Computational Decompositions in Lusztig–Vogan Categories

1. Strategy

Let $R^{W_K} \coloneqq \mathbb{R}[x_1, \dots, x_m] \subseteq R$ for some polynomial ring R, graded in degree 2. Consider some graded Krull–Schmidt (R^{W_K}, R)-bimodule B that is graded free of finite rank as a right R-module with basis $\{b_1, \dots, b_n\}$. We want to determine the indecomposable summands of B. This is equivalent to determining its primitive idempotents.

Let's ignore the grading for a moment. As a free right R-module, we know that $B \cong R^{\oplus n}$ and hence that $\mathrm{End}_{\mathrm{Mod}-R}(B) \cong M_n(R)$. As a result, we can encode the left action of R^{W_K} as a set $\{A_k\}_{k=1}^m$ of $n \times n$ matrices satisfying $A_k f = x_k \cdot f$ for all $f = b_1 \cdot f_1 + \dots + b_n \cdot f_n \in B$. More explicitly, we may write $A_k \coloneqq (a_{ij}^k)_{i,j=1}^n$, where a_{ij}^k is the coefficient of b_i in $x_k \cdot b_j$. Then $\mathrm{End}_{R^{W_K}-\mathrm{Mod}-R}(B)$ is the subset of $M_n(R)$ consisting of matrices that commute with every A_k. Finally, in order to reintroduce the grading, we need only enforce homogeneity on each matrix M; that is, we require some $d \in \mathbb{N}$ such that every Mb_k is homogeneous in degree d. The set of degree d endomorphisms is

$$\mathrm{End}^d(B) \cong \left\{ M = (m_{ij})_{i,j=1}^n \in M_n(R) : \begin{array}{r} MA_k - A_kM,\ \forall k \in \{1, \dots, m\}; \\ \deg(m_{ij}) = d - \deg(b_i) + \deg(b_j),\ \forall m_{ij} \neq 0 \end{array} \right\}.$$

Thus we are interested in the set

$$\mathrm{Idem}(B) \coloneqq \{E \in \mathrm{End}^0(B) : E^2 = E\},$$

where the condition $E^2 = E$ enforces $d = 0$. These conditions give us a system of equations that we can solve using Magma, whence we can find the primitive idempotents manually.

Suppose we have an arbitrary (R, R)-bimodule B that is free of finite rank as an R-module with basis $\{b_1, \dots, b_n\}$ and left action matrices $A_f : g \mapsto f \cdot g$. One can show that, using the basis

$$b_0^s \coloneqq 1 \otimes_{R^s} 1, \qquad b_1^s \coloneqq c_s \coloneqq \frac{1}{2}(\alpha_s \otimes_{R^s} 1 + 1 \otimes_{R^s} \alpha_s),$$

for B_s, the bimodule $B \otimes_R B_s$ admits a basis $\{b_i \otimes_R b_0^s\}_{i=1}^n \cup \{b_i \otimes_R b_1^s\}_{i=1}^n$ and left action matrices

$$A_f^s \coloneqq \begin{pmatrix} s(A_f) & 0 \\ \partial_s(A_f) & A_f \end{pmatrix}$$

for all $f \in R$ (and hence all $f \in R^{W_K}$). All of the modules that we will be looking at are of the form $B = B_{s_1} \otimes_R \dots \otimes_R B_{s_\ell}$. Given a binary word $w = w_1 \cdots w_\ell$ for $w_i \in \{0, 1\}$, let

$$b_w \coloneqq b_{w_1}^{s_1} \otimes_R \dots \otimes_R b_{w_\ell}^{s_\ell}.$$

The set of all b_w corresponding to binary words w of length ℓ is a basis for B. Moreover, if we let $f\varphi_i : g \mapsto fs_i(g)$ if $w_i = 0$ and $f\varphi_i : g \mapsto f\partial_{s_i}(g)$ if $w_i = 1$, then

$$f_0 \otimes_{R^{s_1}} \dots \otimes_{R^{s_\ell}} f_\ell = \sum_w b_w \cdot [f_\ell\varphi_\ell \circ \dots \circ f_1\varphi_1](f_0).$$

Note that in small rank examples, if we have a primitive idempotent E, we can often determine whether or not $1 - E$ is primitive using rank arguments. For instance, if B is rank 4 and we can show that E is the only primitive rank 1 idempotent, then $1 - E$ must necessarily be primitive. This exact situation occurs for $B_s \otimes_R B_t$ and $B_t \otimes_R B_s$ in type G_2.

One optimization we can make is to solve for primitive idempotents one degree at a time. However, solving the system of equations that define $\mathrm{Idem}(B)$ gets computationally expensive quite fast, and even rank 4 computations tend to require some tricks to perform. One significant optimization is to take R over $\mathbb{Q}$ and realize its generators as large primes. This is a significant reduction that appears to make the computations tractable at the cost of a larger solution space. If we already know an idempotent, we can also use it to reduce the dimension of our ideal. Once we have all of the idempotents – and typically there are not too many – determining if an idempotent E is primitive is simply a matter of making sure that $\nexists E_1 \perp E_2 \in \mathrm{Idem}(B)$ such that $E = E_1 + E_2$