Classification of Fusion Categories

1. Prologue

What are fusion categories? What are near-groups, Haagerup–Izumi categories and quadratic categories? What is modular data?

2. The Cuntz Algebra Approach of Izumi

Take Vec_G to be skeletal. Consider an associativity constraint $a_{ghk} : ghk - - > ghk$. Since ghk is a simple object, $\mathrm{Hom}(ghk, ghk) \cong \Bbbk$, whence $a_{ghk} = \lambda_{ghk}\mathrm{id}_{ghk}$ for some $\lambda_{ghk} \in \Bbbk^\times$. Note that the pentagon diagram enforces certain conditions on our choice of λ_{ghk}; in particular, if we look at this diagram, we'll see that $\lambda_{ghk} = \omega(g, h, k)$ for some 3-cocycle ω. By this, we mean a map $\omega : G \times G \times G \to \Bbbk^\times$ satisfying

$$\omega(x, y, zw)\omega(xy, z, w)\omega(y, z, w)^{-1}\omega(x, yz, w)^{-1}\omega(x, y, z)^{-1} = 1$$

for all $x, y, z, w \in G$. We will henceforth denote by Vec_G^ω the category of G-graded vector spaces with associativity constraint $a_{ghk} = \omega(g, h, k)\mathrm{id}_{ghk}$, for all $g, h, k \in G$, and Vec_G the category of G-graded vector spaces with trivial associativity.

Consider the category $\mathrm{End}(M)$, for M a hyperfinite type III factor. This category is strict, as $\rho \otimes \sigma := \rho \circ \sigma$ by definition. Every near-group category with group G contains some copy of Vec_G^ω corresponding to the group-like part. Because every unitary near-group category is a subcategory of $\mathrm{End}(M)$ and is hence itself strict, we know that it will actually contain the "strictification" of some Vec_G^ω. However, Izumi shows that if $\mathcal{C}$ is any fusion category containing a simple object that is fixed under tensor products with invertibles (that is, there exists some simple object X such that $X \otimes g \cong X$ for all invertible g), then it contains a copy of Vec_G, for G the group of isomorphism classes of invertible objects. He shows in addition that if the fusion category is also unitary, then $g \otimes X = X$ (but we may not necessarily have that $X \otimes g = X$). The upshot is that we almost know how objects are tensored, since the group-like part will have trivial associativity (that is, $g \otimes h = gh$). We just need to understand $X \otimes g$ and $X \otimes X$, as well as the morphisms.

In [Izu17], Izumi showed that every unitary near-group category $\mathcal{C}$ with multiplicity m is equivalent to a subcategory of $\mathrm{End}(M)$, where M is the hyperfinite type III_1 factor. In particular, it is generated by a single irreducible endomorphism $\rho \in \mathrm{End}_0(M)$ satisfying the fusion rules

$$\begin{aligned} [\rho] \otimes [\rho] &= \bigoplus_{g \in G} [\alpha_g] \oplus [\rho]^{\oplus m}, \\ [\alpha_g] \otimes [\alpha_h] &= [\alpha_{gh}], \\ [\alpha_g] \otimes [\rho] &= [\rho] \otimes [\alpha_g] = [\rho], \end{aligned}$$

where the map $\alpha : G \to \mathrm{Aut}(M)$ induces an injective homomorphism from G into $\mathrm{Out}(M)$.

The main result of [Izu17] is [Izu17, Theorem 4.9]. Essentially, there is a bijective correspondence between the set of equivalence classes of unitary near-group categories with finite group G and multiplicity parameter m and the set of equivalence classes of admissible tuples $(\mathcal{K}, j_1, j_2, V, U_\mathcal{K}, \chi, l)$ (see [Izu17, Definition 4.8]). Here $\mathcal{K}$ is the finite-dimensional Hilbert space $\mathrm{Hom}(\rho, \rho^2)$, j_1 and j_2 are two antilinear isometries of $\mathcal{K}$, V and $U_\mathcal{K}$ are unitary representations of G on $\mathcal{K}$, $\{\chi_g\}_{g \in G}$ are characters of G and l is a linear map from $\mathcal{K}$ to the set $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{K})$ of bounded operators $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$.

By [Izu17, Theorem 9.1], the unitary near-group categories with finite Abelian group G and $m = |G|$ are completely classified tuples of the form $(\langle \cdot, \cdot \rangle, a, b, c)$, where $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{T}$ is a non-degenerate symmetric bicharacter and where $a : G \to \mathbb{T}$, $b : G \to \mathbb{T}$ and $c \in \mathbb{T}$ satisfy various conditions. When we say that $\langle \cdot, \cdot \rangle$ is a bicharacter, we mean that

$$\langle xy, z \rangle = \langle x, z \rangle \langle y, z \rangle \qquad \text{and} \qquad \langle x, yz \rangle = \langle x, y \rangle \langle x, z \rangle$$

for all $x, y, z \in G$. By non-degenerate, we mean that

$$\langle x, \cdot \rangle = \langle y, \cdot \rangle$$

if and only if $x = y$. This is equivalent to the map $\varphi : G \to \mathrm{Hom}(G, \mathbb{T})$ given by $x \mapsto \langle x, \cdot \rangle$ being an isomorphism.

