Categorical Representation Theory

1. Prologue

Before we get into representation theory, let's *very* briefly review some basic definitions from higher category theory. We will follow [Mal18] and [Str95, §9] as our main references.

Definition 1.1. (Bicategory). *A* bicategory $\mathscr{C}$ *consists of*

- *a class* $\mathrm{Ob}(\mathscr{C})$ *of* objects *(or* 0-cells*);*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, a* Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ *whose objects are called* 1-morphisms *(or* 1-cells*), whose morphisms are called* 2-morphisms *(or* 2-cells*) and where composition of* 2*-morphisms is known as* vertical composition *and denoted* $\circ_v$*;*
- *for each triple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathrm{Ob}(\mathscr{C})$*, a functor* $\circ_h : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ *known as* horizontal composition*;*
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a distinguished* 1*-morphism* $\mathrm{id}_{\mathtt{i}} \in \mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{i}))$ *known as the* identity morphism *for* $\mathtt{i}$*;*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, natural isomorphisms* l *and* r *satisfying*
$$\begin{pmatrix} f \mapsto \mathrm{id}_{\mathtt{j}} \circ_h f \\ \alpha \mapsto \mathrm{id}_{\mathrm{id}_{\mathtt{j}}} \circ_h \alpha \end{pmatrix} \overset{l}{\Rightarrow} \begin{pmatrix} f \mapsto f \\ \alpha \mapsto \alpha \end{pmatrix} \overset{r}{\Leftarrow} \begin{pmatrix} f \mapsto f \circ_h \mathrm{id}_{\mathtt{i}} \\ \alpha \mapsto \alpha \circ_h \mathrm{id}_{\mathrm{id}_{\mathtt{i}}} \end{pmatrix},$$
known respectively as a left *and* right unitor *(whose components* l_f *and* r_f *are* 2*-morphisms);*
- *for each quadruple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} \in \mathrm{Ob}(\mathscr{C})$*, a natural isomorphism* a *between the two horizontal composition functors* $\mathscr{C}(\mathtt{k}, \mathtt{l}) \times \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{l})$ *given by*
$$\begin{pmatrix} h \times g \times f \mapsto (h \circ_h g) \circ_h f \\ \gamma \times \beta \times \alpha \mapsto (\gamma \circ_h \beta) \circ_h \alpha \end{pmatrix} \overset{a}{\Rightarrow} \begin{pmatrix} h \times g \times f \mapsto h \circ_h (g \circ_h f) \\ \gamma \times \beta \times \alpha \mapsto \gamma \circ_h (\beta \circ_h \alpha) \end{pmatrix},$$
known as an associator *(whose components* $a_{h,g,f}$ *are* 2*-morphisms);*

such that the pentagon diagram

$$\begin{tikzcd}
 & ((k \circ_h h) \circ_h g) \circ_h f \arrow[dl, "a_{k,h,g} \circ_h \mathrm{id}_f"'] \arrow[dr, "a_{k \circ_h h, g, f}"] & \\
(k \circ_h (h \circ_h g)) \circ_h f \arrow[d, "a_{k, h \circ_h g, f}"'] & & (k \circ_h h) \circ_h (g \circ_h f) \arrow[d, "a_{k,h,g \circ_h f}"] \\
k \circ_h ((h \circ_h g) \circ_h f) \arrow[rr, "\mathrm{id}_k \circ_h a_{h,g,f}"'] & & k \circ_h (h \circ_h (g \circ_h f))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(g \circ_h \mathrm{id}_{\mathtt{j}}) \circ_h f \arrow[rr, "a_{g, \mathrm{id}_{\mathtt{j}}, f}"] \arrow[dr, "r_g \circ_h \mathrm{id}_f"'] & & g \circ_h (\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[dl, "\mathrm{id}_g \circ_h l_f"] \\
 & g \circ_h f &
\end{tikzcd}$$

commute, for all 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*,* $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k}, \mathtt{l}))$*,* $k \in \mathrm{Ob}(\mathscr{C}(\mathtt{l}, \mathtt{m}))$*.*

A 2-category is a *strict* bicategory; that is, a bicategory whose unitors and associators are all identities. In this case the pentagon and triangle diagrams hold automatically. Observe that (strict) bicategories $\mathscr{C}$ with a single object $\bullet$ are in bijection with (strict) monoidal categories under taking the monoidal delooping; in particular, our monoidal category is nothing but the End-category $\mathscr{C}(\bullet, \bullet)$, where the monoidal product is given by horizontal composition.

We shall henceforth adopt the notation $\mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j}) \coloneqq \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$ and $\mathrm{Mor}^2_{\mathscr{C}}(f, g) \coloneqq \mathrm{Mor}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(f, g)$, for $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ and $f, g \in \mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$. Unfortunately, we will frequently change our notation for 1-morphisms and 2-morphisms depending on what makes sense contextually. For instance, in light of the above remark, 1-morphisms will often be objects of a monoidal category, whence we will write them as X, Y, Z and so on; meanwhile, sometimes they will be realized as functors, in which case we will use F, G, H and so on. The same goes for 2-morphisms. We hope this will not cause any unnecessary confusion!

Definition 1.2. (Pseudofunctor). *A* pseudofunctor F *between bicategories* $\mathscr{C}$ *and* $\mathscr{D}$ *consists of*

- *a map* $\mathrm{F} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$*;*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, a functor* $\mathrm{F} : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{j}))$*;*
- *for each triple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathrm{Ob}(\mathscr{C})$*, a natural isomorphism* m *between the two "composition by* F*" functors* $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{k}))$ *given by*
$$\begin{pmatrix} g \times f \mapsto \mathrm{F}(g) \circ_h \mathrm{F}(f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta) \circ_h \mathrm{F}(\alpha) \end{pmatrix} \overset{m}{\Longrightarrow} \begin{pmatrix} g \times f \mapsto \mathrm{F}(g \circ_h f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta \circ_h \alpha) \end{pmatrix},$$
whose components $m_{g,f}$ *are* 2*-morphisms;*
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, an isomorphism* $i : \mathrm{id}_{\mathrm{F}(\mathtt{i})} \to \mathrm{F}(\mathrm{id}_{\mathtt{i}})$ *in* $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{i})))$*;*

such that the hexagon diagram

$$\begin{tikzcd}
 & (\mathrm{F}(h) \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) \arrow[dl, "m_{h,g} \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] \arrow[dr, "a_{\mathrm{F}(h),\mathrm{F}(g),\mathrm{F}(f)}"] & \\
\mathrm{F}(h \circ_h g) \circ_h \mathrm{F}(f) \arrow[d, "m_{h \circ_h g, f}"'] & & \mathrm{F}(h) \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[d, "\mathrm{id}_{\mathrm{F}(h)} \circ_h m_{g,f}"] \\
\mathrm{F}((h \circ_h g) \circ_h f) \arrow[dr, "\mathrm{F}(a_{h,g,f})"'] & & \mathrm{F}(h) \circ_h \mathrm{F}(g \circ_h f) \arrow[dl, "m_{h, g \circ_h f}"] \\
 & \mathrm{F}(h \circ_h (g \circ_h f)) &
\end{tikzcd}$$

and the squares

$$\begin{tikzcd}
\mathrm{id}_{\mathrm{F}(\mathtt{j})} \circ_h \mathrm{F}(f) \arrow[r, "l_{\mathrm{F}(f)}"] \arrow[d, "i \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & \mathrm{F}(f) \\
\mathrm{F}(\mathrm{id}_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[r, "m_{\mathrm{id}_{\mathtt{j}}, f}"'] & \mathrm{F}(\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[u, "\mathrm{F}(l_f)"']
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
\mathrm{F}(f) \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[r, "r_{\mathrm{F}(f)}"] \arrow[d, "\mathrm{id}_{\mathrm{F}(f)} \circ_h i"'] & \mathrm{F}(f) \\
\mathrm{F}(f) \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}}) \arrow[r, "m_{f, \mathrm{id}_{\mathtt{i}}}"'] & \mathrm{F}(f \circ_h \mathrm{id}_{\mathtt{i}}) \arrow[u, "\mathrm{F}(r_f)"']
\end{tikzcd}$$

commute, for all 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*,* $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k}, \mathtt{l}))$*.*

As before, a 2-functor is a pseudofunctor where m and i are identity. In the same way that bicategories generalize monoidal categories, pseudofunctors generalize strong monoidal functors, preserving both vertical composition (strictly) and horizontal composition (up to isomorphism).

Definition 1.3. (Pseudonatural Transformation). *A* pseudonatural transformation Φ *between pseudofunctors* F *to* G *consists of*

- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a* 1*-morphism* $\Phi_{\mathtt{i}} \in \mathrm{Ob}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{i})))$*;*
- *for each* 1*-morphism* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, a* 2*-morphism* $\Phi_f : \Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \to \mathrm{G}(f) \circ_h \Phi_{\mathtt{i}}$ *in* $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{j})))$*;*

such that

- *for each* 2*-morphism* $\alpha : f \to g$ *in* $\mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, the square*

$$\begin{tikzcd}
\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \arrow[r, "\mathrm{id}_{\Phi_{\mathtt{j}}} \circ_h \mathrm{F}(\alpha)"] \arrow[d, "\Phi_f"'] & \Phi_{\mathtt{j}} \circ_h \mathrm{F}(g) \arrow[d, "\Phi_g"] \\
\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}} \arrow[r, "\mathrm{G}(\alpha) \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"'] & \mathrm{G}(g) \circ_h \Phi_{\mathtt{i}}
\end{tikzcd}$$

  commutes (that is, the 2*-morphisms* Φ_f *are the components of a natural transformation);*
- *for each pair of* 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*, the octagon diagram*

$$\begin{tikzcd}
 & (\Phi_{\mathtt{k}} \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) \arrow[r, "a_{\Phi_{\mathtt{k}}, \mathrm{F}(g), \mathrm{F}(f)}"] \arrow[dl, "\Phi_g \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & \Phi_{\mathtt{k}} \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[dr, "\mathrm{id}_{\Phi_{\mathtt{k}}} \circ_h m_{g,f}"] & \\
(\mathrm{G}(g) \circ_h \Phi_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[d, "a_{\mathrm{G}(g), \Phi_{\mathtt{j}}, \mathrm{F}(f)}"'] & & & \Phi_{\mathtt{k}} \circ_h \mathrm{F}(g \circ_h f) \arrow[d, "\Phi_{g \circ_h f}"] \\
\mathrm{G}(g) \circ_h (\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f)) \arrow[dr, "\mathrm{id}_{\mathrm{G}(g)} \circ_h \Phi_f"'] & & & \mathrm{G}(g \circ_h f) \circ_h \Phi_{\mathtt{i}} \\
 & \mathrm{G}(g) \circ_h (\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}}) & (\mathrm{G}(g) \circ_h \mathrm{G}(f)) \circ_h \Phi_{\mathtt{i}} \arrow[l, "a_{\mathrm{G}(g), \mathrm{G}(f), \Phi_{\mathtt{i}}}"] \arrow[ur, "m_{g,f} \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"'] &
\end{tikzcd}$$

  commutes;
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, the pentagon diagram*

$$\begin{tikzcd}
\Phi_{\mathtt{i}} \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[r, "r_{\Phi_{\mathtt{i}}}"] \arrow[d, "\mathrm{id}_{\Phi_{\mathtt{i}}} \circ_h i"'] & \Phi_{\mathtt{i}} & \mathrm{id}_{\mathrm{G}(\mathtt{i})} \circ_h \Phi_{\mathtt{i}} \arrow[l, "l_{\Phi_{\mathtt{i}}}"'] \arrow[d, "i \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"] \\
\Phi_{\mathtt{i}} \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}}) \arrow[rr, "\Phi_{\mathrm{id}_{\mathtt{i}}}"'] & & \mathrm{G}(\mathrm{id}_{\mathtt{i}}) \circ_h \Phi_{\mathtt{i}}
\end{tikzcd}$$

  commutes.

If each Φ_f *is invertible (they form a natural isomorphism), we call* Φ *a* pseudonatural isomorphism*.*

We say that two bicategories are *biequivalent* if there exists an invertible pseudofunctor between them. For the sections that follow, we will assume that all categories are essentially small, that all bicategories are essentially small and that all fields are algebraically closed.

Note that given a bicategory $\mathscr{C}$ and objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, we have that $\mathscr{C}(X, Y)$ is a $(\mathscr{C}(X, X), \mathscr{C}(Y, Y))$-bimodule category. Check bicategory of bifinite bimodules as in [DGG14].

Examples of bicategories.

Given an algebra (or more generally a ring), one can build a 2-category whose objects are algebras, 1-morphisms are modules (A, B)-bimodules and 2-morphisms are bimodule maps. In subfactor theory, we do this for a unital inclusion $N \subseteq M$ of type II_1 subfactors; we set $\mathrm{Ob}(\mathscr{C}) \coloneqq \{M, N\}$ and $\mathscr{C}(X, Y) \coloneqq \mathsf{Bimod}(X, Y)$ ("bimodule summands of basic constructions" / bifinite (X, Y)-bimodules). This produces a 2-category.

Given a shaded planar algebra P, take $\mathrm{Ob}(\mathscr{C}) \coloneqq \{-, +\}$ and let $\mathscr{C}(\varepsilon, \eta) \coloneqq \mathsf{Rect}_P(\varepsilon, \eta)$ be the subcategory of the rectangular category consisting of those morphisms whose source shading is ε and whose target shading is η, with composition running from bottom to top. In other words,

$$\mathrm{Mor}^1_{\mathscr{C}}(\varepsilon, \eta) \coloneqq \{2k : k \in \mathbb{N} \text{ is even if } \varepsilon = \eta \text{ and odd otherwise}\}$$

and $\mathrm{Mor}^2_{\mathscr{C}}(m, n) \coloneqq P_{m+n,\varepsilon}$, for $m, n \in \mathrm{Mor}^1_{\mathscr{C}}(\varepsilon, \eta)$ ([DGG14]). This gives a $\mathbb{C}$-linear (but unfortunately not additive) 2-category. Look into "The Temperley–Lieb algebra at roots of unity".

Explicit example of multifinitary bicategory: group planar algebra? Maybe we can generalize it, see https://scholars.unh.edu/cgi/viewcontent.cgi?article=1338&context=dissertation.

