Question 2: Parts A - C

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The likelihood function is given by:

$$L(k, c; y_1, y_2, ..., y_n) = \prod_{i=1}^{n} \frac{kcy_i^{c-1}}{(1 + y_i^c)^{k+1}}$$

where c and k are positive numbers, and y_i can be found in "Question2.csv".

1 Part A

Write down the log-likelihood function (expressed as L_o):

$$L_o(k, c) = \sum_{i=1}^{n} ln\left(\frac{kcy_i^{c-1}}{(1 + y_i^c)^{k+1}}\right)$$

The properties of logarithms can be used to reduce this formulation:

$$L_o(k, c) = \sum_{i=1}^{n} ln(kcy_i^{c-1}) - ln((1 + y_i^c)^{k+1})$$

$$= \sum_{i=1}^{n} ln(k) + ln(c) + (c - 1) * ln(y_i) - (k + 1) * ln(1 + y_i^c)$$

2 Part B

Write down the corresponding maximum likelihood formulation:

$$\arg\max_{k,c} \; L_o(k, c)$$

3 Part C

Derive the gradient and Hessian of the log-likelihood function:

3.1 Gradient

The gradient (denoted as ∇) is given as the vector of all first-order partials for the log-likelihood function:

$$\nabla L_o = \left[\frac{\partial L_0}{\partial k}, \frac{\partial L_0}{\partial c}\right]$$

This gradient can be further expressed as:

$$\nabla L_o = \left[\sum_{i=1}^{n} \frac{1}{k} - ln(y_i^c + 1), \sum_{i=1}^{n} \frac{(k+1)y_i^c ln(y_i)}{y_i^{c+1}} + \frac{1}{c} + ln(y_i)\right]$$

3.2 Hessian

The Hessian (denoted as $H|L_o|$) is given as the 2 x 2 matrix of all second-order partials for the log-likelihood function:

$$H|L_o| = \begin{bmatrix} \frac{\partial^2 L_o}{\partial k^2} & \frac{\partial^2 L_o}{\partial k \partial c} \\ \frac{\partial^2 L_o}{\partial c \partial k} & \frac{\partial^2 L_o}{\partial c^2} \end{bmatrix}$$

The Hessian can be further expressed as:

$$H|L_o| = \begin{bmatrix} \sum_{i=1}^{n} -\frac{1}{k^2} & \sum_{i=1}^{n} -\frac{y_i^c ln(y_i)}{y_i^c + 1} \\ \sum_{i=1}^{n} -\frac{y_i^c ln(y_i)}{y_i^c + 1} & \frac{\partial^2 L_o}{\partial c^2} \end{bmatrix}$$

With $\frac{\partial^2 L_o}{\partial c^2}$ being too expansive to express within a matrix.