| < 1$



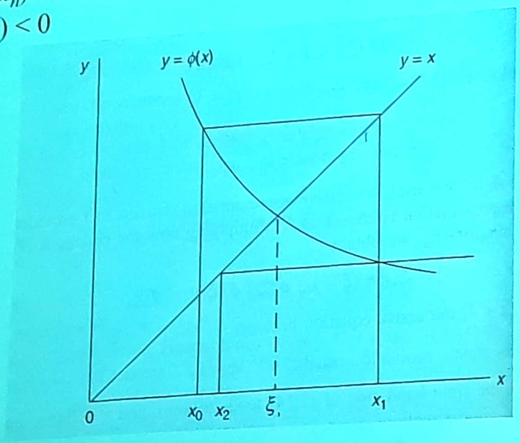
Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ oscillates but ultimately converges, when $|f'(x)| < 1$, but $f'(x) < 0$



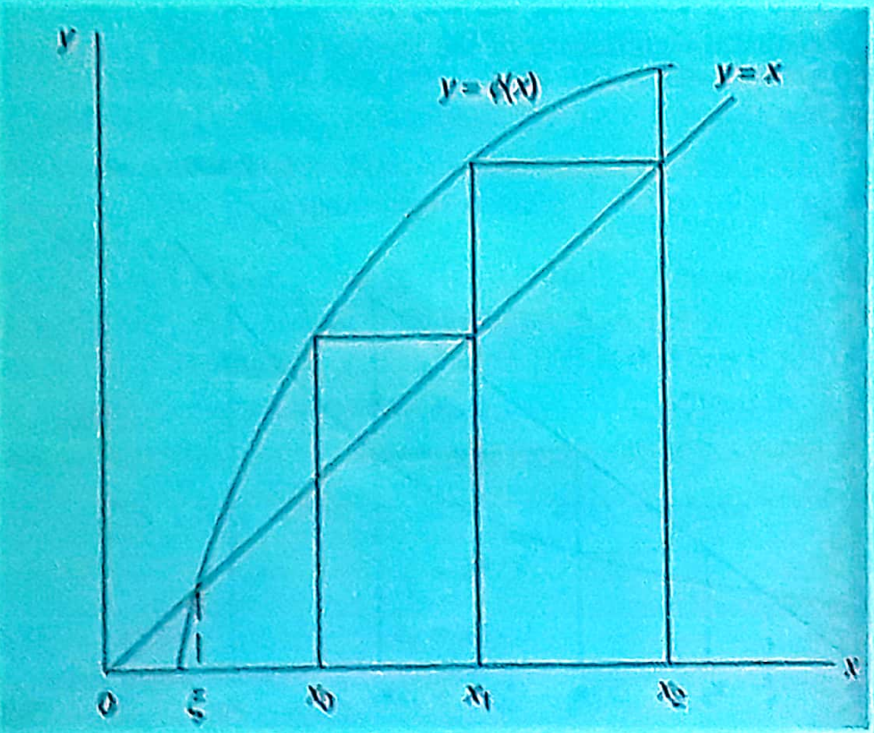
Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ oscillates but ultimately converges, when $|f'(x)| < 1$, but $f'(x) < 0$



Iteration Method: Convergence Conditions

$x_{n+1} = f(x_n)$ diverges, when $f'(x) > 1$



Iteration Method: Example

Solve $x = 2 + sin(x)/2$

Solution

Here $f(x) = 2 + sin(x)/2$

Starting with $x_0 = 2$ we calculate $x_1, x_2,...$

x_0	2
$x_1 = f(x_0)$	2.454648713
$x_2 = f(x_1)$	2.31708862
$x_3 = f(x_2)$	2.367105575
$x_4 = f(x_3)$	2.349674771
$x_5 = f(x_4)$	2.355850929
$x_6 = f(x_5)$	2.353674837
$x_7 = f(x_6)$	2.354443099
$x_8 = f(x_7)$	2.354172058
$x_9 = f(x_8)$	2.354267705
$x_{10} = f(x_9)$	2.354233955
$x_{11} = f(x_{10})$	2.354245864

The Method of False Position Or Regula Falsi

- Like the bisection method, Method of False Position requires two initial guesses x_a and x_b such that $f(x) = 0$ and $f(x_a)$ and $f(x_b)$ has opposite signs.
- Since the graph of $y = f(x)$ crosses the x-axis between these two points, a root must lie in between these points.
- The difference between these two methods is, instead of simply dividing the region in two, it obtains a new point x_1 which is (hopefully, but not necessarily) closer to the root.
- If $f(x_a)$ and $f(x_1)$ has opposite signs, then the new interval to be explored is $[x_a, x_1]$.

The Method of False Position

- The point of intersection in the present case is given by putting $y = 0$ in the equation

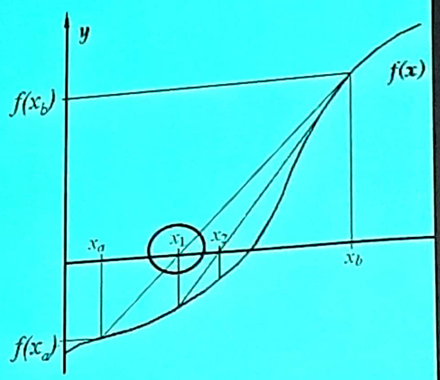
$$\frac{y - f(x_a)}{x - x_a} = \frac{f(x_b) - f(x_a)}{x_b - x_a}$$

- Thus we obtain

$$x = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$

- Hence, the approximate root is

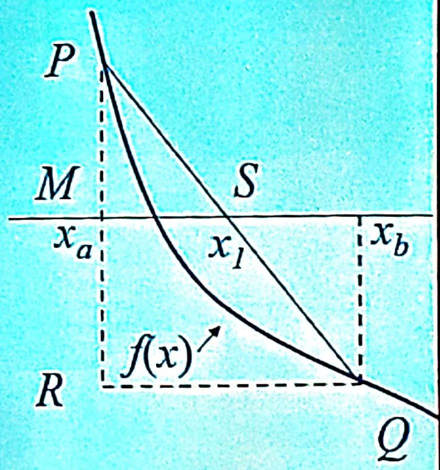
$$x_1 = x_a - \frac{f(x_a)}{f(x_b) - f(x_a)}(x_b - x_a)$$



The Method of False Position : Geometric Significance

Here, for Δ PMS and Δ PRQ

- MS/MP = RQ/RP
- $(x_1 - x_a)/f(x_a) = (x_b - x_a) /(f(x_b)+- f(x_a))$
- $x_1 - x_a = f(x_a)(x_b - x_a) /(f(x_a)+- f(x_b))$
- $x_1 = x_a - f(x_a)(x_b - x_a) /(f(x_b) - f(x_a))$



The Method of False Position: Example

Find the real root of the equation till 2 decimal place

$$f(x)= x^3 - 2x - 5 = 0$$

We observe that $f(2) = -1$ and $f(3) = 16$

And hence a root lies between 2 and 3. Then

x0	x1	x2	f(x0)	f(x1)	f(x2)
2	3	2.058824	-1	16	-0.3908
2.058824	3	2.081264	-0.3908	16	-0.1472
2.081264	3	2.089639	-0.1472	16	-0.05468
2.089639	3	2.09274	-0.05468	16	-0.0202
2.09274	3	2.093884	-0.0202	16	-0.00745

x1	2.059
x2	2.081
x3	2.090
x4	2.093

x_4 is correct to 2 decimal places.

