

# INDECOMPOSABLE SOERGEL BIMODULES OF TYPES $A_1$ AND $A_2$

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The goal of these notes is to explicitly step through the classification of the indecomposable Soergel bimodules of type  $A_2$ , while also covering the simpler  $A_1$  case as a stepping stone. Throughout these notes,  $\mathbb{k}$  will always be taken to be an algebraically closed field of characteristic zero.

**Definition 1.** (Geometric Representation). *Let  $(W, S)$  be a Coxeter system and  $V$  the  $\mathbb{k}$ -vector space with formal basis  $\{\alpha_s : s \in S\}$ . Define a symmetric, bilinear form on  $V$  by linearly extending*

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right), & m_{st} \neq \infty; \\ -1, & m_{st} = \infty. \end{cases}$$

*From this, we define an action of  $s \in S$  on the basis elements  $\alpha_t \in V$  by the linear automorphism*

$$s(\alpha_t) := \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s,$$

*which reflects  $\alpha_t$  across  $\alpha_s$ . The geometric representation of  $(W, S)$  is the representation induced by linearly extending this reflection action to an action of  $W$  on all of  $V$ .*

Recall that by definition,  $R := \bigoplus_{i=0}^{\infty} \text{Sym}^i(V)$ , where  $\text{Sym}^i(V)$  is the quotient of the  $i$ th tensor power  $V^{\otimes i}$  by the action of the symmetric group  $S_i$ , viewed as a  $\mathbb{k}$ -module that is  $\mathbb{Z}$ -graded in degree 2. Our reflection action also extends to an action on  $R$  by taking  $s(fg) := s(f)s(g)$ , for  $f, g \in R$ . It is not too difficult to show that  $\text{Sym}^i(V)$  is isomorphic to the additive subgroup of the polynomial ring  $\mathbb{k}[\alpha_s : s \in S]$  consisting only of homogeneous polynomials of degree  $i$ , meaning we may identify  $R = \mathbb{k}[\alpha_s : s \in S]$ . With this in mind, we have a crucial lemma that we will invoke regularly.

**Lemma 2.** *Suppose that  $M$  is a graded  $(R, R)$ -bimodule that is generated by a homogeneous element  $m \in M$ , in the sense that  $M = RmR$ . Then  $M$  is indecomposable.*

**Proof.** Let  $d$  denote the degree of  $m$ . Because  $M$  is generated by  $m$ , and because  $R^0 = \mathbb{k}$ , we have that  $M^d = R^0 m R^0 = \mathbb{k}m$ . In other words,  $M^d$  is a one-dimensional vector space. Suppose that  $M \cong L \oplus N$ . In this case,  $M^d$  is isomorphic as a  $\mathbb{k}$ -vector space to  $L^d \oplus N^d$ . Assume without loss of generality that  $m \in L^d$ . This forces  $N^d = 0$ , as  $M^d$  is one-dimensional, whence we have that  $M = RmR \subseteq L$ , forcing  $N = 0$ . Thus  $M$  is indecomposable. This completes the proof. ■

From this lemma, it follows that  $R$  itself is indecomposable as an  $(R, R)$ -bimodule, as it is generated by  $1 \in \mathbb{k}$ . Moreover, suppose we define

$$B_s := R \otimes_{R^s} R(1)$$

for any  $s \in S$ , where  $R^s := \{f \in R : s(f) = f\}$ . This is also clearly indecomposable by our lemma, as it is generated as an  $(R, R)$ -bimodule by  $1 \otimes_{R^s} 1$  (where we note that, since  $1 \in R^0 = R(1)^{-1}$ , we have  $1 \otimes_{R^s} 1 \in B_s^{0-1} = B_s^{-1}$ , meaning it is homogeneous of degree  $-1$ ). In particular, it is easy to see that grading shifts of  $R$  and  $B_s$  are all indecomposable too.

