Question 1

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1 Problem 1

Determine if this equation is convex:

$$F(x) = \frac{1}{x}\int_0^x f(t)dt$$

Rewrite the integral as shown below:

$$F(x) = \frac{1}{x}(F(x) - F(0))$$

Convert the differential equation into its functional form:

$$F(x) = -\frac{1}{(x-1)}F(0)$$

Get the 2nd derivative of the functional form with respect to x:

$$\frac{d^2F(x)}{dx^2} = -\frac{2F(0)}{(x-1)^3}$$

The second-order condition for convexity is the below for all values of $x \in R$:

$$\frac{d^2F(x)}{dx^2} \geq 0$$

Therefore, our second derivative must satisfy the condition for all $x \in R$:

$$-\frac{2F(0)}{(x-1)^3} \geq 0$$

This condition can be reduced to:

$$\frac{1}{(x-1)^3} \leq 0$$

$$1 \leq (x-1)^3$$
$$1 \leq (x-1)$$
$$2 \leq x$$

This inequality does not hold for values of $x > 2$; **therefore, this equation does not satisfy the condition for convexity**.

2 Problem 2

Determine if the below equation is convex:

$$f_\theta(x) = \theta^{-1}x^\theta - \theta^{-1}$$

The second derivative for this function is:

$$f_\theta(x) = (\theta - 1)x^{\theta-2}$$

Write the condition for convexity:

$$(\theta - 1)x^{\theta-2} \geq 0$$

From inspection, iven the constraint $0 < \theta \leq 1$, the above second derivative is negative when $\theta < 1$ and $x > 0$. **Therefore, this equation does not satisfy the condition for convexity**.

3 Problem 3

Determine if the below equation is convex when $x_2 > 0$:

$$f(x_1, x_2) = \frac{x_1^2}{x_2}$$

In order to determine this, we must conclude if the Hessian matrix of the function is positive semi-definite for all values in the domain of $x_1, x_2 \in R, R_+$. The Hessian matrix checks for 3 conditions:

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} \geq 0$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} \geq 0$$

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} * \frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} - \left[\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2}\right]^2 \geq 0$$

Determine the second order derivatives for the equation:

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 0$$

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = \frac{2x_1}{x_2^3}$$

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1 \partial x_2} = -\frac{1}{x_2^2}$$

We can now evaluate the condition for x_1 in the Hessian:

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_1^2} = 0 \geq 0$$

We see that this condition holds true for all values in the domains of x_1, x_2. Let us evaluate the condition for x_2 in the Hessian:

$$\frac{\partial^2 f(x_1,x_2)}{\partial x_2^2} = \frac{2x_1}{x_2^3} \geq 0$$

$$\frac{2x_1}{x_2^3} \geq 0$$

$$x_1 \geq 0 \text{ or } x_2 \geq 0 \text{ or } (x_1 \leq 0 \text{ and } x_2 \leq 0)$$

These conditions are not guaranteed to be true because $x_1 \in R$ spans over the domain of all real values. **Therefore, this equation does not satisfy the condition for convexity**.

4 Problem 4

Determine if the below equation is convex when $x_2 > 0$:

$$f(x_1, x_2, x_3) = -e^{-x_1} + x_2 x_3^2$$

The Hessian for the equation is shown below:

$$H(f) = \left[\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial y \partial z}, \frac{\partial^2 f}{\partial z \partial x}, \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z^2}\right]$$

To determine if the equation is convex, we need to determine if the first principal minor is non-negative at all values in the domain $x_1, x_2, x_3 \in R$:

$$|H1| = \frac{\partial^2 f}{\partial x_1^2}$$

Additionally, we need to determine if the second principal minor is non-negative at all points in the domain $x_1, x_2, x_3 \in R$:

$$|H2| = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Finally, we need to determine if the third principal minor is non-negative at all points in the domain $x_1, x_2, x_3 \in R$:

$$|H3| = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

Let us get the first principal:

$$|H1| = \frac{\partial^2 f}{\partial x_1^2} = -e^{-x_1}$$

The first principal is negative for all values in the domain of $x_1 \in R$. **Therefore, this equation does not satisfy the condition for convexity**.