The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place

$$x^3 - 2x^2 + 3x = 5$$ between the points 1 and 2.

Result 1.843734

The Method of False Position: Example

Class Work

Find the real root of the equation till 2 decimal place

$$\sin x + x - 1 = 0.$$

Result 0.510973

Advantages:

1. Simple
2. Brackets the Root

Disadvantages:

1. Can be VERY slow
2. Like Bisection, need an initial interval around the root.

Newton-Raphson Method

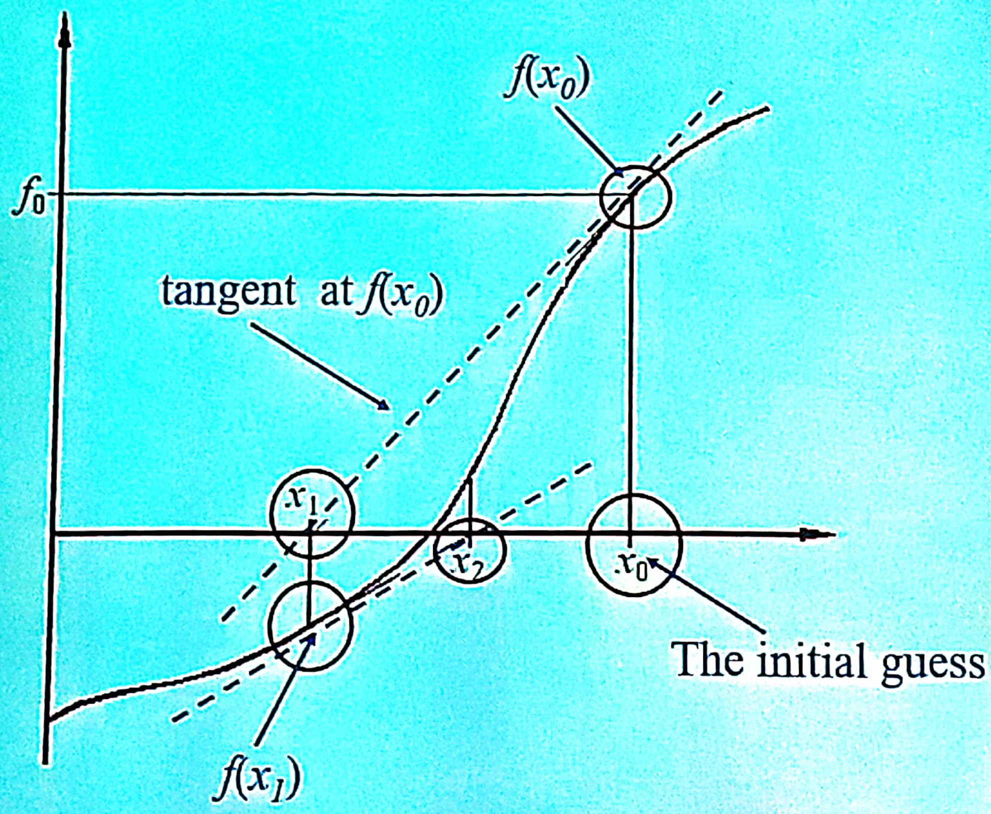
- This method is more efficient than the Bisection and Iteration methods.

If

- x is the real root and x_0 is an initial approximation of the real root of an equation $f(x) = 0$,
- $f'(x_0) \neq 0$,
- $f(x)$ has the same sign between x_0 and x,

Then, the tangent at $f(x_0)$ can lead to the real root x.

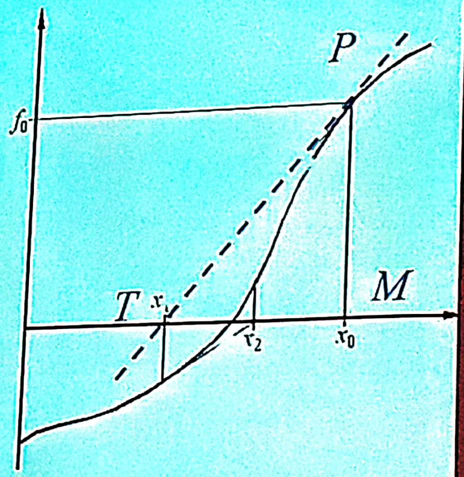
Newton-Raphson Method: Geometric Significance



Newton-Raphson Method: Geometric Significance

Here,

- The slope at x_1 is $tan\ (PTM)$
- $tan\ (PTM) = PM/\text{TM}$
- $tan\ (PTM) = f(x_0)/h$
- Again, $tan\ (PTM) = f'(x_0)$
- Therefore, $f'(x_0) = f(x_0)/h$
- Or, $h = f(x_0)/f'(x_0)$
- $x_1 = x_0 - h$
- Therefore, $x_1 = x_0 - f(x_0)/f'(x_0)$



Newton-Raphson Method

Methodology

- Let x_0 be an approximate root of $f(x) = 0$ and
- Let, x_1 is the correct root such that $x_1 = x_0 + h$ and $f(x_1) = 0$.
- Expanding $f(x_0+h)$ by Taylor's series, we obtain,

