Question 3: Part 1 and 2

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The loss function is given by:

$$L(\beta) = \min_{\beta 1, \beta 2} \sum_{\text{i=1}}^{n} \left(y_i - \frac{y_1 \beta_1}{y_1 + (\beta_1 - y_1) e^{-\beta_2 d_i}} \right)^2$$

1 Part 1

Decompose $L(\beta)$ such that $L(\beta) = g(\beta)^T g(\beta)$:

$$g(\beta) = [g_1(\beta), ..., g_n(\beta)]^T$$

Which can also be expressed as:

$$g(\beta) = \left[y - \frac{y_1 \beta_1}{y_1 + (\beta_1 - y_1) e^{-\beta_2 d}} \right]_{n \times 1}$$

2 Part 2

Derive the Jacobian $J_{g(\beta)}$:

$$J_{g(\beta)} = \left[\frac{\partial g(\beta)}{\partial \beta_1}, \frac{\partial g(\beta)}{\partial \beta_2} \right]_{n \times 2}$$

Which can be further expressed as:

$$J_{g(\beta)} = \left[\frac{y_1^2 e^{\beta_2 d} (e^{\beta_2 d} - 1)}{(\beta_1 + y_1 e^{\beta_2 d} - y_1)^2}, \frac{d \beta_1 y_1 (\beta_1 - y_1) e^{\beta_2 d}}{(\beta_1 + y_1 e^{\beta_2 d} - y_1)^2} \right]_{n \times 2}$$

Which can be expanded outwards to:

$$J_{g(\beta)} = \begin{bmatrix} \frac{\partial g_1(\beta)}{\partial \beta_1}, \frac{\partial g_1(\beta)}{\partial \beta_2} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial g_n(\beta)}{\partial \beta_1}, \frac{\partial g_n(\beta)}{\partial \beta_2} \end{bmatrix}$$