FROM SUBFACTORS TO RICHARD THOMPSON'S GROUPS AND THEIR GENERALIZATIONS

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Abstract

This thesis offers a journey between three seemingly disjoint worlds of mathematics: subfactor theory, chiral conformal field theory and the groups of Richard Thompson. We begin in the realm of subfactors, introducing them and their invariants, before moving onto an important axiomatization of the standard invariant in the form of planar algebras. By attempting to interpret the exterior disc of a planar algebra as the spacetime circle of a conformal field theory, Jones was unexpectedly able to discover some unitary representations of none other than the groups of Richard Thompson, and even more remarkably an entire machine for producing unitary representations of certain groups of fractions. We follow in his footsteps, seeking to construct similarly "discrete" nets but now with a more sophisticated symmetry group arising from the forest-skein formalism of Brothier.

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Chapter 1
Introduction

1.1. A Brief History

Subfactors were introduced by Murray and von Neumann in their study of operator algebras in the 1930s ([MN36]), where they played an important role in classifying isomorphism classes of von Neumann algebras. These von Neumann algebras were central in rigorously formalizing early quantum mechanics, where they played the role of modelling algebras of physical observables (such as the position, momentum and energy of subatomic particles). In particular, they were essential in the creation of quantum statistical mechanics and quantum field theory – the latter being one of the most successful theories in physics in terms of the sheer scale of its predictive ability and accuracy – and continue to be fruitful in these contexts.

On the mathematical side, Murray and von Neumann's work initiated a program sometimes referred to as *quantum mathematics*, which encapsulates somehow the interpretation of classical subjects in the context of non-commutative algebras. Under this umbrella lie, for instance, the C^*-algebras developed by Gelfand and Naimark ([GN43]), the non-commutative geometry developed by Connes ([Con80]) and the compact quantum groups developed by Drinfeld, Jimbo, Woronowicz and others ([Dri85], [Jim85], [Wor87]). These three examples heuristically present "quantum analogues" to topology, differential geometry and (compact) groups, respectively, with von Neumann algebras serving as a kind of "quantum analogue" to measure theory.

Some time after Murray and von Neumann's initial business, the theory of subfactors for their own sake would be developed by Jones, where they were found to have some remarkable properties; this marked the beginning of a prolific period for Jones in the 1980s, where he was not only able to fully classify an important invariant of subfactors ([Jon83]) but also find a deep and unexpected connection between subfactor theory and the seemingly disjoint field of knot theory ([Jon85]). In particular, it was for his discovery of the Jones polynomial that he was awarded the Fields Medal in 1990.

The motivation behind Jones' work, however, always remained rooted in the solid historical ties between subfactors and mathematical physics. A direction that Jones would dedicate much time towards exploring was the relationship between subfactors and quantum field theory. All conformal nets provide subfactors, and some subfactors provide conformal nets; it was Jones' dream to find a way to generate conformal nets from arbitrary subfactors, and in his search he unexpectedly stumbled upon the groups of Richard Thompson ([Jon17]).

Richard Thompson's groups live in a completely different world to the rest of the mathematics we have talked about so far. They are a triplet of infinite, finitely-presented groups $F \subset T \subset V$ first recorded by Thompson in some unpublished notes in the 60s. They have been independently discovered numerous times in a wide range of fields, and have some rather remarkable properties allowing them to appear often as counterexamples to many problems across mathematics. Moreover, T and V are rare examples of infinite simple groups with finite presentations, making them incredibly important case studies in the flowering study of infinite simple groups.

While the initial approach of Jones was ultimately unsuccessful in finding a deeper link between subfactors and conformal field theory, Jones did in the process discover a machine for producing representations of not only Thompson's groups themselves, but of other groups of fractions. The early stages of capitalizing on this technology have been due to Brothier with his forest-skein program ([Bro22]), which hopes to provide a rich source of new and interesting "Thompson-like" groups, as well as potentially even enhance the aforementioned work of Jones in the realm of conformal field theory.

1.2. Thesis Structure

The following chapter of this thesis will serve as a background survey of the relevant material, introducing the reader to subfactor theory, planar algebras, conformal field theory, Thompson's groups and the forest-skein formalism of Brothier. In the next chapter, we first discuss an attempt of Jones to generate a conformal net from a subfactor – a process which involved interpreting the exterior disc of a planar algebra as the spacetime circle of a conformal field theory – and how Thompson's groups naturally arose. We then hope to generalize this approach to Brothier's forest-skein groups. We discuss the challenges involved, as well as the path forward.

1.3. Various Notation

Before ending this chapter, we briefly introduce some notation used throughout the thesis. For the sake of clarity we will try to differentiate between definitions and mere equalities, using the "walrus equals" symbol ":=" for the former. We take $\mathbb{N}$ to be the set of non-negative integers and $\mathbb{Z}_+$ to be the set of positive integers. An action π of a group G on an object X will be denoted by $\pi : G \curvearrowright X$, as opposed to $\pi : G \to \mathrm{End}(X)$. Hilbert spaces will typically be assumed to be *separable* and over the complex numbers unless otherwise specified. The power set of S will be denoted by $\mathscr{P}(S)$. The center of an algebraic structure A on which such a notion makes sense will be denoted $Z(A)$.

Chapter 2

From Subfactors to Richard Thompson's Groups

The goal of this chapter is to serve not only as a survey of the background material, but as a gentle stroll with the reader between two seemingly disjoint worlds of mathematics. The plan is as follows. We begin by defining subfactors and their invariants. This leads into Vaughan Jones' planar algebras, which is the lens through which we view subfactors. An important application of subfactors, and one of the foremost motivations of Jones' work, is in mathematical physics; we take a brief detour here, before concluding with the forest-skein formalism of Arnaud Brothier.

2.1. Subfactor Theory

As promised, this section will introduce the notion of subfactors of von Neumann algebras and some of their invariants. We will be using the great book of Jones and Sunder ([JS97]) as our primary reference. For convenience, we will focus on the theory of type II_1 factors. Although type III_1 factors are more relevant in the context of mathematical physics, their theory mostly parallels the theory of type II_1 factors in some sense ([Pen14], [She12]), just being slightly more technical.

2.1.1. Definition of a Subfactor

The study of subfactors of von Neumann algebras dates back to Murray and von Neumann's seminal work in the 1930s on "rings of operators" ([MN36]). In their original context, von Neumann algebras were very much an analytical beast. Letting $\mathcal{B}(H)$ be the set of all bounded, linear operators on some complex Hilbert space H, a *von Neumann algebra* was traditionally defined as a self-adjoint subalgebra $M \subseteq \mathcal{B}(H)$ containing the identity operator, that is also *weakly closed* (that is, closed with respect to the weak operator topology, defined to be the coarsest topology such that the map $x \mapsto \langle x\xi, \eta\rangle$ is continuous for all $x \in M$ and $\xi, \eta \in H$). However, let $S \subseteq \mathcal{B}(H)$ and define $S' := \{x' \in \mathcal{B}(H) : x'x = xx', \text{for all } x \in S\}$ to be the *commutant of* S. The fundamental *double-commutant theorem* of von Neumann ([Neu30, §II, theorem 8]), which states that a self-adjoint subalgebra $M \subseteq \mathcal{B}(H)$ containing the identity is weakly closed if and only if $M = M''$ (that is, M is equal to its double-commutant, or *von Neumann closure* M''), provides us with an algebraic analogue of our previous definition; in particular, M is a von Neumann algebra if and only if it is a self-adjoint subalgebra of $\mathcal{B}(H)$ satisfying $M = M''$.

Definition 2.1.1.1. (Factor). *A von Neumann algebra M is called a* factor *if it has a trivial centre (a centre containing only scalar multiples of $1_M \in M$, the identity element of M); that is, $Z(M) = M \cap M' = \mathbb{C} \cdot 1_M$. A* subfactor *of M is then nothing but a unital subalgebra $N \subseteq M$ that is itself a factor.*

To be clear, although every factor is a subalgebra containing a unit, we specifically use the terminology *unital subalgebra* to refer to a subalgebra whose unit is the same as the original algebra it's contained in; that is, $1_N = 1_M$. We can more concisely define a subfactor as a *unital inclusion of factors* $N \subseteq M$. Note also that "subfactor" is used to refer to the entire inclusion rather than N by itself, similar to how in a field extension we care more about the way one field lives inside another. Much of the theory of subfactors emerges from the invariants of such inclusions.

It was shown by von Neumann that every von Neumann algebra over a separable Hilbert space could be decomposed as a "direct integral" – a kind of measure theoretic generalization of the direct sum – of its factors, and moreover that this decomposition is unique up to isomorphism ([Neu49]). If one chooses to only consider von Neumann algebras over separable Hilbert spaces, which we will henceforth, we can essentially think of these factors as their "building blocks".

It can also be shown that any von Neumann algebra M is generated by its *orthogonal projections*, or self-adjoint idempotents. In other words, M is the norm-closed linear span of the elements $p \in M$ such that $p = p^2 = p^*$ ([JS97]). The space $\mathcal{P}(M)$ of all such orthogonal projections in M is known as the *lattice of projections*, as it has the structure of a complete lattice when endowed with the following partial ordering ([KR83, §2.5]). An orthogonal projection q is said to be a *subprojection* of p, written $q \leq p$, if $pq = qp = q$. All this is saying is that q projects onto a subspace of the space that p projects onto. Moreover, we have a well-known bijective correspondence between closed subspaces of H and orthogonal projections in $\mathcal{B}(H)$ ([KR83, proposition 2.5.1]). Essentially, if $p : H \to H$ is an idempotent in $\mathcal{B}(H)$, then $\mathrm{Ker}(p) = [1_H - p](H)$, whence we have the orthogonal direct sum $H = p(H) \oplus \mathrm{Ker}(p)$. Conversely, given a closed subspace X, we can write $H = X \oplus Y$ for some Y by [KR83, theorem 2.2.3], whence there exists a unique self-adjoint idempotent $p \in \mathcal{B}(H)$ such that $\mathrm{Ker}(p) = Y$, $p(H) = X$ and $p(x) = x$ for all $x \in X$.

We say that p, q are *Murray-von Neumann equivalent*, written $p \sim q$, if there exists some $u \in M$ – known as a *partial isometry* – such that $u^*u = p$ and $uu^* = q$. An orthogonal projection p is said to be *finite* if there is no Murray-von Neumann equivalent orthogonal projection q such that $q \lneq p$. A non-zero orthogonal projection p is said to be *minimal* if there is no non-zero orthogonal projection q such that $q \leq p$. All minimal orthogonal projections are of course necessarily finite. Moreover, a von Neumann algebra M is said to be *finite* if 1_M is a finite orthogonal projection. Such algebras admit the following nice property due to [Tak79, theorem V.2.6].

Theorem 2.1.1.2. *A von Neumann algebra M is finite if and only if there exists a unique map $\tau : M \to Z(M)$ such that*

- *(trace) $\tau(xy) = \tau(yx)$, for all $x, y \in M$;*
- *(linear) $\tau(ax) = a\tau(x)$, for all $a \in Z(M)$, $x \in M$;*
- *(state) $\tau(1_M) = 1_M$ and $\tau(x^*x) \geq 0$, for all $x \in M$;*
- *(faithful) $\tau(x^*x) \neq 0$, for all non-zero $x \in M$;*
- *$p \sim q$ if and only if $\tau(p) = \tau(q)$, for all $p, q \in \mathcal{P}(M)$.*

Recall that when we say $a \geq 0$ for an element $a \in M$ in the definition of a state, we mean that it is self-adjoint with a non-negative *spectrum*; that is, we have both $a^* = a$ and $\sigma(a) := \{\lambda \in \mathbb{C} : \lambda 1_M - a \notin \mathrm{Inv}(M)\} \subseteq \mathbb{R}_{\geq 0}$, where $\mathrm{Inv}(M)$ denotes the set of invertible elements in M. In addition, the state property here is slightly subtle. A state is a positive, linear functional with unit norm $\|\tau\| = 1$. The Cauchy-Schwarz inequality for positive, linear functionals, combined with boundedness and the definition of the operator norm, then tells us that $\|\tau\| = \tau(1_M)$.

Remark 2.1.1.3. This unique tracial state is also *normal*, in the sense that it is continuous with respect to the σ-weak topology. For a definition of the σ-weak topology, we defer the reader to [EK98, definition 5.1].

The important role of orthogonal projections led Murray and von Neumann to use them in order to classify factors into the following types.

Definition 2.1.1.4. (Types of Factors). *We say that a factor M is of* type

- I, *if M contains at least one minimal orthogonal projection;*
- II, *if M contains at least one non-zero finite orthogonal projection, yet no minimal orthogonal projections;*
- III, *if M contains no non-zero finite orthogonal projections.*

Every factor must be exactly one of type I, II or III. We may also further refine these three types as follows.

If M is a type I factor, then $M \cong \mathcal{B}(H)$ for some separable Hilbert space H. In this case, we say that M is type I_n, where $n = \dim(H)$. Moreover, M is finite if and only if H is finite-dimensional ([JS97]). The theory of type I factors is therefore trivial, as Hilbert spaces with the same dimension are isometrically isomorphic ([KR83, theorem 2.2.12]). The finite-dimensional type I_n factors reduce to complex matrix algebras $M_n(\mathbb{C})$, with all others being of the form $\mathcal{B}(\ell^2(I))$ for some infinite set I.

If a type II factor is finite (that is, its identity element is a finite orthogonal projection), we say that it is of type II_1. Otherwise, it is type II_∞, and all type II_∞ factors are the tensor product of a type II_1 factor and a type I_∞ factor ([Kos93]). Type II_1 subfactors are precisely those type II subfactors admitting a unique linear functional in the sense of theorem 2.1.1.2 (as $Z(M) \cong \mathbb{C}$ for such factors). However, it should be mentioned that II_∞ factors admit some kind of faithful trace of their own, although it is only unique up to rescaling ([JS97]).

Type III factors were also further categorized by Connes into the types III_λ, for $0 \leq \lambda \leq 1$, in his Ph.D. dissertation ([Con73]). Many of the factors that arise naturally in quantum field theory and quantum statistical mechanics are of type III, and in particular type III_1. While the theory of these type III_1 factors is somewhat more involved than that of type II_1, in the context of chiral conformal field theory (where subfactors are typically "finite-depth") they are actually more or less equivalent ([Pen14], [She12]), and hence covering the type II_1 case is sufficient.

Remark 2.1.1.5. If we think back to the definition of a factor for a moment, we see that being a factor is in some sense the "opposite" of being commutative, as by definition the only only commutative component is the underlying field itself, the trivial one-dimensional subalgebra $\mathbb{C}$. Since every commutative von Neumann algebra is isomorphic to $L^\infty(\Gamma)$ for some measure space Γ ([Sak71, proposition 1.18.1]), we can heuristically liken the general theory of von Neumann algebras to some kind of "non-commutative" or "quantum" analogue of measure theory.

Example 2.1.1.6. (Group von Neumann Algebras). Suppose G is a discrete group, and define an action of G on $\ell^2(G)$ by $(g \cdot \xi)(h) := \xi(g^{-1}h)$, for all $g, h \in G$ and $\xi \in \ell^2(G)$. We define the *group von Neumann algebra* of a discrete group G to be $\mathcal{L}G := (\lambda_G(G))''$, where $\lambda_G : G \to \mathcal{U}(\ell^2(G))$ is the *left regular representation of* G *on* $\ell^2(G)$ given by $g \mapsto (\xi \mapsto g \cdot \xi)$. We have that $\mathcal{L}G$ is not only a factor, but a type II_1 factor, if and only if G is an infinite conjugacy class (ICC) group ([JS97, §1.4.1]); that is, a group whose non-trivial conjugacy classes $C(g) := \{hgh^{-1} : h \in G\}$, for $g \neq 1_G$, are all infinite. Some well-known examples of ICC groups are the free groups F_n on $n > 1$ generators and the infinite symmetric group S_∞ of permutations of $\mathbb{N}$ that fix all but finitely many integers.

Remark 2.1.1.7. We remark that $\mathcal{L}S_\infty$ is one way of expressing the *hyperfinite* type II_1 factor R. By hyperfinite, we mean that it is *approximately finite-dimensional*, in the sense that it can be written as the direct limit of a sequence of finite-dimensional von Neumann algebras (direct sums of matrix algebras). A theorem of Connes showed that this factor is in fact unique up to isomorphism, and also admits a variety of equivalent characterizations ([Con76]). It turns out that hyperfiniteness is a rather natural assumption in the world of conformal field theory, and in particular the hyperfinite factor of type III_1 commonly appears in this context.

Example 2.1.1.8. (Fundamental Group). Let M be a factor of type II_1 on H. Then the tensor product $M \otimes \mathcal{B}(H)$ of "infinite matrices over M" is a factor of type II_∞. If τ is any of the aforementioned traces on this II_∞ factor, then we have a group $\mathcal{F}(M) := \{\lambda \in \mathbb{R} : \text{there exists } \alpha \in \mathrm{Aut}(M \otimes \mathcal{B}(H)) \text{ with } \tau \circ \alpha = \lambda\tau\}$ called the *fundamental group of* M, encoding the traces of projections p for which $M \cong pMp$.

Example 2.1.1.9. (Crossed Products). Murray and von Neumann considered type III factors to be pathological, and were originally unable to determine if they truly existed or not ([MN36]). Of course, these "unicorn" factors do certainly exist; in particular, every type III factor is isomorphic to a *crossed product* $M \rtimes_\alpha \mathbb{R}$ of some type II_∞ factor M by a certain action $\alpha : \mathbb{R} \curvearrowright M$ ([Tak03, theorem XII.1.1]). To be precise, if we define a homomorphism $\widetilde{\alpha} : M \to L^\infty(\mathbb{R}, M) \cong M \otimes L^\infty(\mathbb{R})$ by $\widetilde{\alpha} : x \mapsto ((\alpha(z))^{-1}(x))_{z \in \mathbb{R}}$, this crossed product is defined as the von Neumann subalgebra of $M \otimes \mathcal{B}(L^2(\mathbb{R}))$ generated by $\widetilde{\alpha}(M)$ and $1_M \otimes \lambda_\mathbb{R}(\mathbb{R})$, where now $\lambda_\mathbb{R}$ is the left regular representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. These crossed product constructions, while technical, turn out to be an excellent way of constructing factors of types II and III! More details can be found in [Con94, appendix A], [Tak03, p. X.1] and [JS97, §1.4.2], as well as the unfinished notes of Jones ([Jon15]).

2.1.2. Subfactor Invariants

We now present some important invariants of *type II_1 subfactors*; that is, invariants of unital inclusions of type II_1 factors. For the rest of this section, we will focus our attention towards such subfactors, as their property of admitting a unique tracial state allows us to perform a canonical GNS construction. This construction is a well-known fact from the theory of C^*-algebras (norm-closed, self-adjoint subalgebras of $\mathcal{B}(H)$), a generalization of the theory of von Neumann algebras, as every von Neumann algebra is a C^*-algebra (since the weak operator topology is naturally coarser than the norm topology). We reproduce it here for completeness.

Definition 2.1.2.1. (Gelfand-Naimark-Segal Construction; [GN43], [Seg47]). *Let A be a C^*-algebra and $\tau : A \to \mathbb{C}$ a state. Then we have the following construction.*

(i). *The set $Z_\tau := \{x \in A : \tau(x^*x) = 0\}$ is a closed left ideal in A.*

(ii). *The bilinear form defined by $\langle [x], [y] \rangle = \tau(y^*x)$, for equivalence classes $[x], [y] \in A/Z_\tau$, is well-defined and non-degenerate on A/Z_τ. Moreover, the Cauchy completion of A/Z_τ with respect to the norm induced by this inner product is a Hilbert space, denoted $H_\tau := \overline{A/Z_\tau}$.*

(iii). *The vector $\xi_\tau := [1_A] \in A/Z_\tau$ is a cyclic vector for π_τ with $\|\xi_\tau\| = 1$.*

(iv). *For all $x \in A$, the map defined by $[y] \mapsto [xy]$ for all $y \in A$ extends to a unique bounded, linear operator $\pi_\tau(x) \in \mathcal{B}(H_\tau)$ with $\tau(x) = \langle \pi_\tau(x)\xi_\tau, \xi_\tau \rangle$.*

(v). *The function $\pi_\tau : A \to \mathcal{B}(H_\tau)$ is a $*$-representation of A on H_τ, and is faithful (injective) precisely when τ is faithful..*

In other words, given a state τ, we can generate a $*$-representation π_τ along with a cyclic vector ξ_τ. The triple $(H_\tau, \pi_\tau, \xi_\tau)$ is known as the GNS representation *of τ.*

Remark 2.1.2.2. Every type II_1 factor M has a unique tracial state τ that is faithful. Thus $Z_\tau = \{0\}$, whence H_τ is the norm-completion of M under $\|x\|_2 = \sqrt{\tau(x^*x)}$. We therefore identify the M-bimodule $H_\tau = L^2(M, \tau)$. When it is clear from the context, we will write $L^2(M)$ instead. The representation π_τ describing the action of M on $L^2(M)$ is known as the *standard representation* or *standard form* of M.

Definition 2.1.2.3. (Jones' Basic Construction). *Let $N \subseteq M$ be an inclusion of type II_1 factors and let $e_N \in \mathcal{B}(L^2(M))$ be the orthogonal projection of $L^2(M)$ onto $L^2(N)$, taking M to N (known as a* Jones projection*). Identifying M with a subalgebra of $\mathcal{B}(L^2(M))$ via π_τ, we have that $\langle \pi_\tau(M), e_N \rangle := (\pi_\tau(M) \cup \{e_N\})''$ forms a von Neumann algebra. This process of constructing $\langle \pi_\tau(M), e_N \rangle$, and hence the tower of subfactors $\pi_\tau(N) \subseteq \pi_\tau(M) \subseteq \langle \pi_\tau(M), e_N \rangle$, is called the* basic construction.

Remark 2.1.2.4. We will (at first, rather confusingly!) be using M and N to refer to $\pi_\tau(M)$ and $\pi_\tau(N)$, respectively; this is justified due to the standard representation providing a faithful representation of M on $\mathcal{B}(L^2(M))$. This notation becomes practical with large towers of subfactors of the form $M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots$, where $M_{-1} := N$, $M_0 := M$ and $M_{i+1} := \langle M_i, e_{M_{i-1}} \rangle$. Note that these can be obtained canonically by repeated application of the basic construction; as a result, such towers are also invariants of the inclusion $N \subseteq M$.

Another important invariant is the *index*. We first, however, mention conditional expectations, as these are useful for understanding indices in the general setting.

