The Exceptional Lie Groups of Type G_2

1. The Compact Real Form G_2

Consider the 8-dimensional $\mathbb{R}$-vector space with basis $B \coloneqq \{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$. We endow this vector space with the structure of an algebra by defining a multiplication via the following table.

e_ie_j		e_j							
		e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_i	e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
	e_1	e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
	e_2	e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
	e_3	e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
	e_4	e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
	e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
	e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
	e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

(1)

Here e_0 is identified with the scalar 1. This table can also be encoded succinctly via the Fano plane.

Definition 1. (Octonions). *The algebra over $\mathbb{R}$ defined above is known as the* octonions *(or the* Cayley algebra*) and denoted $\mathbb{O}$.*

Let

$$x \coloneqq \sum_{i=0}^{7} x_i e_i \in \mathbb{O}$$

be an octonion whose coordinate vector with respect to our basis B is denoted by

$$[x]_B \coloneqq (x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^8.$$

We define on $\mathbb{O}$ an inner product given by the dot product

$$(x, y) = [x]_B \cdot [y]_B,$$

for any octonion y with coordinate vector $[y]_B$; a conjugation map given by

$$\overline{x} \coloneqq x_0 - \sum_{i=1}^{7} x_i e_i;$$

a modulus given by the Euclidean norm

$$|x| \coloneqq \sqrt{(x, x)};$$

and finally real and imaginary parts $\mathfrak{R}(x) \coloneqq x_0$ and $\mathfrak{I}(x) \coloneqq \sum_{i=1}^{7} x_i e_i$, respectively (where we have $\mathfrak{R}(x) = \frac{1}{2}(x + \overline{x})$ and $\mathfrak{I}(x) = \frac{1}{2}(x - \overline{x})$). We denote $\overline{x}/|x|^2$ by x^{-1}, and observe that $x^{-1}x = 1 = xx^{-1}$. In other words, $\mathbb{O}$ satisfies all of the axioms of a field except for associativity and commutativity.

Remark 2. Octonion multiplication seems scary, but in fact there is some hidden beauty here! As we know, $\mathbb{C} \cong \mathbb{R} \oplus \mathbb{R}$, with multiplication $(a,b)(c,d) := (ac - \overline{d}b, da + b\overline{c})$ and conjugation $\overline{(a,b)} = (\overline{a}, -b)$. Notice that we have written conjugates of real numbers, which is of course superfluous: the reason is because the pattern as we have written here generalizes nicely. For example, the quaternions are given by $\mathbb{H} \cong \mathbb{C} \oplus \mathbb{C}$, with multiplication $(a,b)(c,d) := (ac - \overline{d}b, da + b\overline{c})$ and conjugation $\overline{(a,b)} = (\overline{a}, -b)$. As one may guess, the octonions are then given by $\mathbb{O} \cong \mathbb{H} \oplus \mathbb{H}$, with multiplication and conjugation following this trend. This is known as the *Cayley–Dickson construction* for hypercomplex algebras. To see this explicitly, observe that our definition of the octonions contains the field of quaternions as

$$\mathbb{H} \cong \{x_0 + x_1e_1 + x_2e_2 + x_3e_3 : x_0, x_1, x_2, x_3 \in \mathbb{R}\}.$$

Because any element $x \in \mathbb{O}$ can be written as

$$\begin{aligned} x &= x_0 + x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 + x_5e_5 + x_6e_6 + x_7e_7 \\ &= (x_0 + x_1e_1 + x_2e_2 + x_3e_3) + (x_4 + x_5e_1 + x_6e_2 + x_7e_3)e_4, \end{aligned}$$

for some $x_i \in \mathbb{R}$, it follows that any $x \in \mathbb{O}$ can be written as $x = a + be_4$, for $a, b \in \mathbb{H}$. That is,

$$\mathbb{O} \cong \mathbb{H} \oplus \mathbb{H}e_4.$$

Multiplication, the inner product and conjugation are recovered by

$$\begin{aligned} (a + be_4)(c + de_4) &= (ac - \overline{d}b) + (da + b\overline{c})e_4, \\ (a + be_4, c + de_4) &= (a, c) + (b, d), \\ \overline{a + be_4} &= \overline{a} - be_4. \end{aligned}$$

We will later find it convenient to define also an $\mathbb{R}$-linear map $\gamma : a + be_4 \mapsto a - be_4$, for $a, b \in \mathbb{H}$. It's worth noting too that we have a similar process to show that $\mathbb{O} \cong \mathbb{C} \oplus \mathbb{C}^3$, since

$$x = (x_0 + x_1e_1) + (x_2 + x_3e_1)e_2 + (x_4 + x_5e_1)e_4 + (x_6 + x_7e_1)e_6.$$

Now that we hopefully have a better grasp on the octonions, we can begin to understand G_2. The group G_2 is defined to be the automorphism group of the octonions; that is,

$$G_2 := \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}) := \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathbb{O}) : \alpha(xy) = \alpha(x)\alpha(y),\ \forall x, y \in \mathbb{O}\},$$

where $\mathrm{Iso}_{\mathbb{R}}(\mathbb{O})$ denotes the group of $\mathbb{R}$-vector space isomorphisms of $\mathbb{O}$. We have the following results.

