Categorification and the Lusztig–Vogan Module

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Prologue

In this talk, I would like to motivate categorical representation theory from the perspective of real Lie groups.

I'll start by introducing Kazhdan–Lusztig theory and describing the success of categorification in giving us a new perspective in this realm.

I'll then mention Lusztig and Vogan's real group analogue and where I fit into understanding its categorification.

In particular, I would like to walk through some explicit examples of categorical Jordan–Hölder decompositions.

Kazhdan–Lusztig Theory

In the late 1970s, Kazhdan and Lusztig introduced a family of polynomials now known as Kazhdan–Lusztig polynomials, defined as change-of-basis coefficients between the standard basis and the following self-dual basis for the Iwahori–Hecke algebra $\mathscr{H}(W, S)$.

$$b_x = \sum_{y \in W} h_{y,x}(v)\delta_y$$

Theorem (Kazhdan–Lusztig Conjecture ([KL79]: [BB81], [BK81]))

For all $x, y \in W$, the Jordan–Hölder multiplicity of the composition factor L_y inside the Verma module M_x is given by

$$[M_x : L_y] = h_{y,x}(1).$$

Soergel's Technology

Let $R \coloneqq \mathbb{R}[\alpha_s : s \in S]$ and define an (R, R)-bimodule

$$B_s \coloneqq R \otimes_{R^s} R(1)$$

for each $s \in S$. Given $w \in W$ with expression $\underline{w} = (s_1, \ldots, s_k)$, we define the Bott–Samelson bimodule

$$BS(\underline{w}) \coloneqq B_{s_1} \otimes_R \cdots \otimes B_{s_k}.$$

The Karoubian envelope of the category generated by these (R, R)-bimodules is known as the category of Soergel bimodules and denoted $\mathbb{S}\mathrm{Bim}(W, S)$.

The indecomposables of $\mathbb{S}\mathrm{Bim}(W, S)$ are, up to grading shifts, indexed by W, with B_w denoting the unique indecomposable summand of $BS(\underline{w})$ that does not appear in $BS(\underline{x})$ for any $x < w$.

Soergel's Technology

For all $x \in W$, we define the standard bimodule R_x to be the regular (R,R)-bimodule R whose right action is twisted by reflection by x.

For any Soergel bimodule $B \in \mathrm{Ob}(\mathbb{S}\mathrm{Bim}(W,S))$, we have a so-called Δ-filtration of (R,R)-bimodules

$$0 = B^k \subset B^{k-1} \subset \cdots \subset B^1 \subset B^0 = B$$

with subquotients $B^i/B^{i+1} \cong R_{y_i}^{\oplus h_{y_i}(B)}$, where $h_{y_i}(B) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$.

By definition, $R_y^v \coloneqq R_y(1)$ and $R_y^{v^{-1}} \coloneqq R_y(-1)$.

Soergel's Technology

Theorem (Soergel's Conjecture ([Soe92]: [EW14]))

For all $x, y \in W$, the graded multiplicity of the composition factor R_y in the Δ-filtration of the Soergel bimodule B_x is given by

$$h_y(B_x) = h_{y,x}.$$

Equivalently, the indecomposables of $\mathbb{S}\mathrm{Bim}(W, S)$ categorify the canonical basis for $\mathscr{H}(W, S)$.

On top of this, Soergel bimodules carry also a good deal of homological and geometric structure that isn't seen in the decategorified world.

Real Reductive Groups

The objects we are interested in studying are real reductive groups.

Their full representation theory is totally unmanageable.
Harish-Chandra's approach: study a nice subclass that *is* manageable.

The "right" class of representations are admissible representations.
Irreducible K-admissible representations are in bijection with irreducible $(\mathfrak{g}, K)$-modules, up to some notion of equivalence.

Irreducible $(\mathfrak{g}, K)$-modules can then be realized geometrically as K-equivariant local systems on K-orbits of the flag variety via Beilinson–Bernstein localization.

The Lusztig–Vogan Module

Let G be a connected complex reductive group with holomorphic involution θ. This data is equivalent to choosing a real form $G_{\mathbb{R}}$ of G.