Definition 2.1.2.5. (Conditional Expectation). *Let $N \subseteq M$ be a unital inclusion of von Neumann algebras (that is, $1_N = 1_M$). A* conditional expectation from M to N *is a unit-preserving linear map $E : M \to N$ such that*

(i). *(positive) for all $m \in M$, there exists $n \in N$ such that $E(m^*m) = n^*n$;*
(ii). *(N-N-bimodule map) $E(n_1 m n_2) = n_1 E(m) n_2$, for all $n_1, n_2 \in N, m \in M$.*

A result of Umegaki shows that, given a normal, faithful, unit-preserving trace τ on a von Neumann algebra M, then for any von Neumann subalgebra $N \subseteq M$ we can construct a unique conditional expectation $E : M \to N$ that preserves the trace, in the sense that $\tau \circ E = \tau$ ([Ume54]). In particular, if e_N is a Jones projection from the basic construction, then it is easy to see that the restriction of e_N to M is precisely this unique, trace-preserving conditional expectation, denoted E_N.

Definition 2.1.2.6. (Index). *Let $N \subseteq M$ be an inclusion of type II$_1$ factors. Then by [PP86], M is a finitely-generated projective N-module if and only if $\langle M, e_N \rangle$ is finite (and hence of type II$_1$ by definition, admitting a unique tracial state $\tau_{\langle M, e_N \rangle}$). In this case, we say that N is of* finite index *in M, and define the* index $[M : N]$ of N in M *to be* $[M : N] := (\tau_{\langle M, e_N \rangle}(e_N))^{-1} \in \mathbb{R}$. *Otherwise, we write* $[M : N] := \infty$.

Remark 2.1.2.7. The index is often defined using the *von Neumann dimension* of M with respect to a separable M-module N, in which case $[M : N] := \dim_N(L^2(M))$. However, if N is of finite index in M and $E_M : \mathcal{B}(L^2(M)) \to \mathcal{B}(L^2(N))$ is the conditional expectation corresponding to the restriction of $e_M \in \mathcal{B}(H_{\tau_{\langle M, e_N \rangle}})$ to $\mathcal{B}(L^2(M))$, then we have the *Markov property* $E_M(e_N) = \tau_{\langle M, e_N \rangle}(e_N) \cdot 1_{\mathcal{B}(L^2(M))}$ ([Spe16, §3]). In other words, the language of conditional expectations is sufficient for characterizing the index, and moreover allows the index to be generalized to other types of factors, such as in the case of the Kosaki index ([Kos86]). A wonderful summary of index theory for arbitrary factors is given in [Loi92].

Remark 2.1.2.8. We can actually take the process of constructing towers of type II$_1$ subfactors one step further due to [JS97, lemma 5.1.1]. Given an inclusion $M_0 \subseteq M_1$ of type II$_1$ factors with finite index $[M_1 : M_0] < \infty$, there exists a type II$_1$ subfactor M_{-1} such that $M_0 = \langle M_{-1}, e_0 \rangle$ via the basic construction. In essence, we can actually construct an entire "tunnel" of type II$_1$ subfactors of constant index. Note, however, that while the tower given by the basic construction is canonical, the subfactors $\{N_{-i}\}_{i \geq 1}$ given by the previous lemma are not necessarily unique. That being said, it is still possible to derive from such a tunnel another invariant.

We briefly mention some facts regarding the index. In the context of the basic construction, if $M \subseteq \langle M, e_N \rangle$ is a type II$_1$ subfactor, then $[\langle M, e_N \rangle : M] = [M : N]$ ([JS97, proposition 3.1.2(iv)(a)]). In other words, using the previous notation, we have that if M_1 is a type II$_1$ subfactor, then the entire tower $M_{-1} \subseteq M_0 \subseteq M_1 \subseteq \cdots$ consists strictly of type II$_1$ subfactors, each with index $[M_i, M_{i-1}] = [M_0, M_{-1}]$.

Suppose we now consider the Jones projections $e_i := e_{M_{i-2}}$ obtained from the basic construction. Then [JS97, proposition 3.2.2] tells us that $e_i e_{i\pm 1} e_i = [M : N]^{-1} e_i$ for all e_i, and that $e_i e_j = e_j e_i$ for all $|i - j| \geq 2$. These properties, along with the behaviour of these orthogonal projections under the trace, were used by Jones to obtain a rather surprising result; in particular, Jones was able to *completely categorize every possible index of II₁ subfactors* ([Jon83]). That is, if $N \subseteq M$ is an inclusion of type II_1 factors, then $[M : N] \in \{4\cos^2(\pi/n) : n \geq 3, n \in \mathbb{N}\} \cup [4, \infty]$.

Moreover, we may define a subalgebra $\mathscr{A}_{n,\tau}$ of M_n generated by $\{1, e_1, e_2, \ldots, e_n\}$. When $[M : N] \geq 4$, then this subalgebra is isomorphic to a well-known algebra; namely, the *Temperley-Lieb-Jones algebra* $TL_n(\delta)$, with $\delta = \sqrt{[M : N]}$. For other values of $[M : N]$, it is isomorphic to a *quotient* of this algebra ([Jon91]). The union $\bigcup_{i \geq 0} M_i$ of subfactors emerging from the basic construction admits a unique tracial state tr, whence we have a type II_1 factor $M_\infty := (\pi_{\mathrm{tr}}(\bigcup_{i \geq 0} M_i))''$. The algebra $\mathscr{A}_\tau$ generated by each $\mathscr{A}_{n,\tau}$ is in fact a type II_1 subfactor of M_∞ by [Jon91, §3.6].

Definition 2.1.2.9. (Standard Invariant). *Let $N \subseteq M$ be an inclusion of type II₁ factors of finite index, and construct a tower of type II₁ subfactors via the basic construction. Then the grid of finite-dimensional C^*-algebras*

$$\begin{array}{ccccccccc}
\mathbb{C} = N' \cap N & \subseteq & N' \cap M & \subseteq & N' \cap M_1 & \subseteq & N' \cap M_2 & \subseteq & \cdots \\
 & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
 & & \mathbb{C} = M' \cap M & \subseteq & M' \cap M_1 & \subseteq & M' \cap M_2 & \subseteq & \cdots
\end{array}$$

is invariant of the inclusion $N \subseteq M$. The trace-preserving isomorphism class of this grid is known as the standard invariant *of $N \subseteq M$, and the individual squares in the grid are known as the* canonical commuting squares *of $N \subseteq M$. The finite-dimensional C^*-algebras given by intersections of type II₁ subfactors are known as* relative commutants.

Remark 2.1.2.10. Note that finite-dimensional C^*-algebras are nothing but finite-dimensional von Neumann algebras, which are just direct sums of matrix algebras.

Remark 2.1.2.11. To illustrate what is meant by "trace-preserving isomorphism class", suppose we construct any tunnel $\{N_{-i}\}_{i \geq 1}$ by [JS97, lemma 5.1.1]. Then regardless of the tunnel we choose, after fixing some n, it is actually possible to construct a trace-preserving anti-isomorphism of the commuting squares

$$\begin{pmatrix} M' \cap M_n & \subseteq & M' \cap M_{n+1} \\ \cup\!\mid & & \cup\!\mid \\ M_1' \cap M_n & \subseteq & M_1' \cap M_{n+1} \end{pmatrix} \overset{\text{anti-isomorphism}}{\cong} \begin{pmatrix} M \cap N'_{-(n-1)} & \subseteq & M \cap N'_{-n} \\ \cup\!\mid & & \cup\!\mid \\ N \cap N'_{-(i-n)} & \subseteq & N \cap N_{-n} \end{pmatrix},$$

such that each N_{-i} gets mapped onto M_{i+1} for all $1 \leq i \leq n$. Thus the grid

$$\begin{array}{ccccccccc}
\mathbb{C} = M \cap M' & \subseteq & M \cap N' & \subseteq & M \cap N'_{-1} & \subseteq & M \cap N'_{-2} & \subseteq & \cdots \\
 & & \cup\!\mid & & \cup\!\mid & & \cup\!\mid & & \\
 & & \mathbb{C} = N \cap N' & \subseteq & N \cap N'_{-1} & \subseteq & N \cap N'_{-2} & \subseteq & \cdots
\end{array}$$

is also invariant of $N \subseteq M$, once again regardless of the chosen tunnel ([JS97, §5.6]).

Definition 2.1.2.12. (Extremal Subfactor). *Let $N \subseteq M$ be a type II_1 subfactor with finite index. Then N is said to be* extremal *if the unique tracial states on N' and M (identified as subalgebras of $\mathcal{B}(L^2(M))$) coincide on $N' \cap M$. This implies that the unique traces on N' and M_n coincide on $N' \cap M_n$ for all n ([PP86], [PP88]).*

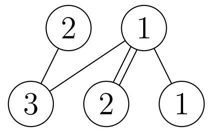
We finally mention one more important invariant of subfactor inclusions; namely the principal graph and its corresponding dual principal graph. In order to understand what the principal graph is, we shall take a brief detour with Bratteli diagrams.

Suppose we have an inclusion $A \subseteq B$ of finite-dimensional C^*-algebras. The *Bratteli diagram* of this inclusion is the bipartite graph whose even vertices are the irreducible representations of A and whose odd vertices are the irreducible representations of B. The number of edges between the ith even vertex and the jth odd vertex correspond to the number of copies of the ith irreducible representation of A in the jth irreducible representation of B's restriction to A. While this initially sounds a bit scary, it is indeed a fact that every finite-dimensional C^*-algebra is semisimple, whence the Artin-Wedderburn theorem allows us to easily classify its irreducible representations. With this in mind, we can perhaps best illustrate what a Bratteli diagram is using the following reproduction of [JS97, example 3.2.1].

Example 2.1.2.13. Let $B \coloneqq M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$ and

$$A \coloneqq \left\{ \left(\begin{pmatrix} X & 0 \\ 0 & y \end{pmatrix}, \begin{pmatrix} y & 0 \\ 0 & y \end{pmatrix}, y \right) : X \in M_2(\mathbb{C}), y \in \mathbb{C} \right\},$$

so that $A \cong M_2(\mathbb{C}) \oplus M_1(\mathbb{C})$. From here, we may construct the Bratteli diagram at a glance. We see that by restricting the representation $M_3(\mathbb{C})$ to A (that is, by restricting the first component of B to matrices of the form of the first component of A), we have one copy of $M_2(\mathbb{C})$ contributed by X and one copy of $\mathbb{C}$ contributed by y. Similarly, the restriction of $M_2(\mathbb{C})$ has two copies of $M_1(\mathbb{C})$ (given by the two y's along the diagonal), while the restriction of $M_1(\mathbb{C})$ has one copy of $M_1(\mathbb{C})$. Thus the Bratteli diagram of $A \subseteq B$ is given by



,

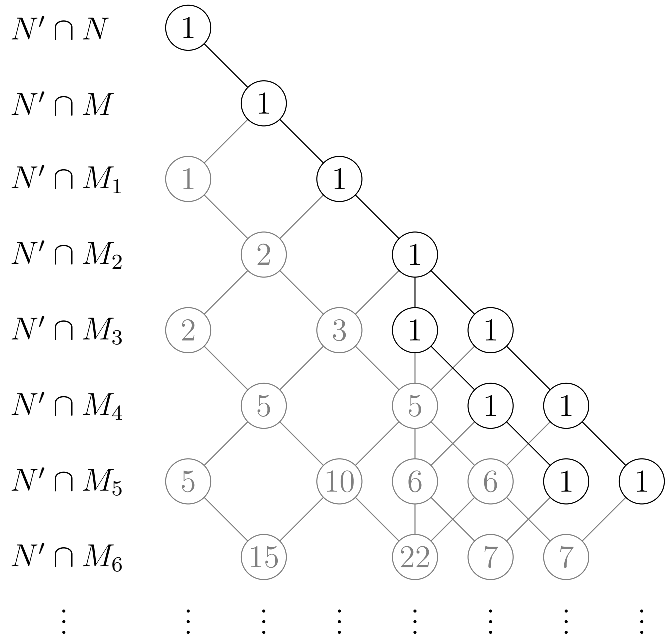
where the numbers on the vertices correspond to the dimensions of the irreducible representations (that is, the dimensions of the matrix algebras). Note that the multiplicities of the vertices on the bottom row are the sums of the multiplicities of the vertices they are connected to. We can extend our definition of a Bratteli diagram to towers of inclusions of the form $A \subseteq B \subseteq C \subseteq \cdots$ as well. Moreover, our observation about the multiplicities holds true in such cases.

As a consequence of [JS97, proposition 2.3.12], the relative commutants in the standard invariant are in fact finite-dimensional C^*-algebras. As a result, we may construct a Bratteli diagram for each inclusion. However, note that because we have the inclusions $e_n \in M'_{n-3} \cap M_{n-1}$ of the Jones projections, $N' \cap M_{n+1}$ contains a copy of the basic construction for the inclusion $(N' \cap M_{n-1}) \subseteq (N' \cap M_n)$.

As a result, we observe a kind of "reflection symmetry" in the Bratteli diagram; that is, the Bratteli diagram for $(N' \cap M_n) \subseteq (N' \cap M_{n+1})$ will contain a "reflection" of the Bratteli diagram for $(N' \cap M_{n-1}) \subseteq (N' \cap M_n)$. We will see what this means shortly.

With the reflection symmetry in mind, we define the *principal graph* invariant of $N \subseteq M$ to be the Bratteli diagram for the tower $\{N' \cap M_n : n \geq -1\}$ of relative commutants, where at stage $(n+1)$ we remove the subgraph given by a reflection of the $(n-1)$th stage about the nth stage. Similarly, the *dual principal graph* is the same idea, but instead for the tower $\{M' \cap M_n : n \geq 0\}$. So far, this definition of a principal graph is all rather abstract; as a result, we will demonstrate precisely what we mean with an example. It is actually trivial to extract from a Bratteli diagram of a subfactor its principal graph, which we shall demonstrate imminently.

Example 2.1.2.14. (Haagerup Subfactor). There exists a subfactor, known as the *Haagerup subfactor*, admitting the following Bratteli diagram:



.

The dark, outer subgraph is precisely the principal graph; note that by reflecting the $(n-1)$th line about the nth line, the result is contained within the $(n+1)$th line. This is what is meant by reflection symmetry. As it happens, this subfactor is of *finite depth* (meaning that its principal graph is finite), as we see that any additional lines will be forced to alternate between the last two lines. This subfactor is actually the unique subfactor with index $\frac{5+\sqrt{13}}{2}$, and is the finite-depth subfactor with the smallest index greater than 4 ([AH99]).

The fact that this Bratteli diagram admits a subfactor is exceptionally non-trivial! Indeed, most Bratteli diagrams are not given by subfactors in this way. However, once we have a principal graph, we can always reconstruct the entire Bratteli diagram, which essentially recovers for us the standard invariant. As a result, it is very common to use the principal graph invariant to represent difficult subfactors.

Early in the history of subfactors, during the 1980s, it was shown that subfactors with quantized index – that is, index less than 4 – are completely classified by their principal graphs, and moreover that these principal graphs are nothing but the Dynkin diagrams A_n, D_{2n}, E_6 and E_8 ([Ocn88]). It was later shown by Popa in [Pop94] that the principal graphs of finite-depth subfactors (subfactors with finite principal graphs) of index exactly 4 are the extended Dynkin diagrams. However, subfactors with small indices greater than 4 have remained largely mysterious. Traditionally, such subfactors have arisen from other mathematical objects, such as groups or quantum groups. Subfactors that do not arise from these familiar objects are often referred to colloquially as *exotic* subfactors.

In 1993, Haagerup proposed a list of candidates for subfactors with small index greater than 4, and later showed that there is a unique subfactor with index $\frac{5+\sqrt{13}}{2}$. This subfactor, known as the *Haagerup subfactor*, is the finite-depth subfactor with the smallest index greater than 4. Asaeda and Haagerup later used a similar process to find the *second*-smallest exotic subfactor, the *Asaeda-Haagerup subfactor*, which is the unique subfactor with index $\frac{5+\sqrt{17}}{2}$ ([AH99]). These two subfactors currently remain rather rare examples of exotic subfactors.

2.2. Planar Algebras

Planar algebras emerged in the 1999 paper of Jones – which was, recently, finally published ([Jon21]) – as a new axiomatization of the standard invariant. They were designed not only for the purpose of studying subfactors, but to provide a rigorous diagrammatic framework for doing subfactor theory, as well as to allow for the construction of new examples of subfactors and actions on them. Because the standard invariant is highly admissible to axiomatizations in the languages of other areas of mathematics, such as knot theory, category theory and conformal and algebraic quantum field theory, care was taken in defining planar algebras such that they would be compatible in these contexts as well ([Jon19]).

2.2.1. Definition of a Planar Algebra

Definition 2.2.1.1. (Planar Tangle). *A* planar tangle T *consists of*

(i). *a smooth* output disc $D_T \subset \mathbb{R}^2$*;*

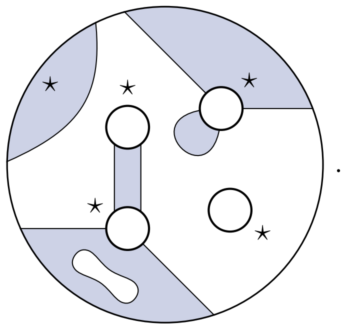
(ii). *a finite set* $\mathfrak{D}_T$ *of smooth, disjoint* input discs *that are each contained within the interior of* D_T*;*

(iii). *a finite set* $\mathfrak{S}_T$ *of smooth, disjoint curves contained within the interior of* D_T *– called* strings *– which intersect neither the output disc* D_T *nor the input discs* $D \in \mathfrak{D}_T$ *apart from at their endpoints (if they have any), which are required to lie on the boundaries of these discs.*

For each disc $D \in \{D_T\} \cup \mathfrak{D}_T$*, the points on the boundary of* D *which meet an endpoint of some string in* $\mathfrak{S}_T$ *are known as the* boundary points *of* D*, where* n_D *denotes the number of boundary points of* D*. Letting* $n := n_{D_T}$*, we may call* T *a* planar n-tangle*. Note that a disc's boundary points partition its boundary into* intervals*; we associate with each disc a* marked interval*, denoted by writing a* $\star$ *near the chosen interval. This orders the boundary points in a clockwise order, starting from the mark.*

Definition 2.2.1.2. (Shaded Planar Tangle). *Let T be a planar tangle. The strings and input discs partition the output disc into a disjoint set of connected components, known as* regions. *If each disc has an even number of boundary points, we may colour each region as either shaded or unshaded, such that no two adjacent regions share the same colouring. In such a case, we say that T is* shaded *(otherwise, we say that T is* vanilla*). The discs of T are coloured as follows. We say that a disc is of kind "$-$" if its marked interval borders a shaded region, and of kind "$+$" if its marked interval borders an unshaded region.*

Example 2.2.1.3. Pictured below is a shaded planar 6-tangle of kind "$-$":



Note the location of the mark ($\star$) beside each disc, as well as that the output disc itself has its own mark that lies, in this instance, within a shaded region.

Definition 2.2.1.4. (Oriented Planar Tangle). *A planar tangle is* oriented *if all of its strings are oriented; that is, all of its strings have a specified direction.*

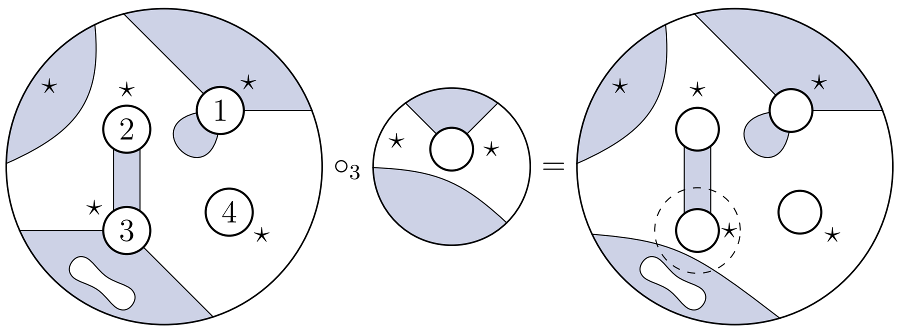
We will let $\delta(D)$ denote the boundary data of a disc D. This depends on the type of planar tangle we're dealing with; for vanilla planar tangles, we write $\delta(D) := n$ for a disc D with n boundary points. For shaded planar tangles, however, we write $\delta(D) := (n, \pm)$ for a disc D of kind "$\pm$" with n boundary points. Finally, for oriented planar tangles, the boundary data $\delta(D)$ of a disc D with n boundary points is given by a function $\alpha_D : \{1, \ldots, n\} \to \{\uparrow, \downarrow\}$, where i maps to $\uparrow$ if the string attached to the ith boundary point of D exits the ith boundary point and i maps to $\downarrow$ otherwise.

Definition 2.2.1.5. (Planar Operad). *A* planar operad $\mathbb{P}$ *is an* operad *(in the sense of [May72]; loosely speaking, a sequence of objects and a collection of n-ary operations between them for each $n \in \mathbb{N}$, along with some notion of composition) whose objects $\mathbb{P}_\iota$ are orientation-preserving diffeomorphism classes of planar tangles with boundary data ι. We denote by $\mathfrak{B}_\mathbb{P}$ the set of boundary data for planar tangles, and denote by $\mathfrak{B}_{\mathbb{P},n}$ the subset of $\mathfrak{B}_\mathbb{P}$ with n boundary points assumed.*

In the case of the operad over vanilla planar tangles, we of course have that $\mathfrak{B}_\mathbb{P} = \mathbb{N}$. For the operad over shaded planar tangles, $\mathfrak{B}_\mathbb{P} = \mathbb{N} \times \{-, +\}$. Finally, for the operad over oriented planar tangles, $\mathfrak{B}_{\mathbb{P},n}$ is the set of functions from $\{1, \ldots, n\}$ to $\{\uparrow, \downarrow\}$. Of course, for planar tangles that are both shaded and oriented, we just let $\mathfrak{B}_\mathbb{P}$ be the Cartesian product of the previous set of boundary data with $\{-, +\}$.