**Definition 3.** (Soergel Bimodule). Let  $(W, S)$  be a Coxeter system and  $\underline{w} := (s_1, \dots, s_k)$  an expression. The Bott–Samelson bimodule corresponding to  $\underline{w}$  is the graded  $(R, R)$ -bimodule

$$\begin{aligned} BS(\underline{w}) &:= B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \\ &= (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R(k). \end{aligned}$$

A Soergel bimodule is any graded  $(R, R)$ -bimodule that is isomorphic to a finite direct sum of grading shifts of direct summands of Bott–Samelson bimodules.

Another useful lemma is that any polynomial in  $R$  can be split into the sum of an  $s$ -invariant component and an  $s$ -antiinvariant component, for any  $s \in S$ .

**Lemma 4.** For any  $s \in S$  and  $f \in R$ , we have that  $f + s(f) \in R^s$  and  $f - s(f) \in R^s\alpha_s$ . In particular, we have a graded  $(R^s, R^s)$ -bimodule splitting  $R \cong R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$  for all  $s \in S$ .

**Proof.** Let  $s \in S$  and  $f := \alpha_t$  for any  $t \in S$ . Observe that  $P_s(f) := \frac{1}{2}(f + s(f)) = \alpha_t - (\alpha_s, \alpha_t)\alpha_s$  is in  $R^s$  (alternatively,  $s(f + s(f)) = s(f) + s^2(f) = s(f) + f$ ). Similarly,  $\partial_s(f)\alpha_s := \frac{1}{2}(f - s(f)) = (\alpha_s, \alpha_t)\alpha_s$  is clearly in  $R^s\alpha_s$  (alternatively,  $s(f - s(f)) = s(f) - s^2(f) = s(f) - f$ ). By linearity of the action of  $S$ , sums will preserve this splitting, so all that remains is to show that products preserve it too. Suppose  $f := f_1 + f_2\alpha_s$  and  $g := g_1 + g_2\alpha_s$ , for  $f_1, f_2, g_1, g_2 \in R^s$ , such that  $fg = f_1g_1 + f_1g_2\alpha_s + f_2g_1\alpha_s + f_2g_2\alpha_s^2$ . We see that  $P_s(fg) = g_1g_2 + h_1h_2\alpha_s^2$  and  $\partial_s(fg)\alpha_s = g_1h_2\alpha_s + g_2h_1\alpha_s$ , whence  $fg$  admits a unique decomposition of the form  $fg = P_s(fg) + \partial_s(fg)\alpha_s$ . Clearly this induces an isomorphism of graded  $(R^s, R^s)$ -bimodules, as for any  $f \in R$  and  $g, h \in R^s$ , we have that  $hfg \mapsto (hP_s(f)g, h\partial_s(f)\alpha_s g)$ . This completes the proof. ■

Note that this is an isomorphism of  $(R^s, R^s)$ -bimodules, not  $(R, R)$ -bimodules! This lemma allows us to completely classify the indecomposables in the type  $A_1$  case.

**Proposition 5.** (Indecomposable Soergel Bimodules of Type  $A_1$ ). The indecomposable Soergel bimodules of type  $A_1$  are, up to grading shift and isomorphism,  $R$  and  $B_s$ .

**Proof.** Let  $W = S_2 = \langle s \rangle$  and  $S = \{s\}$ . By our previous observations, we know that grading shifts of  $R$  and  $B_s$  are indecomposable, as they are generated by the homogeneous unit tensors. The next step is to check the Bott–Samelson  $B_s \otimes_R B_s$ . But observe that

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \tag{Lemma 4} \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

Thus  $B_s \otimes_R B_s$  is decomposable, meaning that we have exhausted all candidates for indecomposables. This completes the proof. ■

**Remark 6.** Observe that the indecomposables are, up to grading shift, in bijection with  $W$ , just as we would expect by the categorification theorem.

One last lemma that we will find useful before we attempt to classify the indecomposable Soergel bimodules of type  $A_2$  is the following.