Fix a finite index subgroup $K \subseteq G^\theta$, a Borel subgroup $B \subset G$, a maximal torus $T \subset K \cap B$ and a non-singular integral infinitesimal character λ.

$$\left\{\begin{array}{c}\text{irreducible } (\mathfrak{g}, K)\text{-modules} \\ \text{with character } \lambda\end{array}\right\} \longleftrightarrow \mathscr{D} := \left\{\begin{array}{c}\text{irreducible } K\text{-equivariant} \\ \text{local systems on } K\text{-orbits} \\ \text{of the flag variety } G/B\end{array}\right\}$$

The free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\mathscr{D}$ is known as the Lusztig–Vogan module and denoted $\mathcal{M}_{LV}$. It admits an action of $\mathscr{H}(W, S)$.

The Lusztig–Vogan module can be thought of as a real group analogue to Kazhdan–Lusztig theory.

The Lusztig–Vogan Module

For each $\delta \in \mathscr{D}$, there exists a "self-dual" element

$$C_\delta = \sum_{\gamma \in \mathscr{D}} P_{\gamma,\delta}(v)\gamma$$

in $\mathcal{M}_{LV}$. The $P_{\gamma,\delta}$ are known as Kazhdan–Lusztig–Vogan polynomials.

Theorem ([LV83])

For all $\delta, \gamma \in W$, the Jordan–Hölder multiplicity of the composition factor L_γ inside the $(\mathfrak{g}, K)$-module M_δ is given by

$$[M_\delta : L_\gamma] = P_{\gamma,\delta}(1).$$

Question: can we learn more about real groups via categorification?

The Lusztig–Vogan Category

In 2022, Larson and Romanov came up with a categorification of the principle block of the Lusztig–Vogan module as a module category over the category of Soergel bimodules.

For the sake of simplicity, we will restrict to the case where K is the connected component containing the identity and T is Abelian.

Theorem ([LR22])

The Grothendieck group of

$$\mathcal{N}_{LV}^0 := \langle R_w \otimes_R X : [w] \in W_K \backslash W,\ X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W,S)) \rangle_{\oplus,\ominus,(1)}$$

is the block of $\mathcal{M}_{LV}$ containing the trivial representation.

The Lusztig–Vogan Category

Victor showed us a diagrammatic calculus for the morphisms.
This is already some new categorical structure.

I'm interested in understanding its (weak) Jordan–Hölder decompositions.

I'd like to use the remainder of the talk to walk through a couple of concrete examples.

Example – Type A_1

Let $G := \mathsf{SL}(2, \mathbb{C})$. This group has two real forms: a compact form $\mathsf{SU}(2)$ and a split form $\mathsf{SU}(1,1)$. Let's look at the latter.

The corresponding Weyl groups are $W = S_2 = \{1, s\}$ and $W_K = \{1\}$, so $W_K \backslash W = \{[1], [s]\}$.

Tensoring standard bimodules with indecomposable Soergel bimodules gives us four initial candidates for indecomposables (up to grading shifts).

$$\begin{aligned} R \ &\otimes_R R \ \cong R, \qquad & R \ &\otimes_R B_s \cong B_s, \\ R_s &\otimes_R R \ \cong R_s, \qquad & R_s &\otimes_R B_s \cong B_s. \end{aligned}$$

All of these are indecomposable, so we're done.

Example – Type A_1

We can graph the relationships between the indecomposables under actions by $\mathbb{S}$Bim as follows.

$$\begin{tikzcd}
 & R \otimes_R B_s \arrow[loop above, "s"] & \\
R \arrow[ur, "s"] & & R_s \arrow[ul, "s"']
\end{tikzcd}$$

The simple transitive module subcategories are generated by the strongly connected components of the graph.

In this instance, we have an equivalence of categories $\mathcal{M}_R \simeq \mathcal{M}_{R_s}$. Thus our composition quotients are $\mathcal{M}_{B_s}$ with multiplicity 1 and $\mathcal{M}_R$ with multiplicity 2. Note that $\mathscr{H}(W,S)$ has two two-sided cells.