Lemma 3. *For any $x, y \in \mathbb{O}$, we have $(x, y) = \frac{1}{2}(\overline{x}y + \overline{y}x)$.*

Proof. Observe that

$$\overline{x}y = \left(x_0 - \sum_{i=1}^{7} x_ie_i\right)\left(y_0 + \sum_{i=1}^{7} y_ie_i\right) = x_0y_0 + \sum_{i=1}^{7} x_0y_ie_i - \sum_{i=1}^{7} x_iy_0e_i - \sum_{i=1}^{7}\sum_{j=1}^{7} x_iy_je_ie_j.$$

It follows that

$$\overline{x}y + \overline{y}x = 2x_0y_0 - \left(\sum_{i=1}^{7}\sum_{j=1}^{7} x_iy_je_ie_j + x_jy_ie_je_i\right).$$

However, suppose we take the table (1) and throw away the first row and column. This induces a matrix that is skew-symmetric, except on the diagonal; in particular, $x_iy_je_ie_j + x_jy_ie_je_i$ is zero whenever $i \neq j$, otherwise it is $-2x_iy_i$. The result follows. This completes the proof. ■

Lemma 4. *For any $\alpha \in G_2$ and $x \in \mathbb{O}$, we have $\overline{\alpha(x)} = \alpha(\overline{x})$.*

Proof. By linearity, it is sufficient to show that $\alpha(1) = 1$ and $\overline{\alpha(e_i)} = -\alpha(e_i)$, for all $1 \leq i \leq 7$. Naturally, $\alpha(1)\alpha(1) = \alpha(1 \cdot 1) = \alpha(1)$, implying $\alpha(1) = 1$. Meanwhile, given $i \neq 0$, we have $\alpha(e_i)\alpha(e_i) = \alpha(e_i e_i) - \alpha(-1) = -1$. Multiplying both sides of this by $-(\alpha(e_i))^{-1}$, it follows that $-\alpha(e_i) = (\alpha(e_i))^{-1} = \overline{\alpha(e_i)}$, and hence that $\overline{\alpha(x)} = \alpha(\overline{x})$ as desired. This completes the proof. ■

Lemma 5. *For any $\alpha \in G_2$ and $x, y \in \mathbb{O}$, we have $(\alpha(x), \alpha(y)) = (x, y)$.*

Proof. It follows from the previous lemmata that

$$\begin{aligned}(\alpha(x), \alpha(y)) &= \frac{1}{2}\Big((\overline{\alpha(x)})(\alpha(y)) + (\overline{\alpha(y)})(\alpha(x))\Big) && \text{(Lemma 3)}\\ &= \frac{1}{2}\left((\alpha(\overline{x}))(\alpha(y)) + (\alpha(\overline{y}))(\alpha(x))\right) && \text{(Lemma 4)}\\ &= \alpha\left(\frac{1}{2}(\overline{x}y + \overline{y}x)\right) = \alpha((x,y)) = (x,y).\end{aligned}$$

This completes the proof. ■

Theorem 6. *The group G_2 is a compact Lie group.*

Proof. Consider the orthogonal group $\mathrm{O}(8)$, which we may write as

$$\mathrm{O}(8) = \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathbb{O}) : (\alpha(x), \alpha(y)) = (x, y),\ \forall x, y \in \mathbb{O}\}.$$

Clearly $G_2 \subset \mathrm{O}(8)$ by Lemma 5. Recall that the orthogonal groups are themselves compact Lie groups; we claim that G_2 is a closed subgroup and hence a compact Lie subgroup of $\mathrm{O}(8)$. Well, note that for any $\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathbb{O})$, we have that $\alpha \in G_2$ if and only if $\alpha(e_i e_j) = \alpha(e_i)\alpha(e_j)$ for all $i, j \in \{0, \ldots, 7\}$. Suppose we fix $i, j \in \{0, \ldots, 7\}$ and let

$$X_{ij} \coloneqq \{\alpha \in \mathrm{O}(8) : \alpha(e_i e_j) = \alpha(e_i)\alpha(e_j)\}.$$

Define a map $f_{ij} : \mathrm{O}(8) \to \mathbb{R}^8$ by

$$f_{ij} : \alpha \mapsto \alpha(e_i e_j) - \alpha(e_i)\alpha(e_j).$$

This map is clearly continuous, and moreover $f_{ij}^{-1}(\{\mathbf{0}\}) = X_{ij}$. In other words, X_{ij} is the continuous preimage of a closed set, and is hence closed. Since G_2 is the finite intersection of each X_{ij}, it too is closed. It follows that G_2 is a compact Lie group, as it is a closed subgroup of the compact Lie group $\mathrm{O}(8)$. This completes the proof. ■

Remark 7. Because $\alpha(1) = 1$ for all $\alpha \in G_2$, it is in fact true that G_2 is a compact subgroup of $\mathrm{O}(7)$. In fact, we will learn later that it is actually a subgroup of $\mathrm{SO}(7)$, although this is not obvious.

We will end this section with two rather nice results that we will not prove. Although their proofs aren't too difficult, they're somewhat involved and ultimately tangential to the purpose of this piece.

Theorem 8. ([Yok25, Theorem 1.9.2, Theorem 1.9.3]). *The Lie group G_2 satisfies $G_2/\mathrm{SU}(3) \cong S^6$, and is hence simply connected.*

Theorem 9. ([Yok25, Theorem 1.11.1]). *The center of G_2 is trivial.*

To summarize, G_2 is a simple, compact, simply connected Lie group given by the automorphism group of the octonions. In Atlas, the compact group G_2 is referred to as `G2_c`.

2. The Real Lie Algebra $\mathfrak{g}_2$

Definition 10. (Derivation). *Let A be any algebra over $\Bbbk$. The* space of derivations of A over $\Bbbk$ *is*

$$\mathrm{Der}_{\Bbbk}(A) := \{D \in \mathrm{End}_{\Bbbk}(A) : D(xy) = D(x)y + xD(y), \ \forall x, y \in A\},$$

where $\mathrm{End}_{\mathbb{R}}(A)$ denotes the set of $\Bbbk$-linear maps on A.