**Lemma 7.** *Let  $s, t \in S$  with  $s \neq t$ . Then  $R^s$  and  $R^t$  generate  $R$  as a ring if and only if  $m_{st} \neq \infty$ .*

**Proof.** Recall that  $R := \text{Sym}(V)$ . As a polynomial ring,  $R$  is generated by its linear terms – those being the vectors in  $V$ . The linear terms in  $R^s$  are precisely the vectors fixed by  $s$ . This “intersection of  $R^s$  with  $V$ ” is nothing but

$$H_s := \{v \in V : s(v) = v\} = \{v \in V : (v, \alpha_s) = 0\}.$$

Note that the map  $\varphi_s : V \rightarrow \mathbb{k}$  given by  $v \mapsto (v, \alpha_s)$  is obviously a surjective linear map, as  $\varphi_s(\lambda \alpha_s)$  always maps to  $\lambda \in \mathbb{k}$ . We therefore have that

$$\dim(H_s) = \dim(\ker(\varphi_s)) = \dim(V) - \dim(\text{rank}(\varphi_s)) = \dim(V) - 1$$

by the rank-nullity theorem. In other words,  $H_s$  is a hyperplane. When  $m_{st} \neq \infty$ , we have that  $H_s$  and  $H_t$  are distinct. Because they are hyperplanes, this means that they must span  $V$ , whence  $R^s$  and  $R^t$  must generate  $R$ . This completes the proof. ■

**Remark 8.** This result becomes trivial when considering that, for any  $s \in S$ , the ring  $R^s$  is generated by the unit,  $\alpha_s^2$  and elements of the form  $\alpha_t - (\alpha_s, \alpha_t)\alpha_s$ , for all  $t \in S \setminus \{s\}$ . It is easy to see that, given fixed  $s, t \in S$  with  $s \neq t$ , we have

$$\alpha_s = \frac{1}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_s - (\alpha_s, \alpha_t)\alpha_t) + \frac{(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_t - (\alpha_s, \alpha_t)\alpha_s)$$

if and only if  $m_{st} \neq \infty$ , and similarly for  $\alpha_t$ . However, showing that these polynomials generate  $R^s$  is non-trivial, following from the Chevalley–Shephard–Todd theorem.

Let  $s, t \in S$  with  $s \neq t$  and  $m_{st} \neq \infty$ . Observe that

$$B_s \otimes_R B_t \cong R \otimes_{R^s} R \otimes_{R^t} R(2) \quad \text{and} \quad B_t \otimes_R B_s \cong R \otimes_{R^t} R \otimes_{R^s} R(2).$$

It follows from Lemma 2 that these are both indecomposable, as by Lemma 7 they are generated by the degree  $-2$  elements  $1 \otimes_{R^s} 1 \otimes_{R^t} 1$  and  $1 \otimes_{R^t} 1 \otimes_{R^s} 1$ , respectively. We shall therefore write  $B_{st} := B_s \otimes_R B_t$  and  $B_{ts} := B_t \otimes_R B_s$  from this point onwards.

**Remark 9.** As it happens,  $B_s \otimes_R B_t$  and  $B_t \otimes_R B_s$  remain indecomposable when  $m_{st} = \infty$ , but proving this is somewhat more involved. In addition to our previous lemma no longer holding (since  $H_s = H_t$  end up being the same 1-dimensional space, generated by  $\alpha_s + \alpha_t$ ), these bimodules are also no longer cyclic, meaning we cannot use Lemma 2!

The remainder of these notes will be in proving the following result, where  $B_{sts} := R \otimes_{R^{s,t}} R(3)$ . This notation will be justified by the following proposition. However, we note that – remarkably – a similar definition can be made for any Soergel bimodule  $B_{w_0}$  corresponding to the longest element  $w_0$  of its associated Coxeter group.

