Introduction to W-Graphs

1. Kazhdan–Lusztig W-Graphs

Recall that, in [KL79], Kazhdan and Lusztig found a pair of self-dual bases over $\mathscr{H}(W,S)$ in terms of the standard basis $\{T_x : x \in W\}$. These are the *Kazhdan–Lusztig* (or *canonical*) *basis*

$$C_x = (-1)^{\ell(x)} q^{\frac{1}{2}\ell(x)} \sum_{y \leq x} (-1)^{\ell(y)} q^{-\ell(y)} \overline{P_{y,x}(q)} T_y$$

and the *dual Kazhdan–Lusztig* (or *dual canonical*) *basis*

$$C'_x = q^{-\frac{1}{2}\ell(x)} \sum_{y \leq x} P_{y,x}(q) T_y.$$

Let $\mathscr{D}_x \coloneqq \{s \in S : sx < x\}$ denote the *left descent set of* x. It was also shown in [KL79] that

$$T_s C_x = \begin{cases} -C_x, & \text{if } s \in \mathscr{D}_x; \\ qC_x + q^{1/2} \sum_{\{y \in W : s \in \mathscr{D}_y\}} \mu(y,x) C_y, & \text{otherwise} \end{cases}$$

and

$$T_s C'_x = \begin{cases} qC'_x, & \text{if } s \in \mathscr{D}_x; \\ -C'_x + q^{1/2} \sum_{\{y \in W : s \in \mathscr{D}_y\}} \mu(y,x) C'_y, & \text{otherwise.} \end{cases}$$

Here $\mu(y,x) \coloneqq c(y,x) + c(x,y)$, where $c(y,x)$ is defined to be the coefficient of $q^{(\ell(x)-\ell(y)-1)/2}$ (the power of q of maximal degree) in the KL polynomial $P_{y,x}(q)$. This motivates the following.

Definition 1. (W-Graph). *Let (W,S) be a Coxeter system and $\mathscr{H}(W,S)$ its associated Iwahori–Hecke algebra. A W*-graph *is a triple (X, I, μ) consisting of a set X of vertices, a function $I : X \to \mathscr{P}(S)$ that assigns to each vertex $x \in X$ a* descent set *$I_x \subseteq S$, and a function*

$$\mu : X \times X \to \mathbb{Z}$$

such that there is an edge $y \to x$ in the graph when the edge weight *$\mu(y,x)$ is non-zero. Moreover, if $E \coloneqq \mathbb{Z}[q^{\pm 1/2}]X$ is the free $\mathbb{Z}[q^{\pm 1/2}]$-module with basis X, we ask that its $\mathbb{Z}[q^{\pm 1/2}]$-endomorphisms*

$$\tau_s(x) \coloneqq \begin{cases} -x, & \textit{if } s \in I_x; \\ qx + q^{1/2} \sum_{\{y \in X : s \in I_y\}} \mu(y,x) y, & \textit{otherwise} \end{cases}$$

for each $s \in S$ satisfy the braid relations

$$\underbrace{\tau_s \tau_t \tau_s \cdots}_{m_{st} \textit{ factors}} = \underbrace{\tau_t \tau_s \tau_t \cdots}_{m_{st} \textit{ factors}},$$

for all $s,t \in S$ such that $m_{st} \neq \infty$. We define dual W-graphs *similarly, where we instead use $\mathbb{Z}[q^{\pm 1/2}]$-endomorphisms of E of the form*

$$\tau'_s(x) \coloneqq \begin{cases} qx, & \textit{if } s \in I_x; \\ -x + q^{1/2} \sum_{\{y \in X : s \in I_y\}} \mu(y,x) y, & \textit{otherwise.} \end{cases}$$

Asking that the A-endomorphisms τ_s defined above satisfy the braid relations is exactly equivalent to asking that E admit a $\mathscr{H}(W,S)$-module structure given by $T_s \cdot x \coloneqq \tau_s(x)$. This is because all $\mathbb{Z}[q^{\pm 1/2}]$-endomorphisms of the form τ_s satisfy the quadratic relation $(\tau_s+1)(\tau_s-q)=0$, provided the sums are finite. To see that this is true, observe that when $s \notin I_x$, we have

$$\begin{aligned}
&(\tau_s+1)(\tau_s-q)x \\
&= \tau_s^2(x) - q\tau_s(x) + \tau_s(x) - qx \\
&= q\tau_s(x) + q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)\tau_s(y) - q\tau_s(x) + qx + q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)y - qx \\
&= q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)\tau_s(y) + q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)y \\
&= q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)(-y) + q^{1/2} \sum_{\{y\in X : s\in I_y\}} \mu(y,x)y \\
&= 0.
\end{aligned}$$

Conversely, when $s \in I_x$, the result follows trivially. It follows that every W-graph (and dual W-graph) corresponds to a representation $\varphi : \mathscr{H}(W,S) \to \mathrm{End}(E)$ given on the generators by $\varphi : T_s \mapsto \tau_s$.