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0$$

- Neglecting the second and higher order derivatives, we have

$$f(x_0) + hf'(x_0) = 0$$

- Which gives

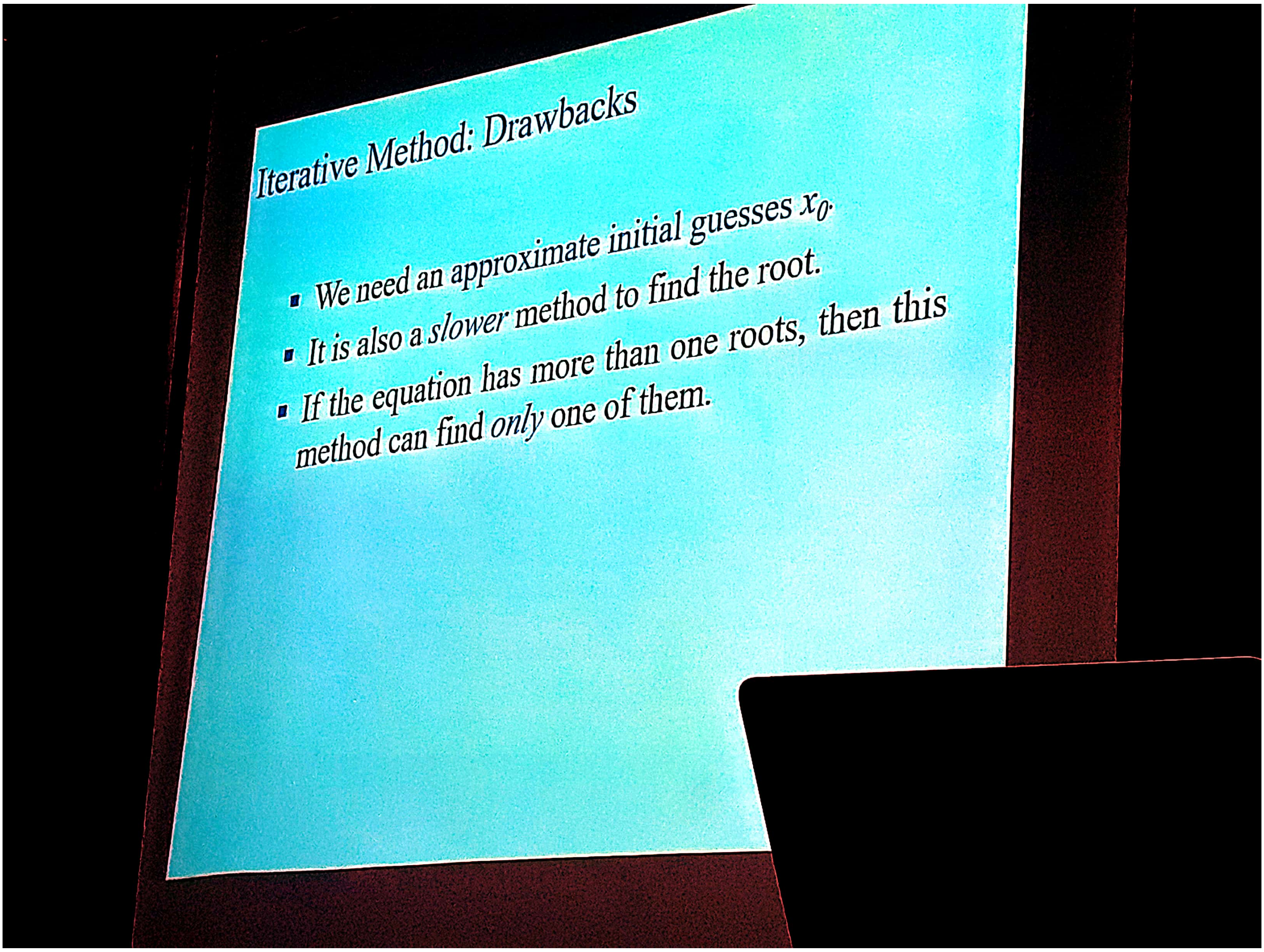
$$h = -\frac{f(x_0)}{f'(x_0)}$$

Newton-Raphson Method (Cont'd.)

- A better approximation than x_0 is therefore given by x_1 where

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)}$$

- Successive approximation are given by $x_2, x_3, \ldots., x_n, x_{n+1}$ where


Iterative Method: Drawbacks
We need an approximate initial guesses x_0.
It is also a *slower* method to find the root.
If the equation has more than one roots, then this method can find *only* one of them.

Acceleration of Convergence : Aitken's Δ^2 Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k|\xi - x_n|, \qquad k<1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x=\xi$ of the equation $x=\phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \qquad (2.22)$$

Acceleration of Convergence : Aitken's Δ^2 Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k\,|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \tag{2.22}$$

Acceleration of Convergence : Aitken's Δ^2 Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \tag{2.22}$$

Acceleration of Convergence : Aitken's Δ^2 Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \qquad (2.22)$$

Acceleration of Convergence : Aitken's Δ^2 Process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k\,|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \qquad (2.22)$$

According to Aitken's

As before,

$$
\begin{array}{lll}
\hline
x_1 = 1.5 & & \\
 & 0.035 & \\
x_2 = 1.535 & & -0.052 \\
 & -0.017 & \\
x_3 = 1.518 & & \\
\hline
\end{array}
$$

Hence we obtain from Eq. (2.23)

$$x_4 = 1.518 - \frac{(-0.017)^2}{-0.052} = 1.524$$

which corresponds to six normal iterations.

$x_{11} = 0.5664147,$ $\qquad$ $x_{12} = 0.5675566,$

$x_{13} = 0.5669089,$ $\qquad$ $x_{14} = 0.5672762,$

$x_{15} = 0.5670679,$ $\qquad$ $x_{16} = 0.567186.$

$x_{17} = 0.567119,$ $\qquad$ $x_{18} = 0.567157.$

$x_{19} = 0.5671354,$ $\qquad$ $x_{20} = 0.5671477.$

19-09-18

Acceleration of convergence: Aitken's Δ^2-process

From the relation

$$|\xi - x_{n+1}| = |\phi(\xi) - \phi(x_n)| \le k\,|\xi - x_n|, \qquad k < 1$$

it is clear that the iteration method is linearly convergent. This slow rate of convergence can be accelerated by using Aitken's method, which is described below.

Let x_{i-1}, x_i, x_{i+1} be three successive approximations to the desired root $x = \xi$ of the equation $x = \phi(x)$. From Section 2.4, we know that

$$\xi - x_i = k(\xi - x_{i-1}), \qquad \xi - x_{i+1} = k(\xi - x_i)$$

Dividing, we obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} = \frac{\xi - x_{i-1}}{\xi - x_i},$$

which gives on simplification

$$\xi = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}. \tag{2.22}$$

If we now define Δx_i and $\Delta^2 x_i$ by the relations

$$\Delta x_i = x_{i+1} - x_i \quad \text{and} \quad \Delta^2 x_i = \Delta(\Delta x_i),$$

then

$$\begin{aligned}
\Delta^2 x_{i-1} &= \Delta(\Delta x_{i-1}) \\
&= \Delta(x_i - x_{i-1}) \\
&= \Delta x_i - \Delta x_{i-1} \\
&= x_{i+1} - x_i - (x_i - x_{i-1}) \\
&= x_{i+1} - 2x_i + x_{i-1}.
\end{aligned}$$

Hence (2.22) can be written in the simpler form

$$\xi = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}, \tag{2.23}$$

which explains the term Δ^2-process.