2. Finitary Birepresentation Theory

Definition 2.1. (Idempotent Complete). *Let $\mathcal{C}$ be a category. An* idempotent *is an endomorphism $p : A \to A$ in $\mathcal{C}$ such that $p \circ p = p$. An idempotent is said to* split *if there is an object B and morphisms $\pi : A \to B$, $\iota : B \to A$ in $\mathcal{C}$ such that $p = \iota \circ \pi$ and* $\mathrm{id}_B = \pi \circ \iota$*. A category is said to be* idempotent complete *(or* idempotent split*) if every idempotent splits.*

Note that the condition $\mathrm{id}_B = \pi \circ \iota$ implies that π is an epimorphism and ι is a monomorphism. To see why, suppose we have morphisms $h, k : B \to C$ with $h \circ \pi = k \circ \pi$. Then $h = h \circ \pi \circ \iota = k \circ \pi \circ \iota = k$, whence π is an epimorphism. Similarly, given morphisms $h, k : C \to B$ with $\iota \circ h = \iota \circ k$, we have that $h = \pi \circ \iota \circ h = \pi \circ \iota \circ k = k$, whence ι is a monomorphism. Thus, because ι is a monomorphism, B is by definition a subobject of A. In other words, a category being idempotent complete means that every idempotent $p : A \to A$ can be seen as a projection onto some subobject B followed by an inclusion back into A. Moreover, in the additive setting we have the following result.

Proposition 2.2. *An idempotent $p : A \to A$ belonging to a preadditive category splits if and only if $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.*

Proof. Suppose $p : A \to A$ is an idempotent that splits. Then by definition we have a subobject I of A together with an epimorphism $\pi_I : A \to I$ and a monomorphism $\iota_I : I \to A$ satisfying $p = \iota_I \circ \pi_I$ and $\mathrm{id}_I = \pi_I \circ \iota_I$. Moreover, because $\mathrm{id}_A - p$ is also idempotent, there similarly exists a subobject K of A together with an epimorphism $\pi_K : A \to K$ and a monomorphism $\iota_K : K \to A$ satisfying $\mathrm{id}_A - p = \iota_K \circ \pi_K$ and $\mathrm{id}_K = \pi_K \circ \iota_K$. Because $\mathrm{id}_A = \iota_I \circ \pi_I + \iota_K \circ \pi_K$, we have the biproduct diagram

$$I \underset{\iota_I}{\overset{\pi_I}{\leftrightarrows}} A \underset{\iota_K}{\overset{\pi_K}{\rightleftarrows}} K.$$

By [Mac13, Theorem VIII.2.2], it follows that $A = I \oplus K$. We claim now that $\mathrm{Im}(p) = I$; we shall prove this by showing that p admits the canonical decomposition

$$K \xrightarrow{\iota_K} A \xrightarrow{\pi_I} I \xrightarrow{\iota_I} A \xrightarrow{\pi_K} K.$$

In particular, we claim that $\mathrm{Ker}(p) = (K, \iota_K)$, that $\mathrm{Coker}(p) = (K, \pi_K)$, that $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and that $\mathrm{Ker}(\pi_K) = (I, \iota_I)$. We show that the first two hold and remark that showing the remaining two is essentially the same. First, observe that

$$p \circ \iota_K = \iota_I \circ \pi_I \circ \iota_K = (\mathrm{id}_A - \iota_K \circ \pi_K) \circ \iota_K = \iota_K - \iota_K \circ \pi_K \circ \iota_K = \iota_K - \iota_K = 0.$$

Moreover, given an object K' together with a morphism $k' : K' \to A$ for which $p \circ k' = 0$, we see that by taking $\ell \coloneqq \pi_K \circ k'$, we have that

$$\iota_K \circ \ell = \iota_K \circ \pi_K \circ k' = (\mathrm{id}_A - p) \circ k' = k'.$$

Thus $\mathrm{Ker}(p) = (K, \iota_K)$. As for $\mathrm{Coker}(p)$, we observe that

$$\pi_K \circ p = \pi_K \circ \iota_I \circ \pi_I = \pi_K \circ (\mathrm{id}_A - \iota_K \circ \pi_K) = \pi_K - \pi_K \circ \iota_K \circ \pi_K = \pi_K - \pi_K = 0,$$

and that for any object C' together with a morphism $c' : A \to C'$ for which $c' \circ p = 0$, taking $\ell \coloneqq c' \circ \iota_K$ gives us

$$\ell \circ \pi_K = c' \circ \iota_K \circ \pi_K = c' \circ (\mathrm{id}_A - p) = c'.$$

That $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and $\mathrm{Ker}(\pi_K) = (I, \iota_I)$ follow similarly, whence $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.

Conversely, suppose that $p : A \to A$ is an idempotent for which $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$. Then we have the canonical decomposition

$$\mathrm{Ker}(p) \xrightarrow{k} A \xrightarrow{\pi} \mathrm{Im}(p) \xrightarrow{\iota} A \xrightarrow{c} \mathrm{Coker}(p).$$

By definition this means that $p = \iota \circ \pi$, so we need only show that $\mathrm{id}_{\mathrm{Im}(p)} = \pi \circ \iota$. But note that π is a cokernel and ι is a kernel, hence they are an epimorphism and a monomorphism, respectively. Thus by the definition of epimorphisms and monomorphisms, we may cancel $p \circ p = p$ on the right by ι and on the left by π, whence we obtain nothing but

$$p \circ p = p \implies \iota \circ \pi \circ \iota \circ \pi = \iota \circ \pi \implies \pi \circ \iota = \mathrm{id}_{\mathrm{Im}(p)}$$

as desired. Thus p splits. This completes the proof. ∎

This result is not only important in its own right, but psychologically helpful: it tells us that split idempotents categorify in some heuristic sense the notion of projections from linear algebra, which always split. Moreover, recall that a preadditive category is said to be *Karoubian* (or *pseudo-Abelian*) if every idempotent admits a kernel (or, equivalently, if every idempotent admits an image, as we may obtain the image by considering $\mathrm{Ker}(\mathrm{id}_A - p)$). We therefore have the following corollary.

Corollary 2.3. *A preadditive category is Karoubian if and only if it is idempotent complete.*

Example 2.4. The category of projective modules over a ring is the Karoubi envelope of its full subcategory of free modules, as a module is projective if and only if it is a direct summand of a free module. In other words, categories of projective modules are idempotent complete in a universal way.

If representation theory can naïvely be described as "group theory in linear sets", birepresentation theory can be described as "group theory in linear categories". We will now make precise the notion of "linear categories" that we will find ourselves working with.

Definition 2.5. (Finitary Category). *An additive,* $\Bbbk$*-linear category* $\mathcal{C}$ *is called* multifinitary *if it is idempotent complete, it has finitely many isomorphism classes of indecomposable objects and it has finite-dimensional* $\Bbbk$*-vector spaces of morphisms. If* $\mathcal{C}$ *is not monoidal, this is equivalent to being* finitary*; otherwise, it is said to be* finitary *if its unit object is indecomposable.*

Remark 2.6. Let $\mathcal{C}$ be an additive category. Then by [Mac13, §VIII.2], its morphisms form a matrix calculus; that is, for any $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $X \cong \bigoplus_{i=1}^m X_i$ and $Y \cong \bigoplus_{j=1}^n Y_j$, we have that

$$f = \sum_{j=1}^{n} \sum_{i=1}^{m} (\iota_{Y_j} \circ f_{i,j} \circ \pi_{X_i})$$

for $f_{i,j} := \pi_{Y_j} \circ f \circ \iota_{X_i}$, where $\pi_{X_i} : X \to X_i$ and $\pi_{Y_j} : Y \to Y_j$ are epimorphisms while $\iota_{X_i} : X_i \to X$ and $\iota_{Y_j} : Y_j \to Y$ are monomorphisms for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

We also have the following folklorish theorem, which has some nice exactness implications in the module category setting. Due to its very technical nature we will only offer a (fallible) sketch of the proof, but the full proof should follow similarly to the proof of the embedding theorem for Abelian categories. Please see the wonderful write-up given in [Jun19] for more details on these subtleties.

Theorem 2.7. (Freyd–Mitchell Embedding Theorem). *A small category is multifinitary if and only if there is an exact, full embedding into the exact category $\mathsf{Mod}_p(A)$ of finitely generated, projective modules over some finite-dimensional, associative $\mathbb{k}$-algebra A.*

Sketch. First, $\mathsf{Mod}_p(A)$ is certainly Karoubi (in fact, the category of projective modules over any ring is the Karoubi envelope of its full subcategory of free modules), and it also has both finitely many isomorphism classes of indecomposable objects and finite-dimensional $\mathbb{k}$-vector spaces of morphisms.

Conversely, let $\mathcal{C}$ be multifinitary. We wish to find a full, exact embedding $\mathcal{C} \to \mathsf{Mod}_p(A)$ for some finite-dimensional, associative $\mathbb{k}$-algebra A. Denote by $\mathcal{L} \coloneqq \mathsf{Fun}_l(\mathcal{C}, \mathsf{Vect}_{\mathbb{k}}^{\text{f.d.}})$ the category of left exact, $\mathbb{k}$-linear functors from $\mathcal{C}$ to $\mathsf{Vect}_{\mathbb{k}}^{\text{f.d.}}$ (which we note are also automatically additive). The contravariant Yoneda embedding $X \mapsto \mathcal{C}(X, -)$ gives us a full, exact embedding $\text{よ} : \mathcal{C}^{\text{op}} \to \mathcal{L}$, as $\mathcal{C}(X, -)$ is left exact, which corresponds to a full, exact, covariant embedding $\text{よ}^{\text{op}} : \mathcal{C} \to \mathcal{L}^{\text{op}}$ by duality[†]. Because $\mathcal{L}$ is complete with injective cogenerator given by the direct product $\prod_{X \in \mathrm{Ob}(\mathcal{C})} \mathcal{C}(X, -) \in \mathrm{Ob}(\mathcal{L})$, it follows that $\mathcal{L}^{\text{op}}$ is cocomplete and admits a corresponding projective generator $P' \in \mathrm{Ob}(\mathcal{L}^{\text{op}})$. Let $\mathcal{L}'$ be the small, exact, full subcategory of $\mathcal{L}^{\text{op}}$ generated by the image of よ, which we note is itself multifinitary by the Yoneda lemma; the final step is to embed $\mathcal{L}'$ into the category of projective modules. Well, suppose we write $I \coloneqq \bigsqcup_{F \in \mathrm{Ob}(\mathcal{L}')} \mathrm{Mor}_{\mathcal{L}'}(P', F)$ and define $P \coloneqq \bigoplus_{i \in I} P'$. Of course $A \coloneqq \mathrm{End}_{\mathcal{L}'}(P)$ is a finite-dimensional, associative $\mathbb{k}$-algebra. Moreover, for any $F \in \mathrm{Ob}(\mathcal{L}')$, we may endow $\mathrm{Mor}_{\mathcal{L}'}(P, F)$ with the structure of a finitely-generated, projective A-module in a canonical way by taking $a \cdot x \coloneqq x \circ a$ for all $a \in A$ and $x \in \mathrm{Mor}_{\mathcal{L}'}(P, F)$. Since P is also a projective generator, we have a full, exact embedding $\mathscr{F} : \mathcal{L}' \to \mathsf{Mod}_p(A)$ sending $F \mapsto \mathrm{Mor}_{\mathcal{L}'}(P, F)$. Thus $\mathscr{F} \circ \text{よ}^{\text{op}} : \mathcal{C} \to \mathsf{Mod}_p(A)$ is itself a full, exact embedding, giving us the desired result. $\square$

Remark 2.8. This result tells us that a small category is multifinitary if and only if it is equivalent to a full subcategory of the category of finitely generated, projective modules over some finite-dimensional, associative $\mathbb{k}$-algebra ([MMM+23]). It may also be worth noting that we can replace "projective" with "injective" in this alternative definition, as these two notions are dual in the sense that being projective is of course the same as being injective in the opposite category.

Theorem 2.9. ([EMTW20, Theorem 11.53]). *Let $\mathcal{C}$ be an additive category. Then the following are equivalent.*

(i). *$\mathcal{C}$ is Krull–Schmidt.*
(ii). *$\mathcal{C}$ is idempotent complete and all endomorphism rings are semiperfect.*

By (ii), we mean that for all $X \in \mathrm{Ob}(\mathcal{C})$, the ring $R \coloneqq \mathrm{End}_{\mathcal{C}}(X)$ admits a complete, orthogonal set of idempotents $p_1, \ldots, p_n$ such that each $p_i R p_i$ is a local ring. This in fact implies that X is indecomposable if and only if its endomorphism ring is local. In the setting where $\mathcal{C}$ is $\mathbb{k}$-linear with finite-dimensional $\mathbb{k}$-vector spaces of morphisms, its endomorphism rings will just be matrix rings, which are always semiperfect (as is any finite-dimensional, associative $\mathbb{k}$-algebra); for such categories, being Krull–Schmidt is equivalent to being idempotent complete.

[†] Since this will inevitably be someone's first time seeing it, よ is the hiragana for "yo" (as in "Yoneda").

Definition 2.10. *Let $\mathfrak{A}_{\Bbbk}^{f}$ denote the 2-category whose objects are multifinitary categories, whose 1-morphisms are $\Bbbk$-linear functors and whose 2-morphisms are natural transformations.*

We briefly remind the reader that any $\Bbbk$-linear functor between additive, $\Bbbk$-linear categories is automatically additive!

For the following definition, we interpret the End-categories $\mathscr{C}(\mathtt{i},\mathtt{i})$ as being monoidal categories with respect to the composition of 1-morphisms.

Definition 2.11. (Finitary Bicategory). *A bicategory $\mathscr{C}$ is said to be* (multi)finitary *if*

- *it has finitely many objects;*
- *for any pair $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, the Hom-category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is (multi)finitary;*
- *horizontal composition of 2-morphisms is $\Bbbk$-bilinear.*

In a multifinitary bicategory, vertical composition will automatically be both biadditive and $\Bbbk$-bilinear as a consequence of the Hom-categories $\mathcal{C}(\mathtt{i},\mathtt{j})$ being additive and $\Bbbk$-linear. However, we genuinely must ask that horizontal composition be biadditive and $\Bbbk$-bilinear.