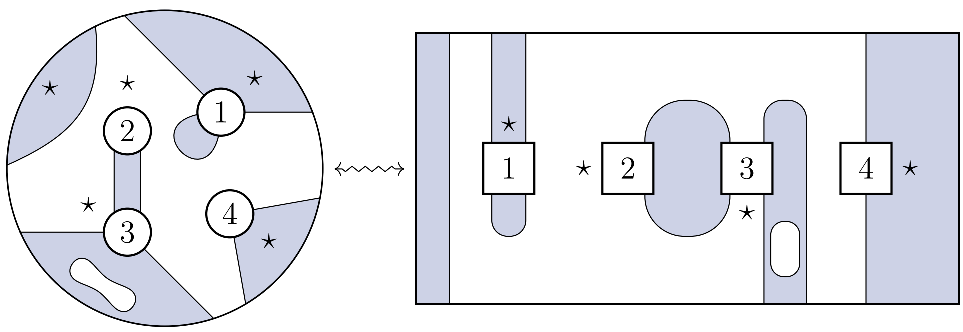
In order to explain how the operadic structure on $\mathbb{P}$ arises from composition, we present an example using shaded planar tangles. The logic carries over easily to operads over other types of planar tangles. Given a planar n-tangle T whose jth input disc is of kind "$\pm$" with m boundary points and a planar m-tangle S of kind "$\pm$", we define $T \circ_j S$ to be the planar n-tangle appearing as T with its jth input disc replaced by S, such that the marked intervals match up. An example of such a composition is presented below.

Example 2.2.1.6. Consider the following composition of planar tangles:



Notice that the marked interval of the third disc in the leftmost planar tangle has been matched up with the marked interval of the operand, keeping the orientation of the inserted disc from being arbitrary. Keep in mind, however, that due to how we define a planar operad, the actual position of the mark on this interval and the arc length of the interval do not matter at all.

Example 2.2.1.7. We can also represent shaded planar $2n$-tangles with rectangles; starting from the left side of the rectangle, we go clockwise around it, putting the first n boundary points (clockwise from the mark) on the top and the remaining n points on the bottom.



Remark 2.2.1.8. Before we define a planar algebra, we make the following remark. Let S and P_s, for each $s \in S$, be sets. Then we may identify the Cartesian product $\bigtimes_{s\in S} P_s$ with the set of functions of the form $f : S \to \bigcup_{s\in S} P_s$, such that $f(s) \in P_s$ for all $s \in S$. With this in mind, consider the following definitions of [Jon19].

Definition 2.2.1.9. (Labelled Tangle). *We call an element* $f \in \bigtimes_{D\in\mathfrak{D}_T} P_{\partial(D)}$ *a* labelling of the planar tangle T, *in which case* T *is called a* labelled planar tangle.

Definition 2.2.1.10. (Planar Algebra). *A* planar algebra P *is a representation of a planar operad* $\mathbb{P}$*; in other words, it is a family of vector spaces* P_ι *(or, more generally, sets) indexed by* $\iota \in \mathfrak{B}_P := \mathfrak{B}_{\mathbb{P}}$*, together with multilinear maps*

$$Z_T : \bigtimes_{D \in \mathfrak{D}_T} P_{\partial(D)} \to P_{\partial(D_T)}$$

for each planar tangle T *with* $\mathfrak{D}_T$ *non-empty, such that*

(i). *if* θ *is some orientation-preserving diffeomorphism of* $\mathbb{R}^2$*, then*

$$Z_{\theta(T)}(f) = Z_T(f \circ \theta);$$

(ii). *(naturality)* $Z_{T \circ S} =$ *"*$Z_T \circ Z_S$*"*,

where $Z_T \circ Z_S$ *is defined as follows. Given a function* f *from* $\mathfrak{D}_{T \circ S}$ *to the appropriate vector space, where* $\mathfrak{D}_{T \circ S} = (\mathfrak{D}_T \setminus \{D_S\}) \cup \mathfrak{D}_S$*, we may define a new function* $\tilde{f}$ *from* $\mathfrak{D}_T$ *to this same appropriate vector space by*

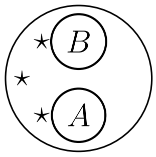
$$\tilde{f} : D \mapsto \begin{cases} f(D), & \text{if } D \neq D_S; \\ Z_S(f|_{\mathfrak{D}_S}), & \text{if } D = D_S. \end{cases}$$

We then define $Z_T \circ Z_S : f \mapsto Z_T(\tilde{f})$*.*

Although this definition may seem a bit abstract, we can think about it as follows. Essentially, the action of a multilinear map Z_T on some element of P_ι formalizes the process of "plugging" elements into the input discs of a planar tangle T to obtain new vectors. In this way, we can think of planar tangles with input discs as representing nothing *but* maps on our vector spaces. The terminology of "input" and "output" discs was specifically chosen due to its evocative nature in this context! Linearity also means that, given some (potentially formal) sum of elements of P_ι, we may act on it by some appropriate Z_T by acting on each term in the sum individually and then summing the results.

The first axiom, 2.2.1.10.(i), simply allows for compatibility with planar tangles being unique up to orientation-preserving diffeomorphisms (that is, shuffling around the components of a planar tangle without changing the ordering of boundary points or shading of intervals does not change the tangle). Meanwhile, 2.2.1.10.(ii) just means that composing the multilinear maps should be exactly the same as the pictorial composition of planar tangles as one would expect. Thus this definition is actually quite natural!

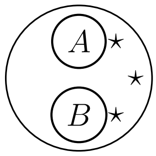
Example 2.2.1.11. To drive this point home, consider [Jon19, lemma 2.2.3]. Let T be a stringless planar tangle with $D_T = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $\mathfrak{D}_T = \{A, B\}$, where $A = \{(x, y) \in \mathbb{R}^2 : x^2 + (y + 1/2)^2 \leq 0.1\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x^2 + (y - 1/2)^2 \leq 0.1\}$. In other words,

$T :=$ 

.

We are asked to show that, if P is a planar algebra, then P_0 becomes a commutative, associative algebra with multiplication $ab = Z_T(f)$, where $f(A) = a$ and $f(B) = b$.

Recall that a planar algebra P will associate with T a multilinear map Z_T that sends all functions of the form $f : \{A, B\} \to P_0$ to P_0. If we define a bilinear product on the vector space P_0 by $ab := Z_T(f_{ab})$, where f_{ab} is the function with $f_{ab}(A) = a$ and $f_{ab}(B) = b$ for $a, b \in P_0$ (or, equivalently, the pair $(a, b) \in P_0 \times P_0$), then P endows P_0 with the structure of an algebra. Suppose we now let θ be the rotation of the plane by π radians. Then because A and B are simply subsets of $\mathbb{R}^2$, we have

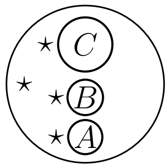
$\theta(T) =$ 

and
$$f_{ab}(\theta(A)) = f_{ab}(B) = b,$$
$$f_{ab}(\theta(B)) = f_{ab}(A) = a.$$

Because the planar operad demands that $T = \theta(T)$, as they are clearly in the same orientation-preserving diffeomorphism class, 2.2.1.10.(i) tells us that

$$ab = Z_T(f_{ab}) = Z_{\theta(T)}(f_{ab}) = Z_T(f_{ab} \circ \theta) = Z_T(f_{ba}) = ba,$$

giving us the commutativity of P_0. Suppose now that we consider the composition $T' \circ_X T$, where T' is a relabelling of T with X replacing A and C replacing B. Then

$T' \circ_X T =$ 

.

Given a function $g_{abc} : \{A, B, C\} \to P_0$ such that $g_{abc}(A) = a$, $g_{abc}(B) = b$ and $g_{abc}(C) = c$, we have by 2.2.1.10.(ii) that

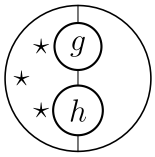
$$\tilde{g}_{abc}(X) = Z_T(g_{abc}|_{\mathfrak{D}_T}) = Z_T(f_{ab}) = ab \quad \text{and} \quad \tilde{g}_{abc}(C) = c,$$
$$\implies Z_{T' \circ_X T}(g_{abc}) = [Z_{T'} \circ_X Z_T](g_{abc}) = Z_{T'}(\tilde{g}_{abc}) = (ab)c.$$

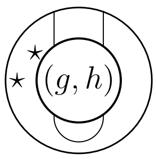
However, $T' \circ_X T$ is in the same diffeomorphism class as $T''' \circ_X T''$, where T''' is a relabelling of T with $B \mapsto X$ and T'' is a relabelling with $A \mapsto C$ and $B \mapsto A$. Appealing to 2.2.1.10.(i) gives us $(ab)c = Z_{T' \circ_X T}(g_{abc}) = Z_{T''' \circ_X T''}(g_{abc}) = a(bc)$.

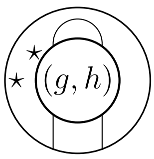
2.2.2. Examples of Planar Algebras

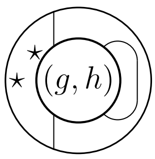
We now briefly introduce some key examples of planar algebras. Most of these examples will benefit from some additional terminology being introduced; as a result, we will often refer to definitions from the next section.

Example 2.2.2.1. Here is nice, albeit contrived, example of a planar algebra for a group G. Let $P_{2n} := \mathbb{C}G^n$ be the group algebra of G^n, the direct product consisting of n copies of G, for $n \in \mathbb{N}$. We define a planar algebraic structure on this as follows.



$:= hg,$ 

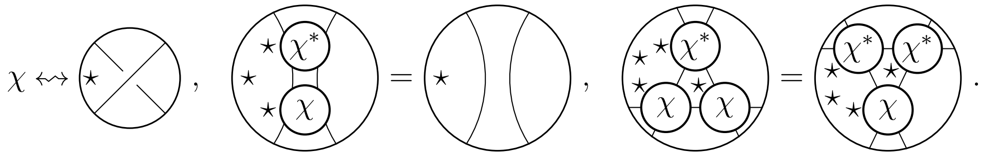
$:= hg,$ 

$:= gh,$ 

$:= g.$

Essentially, the algebraic structure is given by following the strings and multiplying to the left by the group elements we meet. The path starting at the first string clockwise from the marked interval gives us the first component of the result, then the next string (skipping where we ended our previous path, if it happens to be directly clockwise from the first string) gives the second component, and so on.

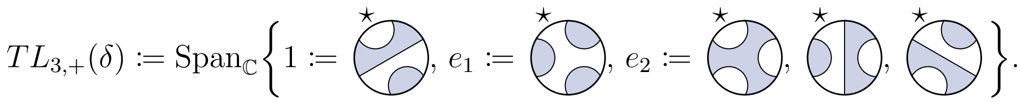
Example 2.2.2.2. (Conway Planar Algebra). One of the most natural examples of a planar algebra is the planar algebra of Conway tangles. We take $P_{2n} \coloneqq C_n$, where C_n is the vector space of n-tangles (in the sense of [Con70]). This planar algebra is generated by the crossing $\chi \in C_2$ under the following Reidemeister relations:



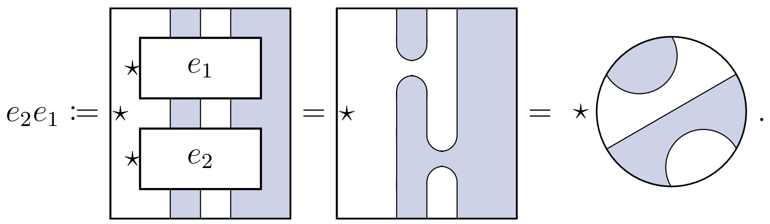
Note that $\chi^* \in C_2$ refers to reflecting vertically the crossing $\chi \in C_2$.

The prototypical example of a planar algebra is the Temperley-Lieb-Jones planar algebra. We will see that it is what is known as a *subfactor planar algebra*; in particular, it is the initial object in the category of subfactor planar algebras, making it arguably the most important example to understand in the context of subfactors!

Example 2.2.2.3. (Temperley-Lieb-Jones Planar Algebra). Suppose we fix a parameter $\delta \in \{2\cos(\pi/n) : n \geq 3, n \in \mathbb{N}\} \cup [2, \infty]$. The *Temperley-Lieb-Jones planar algebra with parameter* δ is derived from the graphical calculus (or *diagram algebra*) of the Temperley-Lieb-Jones algebras $TL_n(\delta)$, introduced by Kauffman in [Kau87]. Let $P_{2n,\pm} \coloneqq TL_{n,\pm}(\delta)$, where $TL_{n,\pm}(\delta)$ is the Temperley-Lieb-Jones algebra on n strings with parameter δ. We interpret $TL_{n,\pm}(\delta)$ as the set of distinct planar tangles of kind $(2n, \pm)$ with no discs or interior loops. For instance,



These five elements are a basis for $TL_{3,+}$, with the span taken over $\mathbb{C}$. The elements 1, e_1 and e_2 in this case correspond to the generators of TL_3; this is easy to see under the lens of example 2.2.1.7, as they reduce to the classical Kauffman diagrams. The remaining two elements are, from left to right, e_1e_2 and e_2e_1, under the canonical product given later in proposition 2.2.3.3. For example,



If we obtain an interior loop (for example, by multiplying e_1 with itself in the aforementioned way), we remove this interior loop and multiply outside by a factor of δ (thus $e_1^2 = \delta e_1$). This relation gives us the Temperley-Lieb-Jones planar algebra.

Example 2.2.2.4. Let $\mathcal{C}$ be a locally small, strict, linear, pivotal category, and consider some distinguished symmetrically self-dual generator $X \in \mathrm{Ob}(\mathcal{C})$. We can construct a planar algebra from this data by taking the vector spaces to be the invariants in $X^{\otimes n}$ – that is, $P_n \coloneqq \mathrm{Mor}_{\mathcal{C}}(1_{\mathcal{C}}, X^{\otimes n})$ – and by taking our composition maps Z_T to be induced by the standard graphical calculus of $\mathcal{C}$. In fact, this gives a classification of planar algebras in terms of such categories ([HPT23], [Yam12]).

2.2.3. Subfactor Planar Algebras

We would now like to introduce some adjectives for planar algebras, with the intent of eventually defining what it means to be a "subfactor planar algebra". Suppose we let $\mathfrak{T}_n$ be the set of all planar n-tangles T with no input discs (that is, $\mathfrak{D}_T = \varnothing$), and define $\mathfrak{T} := \bigcup_n \mathfrak{T}_n$. Then we have the following definitions.

Definition 2.2.3.1. (Unital Planar Algebra). *A planar algebra is said to be* unital *if, for each discless planar tangle $S \in \mathfrak{T}$, there is some planar tangle T such that*

(i). *if θ is some orientation-preserving diffeomorphism of $\mathbb{R}^2$, then*

$$Z_T(\theta(S)) = Z_T(S);$$

(ii). *(naturality) $Z_T(S) = Z_T \circ Z_S$,*

where $Z_T \circ Z_S := Z_T(\tilde{f})$ with

$$\tilde{f} : D \mapsto \begin{cases} f(D), & \text{if } D \neq D_S; \\ Z_T(S), & \text{if } D = D_S. \end{cases}$$

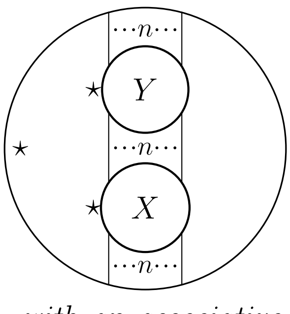
Definition 2.2.3.2. ($*$-Planar Algebra). *A planar algebra P is said to be a* $*$-planar algebra *if each vector space P_ι admits a conjugate-linear involution $*$ such that if θ is an orientation-reversing diffeomorphism of $\mathbb{R}^2$, then*

$$(Z_{\theta(T)}(f))^* = Z_T((f \circ \theta)^*).$$

In practice, the adjoint of a tangle is given by reflecting it vertically and reversing the orientation (as reflecting will reverse the orientation itself, so we wish to undo this).

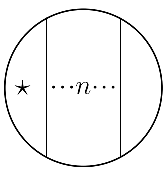
With the previous example in mind, the definition of unital planar algebras should be slightly more clear. If not, then don't worry; we typically won't need such a definition, as shaded planar algebras in particular have very concrete units, as we will see in the following proposition of [Jon19].

Proposition 2.2.3.3. *Let P be a planar algebra. Then any labelled planar $2n$-tangle isotopic to T below defines a multiplication $xy := Z_T(f_{xy})$ on P_{2n}, where*

$T :=$ 

and $\quad f_{xy} : D \mapsto \begin{cases} x, & \text{if } D = X; \\ y, & \text{if } D = Y, \end{cases}$

that endows P_{2n} with an associative algebra structure for each $n \in \mathbb{N}$. Moreover, if P is unital, then the units of P_{2n} are given by

$1_{2n} :=$ 

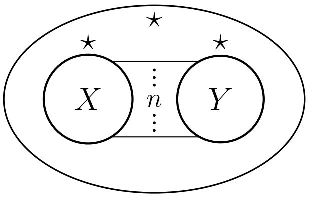
,

and if P is a $*$-planar algebra then each P_{2n} is a $*$-algebra.

The proof for this proposition follows from a straightforward generalization of example 2.2.1.11. Moreover, it applies to other types of planar algebras in the obvious way. In particular, this multiplication endows every vector space $P_{n,\pm}$ of a shaded planar algebra with the structure of an associative algebra.

Definition 2.2.3.4. (C^*-Planar Algebra). *A* C^*-planar algebra *is a $*$-planar algebra P whose associative $*$-algebras P_{2n} all admit norms that turn them into C^*-algebras.*

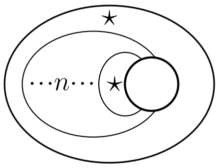
Definition 2.2.3.5. (Measured Planar Algebra). *A planar algebra P is said to be* measured *if there exist non-zero linear maps $\omega_\iota : P_\iota \to \mathbb{C}$ for each $\iota \in \mathfrak{B}_{P,0}$ that are compatible with the composition of planar tangles. Such planar algebras admit a canonical bilinear form $(x, y) \coloneqq \omega(Z_T(f_{xy}))$ on each P_n, where*

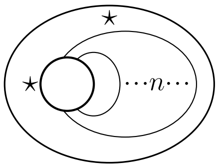
$T \coloneqq$ 

and $\quad f_{xy} : D \mapsto \begin{cases} x, & \text{if } D = X; \\ y, & \text{if } D = Y. \end{cases}$

If P is a $*$-planar algebra, then $\langle x, y\rangle \coloneqq (x, y^*)$ is an inner product on P_n. If this inner product is positive-definite, P is said to be a positive-definite $*$*-planar algebra.*

Definition 2.2.3.6. (Spherical Planar Algebra). *Let P be a measured planar algebra, and consider the planar tangles*

$T_L \coloneqq$ 

, $\quad T_R \coloneqq$ 

.

These are, respectively, known as the left and right ($2n$-)trace tangles. *They define* left and right (Markov) traces $\tau_L \coloneqq \omega \circ Z_{T_L}$ *and* $\tau_R \coloneqq \omega \circ Z_{T_R}$*, respectively, on P_{2n}. The measured planar algebra P is said to be* spherical *if $\tau_L = \tau_R$.*

Once again, these definitions extend easily to other types of planar algebras. In particular, the left and right traces are defined for each $\iota \in \mathfrak{B}_{P,2n}$. Note also that $\tau(xy) = \tau(yx) = (x, y)$; if P is a $*$-planar algebra, then $\langle x, y\rangle = \tau(y^*x)$.

This definition of spherical mirrors nicely the definition of spherical categories. An equivalent, but perhaps more concrete, definition of a spherical planar algebra is a measured planar algebra such that, for all planar 0-tangles T, the corresponding multilinear function $\omega \circ Z_T$ depends only on the isotopy class of T on the 2-sphere compactification of $\mathbb{R}^2$ (that is, the one-point compactification $\mathbb{R}^2 \cup \{\infty\}$ into a sphere, given by the inverse of the stereographic projection). While this initially sounds rather more abstract, we can illustrate it well with an example. If we consider the trace tangles, which are not isotopic in the plane, we see that they *are* isotopically equivalent after being projected onto a 2-sphere; if the disc is projected onto the bottom of the sphere, for instance, we can always isotopically "wrap" the bundle of strings over the top of the sphere and onto the other side of the disc.

Definition 2.2.3.7. (Central Planar Algebra). *A planar algebra P is said to be* central *if* $\dim(P_\iota) = 1$ *for all* $\iota \in \mathfrak{B}_{P,0}$.

Remark 2.2.3.8. Due to [Jon19, proposition 2.6.7], there is a unique bilinear form on unital, central planar algebras that turns them into measured planar algebras.

Definition 2.2.3.9. (Subfactor Planar Algebra). *A* subfactor planar algebra *is a shaded, spherical, central, positive-definite planar algebra.*

Suppose we have an extremal, type II_1 subfactor $N \subseteq M$ of finite index. By taking $P_{n,+} = N' \cap M_{n-1}$ and $P_{n,-} = M' \cap M_n$ from the standard invariant, we obtain a shaded planar algebra where each P_ι is a finite-dimensional C^*-algebra (and in particular, it is central, as $P_{0,\pm} = \mathbb{C}$). Moreover, because the subfactor is extremal, the planar algebra will be spherical and each relative commutant P_ι will admit a canonical positive-definite inner product $\langle x, y\rangle \coloneqq \tau_{P_\iota}(y^* x)$. In other words, all such subfactors give rise to spherical C^*-planar algebras ([Jon21, theorem 4.2.1]), and conversely all spherical C^*-planar algebras give rise to extremal subfactors ([Jon21, theorem 4.3.1]). This correspondence is not necessarily bijective unless we restrict to amenable subfactors, which are in bijection with their standard invariants by [Pop94]. A strengthening of Jones' result was later given in the thesis of Burns ([Bur03]), who gave a canonical construction of rigid C^*-planar algebras from σ-finite subfactors $N \subseteq M$ admitting a normal conditional expectation $E : M \to N$, such that their corresponding Kosaki index (which is equal to the typical Jones index in the case of type II_1 subfactors) is finite, and vice versa. This results in a correspondence between planar algebras and arbitrary type II_1 subfactors.

Example 2.2.3.10. Let us return to the Temperley-Lieb-Jones planar algebra; it forms a subfactor planar algebra as follows. Define our planar $*$-algebra involution by the usual vertical reflection. Both $P_{0,-} \coloneqq \mathbb{C} \cdot \bigcirc$ and $P_{0,+} \coloneqq \mathbb{C} \cdot \bigcirc$ are of course one-dimensional, whence P is central. Because shaded and unshaded circles are both replaced with a factor of δ, we have sphericality. The only tricky condition is positive-definiteness. If $\delta \geq 2$, the canonical bilinear form is positive-definite, giving us a subfactor planar algebra. If instead $\delta = 2\cos(\pi/n)$ for some $n \geq 3$, the bilinear form is positive-semidefinite; we obtain an honest-to-goodness subfactor planar algebra by quotienting by all $x \in TL(\delta)$ such that $\langle x, x\rangle = 0$ ([Pet10]).