The space of derivations is easily seen to be a Lie algebra, with Lie bracket given by the commutator $[D, E] := D \circ E - E \circ D$. We in fact have the following result.

Theorem 11. *Let A be any algebra over $\Bbbk$ whose automorphism group $G := \mathrm{Aut}_{\Bbbk}(A)$ is a Lie group. Then $\mathfrak{g} := \mathrm{Der}_{\Bbbk}(A)$ is the associated Lie algebra of G.*

Proof. Let $\gamma : [0, 1] \to G$ be a tangent vector of G at the identity; that is, a path with $\gamma_0 = \mathrm{id}$, where $\gamma_t := \gamma(t)$. Suppose moreover that the derivative γ_t' with respect to t satisfies $\gamma_0' = D$. Differentiating

$$\gamma_t(xy) = \gamma_t(x)\gamma_t(y)$$

with respect to t, we obtain

$$\gamma_t'(xy) = \gamma_t'(x)\gamma_t(y) + \gamma_t(x)\gamma_t'(y).$$

At $t = 0$, this gives us

$$D(xy) = D(x)y + xD(y).$$

In other words, $D \in \mathrm{Der}_{\Bbbk}(A)$. Conversely, suppose that $D \in \mathrm{Der}_{\Bbbk}(A)$ and let $\gamma_t := \exp(tD)$. Define two maps $\varphi : t \mapsto \gamma_t(xy)$ and $\psi : t \mapsto \gamma_t(x)\gamma_t(y)$. Differentiating φ, we have

$$\varphi'(t) = \gamma_t'(xy) = D(\gamma_t(xy)) = D(\varphi(t)).$$

Meanwhile, differentiating ψ, we have

$$\begin{aligned}
\psi'(t) &= \gamma_t'(x)\gamma_t(y) + \gamma_t(x)\gamma_t'(y) \\
&= D(\gamma_t(x))\gamma_t(y) + \gamma_t(x)D(\gamma_t(y)) \\
&= D(\gamma_t(x)\gamma_t(y)) && \text{(Definition 10)} \\
&= D(\psi(t)).
\end{aligned}$$

Because $\varphi(0) = xy = \psi(0)$ and $\varphi'(0) = \psi'(0)$, it follows that φ and ψ are equivalent as tangent vectors, and hence $\gamma_t(xy) = \gamma_t(x)y + x\gamma_t(y)$. We have thus shown that the tangent space of G at the identity is nothing but $\mathrm{Der}_{\Bbbk}(A)$. This completes the proof. ∎

Corollary 12. *The Lie algebra of G_2 is $\mathfrak{g}_2 := \mathrm{Der}_{\mathbb{R}}(\mathbb{O})$.*

Suppose we define $\mathbb{R}$-linear maps $G_{ij} : \mathbb{O} \to \mathbb{O}$ and $F_{ij} : \mathbb{O} \to \mathbb{O}$ by

$$G_{ij}(e_j) = e_i; \qquad G_{ij}(e_i) = -e_j; \qquad G_{ij}(e_k) = 0, \ \forall k \neq i, j$$

and

$$F_{ij} : x \mapsto \frac{1}{2} e_i(\overline{e}_j x),$$

respectively. Moreover, consider the $\mathbb{R}$-linear map $\pi : \mathfrak{so}(7) \to \mathfrak{so}(7)$ given by

$$\pi : G_{ij} \mapsto F_{ij},$$

where

$$\mathfrak{so}(8) = \{D \in \mathrm{Hom}_{\mathbb{R}}(\mathbb{O}) : (D(x), y) + (x, D(y)) = 0, \ \forall x, y \in \mathbb{O}\},$$
$$\mathfrak{so}(7) = \{D \in \mathfrak{so}(8) : D(1) = 0\}.$$

We now have the following results.

Lemma 13. *The Lie algebra $\mathfrak{g}_2$ is a Lie subalgebra of $\mathfrak{so}(7)$; that is,*

$$\mathfrak{g}_2 = \{D \in \mathfrak{so}(7) : \pi(D) = D\}.$$

Theorem 14. ([Yok25, Theorem 1.4.3]). *Any element of $\mathfrak{g}_2$ is given by some sum of the elements*

$$\lambda_1 G_{23} + \lambda_2 G_{45} + \lambda_3 G_{67}, \qquad -\lambda_1 G_{13} - \lambda_2 G_{46} + \lambda_3 G_{57},$$
$$\lambda_1 G_{12} + \lambda_2 G_{47} + \lambda_3 G_{56}, \qquad -\lambda_1 G_{15} + \lambda_2 G_{26} - \lambda_3 G_{37},$$
$$\lambda_1 G_{14} - \lambda_2 G_{27} - \lambda_3 G_{36}, \qquad -\lambda_1 G_{17} - \lambda_2 G_{24} + \lambda_3 G_{35},$$
$$\lambda_1 G_{16} + \lambda_2 G_{25} + \lambda_3 G_{34},$$

for $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

In particular, the dimension of $\mathfrak{g}_2$ is 14.

Of course, one last result that is particularly important is the following.

Theorem 15. ([Yok25, Theorem 1.6.3]). *The Lie algebra $\mathfrak{g}_2$ is simple, and hence so too is G_2.*

The proofs of these results are again a little involved, so we will not provide them. Proofs can be found in the wonderful book of Yokota as usual.