Proposition 2.12. *A monoidal category $\mathcal{C}$ is (multi)finitary if and only if its monoidal delooping $\mathsf{B}\mathcal{C}$ is (multi)finitary.*

Proof. If $\mathsf{B}\mathcal{C}$ is (multi)finitary, it is obvious that $\mathcal{C}$ is (multi)finitary, since $\mathcal{C} = \mathsf{B}\mathcal{C}(\bullet,\bullet)$, where $\mathrm{Ob}(\mathsf{B}\mathcal{C}) = \{\bullet\}$. Conversely, suppose $\mathcal{C}$ is (multi)finitary. Clearly $\mathsf{B}\mathcal{C}$ has finitely many objects (in particular, it has only one), and $\mathsf{B}\mathcal{C}(\bullet,\bullet) = \mathcal{C}$ is (multi)finitary. Finally, horizontal composition is given by the monoidal product and is hence biadditive and $\Bbbk$-bilinear. This completes the proof. ∎

Definition 2.13. (Birepresentation). *A* birepresentation *of a bicategory $\mathscr{C}$ is a pseudofunctor from $\mathscr{C}$ to Cat, the 2-category of small categories. A* 2-representation *of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to Cat.*

Definition 2.14. (Finitary Birepresentation). *A* (multi)finitary birepresentation *of a (multi)finitary bicategory $\mathscr{C}$ is a covariant, $\Bbbk$-linear pseudofunctor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$. A* (multi)finitary 2-representation *of a (multi)finitary 2-category $\mathscr{C}$ is a covariant, $\Bbbk$-linear 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$.*

Definition 2.15. (Finitary Module Category). *A* (multi)finitary module category *over a (multi)finitary monoidal category $\mathcal{C}$ is a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ for which the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is $\Bbbk$-bilinear.*

Remark 2.16. Recall that a representation of a group G is morally nothing but a functor $F : \mathsf{B}G \to \mathsf{Vect}$, where the action of G on $V := F(\bullet)$ is given by $g \cdot v := [F(g)](v)$ for all $v \in V$ and $g \in \mathrm{Mor}(\mathsf{B}G)$. Analogously, given a 2-representation $\mathrm{M} : \mathscr{C} \to \mathsf{Cat}$, we have a 2-*action* of $\mathscr{C}$ given by $F \cdot X := [\mathrm{M}(F)](X)$ for all $X \in \mathrm{M}(\mathtt{i})$ and $F \in \mathrm{Mor}^{1}_{\mathscr{C}}(\mathtt{i},\mathtt{j})$, where $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. The upshot here is that we should think of the 1-morphisms in our 2-category as our "group elements", with composition becoming "group multiplication".

Proposition 2.17. *Let $\mathcal{C}$ be a monoidal (respectively, strict monoidal) category. There exists a bijection between $\mathcal{C}$-module categories $\mathcal{M}$ and birepresentations (respectively, 2-representations) M of the delooping category $\mathsf{B}\mathcal{C}$. Moreover, $\mathcal{M}$ is (multi)finitary if and only if M is (multi)finitary.*

Proof. Let $\mathcal{C}$ be a monoidal category and $\mathsf{B}\mathcal{C}$ its delooping category. Recall that by [EGNO16, Proposition 7.1.3] there is a bijection between $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors of the form $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$. Such a functor F induces a canonical birepresentation $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ that takes the single object $\bullet \in \mathrm{Ob}(\mathsf{B}\mathcal{C})$ to $\mathcal{M}$ and otherwise acts on the 1-morphisms and 2-morphisms by F. Conversely, let $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ be a birepresentation and write $\mathcal{M} := \mathrm{M}(\bullet)$. This naturally induces a functor $F : \mathcal{C} \to \mathsf{Cat}(\mathcal{M}, \mathcal{M}) = \mathrm{End}(\mathcal{M})$ that acts on objects and morphisms of $\mathcal{C}$ by M. Clearly these two constructions are inverse to each other and (multi)finitarity of both sides is equivalent. It is easy to see that if $\mathcal{C}$ is strict, the result for 2-representations follows similarly. This completes the proof. ∎

We also note that by Remark 2.8, every multifinitary module category is automatically exact, as equivalences of categories preserve projectivity of objects and hence every object must be projective.

Definition 2.18. (Equivalence of Birepresentations). *We say that two birepresentations M and N of $\mathscr{C}$ are* equivalent *if there exists a pseudonatural isomorphism $\Phi : \mathrm{M} \to \mathrm{N}$ such that the component $\Phi_{\mathtt{i}} : \mathrm{M}(\mathtt{i}) \to \mathrm{N}(\mathtt{i})$ is an equivalence of categories for each $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$.*

Definition 2.19. (Yoneda Birepresentation). *Let $\mathscr{C}$ be a bicategory and consider the pseudofunctor $\mathscr{C}(\mathtt{i}, -) : \mathscr{C} \to \mathsf{Cat}$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, such that*

- *objects $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ are sent to the Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$;*
- *1-morphisms of the form $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$ are sent to the "horizontal post-composition by F" functor $F_* : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ given by $G \mapsto F \circ G$ and $(\gamma : G \to G') \mapsto \mathrm{id}_F \circ_h \gamma$;*
- *2-morphisms of the form $\alpha \in \mathrm{Mor}^2_{\mathscr{C}}(F, F')$, for $F, F' \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$, are sent to the "horizontal post-composition by α" natural transformation $\alpha_* : F_* \Rightarrow F'_*$, whose components are given by $(\alpha_*)_G := \alpha \circ_h \mathrm{id}_G$ for each $G \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$.*

We call this the Yoneda *(or* principal*)* birepresentation corresponding to $\mathtt{i}$ *and denote it by $\mathbb{P}_{\mathtt{i}}$.*

If $\mathscr{C}$ is (multi)finitary, then its corresponding Yoneda birepresentations are also all (multi)finitary, and if $\mathscr{C}$ is a (multi)finitary 2-category its Yoneda birepresentations are (multi)finitary 2-representations. As an example, let $\mathscr{C}$ be the monoidal delooping of $\mathcal{C}$. Then $\mathbb{P}_\bullet$ maps $\bullet$ to $\mathcal{C}$, maps 1-morphisms $X \in \mathrm{Ob}(\mathcal{C})$ to the left tensor product functor given by $Y \mapsto X \otimes Y$ and $(f : Y \to Y') \mapsto \mathrm{id}_X \otimes f$, and maps 2-morphisms $f : X \to X'$ to the natural transformation $Y \mapsto f \otimes \mathrm{id}_Y$. This is nothing but the birepresentation corresponding to the regular $\mathcal{C}$-module category $\mathcal{C}$!

Definition 2.20. (Ideal). *A* left *(respectively* right*)* ideal *of a category $\mathcal{C}$ is a collection $\mathcal{I} \coloneqq \{\mathcal{I}(X,Y) : X, Y \in \mathrm{Ob}(\mathcal{C})\}$, where each $\mathcal{I}(X,Y)$ is a non-empty subclass of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, such that $\mathcal{I}$ is stable under post-composition (respectively pre-composition) with morphisms from $\mathcal{C}$. If $\mathcal{C}$ is preadditive, we additionally ask that $\mathcal{I}(X,Y)$ be an Abelian subgroup of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ for all pairs $X, Y \in \mathrm{Ob}(\mathcal{C})$, and if $\mathcal{C}$ is $\Bbbk$-linear we ask that these also be $\Bbbk$-subspaces. We say that $\mathcal{I}$ is a* two-sided *(or* bilateral*)* ideal *if it is both a left ideal and a right ideal, and that it is a subideal of $\mathcal{J}$ if its classes of morphisms are subclasses. An ideal is said to be* proper *if there exists some pair $X, Y \in \mathrm{Ob}(\mathcal{C})$ for which $\mathcal{I}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$, and* maximal *if it is proper and not a subideal of any other proper ideal.*

Let $\mathcal{I}$ be an ideal of a category $\mathcal{C}$. As we have implied previously, if $\mathcal{C}$ is a preadditive, then $\mathrm{End}_{\mathcal{C}}(X)$ is a ring for all $X \in \mathrm{Ob}(\mathcal{C})$, and it follows that the valid choices for $\mathcal{I}(X,X)$ coincide exactly with the ring ideals of $\mathrm{End}_{\mathcal{C}}(X)$. Similarly, if $\mathcal{C}$ is a $\Bbbk$-linear category, then each $\mathrm{End}_{\mathcal{C}}(X)$ is an associative, unital algebra, and the valid choices for $\mathcal{I}(X,X)$ coincide with algebra ideals (that is, a subspace of $\mathrm{End}_{\mathcal{C}}(X)$ that is closed under algebra multiplication).

Example 2.21. Consider a subcategory $\mathcal{T}$ of $\mathsf{Vect}_{\Bbbk}$ whose endomorphisms of $\Bbbk^n$ are the $n \times n$ upper triangular Toeplitz matrices. Then $\mathrm{End}(\Bbbk^2) \cong \Bbbk[x]/\langle x^2 \rangle$. If we consider the sub-semicategory of $\mathcal{T}$ containing only the object $\Bbbk^2$ and the endomorphisms $\Bbbk\{x\}$ (that is, linear scalings of the matrix with 1 in the off-diagonal), we obtain an ideal of $\mathcal{T}$.

Let M be a multifinitary birepresentation of $\mathscr{C}$ for which each $\mathrm{M}(\mathtt{j})$ is additive and idempotent complete, and let $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ for some $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. Consider the *additive closure* (closure under isomorphisms, direct summands and finite direct sums) of the orbit of X under the action of $\mathscr{C}$; that is, the collection of objects

$$\mathscr{C}(\{X\}) \coloneqq \mathsf{add}(\{[\mathrm{M}(F)](X) : \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})\})$$

where the add denotes the aforementioned additive closure. Due to the additivity of the 1-morphisms of $\mathfrak{A}^f_{\Bbbk}$, it follows that $\mathscr{C}(\{X\})$ is itself stable under the action of $\mathscr{C}$. This therefore induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X\})$ of $\mathscr{C}$ by restriction, with each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ sent to

$$\mathscr{C}_{\mathtt{j}}(\{X\}) \coloneqq \mathsf{Add}(\{[\mathrm{M}(F)](X) : F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})\}),$$

the *additive subcategory* (full subcategory that is closed under isomorphisms, direct summands and finite direct sums) of $\mathrm{M}(\mathtt{j})$ generated by the objects of $\mathscr{C}(\{X\})$ that lie in $\mathrm{M}(\mathtt{j})$. Because $\mathrm{M}(\mathtt{j})$ is Karoubian, this is nothing but the Karoubi envelope of the full subcategory generated by the action of $\mathscr{C}$. In principle this process works for any collection $\{X_i : i \in I\}$ with $X_i \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}_i))$, whence

$$\mathscr{C}(\{X_i : i \in I\}) \coloneqq \mathsf{Add}(\{[\mathrm{M}(F)](X_i) : i \in I, \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_i, \mathtt{j})\})$$

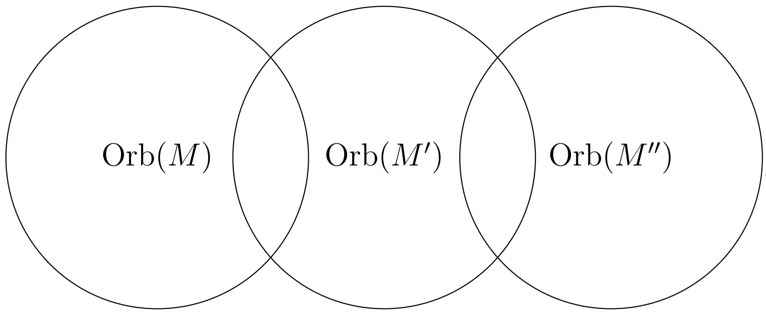
similarly induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X_i : i \in I\})$ of $\mathscr{C}$. In any case, we will only need to consider the single-object situation, as it allows us to make the following evocative definition.

Definition 2.22. (Transitive Birepresentation). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory $\mathscr{C}$. We say that* M *is* transitive *if, for every $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ and non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$, the embedding $\mathscr{C}_{\mathtt{j}}(\{X\}) \hookrightarrow \mathrm{M}(\mathtt{j})$ is an equivalence for all $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$.*

Remark 2.23. Recall that a module M is simple if and only if every cyclic submodule generated by a non-zero element of M is equal to M. This is exactly what we're trying to capture with transitivity! In the birepresentation world, however, things are more involved. We will see more on this shortly.

We say that a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ is transitive if, for all $M \in \mathrm{Ob}(\mathcal{M})$, we have that $\mathrm{Orb}(M) := \mathsf{Add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathcal{M}$ by Proposition 2.17. This is equivalent to it having no full Karoubian subcategories, equivalent to its corresponding birepresentation being transitive and equivalent to its split Grothendieck group $\mathrm{Gr}(\mathcal{M})$ being a simple $\mathrm{Gr}(\mathcal{C})$-module.

Being transitive is clearly a much stronger condition for a module category than being indecomposable (in the sense that it is not the direct sum of two non-zero multifinitary module subcategories). Given any $M \in \mathrm{Ob}(\mathcal{M})$ and $M' \in \mathrm{Ob}(\mathrm{Orb}(M))$, it's possible that the orbit of M and the orbit of some $M'' \in \mathrm{Ob}(\mathrm{Orb}(M'))$ may have trivial intersection. For instance,



.

This picture becomes much more insightful in light of Lemma 2.30 and Proposition 2.31.