Example – Type A_2

Let $G \coloneqq \mathrm{SL}(3, \mathbb{C})$. This group has three real forms: a compact form $\mathrm{SU}(3)$, a split form $\mathrm{SL}(3, \mathbb{R})$ and a quasi-split form $\mathrm{SU}(2, 1)$.
We'll look at $\mathrm{SU}(2, 1)$.

The corresponding Weyl groups are $W = S_3 = \langle s, t : s^2 = t^2 = (st)^3 = 1\rangle$ and $W_K = S_2 = \{1, s\}$, so $W_K \backslash W = \{[1], [t], [ts]\}$.

Not every product of a standard bimodule with an indecomposable Soergel bimodule is indecomposable, so we need to look for direct summands. Our indecomposables are, up to grading shifts,

$$R, \quad R_t, \quad R_{ts}, \quad R \otimes_R B_t, \quad R_t \otimes_R B_s \quad \text{and} \quad R^s \otimes_{R^{s,t}} R.$$

Example – Type A_2

The graph of the action preorder is

$$
\begin{tikzcd}
 & & R^s \otimes_{R^{s,t}} R \arrow[loop above, "{s,t}"] & & \\
 & R \otimes_R B_t \arrow[loop above, "t"] \arrow[ur, "s"] \arrow[dl, shift right, "s"'] & & R_t \otimes_R B_s \arrow[loop above, "s"] \arrow[ul, "t"'] \arrow[dr, shift left, "t"'] & \\
R \arrow[loop below, "s"] \arrow[ur, shift right, "t"'] & & R_t \arrow[ul, "t"] \arrow[ur, "s"'] & & R_{ts} \arrow[loop below, "t"] \arrow[ul, shift left, "s"']
\end{tikzcd}
.
$$

We have three simple quotients, where $\mathcal{M}_R \simeq \mathcal{M}_{R_{ts}}$ has multiplicity 2.

Epilogue

We have just seen two examples for type A Coxeter systems.

In this case, the action preorder is given by the W-graph (boring).

For some Coxeter system (W, S) of type A_n, there is one equivalence class of simple transitive 2-representations of $\mathbb{S}\mathrm{Bim}(W, S)$ for each two-sided cell (where the number of two-sided cells is equal to the number of partitions of $n+1$), and these are all cell 2-representations ([MMM+23]).

Epilogue

There are already lots of questions!

- Are there Coxeter systems for which the the action preorder for the Lusztig–Vogan category is not predicted by the W-graph?
- Can we build any interesting module categories from "extensions" of Lusztig–Vogan categories?
- Is there a real group analogue to Soergel's conjecture?
- What genuinely new information can we learn about real groups?

References

[BB81] Beilinson, A. and Bernstein, J., *Localisation de $\mathfrak{g}$-modules*, C. R. Acad. Sci. Paris Sér. I Math. 292.**1** (1981), pp. 15–18.

[BK81] Brylinski, J.-L. and Kashiwara, M., *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. 64.**3** (1981), pp. 387–410.

[EW14] Elias, B. and Williamson, G., *The Hodge theory of Soergel bimodules*, Ann. Math. 180 (2014), pp. 1089–1136.

[KL79] Kazhdan, D. and Lusztig, G., *Representations of Coxeter Groups and Hecke Algebras*, Invent. Math. **53** (1979), pp. 165–184.

[LR22] Larson, S. and Romanov, A., *A categorification of the Lusztig–Vogan module*, (2022).

[LV83] Lusztig, G. and Vogan Jr., D. A., *Singularities of closures of K-orbits on flag manifolds*, Invent. Math. 71.**2** (1983), pp. 365–379.

[MMM+23] Mackaay, M. et al., *Simple transitive 2-representations of Soergel bimodules for finite Coxeter types*, Proc. Lond. Math. Soc. 126.**5** (2023), pp. 1585–1655.

[Soe92] Soergel, W., *The combinatorics of Harish-Chandra bimodules*, J. Reine Angew. Math. **429** (1992), pp. 49–74.

Thank you for your attention!