The general strategy so far has been to understand G_2 and its Lie algebra by looking at familiar objects that they live inside and familiar objects that live inside them. For $\mathfrak{g}_2$, we have done this by viewing it as a subalgebra of $\mathfrak{so}(7)$ and by looking at its subalgebra $\mathfrak{su}(3)$. In particular, $\mathfrak{su}(3)$ is isomorphic to $\{D \in \mathfrak{g}_2 : D(e_1) = 0\}$ – a fact that lifts to G_2 and gives Theorem 8 – and moreover $\mathfrak{g}_2$ decomposes into a direct sum of $\mathfrak{su}(3)$ with some algebra $\mathfrak{S}$, giving a way of attacking Theorem 15. More on this will be revealed later on.

3. The Complex Form $G_2^{\mathbb{C}}$

By definition, the complexification of $\mathbb{O}$ is given by

$$\mathbb{O}^{\mathbb{C}} \coloneqq \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C} = \{a + ib : a, b \in \mathbb{O}\}.$$

In this section, we would like to show that

$$G_2^{\mathbb{C}} \coloneqq \{\alpha \in \mathrm{Iso}_{\mathbb{C}}(\mathbb{O}^{\mathbb{C}}) : \alpha(xy) = \alpha(x)\alpha(y), \forall x, y \in \mathbb{O}^{\mathbb{C}}\}$$

is a simple, simply connected, complex Lie group of type G_2. We know it is an algebraic subgroup of $\mathrm{Iso}_{\mathbb{C}}(\mathbb{O}^{\mathbb{C}}) = \mathrm{GL}(8, \mathbb{C})$. Theorem 11 therefore tells us that its corresponding Lie algebra is

$$\mathfrak{g}_2^{\mathbb{C}} \coloneqq \mathfrak{g}_2 \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Der}_{\mathbb{C}}(\mathbb{O}^{\mathbb{C}}).$$

From here, Theorem 15 is easily adapted to $\mathfrak{g}_2^{\mathbb{C}}$, making $G_2^{\mathbb{C}}$ a simple, complex Lie group (alternatively, a real, compact Lie algebra is simple if and only if its complexification is simple). All that remains is to show that $G_2^{\mathbb{C}}$ is simply connected and indeed the complexification of G_2.

For what follows, we define conjugation by $\overline{a + ib} \coloneqq \overline{a} + i\overline{b}$ and let $\tau \in G_2^{\mathbb{C}}$ denote the complex conjugation map taking $a + ib \in \mathbb{O}^{\mathbb{C}}$ to $a - ib$. Then $\mathbb{O}^{\mathbb{C}}$ inherits multiplication and an inner product

$$(a + ib, c + id) \coloneqq (a, c) + i(a, d) + i(b, c) - (b, d)$$

from $\mathbb{O}$, where it can be shown that the latter satisfies $\mathbb{O}^{\mathbb{C}}$-analogues of Lemma 3, Lemma 4 and Lemma 5 (see [Yok25, Lemma 1.12.1]). This induces the canonical inner product

$$\langle a + ib, c + id\rangle \coloneqq (\tau(a + ib), c + id) = (a, c) + i(a, d) - i(b, c) + (b, d).$$

Given $\alpha \in G_2^{\mathbb{C}}$, we will define its adjoint to be the map α^* such that $\langle \alpha^*(x), y\rangle = \langle x, \alpha(y)\rangle$.

Lemma 16. *For any $\alpha \in G_2^{\mathbb{C}}$, we have $\alpha^* = \tau\alpha^{-1}\tau \in G_2^{\mathbb{C}}$.*

Proof. For all $x, y \in \mathbb{O}^{\mathbb{C}}$, we have

$$\langle \alpha^*(x), y\rangle = \langle x, \alpha(y)\rangle = (\tau(x), \alpha(y)) = ([\alpha^{-1}\tau](x), y) = \langle [\tau\alpha^{-1}\tau](x), y\rangle.$$

This completes the proof. ■

The upshot from this lemma is that $G_2^{\mathbb{C}}$ is closed under taking conjugate transposes. Moreover, $\alpha \in G_2^{\mathbb{C}}$ is unitary if and only if it commutes with our complex conjugation map τ. We can actually say a bit more than this. Given $\alpha \in G_2$, we have a unique complexification $\alpha^{\mathbb{C}} \in G_2^{\mathbb{C}}$ that takes $x \otimes_{\mathbb{R}} z \in \mathbb{O} \otimes_{\mathbb{R}} \mathbb{C}$ to $\alpha(x) \otimes_{\mathbb{R}} z = z\alpha(x)$. In fact, we can do this process in reverse.

Lemma 17. *The group G_2 lives inside $G_2^{\mathbb{C}}$ as the fixed-point subgroup $(G_2^{\mathbb{C}})^\tau = \{\alpha \in G_2^{\mathbb{C}} : \tau\alpha = \alpha\tau\}$.*

Proof. Suppose $\alpha \in G_2^{\mathbb{C}}$ satisfies $\tau\alpha = \alpha\tau$. Given $x \in \mathbb{O}$, we have that $[\tau\alpha](x) = [\alpha\tau](x) = \alpha(x)$, and hence $\alpha(x) \in \mathbb{O}$. Thus α restricts to an $\mathbb{R}$-transformation $\alpha|_{\mathbb{O}} \in G_2$, which satisfies $\alpha = (\alpha|_{\mathbb{O}})^{\mathbb{C}}$. This completes the proof. ■

From this lemma, it follows that by identifying $G_2^{\mathbb{C}}$ with a subgroup of $\mathrm{GL}(8,\mathbb{C})$, we have that $G_2^{\mathbb{C}} \cap U(8) = G_2$, where $U(8)$ is the set of 8×8 unitary matrices. This fact allows us to determine the polar decomposition of $G_2^{\mathbb{C}}$.