**Proposition 10.** (Indecomposable Soergel Bimodules of Type  $A_2$ ). *The indecomposable Soergel bimodules of type  $A_2$  are, up to grading shift and isomorphism,  $R$ ,  $B_s$ ,  $B_t$ ,  $B_{st}$ ,  $B_{ts}$  and  $B_{sts}$ .*

**Proof.** Let  $W = S_3 = \{1, s, t, st, ts, sts\}$  and  $S = \{s, t\}$ . We have already seen that  $R$ ,  $B_s$ ,  $B_t$ ,  $B_{st}$  and  $B_{ts}$  are indecomposable Soergel bimodules. Certainly  $B_{sts}$  is indecomposable as an  $(R, R)$ -bimodule by Lemma 2, as it is generated by the degree  $-3$  element  $1 \otimes_{R^{s,t}} 1$ , but it remains to be shown that it is indeed a Soergel bimodule. We will begin by showing that it appears as a direct summand of both  $B_s \otimes_R B_t \otimes_R B_s$  and  $B_t \otimes_R B_s \otimes_R B_t$ . Once we've done this, we will show that these indecomposables are exhaustive.

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In order to show that  $B_{sts}$  is a Soergel bimodule, we first claim that

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \quad \text{and} \quad B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t.$$

To this end, let's define an  $(R, R)$ -bimodule homomorphism  $\iota_{sts} : B_{sts} \rightarrow B_s \otimes_R B_t \otimes_R B_s$  by extending

$$\iota_{sts} : 1 \otimes_{R^{s,t}} 1 \mapsto 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

This is clearly a well-defined homomorphism of graded  $(R, R)$ -bimodules, as  $f \otimes_{R^{s,t}} 1 = 1 \otimes_{R^{s,t}} f$  if and only if  $f \in R^{s,t}$  if and only if  $f \in R^s, R^t$ . In fact, this reasoning also shows that it is injective. Suppose we define another  $(R, R)$ -bimodule homomorphism  $\iota_s : B_s \rightarrow B_s \otimes_R B_t \otimes_R B_s$  by extending

$$\iota_s : 1 \otimes_{R^s} 1 \mapsto \frac{1}{2}(1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1).$$

In order to show that this is well-defined, we need to show that

$$f \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1 = 1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} f$$

for all  $f \in R^s$ . By Lemma 4, for any  $f \in R$  we have

$$f = P_t(f) + \partial_t(f)\alpha_t,$$

where  $P_t(f), \partial_t(f) \in R^t$ . If  $f \in R^s$ , then

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 &= 1 \otimes_{R^s} f \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} (P_t(f) + \partial_t(f)\alpha_t) \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= (1 \otimes_{R^s} P_t(f) \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1) + (1 \otimes_{R^s} \partial_t(f) \alpha_t^2 \otimes_{R^t} 1 \otimes_{R^s} 1) \\ &= (1 \otimes_{R^s} \alpha_t \otimes_{R^t} P_t(f) \otimes_{R^s} 1) + (1 \otimes_{R^s} 1 \otimes_{R^t} \partial_t(f) \alpha_t^2 \otimes_{R^s} 1), \end{aligned}$$

and similarly

$$f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 = (1 \otimes_{R^s} 1 \otimes_{R^t} P_t(f) \alpha_t \otimes_{R^s} 1) + (1 \otimes_{R^s} \alpha_t \otimes_{R^t} \partial_t(f) \alpha_t \otimes_{R^s} 1).$$

Summing these and combining the “even” terms  $P_t(f)$  with the “odd” terms  $\partial_t(f)\alpha_t$ , it follows that

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 + f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} f \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} f \alpha_t \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} f + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} f. \end{aligned}$$

Thus  $\iota_s$  is well-defined, and like  $\iota_{sts}$  it is a monomorphism. Because  $\langle 1 \otimes_{R^t} 1 \rangle$  and  $\langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle$  clearly generate disjoint  $(R^s, R^s)$ -bimodules,  $\iota_{sts}$  and  $\iota_s$  are themselves disjoint. If we can show

$$R \otimes_{R^t} R \cong \langle 1 \otimes_{R^t} 1 \rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle,$$

then  $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$ , with a similar computation yielding  $B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t$ .