Definition 2.24. ($\mathscr{C}$-Stable Ideal). *Let* M *be a birepresentation of* $\mathscr{C}$. *A* $\mathscr{C}$-stable ideal I *of* M *is a collection* $\mathrm{I} := \{\mathrm{I}(\mathtt{i}) : \mathtt{i} \in \mathrm{Ob}(\mathscr{C})\}$, *where each* $\mathrm{I}(\mathtt{i})$ *is a two-sided ideal of* $\mathrm{M}(\mathtt{i})$ *such that* $[\mathrm{M}(F)](\mathrm{I}(\mathtt{i}))$ *is a subclass of* $\mathrm{I}(\mathtt{j})$ *for all* 1*-morphisms* $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$. *A* $\mathscr{C}$*-stable subideal* I *of a* $\mathscr{C}$*-stable ideal* J *is a* $\mathscr{C}$*-stable ideal for which* $\mathrm{I}(\mathtt{i})$ *is a subideal of* $\mathrm{J}(\mathtt{i})$ *for all* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. *We say that* I *is* proper *if there exists some* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *for which* $\mathrm{I}(\mathtt{i})$ *is proper, and* maximal *if it is proper and not a* $\mathscr{C}$*-stable subideal of any other proper* $\mathscr{C}$*-stable ideal.*

Definition 2.25. (Simple Birepresentation). *A multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *is said to be* simple *if it admits no proper, non-zero* $\mathscr{C}$*-stable ideals.*

Similarly to before, we say that a $\mathcal{C}$-module category $\mathcal{M}$ is simple if its corresponding birepresentation is simple. In other words, given non-zero $f, g \in \mathrm{Mor}(\mathcal{M})$, we can obtain g by composing f with other morphisms in $\mathcal{M}$ and acting via $\mathcal{C}$ (by taking the left monoidal product with identity morphisms).

In light of the module category picture, we see that the "right" way to think about transitivity and simplicity is to observe that transitivity is really just asking that your birepresentation is cyclically generated by its objects ($\mathsf{add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathrm{Ob}(\mathcal{M})$ for all $X \in \mathrm{Ob}(\mathcal{M})$), while simplicity is asking that it is cyclically generated by its morphisms ($\{g \otimes f : g \in \mathrm{Mor}(\mathcal{C})\} = \mathrm{Mor}(\mathcal{M})$ for all $f \in \mathrm{Mor}(\mathcal{M})$)! This perspective, in addition to the following result, really elucidates our notion of simplicity for birepresentations.

Proposition 2.26. *Every simple birepresentation is transitive.*

Proof. Let M be a simple birepresentation of a multifinitary bicategory $\mathscr{C}$ and take $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero. Certainly $\mathrm{G}_{\mathrm{M}}(\{X\})$ is non-zero (as it contains X) and hence induces a non-proper $\mathscr{C}$-stable ideal of M by simplicity. Thus for each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, we know that $\mathrm{Mor}(\mathscr{C}_{\mathtt{j}}(\{X\}))$ cannot be proper and so $\mathscr{C}_{\mathtt{j}}(\{X\})$ must be equivalent to $\mathrm{M}(\mathtt{j})$. In other words, M is transitive. This completes the proof. ∎

It is not necessarily the case that transitive birepresentations are simple! This will become clear when we look at certain subcategories of the Lusztig–Vogan module categories. We do, however, have the following proposition, which essentially states that transitive birepresentations can be made simple by leaving the objects alone and just throwing away some collection of morphisms. This result will end up being important in formulating the categorical version of the Jordan–Hölder theorem.

Proposition 2.27. *Let* M *be a transitive birepresentation of a multifinitary bicategory* $\mathscr{C}$. *Then* M *admits a unique maximal* $\mathscr{C}$*-stable ideal* I, *and moreover each* $\mathrm{I}(\mathtt{i})$ *contains no identity morphisms apart from the one corresponding to the zero object.*

Proof. Let I be the sum (as vector spaces) of all $\mathscr{C}$-stable ideals of M that do not contain id_X for any non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and any $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This is certainly itself a $\mathscr{C}$-stable ideal by construction. Moreover, because $U, V \subseteq U + V$ for vector spaces U, V, it follows that a sum of ideals is itself an ideal containing each ideal being summed, whence I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms. To see that it is genuinely maximal, suppose J is a $\mathscr{C}$-stable ideal containing I. Because I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms, J must contain at least one identity morphism, say id_X for some non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$. Given any non-zero $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, the transitivity of M tells us that Y is either isomorphic to a direct summand of $[\mathrm{M}(F)](X)$, for some $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$, or isomorphic to a direct sum $[\mathrm{M}(F_1)](X) \oplus \cdots \oplus [\mathrm{M}(F_n)](X)$, for some $F_1, \ldots, F_n \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$. We claim that id_Y must lie in $\mathrm{J}(\mathtt{j})$ in both cases.

Let $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$ with $[\mathrm{M}(F)](X) = X_1 \oplus \cdots \oplus X_n$ and suppose that $\varphi : Y \to X_k$ is an isomorphism for some $1 \leq k \leq n$. Then $[\mathrm{M}(F)](\mathrm{id}_X) = \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \in \mathrm{J}(\mathtt{j})$ by $\mathscr{C}$-stability. But by pre-composing with $\iota_{X_k} \circ \varphi^{-1} : Y \to X_k \to X_1 \oplus \cdots \oplus X_n$ and post-composing with $\varphi \circ \pi_{X_k} : X_1 \oplus \cdots \oplus X_n \to X_k \to Y$, we obtain that $\mathrm{id}_Y = (\varphi \circ \pi_{X_k}) \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ (\iota_{X_k} \circ \varphi^{-1}) \in \mathrm{J}(\mathtt{j})$.

Suppose now that $\varphi : Y \to X_1 \oplus \cdots \oplus X_n$ is an isomorphism, where for each $1 \leq k \leq n$ we have $X_k = [\mathrm{M}(F_k)](X)$ for some $F_k \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$. As before, $[\mathrm{M}(F_k)](\mathrm{id}_X) = \mathrm{id}_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$ by $\mathscr{C}$-stability. Moreover, because $\mathrm{J}(\mathtt{j})$ is an ideal, $\iota_{X_k} \circ \mathrm{id}_{X_k} \circ \pi_{X_k} = \iota_{X_k} \circ \pi_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$. Thus by the definition of an additive category, $\mathrm{id}_{X_1 \oplus \cdots \oplus X_n} = \iota_{X_1} \circ \pi_{X_1} + \cdots + \iota_{X_n} \circ \pi_{X_n} \in \mathrm{J}(\mathtt{j})$, whence it follows that $\mathrm{id}_Y = \varphi^{-1} \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ \varphi \in \mathrm{J}(\mathtt{j})$.

We have thus shown that J must contain *all* identity morphisms and therefore cannot be proper, meaning that I must in fact be maximal as claimed. The uniqueness of I follows by construction. This completes the proof. ∎

Let $\mathcal{C}$ be a preadditive category admitting a two-sided ideal $\mathcal{I}$. We define a congruence relation $\sim_{\mathcal{I}}$ on each $\mathcal{C}(X, Y)$ by $f \sim_{\mathcal{I}} g$ if and only if $f - g \in \mathcal{I}(X, Y)$. This motivates the following definition.

Definition 2.28. (Quotient Category). *Let $\mathcal{C}$ be a preadditive category admitting a two-sided ideal $\mathcal{I}$. We define the* quotient category $\mathcal{C}/\mathcal{I}$ *to be the subcategory whose classes of morphisms are given by $\mathcal{C}/\mathcal{I}(X,Y) := \mathcal{C}(X,Y)/\sim_{\mathcal{I}}$, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.*

Proposition-Definition 2.29. (Simple Quotient). *A transitive birepresentation* M *of a multifinitary bicategory $\mathscr{C}$ is simple if and only if the unique maximal $\mathscr{C}$-stable ideal* I *from Proposition 2.27 is the zero ideal. The simple sub-birepresentation $\underline{\mathrm{M}}$ of* M *that sends each object $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ to the quotient subcategory* $\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i})$ *is known as the* simple quotient *of* M.

Proof. Naturally if I – the sum of all $\mathscr{C}$-stable ideals without non-zero identity morphisms – is the zero ideal, then M must contain no proper, non-zero $\mathscr{C}$-stable ideals. Conversely, if M is simple, then because I is the sum of proper $\mathscr{C}$-stable ideals, they must all be zero. Finally, because I is maximal, every morphism $f \in \mathrm{Mor}(\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i}))$ must generate either $\{0\}$ or $\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i})$ under multiplication by $\mathscr{C}$, whence it follows that $\underline{\mathrm{M}}$ is simple. This completes the proof. ■

Let M be a multifinitary birepresentation of $\mathscr{C}$. We denote by Ind(M) the set of isomorphism classes of indecomposable objects in every $\mathrm{M}(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$; that is,

$$\mathrm{Ind}(\mathrm{M}) = \bigsqcup_{\mathtt{i} \in \mathrm{Ob}(\mathscr{C})} \{[X] \in \mathrm{M}(\mathtt{i}) : X \text{ is indecomposable}\}.$$

Note that Ind(M) is clearly finite, as $\mathscr{C}$ has finitely many objects and each category $\mathrm{M}(\mathtt{i}) \in \mathrm{Ob}(\mathfrak{A}_{\Bbbk}^{f})$ has finitely many isomorphism classes of indecomposable objects.

For $X, Y \in \mathrm{Ind}(\mathrm{M})$, where for instance $X \in \mathrm{M}(\mathtt{i}_X)$ and $Y \in \mathrm{M}(\mathtt{i}_Y)$, we write $X \geq Y$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_X, \mathtt{i}_Y)$ such that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$.

Lemma 2.30. *Let* M *be a multifinitary birepresentation. The binary relation $\geq$ defined above defines a preorder on* Ind(M) *known as the* action preorder.

Proof. Clearly $\geq$ is reflexive, as we can just take $F := \mathrm{id}_{\mathtt{i}}$. Moreover, suppose that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$ and Y is isomorphic to a direct summand of $[\mathrm{M}(G)](Z)$; that is,

$$[\mathrm{M}(F)](Y) \cong X \oplus X_1 \oplus X_2 \oplus \cdots ,$$
$$[\mathrm{M}(G)](Z) \cong Y \oplus Y_1 \oplus Y_2 \oplus \cdots .$$

In order to show transitivity, we would like to show that X is isomorphic to a direct summand of $[\mathrm{M}(FG)](Z)$. Well, because the morphisms of $\mathfrak{A}_{\Bbbk}^{f}$ are additive, we simply observe that

$$\begin{aligned}[\mathrm{M}(FG)](Z) &\cong [\mathrm{M}(F)](Y) \oplus [\mathrm{M}(F)](Y_1) \oplus [M(F)](Y_2) \oplus \cdots \\ &\cong X \oplus X_1 \oplus X_2 \oplus \cdots \oplus [\mathrm{M}(F)](Y_1) \oplus [\mathrm{M}(F)](Y_2) \oplus \cdots .\end{aligned}$$

This completes the proof. ■

Suppose we define an equivalence relation $\sim$ given by $X \sim Y$ if and only if $X \geq Y$ and $Y \geq X$. Obviously $\geq$ extends to a partial order on $\mathrm{Ind}(\mathrm{M})/\sim$. In particular, we have the following result.

Proposition 2.31. *Let* M *be a multifinitary birepresentation. Then* M *is transitive if and only if* $\mathrm{Ind}(\mathrm{M})/\sim$ *has only one element.*

Proof. Suppose $\mathrm{Ind}(\mathrm{M})/\sim$ is a singleton and take any $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero as a representative. Then for any indecomposable $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, there exists some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$ for which Y is isomorphic to a direct summand of $[M(F)](X)$, since $Y \geq X$. In other words, the additive subcategory $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, as by definition it is closed under direct summands. Thus M is transitive.

Conversely, suppose M is transitive, and consider any pair of indecomposables $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$. Because $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, we know by the definition of $\mathscr{C}_{\mathtt{j}}(\{X\})$ that Y is isomorphic to a direct summand of $[M(F)](X)$ for some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$; that is, $Y \geq X$. The same argument applied to $\mathscr{C}_{\mathtt{i}}(\{Y\})$ shows us that $X \geq Y$, whence $\mathrm{Ind}(\mathrm{M})/\sim$ has only one element. This completes the proof. ■

Definition 2.32. (Directed Order Ideal). *A* directed order coideal *of a partially ordered set* $(P, \geq)$ *is a non-empty subset* I *such that*

- *for all* $x \in I$ *and* $y \in P$*,* $y \geq x$ *implies that* $y \in I$ *(upper set);*
- *for all* $x, y \in I$*, there is some* $z \in I$ *such that* $x \geq z$ *and* $y \geq z$ *(downward directed set).*

Remark 2.33. This definition is slightly unusual. It is more typical to talk of *directed order ideals* of partially ordered sets $(P, \leq)$, which are non-empty subsets that are lower sets and upward directed sets. We have chosen to use coideals rather than ideals due to how we have defined our partial order; the standard notion of a directed order ideal goes "downwards", while we want to go "upwards". To more explicitly illustrate why we have made this choice, suppose we have some $\mathcal{C}$-module category of R-modules with indecomposables $R, X \in \mathrm{Ob}(\mathcal{C})$. Naturally we would expect $X \geq R$, and indeed with our setup this will be true, as we can always consider the functor $\mathrm{M}(X) = - \otimes_R X$.

Let M be a multifinitary birepresentation of $\mathscr{C}$ and Q a directed order coideal of $\mathrm{Ind}(\mathrm{M})/\sim$. For $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, define $\mathrm{M}_Q(\mathtt{i})$ to be the additive subcategory of $\mathrm{M}(\mathtt{i})$ generated by every indecomposable object $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ whose equivalence class lies in Q. Then $\mathrm{M}_Q : \mathtt{i} \mapsto \mathrm{M}_Q(\mathtt{i})$ induces a multifinitary sub-birepresentation of M, known as the sub-birepresentation of M associated to Q.

Let $Q \subset R$ be a pair of directed order coideals in $\mathrm{Ind}(\mathrm{M})/\sim$ and let $\mathrm{I}_Q(\mathtt{i})$ denote the ideal in $\mathrm{M}_R(\mathtt{i})$ generated by the identity morphisms in $\mathrm{M}_Q(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This collection of ideals is $\mathscr{C}$-stable, whence the multifinitary birepresentation M_R induces a multifinitary birepresentation $\mathrm{M}_{R/Q} : \mathtt{i} \mapsto \mathrm{M}_R(\mathtt{i})/\mathrm{I}_Q(\mathtt{i})$. This is known as the quotient of M associated to $Q \subset R$. Note that if $|R \setminus Q| = 1$, then $|\mathrm{Ind}(\mathrm{M}_{R/Q})/\sim| = 1$, so $\mathrm{M}_{R/Q}$ will be transitive by Proposition 2.31.

