HW5: Question 1.1

Ian Dover

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Solve the following problem using ADMM:

$$\underset{x}{min}\left(\frac{1}{2}x^TPx + q^Tx + \frac{\lambda}{2}||z||_2^2\right) \text{ s.t. } x = z \text{ and } a \leq z \leq b$$

1 Part 1

Write the augmented Lagrangian function (the scaled form) and drive the ADMM updates.

$$f(x) = \frac{1}{2}x^TPx + q^Tx$$

$$g(z) = \frac{\lambda}{2}||z||_2^2$$

The augmented Lagrangian can be written as:

$$\underset{x}{min}\left(f(x) + g(z) + \frac{\rho}{2}||x - z + w||_2^2 + \frac{\rho}{2}||w||_2^2\right) \text{ s.t. } a \leq z \leq b$$

$$\text{where } w = \frac{u}{p}$$

Below is how we would drive the updates:

ADMM Algorithm

Select ρ and ϵ
Initialize w_0 and $z_0; w_0 = \frac{u_0}{\rho}$
for k = 1 : K do
  Compute $x_{k+1} = \underset{x}{argmin}\left[\frac{1}{2}x^T Px + q^T x + \frac{\rho}{2}||x - z_k + w_k||_2^2\right]$
  Compute $z_{k+1} = \underset{z}{argmin}\left[\frac{\lambda}{2}||z||_2^2 + \frac{\rho}{2}||x_{k+1} - z + w_k||_2^2\right]$
  if $z_{k+1} > b$
    $z_{k+1} = b$
  end if

  if $z_{k+1} < a$
    $z_{k+1} = a$
  end if

  Compute $w_{k+1} = w_k + x_{k+1} - z_{k+1}$

  if $||x_{k+1} - z_{k+1}||_2 < \epsilon$
    Break
  end if
end for

The minimization of both x and z can be determined by setting the partial derivatives to 0. This can be formally expressed as seen below:

Solve for x (1):

$$\frac{\partial\left[\frac{1}{2}x^T Px + q^T x + \frac{\rho}{2}||x - z_k + w_k||_2^2\right]}{\partial x} = 0$$

and

Solve for z (2):

$$\frac{\partial\left[\frac{\lambda}{2}||z||_2^2 + \frac{\rho}{2}||x_{k+1} - z + w_k||_2^2\right]}{\partial z} = 0$$

(1) can be rewritten:

$$\frac{1}{2}(P + P^T)x + q^T + \rho||x - z_k + w_k||_2 = 0$$

$$\frac{1}{2}(P + P^T)x + q^T + \rho(x - z_k + w_k) = 0$$
$$\frac{1}{2}(P + P^T + 2\rho I)x + q^T + \rho(-z_k + w_k) = 0$$

where I represents the identity matrix of size $n \times n$.

Solving for x, we get:

$$x = -\frac{q^T + \rho(-z_k + w_k)}{\frac{1}{2}(P + P^T + 2\rho I)}$$

Another form of this solution is:

$$x = -(q^T + \rho(-z_k + w_k)) \times 2(P + P^T + 2\rho I)^{-1}$$

(2) can be rewritten:

$$\lambda||z||_2 + \rho||x_{k+1} - z + w_k||_2 = 0$$

$$\lambda(z) + \rho(x_{k+1} - z + w_k) = 0$$
$$(\lambda - \rho)z + \rho(x_{k+1} + w_k) = 0$$

Solving for z, we get:

$$z = -\frac{\rho(x_{k+1} + w_k)}{(\lambda - \rho)}$$

The ADMM algorithm can now be rewritten:

ADMM Algorithm

01. Select ρ and ϵ
02. Initialize w_0 and $z_0; w_0 = \frac{u_0}{\rho}$
03. **for** k = 1 : K do
04. Compute $x_{k+1} = -(q^T + \rho(-z_k + w_k)) \times 2(P + P^T + 2\rho I)^{-1}$
05. Compute $z_{k+1} = -\frac{\rho(x_{k+1}+w_k)}{(\lambda-\rho)}$
06. **if** $z_{k+1} > b$
07. $z_{k+1} = b$
08. **end if**

09. **if** $z_{k+1} < a$
10. $z_{k+1} = a$
11. **end if**

12. Compute $w_{k+1} = w_k + x_{k+1} - z_{k+1}$

13. **if** $||x_{k+1} - z_{k+1}||_2 < \epsilon$

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        Break
    end if
end for
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