In any numerical application, the values of the following underlined quantities must be obtained.

x_{i-1}		
	Δx_{i-1}	
x_i		$\Delta^2 x_{i-1}$
	Δx_i	
x_{i+1}		

***Example* 2.9** We consider again Example 2.7, viz., the equation

$$x = \frac{1}{2}(3 + \cos x)$$

As before,

In any numerical application, the values of the following underlined quantities must be obtained.

x_{i-1}		
	Δx_{i-1}	
x_i		$\Delta^2 x_{i-1}$
	Δx_i	
x_{i+1}		

Example 2.9 We consider again Example 2.7, viz., the equation

$$x = \frac{1}{2}(3 + \cos x)$$

As before,

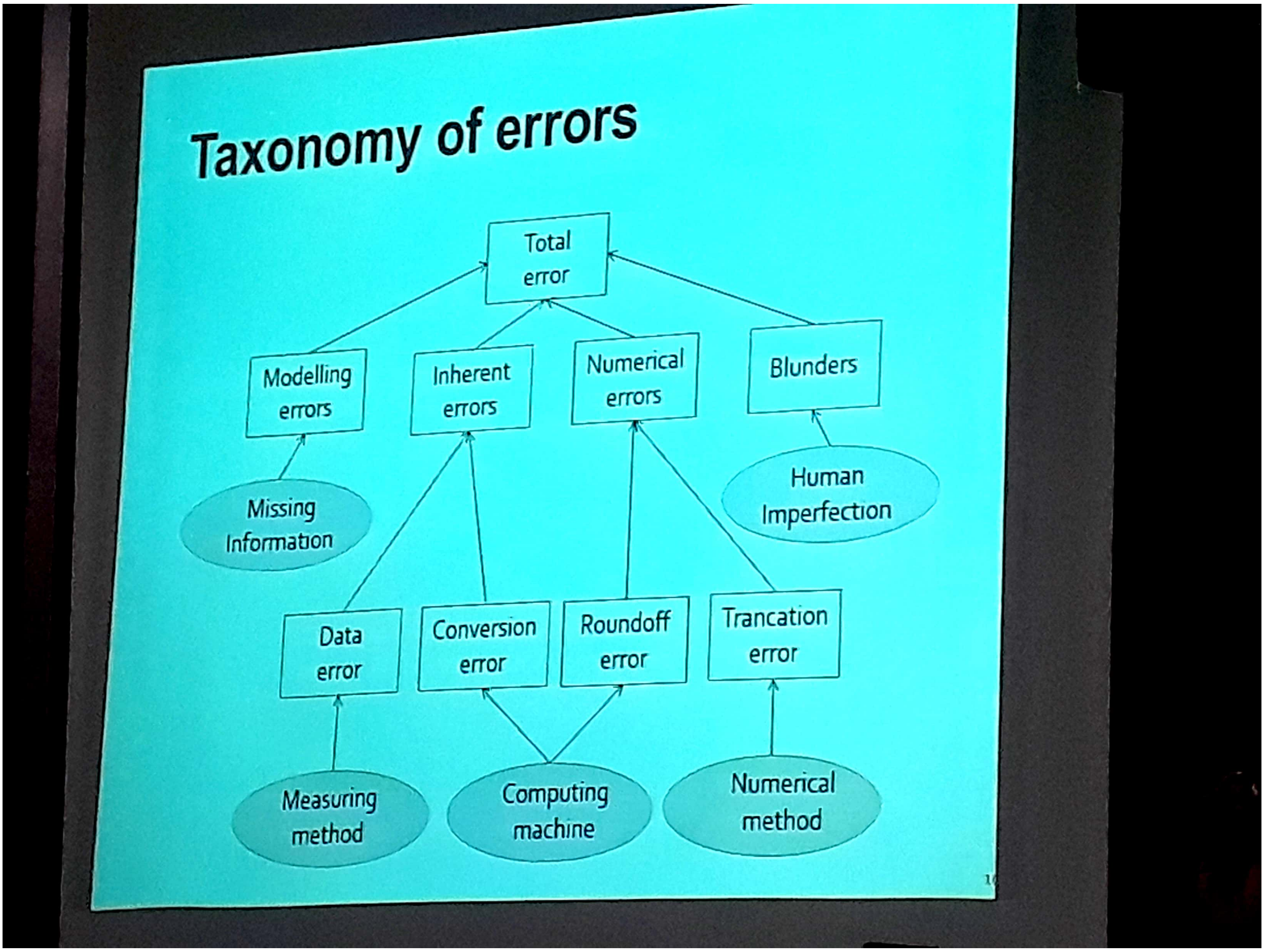
$x_1 = 1.5$		
	0.035	
$x_2 = 1.535$		−0.052
	−0.017	
$x_3 = 1.518$		

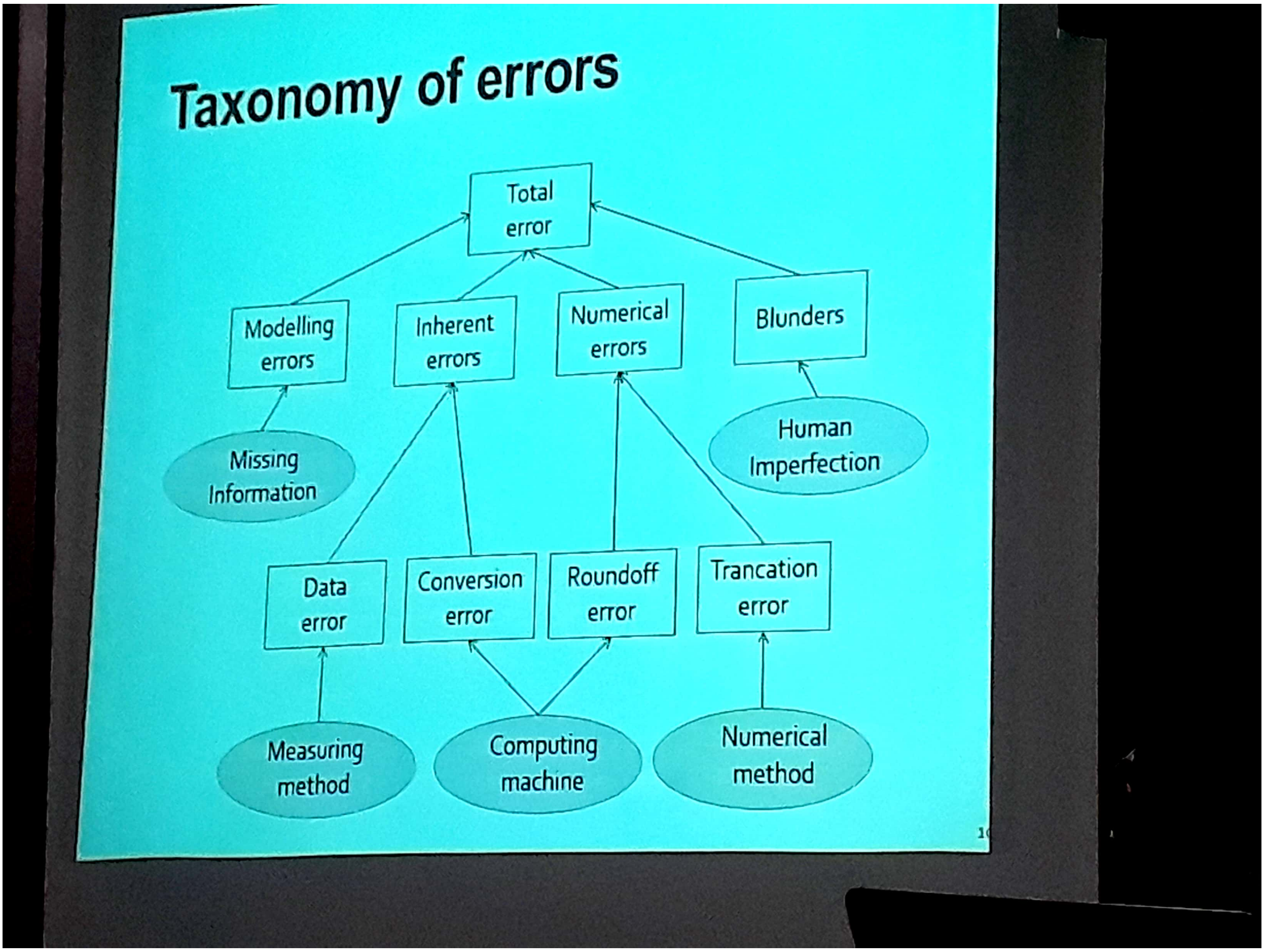
Hence we obtain from Eq. (2.23)