Choose $r \in \mathrm{Ind}(\mathrm{M})/\sim$ and let X_r be the maximal directed order coideal in $\mathrm{Ind}(\mathrm{M})/\sim$ that does not contain r. In other words, $(\mathrm{Ind}(\mathrm{M})/\sim) \setminus X_r$ – the complement of X_r – has maximal element r. Thus we also obtain a directed order coideal $Y_r := X_r \cup \{r\}$, as r being maximal in the complement means Y_r will be an upper set, whence the associated quotient M_{Y_r/X_r} is transitive by Proposition 2.31. We henceforth let $\underline{\mathrm{M}}_r$ denote the simple quotient $\underline{\mathrm{M}}_{Y_r/X_r}$.

Consider a filtration of directed order coideals

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim$$

such that $|Q_i \backslash Q_{i-1}| = 1$ for all $i \in \{1, \ldots, n\}$. We call this a *complete filtration.* As shown previously, from such a filtration we have a corresponding *weak Jordan–Hölder series*

$$\{0\} = \mathrm{M}_{Q_0} \subset \mathrm{M}_{Q_1} \subset \cdots \subset \mathrm{M}_{Q_n} = \mathrm{M}$$

consisting of sub-birepresentations whose *weak composition quotients* $\mathrm{L}_i := \underline{\mathrm{M}}_{Q_i/Q_{i-1}}$ are simple birepresentations for all $i \in \{1, \ldots, n\}$. With this, we have the following result.

Theorem 2.34. (Weak Jordan–Hölder Theorem). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *admitting the two complete filtrations*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim,$$

$$\varnothing = Q'_0 \subset Q'_1 \subset \cdots \subset Q'_m = \mathrm{Ind}(\mathrm{M})/\sim,$$

with weak composition quotients $\{\mathrm{L}_i\}_{i=1}^n$ *and* $\{\mathrm{L}'_j\}_{j=1}^m$ *respectively. Then* $m = n$*, and moreover there exists a permutation* $\sigma \in S_n$ *such that* L_i *and* $\mathrm{L}_{\sigma(i)}$ *are equivalent for all* $i \in \{1, \ldots, n\}$.

Proof. We clearly have $m = n = |\mathrm{Ind}(\mathrm{M})/\sim|$ by the definition of a complete filtration. Suppose now that $r \in \mathrm{Ind}(\mathrm{M})/\sim$; then there exist unique $i, j \in \{1, 2, \ldots, n\}$ for which $Q_i \backslash Q_{i-1} = Q'_j \backslash Q_{j-1} = \{r\}$. If we can show that the birepresentations L_i and L'_j are both equivalent to $\underline{\mathrm{M}}_r$, then we are done. In particular, by symmetry it is enough to show that L_i is equivalent to $\underline{\mathrm{M}}_r$.

Let I_{X_r} be the $\mathscr{C}$-stable ideal in M_{Y_r} for which $\mathrm{M}_{Y_r/X_r} = \mathrm{M}_{Y_r}/\mathrm{I}_{X_r}$ and $\mathrm{I}_{Q_{i-1}}$ the $\mathscr{C}$-stable ideal in M_{Q_i} for which $\mathrm{M}_{Q_i/Q_{i-1}} = \mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$. Since $\{r\} = Q_i \setminus Q_{i-1}$, we know by construction that $Q_{i-1} \subsetneq X_r$, as X_r is by definition the maximal directed order coideal not containing r; similarly, $Q_i \subsetneq Y_r$. This second inclusion induces a pseudonatural isomorphism from M_{Q_i} to M_{Y_r} (a collection of natural isomorphisms from functors between Hom-categories to functors between Hom-subcategories), whence the first inclusion induces a pseudonatural isomorphism $\sigma : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Y_r/X_r}$ by taking the quotient. Now, $\mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$ contains only the objects generated by indecomposables in the equivalence class r. But for any such pair of indecomposable objects $X, Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$ lying in the equivalence class r, we have that $\mathrm{I}_{X_r}(X, Y) \subsetneq \mathrm{I}_{Q_{i-1}}(X, Y)$ by the aforementioned inequalities. Thus the pseudonatural isomorphism σ factors through $\mathrm{M}_{Q_i/Q_{i-1}}$, in the sense that there exist pseudonatural transformations $\sigma_1 : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Q_i/Q_{i-1}}$ and $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ such that $\sigma = \sigma_2 \circ \sigma_1$. In particular, this gives us a pseudonatural transformation $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ that is obviously surjective on morphisms by fullness; therefore, because $\mathrm{M}_{Q_i/Q_{i-1}}$ and M_{Y_r/X_r} are both transitive, taking their simple quotients via proposition 2.29 induces an equivalence between L_i and $\underline{\mathrm{M}}_r$ as desired, whence the result follows. This completes the proof. ■

This proof feels kind of handwavey, need to double-check it slowly. Also, in what sense is this weak Jordan–Hölder theorem "weak"? The decategorifications of these simple quotients are "transitive $\mathbb{N}$-modules" and usually not simple. Should also do some examples here.

3. Categories of Soergel Bimodules

Throughout this chapter and the next, we will explore one of the main examples that has motivated the theory from the previous chapter: namely, categories of Soergel bimodules. Given a Coxeter system (W, S), we may define a corresponding Iwahori–Hecke algebra. These algebras appear all over mathematics, playing important roles in areas such as the representation theory of Lie groups and quantum groups, knot theory and statistical mechanics, among others. From this same Coxeter system, we may also construct a category of so-called Soergel bimodules, which is an algebraic categorification of the corresponding Iwahori–Hecke algebra. These categories were fundamental to the recent, purely algebraic proofs of the Kazhdan–Lusztig Conjecture ([KL79, Conjecture 1.5]) and Kazhdan–Lusztig Positivity Conjecture ([KL79, p. 166]) by Elias and Williamson ([EW14, Theorem 1.1 and Corollary 1.2, respectively]), and have since become an indispensable tool in Lie theory.

Before introducing categories of Soergel bimodules, we will briefly give definitions of Coxeter systems and Iwahori–Hecke algebras. We will not give much insight into these objects; for a more detailed survey, please see the wonderful book of Elias, Makisumi, Thiel and Williamson ([EMTW20]).

Definition 3.1. (Coxeter System). *Let S be a finite set and $(m_{st})_{s,t\in S}$ a matrix satisfying*

- $m_{ss} = 1$, *for each* $s \in S$;
- $m_{st} = m_{ts} \in \{2, 3, \dots\} \sqcup \{\infty\}$, *for* $s \neq t \in S$.

Consider now the subgroup W of F_S, the free group over S, with presentation

$$W = \langle s \in S : (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ with } m_{st} < \infty\rangle.$$

The pair (W, S) is known as a Coxeter system*, while W is known as a* Coxeter group.

We call the generating set S the set of *simple reflections* and $(m_{st})_{s,t\in S}$ a *Coxeter matrix.* Because $s^2 = 1$ for all $s \in S$, the relation $(st)^{m_{st}} = 1$ is equivalent to the braid relation $sts\cdots = tst\cdots$ for all $s, t \in S$ with $m_{st} < \infty$, where both sides are the product of m_{st} simple reflections.

Definition 3.2. (Expression). *Let (W, S) be a Coxeter system with $w \in W$. An* expression for w of length k *is any sequence $\underline{w} := (s_1, \dots, s_k)$, for some not necessarily unique choice of $s_1, \dots, s_k \in S$, such that $w = s_1 s_2 \cdots s_k$. The* length *of w, denoted $\ell(w)$, is the length of its shortest expression, and any expression for w of length $\ell(w)$ is said to be* reduced. *Moreover, $\ell(w) = 0$ if and only if $w = 1$.*

Definition 3.3. (Bruhat Order). *Let (W, S) be a Coxeter system and $\Phi \coloneqq \{wsw^{-1} : s \in S, w \in W\}$ the set of conjugates of simple reflections in W. For $x, y \in W$, we write $x \leq y$ if and only if there exists a chain $x = x_0, x_1, \dots, x_k = y$ such that $\ell(x_i) < \ell(x_{i+1})$ and $x_i^{-1}x_{i+1} \in \Phi$ for all $0 \leq i < k$. This defines a partial order on W, known as the* Bruhat order.

Theorem 3.4. (Matsumoto's Theorem). *For any two reduced expressions of an element of a Coxeter group, the first can always be transformed into the second by repeatedly applying the braid relation.*

This result was first shown in [Mat64]. A sketch is given in [EMTW20, Theorem 2.20], and the full proof can be found in [GP00, Theorem 1.2.2].

Definition 3.5. (Iwahori–Hecke Algebra). *Let (W, S) be a Coxeter system and q a formal variable. The* (one-parameter) Iwahori–Hecke algebra *corresponding to (W, S) is the unital, associative $\mathbb{Z}[q, q^{-1}]$-algebra $\mathscr{H}(W, S)$ with generators $\{T_s : s \in S\}$ and relations*

- *(braid relation) $T_sT_tT_s \cdots = T_tT_sT_t \cdots$ for all $s, t \in S$ with $m_{s,t} < \infty$, where both sides are the product of $m_{s,t}$ generators;*
- *(quadratic relation) $(T_s - q^{-1})(T_s + q) = 0$, for all $s \in S$.*

Note that we can expand and rearrange the quadratic relation as $1 = T_s^2 + (q - q^{-1})T_s$, whence by multiplying through by T_s^{-1} we obtain $T_s^{-1} = T_s + (q - q^{-1})$.

Remark 3.6. If we identify our parameter q with 1, the Iwahori–Hecke algebra reduces to the group algebra $\mathbb{Z}W$. In other words, it is a deformation of the group algebra of its associated Coxeter group.

Remark 3.7. We have defined above the *one-parameter* Iwahori–Hecke algebra, as opposed to the more general *multiparameter* Iwahori–Hecke algebra, where instead of taking $\mathscr{H}(W, S)$ to be over the ring of one-parameter Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ we consider a family of units $\{q_s : s \in S\}$ and take $\mathscr{H}(W, S)$ to be over the ring $\mathbb{Z}[q_s^{\pm 1} : s \in S]$.

Let (W, S) be a Coxeter system, and take $(s_1, \ldots, s_\ell)$ and $(t_1, \ldots, t_\ell)$ to be two reduced expressions for $w \in W$. Then $T_{s_1}T_{s_2} \cdots T_{s_\ell} = T_{t_1}T_{t_2} \cdots T_{t_\ell} =: T_w$ by Matsumoto's Theorem and the braid relation. In particular, by [EMTW20, Theorem 3.5], the set $\{T_w : w \in W\}$ forms a $\mathbb{Z}[q, q^{-1}]$-basis for $\mathscr{H}(W, S)$ with $T_{\mathrm{id}} = 1$, known as the *standard basis*. Another important basis is the Kazhdan–Lusztig basis.

Definition 3.8. (Kazhdan–Lusztig Involution). *Let $\mathscr{H}(W, S)$ be an Iwahori–Hecke algebra. The* Kazhdan–Lusztig involution *is the $\mathbb{Z}$-linear automorphism $h \mapsto \overline{h}$ on $\mathscr{H}(W, S)$, defined by*

$$\overline{T_s} := T_s^{-1} = T_s + (q - q^{-1})$$

on generators and by $\overline{q} = q^{-1}$ on Laurent polynomials. The Kazhdan–Lusztig anti-involution *is the $\mathbb{Z}$-linear anti-automorphism $h \mapsto \omega(h)$ defined similarly on generators and Laurent polynomials.*

Given $w \in W$ admitting a reduced expression $(s_1, \ldots, s_\ell)$, we find that

$$\overline{T_w} = \overline{T_{s_1}} \cdots \overline{T_{s_\ell}} = T_{s_1}^{-1} \cdots T_{s_\ell}^{-1} = (T_{w^{-1}})^{-1},$$
$$\omega(T_w) = \overline{T_{s_\ell}} \cdots \overline{T_{s_1}} = T_{s_\ell}^{-1} \cdots T_{s_1}^{-1} = T_w^{-1}.$$

With this, we are ready to define the Kazhdan–Lusztig basis.

Theorem-Definition 3.9. (Kazhdan–Lusztig Basis). *Let $\mathscr{H}(W, S)$ be an Iwahori–Hecke algebra. Then it admits a unique $\mathbb{Z}[q, q^{-1}]$-basis $\{b_w : w \in W\}$ such that each b_w satisfies*

- *(self-duality) $\overline{b_w} = b_w$,*
- *(degree bound) $b_w = \sum_{y \in W} P_{y,w}T_y$,*

for some $P_{y,w} \in \mathbb{Z}_{\geq 0}[q]$ with $P_{w,w} := 1$ and $P_{y,w} := 0$ whenever $y \not\leq w$ under the Bruhat order. This basis is known as the Kazhdan–Lusztig basis*, and the coefficients $P_{y,w}$ are called* Kazhdan–Lusztig polynomials.

A proof for existence can be found in [EMTW20, Theorem 3.25], while a proof for uniqueness can be found in [EMTW20, Lemma 3.10]. A helpful example for $W = S_3$ is given in [EMTW20, §3.3.1]. Note that all coefficients of Kazhdan–Lusztig polynomials are non-negative; this is the aforementioned *Kazhdan–Lusztig Positivity Conjecture*, which was first proven for general Coxeter systems by Elias and Williamson in [EW14, Corollary 1.2]. Observe also that *any* set $\{b_w : w \in W\}$ satisfying the degree bound condition will be a basis; in particular, the Kazhdan–Lusztig polynomials induce a triangular change of basis matrix with 1's along the diagonal that maps the standard basis to $\{b_w : w \in W\}$, giving an isomorphism between the two sets.

