Basics of Module Categories

Past 10 weeks

- Lots of examples of monoidal and tensor categories.
- Lots of technology for studying them.

Today

- What are module categories?
- Why should I care?
- How might we classify them?

Definition (Module Category).

A **left module category** over a monoidal category $\mathcal{C}$ "$= (\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$" consists of

- a category $\mathcal{M}$;
- a **bifunctor** $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$; "left action"
- a **natural isomorphism** $m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$; "module associativity constraint"
- a **natural isomorphism** $l_M : \mathbb{1} \otimes M \xrightarrow{\sim} M$; "unit constraint"

for $X, Y \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M})$, such that the following commute for $X, Y, Z \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M})$:

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes id_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"'] & & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X,Y,Z \otimes M}"] \\
X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "id_X \otimes m_{Y,Z,M}"'] & & X \otimes (Y \otimes (Z \otimes M))
\end{tikzcd}$$

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes M \arrow[rr, "m_{X,\mathbb{1},M}"] \arrow[dr, "r_X \otimes id_M"'] & & X \otimes (\mathbb{1} \otimes M) \arrow[dl, "id_X \otimes l_M"] \\
 & X \otimes M &
\end{tikzcd}$$

If $\mathcal{C}$ is a tensor category, we ask
- that $\mathcal{M}$ be locally finite, Abelian;
- that $\otimes$ be bilinear on morphisms, exact in the first variable (why?).

Regular representation: $\mathcal{C}$ is a $\mathcal{C}$-module category with left action $\otimes_{\mathcal{C}}$, $m := a$ and unit constraint l.

$\mathcal{C} := \mathrm{Vec}$: every locally finite, Abelian $\mathcal{M}$ admits a $\mathcal{C}$-module structure given by $X \otimes M := M^{\oplus n}$, for $X \in [\Bbbk^n]$, $M \in \mathrm{Ob}(\mathcal{M})$.

$\mathcal{C} := \mathrm{Vec}_G$: module categories are Abelian categories $\mathcal{M}$ admitting a monoidal functor $F : \mathrm{Cat}(G) \to \mathrm{Aut}_{\otimes}(\mathcal{M})$.

Such an "action of G on $\mathcal{M}$" can always be constructed by taking $F(g) : M \to \delta_g \otimes M$.

If $F : \mathcal{C} \to \mathcal{M}$ is a tensor functor, $X \otimes Y := F(X) \otimes Y$ defines a $\mathcal{C}$-module structure on $\mathcal{M}$.

E.g. $\mathrm{Res} : \mathrm{Rep}(G) \to \mathrm{Rep}(H)$ induces a $\mathrm{Rep}(G)$-module structure on $\mathrm{Rep}(H)$.

<u>Proposition</u>. There is a bijection between $\mathcal{C}$-module structures on $\mathcal{M}$ and monoidal functors $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$.

(what about for tensor categories?)

$\mathcal{M} := \mathrm{Vec}$: there is a tensor equivalence $\mathcal{M} \cong \mathrm{End}_{\ell}(\mathcal{M})$ induced by the covariant Yoneda embedding (i.e., $\text{よ} : V \mapsto \mathrm{Hom}_{\mathcal{M}}(V, -)$).

$\Longrightarrow$ Vec admits a $\mathcal{C}$-module structure <u>if and only if</u> there is a fibre functor on $\mathcal{C}$!

Definition. A module subcategory is a full subcategory that is closed under $\otimes$.

Definition. A $\mathcal{C}$-module functor consists of

- a functor $F : \mathcal{M} \longrightarrow \mathcal{N}$;
- a natural isomorphism $s_{X,M} : F(X \otimes M) \to X \otimes F(M)$, $X \in Ob(\mathcal{C})$, $M \in Ob(\mathcal{M})$;

satisfying the obvious coherence conditions.

Definition. A module equivalence is a module functor that gives an equivalence of categories.