$$x_4 = 1.518 - \frac{(-0.017)^2}{-0.052} = 1.524,$$

which corresponds to six normal iterations.

2.5 NEWTON–RAPHSON METHOD

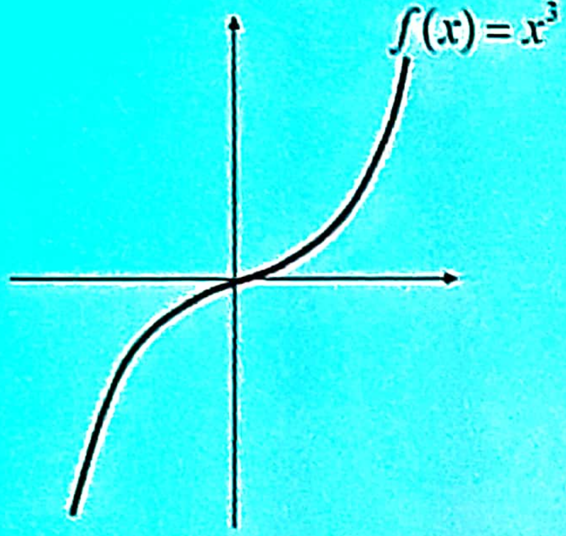

Taxonomy of errors
Total error
Modelling errors
Inherent errors
Numerical errors
Blunders
Missing Information
Human Imperfection
Data error
Conversion error
Roundoff error
Trancation error
Measuring method
Computing machine
Numerical method

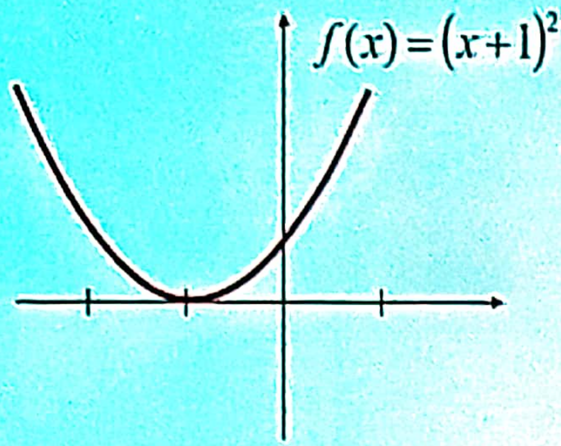

Taxonomy of errors
Total error
Modelling errors
Inherent errors
Numerical errors
Blunders
Missing Information
Human Imperfection
Data error
Conversion error
Roundoff error
Trancation error
Measuring method
Computing machine
Numerical method

Newton-Raphson Method: Drawbacks

- The Newton-Raphson method requires the calculation of the *derivative* of a function, which is not always easy.
- If f' vanishes at an iteration point, then the method will fail to converge.
- When the step is too large or the value is oscillating, other more conservative methods should take over the case.

Multiple Roots





f(x) has three zeros at x = 0

f(x) has two zeros at x = -1

Generalized Newton's Method

- If ξ is a root of $f(x)=0$ with multiplicity p, then the generalized Newton's formula is

$$x_{n+1} = x_n - p\frac{f(x)}{f'(x)}$$

- Since ξ is a root of $f(x) = 0$ with multiplicity p, it follows that ξ is a root of $f'(x) = 0$ with multiplicity $(p - 1)$, of $f''(x) = 0$ with multiplicity $(p - 2)$, and so on.

Example

Example Find a double root of the equation

$$f(x) = x^3 - x^2 - x + 1 = 0.$$

Here $f'(x) = 3x^2 - 2x - 1$, and $f''(x) = 6x - 2$. With $x_0 = 0.8$, we obtain

$$x_0 - 2\frac{f(x_0)}{f'(x_0)} = 0.8 - 2\frac{0.072}{-(0.68)} = 1.012,$$

and

$$x_0 - \frac{f'(x_0)}{f''(x_0)} = 0.8 - \frac{-(0.68)}{2.8} = 1.043,$$

Example

Example Find a double root of the equation

$$f(x)=x^3-x^2-x+1=0.$$

Here $f'(x)=3x^2-2x-1$, and $f''(x)=6x-2$. With $x_0=0.8$, we obtain

$$x_0-2\frac{f(x_0)}{f'(x_0)}=0.8-2\frac{0.072}{-(0.68)}=1.012,$$

and

$$x_0-\frac{f'(x_0)}{f''(x_0)}=0.8-\frac{-(0.68)}{2.8}=1.043,$$

Newton's Method (Review)

Assumptions: $f(x)$, $f'(x)$, x_0 *are available*,

$f'(x_0) \neq 0$

Newton's Method new estimate:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Problem:

$f'(x_i)$ is not available,

or difficult to obtain analytically.