Definition 3.10. (Standard Form). *Let $\mathscr{H}(W,S)$ be an Iwahori–Hecke algebra. The* standard trace $\tau : \mathscr{H}(W,S) \to \mathbb{Z}[q,q^{-1}]$ *is the map that extracts the coefficient of T_{id}; that is, it is the $\mathbb{Z}[q,q^{-1}]$-linear map for which $\tau(T_{\mathrm{id}}) = 1$ and $\tau(T_w) = 0$ for all $w \neq \mathrm{id}$. We then define the* standard form *on $\mathscr{H}(W,S)$ to be the sesquilinear form given by $(a,b) \coloneqq \tau(\omega(a)b)$, for all $a, b \in \mathscr{H}(W,S)$*

Definition 3.11. (Geometric Representation). *Let (W,S) be a Coxeter system and V the $\Bbbk$-vector space with basis $\{\alpha_s : s \in S\}$, where* $\mathrm{char}(\Bbbk) = 0$. *Define a symmetric, bilinear form on V by*

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right), & m_{s,t} \neq \infty; \\ -1, & m_{s,t} = \infty. \end{cases}$$

From this, we define an action of $s \in S$ on the basis elements $\alpha_t \in V$ by the linear automorphism

$$s(\alpha_t) \coloneqq \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s,$$

which reflects α_t across α_s. The geometric representation *of (W,S) is the representation induced by linearly extending this reflection action to an action of W on all of V.*

Let (W,S) be a Coxeter system with geometric representation $V \coloneqq \mathrm{span}_{\Bbbk}\{\alpha_s : s \in S\}$, and let $R \coloneqq \mathrm{Sym}(V) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V)$ be the symmetric algebra of V, where $\mathrm{Sym}^i(V)$ is the *ith symmetric power of V*. We define this to be the quotient of the ith tensor power $V^{\otimes i}$ by the action of the symmetric group S_i, viewed as a $\Bbbk$-module. For technical reasons we will take R to be *graded in degree* 2; that is, we will take all odd graded pieces to be $\{0\}$ and label $\mathrm{Sym}^i(V)$ as its $2i$th graded piece rather than its ith graded piece. We will comment more on why we do this in Remark 3.22 at the end of this chapter.

For the time being, let's try to understand what $\mathrm{Sym}^i(V)$ actually looks like. Let $S = \{s_1, s_2, \dots\}$, and observe that $V^{\otimes i}$ admits the basis $\{\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}} : s_{j_1}, \dots, s_{j_i} \in S\}$. We define an action of the symmetric group S_i on basis elements in $V^{\otimes i}$ by $\sigma \cdot (\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}}) = (\alpha_{s_{\sigma(j_1)}} \otimes \cdots \otimes \alpha_{s_{\sigma(j_i)}})$, for $\sigma \in S_i$. This extends linearly to an action of S_i on all of $V^{\otimes i}$. Thus quotienting by the action of the symmetric group has the effect of making $\otimes$ commutative; this can be seen clearly by considering finite S and small i, for instance. Therefore, by linearly extending the map that sends basis elements $(\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}})$ to degree i monomials $\alpha_{s_{j_1}} \cdots \alpha_{s_{j_i}}$, we obtain a $\Bbbk$-module isomorphism from $\mathrm{Sym}^i(V)$ to the additive subgroup of the polynomial ring $\Bbbk[\alpha_s : s \in S]$ consisting only of homogeneous degree i polynomials. The upshot is that we may identify R with the $\mathbb{Z}$-graded polynomial ring $\Bbbk[\alpha_s : s \in S]$, whose ith graded piece is the additive subgroup of homogeneous polynomials of degree i for all even, non-negative i and $\{0\}$ for all negative or odd i.

Remark 3.12. Note that the reflection action of W on V given in Definition 3.11 induces an action of W on R. This is given on monomials by $w(\alpha_{s_{j_1}} \cdots \alpha_{s_{j_i}}) = w(\alpha_{s_{j_1}}) \cdots w(\alpha_{s_{j_i}})$ and extended linearly to an action on all of R. Given a Coxeter group (W, S) and some $I \subseteq S$, we denote by W_I the subgroup of W generated by I, known as the *(standard) parabolic subgroup generated by* I. We say that I is *finitary* if W_I is a finite group, and write $R^I := \{r \in R : w(r) = r \text{ for all } w \in W_I\}$ for the set of W_I-invariant polynomials in R. Given $s \in S$, we will typically write R^s rather than $R^{\{s\}}$; naturally, since $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ for any element g of an arbitrary group G, we have that $W_{\{s\}} = \{1, s\}$. Thus $R^s = \{r \in R : s(r) = r\}$, justifying our simplified notation.

We are just about ready to define Soergel bimodules. Before that, we will introduce Bott–Samelson bimodules. The category of Soergel bimodules is in fact the graded closure of the Karoubian envelope of the category of Bott–Samelson bimodules. The easiest non-trivial example of a Bott–Samelson bimodule (and hence the easiest example of a Soergel bimodule) is, given a Coxeter system (W, S) and any $s \in S$, the graded (R, R)-bimodule

$$B_s := R \otimes_{R^s} R(1),$$

where (1) denotes a grading shift by 1. By this we mean that $R(1)$ is a copy of R whose ith graded piece is the $(i+1)$th graded piece of R. More generally, letting R^i denote the ith graded piece of R, we have that $R(k)^i := R^{i+k}$. Note that grading shifts commute with tensor products, and hence

$$B_s = R \otimes_{R^s} R(1) = (R \otimes_{R^s} R)(1) = R(1) \otimes_{R^s} R.$$

Definition 3.13. (Bott–Samelson Bimodule). *Let (W, S) be a Coxeter system and $\underline{w} := (s_1, \ldots, s_k)$ an expression. The* Bott–Samelson bimodule corresponding to $\underline{w}$ *is the graded (R, R)-bimodule*

$$\begin{aligned} BS(\underline{w}) &:= B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \\ &= (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R(k). \end{aligned}$$

We take the convention that if $\varnothing$ is the empty expression, then $BS(\varnothing) := R$. Denote by $\mathbb{BS}\mathsf{Bim}(W, S)$ the category of Bott–Samelson bimodules corresponding to (W, S), *whose objects are direct sums of Bott–Samelson bimodules corresponding to expressions in (W, S) and whose morphisms are graded (R, R)-bimodule homomorphisms; that is, (R, R)-bimodule homomorphisms $\varphi : X \to Y$ that respect gradings, in the sense that $\varphi(X^i) \subseteq Y^i$ for all $i \in \mathbb{Z}$.*

Before we get too ahead of ourselves, a natural question we may ask is where these B_s bimodules come from. Well, for any $s \in S$, one can verify that R^s is generated by α_s^2 and elements of the form $\alpha_t - (\alpha_s, \alpha_t)\alpha_s$, for $t \in S \setminus \{s\}$. This gives us a decomposition of R into s-invariants and s-antiinvariants; that is, $R = R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$. With this, we observe that

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

If we now look at the Kazhdan–Lusztig basis element b_s, it follows from uniqueness that $b_s = T_s + q$, as this is certainly a self-dual element satisfying the degree bound conditions for s. Thus

$$\begin{aligned} b_s^2 &= T_s^2 + 2qT_s + q^2 \\ &= (1 + (q^{-1} - q)T_s) + 2qT_s + q^2 \\ &= 1 + q^{-1}T_s + qT_s + q^2 \\ &= q(T_s + q) + q^{-1}(T_s + q) \\ &= qb_s + q^{-1}b_s. \end{aligned}$$

Comparing the expressions for B_s and b_s, we see some striking similarities! This Bott–Samelson bimodule arising from the generator s happens to behave just like an element of the Kazhdan–Lusztig basis, with grading shifts corresponding to multiplication by powers of our formal variable q. But what about the basis elements corresponding $w \in W \setminus S$? For these, Bott–Samelsons alone are insufficient. We therefore introduce Soergel bimodules.

Definition 3.14. (Soergel Bimodule). *Let (W, S) be a Coxeter system. A* Soergel bimodule *is a direct summand of a finite direct sum of grading shifts of Bott–Samelson bimodules corresponding to expressions in (W, S). Denote by $\mathbb{S}\mathsf{Bim}(W, S)$ the* category of Soergel bimodules corresponding to *(W, S), which we define as the "graded closure" of the Karoubian envelope of $\mathbb{BS}\mathsf{Bim}(W, S)$; in other words, it is the closure of the category of Bott–Samelson bimodules corresponding to (W, S) under isomorphisms, direct summands, finite direct sums and grading shifts.*

Remark 3.15. When we constructed R, we allowed $\Bbbk$ to be any field of characteristic zero. In principle, this choice is completely arbitrary for $\mathbb{S}\mathsf{Bim}$; this is one of the powers of Soergel bimodules! The field we choose to work over can be decided based on the context. In [EMTW20], this field is taken to be $\mathbb{R}$, as every symmetric $\mathbb{R}$-bilinear form is determined by its signature, a fundamental fact for doing Hodge theory. Alternatively, we will often be in the setting where we have some complex, semisimple Lie algebra $\mathfrak{g}$, along with a choice of Cartan subalgebra $\mathfrak{h}$ that determines a Weyl group W and Borel subalgebra $\mathfrak{b}$ that determines a set of simple reflections S. In this setting, we will take R to be the ring of regular functions on $\mathfrak{h}^*$, whence $R = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\alpha_s^\vee : s \in S]$ for coroots $\alpha_s^\vee \in \mathfrak{h}$ and $\mathfrak{h}^* = \mathrm{spec}(R)$. In this case, it is a result of Serre's theorem for quasicoherent sheaves that we can view $\mathbb{S}\mathsf{Bim}(W, S)$ as a subcategory of the category of $\mathbb{C}^\times$-equivariant coherent sheaves of $\mathscr{O}_X$-modules on X, where $\mathscr{O}_X$ is the sheaf of regular functions on the pullback $X := \mathfrak{h}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^* = \mathrm{spec}(R \otimes_{R^W} R)$.

Theorem 3.16. (Soergel's Categorification Theorem I). *There is a unique $\mathbb{Z}[q, q^{-1}]$-algebra homomorphism $c : \mathscr{H}(W, S) \to \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W, S))$ sending $q^k b_s$ to the isomorphism class $[B_s(k)]$, for each $s \in S$. Moreover, the Δ-character function $\mathrm{ch}_\Delta : \mathbb{S}\mathsf{Bim}(W, S) \to \mathscr{H}(W, S)$ descends to a homomorphism $\mathrm{ch} : \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W, S)) \to \mathscr{H}(W, S)$ that is inverse to c, making these isomorphisms.*

Theorem 3.17. (Soergel's Categorification Theorem II). *Let (W, S) be a Coxeter system and $\underline{w}$ a reduced expression for $w \in W$. Then $BS(\underline{w})$ contains, up to isomorphism, a unique indecomposable summand B_w, which does not appear in $BS(\underline{x})$ for any $x < w$ and depends only on w, not the reduced expression. Moreover, all other direct summands are grading shifts of B_x, for $x < w$.*

A statement of and reference for these results can be found in [EMTW20, Theorem 5.24], as well as an algorithm for finding the indecomposables B_w. The precise definition of the Δ-character function is $\mathrm{ch}_\Delta : B \mapsto \sum_{y\in W} q^{\ell(y)} P_y(B) T_y$, where $P_y(B) \in \mathbb{Z}_{\geq 0}[q, q^{-1}]$ is some kind of graded multiplicity (this is slightly technical, see [EMTW20, Theorem 5.10] for details). For any indecomposable Soergel bimodule B_w, the graded multiplicities satisfy $P_y(B_w(k)) = q^k P_{y,w}$ by Soergel's Conjecture.

Corollary 3.18. *The isomorphism classes of indecomposables of* $\mathbb{S}\mathsf{Bim}(W, S)$ *are in bijection with* $W \times \mathbb{Z}$.

Proof. This follows from Theorem 3.17, together with the fact that grading shifts preserve indecomposability. In particular, the indecomposable objects are, up to isomorphism, grading shifts of B_w, for $w \in W$. This completes the proof. ∎

For any pre-additive category $\mathcal{C}$ with a *shift functor* – that is, an additive, invertible endofunctor $(1) : \mathcal{C} \to \mathcal{C}$ – we have that $\mathrm{Mor}_\mathcal{C}(X(j), Y(i)) = \mathrm{Mor}_\mathcal{C}(X, Y(i-j))$, where $X(k) := [(1)^k](X)$. We define the corresponding *graded Hom-space* by $\mathrm{Mor}^\bullet_\mathcal{C}(X, Y) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Mor}^i_\mathcal{C}(X, Y)$, where the graded pieces are given by $\mathrm{Mor}^i_\mathcal{C}(X, Y) := \mathrm{Mor}_\mathcal{C}(X, Y(i))$. In other words, this process endows $\mathcal{C}$ with the structure of a *graded category* – a category whose Hom-spaces are $\mathbb{Z}$-graded Abelian groups. Similarly, we can turn any graded category into a category with a shift functor, and in fact by [EMTW20, Proposition 11.9] we have a pair of mutually adjoint pseudofunctors

$$\begin{tikzcd} \left\{\begin{matrix}\text{categories with}\\ \text{a shift functor}\end{matrix}\right\} \arrow[rr, bend left, "(-)^{\mathrm{gr}}"] & & \left\{\begin{matrix}\text{graded}\\ \text{categories}\end{matrix}\right\} \arrow[ll, bend left, "(-)^{\mathrm{sh}}"] \end{tikzcd}$$

In other words, we can always think of the Hom-spaces of $\mathbb{S}\mathsf{Bim}$ as being graded!

Now, given a finite-dimensional $\mathbb{Z}$-graded $\Bbbk$-vector space $V = \oplus V^i$, we define its *graded dimension* to be the Hilbert–Poincaré series $\mathrm{gdim}(V) := \sum_{i\in\mathbb{Z}} \dim(V^i) q^i$. Then given a free, finitely-generated left (respectively, right) graded R-module M, we define its *graded rank* to be $\mathrm{grk}(M) := \mathrm{gdim}(\Bbbk \otimes_R M)$ (respectively, $\mathrm{grk}(M) := \mathrm{gdim}(M \otimes_R \Bbbk)$). Note that the purpose of tensoring with the underlying field here is to promote M to a finite-dimensional $\Bbbk$-vector space; since M is finitely-generated, it admits a finite generating set $\{m_1, \dots, m_n\}$, whence $\{1 \otimes m_1, \dots, 1 \otimes m_n\}$ is a $\Bbbk$-basis for $\Bbbk \otimes_R M$.

Theorem 3.19. (Soergel Hom Formula). *For any two Soergel bimodules* $B, B' \in \mathbb{S}\mathsf{Bim}(W, S)$, $\mathrm{Mor}^\bullet_{\mathbb{S}\mathsf{Bim}}(B, B')$ *is free as a left graded R-module and as a right graded R-module, both with* $\mathrm{grk}(\mathrm{Mor}^\bullet_{\mathbb{S}\mathsf{Bim}}(B, B')) = (\mathrm{ch}_\Delta(B), \mathrm{ch}_\Delta(B'))$, *where* $(-,-)$ *denotes the standard form on* $\mathscr{H}(W, S)$. *In particular, the dimension of* $\mathrm{Mor}_{\mathbb{S}\mathsf{Bim}}(B, B'(i))$ *is given by the coefficient of* q^i *in* $(\mathrm{ch}_\Delta(B), \mathrm{ch}_\Delta(B'))$.