Theorem 18. *The polar decomposition of $G_2^{\mathbb{C}}$ is given by the topological product*

$$G_2^{\mathbb{C}} \cong G_2 \times \mathbb{R}^{14}.$$

In particular, $G_2^{\mathbb{C}}$ is simply connected.

Proof. Recall that invertible matrices $A \in \mathrm{GL}(n,\mathbb{C})$ admit a unique polar decomposition of the form $A = UP$, where $U \in U(n)$ is a unitary matrix and $P \in H(n)$ is a positive-definite Hermitian matrix. In other words, by [Che46, Proposition I.V.3], we have a homeomorphism

$$\mathrm{GL}(n,\mathbb{C}) \cong U(n) \times H(n).$$

Note that we may identify the set $H(n)$ of positive-definite Hermitian matrices with $\mathbb{R}^d$, where

$$d \coloneqq \dim(\mathrm{GL}(n,\mathbb{C})) - \dim(U(n)) = 2n^2 - n^2 = n^2.$$

Because $G_2^{\mathbb{C}}$ is an algebraic subgroup of $\mathrm{GL}(8,\mathbb{C})$, it follows from Lemma 17 that

$$G_2^{\mathbb{C}} \cong (G_2^{\mathbb{C}} \cap U(8)) \times \mathbb{R}^d = G_2 \times \mathbb{R}^d,$$

where $d = \dim(G_2^{\mathbb{C}}) - \dim(G_2) = 2\times 14 - 14 = 14$ by Theorem 14. This completes the proof. ■

We have now seen the complex form $G_2^{\mathbb{C}}$, which is a simple, simply connected, complex Lie group of type G_2. In Atlas, this complex group is referred to as `G2_ic`. It admits two real forms; a simple, simply connected, compact form, and a split form that we will see momentarily. Before that, however, I would like to look more at $\mathfrak{g}_2^{\mathbb{C}}$.

4. The Complex Lie Algebra $\mathfrak{g}_2^{\mathbb{C}}$

Let's find the Dynkin diagram for $\mathfrak{g}_2^{\mathbb{C}}$. We shall begin by determining its Killing form. We first introduce some notation for the sake of convenience.

$$\begin{aligned} &H_1 = -G_{23} + G_{45}, \qquad H_2 = -G_{45} + G_{67},\\ &L_{12} = G_{24} + G_{35}, \qquad L_{21} = -G_{25} + G_{34}, \qquad L_{13} = G_{26} + G_{37},\\ &L_{21} = -G_{27} + G_{36}, \qquad L_{23} = G_{46} + G_{57}, \qquad L_{32} = -G_{47} + G_{56};\end{aligned}$$

$$\begin{aligned} &S_1 = 2G_{12} - G_{47} - G_{56}, \qquad S_2 = 2G_{13} - G_{46} + G_{57}, \qquad S_3 = 2G_{14} + G_{27} + G_{36},\\ &S_4 = 2G_{15} + G_{26} - G_{37}, \qquad S_5 = 2G_{16} - G_{25} - G_{34}, \qquad S_6 = 2G_{17} - G_{24} + G_{35}.\end{aligned}$$

Note that H_1, H_2 and the L_{ij} span $\mathfrak{su}(3)$, while the S_k span the algebra $\mathfrak{S}$ for which $\mathfrak{g}_2 \cong \mathfrak{su}(3) \oplus \mathfrak{S}$; it is easy to see by Theorem 14 that these indeed span $\mathfrak{g}_2$.

Theorem 19. *The Killing form B of $\mathfrak{g}_2^{\mathbb{C}}$ is given by*

$$B(D_1, D_2) = 4\mathrm{Tr}(D_1 D_2),$$

for all $D_1, D_2 \in \mathfrak{g}_2^{\mathbb{C}}$.

Proof. Recall that Schur's lemma applied to the adjoint representation tells us that any invariant, symmetric, bilinear form on a simple complex Lie algebra is a scalar multiple of the Killing form. Because $(D_1, D_2) \mapsto \mathrm{Tr}(D_1 D_2)$ is such a form, it follows that there exists some $z \in \mathbb{C}$ for which

$$B(D_1, D_2) = z\mathrm{Tr}(D_1 D_2).$$

Suppose we let $D_1 = D_2 = H_1$. It can be shown that

$$\begin{aligned}
[H_1,[H_1,L_{12}]] &= [H_1, 2L_{21}] = -4L_{12}, & [H_1,[H_1,L_{21}]] &= [H_1,-2L_{12}] = -4L_{21},\\
[H_1,[H_1,L_{13}]] &= [H_1, L_{31}] = -L_{13}, & [H_1,[H_1,L_{31}]] &= [H_1,-L_{13}] = -L_{31},\\
[H_1,[H_1,L_{23}]] &= [H_1, -L_{32}] = -L_{23}, & [H_1,[H_1,L_{32}]] &= [H_1,L_{23}] = -L_{32};\\
[H_1,[H_1,S_1]] &= [H_1, S_2] = -S_1, & [H_1,[H_1,S_2]] &= [H_1,-S_1] = -S_2,\\
[H_1,[H_1,S_3]] &= [H_1, -S_4] = -S_3, & [H_1,[H_1,S_4]] &= [H_1,S_3] = -S_4,\\
[H_1,[H_1,S_5]] &= 0, & [H_1,[H_1,S_2]] &= 0.
\end{aligned}$$

It follows that

$$B(H_1, H_1) := \mathrm{Tr}(\mathrm{ad}(H_1) \circ \mathrm{ad}(H_1)) = (-4) \times 2 + (-1) \times 8 = -16.$$