While we could figure out how to write  $f \otimes_{R^t} g \in \langle 1 \otimes_{R^t} 1 \rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle$  for all  $f, g \in R$ , there is perhaps a more informative way to prove this result. This is to find explicit projections  $\pi_s : B_s \otimes_R B_t \otimes_R B_s \rightarrow B_s$  and  $\pi_{sts} : B_s \otimes_R B_t \otimes_R B_s \rightarrow B_{sts}$  for which

$$\begin{aligned}\pi_{sts} \circ \iota_{sts} &= \text{id}_{B_{sts}}, \\ \pi_s \circ \iota_s &= \text{id}_{B_s}, \\ \iota_{sts} \circ \pi_{sts} + \iota_s \circ \pi_s &= \text{id}_{B_s \otimes_R B_t \otimes_R B_s}.\end{aligned}$$

Note that this is precisely what  $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$  means by definition. It is not too hard to find  $\pi_s$ . One can easily show that it is nothing but

$$\pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -2\delta_s(fg) \otimes_{R^s} 1,$$

where we remind the reader that we are using the relatively non-standard normalization  $\delta_s : f \mapsto (f - s(f))/2\alpha_s$  of the Demazure operator. But how can we find  $\pi_{sts}$ ? Well, let

$$e := \iota_s \circ \pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -\delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

As we would expect,  $e_s$  is an idempotent, as

$$e_s^2 = \iota_s \circ \pi_s \circ \iota_s \circ \pi_s = \iota_s \circ \text{id}_{B_s} \circ \pi_s = e_s.$$

As mentioned above, we want to find  $e_{sts} = \iota_{sts} \circ \pi_{sts}$  satisfying

$$e_{sts} + e_s = \text{id}_{B_s \otimes_R B_t \otimes_R B_s}.$$

This gives us an explicit formula for  $e_{sts}$ ; namely,  $e_{sts} = \text{id}_{B_s \otimes_R B_t \otimes_R B_s} - e_s$ , whence

$$e_{sts} = 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 + \delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

We can use this to determine  $\pi_{sts}$ . First, we know that  $\pi_{sts}$  will be some map of the form

$$\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto \sum_i h_i \otimes_{R^{s,t}} k_i.$$

Thus we must rearrange the image of  $e_{sts}$  such that the inner tensors are 1. Let's write

$$\begin{aligned}1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 &= 1 \otimes_{R^s} (P_s(f) + \delta_s(f)\alpha_s) \otimes_{R^t} (P_s(g) + \delta_s(g)\alpha_s) \otimes_{R^s} 1 \\ &= P_s(f) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad P_s(f) \otimes_{R^s} 1 \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g).\end{aligned}$$

This greatly simplifies our work, as now we need only compute how  $e_{sts}$  acts on these four tensors. The first one is simple, since  $\delta_s(1) = 0$ :

$$e_{sts}(1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1) = 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

As for the next two, observe that

$$(\alpha_s + \alpha_t) - \frac{2}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) = \frac{2}{3} \left( \alpha_s + \frac{1}{2} \alpha_t \right).$$

Hence

$$\begin{aligned}
e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} 1) &= 1 \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} (\alpha_s + \alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} \frac{1}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 + \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{1}{3}(2\alpha_s + 4\alpha_t) \otimes_{R^s} 1 + 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}(\alpha_s + 2\alpha_t) + \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.
\end{aligned}$$

The result for  $1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_s \otimes_{R^s} 1$  follows by symmetry. We can then play a similar game for the remaining tensor. We begin by observing that

$$\alpha_s + \frac{2}{3} \left( \frac{1}{2}\alpha_s + \alpha_t \right) = \frac{4}{3} \left( \alpha_s + \frac{1}{2}\alpha_t \right).$$