Remark 3.20. Any category of Soergel bimodules is naturally a strict monoidal category. Although they are not Abelian, they have finite-dimensional vector spaces of morphisms by the Soergel Hom Formula and are therefore Krull–Schmidt by Theorem 2.9. Unfortunately, however, by Corollary 3.18 it will only be *graded multifinitary*, as we only have finitely many indecomposable objects under both isomorphisms and grading shifts. Fortunately, the only time having finitely many isomorphism classes of indecomposable objects comes into play is with the weak Jordan–Hölder Theorem, and in the case of $\mathbb{S}\mathsf{Bim}(W, S)$ we have that $\mathrm{Ind}(\mathrm{M})/\sim$ is finite for any module category M, since $B(i) \otimes \mathbb{1}(j) = B(i + j) \otimes \mathbb{1} \cong B(i + j)$ for all $B \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W, S))$ and $i, j \in \mathbb{Z}$, and hence $B \sim B(k)$ for all $k \in \mathbb{Z}$.

Theorem 3.21. (Soergel's Conjecture). *The isomorphism* $\mathrm{ch} : \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W, S)) \to \mathscr{H}(W, S)$ *from Theorem 3.16 sends* $B_w(k)$ *to the isomorphism class* $q^k b_w$*, for each* $w \in W$*.*

This result, which was first proven by Elias and Williamson ([EW14, Theorem 1.1]), generalizes the Kazhdan–Lusztig Conjecture and finally gives us an algebraic categorification of the Iwahori–Hecke algebra. This also means that our categories of Soergel bimodules are actually deceptively easy to work with; up to isomorphism, we can manipulate the objects by viewing them not as bimodules but as elements in the corresponding Iwahori–Hecke algebra. That said, Soergel bimodules also give us a collection of morphisms to look at. In fact, from the categorical perspective, the morphisms are really the items of interest – compared to the Iwahori–Hecke algebra, they provide us with genuinely new information! As it turns out, even these morphisms are nice to work with, thanks to Theorem 3.19 and the fact that they admit a graphical calculus.

Remark 3.22. In the following chapters, we will typically have a sequence of subgroups $T \subset B \subset G$, where G is a connected, complex, reductive algebraic group, B is a Borel subgroup and T is a maximal torus. These inclusions induce a Weyl group $W := N_G(T)/T$, where

$$N_G(T) := \{g \in G : gtg^{-1} \in T, \text{ for all } t \in T\};$$

a set $\Phi = \Phi(G, T)$ of roots of G relative to T; a set S of simple roots of W corresponding to B, such that (W, S) is a Coxeter system (it is a fact that the set of Borel subgroups of G containing T are in bijection with the set of bases for Φ); and a complex flag manifold (or generalized flag variety) G/B. Suppose now that we choose, for each $w \in W$, a representative $g_w \in N_G(T)$ for which $w = g_wT$. Letting $C_w := Bg_wB$, which is well-defined since T is a subgroup of B, we have the following stratification of G into a disjoint union of (B, B)-double cosets:

$$G = \bigsqcup_{w \in W} C_w.$$

This is known as a *Bruhat decomposition* of G, and the double cosets C_w are known as *Bruhat cells*. Importantly, this descends to a stratification of the flag variety G/B in terms of *Schubert cells* of the form $X_w := C_w/B$:

$$G/B = \bigsqcup_{w \in W} X_w.$$

The cells X_w are locally closed subvarieties of G/B. We refer to the Zariski closure $\overline{X_w}$ as the *Schubert variety of* w, and remark that $\overline{X_y} \subseteq \overline{X_w}$ if and only if $y \leq w$. As an aside, we have that

$$P_{y,w}(q) = \sum_{i \geq 0} q^i \dim_{\mathbb{C}}(IH^{2i}_{X_y}(\overline{X_w}, \mathbb{C})),$$

where $P_{y,w}$ is a Kazhdan–Lusztig polynomial for the Kazhdan–Lusztig basis of $\mathscr{H}(W, S)$. The term $IH^{2i}_{X_y}(X_w)$ is the stalk of the ith hypercohomology of IC_w at a (equivalently, any) point of X_y, where IC_w denotes the intersection cohomology sheaves of $\overline{X_w}$ ([EMTW20, p. 13.13]). This result was first shown in the appendix to [KL79] (see Remark A11). One thing we may observe here is that we only sum over even graded pieces of the local intersection cohomology; the reason for this is that the odd graded pieces are all zero. In fact, because we have a left action of B on G/B, we can consider the B-equivariant cohomology $H^*_B(G/B)$, which we will find is a ring that is also graded in degree 2. As it turns out, $H^*_B(G/B) \cong R \otimes_{R^W} R$, which is exactly why we grade R the way that we do!

Go through the SL_2 and SL_3 examples in detail to put everything together, it'd be good to see an example of the Kazhdan–Lusztig basis of a Iwahori–Hecke algebra and compare it to the indecomposables in the corresponding category of Soergel bimodules. I should also start typing up Lie theory notes.

4. Lusztig–Vogan Module Categories

This chapter will aim to summarize the recent work of Larson and Romanov in their Soergel bimodule approach for algebraically categorifying the trivial block of the Lusztig–Vogan module ([LR22]), which provides interesting examples of module categories over the category of Soergel bimodules. In the most general setting, the construction of a Lusztig–Vogan module category takes as ingredients a connected, complex, reductive algebraic group G, a Borel subgroup B of G, a holomorphic involution θ of G and a finite-index subgroup K of the fixed-point subgroup $G^\theta \coloneqq \{g \in G : \theta(g) = g\}$. We will also make the additional assumption that K is the identity component of G^θ (see [LR22] for details).

Let $P \coloneqq \mathrm{Sym}(\mathrm{span}_{\Bbbk}(X(T_K)))$ be the symmetric algebra on the $\Bbbk$-span of the character lattice $X(T_K) \coloneqq \mathrm{Hom}(T_K, \mathbb{C}^\times)$ of a maximal torus T_K of K contained in $B_K \coloneqq B \cap K$, which we grade in degree 2, and similarly let $R \coloneqq \mathrm{Sym}(\mathrm{span}_{\Bbbk}(X(T)))$ be the symmetric algebra on the $\Bbbk$-span of the character lattice of the unique maximal, θ-stable torus $T \coloneqq Z_G(T_K)$ of G containing T_K (see [LR22, Lemma 6.1.2]), which we once again grade in degree 2. Denote by $W \coloneqq N_G(T)/T$ the Weyl group of G corresponding to T, by W^θ the set of elements of W fixed under the involution induced by θ and by $W_K \coloneqq N_K(T_K)/T_K$ the Weyl group of K corresponding to T_K, where we write $P^{W_K} \coloneqq \{p \in P : wp = p \text{ for all } w \in W_K\}$. Note that $W_K \subseteq W^\theta \subseteq W$, and by choosing the set of simple roots in W corresponding to B we obtain a Coxeter system (W, S). Finally, let $\phi : R \to P$ be the algebra homomorphism extending the restriction map $X(T) \to X(T_K)$. For each $w \in W$, we define the *w-standard bimodule* P_w to be the (P_{W_K}, R)-bimodule given by P as a vector space with left action given by left multiplication and right action given by $s \cdot_w r \coloneqq s\phi(w(r))$ for all $s \in P$ and $r \in R$, where $w(r)$ denotes w acting on r via the reflection action given in Remark 3.12. We now define

$$\mathcal{N}^0_{LV} \coloneqq \langle P_w \otimes_R X : w \in W^\theta, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W, S))\rangle_{\oplus,\ominus,(1)}$$

to be the category of (P^{W_K}, R)-bimodules generated by standard bimodules under the right action of Soergel bimodules and closed under direct sums, direct summands and gradning shifts. By [LR22, Theorem 1.3.1], this categorifies the trivial block of the associated module of Lusztig and Vogan.

To study this in full generality involves some deep results from Lie theory; thus for the time being we will restrict our attention to the case where the tori T_K and T are of equal rank (that is, where $T_K = T$ and hence $P = R$). In this situation the picture is much simpler. Let (W, S) be a Coxeter system with finite index subgroup $W_K \subseteq W$. Given the collection $\{\alpha_s : s \in S\}$ of simple roots, we define a polynomial algebra $R \coloneqq \Bbbk[\alpha_s : s \in S]$, which we recall is isomorphic to the symmetric algebra of the vector space associated with the geometric representation of (W, S) given in Definition 3.11. We therefore have an action of W on R, given on generators $s \in S$ and simple roots $\alpha_t \in R$ by

$$s(\alpha_t) = \alpha_t + 2\cos\left(\frac{\pi}{m_{st}}\right)\alpha_s.$$

We define R^{W_K} to be the polynomials in R that are invariant under action by W_K. With this, given a generator w representing a coset in W/W_K, we define the w-standard bimodule R_w to be the (R^{W_K}, R)-bimdoule given by R as a vector space with left action given by left multiplication and right action given by $s \cdot_w r \coloneqq sw(r)$, for all $s \in R_w$ and $r \in R$. Thus, just as before, the corresponding algebraic Lusztig–Vogan module category is

$$\mathcal{M}^0_{LV} = \langle R_w \otimes_R X : [w] \in W/W_K, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W, S))\rangle_{\oplus,\ominus,(1)}.$$

Remark 4.1. Suppose G is a connected, complex, reductive algebraic group and let $\sigma : G \to G$ be an antiholomorphic involution of G. Then G^σ, the fixed-point subgroup of σ, has the structure of a real Lie group whose complexification is G. Moreover, σ' is G-conjugate to σ (that is, there exists some $h \in G$ for which $\sigma' = \mathrm{int}_h \circ \sigma \circ \mathrm{int}_h^{-1}$, where $\mathrm{int}_h : g \mapsto hgh^{-1}$ is known as the *inner automorphism associated to* h) if and only if $G^{\sigma'}$ and G^σ are isomorphic as real Lie groups. We will call such an isomorphism class a *real form* of G. Now, a classical result of Cartan tells us the following. First, a connected, complex algebraic group is reductive if and only if it is the complexification of a unique connected, compact, real Lie group ([Kam11, p. 34]); thus there is, up to G-conjugation, only one antiholomorphic involution σ_c whose fixed-point subgroup is compact. This is known as the *compact form* of G. Second, let Θ be any G-conjugacy class of holomorphic involutions of G. Then there exists some holomorphic involution $\theta \in \Theta$ that commutes with σ_c, and furthermore each G-conjugacy class of antiholomorphic involutions of G contains $\sigma = \theta \circ \sigma_c$ for some unique choice of initial Θ ([Ada14]). Putting everything together, we have bijections

$$\{\text{real forms of } G\} \longleftrightarrow \left\{\begin{matrix}\text{antiholomorphic}\\ \text{involutions of } G\end{matrix}\right\}/\!\sim \;\longleftrightarrow \left\{\begin{matrix}\text{holomorphic}\\ \text{involutions of } G\end{matrix}\right\}/\!\sim,$$

where the equivalence relations are given by conjugation by G. The holomorphic involution corresponding to a real form is known as the *Cartan involution* of that real form. An immediate corollary is that a real form is compact if and only if its Cartan involution is the identity. For any real form, we also have a diamond

$$\begin{tikzcd}[column sep=small]
 & G \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\theta"] & \\
G_{\mathbb{R}} \coloneqq G^\sigma \arrow[dr, dash, "\theta"'] & & K \coloneqq G^\theta \arrow[dl, dash, "\sigma"] \\
 & K_{\mathbb{R}} \coloneqq G^{\sigma_c} &
\end{tikzcd}$$

where $G_{\mathbb{R}}$ is a real Lie group, K is a complex Lie group and $K_{\mathbb{R}}$ is the maximal compact subgroup of $G_{\mathbb{R}}$. Note that K is not in general compact, although it *will* be the complexification of $K_{\mathbb{R}}$.

Example 4.2. Let's work through an example with $G \coloneqq \mathrm{SL}(2,\mathbb{C})$. Recall that G admits two real forms: a compact form $\mathrm{SU}(2)$ and a split form $\mathrm{SL}(2,\mathbb{R})$. The former is the fixed-point subgroup of the antiholomorphic involution $\sigma_c : g \mapsto ((\overline{g})^T)^{-1}$, while the latter is the fixed-point subgroup of the antiholomorphic involution $\sigma_s : g \mapsto \overline{g}$. By Remark 4.1, the Cartan involution of $\mathrm{SU}(2)$ will be trivial; this is a bit boring, so let's look at $\mathrm{SL}(2,\mathbb{R})$ instead. It admits the Cartan involution $\theta' : g \mapsto (g^T)^{-1}$ with fixed-point subgroup

$$G^{\theta'} = \mathrm{SO}(2,\mathbb{C}) \coloneqq \left\{\begin{pmatrix} a & b \\ -b & a\end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1\right\}.$$

Note that this is homeomorphic to $\mathbb{C}^\times \coloneqq \mathbb{C}\backslash\{0\}$ and is hence a maximal torus for G as a complex algebraic group (that is, a subgroup that is maximal among subgroups homeomorphic to $(\mathbb{C}^\times)^{\oplus k}$), as there are no groups H for which $G^{\theta'} \subset H \subset G$. In fact, it is the complexification of $\mathrm{SO}(2,\mathbb{R})$, the maximal torus of $\mathrm{SL}(2,\mathbb{R})$ as a real Lie group (that is, the subgroup that is maximal among subgroups homeomorphic to $(S^1)^{\oplus k}$). Since $G^{\theta'}$ is connected, the identity component is the entire group. However, $G^{\theta'}$ is awkward to work with; therefore, instead of using $G^{\theta'}$, consider the following.