Example 2.1. (Fibonacci Category). Let's look at the Fibonacci category. This is the near-group with $G = \{0\}$ and $m = 1$. Our choice for $\langle \cdot, \cdot \rangle$ is obvious, and [Izu17, Lemma 7.1] tells us that

$$c^3 a(0) = \sqrt{n} = 1 \implies a(0) = c^{-3}.$$

Moreover, [Izu17, Theorem 9.1] tells us that b is defined by $b : 0 \mapsto -1/d$, where d corresponds to the dimension of our irreducible generator ρ. Let's determine c and d. Because b is equal to its own Fourier transform, [Izu17, Theorem 9.1] tells us that

$$b(0) = ca(0)b(0) \implies a(0) = c^{-1}.$$

In order for $c^{-1} = c^{-3}$, we require $c = \pm 1$. Finally, [Izu17, Equation 9.5] tells us that

$$\begin{aligned} b(0)b(0)b(0) &= b(0)b(0) \mp \frac{1}{d}, \\ \implies -\frac{1}{d^3} &= \frac{1}{d^2} \mp \frac{1}{d}, \\ \implies \pm d^2 - d - 1 &= 0. \end{aligned}$$

This only has a real solution when $c = 1$, whence d is nothing but the golden ratio (as it cannot be negative in the unitary case - this alternative solution, known as the Galois dual, corresponds to a non-unitary near-group in this case and many others). This is exactly what we would expect, as d is the dimension of X (where $d^2 = 1 + d$ comes from the fusion rule $X^2 = \mathbb{1} \oplus X$).

Example 2.2. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's look at the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $m = 2$. This near-group corresponds to the even part of the type A_4 subfactor. We know the dimension is

$$d_\pm := \frac{m \pm \sqrt{m^2 + 4n}}{2} = 1 \pm \sqrt{3}.$$

In the unitary setting, we of course ask that d be positive, and hence we choose $d = d_+$. The only possibility for a non-degenerate bicharacter is

$$\langle 0, 0 \rangle = 1, \qquad \langle 0, 1 \rangle = \langle 1, 0 \rangle = 1 \qquad \text{and} \qquad \langle 1, 1 \rangle = -1.$$

From [Izu17, Equation 7.8], it follows that

$$a(0) = 1 \qquad \text{and} \qquad a(1) = \pm i.$$

Meanwhile, [Izu17, Equation 9.4] tells us that

$$\overline{b(1)} = \pm i b(1) \implies \mathfrak{R}(b(1)) = \mp \mathfrak{I}(b(1)),$$

whence [Izu17, Equation 9.3] gives us

$$\mathfrak{R}(b(1))^2 + \mathfrak{I}(b(1))^2 = (b(1)\overline{b(1)})^2 = \frac{1}{2} \implies b(1) = \frac{1 - a(1)}{2}.$$

It then follows from evaluating [Izu17, Equation 9.1] with $g = 0$ and rearranging for c that

$$c = \frac{1 - \sqrt{3} + a(1)(1 + \sqrt{3})}{2\sqrt{2}}.$$

Note that we may choose either $a(1) = i$ or $a(1) = -i$; both of these lead to solutions. Moreover, in the non-unitary setting, we may take the Galois conjugate of d.

Example 2.3. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's determine the Haagerup–Izumi categories with $G = \mathbb{Z}/2\mathbb{Z}$. Let

$$d_\pm := \frac{n \pm \sqrt{n^2+4}}{2},$$

where in this example $d := 1 + \sqrt{2}$. Izumi's classification involves a triplet $(\epsilon_h(g), \omega(g), A_{h,k}(g))$, where $\epsilon_h(g) \in \{-1, 1\}$, $\omega(g) \in \mathbb{T}$ and $A_{h,k}(g) \in \mathbb{C}$ satisfy [Izu18, Equations 4.1–4.9]. Well, we know

$$\epsilon_0(0) = \epsilon_1(0) = 1 \qquad \text{and} \qquad \epsilon_0(1) = \epsilon_0(1)\epsilon_0(1) \implies \epsilon_0(1) = 1.$$

By [Izu18, Equation 4.7],

$$A_{0,0}(g) = A_{0,0}(g)\omega(g),$$

which tells us that either $\omega(g) = 1$ or $A_{0,0}(g) = 0$ for each $g \in G$. Let's fix any $g \in G$ and consider the case when $A_{0,0}(g) = 0$. In this case, however, [Izu18, Equations 4.3 and 4.4] give us

$$A_{1,0}(g)\overline{A_{\delta_{g,0}-g,0}(g)} = 1 - \frac{|\omega(g)|}{d} \implies \text{“}\frac{1}{d^2} = 1 - \frac{1}{d}\text{”}.$$