Secant Method

- We have seen that the Newton-Raphson method requires the evaluation of derivatives of the function and this is not always possible, particularly in the case of functions arising in practical problems.
- In the secant method, the derivative at x_n is approximated by the formula

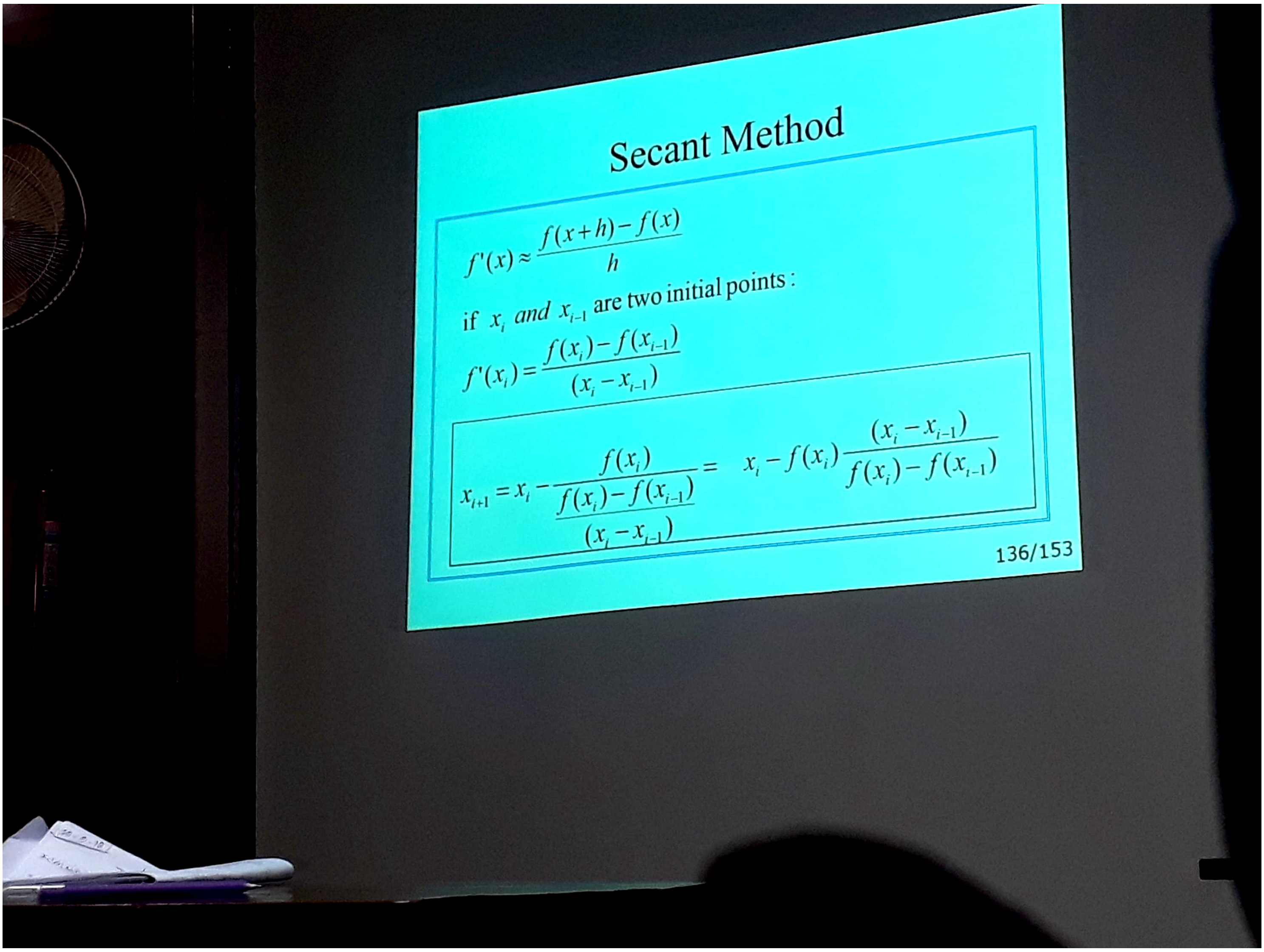
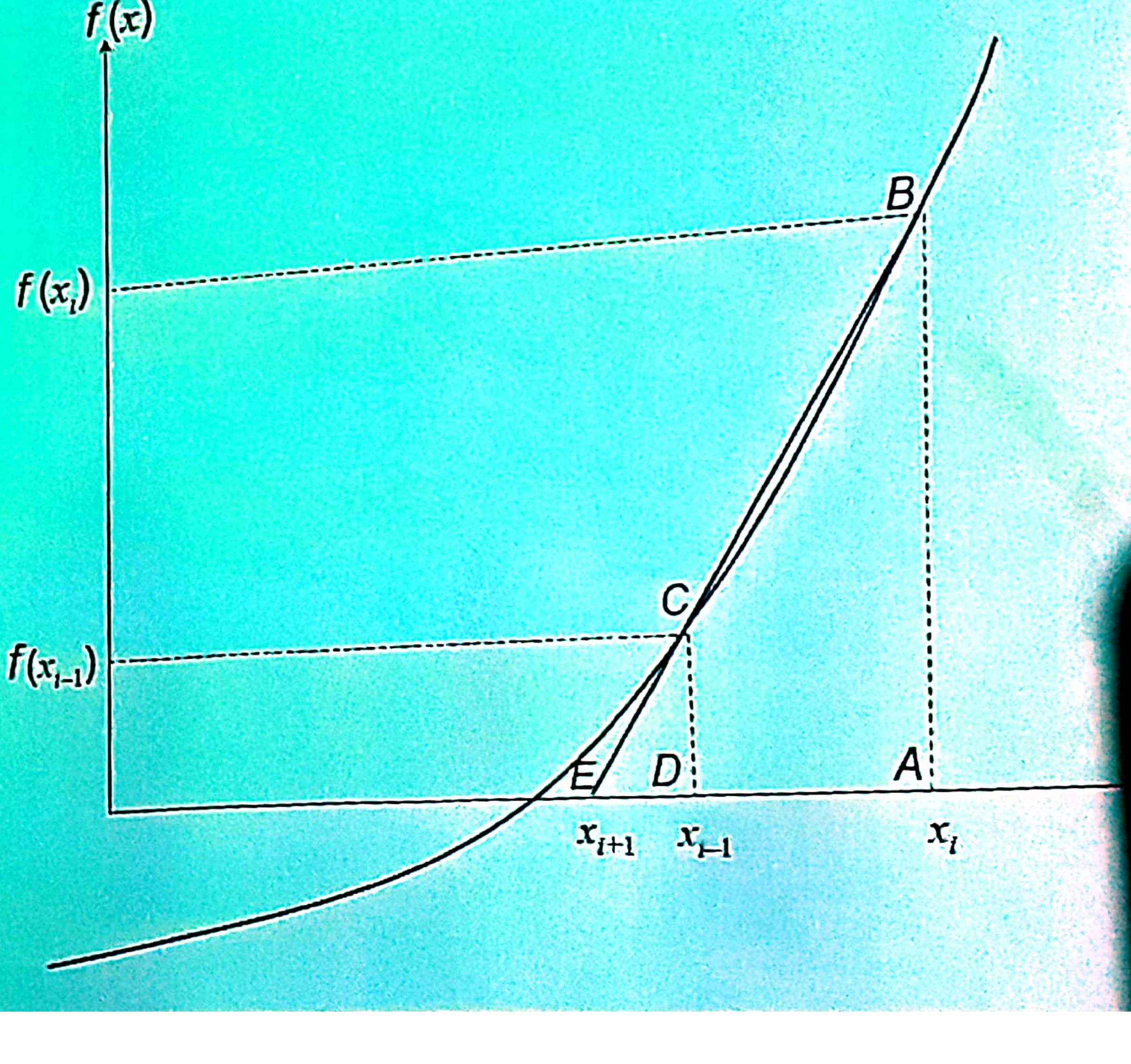