Conversely,

$$\begin{aligned}
H_1H_1e_2 &= H_1e_3 = -e_2, & H_1H_1e_3 &= -H_1e_2 = -e_3,\\
H_1H_1e_4 &= -H_1e_5 = -e_4, & H_1H_1e_5 &= H_1e_4 = -e_5,
\end{aligned}$$

with $H_1H_1e_i = 0$ otherwise. Thus $\mathrm{Tr}(H_1H_1) = -4$, whence $z = 4$. This completes the proof. ■

Now, note that there is a Lie algebra isomorphism $f_* : \mathfrak{sl}(3, \mathbb{C}) \to \mathfrak{su}(3)^{\mathbb{C}}$ of the form

$$f_* : A \mapsto \varepsilon A - \bar{\varepsilon} A^T,$$

for $\varepsilon := \frac{1}{2}(1 + ie_1)$. We also have an embedding $\varphi_* : \mathfrak{su}(3)^{\mathbb{C}} \to \mathfrak{g}_2^{\mathbb{C}}$ of the form

$$\varphi_*(D) : a + m \mapsto D(m),$$

where $a + m \in \mathbb{C}^{\mathbb{C}} \oplus (\mathbb{C}^3)^{\mathbb{C}} \cong \mathbb{O}^{\mathbb{C}}$ by our observation at the end of Remark 2. We view $\mathfrak{sl}(3, \mathbb{C})$ as a subalgebra of $\mathfrak{g}_2^{\mathbb{C}}$ via $\varphi_* \circ f_*$. The Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ admits a Cartan subalgebra

$$\mathfrak{h} := \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C},\ \lambda_1 + \lambda_2 + \lambda_3 = 0 \right\}$$

with corresponding roots $\pm(\lambda_k - \lambda_l)$ and root vectors E_{kl} for $1 \leq k < l \leq 3$, where E_{kl} is the 3×3 matrix whose (k, l)-th entry 1 with all other entries 0. With this, we have the following results.

Theorem 20. *The Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$ admits a Cartan subalgebra*

$$\mathfrak{h} := \{-i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{C},\ \lambda_1 + \lambda_2 + \lambda_3 = 0\}$$

with corresponding roots and root vectors

$$\begin{aligned}
\pm(\lambda_1 - \lambda_2) \quad &\leftrightsquigarrow \quad \pm(G_{24} + G_{35}) + i(-G_{25} + G_{34}),\\
\pm(\lambda_1 - \lambda_3) \quad &\leftrightsquigarrow \quad \pm(G_{26} + G_{37}) + i(-G_{27} + G_{36}),\\
\pm(\lambda_2 - \lambda_3) \quad &\leftrightsquigarrow \quad \pm(G_{46} + G_{57}) + i(-G_{47} + G_{56}),\\
\pm\lambda_1 \quad &\leftrightsquigarrow \quad (2G_{12} - G_{47} - G_{56}) \pm i(2G_{13} - G_{46} + G_{57}),\\
\pm\lambda_2 \quad &\leftrightsquigarrow \quad (2G_{14} - G_{27} - G_{36}) \pm i(2G_{15} - G_{26} + G_{37}),\\
\pm\lambda_3 \quad &\leftrightsquigarrow \quad (2G_{16} - G_{25} - G_{34}) \pm i(2G_{17} - G_{24} + G_{35}).
\end{aligned}$$

Corollary 21. *The root system from Theorem 20 induces a fundamental root system*

$$\alpha_1 := \lambda_2, \qquad \alpha_2 := \lambda_1 - \lambda_2$$

with highest root

$$\mu := 3\alpha_1 + 2\alpha_2.$$

Proof. The positive roots from Theorem 20 can be written as

$$\begin{aligned}
\lambda_2 &= \alpha_1, & \lambda_1 - \lambda_3 &= 3\alpha_1 + 2\alpha_2, & \lambda_2 - \lambda_3 &= 3\alpha_1 + \alpha_2,\\
\lambda_1 &= \alpha_1 + \alpha_2, & \lambda_1 - \lambda_2 &= \alpha_2, & -\lambda_3 &= 2\alpha_1 + \alpha_2,
\end{aligned}$$

where we have used the fact that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This completes the proof. ∎

Theorem 22. ([Yok25, Theorem 1.8.2]). *The Cartan matrix for $\mathfrak{g}_2^{\mathbb{C}}$ is*

$$A := \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

Proof. The real part of $\mathfrak{h}$ is

$$\mathfrak{h}_{\mathbb{R}} = \{-i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R},\ \lambda_1 + \lambda_2 + \lambda_3 = 0\},$$

with the Killing form restricting to $\mathfrak{h}_{\mathbb{R}}$ as

$$B(H, H') = 8 \sum_{j=1}^{3} \lambda_j \lambda_j',$$

for

$$\begin{aligned}
H &:= -i\lambda_1 G_{23} - i\lambda_2 G_{45} - i\lambda_3 G_{67},\\
H' &:= -i\lambda_1' G_{23} - i\lambda_2' G_{45} - i\lambda_3' G_{67}.
\end{aligned}$$

Now, suppose we consider the canonical element $H_{\alpha_i} \in \mathfrak{h}_{\mathbb{R}}$ such that $B(H_{\alpha_i}, H) = \alpha_i(H)$ for all $H \in \mathfrak{h}_{\mathbb{R}}$. These can be computed for our simple roots α_1 and α_2 as

$$H_{\alpha_1} = \frac{1}{24} i G_{23} - \frac{1}{12} i G_{45} + \frac{1}{24} i G_{67} \qquad \text{and} \qquad H_{\alpha_2} = -\frac{1}{8} i G_{23} + \frac{1}{8} i G_{45}.$$

