Classification of Fusion Categories

1. Prologue

What are fusion categories? What are near-groups?

2. The Cuntz Algebra Approach of Izumi

Take Vec_G to be skeletal. Consider an associativity constraint $a_{ghk} : ghk - - > ghk$. Since ghk is a simple object, $\mathrm{Hom}(ghk, ghk) \cong \Bbbk$, whence $a_{ghk} = \lambda_{ghk}\mathrm{id}_{ghk}$ for some $\lambda_{ghk} \in \Bbbk^\times$. Note that the pentagon diagram enforces certain conditions on our choice of λ_{ghk}; in particular, if we look at this diagram, we'll see that $\lambda_{ghk} = \omega(g, h, k)$ for some 3-cocycle ω. By this, we mean a map $\omega : G \times G \times G \to \Bbbk^\times$ satisfying

$$\omega(x, y, zw)\omega(xy, z, w)\omega(y, z, w)^{-1}\omega(x, yz, w)^{-1}\omega(x, y, z)^{-1} = 1$$

for all $x, y, z, w \in G$. We will henceforth denote by Vec_G^ω the category of G-graded vector spaces with associativity constraint $a_{ghk} = \omega(g, h, k)\mathrm{id}_{ghk}$, for all $g, h, k \in G$, and Vec_G the category of G-graded vector spaces with trivial associativity.

Consider the category $\mathrm{End}(M)$, for M a hyperfinite type III factor. This category is strict, as $\rho \otimes \sigma \coloneqq \rho \circ \sigma$ by definition. Every near-group category with group G contains some copy of Vec_G^ω corresponding to the group-like part. Because every unitary near-group category is a subcategory of $\mathrm{End}(M)$ and is hence itself strict, we know that it will actually contain the "strictification" of some Vec_G^ω. However, Izumi shows that if $\mathcal{C}$ is any fusion category containing a simple object that is fixed under tensor products with invertibles (that is, there exists some simple object X such that $X \otimes g \cong X$ for all invertible g), then it contains a copy of Vec_G, for G the group of isomorphism classes of invertible objects. He shows in addition that if the fusion category is also unitary, then $g \otimes X = X$ (but we may not necessarily have that $X \otimes g = X$). The upshot is that we almost know how objects are tensored, since the group-like part will have trivial associativity (that is, $g \otimes h = gh$). We just need to understand $X \otimes g$ and $X \otimes X$, as well as the morphisms.

In [Izu17], Izumi showed that every unitary near-group category $\mathcal{C}$ with multiplicity m is equivalent to a subcategory of $\mathrm{End}(M)$, where M is the hyperfinite type III_1 factor. In particular, it is generated by a single irreducible endomorphism $\rho \in \mathrm{End}_0(M)$ satisfying the fusion rules

$$\begin{aligned} [\rho] \otimes [\rho] &= \bigoplus_{g \in G} [\alpha_g] \oplus [\rho]^{\oplus m}, \\ [\alpha_g] \otimes [\alpha_h] &= [\alpha_{gh}], \\ [\alpha_g] \otimes [\rho] = [\rho] \otimes [\alpha_g] &= [\rho], \end{aligned}$$

where the map $\alpha : G \to \mathrm{Aut}(M)$ induces an injective homomorphism from G into $\mathrm{Out}(M)$.

The main result of [Izu17] is [Izu17, Theorem 4.9]. Essentially, there is a bijective correspondence between the set of equivalence classes of unitary near-group categories with finite group G and multiplicity parameter m and the set of equivalence classes of admissible tuples $(\mathcal{K}, j_1, j_2, V, U_{\mathcal{K}}, \chi, l)$ (see [Izu17, Definition 4.8]). Here $\mathcal{K}$ is the finite-dimensional Hilbert space $\mathrm{Hom}(\rho, \rho^2)$, j_1 and j_2 are two antilinear isometries of $\mathcal{K}$, V and $U_{\mathcal{K}}$ are unitary representations of G on $\mathcal{K}$, $\{\chi_g\}_{g \in G}$ are characters of G and l is a linear map from $\mathcal{K}$ to the set $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{K})$ of bounded operators $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$.