Thus

$$\begin{aligned}
e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1) &= 1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^s} 1 + \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(8\alpha_s^2 + 8\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(8\alpha_s^2 + 8\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t).
\end{aligned}$$

From here, we note that

$$R^{s,t} = \mathbb{R}[\alpha_s^2 + \alpha_s\alpha_t + \alpha_t^2];$$

in particular,

$$\alpha_s^2 + \alpha_s\alpha_t + \alpha_t^2 \in R^t.$$

We may hence continue with

$$\begin{aligned} e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1) &= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(6\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \\ &= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\ &= 1 \otimes_{R^s} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\ &= 1 \otimes_{R^s} \frac{2}{9}(\alpha_s^2 - 2\alpha_s\alpha_t - 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 \\ &= \frac{4}{9}(-\alpha_s^2 - \alpha_s\alpha_t - \alpha_t^2) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 + \\ &\quad \frac{2}{3}\alpha_s^2 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1. \end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 &\mapsto P_s(f) \otimes_{R^{s,t}} P_s(g) + \\
&\quad \partial_s(f) \frac{2}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} P_s(g) + \\
&\quad \partial_s(f) \otimes_{R^{s,t}} \frac{4}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) P_s(g) + \\
&\quad P_s(f) \otimes_{R^{s,t}} \frac{2}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) \partial_s(g) + \\
&\quad P_s(f) \frac{4}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} \partial_s(g) + \\
&\quad \partial_s(f) \frac{1}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} \frac{1}{3} \left( \frac{1}{2} \alpha_s + \alpha_t \right) \partial_s(g) + \\
&\quad \partial_s(f) \otimes_{R^{s,t}} \frac{2}{3} \alpha_s^2 \partial_s(g) + \\
&\quad \partial_s(f) \frac{2}{3} \alpha_s^2 \otimes_{R^{s,t}} \partial_s(g) - \\
&\quad \partial_s(f) \frac{4}{9} (\alpha_s^2 + \alpha_s \alpha_t + \alpha_t^2) \otimes_{R^{s,t}} \partial_s(g).
\end{aligned}$$

Phew!!

All that's left is to clean up a few loose ends. First, we see that  $B_{st}$  and  $B_{ts}$  are not isomorphic, as

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \not\cong B_{st}(-1) \oplus B_{st}(1) \cong B_s \otimes_R B_s \otimes_R B_t.$$

Note that we have used the fact here that direct sum decompositions are unique. Finally, in order to show that all indecomposables have been exhausted, we claim that

$$B_{sts} \otimes_R B_s \cong B_s \otimes_R B_{sts} \cong B_{sts}(1) \oplus B_{sts}(-1) \cong B_t \otimes_R B_{sts} \cong B_{sts} \otimes_R B_t.$$

Recall that in Lemma 4, we found an isomorphism  $R \cong R^s \oplus R^s(-2)$  of  $(R^s, R^s)$ -bimodules. It is easy to see that this restricts to an isomorphism of  $(R^{s,t}, R^s)$ -bimodules, whence

$$\begin{aligned}
B_{sts} \otimes_R B_s &\cong (R \otimes_{R^{s,t}} R(3)) \otimes_R (R \otimes_{R^s} R(1)) \\
&\cong R \otimes_{R^{s,t}} R \otimes_{R^s} R(4) \\
&\cong R \otimes_{R^{s,t}} (R^s \oplus R^s(-2)) \otimes_{R^s} R(4) \tag{Lemma 4} \\
&\cong (R \otimes_{R^{s,t}} R(4)) + (R \otimes_{R^s} R(2)) \\
&\cong B_{sts}(1) \oplus B_{sts}(-1).
\end{aligned}$$

The other isomorphisms follow similarly. We have thus shown that we obtain no new indecomposables by tensoring, meaning we are done with our classification. This completes the proof.  $\blacksquare$