Secant Method
$f'(x) \approx \frac{f(x+h)-f(x)}{h}$
if x_i *and* x_{i-1} are two initial points :
$f'(x_i) = \frac{f(x_i)-f(x_{i-1})}{(x_i - x_{i-1})}$
$x_{i+1} = x_i - \frac{f(x_i)}{\frac{f(x_i)-f(x_{i-1})}{(x_i - x_{i-1})}} = x_i - f(x_i)\frac{(x_i - x_{i-1})}{f(x_i)-f(x_{i-1})}$
136/153

Figure 1 Geometrical represe[...] of the secant method



Derivation of Secant Method (continued)

- The secant method can also be derived from geometry, as shown in Figure 1. Taking two initial guesses, and , one draws a straight line between and passing through the x-axis at x_i. ABE and DCE are similar triangles.
- Hence

$$\frac{AB}{AE}=\frac{DC}{DE}$$

$$\frac{f(x_i)}{x_i-x_{i+1}}=\frac{f(x_{i-1})}{x_{i-1}-x_{i+1}}$$

- On rearranging, the secant method is given as

$$x_{i+1}=x_i-\frac{f(x_i)(x_i-x_{i-1})}{f(x_i)-f(x_{i-1})}$$

Apply Secant Method in the floating ba
problem

$$f(x) = x^3 - 0.165x^2 + 3.993 \times 10^{-4}$$

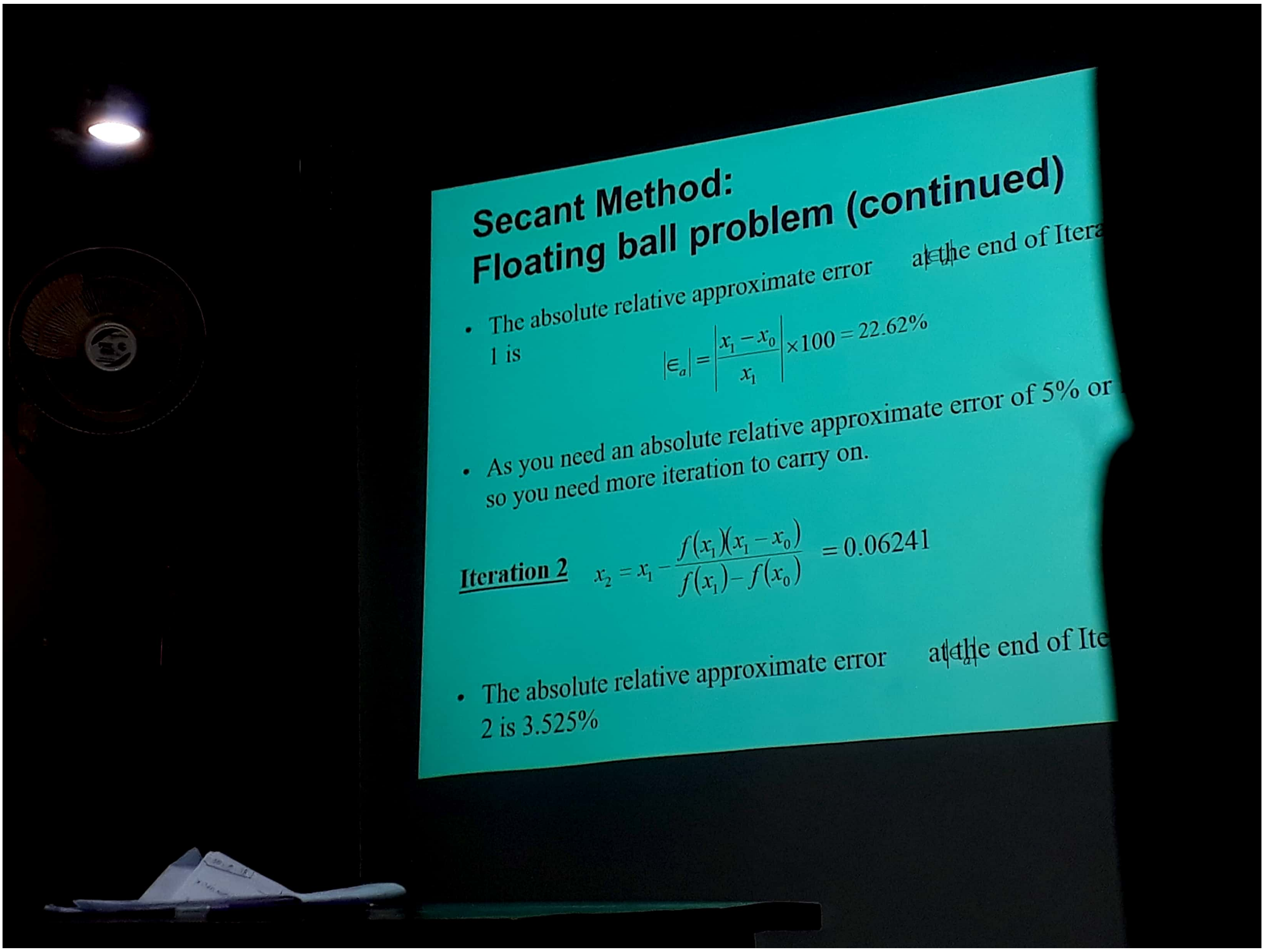
- Let us assume the initial guesses of the root of $f(x)=0$ as $x_{-1}=0.02$ and $x_0=0.05$

$$x_1 = x_0 - \frac{f(x_0)(x_0 - x_{-1})}{f(x_0) - f(x_{-1})}$$

$$= x_0 - \frac{(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4})\times(x_0 - x_{-1})}{(x_0^3 - 0.165x_0^2 + 3.993\times10^{-4}) - (x_{-1}^3 - 0.165x_{-1}^2 + 3.993\times10}$$