Proposition. Given two $\mathcal{C}$-module categories $\mathcal{M}_1$ and $\mathcal{M}_2$, we have a natural notion of a direct sum $\mathcal{M} := \mathcal{M}_1 \oplus \mathcal{M}_2$ that is itself a $\mathcal{C}$-module category.

Definition. Indecomposable means there is no module equivalence with a non-trivial direct sum. Simple means no non-trivial module subcategories.

We have seen some examples of module categories, but there are some issues:

- our module functors do not descend to morphisms of Grothendieck group modules;
- even if we think back to Vec, <u>there is just no hope of classifying all of its module categories</u>!

Definition. Let $\mathcal{C}$ be a tensor category with enough projectives. A $\mathcal{C}$-module category $\mathcal{M}$ is said to be exact if $P \otimes M$ is projective for all $M \in \mathrm{Ob}(\mathcal{M})$ and projective $P \in \mathrm{Ob}(\mathcal{C})$.

Somehow exact categories play an analogous role to projective modules.	$\mathcal{C} := \mathrm{Vec}$: $\mathcal{M}$ is exact if and only if it is semisimple.
Any tensor category with enough projectives is exact over itself.	If $\mathcal{C}$ is semisimple, then its exact module categories are semisimple.

$\mathcal{C} := \mathrm{Rep}(G)$: assume semisimple (i.e., $\Bbbk$ does not divide $|G|$).

Every exact, indecomposable $\mathcal{C}$-module is equivalent to some $\mathrm{Rep}^1(\tilde{H})$, where $\tilde{H}$ is the central extension of some $H \leq G$ by $\Bbbk^* := \Bbbk \setminus \{0\}$ and $\mathrm{Rep}^1(\tilde{H})$ is the category of $\tilde{H}$-modules that $\Bbbk^*$ acts on by scalar multiplication.

Let $G := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. We have five subgroups: G, $\{0\}$ and three isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

This gives us five central extensions, all of the form $\Bbbk^* \times H$. Not every central extension is of this form, however; in particular, because $Q_8/Z(Q_8) \cong G$, G has a non-trivial central extension by the quaternion group Q_8, given by the sign homomorphism $\iota : \pm\lambda \mapsto \pm 1$.

__Lemma__. Exact module categories have enough projectives.

It follows that all exact module categories with finitely many isomorphism classes of simple objects are finite

__Lemma__. An object in an exact module category is projective if and only if it is injective.

Moreover, such categories are quasi-Frobenius and hence amenable to homological techniques.

Proposition. An indecomposable, exact $\mathcal{C}$-module category decategorifies to a simple module over $Gr(\mathcal{C})$.

If $\mathcal{C}$ is a finite tensor category, there are only finitely-many $\mathbb{Z}_+$-modules over $Gr(\mathcal{C})$ of the form $Gr(\mathcal{M})$.

Characterization Theorem #1

Let $\mathcal{M}_1, \mathcal{M}_2$ be $\mathcal{C}$-module categories. Then

- If $\mathcal{M}_1$ is exact, then every additive module functor $F: \mathcal{M}_1 \to \mathcal{M}_2$ is exact.
- If every module functor $F: \mathcal{M}_1 \to \mathcal{M}_2$ is exact, then $\mathcal{M}_1$ is exact.

One great implication of this theorem is that exact module functors descend to morphisms on Grothendieck modules!

We would now like to ~~transition~~ towards algebra objects and how they correspond to certain module categories.

Recall that an algebra in $\mathcal{C}$ consists of

- an object $A \in \mathcal{O}b(\mathcal{C})$;
- a unit morphism $u : \mathbb{1} \to A$;
- a multiplication morphism $m : A \otimes A \to A$;

satisfying certain coherence conditions.

$\mathcal{C} := \mathrm{Vec}$: an algebra object is a finite-dimensional, associative, unital $\Bbbk$-algebra.

$\mathcal{C} := \mathrm{Vec}_G$: an algebra object is a G-graded algebra. Moreover, $\Bbbk L$ is an algebra for $L \subseteq G$.