This example demonstrates explicitly the relationship between subfactor planar algebras and the standard invariant. We can go between a subfactor planar algebra P and a standard invariant of index $[M : N]$ by identifying $P_{n,+} = N' \cap M_{n-1}$ and $P_{n,-} = M' \cap M_n$, where the index comes from squaring the parameter δ that all subfactor planar algebras inherit due to sphericality ([Liu15]).

Example 2.2.3.11. (Haagerup Planar Algebra). In her Ph.D. thesis, Emily Peters gave a new, planar algebraic proof for the existence of the Haagerup subfactor by finding a subfactor planar algebra with the principal graph H of the Haagerup subfactor within the graph planar algebra corresponding to H ([Pet10]).

2.3. Conformal Field Theory

In the language of physics, a quantum field theory (QFT) refers to the quantum theory of fields, or rather the application of quantum mechanics to classical field theory. It involves the quantization of classical field theories, where "quantization" here refers not to the discretization of fields, but to their interpretation as operators. There are two main approaches; the historic algebraic (or "axiomatic") quantum field theory (AQFT) and the more modern functorial quantum field theory (FQFT). The former, which is sometimes said to formalize the "Heisenberg picture" of quantum mechanics, involves some algebraic structure subject to a set of axioms derived from physics (such as the axioms of [SW64] or [HK64]). The latter, formalizing the "Schrödinger picture", models a field theory as a symmetric monoidal functor from the category of cobordisms to the category of topological vector spaces, sometimes subject to some additional conditions ([Seg88], [Lur09]).

Quantum field theory has a variety of important applications in many areas of theoretical and mathematical physics, such as particle physics, quantum physics, condensed matter physics, statistical mechanics and string theory. Possibly the most well-known example of a quantum field theory is the standard model of particle physics, which is a functorial quantum field theory (in particular, it is formulated using the path integral method, which functorial quantum field theory formalizes).

In this section we will talk briefly about two-dimensional algebraic conformal field theories (CFTs), and more specifically their decompositions into one-dimensional chiral algebraic CFTs. These are defined rather nicely under the FQFT formalism as "weakly conformal" Segal CFTs, or under the AQFT formalism in terms of either vertex operator algebras or "nets" of von Neumann algebras arising from the Haag-Kastler axioms ([HK64]). The notions of FQFT and AQFT are conjectured to be equivalent after adding some additional conditions, such as unitarity or rationality ([Hen18]), although this will not be the focus of this thesis.

We find interest in Haag-Kastler nets for the following reason. From conformal nets, one naturally obtains type III_1 subfactors ([LR95], [Bis17]). In fact, CFTs have historically been an incredibly rich source of new and interesting subfactors ([Xu18]). As a result, a fundamental question arose that Jones dedicated much effort towards answering: do all finite index subfactors of type III_1 come from quantum field theory in such a way ([Jon17], [Bis17])? The goal of this section will be motivate conformal nets and familiarize the reader with them by mentioning key examples, as well as their relation to subfactor theory.

Definition 2.3.1. (Minkowski Spacetime). *The space* $\mathbb{R}^n$ *equipped with the* Lorentz bilinear form $\langle (u_i)_{i=0}^{n-1}, (v_j)_{j=0}^{n-1} \rangle := -u_0 v_0 + \sum_{k=1}^{n-1} u_k v_k$ *and endowed with the standard Euclidean topology is called the* n-dimensional Minkowski space(time) *and denoted* $\mathbb{R}^{1,n-1}$*, where* $(1, n-1)$ *refers to the metric signature* $(-, +, \dots, +)$*. The symmetry group of* $\mathbb{R}^{1,3}$ *is called the* Poincaré group $\mathcal{P}$*, and consists of all bijective isometries of* $\mathbb{R}^{1,3}$ *with respect to the induced* Lorentz metric*.*

The Poincaré group is essentially the minimal subgroup of the affine group $\mathrm{Aff}(4, \mathbb{R})$ containing the group of translations $\mathbb{R}^4$ and the group of Lorentz transformations $\mathcal{L} \cong \mathrm{O}(1, 3)$. As a result, we write it as the semidirect product $\mathcal{P} \cong \mathbb{R}^4 \rtimes \mathcal{L}$. In relativity, however, we typically care only about Lorentz transformations that preserve both the orientation and direction of time. The subgroup of such transformations is known as the *proper, orthochronous Lorentz group* and denoted $\mathcal{L}_+^\uparrow \cong \mathrm{SO}_+^\uparrow(1, 3)$; we remark that this is in fact isomorphic to the *Möbius group* $\text{Möb} \cong \mathrm{PSU}(1, 1) \cong \mathrm{PSL}(2, \mathbb{R})$! Moreover, because elements of a quantum Hilbert space that differ only by some phase (multiplication by a constant) represent the same physical state, we are usually interested in *projective* unitary representations of our spacetime symmetry group. By Bargmann's theorem ([Sim71]), every (strongly continuous) *projective* unitary representation of our *proper, orthochronous Poincaré group* $\mathcal{P}_+^\uparrow$ lifts to a unitary representation of its universal cover. As it is a Lie group, $\mathcal{L}_+^\uparrow$ has the unique universal cover $\tilde{\mathcal{L}}_+^\uparrow \cong \widetilde{\text{Möb}} \cong \mathrm{SL}(2, \mathbb{C})$, and hence it is sufficient to consider unitary representations of $\tilde{\mathcal{P}}_+^\uparrow \cong \mathbb{R}^4 \rtimes \mathrm{SL}(2, \mathbb{C})$. With this in mind, we are ready to introduce causally local Haag-Kastler nets in Minkowski space, which we shall do with the help of the informative proceedings of Kawahigashi ([Kaw03]).

Definition 2.3.2. (Local Poincaré Covariant Net). *A local Poincaré covariant net $\mathcal{A}$ is a map $\mathcal{A} : \mathcal{O} \to \mathscr{P}(\mathcal{B}(H))$ from the set $\mathcal{O}$ of open, bounded subsets of $\mathbb{R}^{1,3}$ into a family of von Neumann algebras $\{\mathcal{A}(O)\}_{O \in \mathcal{O}}$ on a Hilbert space H satisfying*

(i). *(isotony) $O_1 \subset O_2$ implies $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$, for all $O_1, O_2 \in \mathcal{O}$;*

(ii). *(locality) $O_1 \cap O_2 = \varnothing$ implies $\mathcal{A}(O_1) \subset (\mathcal{A}(O_2))'$, for all $O_1, O_2 \in \mathcal{O}$ (that is, $[\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$, where the brackets denote the commutator);*

(iii). *(Poincaré covariance) there is a strongly continuous unitary representation $U : \tilde{\mathcal{P}}_+^\uparrow \to \mathcal{U}(H)$ with $U(\gamma)\mathcal{A}(O)U(\gamma)^* = \mathcal{A}(\gamma \cdot O)$ for all $\gamma \in \tilde{\mathcal{P}}_+^\uparrow$, $O \in \mathcal{O}$;*

(iv). *(positivity of energy) the joint spectrum of the image under U of the four generators of the translation subgroup is contained in the closed, forward lightcone $\overline{V}_+ := \{v \in \mathbb{R}^4 : \langle v, v\rangle \leq 0, v_0 \geq 0\}$;*

(v). *(existence of vacuum) there exists a unit, vector $\Omega \in H$ called the* vacuum vector *that is U-invariant (in the sense that $U(\gamma)\Omega = \Omega$ for all $\gamma \in \tilde{\mathcal{P}}_+^\uparrow$) and cyclic for the von Neumann algebra $\bigvee_{O \in \mathcal{O}} \mathcal{A}(O) := (\bigcup_{O \in \mathcal{O}} \mathcal{A}(O))''$.*

Remark 2.3.3. The translation subgroup of $\tilde{\mathcal{P}}_+^\uparrow$ forms a Lie group generated by four translation operators $(P_i)_{i=0}^4$, one for each component of $\mathbb{R}^4$. When we say the joint spectrum lies in $\overline{V}_+$, we mean that $\{(\lambda_i)_{i=0}^4 \in \mathbb{R}^{1,3} : \lambda_i \in \sigma(P_i) \text{ for } 0 \leq i \leq 3\} \subseteq \overline{V}_+$.

Such nets describe the Haag-Kastler approach to AQFT on four-dimensional Minkowski space, and we observe that they extend naturally to produce QFTs on higher-dimensional Minkowski space. An exciting discovery regarding these four-dimensional AQFTs is that their structure is in some sense rather rigid; in particular, the local algebras $\mathcal{A}(O)$ will always be type III_1 factors, and under additional physically-motivated constraints will in fact be isomorphic to the (unique up to equivalence, due to [Haa87]) hyperfinite factor of type III_1, sometimes known as the *Araki-Woods factor of type III$_1$*. What separates such models is how these factors are embedded in each other via the net structure afforded by $\mathcal{A}$ ([Yng05]).

The natural question that arises is whether or not the nets themselves can be entirely classified. The first steps under this program have been in the direction of answering a simpler problem: namely, the classification of certain classes of nets, such as conformal nets. Rather than look at full two-dimensional chiral conformal field theories, however, one can often look instead at their chiral decompositions.

Let $\mathcal{I}$ denote the set of open, connected, non-dense, non-empty intervals of the circle S^1. Given a fixed Hilbert space H, some index set J and a family of von Neumann algebras $\{M_j\}_{j\in J}$ with $M_j \subset \mathcal{B}(H)$, we define the von Neumann algebra generated by $\{M_j\}_{j\in J}$ to be $\bigvee_{j\in J} M_j \coloneqq (\bigcup_{j\in J} M_j)''$. Finally, let $\mathrm{Diff}^+(S^1)$ denote the group of smooth, *orientation-preserving* diffeomorphisms of S^1 and let $\mathrm{Diff}^+_0(I')$ for $I \in \mathcal{I}$ be the subgroup of $\mathrm{Diff}^+(S^1)$ such that $\gamma \cdot z = z$ for all $\gamma \in \mathrm{Diff}^+_0(I')$ and $z \in I'$, where $I' \coloneqq \mathrm{Int}(S^1 \setminus I)$ is the interior of the complement of I in S^1.

Definition 2.3.4. (Local Conformal Net). *A* local Möbius covariant net $\mathcal{A}$ *is a map $\mathcal{A} : \mathcal{I} \to \mathscr{P}(\mathcal{B}(H))$ from the set of intervals $\mathcal{I}$ into a family of von Neumann algebras $\{\mathcal{A}(I)\}_{I\in\mathcal{I}}$ on a Hilbert space H satisfying*

(i). *(isotony) $I_1 \subset I_2$ implies $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$, for all $I_1, I_2 \in \mathcal{I}$;*

(ii). *(locality) $I_1 \cap I_2 = \varnothing$ implies $\mathcal{A}(I_1) \subset (\mathcal{A}(I_2))'$ for all $I_1, I_2 \in \mathcal{I}$ (that is, it implies $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$, where the brackets denote the commutator);*

(iii). *(Möbius covariance) there is a strongly continuous unitary representation $U : \widetilde{\mathrm{M\ddot{o}b}} \to \mathcal{U}(H)$ with $U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma \cdot I)$ for all $\gamma \in \widetilde{\mathrm{M\ddot{o}b}}$, $I \in \mathcal{I}$;*

(iv). *(positivity of energy) the infinitesimal generator L_0 (known as the* conformal Hamiltonian*) of the one-parameter rotation subgroup $\theta \mapsto U(R_\theta)$ has a positive spectrum, where $R_\theta \in \widetilde{\mathrm{M\ddot{o}b}}$ denotes the rotation of $\mathbb{C}$ by θ;*

(v). *(existence of vacuum) there exists a U-invariant unit vector $\Omega \in H$ called the* vacuum vector *that is cyclic for the von Neumann algebra $\bigvee_{I\in\mathcal{I}} \mathcal{A}(I)$.*

Such a net is said to be a local diffeomorphism covariant net *– or simply a* local conformal net *– if, additionally,*

(vi). *(diffeomorphism covariance) U extends to a strongly continuous, projective, unitary representation $U : \mathrm{Diff}^+(S^1) \to \mathrm{PGL}(H)$ such that, for all $I \in \mathcal{I}$,*
- *$U(\gamma)\mathcal{A}(I)U(\gamma)^* = \mathcal{A}(\gamma \cdot I)$, for all $\gamma \in \mathrm{Diff}^+(S^1)$;*
- *$U(\gamma)XU(\gamma)^* = X$, for all $X \in \mathcal{A}(I)$ and $\gamma \in \mathrm{Diff}^+_0(I')$.*

Remark 2.3.5. In the literature, "conformal net" is occasionally used to refer to Möbius covariant nets rather than diffeomorphism covariant nets ([CKLW18]).

Given two local diffeomorphism covariant nets $\mathcal{A}_-$ and $\mathcal{A}_+$ whose sets of intervals $\mathcal{I}_\pm$ are each taken to be open, bounded intervals in $L_\pm = \{(v_i)_{i=0}^3 \in \mathbb{R}^{1,1} : v_0 \pm v_1 = 0\}$, respectively (which extend to intervals of S^1 under a one-point compactification), we can reconstruct a two-dimensional conformal field theory on $\mathbb{R}^{1,1}$ by constructing all open, bounded regions from open 2-cells of the form $I_- \times I_+$ and using the process described in [Kaw03] and [Tan12]. In this way we can view a full two-dimensional algebraic CFT as a product of two one-dimensional chiral algebraic CFTs. Thus it is in some sense sufficient to study these one-dimensional chiral components.

It is indeed a fact that, outside of the trivial case where $\mathcal{A}(I) = \mathbb{C}$, these local algebras $\mathcal{A}(I)$ will once again always be type III_1 factors ([CKLW18], see [GF93, lemma 2.9] or [DLR01, corollary 2.6]; under additional constraints, they will be hyperfinite by [GF93, theorem 2.13]), and hence from conformal nets we obtain subfactors due to axiom (ii) as mentioned previously ([Jon17]). One of the greatest challenges towards understanding these CFTs – and AQFTs in general – lies in the great difficulty behind constructing explicit, non-trivial examples. Some of the most interesting examples are due to the collaboration of Jones and Wassermann, arising from loop groups ([Was98]). To illustrate this, consider the following example.

Example 2.3.6. (Jones-Wassermann Nets). Suppose we fix a *level* $\ell \in \mathbb{Z}_+$ and let G be a simple, simply-connected, simply-laced (in the sense of [PS86, §2.5]) compact Lie group (such as $SU(n)$, for instance; this was the first example described by Wassermann in [Was98]). From this pair, we would like to construct a conformal net $\mathcal{A}_{G,\ell}$, known as a *loop group conformal net*. We begin by writing $LG := C^\infty(S^1, G)$ to be the topological group consisting of smooth maps from the circle S^1 to G under pointwise multiplication, endowed with the compact-open topology. We call this the *(smooth, free) loop group of* G. Because G is simple and simply-connected, LG admits a *basic central extension* $\widetilde{LG}$ by the circle group $U(1)$ ([PS86, proposition 4.4.6], [Hen19]), known as the *level* 1 *central extension of* LG:

$$1 \longrightarrow U(1) \longrightarrow \widetilde{LG} \longrightarrow LG \longrightarrow 1.$$

Moreover, if we let $\widetilde{LG}^\ell := \widetilde{LG}/\omega_\ell$ for $\omega_\ell \subset U(1) \subseteq Z(\widetilde{LG})$ the central subgroup of all ℓth roots of unity, then because $U(1) \cong U(1)/\omega_\ell$ by the first isomorphism theorem we have another central extension

$$1 \longrightarrow U(1) \longrightarrow \widetilde{LG}^\ell \longrightarrow LG \longrightarrow 1,$$

known as the *level* ℓ *central extension of* LG. If we now let

$$L_IG := \{\gamma \in LG : \gamma(z) = e \text{ for all } z \notin I\} = \{\gamma \in LG : \mathrm{Supp}(\gamma) \subseteq I\}$$

for each $I \in \mathcal{I}$, where $e \in G$ denotes the group identity, then we have a restriction $\widetilde{L_IG}^\ell$ of the level ℓ central extension:

$$1 \longrightarrow U(1) \longrightarrow \widetilde{L_IG}^\ell \longrightarrow L_IG \longrightarrow 1.$$

At this point we construct some Hilbert space H and a projective unitary representation $\pi : \widetilde{LG}^\ell \to \mathcal{U}(H)$, whence taking $\mathcal{A}_{G,\ell}(I) := (\pi(\widetilde{L_IG}^\ell))''$ for each $I \in \mathcal{I}$ gives us a conformal net. Such conformal nets are called *Jones-Wasserman nets*, or occasionally *affine Kac-Moody nets*. The actual process for constructing the representation π is a bit involved; details can be found in [GF93]. We also remark that by [GF93, theorem 3.3], each $\mathcal{A}_{G,\ell}(I)$ will be isomorphic to the hyperfinite factor of type III_1.

There are a variety of other examples of conformal nets that admit similar constructions arising from loop groups: see [Sha17, §1.2.1], for instance, which briefly describes so-called *Heisenberg nets* and *lattice nets*.

2.4. Richard Thompson's Groups

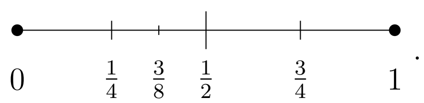
In another world entirely lie the groups of Richard Joseph Thompson, sometimes referred to as the "chameleon groups". This triplet of countably-infinite groups $F \subset T \subset V$ was originally introduced by Thompson in some unpublished notes during the 60's. Thompson's group F is considered to be historically the first potential counterexample to the so-called *von Neumann conjecture*, which proposed that a group is non-amenable if and only if it contains the free group on two generators as a subgroup (which, while named after von Neumann, is suspected to have its earliest appearance in the literature due to [Day57, p. 520]). While this conjecture was later proven false by Olshansky with the construction of his Tarski monster groups ([Ols80]), in fact the question of whether or not Thompson's groups are amenable remain open to this day (although it is at least known that F is not *elementary amenable*; see [CFP96, theorem 4.10]). Surprisingly, despite appearing so inoffensive on the surface, proving almost anything about these groups remains quite a challenge!

What makes them so interesting to study in spite of this is the bizarre and rare properties that they have been shown to satisfy. For instance, T and V were historically the first examples of finitely-presented infinite simple groups, making them an important step towards understanding infinite simple groups in general. Moreover, the three groups – especially F – appear naturally in a variety of often unexpected contexts, such as in topology, logic and cryptography, among others.

In this section, we will introduce the groups F, T and V, along with various ways that we can understand them. We will find T particularly interesting due to how it encodes discrete transformations of the circle, a fact that we will later try to liken to the diffeomorphism covariance of a chiral conformal net using the technology of Jones. For a more detailed discussion of Thompson's groups and their various properties, we encourage the reader to consult [Bel04] and [CFP96].

2.4.1. Thompson's Group F

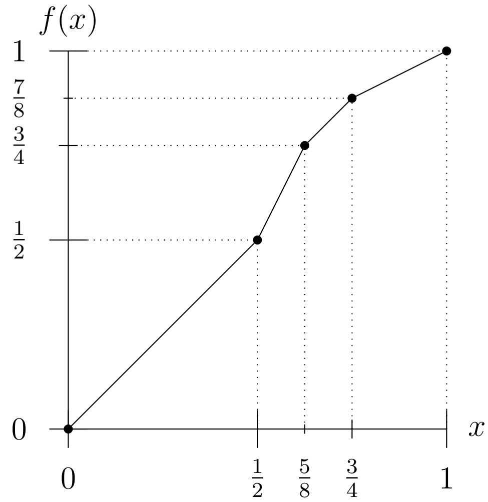
A *dyadic subdivision* of $[0,1]$ is a subdivision of the interval $[0,1]$ obtained by repeatedly cutting intervals in half a finite number of times. More precisely, it is a partition of $[0,1]$ consisting only of finitely many *dyadic intervals*; intervals of the form $I_{j,k} = [j/2^k, (j+1)/2^k]$, for $j, k \in \mathbb{N}$. The following partition, for instance, is a dyadic subdivision consisting of five dyadic intervals:



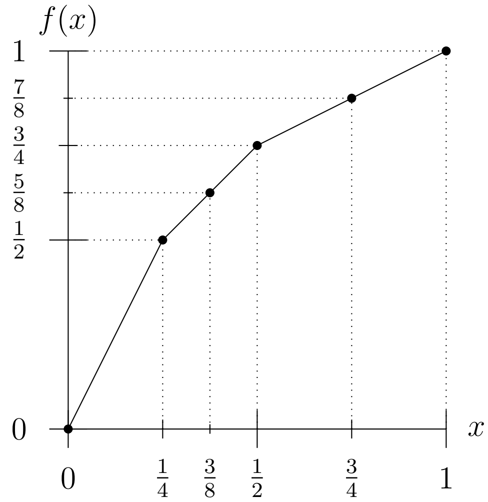
Suppose we have two dyadic subdivisions $\mathcal{I} := (I_{j_i,k_i})_{i=1}^n$ and $\mathcal{J} := (I_{l_i,m_i})_{i=1}^n$, both composed of the same number n of dyadic intervals. We can then define a piecewise affine homeomorphism $f : [0,1] \to [0,1]$ for which the ith interval of $\mathcal{I}$ is mapped linearly onto the ith interval of $\mathcal{J}$, by defining the restriction of f to each dyadic interval $I_{j_i,k_i} \in \mathcal{I}$ to be the linear interpolation

$$f|_{I_{j_i,k_i}} : x \mapsto \frac{l_i}{2^{m_i}} + (2^{k_i}x - j_i)\frac{1}{2^{m_i}}.$$

Such a map is known as a *dyadic rearrangement* of $[0,1]$. Examples of two such maps are provided as follows, in order to demonstrate how they might be visualized.



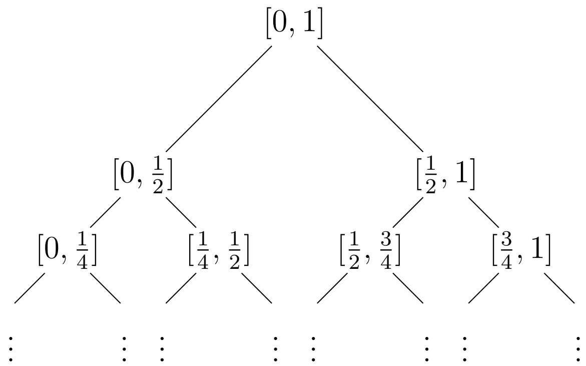
,



.