We therefore have

$$(\alpha_1, \alpha_1) = B(H_{\alpha_1}, H_{\alpha_1}) = 8\left(\left(-\frac{1}{24}\right)^2 + \left(\frac{1}{12}\right)^2 + \left(-\frac{1}{24}\right)^2\right) = \frac{1}{12},$$

$$(\alpha_2, \alpha_2) = B(H_{\alpha_2}, H_{\alpha_2}) = 8\left(\left(\frac{1}{8}\right)^2 + \left(-\frac{1}{8}\right)^2\right) = \frac{1}{4},$$

$$(\alpha_1, \alpha_2) = B(H_{\alpha_1}, H_{\alpha_2}) = 8\left(\left(-\frac{1}{24}\right)\left(\frac{1}{8}\right) + \left(\frac{1}{12}\right)\left(-\frac{1}{8}\right)\right) = -\frac{1}{8}.$$

The entries of the Cartan matrix are given by

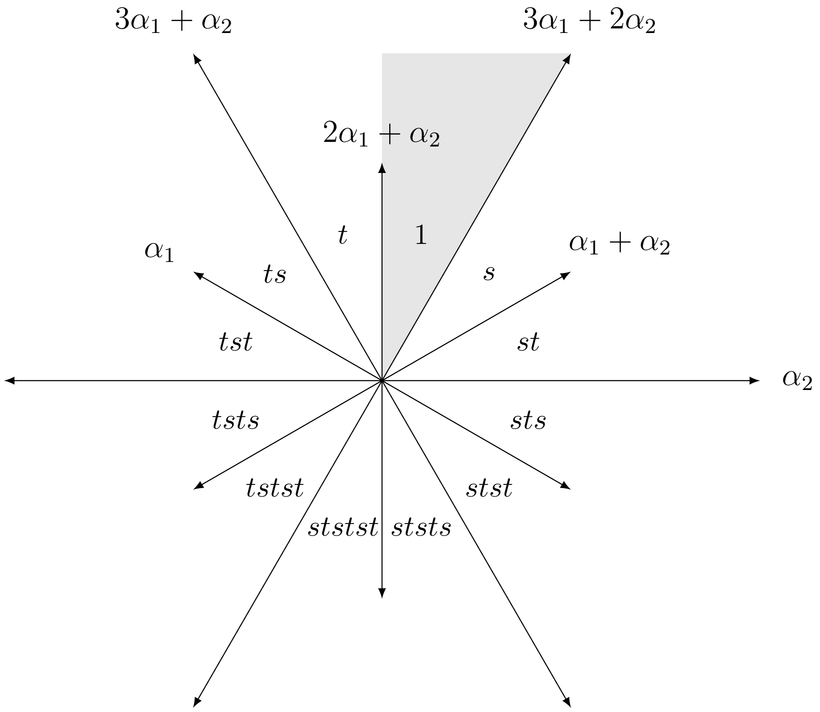
$$A_{ij} \coloneqq 2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)},$$

whence the result follows. This completes the proof. ■

Note that $(\alpha_1, \alpha_1) = 1/12$ tells us that $|\alpha_1| = 1/\sqrt{12}$, and similarly $|\alpha_2| = 1/2$. Meanwhile, from $(\alpha_1, \alpha_2) = -1/8$ we may deduce that the angle between α_1 and α_2 is

$$\theta = \cos^{-1}\left(-\frac{|\alpha_1||\alpha_2|}{8}\right) = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}.$$

Thus we can draw our root system as follows.



This diagram immediately tells us that the Weyl group for G_2 is the dihedral group of order 12,

$$W = D_6 \coloneqq \langle s, t : s^2 = t^2 = (st)^6 = 1\rangle.$$

In particular, the simple reflections are given by

$$s_i(v) \coloneqq v - 2\frac{(v,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i,$$

for $i \in \{1,2\}$; this is nothing but the reflection across the line perpendicular to α_i. Here $s \coloneqq s_1$ is the reflection across $3\alpha_1 + 2\alpha_2$, while $t \coloneqq s_2$ is the reflection across $2\alpha_1 + \alpha_2$. The *fundamental Weyl chamber* is by definition $\{v \in \mathbb{R}^2 : (v,\alpha_1) > 0, (v,\alpha_2) > 0\}$; this region has been shaded in the diagram above. For G_2, the fundamental Weyl chamber is the same as the *fundamental Weyl alcove* $\{v \in \mathbb{R}^2 : (v,\alpha_1) > 0, (v,\alpha_2) > 0, (v,\mu) < 1\}$, where we recall that the highest root is $\mu \coloneqq 3\alpha_1 + 2\alpha_2$. We have also labelled the fundamental alcove by 1, and the other alcoves by the element of the Weyl group for which they are the image of the fundamental alcove under.