$\mathcal{C} := \mathrm{Rep}(G)$: the algebra of functionals $G \to \Bbbk$ is an algebra, with G acting as right translation.

A right module over an algebra A is an object $M \in \mathrm{Ob}(\mathcal{C})$ together with a morphism $\rho: A \otimes M \to M$, subject to certain coherence conditions.

These A-modules form a left $\mathcal{C}$-module category $\mathrm{Mod}_{\mathcal{C}}(A)$.

If $\mathcal{C}$ has enough projectives (resp. is finite), then $\mathrm{Mod}_{\mathcal{C}}(A)$ has enough projectives (resp. is finite).

$\mathcal{C} := \mathrm{Vec}$, $\mathcal{M} := \mathrm{Vec}$: by our last remark, $\mathcal{M}$ cannot appear as some $\mathrm{Mod}_{\mathcal{C}}(A)$.

Suppose $\mathcal{C}$ is rigid: then $\mathbb{1}$ is an algebra. So is $A := X \otimes X^*$ for all $X \in \mathrm{Ob}(\mathcal{C})$, with unit $u := \mathrm{coev}_X$ and multiplication $m := \mathrm{id}_X \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^*}$. For all $Y \in \mathrm{Ob}(\mathcal{C})$, $Y \otimes X^*$ is a module over A, and $Y \mapsto Y \otimes X^*$ is a module equivalence $\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}(A)$.

Let $\mathcal{M}$ be a module category over a finite tensor category $\mathcal{C}$ and fix $M_1, M_2 \in \mathrm{Ob}(\mathcal{M})$. Then the functor $X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)$ is represented by $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathrm{Ob}(\mathcal{C})$.

This representative is known as the internal Hom. The Yoneda lemma gives us a $\mathcal{C}$-module functor $\underline{\mathrm{Hom}}(-,-) : (M_1, M_2) \mapsto \underline{\mathrm{Hom}}(M_1, M_2)$.

Characterization Theorem #2

Let $\mathcal{M}$ be a $\mathcal{C}$-module category.

(i). If $\mathcal{M}$ is finite, then $\mathcal{M} \cong \mathrm{Mod}_{\mathcal{C}}(A)$ for some algebra $A \in \mathrm{Ob}(\mathcal{C})$.

(ii). If $\mathcal{M}$ is exact and $M \in \mathrm{Ob}(\mathcal{M})$ is an object such that the isomorphism class $[M]$ generates $\mathrm{Gr}(\mathcal{M})$ as a $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$, then $\mathcal{M} \cong \mathrm{Mod}_{\mathcal{C}}(A)$ for $A := \underline{\mathrm{Hom}}(M, M)$.

Thus we are able to classify all finite and all exact, "cyclic" module categories in terms of algebras!

This is the part of the talk where I try to sell you subfactors.
I owe this section to Pinhas and Ammanuel.

One of the most extensively studied "exotic" subfactors is the Haagerup subfactor $N \subseteq M$. Let $M = N \oplus X$ (that is, X is in some sense the "complement" of N in M). We find that as an (N,N)-bimodule,

$$X \otimes X \cong N \oplus X \oplus (g \otimes X) \oplus (g^2 \otimes X),$$

where g is a one-dimensional (N,N)-bimodule satisfying

$$g^3 \cong N \qquad \text{and} \qquad g \otimes X \cong X \otimes g^2.$$

This gives us the Haagerup category. In other words, it is the fusion category with simple objects N, g, g^2, X, $g\otimes X$ and $g^2\otimes X$, subject to the fusion rules

$$g^3 \cong N,$$
$$g\otimes X \cong X\otimes g^2,$$
$$X\otimes X \cong N\oplus X\oplus (g\otimes X)\oplus (g^2\otimes X).$$

It was shown by Pinhas and Noah in 2012 that this category admits three "simple" module categories, given by the algebra objects N, $N\oplus X$ and $N\oplus g\oplus g^2$.

Thank you very much for your time!