Mid-Term: Question 4, Part 1

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March 2023

1 Part 1-1

For each of the following functions, determine whether it is convex, concave, or neither. Support your argument.

1.1 Part A

The equation we are analyzing is:

$$f(x) = e^{x^2} \text{ on dom } f = R$$

Let us take the second derivative of this equation:

$$\frac{\partial^2 f(x)}{\partial x} = 2e^{x^2}(1 + 2x^2)$$

To determine if this equation is convex, we must determine if this second derivative is positive (and non-zero) on all values of the domain. This can be conditionally expressed as:

$$2e^{x^2}(1 + 2x^2) \geq 0$$

To prove this conditional statements holds true, we must show that each individual factor is positive for all values of the domain:

$$2e^{x^2} \geq 0$$

$$(1 + 2x^2) \geq 0$$

From simple inspection, we can show that any real number squared is positive and greater than zero. Therefore, both factors are positive (and non-zero) on all values of the domain. If all factors are positive, then their product must also be positive. **This means that the equation satisfies the second derivative condition for convexity.**

1.2 Part B

The equation we are analyzing is:

$$f(x) = log(1 + e^x) \text{ on dom } f = R$$

Let us take the second derivative of this equation:

$$\frac{\partial^2 f(x)}{\partial x} = \frac{e^x}{(1 + e^x)^2}$$

To determine if this equation is convex, we must determine if this second derivative is positive (and non-zero) on all values of the domain. This can be conditionally expressed as:

$$\frac{e^x}{(1 + e^x)^2} \geq 0$$

From inspection, we can see that all factors containing x are positive and non-zero on all values of the domain. Additionally, as the domain approaches ∞, the product approaches 0 from the positive direction. This can be expressed as:

$$\lim_{\text{x}\to\infty} \frac{e^x}{(1 + e^x)^2} = 0_+$$

Conversely, as the domain approaches $-\infty$, it also converges to 0 from the positive direction as expressed below:

$$\lim_{\text{x}\to-\infty} \frac{e^x}{(1 + e^x)^2} = 0_+$$

This proves that the second-order derivative is positive at all values of the domain. **Therefore, this equation satisfies the second-order condition for convexity.**

1.3 Part C

The equation we are analyzing is:

$$f(x_1, x_2) = \tfrac{x_1}{x_2} \text{ on dom } f = R^2_{++}$$

To determine if this equation is convex or concave, we must evaluate the determinant of the Hessian and determine if it it positive at all points in the domain of x_1 and x_2; next, we must determine if the equation is either concave or convex by the value of $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}$.

To construct the Hessian of the equation, we must first derive the second-order partial derivatives with respect to both variables in the domain of the function.

The Hessian can be expressed as:

$$H|f(x_1, x_2)| = \begin{bmatrix} \frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} & \frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f(x_1,x_2)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} \end{bmatrix}$$

Now we must evaluate the Hessian at each element:

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = 0$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = -\frac{1}{x_2^2}$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2 \partial x_1} = \frac{2x_1}{x_2^3}$$

This allows us to construct the Hessian:

$$H|f(x_1, x_2)| = \begin{bmatrix} 0 & -\frac{1}{x_2^2} \\ -\frac{1}{x_2^2} & \frac{2x_1}{x_2^3} \end{bmatrix}$$

Because then Hessian is 0 at $\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2}$, we can deduce that the equation is neither convex nor concave. **Therefore, the equation does not satisfy the conditions for convexity nor concavity.**

2 Part 1-2

2.1 Part A

Given that $f_1(x)$ and $f_2(x)$, prove that $f(x) = max\{f_1(x), f_2(x)\}$ is convex.

Use the equation for convexity given that t is in [0, 1] and for all x, y:

$$f(z) = f(tx + (1-t)y) \leq t * f(x) + (1-t) * f(y)$$

Since $f(x) = max\{f_1(x), f_2(x)\}$, then we must solve these two inequalities:

$$f_1(z) = f_1(tx + (1-t)y) \leq t * f_1(x) + (1-t) * f_1(y)$$

and

$$f_2(z) = f_2(tx + (1-t)y) \leq t * f_2(x) + (1-t) * f_2(y)$$

Since we know these two inequalities are true due to the convexity of each, we can hereby show that the maximum of the two is convex.

2.2 Part B

It is not necessarily true that the $f(x) = min\{f_1(x), f_2(x)\}$ is convex. In fact, this is ONLY true when one convex function is fully encompassed by the other convex function. This is shown in the visual proof below:

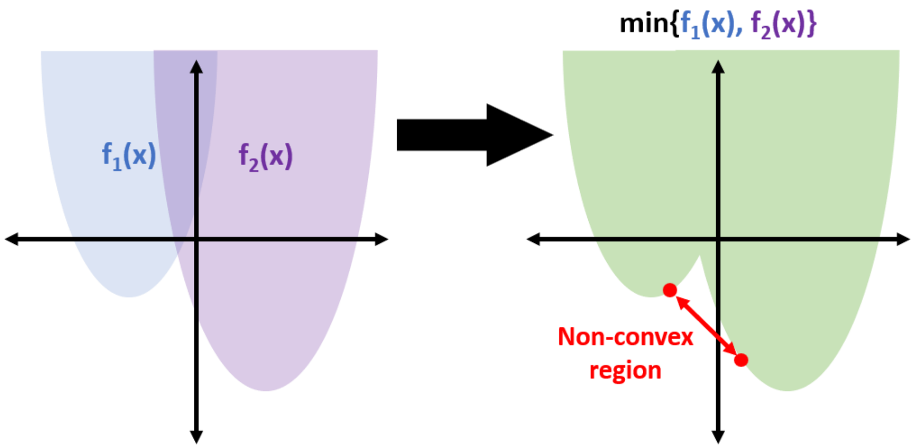

Figure 1: Visual proof showing that the minimum of two convex functions is not necessarily convex.