This "equality" is nonsense; we must therefore have $\omega(g) = 1$ for all $g \in G$. Suppose now that $\epsilon_1(1) = 1$. Then [Izu18, Equation 4.7] gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \qquad \text{and} \qquad A_{0,1}(1) = A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(0) = A_{1,1}(1)$. Now, [Izu18, Equations 4.4 and 4.6] tell us

$$A_{0,1}(0)A_{1,1}(1) + A_{1,1}(0)A_{1,0}(1) = 0.$$

Thus $A_{0,1}(g) = A_{1,1}(g) = A_{1,0}(g) = 0$ and hence $A_{0,0}(g) = -1/d$ by [Izu18, Equation 4.3]. However, in this case we cannot satisfy [Izu18, Equation 4.9]. Suppose instead that $\epsilon_1(1) = -1$. With this new 2-cocycle, [Izu18, Equation 4.7] now gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \qquad \text{and} \qquad A_{0,1}(1) = -A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(1) = -A_{1,1}(0)$. We then see by [Izu18, Equation 4.4] that

$$A_{1,0}(0) = \pm\frac{1}{\sqrt{2}} = \pm\frac{1}{d-1},$$

and

$$A_{0,0}(0)A_{1,0}(0)^2 = A_{1,0}(0)^2 + A_{1,0}(0)^3 \implies A_{0,0}(0) = 1 + A_{1,0}(0) = \frac{d-1\pm 1}{d-1}$$

by [Izu18, Equation 4.9]. Finally, [Izu18, Equation 4.3] allows us to deduce that

$$A_{1,0}(0) = -\frac{1}{d-1},$$

whence

$$A(0) = \frac{1}{d-1}\begin{pmatrix} d-2 & -1 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad A(1) = \frac{1}{d-1}\begin{pmatrix} d-2 & -1 \\ -1 & 1 \end{pmatrix}.$$

Remark 2.4. Suppose that $|G|$ is odd. Then [Izu18, Equation 4.1] tells us that $\epsilon_h(g) = 1$, while [Izu18, Equation 4.2] tells us that $\omega(g)$ does not depend on g. Moreover, $A_{h,k}(g)$ cannot depend on g by [Izu18, Equation 4.5], and either $\omega = 1$ or $A_{0,0} = 0$ by [Izu18, Equation 4.7]. In this case, [Izu18, Equations 4.1–4.9] reduce to the following four equations.

$$A_{h,k} = A_{-k,h-k}\omega = A_{k-h,-h}\overline{\omega},$$

$$\sum_{h\in G} A_{h,0} = -\frac{\overline{\omega}}{d_\pm},$$

$$\sum_{h\in G} A_{h-g,k}A_{k,h-g'} = \delta_{g,g'} - \frac{\delta_{k,0}}{d_\pm},$$

$$\sum_{l\in G} A_{x+y,l}A_{-x,l+p}A_{-y,l+q} = A_{p+x,q+x+y}A_{q+y,p+x+y} - \frac{\delta_{x,0}\delta_{y,0}}{d_\pm}.$$

The first three equations above are precisely [EG17, Equations 4.7, 4.8 and 4.9]! In particular, to see that our third equation is equivalent to [EG17, Equation 4.9], we simply make the change of variables $\hat{g} := g' - g$ and $\hat{h} := h - g'$, whence we obtain

$$\sum_{\hat{h}\in G} A_{\hat{h}+\hat{g},k}A_{k,\hat{h}} = \delta_{\hat{g},0} - \frac{\delta_{k,0}}{d_\pm}.$$

Similarly, using our first equation while making the change of variables $\hat{l} := l - x - y$, $\hat{p} := p + x + y$, $\hat{q} := q + x + y$, $\hat{x} := -x$ and $\hat{y} := -y$, our fourth equation becomes

$$\overline{\omega}\sum_{\hat{l}\in G} A_{\hat{l},\hat{x}+\hat{y}}A_{\hat{x},\hat{l}+\hat{p}}A_{\hat{y},\hat{l}+\hat{q}} = A_{\hat{y}+\hat{p},\hat{q}}A_{\hat{x}+\hat{q},\hat{p}} - \frac{\delta_{\hat{x},0}\delta_{\hat{y},0}}{d_\pm},$$

showing that it is equivalent to [EG17, Equation 4.11].

3. The Leavitt Algebra Approach of Evans–Gannon

The important result is [EG17, Theorem 2].

Example 3.1. (Yang–Lee Category). Let $G = \{0\}$. Then [EG17, Equation 4.7] demands that

$$A_{0,0} = \omega A_{0,0} = \overline{\omega} A_{0,0} \implies \omega = 1,$$

whence [EG17, Equation 4.8] tells us that

$$A_{0,0} = -\frac{1}{d_\pm}.$$

The rest of [EG17, Equations 4.7–4.10] are satisfied by these choices. Hence by [EG17, Theorem 2], we have two fusion categories for $G = \{0\}$; a unitary one with $\pm = +$ (the Fibonacci category) and a non-unitary one with $\pm = -$ (the Yang–Lee category).

Example 3.2. ($G = \mathbb{Z}/2\mathbb{Z}$). In [EG17], the classification was only given for $|G|$ odd. Mimicking [Izu18], however, we can extend it to the case when $|G|$ is even. Equations

To illustrate this, suppose we let $G = \mathbb{Z}/2\mathbb{Z}$.

References

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