$$= 0.05 - \frac{[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}]\times[0.05 - 0.02]}{[0.05^3 - 0.165(0.05)^2 + 3.993\times10^{-4}] - [0.02^3 - 0.165(0.02)^2 + 3.993\times10}$$

$$= 0.06461$$

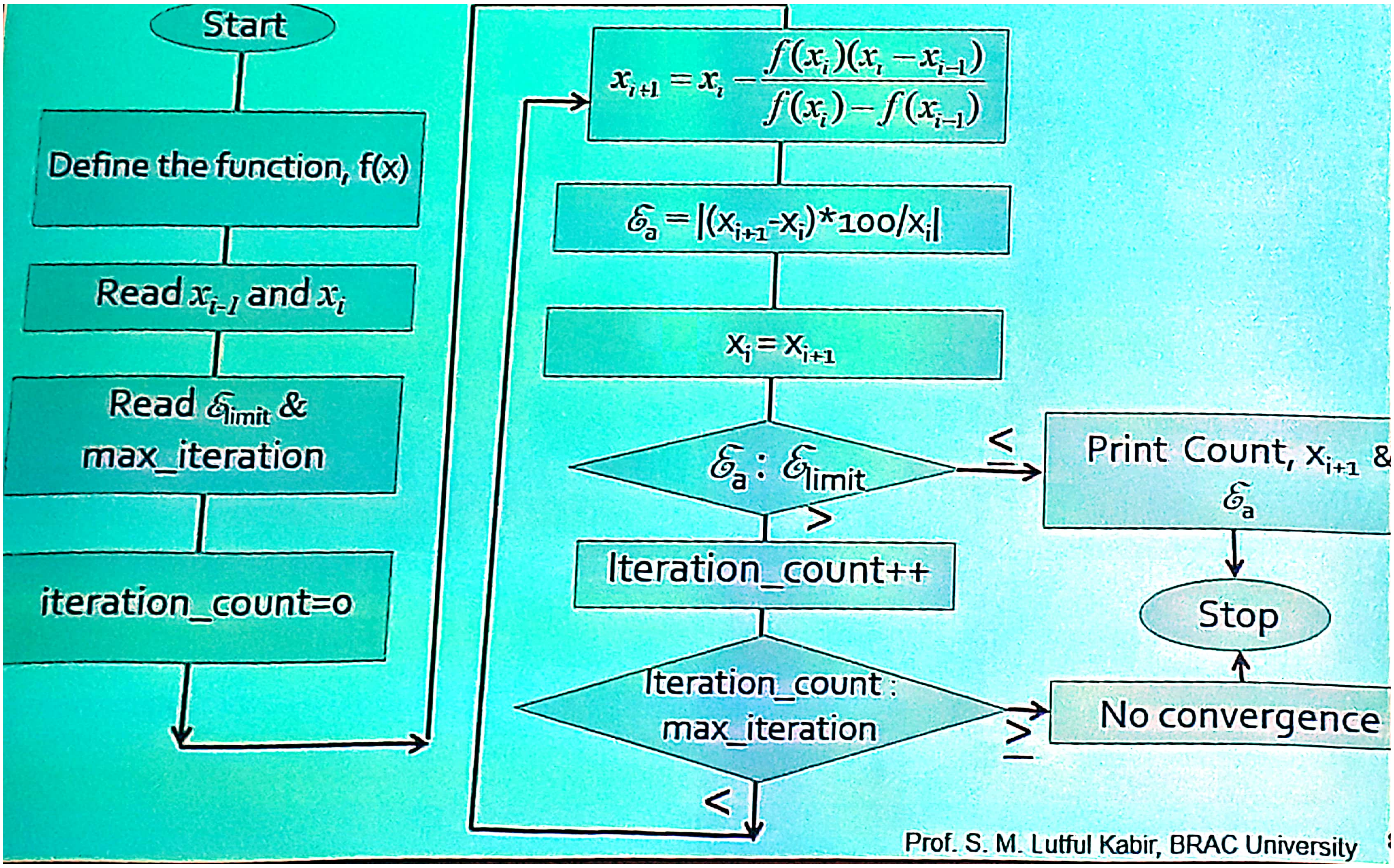

Secant Method:
Floating ball problem (continued)
The absolute relative approximate error $|\epsilon_a|$ at the end of Itera
1 is
$|\epsilon_a| = \left|\frac{x_1 - x_0}{x_1}\right| \times 100 = 22.62\%$
As you need an absolute relative approximate error of 5% or
so you need more iteration to carry on.
Iteration 2
$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 0.06241$
The absolute relative approximate error $|\epsilon_a|$ at the end of Ite
2 is 3.525%

Iteration 3 for the floating ball problem (Secant Method)

- $X_3 = 0.06238$
- The absolute relative approximate error at the end of Iteration 3 is 0.0595%
- Table 1 shows the secant method calculations for the results from the above problem.

Table 1: Secent Method Result as Function of Iteration

Iteration Number, I	x_{i-1}	x_i	x_{i+1}	$\lvert\in_a\rvert\%$	$f(x_{i+1})$
1	0.02	0.05	0.06461		$-1.9812\text{X}10^{-5}$
2	0.05	0.06461	0.06241	22.62	$-3.2852\text{X}10^{-7}$
3	0.06461	0.06241	0.06238	3.525	$2.0252\text{X}10^{-9}$
4	0.06241	0.06238	0.06238	0.0595	$-1.8576\text{X}10^{-13}$


Start
Define the function, f(x)
Read x_{i-1} and x_i
Read $\mathcal{E}_{limit}$ & max_iteration
iteration_count=0
$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$
$\mathcal{E}_a = |(x_{i+1}-x_i)*100/x_i|$
$x_i = x_{i+1}$
$\mathcal{E}_a : \mathcal{E}_{limit}$
$\leq$
$>$
Print Count, x_{i+1} & $\mathcal{E}_a$
Stop
Iteration_count++
Iteration_count : max_iteration
$\geq$
$<$
No convergence
Prof. S. M. Lutful Kabir, BRAC University

Advantages of Secant Method

1. It converges at faster than a linear rate, so that it is more rapidly convergent than the bisection method.
2. It does not require use of the derivative of the function, something that is not available in a number of applications.
3. It requires only one function evaluation per iteration, as compared with Newton's method which requires two.

Disadvantages of Secant Method

1. It may not converge.
2. There is no guaranteed error bound for the computed iterates.
3. It is likely to have difficulty if $f'(\alpha) = 0$. This means the x-axis is tangent to the graph of $y = f(x)$ at $x = \alpha$.
4. Newton's method generalizes more easily to new methods for solving simultaneous systems of nonlinear equations.