5. The Split Real Form G_2^s

Recall that in Remark 2 we gave an algorithm for building a family of hypercomplex algebras. A slight modification of this construction – namely, replacing the minus sign with a plus sign in the definition of multiplication – yields a new family of so-called *split* hypercomplex algebras. Let $\mathbb{O}^s$ therefore denote the split-octonions, given by the direct sum of two copies of the split-quaternions. As a complex vector space, we alternatively have

$$\mathbb{O}^s \cong \{x \in \mathbb{O}^{\mathbb{C}} : [\tau\gamma](x) = x\},$$

where $\gamma : a + be_4 \mapsto a - be_4$ for $a, b \in \mathbb{H}$. We define G_2^s to be the automorphism group

$$G_2^s \coloneqq \mathrm{Aut}_{\mathbb{R}}(\mathbb{O}^s) \coloneqq \{\alpha \in \mathrm{Iso}_{\mathbb{R}}(\mathbb{O}^s) : \alpha(xy) = \alpha(x)\alpha(y),\ \forall x, y \in \mathbb{O}^s\}$$

of the split-octonions. Alternatively, we can view this as the centralizer of $\tau\gamma$ in $G_2^{\mathbb{C}}$,

$$G_2^s = (G_2^{\mathbb{C}})^{\tau\gamma} = \{\alpha \in G_2^{\mathbb{C}} : \tau\gamma\alpha = \alpha\tau\gamma\}.$$

Theorem 23. ([Yok25, Theorem 1.13.1]). *The polar decomposition of G_2^s is given by*

$$G_2^s \cong \mathrm{SO}(4) \times \mathbb{R}^8,$$

where $\mathrm{Sp}(1) = \{a \in \mathbb{H} : |a| = 1\}$ *denotes the symplectic group of unit quaternions.*

This result follows similarly to Theorem 18, where $(G_2)^\gamma \cong (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2$ as shown in [Yok25, Theorem 1.10.1] and $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbb{Z}_2 \cong \mathrm{SO}(4)$. Finally, we have one last result.

Theorem 24. ([Yok25, Theorem 1.13.2]). *The center of G_2^s is trivial.*

In Atlas, the split real form is referred to as `G2_s`.

Appendix

This appendix contains an assortment of observations and remarks that would not fit anywhere else in this piece. I will likely continue to add to this over time.

Recall that the Cayley–Dickson construction gives us an infinite family of hypercomplex algebras, sometimes referred to as the "Cayley–Dickson algebras". As a result, we would naturally expect to obtain an infinite family of Lie groups given by the automorphism groups of Cayley–Dickson algebras. How does G_2, as an exceptional Lie group, fit into this infinite family? Well, suppose we let A_n denote the Cayley–Dickson algebra of real dimension 2^n (that is, $A_0 := \mathbb{R}$, $A_1 := \mathbb{C}$, $A_2 := \mathbb{H}$ and so on). For the first few cases, we have $\mathrm{Aut}_{\mathbb{R}}(\mathbb{R}) = \{0\}$, $\mathrm{Aut}_{\mathbb{R}}(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Aut}_{\mathbb{R}}(\mathbb{H}) = \mathrm{SO}(3)$. However, beyond this, something quite remarkable happens. In particular, for all $n \geq 3$, we have

$$\mathrm{Aut}_{\mathbb{R}}(A_{n+1}) = \mathrm{Aut}_{\mathbb{R}}(A_n) \oplus S_3.$$

In other words, for the octonions and beyond, we just have a copy of G_2 and $n-3$ copies of the symmetric group S_3 ([ES90]).

On a separate note, let G be a connected, complex, reductive algebraic group. Recall from structure theory that we have bijections of the form

$$\{\text{real forms of } G\} \longleftrightarrow \left\{\begin{matrix}\text{antiholomorphic}\\ \text{involutions of } G\end{matrix}\right\}/\!\sim\; \longleftrightarrow \left\{\begin{matrix}\text{holomorphic}\\ \text{involutions of } G\end{matrix}\right\}/\!\sim.$$

In particular, let σ_c be the unique (up to G-conjugation) antiholomorphic involution of G for which G^{σ_c} is compact. If we let Θ be any G-conjugacy class of holomorphic involutions of G, then there exists some holomorphic involution $\theta \in \Theta$ that commutes with σ_c, and furthermore each G-conjugacy class of antiholomorphic involutions of G contains $\sigma = \sigma_c \circ \theta$ for some unique choice of initial Θ. In the case of $G_2^{\mathbb{C}}$, we have that $\sigma_c = \tau$. Meanwhile, the antiholomorphic involution for the split real form G_2^s is $\sigma = \tau\gamma$, so the Cartan involution corresponding to G_2^s is $\theta = \tau\tau\gamma = \gamma$. The maximal compact subgroup of G_2^s is $(G_2^s)^\gamma \cong \mathrm{SO}(4)$, hence $(G_2^{\mathbb{C}})^\gamma$ is isomorphic to $\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$, the complexification of $\mathrm{SO}(4)$.

Going one step further, let $K := (G_2^{\mathbb{C}})^\gamma \cong \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$. This corresponds to a Weyl group $W_K = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. There are three distinct copies of the Klein four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ living inside D_6; these are

$$\begin{aligned}\langle s, tstst\rangle &= \{1, s, tstst, ststst\},\\ \langle sts, tst\rangle &= \{1, sts, tst, ststst\},\\ \langle t, ststs\rangle &= \{1, t, ststs, ststst\}.\end{aligned}$$

The generators we have chosen correspond to orthogonal roots; in particular, s is a reflection across $3\alpha_1 + 2\alpha_2$, t is a reflection across $2\alpha_1 + \alpha_2$, sts is a reflection across $\alpha_1 + \alpha_2$, tst is a reflection across $3\alpha_1 + \alpha_2$, $ststs$ is a reflection across α_2 and $tstst$ is a reflection across α_1. We could alternatively have found these candidates by observing that W_K must be generated by orthogonal roots – as it is given by two commuting copies of A_1 – and these candidates are easily seen to be the only three sets of orthogonal roots. The Atlas of Lie Groups and Representations software is then able to tell us that the generators for W_K correspond to the roots $3\alpha_1 + \alpha_2$ and $\alpha_1 + \alpha_2$, meaning $W_K = \langle sts, tst\rangle$.

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