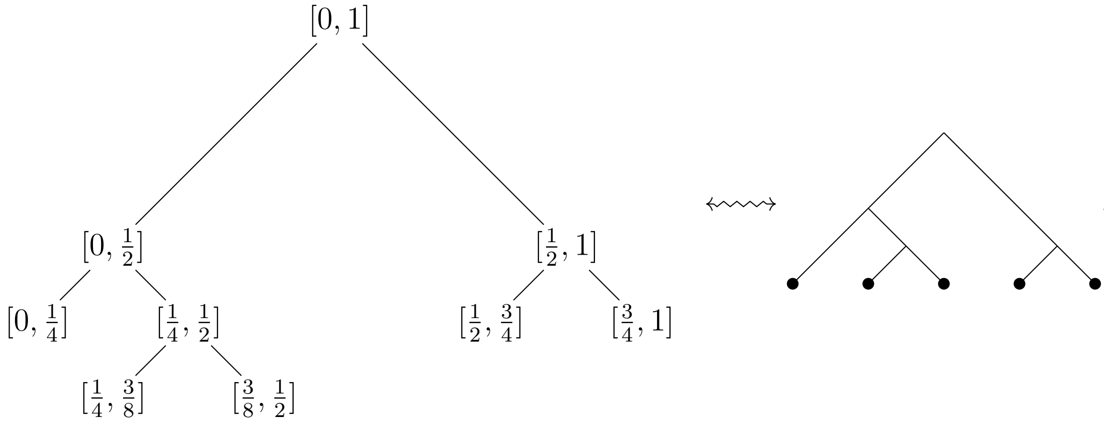
Proposition-Definition 2.4.1.1. (Thompson's Group F). *The set F of all dyadic rearrangements forms a group under composition, known as* Thompson's group F.

A proof of this fact may be found in [CFP96, §1]. A convenient way of studying F is through the use of tree diagrams. To this end, we begin by first viewing each dyadic subdivision as a bifurcating tree; in particular, we will represent each subdivision by a *finite subtree* of the *maximal infinite bifurcating tree* of dyadic intervals:



.

For instance, suppose we consider the subdivision given at the beginning of this section; this may be represented by the following finite subtree:



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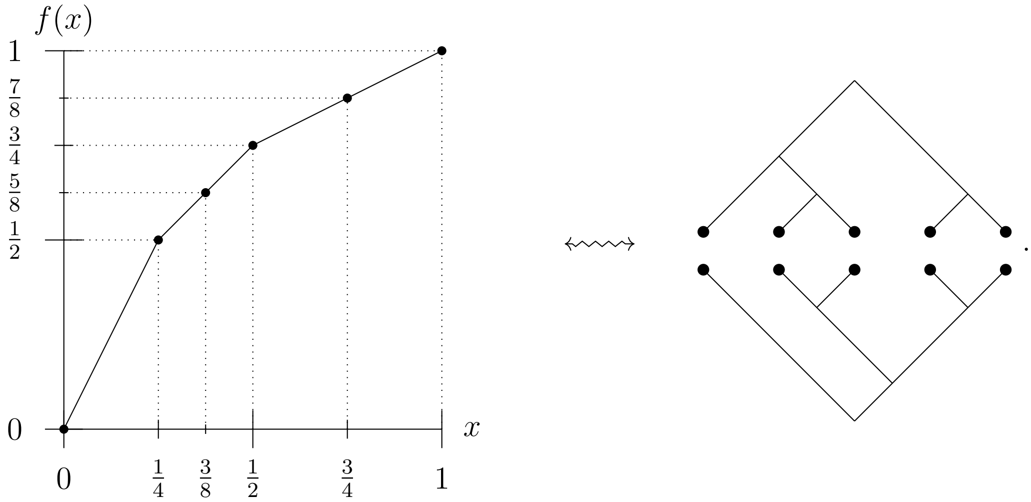
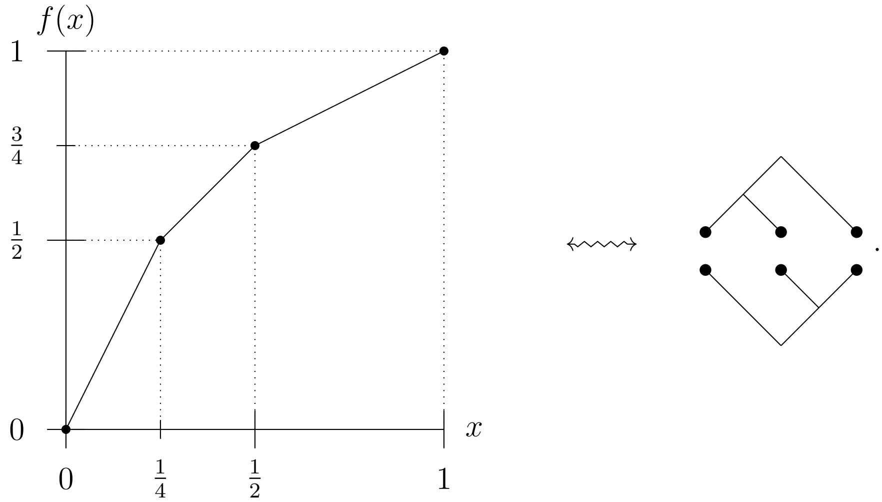
The number of dyadic intervals of a particular dyadic subdivision corresponds precisely to the number of leaves in its corresponding bifurcating tree. In order to represent dyadic rearrangements, we introduce *tree diagrams*:



These tree diagrams consist of the bifurcating tree for the target subdivision being reflected across the horizontal axis and placed on top of the bifurcating tree for the source subdivision, such that the leaves match up. A subtle observation is that the dyadic rearrangement given above is not unique: in particular, we have two vertices – one at $(\frac{3}{8}, \frac{5}{8})$ and another at $(\frac{3}{4}, \frac{7}{8})$ – that contribute no new information to the map. These manifest as opposing pairs of carets in the tree diagram. The process of collapsing these carets to a single point is known as *reduction*, and a tree diagram with no such opposing pairs of carets is referred to as *reduced*. Reducing a tree diagram in this way does not change the element of F described by the diagram. For instance, the following map is clearly equivalent to the one printed above, and hence so too is the tree diagram corresponding to it.



In particular, we have the following result. This is essentially a consequence of the bijective correspondence between opposing pairs of carets and "excess" vertices in the dyadic rearrangements, and will give us precisely the correspondence we desire between these two pictures for viewing F.

Theorem 2.4.1.2. *Every element of F has a unique reduced tree diagram.*

Proof. We first note that carets are, by definition, dyadic intervals that have been split in half. Suppose now that we have two pairs of half-intervals $\{[a, \frac{a+b}{2}], [\frac{a+b}{2}, b]\}$ and $\{[c, \frac{c+d}{2}], [\frac{c+d}{2}, d]\}$ corresponding to opposing pairs of carets. Note that by mapping these half-intervals to each other, we obtain two intervals with slope $\frac{d-c}{b-a}$, meaning we will have a "useless" vertex between them. In other words, opposing pairs of carets correspond bijectively to excess vertices in the dyadic rearrangement. Such vertices can always be removed by rejoining the offending half-intervals. Because there are only a finite number of vertices, it is certainly possible to remove all such excess vertices in this way. The result will be a unique, minimal representation of the dyadic rearrangement with vertices only between linear segments of differing slopes. The result follows by our aforementioned bijective correspondence between excess vertices and carets. This completes the proof. ■

We now mention some final details regarding the tree diagram formalism. If we let s denote the bifurcating tree of the source and t the bifurcating tree of the target, we may denote tree diagrams by the pair $[s, t]$. Given two elements $f, g \in F$ with corresponding tree diagrams $[s, t]$ and $[t, u]$, it follows that $[s, u]$ is a tree diagram for the product $gf := g \circ f$. Moreover, for any two elements $f, g \in F$, we can always find tree diagrams such that the top tree of f is congruent to the bottom tree of g by expanding the reduced tree diagrams (that is, by introducing pairs of opposing carets in the source and target bifurcating trees); in this way, we can take products of any two tree diagrams. Finally, note that dyadic rearrangements are trivially invertible by swapping the source and target subdivisions. Thus if $[s, t]$ is a tree diagram for $f \in F$, then $[t, s]$ must be a tree diagram for f^{-1}.

2.4.2. Thompson's Groups T and V

We can define Thompson's group T to be the group containing the same tree diagrams of F, but where we allow cyclic permutations between the leaves of the two bifurcating trees. The story for V is similar, but where we instead allow any permutation between leaves. Note that F may be viewed in terms of its canonical action on the interval $[0, 1]$ as the group of piecewise affine homeomorphisms of $[0, 1]$ that send dyadic rationals to dyadic rationals and are differentiable except at finitely many dyadic rationals, with the derivatives being powers of 2 where they exist ([Bel04, theorem 1.1.2]). With this in mind, and by identifying the endpoints of $[0, 1]$ to obtain the circle S^1, we may then think of T as the group of piecewise affine homeomorphisms of S^1 that send dyadic rationals to dyadic rationals and are differentiable except at finitely many dyadic rationals, with the derivatives being powers of 2 where they exist. Meanwhile, V admits a similar description in terms of piecewise affine homeomorphisms of the Cantor space of infinite binary sequences.

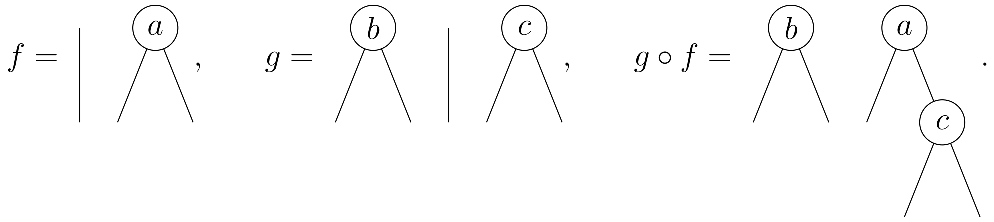
In this way, we can view Thompson's groups as either groups of tree diagrams or via their canonical actions on algebraic structures such as the unit interval, unit circle and Cantor set. In the following section, we will look at how we may generalize these groups to find a new family of groups with similar exceptional properties.

2.5. Forest-Skein Formalism

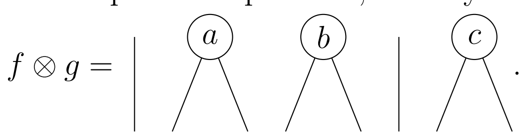
We believe the example from the previous section will be helpful in understanding and motivating what follows; a generalization of the tree-based formalism designed for the purpose of understanding and constructing a broader variety of so-called "Thompson-like groups", which share some of the unusual properties of Thompson's groups. This section is dedicated to introducing to the reader the forest-skein formalism, summarizing some of the important definitions from [Bro22, §1].

2.5.1. Forest-Skein Categories

A *forest* $f := (f_1, \dots, f_n)$ is an ordered list of finite, rooted bifurcating trees, where each vertex has zero or two children. We let $\mathrm{Leaf}(f)$, $\mathrm{Root}(f)$ and $\mathrm{Ver}(f)$ denote the sets of leaves, roots and interior vertices (vertices with exactly two children), respectively, of the forest f. The term *tree* shall henceforth be used to refer to a forest with only a single root. If $\mathcal{S}$ is a non-empty set, a *coloured forest* is a pair (f, c), where f is a forest and $c : \mathrm{Ver}(f) \to \mathcal{S}$ is a *colouring map* that assigns to each interior vertex a *colour* from the *set of colours* $\mathcal{S}$. We will typically let I denote the trivial tree with no interior vertices, and let Y_x denote the caret coloured by x. For two forests f and g with $|\mathrm{Leaf}(f)| = |\mathrm{Root}(g)|$, we define their composition $g \circ f$ to be the *vertical* concatenation of g onto f, where the ith leaf of f is replaced by the ith root of g. For instance, given $\mathcal{S} := \{a, b, c\}$ with $f := (I, Y_a)$ and $g := (Y_b, I, Y_c)$,



For any two forests f and g, we also define their monoidal product $f \otimes g$ to be the *horizontal* concatenation of g to the right of f; that is, if $f := (f_1, \dots, f_n)$ and $g := (g_1, \dots, g_m)$, we have $f \otimes g := (f_1, \dots, f_n, g_1, \dots, g_m)$. Using the forests from the above example of the composition operation, we may write



This leads us to the following definition.

Definition 2.5.1.1. (Free Forest-Skein Category). *Let $\mathcal{S}$ be a non-empty set. The* free forest-skein category over $\mathcal{S}$ *(also called the* universal forest-skein category over $\mathcal{S}$*) is the small monoidal category $\mathcal{UF}\langle\mathcal{S}\rangle$, endowed with the composition and monoidal product operations defined previously, where objects are given by non-negative integers and the set of morphisms from m to n is given by the set of forests, coloured by $\mathcal{S}$, with m roots and n leaves. If $\mathcal{S} = \{x\}$ is a singleton, we may instead denote the category by $\mathcal{UF}\langle x\rangle$ and refer to it as the* monochromatic free forest-skein category with colour x. *If it is clear from the context, we may in general denote the category by $\mathcal{UF}$, with $\mathcal{T} \subset \mathcal{UF}$ denoting the set of trees in $\mathcal{UF}$.*

A *skein relation in* $\mathcal{UF}\langle\mathcal{S}\rangle$ is a pair of trees $(t, t') \in \mathcal{T}\langle\mathcal{S}\rangle \times \mathcal{T}\langle\mathcal{S}\rangle$ with the same number of leaves. A *skein presentation* is then a pair $(\mathcal{S}, \mathcal{R})$, where $\mathcal{R}$ is a set of skein relations in $\mathcal{UF}\langle\mathcal{S}\rangle$. Such a set $\mathcal{R}$ generates equivalence relation on $\mathcal{UF}\langle\mathcal{S}\rangle$, where we define $t \sim t'$ for all $(t, t') \in \mathcal{R}$, and demand that $\sim$ be closed under taking compositions and monoidal products. This congruence relation generated by $\mathcal{R}$ then gives us a quotient set of the set $\mathcal{UF}\langle\mathcal{S}\rangle$. Informally, we can think of this equivalence relation as identifying a tree t with another tree t', such that substituting a subtree t in a forest f with its equivalent subtree t' does not modify the forest $[f]$ in the quotient set. Note that compositions and monoidal products thus extend in the obvious way to the quotient set: in particular, we may define $[g] \circ [f] := [g \circ f]$ and $[h] \otimes [k] := [h \otimes k]$, for $h, k \in \mathcal{UF}\langle\mathcal{S}\rangle$ and composable $f, g \in \mathcal{UF}\langle\mathcal{S}\rangle$. This allows us to define more general, *presented* forest-skein categories.

Definition 2.5.1.2. (Forest-Skein Category). *Let* $(\mathcal{S}, \mathcal{R})$ *be a skein presentation in* $\mathcal{UF}\langle\mathcal{S}\rangle$. *The* forest-skein category presented by $(\mathcal{S}, \mathcal{R})$ *is the small monoidal category* $\mathcal{F} := \mathrm{ForC}\langle\mathcal{S}|\mathcal{R}\rangle$ *defined by taking the quotient of the set* $\mathcal{UF}\langle\mathcal{S}\rangle$ *with respect to the equivalence relation generated by* $\mathcal{R}$.

In addition to forest-skein categories, skein presentations also define what we will call forest-skein monoids, which are the inductive limits of forest-skein categories.

Definition 2.5.1.3. (Forest-Skein Monoid). *Let* $\mathcal{F} := \mathrm{ForC}\langle\mathcal{S}|\mathcal{R}\rangle$ *be a forest-skein category. The* forest-skein monoid associated with $\mathcal{F}$ *is the monoid* $\mathcal{F}_\infty := \mathrm{ForM}\langle\mathcal{S}|\mathcal{R}\rangle$ *consisting of* infinite forests in $\mathcal{F}$*; that is, infinite sequences of trees in* $\mathcal{F}$. *The binary operation of* $\mathcal{F}_\infty$ *is given by the composition operation of* $\mathcal{F}$ *extended in the obvious way to infinite sequences of trees. If* $\mathcal{R} = \varnothing$, *we may write* $\mathcal{UF}_\infty := \mathrm{ForM}\langle\mathcal{S}\rangle$.

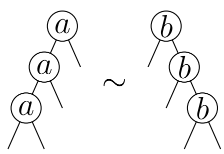
Just like with Thompson's group F, we can generalize these forest-skein categories further by introducing various decorations on the leaves. We will henceforth refer to the forest-skein categories we have seen so far, without any decorations, as *F-like forest-skein categories*. Continuing this analogy, we will call forest-skein categories where the leaves are subject to cyclic permutations *T-like forest-skein categories*, or *V-like forest-skein categories* in the relaxed case where any permutation is allowed. If the version (that is, F-like, T-like or V-like) is not mentioned in the following text, one may safely assume that it does not matter. Otherwise, we will use a superscript to indicate the version, writing for instance $\mathcal{F}^T \in \mathrm{ForC}^T\langle\mathcal{S}|\mathcal{R}\rangle$.

Example 2.5.1.4. (Generalized Cleary Groups). Consider the categories

$$\mathcal{C}_n := \mathrm{ForC}^F\left\langle a, b : \bigcirc_{i=1}^{n} Y_a \otimes I^{\otimes n-i} \sim \bigcirc_{i=1}^{n} I^{\otimes n-i} \otimes Y_b \right\rangle,$$

where, for example, we have in the $n = 3$ case that

$$\mathcal{C}_3 = \mathrm{ForC}^F\left\langle a, b : \right.$$



$$\left.\right\rangle.$$

These admit localizations $G_{\mathcal{C}_n}$ ([Bro22]), generalizing the Cleary group $G_{\mathcal{C}_2}$.

2.5.2. Localization of Forest-Skein Categories

We have already seen how bifurcating trees associated with dyadic rearrangements admit a group structure; namely, Thompson's group F. Some forest-skein categories admit group structures of their own in a similar fashion. We will briefly explain how this may be done, describing the sufficient conditions for such a group to exist. In particular, we will generate these kinds of groups from forest-skein categories admitting a so-called "calculus of fractions" in the sense of [GZ67] and [Bro23].

Definition 2.5.2.1. (Calculus of Fractions). *A category $\mathcal{C}$ together with a fixed object $e \in \mathrm{Ob}(\mathcal{C})$ is said to admit a* calculus of (left) fractions at e *if*

(i). *(left Ore condition at e) for all pairs of morphisms $f, g \in \mathrm{Mor}(\mathcal{C})$ sharing the same source e, there exist $h, k \in \mathrm{Mor}(\mathcal{C})$ such that $h \circ f = k \circ g$;*

(ii). *(weak right-cancellability at e) for all morphisms $f \in \mathrm{Mor}(\mathcal{C})$ with source e and all $h, k \in \mathrm{Mor}(\mathcal{C})$ such that $h \circ f = k \circ f$, there exists $g \in \mathrm{Mor}(\mathcal{C})$ such that $g \circ h = g \circ k$.*

If $\mathcal{C}$ admits a calculus of (left) fractions in all of its objects, it is called an Ore category*, or said to simply admit a* calculus of (left) fractions*.*

These two conditions may be summarized by the following commutative diagrams.

$$\text{(i).}\quad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "g"'] & \bullet \arrow[d, dashed, "\exists\, h"] \\ \bullet \arrow[r, dashed, "\exists\, k"'] & \bullet \end{tikzcd}, \qquad \text{(ii).}\quad \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f"'] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r, "k"'] & \bullet \end{tikzcd} \Longrightarrow \begin{tikzcd} \bullet \arrow[r, "h"] \arrow[d, "k"'] & \bullet \arrow[d, dashed, "\exists\, g"] \\ \bullet \arrow[r, dashed, "\exists\, g"'] & \bullet \end{tikzcd}.$$

The consequence of considering categories that admit a calculus of fractions is that these categories support the following localization of [GZ67, chapter I.2], stemming from the work of Øystein Ore in the 1930s in the context of monoids and groups. Suppose $\mathcal{C}$ is a category admitting a calculus of fractions, and define $\mathcal{P}_{\mathcal{C}}$ to be the pairs of morphisms (s, t) in $\mathcal{C}$ such that s and t have the same target. Then we define the *fraction groupoid* $\mathrm{Frac}(\mathcal{C})$ (also written $\mathcal{C}[\mathrm{Mor}(\mathcal{C})^{-1}]$, alluding to the general case where we define a calculus of fractions on a class of morphisms from $\mathcal{C}$) to be the quotient of $\mathcal{P}_{\mathcal{C}}$ with respect to the smallest equivalence relation $\sim$ generated by

$$(s, t) \sim (f \circ s, f \circ t),$$

for all $(s, t) \in \mathcal{P}_{\mathcal{C}}$ and composable $f \in \mathrm{Mor}(\mathcal{C})$. We will denote the equivalence class of (s, t) by $[s, t] \in \mathrm{Frac}(\mathcal{C})$. We may realize $\mathrm{Frac}(\mathcal{C})$ as a groupoid by endowing it with the following partially defined binary operation. Suppose we consider two elements with representatives $[s, t], [s', t'] \in \mathrm{Frac}(\mathcal{C})$ sharing the same source. Then by Ore's condition, there exist $f, f' \in \mathrm{Mor}(\mathcal{C})$ satisfying $f \circ t = f' \circ s'$; we thus define

$$[s, t] \circ [s', t'] := [f \circ s, f' \circ t'].$$

To see that this is well-defined, depending only on $[s, t]$ and $[s', t']$, suppose we consider another pair of morphisms $g, g' \in \mathrm{Mor}(\mathcal{C})$ satisfying $gt = g's'$, where we take multiplication to be composition for convenience. By Ore's condition, there exist $h, k \in \mathrm{Mor}(\mathcal{C})$ such that $hft = kgt$. In particular, $hf's' = hft = kgt = kg's'$.

It therefore follows by the weak cancellation property that there exists some $p \in \mathrm{Mor}(\mathcal{C})$ such that $phf' = pkg'$. Moreover, because $hft = kgt$, we have $phft = pkgt$, whence the weak cancellation property gives us some $q \in \mathrm{Mor}(\mathcal{C})$ for which $qphf = qpkg$. We thus obtain the equalities $phf' = pkg'$ and $qphf = qpkg$, whence it follows that

$$\begin{aligned}[fs, f't'] &= [phfs, phf't'] = [phfs, pkg't'] \\ &= [qphfs, qpkg't'] = [qpkgs, qpkg't'] \\ &= [gs, g't'].\end{aligned}$$

Under this newly endowed groupoid structure, we denote the *formal inverse* of an element $[s,t] \in \mathrm{Frac}(\mathcal{C})$ by $[s,t]^{-1} := [t,s] \in \mathrm{Frac}(\mathcal{C})$. When $\mathcal{C}$ is small, this fraction groupoid admits an isotropy group $\mathrm{Frac}(\mathcal{C}, e)$ for each object $e \in \mathrm{Ob}(\mathcal{C})$:

$$\mathrm{Frac}(\mathcal{C}, e) := \{[s,t] \in \mathrm{Frac}(\mathcal{C}) : s, t \in \mathrm{Mor}_{\mathcal{C}}(e, c), \text{ for some } c \in \mathrm{Ob}(\mathcal{C})\}.$$

That is, the *group of fractions* $\mathrm{Frac}(\mathcal{C}, e)$ is just the set of equivalence classes of pairs of morphisms with source e and shared target. We see that these isotropy groups are indeed groups, as locally composition is no longer only partially defined. In particular, even if $\mathcal{C}$ only admits a calculus of fractions at the object $e \in \mathrm{Ob}(\mathcal{C})$, we can consider $\mathcal{P}_{\mathcal{C}}$ being the pairs of morphisms with source e and shared target, whence the same process as before gives us directly the isotropy subgroup $\mathrm{Frac}(\mathcal{C}, e)$.