By [Izu17, Theorem 9.1], the unitary near-group categories with finite Abelian group G and $m = |G|$ are completely classified tuples of the form $(\langle \cdot, \cdot \rangle, a, b, c)$, where $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{T}$ is a non-degenerate symmetric bicharacter and where $a : G \to \mathbb{T}$, $b : G \to \mathbb{T}$ and $c \in \mathbb{T}$ satisfy various conditions. When we say that $\langle \cdot, \cdot \rangle$ is a bicharacter, we mean that

$$\langle xy, z \rangle = \langle x, z \rangle \langle y, z \rangle \qquad \text{and} \qquad \langle x, yz \rangle = \langle x, y \rangle \langle x, z \rangle$$

for all $x, y, z \in G$. By non-degenerate, we mean that

$$\langle x, \cdot \rangle = \langle y, \cdot \rangle$$

if and only if $x = y$. This is equivalent to the map $\varphi : G \to \mathrm{Hom}(G, \mathbb{T})$ given by $x \mapsto \langle x, \cdot \rangle$ being an isomorphism.

Example 2.1. (Fibonacci Category). Let's look at the Fibonacci category. This is the near-group with $G = \{0\}$ and $m = 1$. Our choice for $\langle \cdot, \cdot \rangle$ is obvious, and [Izu17, Lemma 7.1] tells us that

$$c^3 a(0) = \sqrt{n} = 1 \implies a(0) = c^{-3}.$$

Moreover, [Izu17, Theorem 9.1] tells us that b is defined by $b : 0 \mapsto -1/d$, where d corresponds to the dimension of our irreducible generator ρ. Let's determine c and d. Because b is equal to its own Fourier transform, [Izu17, Theorem 9.1] tells us that

$$b(0) = ca(0)b(0) \implies a(0) = c^{-1}.$$

In order for $c^{-1} = c^{-3}$, we require $c = \pm 1$. Finally, [Izu17, Equation 9.5] tells us that

$$\begin{aligned} b(0)b(0)b(0) &= b(0)b(0) \mp \frac{1}{d}, \\ \implies -\frac{1}{d^3} &= \frac{1}{d^2} \mp \frac{1}{d}, \\ \implies \pm d^2 - d - 1 &= 0. \end{aligned}$$

This only has a real solution when $c = 1$, whence d is nothing but the golden ratio (as it cannot be negative in the unitary case - this alternative solution, known as the Galois dual, corresponds to a non-unitary near-group in this case and many others). This is exactly what we would expect, as d is the dimension of X (where $d^2 = 1 + d$ comes from the fusion rule $X^2 = \mathbb{1} \oplus X$).

Example 2.2. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's look at the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $m = 2$. This near-group corresponds to the even part of the type A_4 subfactor. We know the dimension is

$$d_\pm := \frac{m \pm \sqrt{m^2 + 4n}}{2} = 1 \pm \sqrt{3}.$$

In the unitary setting, we of course ask that d be positive, and hence we choose $d = d_+$. The only possibility for a non-degenerate bicharacter is

$$\langle 0, 0 \rangle = 1, \qquad \langle 0, 1 \rangle = \langle 1, 0 \rangle = 1 \qquad \text{and} \qquad \langle 1, 1 \rangle = -1.$$

From [Izu17, Equation 7.8], it follows that

$$a(0) = 1 \qquad \text{and} \qquad a(1) = \pm i.$$

Meanwhile, [Izu17, Equation 9.4] tells us that

$$\overline{b(1)} = \pm i b(1) \implies \mathfrak{R}(b(1)) = \mp \mathfrak{I}(b(1)),$$

whence [Izu17, Equation 9.3] gives us

$$\mathfrak{R}(b(1))^2 + \mathfrak{I}(b(1))^2 = (b(1)\overline{b(1)})^2 = \frac{1}{2} \implies b(1) = \frac{1 - a(1)}{2}.$$

It then follows from evaluating [Izu17, Equation 9.1] with $g = 0$ and rearranging for c that

$$c = \frac{1 - \sqrt{3} + a(1)(1 + \sqrt{3})}{2\sqrt{2}}.$$

Note that we may choose either $a(1) = i$ or $a(1) = -i$; both of these lead to solutions. Moreover, in the non-unitary setting, we may take the Galois conjugate of d.

3. The Leavitt Algebra Approach of Evans–Gannon

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References

[Izu17] Izumi, M., *A Cuntz algebra approach to the classification of near-group categories*, Proceedings of the 2014 Maui and 2015 Qinhuangdao Conferences in Honour of Vaughan F. R. Jones' 60th Birthday, vol. 46, Proc. Centre Math. Appl. Austral. Nat. Univ. Australian National University, Centre for Mathematics and its Applications, Canberra, 2017, pp. 222–343.