Graph Neural Networks

From First Principles

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Contents

An example reference: In the beginning, there was [1]

Set Theory

Definitions

A set is simply a collection of objects where order doesn't matter.

$$A := \{1, 2, 3, 4\}$$
$$B := \{4, 2, 3, 1\}$$

As order doesn't matter, $A = B$

The elements of a set are assumed to be distinct, even if duplicates exist. Therefore:

$$C := \{1, 1, 2, 3, 4\}$$
$$A = C$$

A large set, or an infinite set can be defined with a vertical bar that reads "such that", followed by a condition

$$D := \{x | x \text{ is a positive, even integer}\}$$

It is also possible to have a set of sets: $E = \{\mathbb{R}, \mathbb{Z}\}$

The **cardinality** of a set is denoted $|A|$ and gives the number of elements in the set. ie) $|A| = 4$ and $|\mathbb{Z}| = \text{infinity}$

An empty set is referred to as the **null** set or the **void** set and is denoted with $\emptyset$, where $\emptyset = \{\}$.

An ordered set is denoted using ()

$$P = (a, b)$$
$$Q = (c, d)$$
$$(a, b) \neq (b, a)$$
$$P = Q \text{ iff } a = c \text{ and } b = d$$

Equality and Membership

If x belongs to the set X then $x \in X$, else $x \notin X$

For $X = Y$ both the following must be true:

For every x, if $x \in X$, then $x \in Y$

AND

For every x, if $x \in Y$, then $x \in X$

Subsets

When a set contains all the elements of a subset in addition to others, the second set is said to be a subset of the first set.

$$A = \{1, 2, 3, 4\} B = \{3, 2\} B \subseteq A$$

Note: Any set X is a subset of itself. The null set is a subset of all sets

If X is a subset of Y and $X \neq Y$, then X is a proper subset of Y, denoted $X \subset Y$

The set of subsets of a set X, regardless of whether they are proper or not, is called the **power set** and denoted $\mathcal{P}(X)$

$$A = \{a, b, c\}$$
$$\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$$

It is worth noting that in general $|\mathcal{P}(X)| = 2^{|X|}$

Set Operations

Union $\cup$

The union operator creates a new set consisting of all the elements belonging to either of the sets being joined or elements that belong to both of them.

Formally: $X \cup Y = \{x | x \in X \text{ or } x \in Y\}$

$$A = \{1, 3, 4\}$$
$$B = \{2, 5\}$$
$$A \cup B = \{1, 3, 4, 2, 5\}$$

Intersection $\cap$

The intersection operator creates a new set containing only the elements that belong to **both** sets.

Formally: $X \cap Y = \{x | x \in X \text{ and } x \in Y\}$

$$A = \{2, 3\}$$
$$B = \{1, 2\}$$
$$A \cap B = \{2\}$$

Difference (Relative Complement) $-$

The difference operator returns the set of elements that are in the first set but not in the second. Formally: $X - Y = \{x | x \in X \text{ and } x \notin Y\}$

$$A = \{1, 2, 3\}$$
$$B = \{1, 2\}$$
$$A - B = \{3\}$$

Note: $A - B \neq B - A$

Cartesian Product $\times$

The cartesian product of X and Y gives the set of all ordered pairs (x, y) where

$x \in X$ and $y \in Y$

$$X = \{1, 2, 3\}$$
$$Y = \{a, b\}$$
$$X \times Y = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$$
$$Y \times X = \{(a, 1), (b, 1), (a, 2), (b, 2), (a, 3), (b, 3)\}$$
$$X \times Y \neq Y \times X$$

Note: $|X \times Y| = |X| \cdot |Y|$

Disjoint and Complement Sets

A pair of sets are said to be **disjoint** if no elements belong to both sets, in other words:

$X \cap Y =$

Given a set of sets $\mathcal{S}$, it is said to be **pairwise disjoint** if given two distinct sets X and Y, then $X \cap Y =$. ie) $\mathcal{S} = \{\{1, 4, 5\}, \{2, 6\}, \{3\}\}$ is pairwise disjoint.

If working with sets that are all a subset of some larger set, U, this larger set is referred to as the **universal set**. It must be explicitly given or inferred through context.

If $X \subset U$, the **complement** of X is given as $\bar{X} = U - X$

Linear Algebra

Vectors

alalallala

Matrices

alalallala in $\mathbb{R}^{m \times n}$

Calculus

Graph Theory

References

[1] S. Russell and P. Norvig, "AI a modern approach," *Learning*, vol. 2, no. 3, p. 4, 2005.