Remark 2.5.2.2. A generalization of the concept of the fraction groupoid based on universal properties exists due to Quillen ([Qui73]); every category $\mathcal{C}$ admits an "enveloping groupoid" that reduces to $\mathrm{Frac}(\mathcal{C})$ when $\mathcal{C}$ admits a calculus of fractions.

Example 2.5.2.3. (One-Object Groupoid). Recall that, given a group G, we may define a *one-object groupoid* $\mathsf{B}G$ consisting of a unique object $\mathrm{Ob}(\mathsf{B}G) := \{\bullet\}$ and morphisms $\mathrm{Mor}(\mathsf{B}G) := G$ corresponding to the elements of G. Such a category trivially admits a calculus of fractions at $\bullet$; in particular, the group of fractions of the one-object groupoid of a group G simply recovers for us the original group G.

In the context of an Ore forest-skein category $\mathcal{F}$ – which is guaranteed to be a small monoidal category – we can interpret the equivalence class $[s,t] \in \mathrm{Frac}(\mathcal{F})$ as "s/t" or "$s \circ t^{-1}$", represented by drawing s and then placing t upside down below it, such that the ith leaf of s is joined with the ith leaf of t. This should be rather evocative of what we did with Thompson's group F! In particular, consider the following definition.

Definition 2.5.2.4. (Forest-Skein Group). *Suppose $\mathcal{F}$ is a forest-skein category admitting a calculus of fractions in $1 \in \mathrm{Ob}(\mathcal{F})$. The* forest-skein group associated *with $\mathcal{F}$ is the fraction group $G_{\mathcal{F}} := \mathrm{Frac}(\mathcal{F}, 1)$; that is, the isotropy group of trees.*

From this definition, it is hopefully clear that Thompson's group F (respectively T, V) is nothing but the forest-skein group associated with the monochromatic, free, F-like (T-like, V-like) forest-skein category! In particular, these are the *only* forest-skein groups that arise from free forest-skein categories; in order to satisfy Ore's condition, a free forest-skein category must necessarily be monochromatic.

Remark 2.5.2.5. Some authors define forest-skein categories as the opposite category to our definition, where morphisms go from leaves to roots. This requires a dual notion of a calculus of fractions; they define a calculus of right fractions using the right Ore condition and left-cancellability, defined in the obvious ways. $\mathcal{C}^{\mathrm{op}}$ admits a calculus of right fractions if and only if $\mathcal{C}$ admits a calculus of left fractions.

2.6. Jones' Technology

In 2018, Jones discovered a machine that is capable of providing actions of groups of fractions on arbitrary categories ([Jon18]). In particular, this allows us to construct a rich representation theory for our forest-skein categories. In the context of this thesis, we will use Jones' technology to produce a so-called Jones representation: a Hilbert space along with a unitary representation of a group of fractions on it. This will correspond to the Hilbert space and unitary representation with which we might hope to build our "discrete" CFTs.

Suppose we have a small category $\mathcal{C}$ admitting a calculus of fractions in $e \in \mathrm{Ob}(\mathcal{C})$, a category $\mathcal{D}$ whose objects are sets† and a covariant functor $\Phi : \mathcal{C} \to \mathcal{D}$. Consider the set of morphisms with source e, which we endow with the relation $\lesssim$ defined by $s \lesssim t$ if and only if there exists some $f \in \mathrm{Mor}(\mathcal{C})$ such that $t = f \circ s$. Note that this f is not necessarily unique unless $\mathcal{C}$ is strongly right-cancellative at e. This defines not only a preorder, but a directed set, as we have for any s, t with source e that there exist morphisms h, k such that $s \lesssim h \circ s = k \circ t \gtrsim t$ by the left Ore condition.

We now define a direct system $X \coloneqq \langle X_s, \iota_f^s \rangle$ consisting of sets $X_s \coloneqq \Phi(\mathrm{Target}(s))$ and maps $\iota_f^s : X_s \to X_f$ given by $\iota_f^s \coloneqq \Phi(f)$, for $s \lesssim f \circ s$ from our directed set. Finally, we denote the direct limit by $\mathscr{X}_\Phi \coloneqq \varinjlim X$, which is by definition given by

$$\mathscr{X}_\Phi = \{(r, x) : r \in \mathrm{Mor}_{\mathcal{C}}(e, \mathrm{Target}(r)), x \in \Phi(\mathrm{Target}(r))\}/\sim$$

with respect to the equivalence relation $\sim$ defined by $(r, x) \sim (f \circ r, [\Phi(f)](x))$ for any composable $f \in \mathrm{Mor}(\mathcal{C})$. We write "$r/x$" for the equivalence class of (r, x), calling it a *fraction*. Note that this construction is very similar to the construction of fraction groupoids seen before, and in fact one may choose to construct these groups via a direct limit as well. With this, we introduce the technology of Jones.

Definition 2.6.1. (Jones Action). *Let $\mathcal{C}$ be a small category admitting a calculus of fractions at e and $G_{\mathcal{C}} \coloneqq \mathrm{Frac}(\mathcal{C}, e)$ its group of fractions. Given a covariant functor $\Phi : \mathcal{C} \to \mathcal{D}$, where the objects of $\mathcal{D}$ are sets†, we define an action $\pi_\Phi : G_{\mathcal{C}} \curvearrowright \mathscr{X}_\Phi$ by*

$$\Big[\pi_\Phi\Big(\frac{s}{t}\Big)\Big]\Big(\frac{r}{x}\Big) \coloneqq \frac{p \circ s}{[\Phi(q)](x)},$$

for $p, q \in \mathrm{Mor}(\mathcal{C})$ such that $p \circ t = q \circ r$. Such an action is known as a Jones action.

To see that this is well-defined, consider $p', q' \in \mathrm{Mor}(\mathcal{C})$ such that $p't = q'r$. By Ore's property and weak cancellability, we may find morphisms $f, g, h, k \in \mathrm{Mor}(\mathcal{C})$ such that $fhq = fkq'$ and $gfhp = gfkp'$. It follows by functoriality that

$$\frac{ps}{[\Phi(q)](x)} = \frac{gfhps}{[\Phi(gfhq)](x)} = \frac{gfkp's}{[\Phi(gfkq')](x)} = \frac{p's}{[\Phi(q')](x)}.$$

This is nothing but the process we used to show that multiplication was well-defined in the fraction groupoid! That π_Φ indeed defines an action is also easily verified.

† *Remark* 2.6.2. Recall that we have assumed in constructing our directed system that $\mathcal{D}$ is a category with sets as objects. We may lift this restriction by instead insisting that $\mathcal{D}$ is locally small, taking $X_s := \mathrm{Mor}_{\mathcal{D}}(\Phi(e), \Phi(\mathrm{Target}(s)))$ and defining maps $\iota_t^s(v) := \Phi(f) \circ v$. This was in fact the original idea of Jones ([Jon18]). The reason why we will tend to prefer $X_s := \Phi(\mathrm{Target}(s))$ is because this way *allows us to inherit the structure of the objects of* $\mathcal{D}$ *in the colimit* $\mathscr{X}_\Phi$.

Proposition-Definition 2.6.3. (Jones Representation). *Let* $\mathcal{C}$ *be a small category admitting a calculus of fractions at* e *and* $G_{\mathcal{C}} := \mathrm{Frac}(\mathcal{C}, e)$ *its group of fractions. Consider a covariant functor* $\Phi : \mathcal{C} \to \mathsf{Vect}_{\mathbb{K}}$*, where* $\mathsf{Vect}_{\mathbb{K}}$ *is the monoidal category with* $\mathbb{K}$*-vector spaces as objects, linear maps as morphisms and the tensor product as its monoidal product. The associated Jones action* $\pi_\Phi : G_{\mathcal{C}} \curvearrowright \mathscr{X}_\Phi$ *then induces a linear representation* $V_\Phi : g \mapsto \pi_\Phi(g)$ *of* $G_{\mathcal{C}}$ *on* $\mathscr{X}_\Phi$ *known as a* Jones representation.

Proof. Recall our definition of the direct limit $\mathscr{X}_\Phi$. We define a vector space structure on it as follows. Given equivalence classes $\xi, \eta \in \mathscr{X}_\Phi$ and a scalar $\lambda \in \mathbb{K}$, choose representatives $(r_1, x_1) \in \xi$ and $(r_2, x_2) \in \eta$. Then there exists by directedness some $r \in \mathrm{Mor}_{\mathcal{C}}(e, \mathrm{Target}(r))$ such that $f_1 \circ r_1 = r = f_2 \circ r_2$, for morphisms f_1, f_2 (that is, such that $r \gtrsim r_1, r_2$). We thus define

$$\lambda\xi + \eta = \lambda\frac{r_1}{x_1} + \frac{r_2}{x_2} := \frac{r_1}{\lambda x_1} + \frac{r_2}{x_2} := \frac{r}{\lambda[\Phi(f_1)](x_1)} + \frac{r}{[\Phi(f_2)](x_2)} = \frac{r}{x}$$

for $x := \lambda[\Phi(f_1)](x_1) + [\Phi(f_2)](x_2)$, where we have used the fact that Φ takes morphisms to $\mathbb{K}$-linear maps. This is indeed well-defined; for any representatives $(r_1', x_1') \in \xi$ and $(r_2', x_2') \in \eta$ we choose, and for any morphisms r', f_1', f_2' such that $f_1' \circ r_1' = r' = f_2' \circ r_2'$, we will always have

$$\frac{r'}{[\Phi(f_1')](x_1')} = \frac{r_1'}{x_1'} = \frac{r_1}{x_1} = \frac{r}{[\Phi(f_1)](x_1)},$$

and similarly for r_2/x_2 and r_2'/x_2', whence it follows that we must have $r'/x' = r/x$ for $x' := \lambda[\Phi(f_1')](x_1') + [\Phi(f_2')](x_2')$. We therefore have for any $s/t \in G_{\mathcal{C}}$ that

$$\begin{aligned}\Big[\pi_\Phi\Big(\frac{s}{t}\Big)\Big](\lambda\xi + \eta) = \frac{p \circ s}{[\Phi(q)](x)} &= \lambda\frac{p \circ s}{[\Phi(q)]([\Phi(f_1)](x_1))} + \frac{p \circ s}{[\Phi(q)]([\Phi(f_2)](x_2))} \\ &= \lambda\Big[\pi_\Phi\Big(\frac{s}{t}\Big)\Big](\xi) + \Big[\pi_\Phi\Big(\frac{s}{t}\Big)\Big](\eta)\end{aligned}$$

for morphisms p, q such that $p \circ t = q \circ r$, where we have used the equivalences of r_1/x_1 and r_2/x_2 outlined above, as well as the fact that Φ takes morphisms to $\mathbb{K}$-linear maps. Thus the Jones action induces a linear representation as claimed. This completes the proof. ∎

We will often like to consider a more interesting class of Jones representations. It is actually the case that with a functor that sends objects to inner product spaces and morphisms to linear isometries, we can construct a Hilbert space and unitary representations on it. To this end, we introduce the following lemma.

Lemma 2.6.4. *Let $\langle I, \leq \rangle$ be a directed set and $H := \langle H_i, T_j^i \rangle$ a directed system over I. If H_i is a pre-Hilbert space for all $i \in I$ and $T_j^i : H_i \to H_j$ is a linear isometry for all $i, j \in I$ with $i \leq j$, then the direct limit $\mathscr{H} := \varinjlim H$ is a pre-Hilbert space.*

Proof. Recall that we define $\mathscr{H}$ by the quotiented disjoint union

$$\mathscr{H} := \Big(\bigsqcup_{i \in I} H_i \Big) / \sim,$$

where for any $\xi_i \in H_i$ and $\eta_j \in H_i$ with $i, j \in I$, we have $\xi_i \sim \eta_j$ if and only if there exists some $k \in I$ with $i, j \leq k$ such that $T_k^i(\xi_i) = T_k^j(\eta_j)$. We wish to endow this with the structure of a vector space and an inner product. Well, suppose we let $\xi_i \in H_i$ for some $i \in I$. Then observe that for any $k \in I$ such that $i \leq k$, we trivially have that $\xi_i \sim T_k^i(\xi_i)$. It follows that, given two equivalence classes $\xi, \eta \in \mathscr{H}$, we may define vector addition in $\mathscr{H}$ by the equivalence class

$$\xi + \eta := [T_{k_{ij}}^i(\xi_i) + T_{k_{ij}}^j(\eta_j)],$$

where $\xi_i \in H_i$ and $\eta_j \in H_j$, for $i, j \in I$, are any elements from the equivalence classes ξ and η respectively, and where $k_{ij} \in I$ is chosen such that $i, j \leq k_{ij}$. This is in fact well-defined due to our previous observation; if we instead take $\xi_{i'} \in \xi$ and $\eta_{j'} \in \eta$ with $\xi_{i'} \in H_{i'}$ and $\eta_{j'} \in H_{j'}$, where $i', j' \leq k_{i'j'}$, we will always have that

$$T_{k_{ij}}^i(\xi_i) \sim \xi_i \sim \xi_{i'} \sim T_{k_{i'j'}}^{i'}(\xi_{i'}),$$

and similarly for $T_{k_{ij}}^j(\eta_j) \sim T_{k_{i'j'}}^{j'}(\eta_{j'})$. Linearity follows easily from the fact that we have linear isometries. We now wish to construct an inner product on $\mathscr{H}$. Suppose we have $\xi_k, \eta_k \in H_k$ and $\xi_{k'}, \eta_{k'} \in H_{k'}$ for $k, k' \in I$, such that $\xi_k \sim \eta_k$ and $\xi_{k'} \sim \eta_{k'}$. Moreover, consider any $K \in I$ such that $k, k' \leq K$. Then it follows from the fact that the maps T_K^k and $T_K^{k'}$ are isometries, as well as our initial observation, that

$$\langle \xi_k, \eta_k \rangle_k = \langle T_K^k(\xi_k), T_K^k(\eta_k) \rangle_K = \langle T_K^{k'}(\xi_{k'}), T_K^{k'}(\eta_{k'}) \rangle_K = \langle \xi_{k'}, \eta_{k'} \rangle_{k'},$$

where $\langle \cdot, \cdot \rangle_i$ denotes the inner product of H_i. Linearity follows easily once more, where we again use the linearity of T_K^k and $T_K^{k'}$. In other words, we have shown that

$$\langle \xi, \eta \rangle := \langle T_{k_{ij}}^i(\xi_i), T_{k_{ij}}^j(\eta_j) \rangle_{k_{ij}}$$

is a well-defined inner product, given the same notation as when we showed that $\mathscr{H}$ was a vector space. Thus $\mathscr{H}$ is a pre-Hilbert space. This completes the proof. ■

If we have a functor $\Phi : \mathcal{C} \to \mathsf{Inn}$, where Inn is the monoidal category whose objects are complex inner product spaces and whose morphisms are *linear isometries*, then by lemma 2.6.4 the direct limit $\mathscr{K}_\Phi$ will be an inner product space. The Jones action extends to an action on its Hilbert space completion $\mathscr{H}_\Phi$, which induces a unitary representation $U_\Phi : G_\mathcal{C} \to \mathcal{U}(\mathscr{H}_\Phi)$. The proof follows from the fact that Φ takes morphisms of $\mathcal{C}$ to linear isometries and hence the Jones action will preserve the inner product, as it will act on elements of $\mathscr{H}_\Phi$ via linear isometries. Of course, since $\pi_\Phi(t/s) = (\pi_\Phi(s/t))^{-1}$ by the fact that $G_\mathcal{C}$ is a group, these will be unitary transformations. This is true even if the underlying functor Φ does not necessarily take morphisms of $\mathcal{C}$ to unitary maps! These *unitary Jones representations* are the original definition of Jones representations due to Brothier ([Bro23]).

Consider now the following observation. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories and Φ is a strict monoidal functor. Suppose moreover that $\mathcal{C}$ is generated by some set of objects and morphisms with respect to the monoidal product and morphism composition. Then Φ is in fact completely determined by how it acts on these generating objects and morphisms. Such is the case for a forest-skein monoid $\mathcal{F}$, which is generated by the object $1 \in \mathrm{Ob}(\mathcal{F})$ and the trees $Y_a \in \mathrm{Mor}(\mathcal{F})$ for each colour a. It is easy to see then that the functor Φ is completely determined by how $\Phi(1) = x$ and $\Phi(Y_a) \in \mathrm{Mor}_{\mathcal{D}}(x, x \oplus x)$, for each colour a, are chosen. For unitary Jones representations of forest-skein categories, we will have $\Phi(1) = H \subseteq \mathscr{H}_\Phi$ a pre-Hilbert space and $R_a : H \to H \oplus H$ a linear isometry for each colour a. This is the setting that we will find ourselves in during the next chapter. Note that if our monoidal functor were not strict, we would in general have to also specify natural transformations for every pair of objects.

2.6.1. Canonical Action of Forest-Skein Groups

One of the primary motivations of forest-skein categories is to be able to capitalize on the technology of Jones. We shall hence mention, before concluding this chapter, a rather nice application of Jones' technology to the forest-skein formalism due to Brothier ([Bro22, §6]). In analogy to the canonical actions of Thompson's groups on the Cantor set, Jones' technology in fact gives us a canonical action of *any* forest-skein group on a corresponding "Cantor-like" space.

Let Set be the category of sets with monoidal structure given by disjoint union. For any forest-skein category $\mathcal{F}$, there in fact exists (due to the universal property of forest-skein categories, see [Bro22, §1.4]) a unique covariant strict monoidal functor $\Phi : \mathcal{F} \to \mathsf{Set}$ that acts on objects as $\Phi(n) := \{1, \dots, n\}$ and on trees such that $[\Phi(t)](1) := 1$. Note that, given a forest $f := (f_1, \dots, f_r)$, we have for instance that $\Phi(f)$ will act on the ordered r-tuple $\{1\} \sqcup \cdots \sqcup \{1\}$, where the jth copy of 1 (that we identify with the integer j for convenience) is mapped to the integer corresponding to the first leaf of f_j. Thus it follows that $\Phi(f) := (j \mapsto |\mathrm{Leaf}(f_1, \dots, f_{j-1})| + 1)$. In other words, $\Phi(f)$ takes $j \in \{1, \dots, r\}$ to the integer corresponding to the first leaf of the jth tree of f.

This functor gives us, for each $t \in \mathrm{Mor}_{\mathcal{F}}(1, n)$ with $n \in \mathbb{Z}_+$, a space

$$X_t := \Phi(\mathrm{Target}(t)) = \{1, \dots, n\},$$

whence the disjoint union $X := \bigsqcup_{t \in \mathcal{T}_{\mathcal{F}}} X_t$ is the set of all *pointed trees of* $\mathcal{F}$ (pairs carrying a tree from $\mathcal{F}$ along with a distinguished leaf); in other words,

$$X := \{(t, j) : t \in \mathrm{Mor}_{\mathcal{F}}(1, n), j \in \{1, \dots, n\}, n \in \mathbb{Z}_+\}.$$

If we now insist that $\mathcal{F}$ is an Ore forest-skein category with its set of trees $\mathcal{T}_{\mathcal{F}}$ countable and define the direct limit $Q_{\mathcal{F}} := X/\sim$, where $(t, j) \sim (f \circ t, [\Phi(f)](j))$ for any composable forest f, we of course induce a Jones action $G_{\mathcal{F}} \curvearrowright Q_{\mathcal{F}}$. In the monochromatic, free, F-like case, we can think of each $p \in Q_{\mathcal{F}}$ as encoding a *finitely-terminating infinite path*; that is, a finite path in the maximal infinite bifurcating tree, provided by the unique pair $(t, j) \in p$ containing the smallest rooted subtree t, followed by an infinite path travelling left appended to the leaf j.

The space $Q_{\mathcal{F}}$ has been suspected of being "Cantor-like", which we now give meaning to, as it is currently absent from the literature. I am extremely grateful to Ryan Seelig for his guidance and patience in helping me understand some of the technicalities in what follows. Let $s, t \in \mathcal{T}_{\mathcal{F}}$ with $s \lesssim t$, and write $\pi_t^s : X_t \to X_s$ for the map sending each leaf of t to the leaf of s that grows into it. We then define

$$\mathrm{Cone}(t, j) \coloneqq \{(s, j_s) \in X : s \gtrsim t,\, \pi_s^t(j_s) = j\};$$

that is, $\mathrm{Cone}(t, j)$ is the set of pointed trees that are grown from the jth leaf of t. We call the topology on X generated by the set of all cones in the *topology of cones*, and this induces a topology of cones on $Q_{\mathcal{F}}$. We now endow $Q_{\mathcal{F}}$ with a uniform structure as follows. Associate with each $t \in \mathrm{Mor}_{\mathcal{F}}(1, n)$ an entourage of the form

$$U_t \coloneqq \{(p_1, p_2) \in Q_{\mathcal{F}} \times Q_{\mathcal{F}} : p_1, p_2 \in \mathrm{Cone}(t, j)/\!\sim \text{ for some } j \in \{1, \ldots, n\}\}.$$

Note that this is indeed an entourage; if we fix $t \in \mathcal{T}_{\mathcal{F}}$, then by the left Ore condition every equivalence class $p \in Q_{\mathcal{F}}$ will contain a tree that has t as a rooted subtree, and hence the diagonal is contained in U_t. We then define the uniformity $\mathcal{U}_{\mathcal{F}}$ for $Q_{\mathcal{F}}$ to be the uniformity generated by the base of entourages $\{U_t\}_{t \in \mathcal{T}_{\mathcal{F}}}$; that is,

$$\mathcal{U}_{\mathcal{F}} \coloneqq \{V \subseteq Q_{\mathcal{F}} \times Q_{\mathcal{F}} : U_t \subseteq V \text{ for some } t \in \mathcal{T}_{\mathcal{F}}\}.$$

This uniform structure is compatible with the topology of cones in the following sense: if $U[p] \coloneqq \{p' \in Q_{\mathcal{F}} : (p, p') \in U\}$ and $\mathcal{U}_{\mathcal{F}}[p] \coloneqq \{U[p] : U \in \mathcal{U}_{\mathcal{F}}\}$, then $\{\mathcal{U}_{\mathcal{F}}[p] : p \in Q_{\mathcal{F}}\}$ is the system of neighbourhood filters for the topology of cones.