Suppose we conjugate θ' by the inner automorphism associated to the matrix

$$h := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C});$$

that is, let $\theta := \mathrm{int}_h \circ \theta' \circ \mathrm{int}_h^{-1}$. Realizing θ' as the inner automorphism of the matrix with 1 and -1 on its off-diagonals, we have

$$\begin{aligned}\theta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) &:= \frac{1}{4}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}.\end{aligned}$$

This holomorphic involution admits the fixed-point subgroup

$$G^\theta = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times\right\},$$

which is of course homeomorphic to $\mathbb{C}^\times$. Once again, since G^θ is connected, the identity component is just the entire group, and hence we will take $K := G^\theta$. Once again, G^θ is a maximal torus, and is given by the intersection of the pair of opposite Borel groups

$$B := \left\{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\} \qquad \text{and} \qquad B' := \left\{\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\}.$$

Therefore, let $T_K := G^\theta$. The Weyl group corresponding to T_K is just $W_K := N_K(T_K)/T_K = \{1\}$, while the Weyl group corresponding to $T = T_K$ is $W := N_G(T)/T = S_2 = \{1, s\}$, since

$$N_G(T) = \{g \in G : gtg^{-1} \in T, \text{ for all } t \in T\} = T \sqcup \left\{\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix} : b \in \mathbb{C}^\times\right\}.$$

The only choice of simple roots we can make is $S = \{s\}$, whence $R = \mathbb{R}[\alpha_s]$. The action of W on R is given by $1 : r \mapsto r$ and $s : r \mapsto -r$, for $r \in R$, giving us $R^{W_K} = R$ and an easy (R^{W_K}, R)-bimodule structure for R. Note that $W^\theta = W$, since for all $g \in N_G(T)$ we have that $\theta(g) \in gT$. This gives us a very explicit right module category $\mathcal{M}^0_{LV}$ over $\mathbb{S}\mathsf{Bim}(W,S)$! A standard exercise we can now do is to compute the Jordan–Hölder filtrations of the corresponding birepresentation M that maps $\bullet$ to $\mathcal{M}^0_{LV}$ and $X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W,S))$ to functors $- \otimes_R X$.

Recall that the indecomposable objects in $\mathbb{S}\mathsf{Bim}(W,S)$ are, up to grading shifts and isomorphisms, given by Bott–Samelson bimodules of the form $B_w := R \otimes_{R^w} R(1)$, for $w \in W$. Because module products distribute over direct sums, the indecomposables of $\mathcal{M}^0_{LV}$ are of the form $R_{w_1} \otimes_R B_{w_2}$, for $[w_1] \in W/W_K$ and $w_2 \in W$; that is, products of generators of $\mathcal{M}^0_{LV}$ with indecomposable Soergel bimodules. In our case with $\mathrm{SL}(2,\mathbb{C})$, we have four candidates, which are evaluated as follows:

$$\begin{aligned} R_1 \otimes_R B_1 &\cong (R \otimes_R R) \otimes_R R(1) \cong R, & R_1 \otimes_R B_s &\cong (R \otimes_R R) \otimes_{R^s} R(1) \cong B_s, \\ R_s \otimes_R B_1 &\cong (R_s \otimes_R R) \otimes_R R(1) \cong R_s, & R_s \otimes_R B_s &\cong (R_s \otimes_R R) \otimes_{R^s} R(1) \cong B_s,\end{aligned}$$

where we note that $R_1 \cong R \cong B_1$. We therefore have, up to grading shifts and isomorphisms, three indecomposables: R, R_s and B_s. Just like categories of Soergel bimodules, our Lusztig–Vogan module categories are not multifinitary, as they have infinitely many isomorphism classes of indecomposable objects. Luckily, don't quite need this and instead only need $\mathrm{Ind}(\mathrm{M})/\sim$ to be finite, which is true by Remark 3.20. So we are free to continue.

Of course, we also know that $B_s \geq R, R_s$, since B_s is isomorphic to both $[\mathrm{M}(B_s)](R) = R \otimes_R B_s$ and $[\mathrm{M}(B_s)](R_s) = R_s \otimes_R B_s$; thus we obtain the two complete filtrations of isomorphism classes

$$\{[B_s]\} \subset \{[R_s], [B_s]\} \subset \{[R], [R_s], [B_s]\} \qquad \text{and} \qquad \{[B_s]\} \subset \{[R], [B_s]\} \subset \{[R], [R_s], [B_s]\}.$$

The corresponding weak composition quotients of these two complete filtrations will of course be equivalent, so we will just compute them for the filtration given by $Q_1 \coloneqq \{[B_s]\}$, $Q_2 \coloneqq \{[R_s], [B_s]\}$ and $Q_3 \coloneqq \{[R], [R_s], [B_s]\}$. These give rise to the additive $\mathcal{M}^0_{LV}$ subcategories

$$\mathcal{M}_{Q_1} = \langle B_s \rangle_{\oplus,(1)}, \qquad \mathcal{M}_{Q_2} = \langle R_s, B_s \rangle_{\oplus,(1)}, \qquad \mathcal{M}_{Q_3} = \langle R, R_s, B_s \rangle_{\oplus,(1)} \simeq \mathcal{M}^0_{LV},$$

as well as two-sided ideals $\mathcal{I}_{Q_i}$ of $\mathcal{M}^0_{LV}$ generated by the identity morphisms in Q_i. Of course, one of the resulting quotients is trivial, so the subcategory $\mathcal{M}_{Q_1/Q_0} = \mathcal{M}_{Q_1}$ is already transitive. For the remaining transitive quotient subcategories, we observe that $\mathcal{I}_{Q_i}$ consists of all morphisms in $\mathcal{M}^0_{LV}$ that factor through objects in Q_i (with all other $\mathcal{I}_{Q_i}(X, Y)$ containing, by definition, only the unique zero morphism). The idea here is that the quotient functor from $\mathcal{M}_{Q_2}$ to $\mathcal{M}_{Q_2/Q_1}$ sends B_s to 0, while the quotient functor from $\mathcal{M}_{Q_3}$ to $\mathcal{M}_{Q_3/Q_2}$ sends both R_s and B_s to 0. As it happens, the functor sending R_s to R defines an equivalence of module categories $\mathcal{M}_{Q_2/Q_1} \simeq \mathcal{M}_{Q_3/Q_2}$. It is easy to be fooled by the fact that $R_s \otimes_R R_s = R$, but one must remember that our Lusztig–Vogan module categories are *not* monoidal, and thus we cannot tensor by R_s, since it doesn't live in $\mathbb{S}\mathsf{Bim}(W, S)$!

In order to compute the weak composition quotient L_i, we must quotient the sub-birepresentation $\mathrm{M}_{Q_i/Q_{i-1}} : \bullet \mapsto \mathcal{M}_{Q_i/Q_{i-1}}$ by the unique maximal $\mathbb{S}\mathsf{Bim}(W, S)$-stable ideal given by Proposition 2.27. Show that there are, up to equivalence, only two weak composition quotients. Draw Young tabeleaux.

References

[Ada14] Adams, J., *Real Forms*, University of Maryland, 2014, available at https://math.umd.edu/ jda/744/744.real_forms.pdf.

[DGG14] Das, P., Ghosh, S. K., and Gupta, V. P., *Perturbations of Planar Algebras*, Math. Scand. 114.**1** (2014), pp. 38–85.

[Ela20] Elagin, A. D., *Гомологические методы в теории представлений конечномерных алгебр*, Moscow Center for Continuous Mathematical Education, 2020, available at https://ium.mccme.ru/f20/f20-Elagin.html.

[EMTW20] Elias, B., Makisumi, S., Thiel, U., and Williamson, G., *Introduction to Soergel Bimodules*, vol. 5, RSME Springer Series, Springer Nature Switzerland, 2020.

[EW14] Elias, B. and Williamson, G., *The Hodge theory of Soergel bimodules*, Ann. Math. 180 (2014), pp. 1089–1136.

[EGNO16] Etingof, P., Gelaki, S., Nikshych, D., and Ostrik, V., *Tensor Categories*, vol. 205, Mathematical Surveys and Monographs, American Mathematical Society, 2016.

[GP00] Geck, M. and Pfeiffer, G., *Characters of Finite Coxeter Groups and Iwahori–Hecke Algebras*, vol. 21, London Mathematical Society Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[Jun19] Junhan, A. T., *The Freyd-Mitchell Embedding Theorem* (2019), DOI: `10.48550/arXiv.1901.08591`.

[Kam11] Kamnitzer, J., *Representation theory of compact groups and complex reductive groups*, University of Toronto, 2011, available at https://www.math.toronto.edu/jkamnitz/courses/reptheory/reptheory.pdf.

[KL79] Kazhdan, D. and Lusztig, G., *Representations of Coxeter Groups and Hecke Algebras*, Invent. Math. **53** (1979), pp. 165–184.

[LR22] Larson, S. and Romanov, A., *A categorification of the Lusztig–Vogan module* (2022), DOI: `10.48550/arXiv.2203.09007`.

[Mac13] Mac Lane, S., *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer New York, 2013.

[Mat64] Matsumoto, H., *Générateurs et relations des groupes de Weyl généralisés*, C. R. Acad. Sci. Paris 258 (1964), pp. 3419–3422.

[MMM+23] Mackaay, M. et al., *Simple transitive* 2*-representations of Soergel bimodules for finite Coxeter types*, Proc. Lond. Math. Soc. 126.**5** (2023), pp. 1585–1655.

[Mal18] Malkiewich, C., *Bicategories, pseudofunctors, shadows: a cheat sheet*, 2018, available at https://people.math.binghamton.edu/malkiewich/category_theory_cheat_sheet.pdf.

[Sha23] Shah, A., *Krull-Remak-Schmidt decompositions in Hom-finite additive categories*, Expo. Math. 41 (2023), pp. 220–237.

[Str95] Street, R., *Categorical Structures*, Handbook of Algebra, ed. by Hazewinkel, M., vol. 1, Elsevier North-Holland, 1995.

Page intentionally left blank.

Motivation

- What is finitary birepresentation theory? My take: a "categorification" of the representation theory of finite-dimensional algebras, arising from the study of knot invariants, tensor categories and operator algebras. In other words, Jones mathematics (lol).
- Stealing some slogans from one of Dani's talks: representation theory is group theory in vector spaces, while birepresentation theory is group theory in linear categories.
- Why is it important? Basically for (somehow) studying the fields I just mentioned. Hopefully by the end of this little series we can figure out the answer to this together.

Plan

- Today I'd like to introduce one of the main theorems of birepresentation theory: the "weak Jordan–Hölder theorem". This theorem is a nice (abstract) motivation for the classification of simple birepresentations.
- In principle I'm not sure how helpful this talk will be. It will mostly be walking you through what I've been thinking about over the past couple of weeks. The main theorem is, I think, quite nice, but getting there will likely be dry. Next time I'd like to focus more on examples (especially Soergel bimodules and their classification) if I'm capable enough.
- Addressing a question: why am I talking about birepresentations? The people were promised 2-representations! In classifying 2-representations of Soergel bimodules, it was quickly found that the language of 2-representations was too restrictive. Later on in this series I will probably transition to 2-representations, but at least in this talk I'd like to state things as generally as possible to avoid issues later on.
- As a reminder to myself, I'd like to maintain a little definition bank so you guys can keep track of definitions.

Multifinitary Categories

- What are multifinitary 1-categories? Add this to the definition bank.
- In linear algebra, all idempotents split. Idempotent complete categories live somewhere between additive categories and Abelian categories. In particular, *all pseudo-Abelian categories are idempotent complete!* These remarks are what "unlocked" idempotent completeness for me.
- Examples of finitary categories (that I can't write down because I haven't prepared any concrete examples): (semi)groups and their representations, quantum groups and their categorifications, tensor categories, fusion categories, modular (tensor) categories, 2-Kac-Moody categories, etc.
- $\mathsf{Vect}_{\Bbbk}$ is a helpful example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$.
- Maybe mention example 2.8? Mazorchuk and Miemietz would have you believe it's a useful example but man is it impenetrable.
- Also, $\mathbb{S}\mathsf{Bim}(W, S)$ lololol
- Define the category of multifinitary categories.
- Finitary bicategories. Add this to the definition bank. We can generate these from finitary monoidal categories: in particular, there is a bijection between monoidal categories and one-object bicategories, with delooping taking us upstairs and Hom-categories taking us downstairs. Mention strict case.

Finitary Birepresentations

- What are finitary birepresentations? There is a bijection between module categories and birepresentations of one-object bicategories. Example: Yoneda birepresentations.
- Equivalence of birepresentations. I don't understand these very well yet.

Simple Birepresentations

- Walk through next block: essentially, I want to define what it means for a birepresentation to be transitive. If you're like me and you keep forgetting what this adjective means, an action of G on a set X is said to be transitive if, for any $x, y \in X$, there is some $g \in G$ for which $g \cdot x = y$.
- Define transitive birepresentations. Add this to the definition bank. What are we trying to capture with transitivity?
- There is a "better" notion of simplicity.
- What are ideals of finitary 1-categories? Mention that I've talked to Dani about this and they gave a somewhat confusing answer, but I think I've figured it out. I'm mentioning this in case I'm actually stupid.
- Give an example – $\mathsf{Vect}_{\Bbbk}$ yippeeee!!
- What are $\mathscr{C}$-stable ideals of birepresentations?
- What are simple birepresentations? Add this to the definition bank. Why are they "better"? They somehow more completely characterize what it means for a representation to be simple. Unfortunately, I don't really have an example of a non-simple transitive birepresentation for you right now asides from Lusztig–Vogan module categories, but I'll hopefully have one next week. Many of the naïve examples of transitive birepresentations end up being simple.
- Result I'll need: transitive birepresentations admit a unique maximal ideal, and quotienting by this ideal gives you a simple birepresentation. We call this the simple quotient of M. Run through a sketch of the proof.

Weak Jordan–Hölder

- Define Ind(M), note that it is finite. Define preorder, outline the proof. Recall: preorder means $X \geq X$ and $X \geq Y, Y \geq Z \implies X \geq Z$. Quotienting by $X \sim Y \iff X \geq Y, Y \geq X$ gives us an honest partial order (pretty much by definition).
- Proposition: M is transitive if and only if $|\mathrm{Ind}(\mathrm{M})/\!\sim| = 1$..
- Define poset ideal and coideal.
- Maybe now would be a good time to remind everyone of the classical Jordan–Hölder theorem. Given a representation M, we would like a sensible notion of composition series.
- How do coideals induce sub-birepresentations?
- How do pairs of coideals induce transitive birepresentations?
- How can we make these birepresentations simple?
- State the theorem and run through a sketch of the proof.