We now define a notion of completeness in the canonical way. We say that a proper filter $\mathfrak{F}$ in $Q_{\mathcal{F}}$ is Cauchy if, for each entourage $U \in \mathcal{U}_{\mathcal{F}}$, there is some $A \in \mathfrak{F}$ such that $A \times A \subseteq U$. A filter $\mathfrak{F}$ is then said to converge to a limit point $p \in Q_{\mathcal{F}}$ if the filter $\mathfrak{F} \cap \mathfrak{U}(p)$ is a Cauchy filter, where $\mathfrak{U}(p) \coloneqq \{S \subseteq Q_{\mathcal{F}} : p \in S\}$ is the ultrafilter fixed at p. Thus the Cauchy completion of a uniform space is a space whose Cauchy filters all converge that contains a dense subspace isomorphic to the original space.

Proposition 2.6.1.1. *The space $Q_{\mathcal{F}}$ is a dense subset of a Cantor space.*

Proof. Consider the projective system consisting of the same spaces X_t from our direct limit, where we now use the maps $\pi_t^s : X_t \to X_s$ sending each leaf of t to the leaf of s that grows into it. Then we have the projective limit

$$C_{\mathcal{F}} \coloneqq \left\{ (t, j_t)_{t \in \mathcal{T}_{\mathcal{F}}} \in \prod_{t \in \mathcal{T}_{\mathcal{F}}} X_t : (s, j_s) \in \mathrm{Cone}(t, j_t) \text{ for all } s \gtrsim t \right\}.$$

We endow $C_{\mathcal{F}}$ with the topology induced by assigning each X_t with the discrete topology, or in other words the coarsest topology for which the canonical projections $\pi_t : C_{\mathcal{F}} \to X_t$ are continuous. We claim that $C_{\mathcal{F}}$ is the Cauchy completion of $Q_{\mathcal{F}}$. First, observe that given a pair $(t, j) \in X$ and any tree $s \in \mathcal{T}_{\mathcal{F}}$, we have by the left Ore condition that there exists a tree $t' \gtrsim s, t$. Choose $j' \coloneqq [\Phi(f)](j)$ for f satisfying $t' = f \circ t$, such that $(t', j') \sim (t, j)$. Then there is a unique leaf $\ell_s \coloneqq \pi_{t'}^s(j')$. What we are essentially doing here is making choices of ℓ_s that trace a path from the root to j, before constantly travelling left. It is thus easy to see that the map $\sigma : [t, j] \mapsto (s, \ell_s)_{s \in \mathcal{T}_{\mathcal{F}}}$ is well-defined, and it is also obviously a faithful embedding.

Let's think more about the topology of $C_{\mathcal{F}}$. By definition any preimage of $\pi_{t'}$, for any $t' \in \mathcal{T}_{\mathcal{F}}$, is open. For any non-empty $A_{t'} \subseteq X_{t'}$, we have an open set of the form

$$\pi_{t'}^{-1}(A_{t'}) = \{(t, j_t)_{t\in\mathcal{T}_{\mathcal{F}}} \in C_{\mathcal{F}} : j_{t'} \in A_{t'}\}.$$

By taking intersections, we find that every open (and closed) set in $C_{\mathcal{F}}$ is of the form

$$\{(t, j_t)_{t\in\mathcal{T}_{\mathcal{F}}} \in C_{\mathcal{F}} : t_i \in A_{t_i} \text{ for all } 1 \leq i \leq n\}.$$

for finite sets $\{t_1, \ldots, t_n\} \subset \mathcal{T}_{\mathcal{F}}$ and $\{A_{t_1}, \ldots, A_{t_n}\}$ with $A_t \subset X_t$ and $n \in \mathbb{N}$. Suppose we now choose some $p \in C_{\mathcal{F}}$ and consider any neighbourhood N, which will contain some open set containing p characterized by finite sets $\{t_1, \ldots, t_n\}$ and $\{A_{t_1}, \ldots, A_{t_n}\}$ as before. Suppose that, for each t_i, we choose some $\ell_{t_i} \in A_{t_i}$ with

$$N' := \left\{(t, j_t)_{t\in\mathcal{T}_{\mathcal{F}}} \in \prod_{t\in\mathcal{T}_{\mathcal{F}}} X_t : j_{t_i} = \ell_{t_i} \text{ for all } 1 \leq i \leq n\right\} \subseteq C_{\mathcal{F}}.$$

Then the structure of $C_{\mathcal{F}}$ forces $\sigma^{-1}(N) \neq \varnothing$, as by our previous argument (the directedness of $\lesssim$) we can always find some tree $t' \gtrsim t_1, \ldots, t_n$ and leaf j' such $(t', j') \sim (t_i, \ell_{t_i})$ for some i, whence $\sigma([t_i, \ell_{t_i}]) \in N' \subseteq N$. In other words, we have shown that $Q_{\mathcal{F}}$ is dense in $C_{\mathcal{F}}$. Moreover, because every X_t is discrete and finite, they are naturally compact, whence it follows by Tychonoff's theorem ([Bou95, chapter I, §9.5, theorem 3]) that their product is also compact. Because our X_t are all discrete and hence Hausdorff, we not only have that $C_{\mathcal{F}}$ is Hausdorff by [Bou95, chapter I, §8.2, proposition 7], but that it is closed by [Dug66, p. 429] and hence compact. This tells us something wonderful: since $C_{\mathcal{F}}$ is compact, [Bou95, chapter II, §4.1, theorem 1] tells us that there is a unique uniform structure that is compatible with its topology, and moreover that $C_{\mathcal{F}}$ is complete with respect to this uniform structure. Thus $C_{\mathcal{F}}$ is the completion of $Q_{\mathcal{F}}$, and all that remains is to show that it is a Cantor space.

We have shown already that $C_{\mathcal{F}}$ is a compact Hausdorff space. It also contains no isolated points, as for instance no singleton is open by our previous observations. Finally, since each X_t is second-countable and by hypothesis $\mathcal{T}_{\mathcal{F}}$ is countable, $C_{\mathcal{F}}$ is a subspace of a countable product of second-countable spaces and is therefore itself second-countable, and moreover any countable base will consist of sets that are both open and closed. It thus follows by Brouwer's theorem (see [Fra12], for instance) that $C_{\mathcal{F}}$ is homeomorphic to the Cantor set. This completes the proof. ■

It can be shown that the embedding σ induces an isomorphism with respect to the uniform structures given on $Q_{\mathcal{F}}$ (compatible with the topology of cones) and the Cantor space $C_{\mathcal{F}}$, in the sense that $\sigma : Q_{\mathcal{F}} \to \mathrm{Im}(\sigma)$ is uniformly continuous with a uniformly continuous inverse. This example not only serves as a nice example of the Jones action, but illustrates a beautiful symmetry between the canonical actions of Richard Thompson's group and of forest-skein groups.

With this, we conclude the present chapter. In the following chapter, we will discuss how Jones arrived at Richard Thompson's groups in his efforts to more deeply link subfactor theory with conformal field theory.

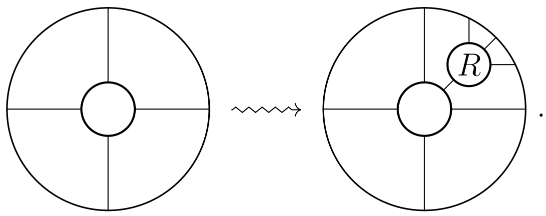
Chapter 3

The Road Towards “Forest-Skein Covariant Nets”

We briefly recall the original attempt of Jones in constructing a kind of “discrete” conformal field theory with Thompson’s group symmetry from a subfactor planar algebra ([Jon17]). We then view it under the lens of the forest-skein formalism.

3.1. The Intuitive Picture of Jones

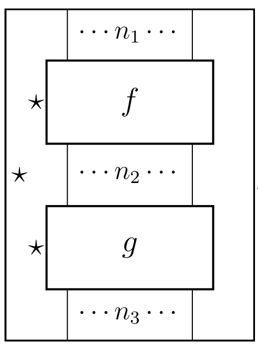
Jones’ original idea was as follows. Consider a subfactor planar algebra P and some fixed isometric element $R \in P_4$; that is, an element such that $R^*R = 1$. Given a planar tangle, we can refine it by inserting “tridents” containing R as follows.

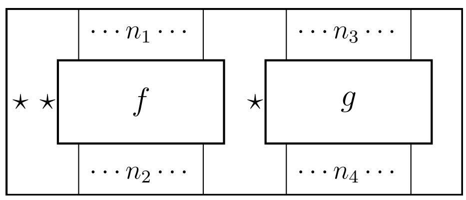


By interpreting the boundary of the output disc as the spacetime circle of a CFT, we obtain in the limit a kind of “discrete CFT”, with Thompson’s group T replacing the diffeomorphism group ([Bro20]). This is essentially what we wish to formalize.

3.1.1. First Steps: Building Unitary Representations of F

Definition 3.1.1.1. (Rectangular Category). *The* rectangular category $\mathrm{Rect}(P)$ *of a planar algebra P is the linear category with objects* $\mathrm{Ob}(\mathrm{Rect}(P)) := \mathfrak{B}_P$ *and morphisms* $\mathrm{Mor}(\iota_1, \iota_2) := P_{n_1+n_2}$, *for ι_1 and ι_2 in $\mathfrak{B}_P$ with n_1 and n_2 boundary points, respectively, and identical shading. We define composition of morphisms $f \in \mathrm{Mor}(\iota_1, \iota_2)$ and $g \in \mathrm{Mor}(\iota_2, \iota_3)$, for $\iota_1, \iota_2, \iota_3 \in \mathfrak{B}_P$ with n_1, n_2 and n_3 boundary points and identical shading, according to the concatenation tangle placed below and to the left, where shading and other boundary information is implied. This endows each P_ι with the structure of an algebra. Moreover, we make* $\mathrm{Rect}(P)$ *into a monoidal category by defining a monoidal product between $f \in \mathrm{Mor}(\iota_1, \iota_2)$ and $g \in \mathrm{Mor}(\iota_3, \iota_4)$ as below and to the right.*

$gf := g \circ f :=$ 

, $f \otimes g :=$ 

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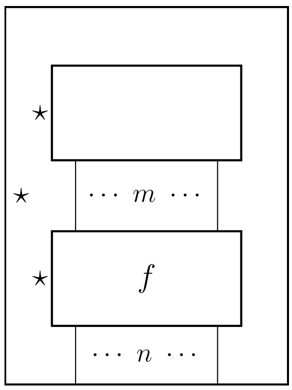
Recall that a representation of a group G is just a functor from the one-object groupoid $\mathsf{B}G$ – which consists of a unique object $\mathrm{Ob}(\mathsf{B}G) := \{\bullet\}$ and morphisms $\mathrm{Mor}(\mathsf{B}G) := G$ corresponding to the elements of G – to the category of vector spaces Vect, where the unique object of $\mathsf{B}G$ is sent to the vector space V for the representation and the action of G on V is defined by how the functor maps morphisms (elements of G). With this in mind, consider the following definition.

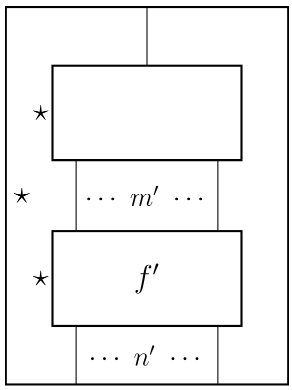
Definition 3.1.1.2. (Rectangular Representation). *A rectangular representation of P is a linear representation of* $\mathrm{Rect}(P)$ *– that is, a functor $V : \mathrm{Rect}(P) \to \mathsf{Vect}$.*

The *regular representation* is, in the unshaded case, simply the trivial representation taking $n \in \mathfrak{B}_P$ to $V_n := V(n) = P_n$, with the action given by the concatenation of rectangular tangles. In the shaded case, however we need to be a bit more careful; in particular, since $\mathrm{Mor}((m, \pm), (n, \pm))$ is zero unless $m + n$ is even (that is, m and n have identical parity), we choose

$$V_{n,\pm} := \begin{cases} P_{n,\pm}, & \text{if } n \text{ is even;} \\ P_{n+1,\pm}, & \text{if } n \text{ is odd.} \end{cases}$$

In the case when m and n are both even, we define the action of $f \in \mathrm{Mor}(m, n)$ on $v \in V_m$ in the shaded case by $[V(f)](v) := Z_T(v)$, for T defined below. In the odd parity case, with n' and m' both odd, we instead define the action of $f' \in \mathrm{Mor}(m', n')$ on $v' \in V_{m'} := P_{m'+1}$ by $[V(f')](v') := Z_{T'}(v')$, for T' defined below. Note that we view f and f' as elements of P_{m+n} and $P_{m'+n'}$, respectively.

$T :=$ 

, $T' :=$ 

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Suppose P is a shaded planar algebra with V its regular representation, and let $v \in V_1$ be an element of P_2. Consider the subrepresentation W generated by v; that is, the representation whose vector spaces consist of the image of v under various morphisms from our representation as above. Note that every such morphism acting on v must lie in $\mathrm{Mor}(1, n)$ for n necessarily odd. We see then that we must insist that $W_k = \{0\}$ whenever k is even, as the images all lie within odd-indexed spaces.

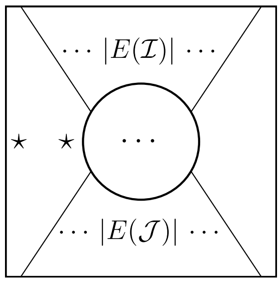
Suppose we let $\mathscr{D}$ be the set of all dyadic subdivisions of $[0, 1]$. This is a directed set under the preorder $\lesssim$, defined for partitions $\mathcal{I}, \mathcal{J} \in \mathscr{D}$ by $\mathcal{I} \lesssim \mathcal{J}$ if and only if

$$I = \bigcup_{\substack{J \in \mathcal{J} \\ J \subseteq I}} J$$

for each dyadic interval $I \in \mathcal{I}$. We say in this instance that $\mathcal{J}$ is a *refinement* of $\mathcal{I}$.

In this way, $\langle \mathscr{D}, \lesssim \rangle$ defines a small category. For each partition $\mathcal{I} \in \mathscr{D}$, we denote by $M(\mathcal{I})$ the set of midpoints of dyadic intervals in $\mathcal{I}$, and write $E(\mathcal{I}) \coloneqq M(\mathcal{I}) \cup e(\mathcal{I})$, where $e(\mathcal{I})$ is the set of all endpoints of intervals in $\mathcal{I}$ excluding 0 and 1.

Definition 3.1.1.3. *Let P be a planar algebra. We define $\mathrm{Cat}(P)$ to be the category with $\mathrm{Ob}(\mathrm{Cat}(P)) \coloneqq \mathscr{D} \times \{-,+\}$ (or similarly for other boundary data) and morphisms $T_{\mathcal{J}}^{\mathcal{I}} : (\mathcal{I}, \pm) \to (\mathcal{J}, \pm)$ for $\mathcal{I} \lesssim \mathcal{J}$ of the form*

$T_{\mathcal{J}}^{\mathcal{I}} =$ 

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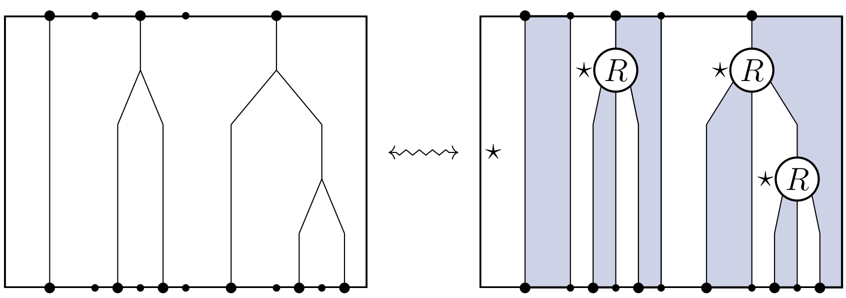
where the marked region of the box has shading $\pm$, the boundary points on the top and bottom sides of the rectangle are placed at $E(\mathcal{I})$ and $E(\mathcal{J})$, respectively (after identifying these sides with the unit interval), and where the planar tangle placed in the center determines the exact morphism.

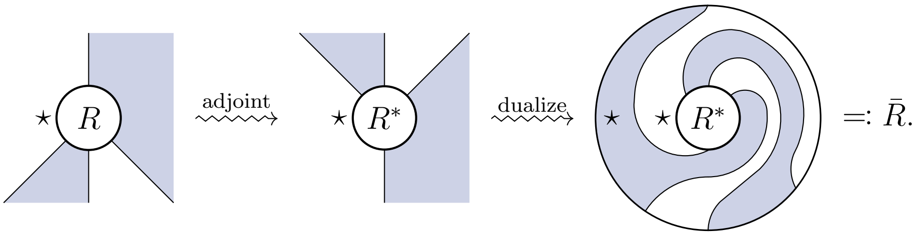
Note that $E(\mathcal{I})$ and $E(\mathcal{J})$ will always be odd, hence the rectangular tangles that represent morphisms of $\mathrm{Cat}(P)$ will always have an even number of boundary points as we expect from subfactor planar tangles. This is precisely the reason for considering $E(\mathcal{I})$ and $E(\mathcal{J})$! We will see imminently how we may interpret these boundary points. In essence, the boundary points in $M(\mathcal{I})$ and $M(\mathcal{J})$ can naturally be interpreted as roots and leaves of forests, with "tridents" containing an element R providing branches.

To illustrate this construction, consider the following diagram. We have on the left a dyadic rearrangement between the partition $\mathcal{I} \coloneqq ([0,1/4],[1/4,1/2],[1/2,1])$ and some refinement $\mathcal{J} \coloneqq ([0,1/4],[1/4,3/8],[3/8,1/2],[1/2,3/4],[3/4,7/8],[7/8,1])$, depicted using forests. We might denote this $F_{\mathcal{J}}^{\mathcal{I}}$, for instance. By replacing each trivalent vertex with a trident containing R, we obtain the tangle corresponding to a particular morphism $T_{\mathcal{J}}^{\mathcal{I}} \in \mathrm{Mor}_{\mathrm{Cat}(P)}((\mathcal{I},+),(\mathcal{J},+))$ representing $F_{\mathcal{J}}^{\mathcal{I}}$.

$F_{\mathcal{J}}^{\mathcal{I}} \coloneqq$ 

$\eqqcolon T_{\mathcal{J}}^{\mathcal{I}}$.

Note that with the marked interval chosen as in the previous diagram, the shading of the tangle R will always match the shading of the rectangular tangle $T^{\mathcal{I}}_{\mathcal{J}}$. This is essentially because the shading to the left of a root or leaf (boundary point representing the midpoint of an interval) will always match the overall shading of the rectangular tangle $T^{\mathcal{I}}_{\mathcal{J}}$. Because we want $T^{\mathcal{I}}_{\mathcal{J}}$ to be an isometry for any shading, we actually need two tangles, one whose marked interval is shaded and another whose marked interval is unshaded. To this end, we consider in the shaded case some isometric $\bar{R}$ given by R^* with its marked interval moved into the shaded region by composing it with some "dualizing tangle". In other words, we insist that



It is easy (yet necessary!) to check that $\bar{R}$, as a morphism in $\mathrm{Mor}_{\mathrm{Rect}(P)}(1,3)$, is isometric in the sense that $\bar{R}^*\bar{R} = 1$. This follows from a simple substitution, as well as the diffeomorphism invariance of planar tangles.

Suppose now that we have a planar algebra P, a rectangular representation V with $V_\iota := V(\iota)$ and an isometric element $R \in P_4$. Define a diagram $H : \mathrm{Cat}(P) \to \mathsf{Vect}$ taking objects $(\mathcal{I}, \pm) \mapsto V(|E(\mathcal{I})|, \pm)$ and morphisms $T^{\mathcal{I}}_{\mathcal{J}} \mapsto V(T^{\mathcal{I}}_{\mathcal{J}})$, with each $T^{\mathcal{I}}_{\mathcal{J}}$ constructed with respect to R and interpreted as a morphism in $\mathrm{Rect}(P)$; that is, an element of $\mathrm{Mor}_{\mathrm{Rect}(P)}(|E(\mathcal{I})|, |E(\mathcal{J})|)$. Of course, this is fine since we have an obvious functor from $\mathrm{Cat}(P)$ to $\mathrm{Rect}(P)$. In fact, what we end up with is nothing but a directed system of vector spaces, since $T^{\mathcal{J}}_{\mathcal{K}} \circ T^{\mathcal{I}}_{\mathcal{J}} = T^{\mathcal{I}}_{\mathcal{K}}$ when $\mathcal{I} \lesssim \mathcal{J}$ and $\mathcal{J} \lesssim \mathcal{K}$.

If P is a subfactor planar algebra, then each $H(\mathcal{I})$ will naturally be a pre-Hilbert space with inner product $\langle V(f), V(g)\rangle := V(Z_{T_L}(g^* \circ f))$, for $f, g \in P_{|E(\mathcal{I})|,\pm}$ and T_L as given in the definition for spherical planar algebras. This is because $P_{0,\pm}$ is one-dimensional and hence we require $V(P_{0,\pm}) \cong \mathbb{C}$. Moreover, if R is isometric in the aforementioned sense as a morphism in $\mathrm{Mor}_{\mathrm{Rect}(P)}(1,3)$, then $T^{\mathcal{I}}_{\mathcal{J}}$ is isometric as a morphism in $\mathrm{Rect}(P)$. It in fact follows that $V(T^{\mathcal{I}}_{\mathcal{J}})$ must then be an honest-to-goodness isometry of pre-Hilbert spaces; this may be seen by the functoriality of V together with the previous illustrations, as we must always have that both $T^{\mathcal{I}}_{\mathcal{I}} = \mathrm{id}_{\mathcal{I}} \in \mathrm{Mor}_{\mathrm{Rect}(P)}(|E(\mathcal{I})|, |E(\mathcal{I})|)$ and $(T^{\mathcal{I}}_{\mathcal{J}})^* \circ T^{\mathcal{I}}_{\mathcal{J}} = \mathrm{id}_{\mathcal{I}}$.

With this data, we may take a colimit (direct limit) and complete to obtain a full Hilbert space. This is the space on which we will define our representation.

Definition 3.1.1.4. (Dyadic Hilbert Space). *Let V be a rectangular representation of a planar algebra P and $\mathcal{V}^F_{V,R}$ the direct limit of the previously defined directed system $(H(\mathcal{I}), H(T^{\mathcal{I}}_{\mathcal{J}}))$ induced by V and some fixed isometry $R \in P_4$. We call this the* dyadic limit of the rectangular representation *V. If P is a subfactor planar algebra, then $\mathcal{V}^F_{V,R}$ becomes a pre-Hilbert space by lemma 2.6.4. Its Hilbert space completion is known as the* dyadic Hilbert space $\mathscr{H}^F_{V,R}$ associated with R and V.

After constructing a dyadic Hilbert space $\mathscr{H}^F_{V,R}$, Jones went on to show that the action of Thompson's group F on $\mathscr{H}^F_{V,R}$ induces a unitary representation. We briefly summarize his process from [Jon17]. We begin by defining an action of dyadic rearrangements $g \in F$ on each dyadic partition $\mathcal{I} \in \mathscr{D}$, then show that it defines an action ρ on $H(\mathcal{I})$ and finally extends to a unitary representation on the direct limit.

Definition 3.1.1.5. *A dyadic partition $\mathcal{I} \in \mathscr{D}$ is said to be* good for the dyadic rearrangement $g \in F$ *if $g(\mathcal{I}) \in \mathscr{D}$; that is, g sends $\mathcal{I}$ to a dyadic partition. In this case, we define a rectangular tangle $g_{\mathcal{I}} \in \mathrm{Mor}_{\mathrm{Cat}(P)}(\mathcal{I}, g(\mathcal{I}))$ consisting of straight lines connecting each point $k \in E(\mathcal{I})$ on the top to the point $g(k)$ on the bottom.*

Of course it is obvious from the definition of F that for each dyadic rearrangement, there exists a dyadic partition that is good for it. In fact, it is easy to see that given *any* dyadic rearrangement $g \in F$ and dyadic partition $\mathcal{I}$, we can always refine $\mathcal{I}$ into a dyadic partition $\mathcal{J} \gtrsim \mathcal{I}$ that is good for g by making sure that $\mathcal{J}$ is also a refinement of the source partition of g.

Lemma 3.1.1.6. *If $\mathcal{I}$ is good for $g \in F$ and $\mathcal{J} \gtrsim \mathcal{I}$ is a refinement of $\mathcal{I}$, then $\mathcal{J}$ is also good for g. Moreover, the morphism $g_{\mathcal{J}} \circ T^{\mathcal{I}}_{\mathcal{J}} \in \mathrm{Mor}_{\mathrm{Cat}(P)}(\mathcal{I}, g(\mathcal{J}))$ is isotopic as a labelled tangle to $T^{g(\mathcal{I})}_{g(\mathcal{J})} \circ g_{\mathcal{I}}$.*

The proof of this lemma is "intuitively obvious" from the pictorial setup, and proving it rigorously is not much more difficult. We now have an action of each dyadic rearrangement $g \in F$ on compatible dyadic partitions that is preserved under the direct limit. With these considerations in mind, suppose take $g \in F$ and $v \in H(\mathcal{I})$, for some $\mathcal{I} \in \mathscr{D}$. By our previous remark, we can always find $\mathcal{J} \gtrsim \mathcal{I}$ that is good for g. We thus define $\rho^F_{\mathcal{J}}(v) := [H(g_{\mathcal{J}} \circ T^{\mathcal{I}}_{\mathcal{J}})](v) \in H(g(\mathcal{J}))$. It follows from [Jon17] (and is not too difficult to see) that $\rho^F_{\mathcal{J}}(v)$ is in fact independent of the chosen refinement $\mathcal{J}$ in the direct limit. In particular, given any $v \in H(\mathcal{I})$ and $v' \in H(\mathcal{I}')$ that are equal in the direct limit, we also have that $\rho^F_{\mathcal{K}}(v) = \rho^F_{\mathcal{K}}(v')$ in the direct limit, for appropriate $\mathcal{K} \gtrsim \mathcal{I}, \mathcal{I}'$. We thus make the following definition.

Proposition-Definition 3.1.1.7. *For $g \in F$ and $v \in \mathcal{V}^F_{V,R}$, we define an action $\pi^T_{V,R} : F \curvearrowright \mathcal{V}^F_{V,R}$ by $[\pi^F_{V,R}(g)](v) := \rho^F_{\mathcal{I}}(w)$, where we choose any $\mathcal{I}$ and $w \in H(\mathcal{I})$ such that w represents v in the direct limit. Then the map $U^F_{V,R} : g \mapsto \pi^F_{V,R}(g)$ defines a linear representation, and if P is a subfactor planar algebra it preserves the inner product. Thus it extends to a unitary representation of F on $\mathscr{H}^F_{V,R}$.*

3.1.2. Building Projective Unitary Representations of T

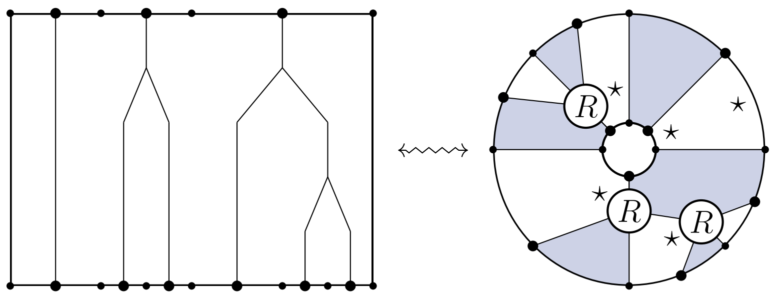
In addition to providing unitary representations of F, Jones also managed to obtain from his construction unitary representations of Thompson's group T. The process is quite similar to before, but we now consider affine representations of P; these are an annular equivalent to our rectangular representations. In his paper, Jones actually did something slightly more general, producing representations of the central extension $\widetilde{T}$ that, in certain cases, induce genuine representations of T.

Definition 3.1.2.1. (Affine Category). *The* affine category $\mathrm{Aff}(P)$ *of a planar algebra* P *is the linear category with objects* $\mathrm{Ob}(\mathrm{Aff}(P)) \coloneqq \mathfrak{B}_P$ *and morphism sets described as follows. Let* $\iota_1, \iota_2 \in \mathfrak{B}_P$ *with* n_1 *and* n_2 *boundary points, respectively, and identical shading. Then* $\mathrm{Mor}(\iota_1, \iota_2)$ *is the vector space consisting of formal linear combinations of labelled* n_2*-tangles of* P*, each containing a single central input disc with* n_1 *boundary points, where we now insist that the boundary of this input disc is no longer subject to diffeomorphism invariance (as is usually only the case for the boundary of the output disc of a tangle from a planar operad). The composition of morphisms* $g \circ f$ *is given by inserting the tangle for* f *into the input disc of* g*.*

We define *affine representations of* P in much the same way as we define rectangular representations, except with our action of the corresponding (now irreducible) trivial representation involving just plugging a vector into the input disc of the morphism.

We also define maps $T^{\mathcal{I}}_{\mathcal{J}}$ similarly to before, except now they are given an annular representation rather than a rectangular one. Let $\bar{E}(\mathcal{I}) \coloneqq M(\mathcal{I}) \cup \bar{e}(\mathcal{I})$ for $\bar{e}(\mathcal{I})$ consisting of all endpoints of intervals in $\mathcal{I}$, *including now the endpoint* $0 \sim 1$. Given a map $F^{\mathcal{I}}_{\mathcal{J}}$, we define a tangle $T^{\mathcal{I}}_{\mathcal{J}} \in \mathrm{Mor}_{\mathrm{Aff}(P)}((\bar{E}(\mathcal{I}), \pm), (\bar{E}(\mathcal{J}), \pm))$ by simply taking the corresponding tangle $T^{\mathcal{I}}_{\mathcal{J}}$ from $\mathrm{Cat}(P)$ and gluing the left and right edges into an annulus. To illustrate this, consider $\mathcal{I} \coloneqq ([0, 1/4], [1/4, 1/2], [1/2, 1])$ along with the refinement $\mathcal{J} \coloneqq ([0, 1/4], [1/4, 3/8], [3/8, 1/2], [1/2, 3/4], [3/4, 7/8], [7/8, 1])$ as before. Since F is a subgroup of T, we may consider the same map $F^{\mathcal{I}}_{\mathcal{J}}$ as before with no cyclic permutations. Then $T^{\mathcal{I}}_{\mathcal{J}} \in \mathrm{Mor}_{\mathrm{Aff}(P)}((\bar{E}(\mathcal{I}), +), (\bar{E}(\mathcal{J}), +))$ is given by

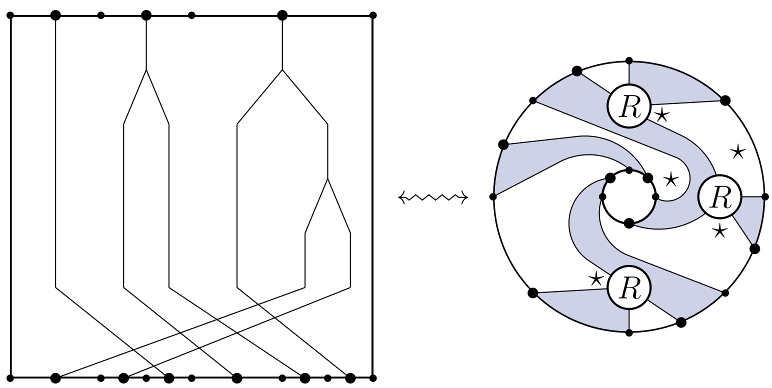
$$F^{\mathcal{I}}_{\mathcal{J}} \coloneqq$$



$$\eqqcolon T^{\mathcal{I}}_{\mathcal{J}}.$$

To see how this works with a slightly more interesting map, let $F^{\mathcal{I}}_{\mathcal{J}}$ be as before but composed with the cyclic permutation that shifts the leaves right twice; this is a map we could not represent before. Then $T^{\mathcal{I}}_{\mathcal{J}} \in \mathrm{Mor}_{\mathrm{Aff}(P)}((\bar{E}(\mathcal{I}), +), (\bar{E}(\mathcal{J}), +))$ becomes

$$F^{\mathcal{I}}_{\mathcal{J}} \coloneqq$$



$$\eqqcolon T^{\mathcal{I}}_{\mathcal{J}}.$$

Suppose we now let V be an affine representation of a planar algebra P. We define a new directed system just like before, using now the diagram $K : \mathrm{Aff}(P) \to \mathsf{Vect}$ that acts on objects as $(\mathcal{I}, \pm) \mapsto V(\left|\bar{E}(\mathcal{I})\right| + 1, \pm)$ and on morphisms as $T_{\mathcal{J}}^{\mathcal{I}} \mapsto V(T_{\mathcal{J}}^{\mathcal{I}})$, with each $T_{\mathcal{J}}^{\mathcal{I}}$ constructed with respect to some fixed isometry $R \in P_4$. Taking the direct limit, we obtain a new kind of dyadic Hilbert space.

Definition 3.1.2.2. (Dyadic Hilbert Space). *Let V be an affine representation of a planar algebra P and $\mathcal{V}_{V,R}^{T}$ be the direct limit of the previously defined directed system $(K(\mathcal{I}), K(T_{\mathcal{J}}^{\mathcal{I}}))$ induced by V and some fixed isometry $R \in P_4$. We call this the* dyadic limit of the affine representation V*. If P is a subfactor planar algebra, the Hilbert space completion of $\mathcal{V}_{V,R}^{T}$ is known as the* dyadic Hilbert space $\mathscr{H}_{V,R}^{T}$ associated with R and V.

Suppose we now define $\widetilde{T}$ to be the group of piecewise affine homeomorphisms g of $\mathbb{R}$ with $g(x) = g(x+1) - 1$ that send dyadic rationals to dyadic rationals and are differentiable on intervals with dyadic rational endpoints, with the derivatives being powers of 2 where they exist. Note that $\widetilde{T}$ is the central extension of T by $\mathbb{Z}$ with monomorphism $\widetilde{\iota} : n \mapsto (x \mapsto x + n)$ and epimorphism $\widetilde{\pi} : g \mapsto (x \mapsto g(x) \bmod 1)$, giving us the isomorphism $T \cong \widetilde{T}/\mathrm{Im}(\widetilde{\iota}) = \widetilde{T}/\mathrm{Ker}(\widetilde{\pi})$.

By adapting the process that we used to obtain unitary representations of F, we may construct an action $\pi_{V,R}^{T} : \widetilde{T} \curvearrowright \mathcal{V}_{V,R}^{T}$ that extends to a unitary representation $U_{V,R}^{T} : \widetilde{T} \to \mathcal{U}(\mathscr{H}_{V,R}^{T})$. We shall briefly discuss how it may be adapted. Similarly to before, we say that an element $\mathcal{I} \in \mathscr{D}$ is *good for* $g \in \widetilde{T}$ if $[\widetilde{\pi}(g)](\mathcal{I})$ is a dyadic partition. In this case, we define an affine tangle $g_{\mathcal{I}} \in \mathrm{Mor}_{\mathrm{Aff}(P)}(\mathcal{I}, [\widetilde{\pi}(g)](\mathcal{I}))$ in the obvious way, by drawing a curve from the point $\exp(2\pi i k)$ on the inner circle to the point $\exp(2\pi i[\widetilde{\pi}(g)](k))$ on the outer circle, for each $k \in \bar{E}(\mathcal{I})$. By defining $\rho_{\mathcal{J}}^{T}(v) \coloneqq [K(g_{\mathcal{J}} \circ T_{\mathcal{J}}^{\mathcal{I}})](v) \in K(g(\mathcal{J}))$ similarly to how we did in the case of F, we have the following theorem of [Jon17].

Theorem 3.1.2.3. *For $g \in \widetilde{T}$ and $v \in \mathcal{V}_{V,R}^{T}$, we define an action $\pi_{V,R}^{T} : \widetilde{T} \curvearrowright \mathcal{V}_{V,R}^{T}$ by $[\pi_{V,R}^{T}(g)](v) \coloneqq \rho_{\mathcal{I}}^{T}(w)$, where we choose any $\mathcal{I}$ and $w \in K(\mathcal{I})$ such that w represents v in the direct limit. Then the map $U_{V,R}^{T} : g \mapsto \pi_{V,R}^{T}(g)$ defines an affine representation, and if P is a subfactor planar algebra it preserves the inner product. Thus it extends to a unitary representation of $\widetilde{T}$ on $\mathscr{H}_{V,R}^{T}$.*

In this way, Jones showed that affine representations provide unitary representations of $\widetilde{T}$. If we insist that our representation is irreducible and that 2π rotations act as scalar multiples of the identity – that is, $V(T_{2\pi}) = \lambda\mathrm{id}$ for $T_{2\pi} \in \mathrm{Mor}(\mathrm{Aff}(P))$ the 2π rotation tangle and $\lambda \in \mathbb{C}$, we obtain a projective unitary representation of T ([Jon18]). Of course if $\lambda = 1$, this becomes an honest-to-goodness unitary representation of T as one would expect, and we refer to affine representations with this property as *annular representations*. The restriction of the resulting unitary representations to F also recover the representations of F we constructed previously.

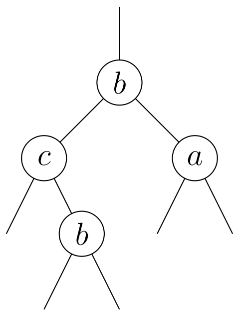
Given these unitary representations of F and T, it is likely that one could define a notion of support and obtain "discrete" CFTs, where Thompson's groups played the role of the chiral conformal symmetry group. Unfortunately, even ignoring the fact that we only have representations of F and T rather than of $\mathrm{Diff}^+(S^1)$, we also have the issue that our representations don't actually use very much data from the planar algebra, as even forcing them to use the shading is contrived. An earlier attempt of Jones that involved attaching isometric elements $R \in P_1$ to the output disc of an annular tangle did manage to obtain projective unitary representations of $\mathrm{Diff}^+(S^1)$ in the direct limit, but unfortunately the dyadic Hilbert space was not separable and the action of the diffeomorphism group was unable to be made continuous; moreover, it used even less of the subfactor data than his later attempts!

3.2. Jones Representations of Forest-Skein Groups

We conclude the thesis by briefly discussing how we could apply the technology that Jones discovered to other Thompson-like groups, such as the forest-skein groups of Brothier. Optimistically we might like to eventually find a way of building CFTs (or "CFT-like" constructions), although we will see that this proves rather challenging. Let P be a subfactor planar algebra admitting an annular representation V and $\mathcal{F} := \mathrm{ForC}\langle \mathcal{S} | \mathcal{R} \rangle$ a forest-skein category of type F or T admitting a calculus of fractions at 1. Fix $k \geq 2$, and for each colour $a \in \mathcal{S}$ define a distinct isometry $R_a \in \mathrm{Mor}_{\mathrm{Rect}(P)}((k,-),(2k,-))$. Moreover, let $R_I \in \mathrm{Mor}_{\mathrm{Rect}(P)}((k,-),(k,-))$ denote the unit tangle consisting of k strings whose endpoints all lie on the outer disc. Suppose now that we let $t \in \mathrm{Mor}(\mathcal{UF}^F\langle \mathcal{S} \rangle)$, and observe that we may write

$$t = f_\ell \circ \cdots \circ f_1$$

for composable forests $f_1, \ldots, f_\ell$, each consisting of nothing but a finite monoidal product of trivial trees and coloured carets. For instance,

$t =$ 

$= (I \otimes Y_b \otimes I \otimes I) \circ (Y_c \otimes Y_a) \circ Y_b.$

We will call this the *monoidal decomposition* of a tree, and note that by placing the diagram in a circle and replacing the coloured nodes with their corresponding isometries, we obtain a planar tangle T_t. We thus construct a strict monoidal functor $\phi : \mathcal{UF}\langle \mathcal{S} \rangle \to \mathsf{Vect}$ acting on objects by taking $n \mapsto V(P_{kn,-})$ and on morphisms by taking $I \mapsto V(R_I)$ and $Y_a \mapsto V(R_a)$ for each colour $a \in \mathcal{S}$. For each relation $(t,t') \in \mathcal{R}$ and planar tangles $T_t, T_{t'} \in \mathrm{Mor}_{\mathrm{Aff}(P)}(1, \mathrm{Leaf}(t))$ representing t and t', respectively, we quotient $P_{\mathrm{Leaf}(t)}$ by $T_t - T_{t'}$ to obtain a new planar algebra $\widetilde{P}$. At this point, we need to put some constraints on our annular representation, as taking a quotient of V will only serve to destroy the entire structure of the representation. In particular, we need a new representation $\widetilde{V}$ with the additional constraints that $\widetilde{V}(T_t) = \widetilde{V}(T_{t'})$ for each $(t,t') \in \mathcal{R}$.

Given such a setup, we can in theory build up a unitary representation of $\mathcal{F}$ on some Hilbert space, but developing in full generality the notion of a "forest-skein covariant net" that is analogous to the "discrete CFTs" proposed by Jones seems quite challenging. In order for this to be done, we must define some notion of support, and we would also ideally like to understand what kinds of algebras of observables and representations we obtain. Undeterred, we recognize that the first step forward along this path is to understand precisely how skein relations can translate into relations in our subfactor planar algebras.

To this end, let us proceed by forgetting about the intermediate subfactor planar algebras for the moment and focus on building directly a strict monoidal functor $\Phi : \mathcal{F} \to \mathsf{Inn}$. Given such a functor, we can reinterpret it as passing through a planar algebra later on. If we let $V := \Phi(1)$ and have an isometry $R : V \to V \oplus V$, the pair of endomorphisms (A, B) for which $R = A \oplus B$ must satisfy

$$\langle x, x\rangle = \langle R(x), R(x)\rangle = \langle x, [A^*A + B^*B](x)\rangle \iff A^*A + B^*B = \mathrm{id}.$$

We call such a pair (A, B) with $A^*A+B^*B = \mathrm{id}$ a *Pythagorean pair*. This essentially characterizes what our isometric elements R corresponding to the carets of $\mathcal{F}$ must look like. In general, these Pythagorean pairs are difficult to get a grasp on. As a result, let us instead consider an explicit example. Recall the forest-skein category $\mathcal{C}_3$ underlying one of the generalized Cleary groups, as given in example 2.5.1.4, and let $R_a := u_a \oplus v_a$ and $R_b := u_b \oplus v_b$ be our caret isometries. Under the skein relations of $\mathcal{C}_3$, these maps must satisfy

$$(R_a \oplus \mathrm{id}_V \oplus \mathrm{id}_V) \circ (R_a \oplus \mathrm{id}_V) \circ R_a = (\mathrm{id}_V \oplus \mathrm{id}_V \oplus R_b) \circ (\mathrm{id}_V \oplus R_b) \circ R_b.$$

This gives us an equation for each leaf, resulting in the system

$$u_a^3 = u_b, \qquad v_a u_a^2 = u_b v_b, \qquad v_a u_a = u_b v_b^2, \qquad v_a = v_b^3.$$

Substitution thus gives us

$$v_a u_a^2 v_b = u_b v_b^2 = v_a u_a, \qquad u_b v_b^2 u_a = v_a u_a^2 = u_b v_b.$$

If we assume that $v_a u_a$ and $u_b v_b$ are injective – that is, they have left-inverses – then

$$v_a u_a^2 v_b = v_a u_a \implies u_a v_b = \mathrm{id}, \qquad u_b v_b^2 u_a = u_b v_b \implies v_b u_a = \mathrm{id}.$$

In other words, u_a and v_b are in fact mutually inverse. Thus $u_b = u_a^{-3} = v_b^3$ and $v_a = v_b^{-3} = u_a^3$, meaning $R_a = r \oplus r^{-3}$ and $R_b = r^3 \oplus r^{-1}$ for some linear bijection r. This is actually quite exciting; under a rather unintrusive assumption, we are able to write both R_a and R_b in terms of a single *bijection* despite never insisting directly that they be bijective, and moreover the explicit forms of R_a and R_b seem to reflect the diagrammatic skein relations in a transparent way. In particular, if we write $R_a = \frac{1}{\sqrt{2}} u_a \oplus \frac{1}{\sqrt{2}} u_b$ for isometries u_a and u_b, and similarly for R_b, the structure of our skein relation tells us that R_a and R_b must be unitary maps.

Unfortunately, we observe that a similar process seems to fail – or at least not be so straightforward – for the other Cleary groups. In general, the structure of our caret isometries seems to be usually rather unintuitive. We believe the way forward from here would be to better understand when exactly a forest-skein category admits a calculus of fractions, as well as what these Pythagorean pairs look like.

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