It is sometimes convenient to look at the transposes of τ_s and τ_s'. In particular, suppose we fix $s \in S$ and choose an ordering $X = \{x_1, \dots, x_n\}$ such that $s \in I_{x_i}$ for all $1 \le i \le k$ and $s \notin I_{x_j}$ for all $j \ge k$. Then we can express τ_s and τ_s' as the matrices

$$\tau_s = \begin{pmatrix} -I_k & 0 \\ * & qI_{n-k} \end{pmatrix} \qquad \text{and} \qquad \tau_s' = \begin{pmatrix} qI_k & 0 \\ * & -I_{n-k} \end{pmatrix},$$

where I_k is the $k \times k$ identity matrix and the block labelled by the asterisk $*$ has elements that are either 0 or of the form $q^{1/2}\mu(y,x)$. The transposes of τ_s and τ_s' thus involve moving only the sums, giving us

$$\tau_s^T(x) = \begin{cases} -x + q^{1/2} \sum_{\{y\in X : s\notin I_y\}} \mu(y,x)y, & \text{if } s \in I_x; \\ qx, & \text{otherwise} \end{cases}$$

and

$$(\tau_s')^T(x) = \begin{cases} qx + q^{1/2} \sum_{\{y\in X : s\notin I_y\}} \mu(y,x)y, & \text{if } s \in I_x; \\ -x, & \text{otherwise,} \end{cases}$$

respectively. These also produce representations of $\mathscr{H}(W,S)$, and hence W-graphs, of their own.

The (one-sided) W-graph constructed in [KL79] is defined by taking

$$X \coloneqq W, \qquad I_x \coloneqq \mathscr{D}_x \qquad \text{and} \qquad \mu(y,x) \coloneqq c(y,x) + c(x,y).$$

Note that either $c(y,x) = 0$ or $c(x,y) = 0$ (usually both), so this W-graph is not directed. In particular, it corresponds to the left regular representation of $\mathscr{H}(W,S)$.

From now on, we will be working with the normalized standard basis $\delta_w := v^{\ell(w)} T_w$ for $\mathscr{H}(W,S)$, where $v := q^{-1/2}$. Normalizing the dual Kazhdan–Lusztig basis, we obtain

$$b_x = v^{\ell(x)} \sum_{y \leq x} P_{y,x}(v^{-2}) v^{-\ell(y)} \delta_y = \sum_{y \leq x} h_{y,x}(v) \delta_y,$$

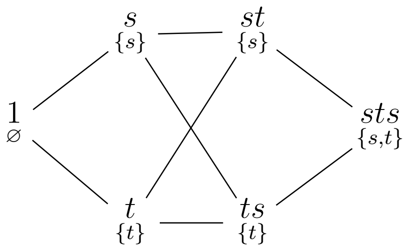
where $h_{y,x} : v \mapsto v^{\ell(x)-\ell(y)} P_{y,x}(v^{-2})$. Recall that $P_{y,x}(q)$ has maximal degree $\frac{1}{2}(\ell(x)-\ell(y)-1)$ in q. It therefore has degree greater than or equal to $1+\ell(y)-\ell(x)$ in v, meaning $h_{y,x}(v) \in v\mathbb{Z}[v]$, where the coefficient of $q^{\frac{1}{2}(\ell(x)-\ell(y)-1)}$ in $P_{y,x}(q)$ is now the coefficient of v in $h_{y,x}(v)$. Because $\tau'_s(x)$ corresponds to $T_s \cdot x$, converting to the normalized basis involves multiplying by a factor of $v^{\ell(s)} = v$, whence

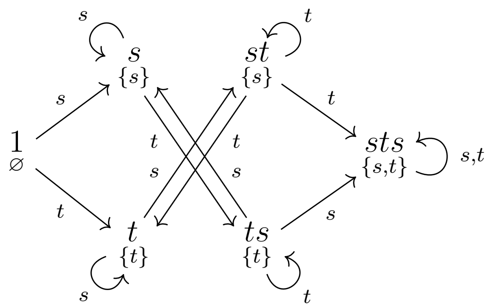
$$\delta_x \cdot x := \begin{cases} v^{-1}x, & \text{if } s \in I_x; \\ -vx + \sum_{\{y \in X : s \in I_y\}} \mu(y,x)y, & \text{otherwise.} \end{cases}$$

Using $b_s = \delta_s + v$, we have an action of the normalized Kazhdan–Lusztig basis given pleasantly by

$$b_s \cdot x := \begin{cases} (v+v^{-1})x, & \text{if } s \in I_x; \\ \sum_{\{y \in X : s \in I_y\}} \mu(y,x)y, & \text{otherwise.} \end{cases}$$

Example 2. ($W = S_3$). In this case, $P_{y,x}(q) \in \{0,1\}$ for all $x, y \in W$, so the edges all have unit weight. Perhaps unsurprisingly, the graph – pictured on the left – recovers the Bruhat order.





The action of $\{b_s, b_t\}$ induces a subgraph structure – pictured on the right – with the strongly connected components corresponding to the left cells $\{1\}$, $\{s, ts\}$, $\{t, st\}$ and $\{sts\}$ of W. In the language of Soergel bimodules, an arrow $x \xrightarrow{s} y$ (respectively $x \xrightarrow{t} y$) indicates that $B_y \geq_L B_x$; that is, B_y is isomorphic to a direct summand of $B_s \otimes_R B_x$ (respectively $B_t \otimes_R B_x$).

2. The Harish-Chandra Picture

Let's briefly mention the connection between admissible representations and $(\mathfrak{g}, K)$-modules. Good resources for what follows are [Vog81] and [Bin10].

Definition 3. (Continuous Representation). *Let G be a Lie group. A* continuous representation *of G is a pair $(\pi, \mathcal{H})$, where $\mathcal{H}$ is a complex Hilbert space and $\pi : G \to \mathcal{B}(\mathcal{H})$ is a continuous homomorphism of G into the semigroup $\mathcal{B}(\mathcal{H})$ of bounded operators on $\mathcal{H}$, where $\mathcal{B}(\mathcal{H})$ is endowed with the weak topology. An* invariant subspace *of $(\pi, \mathcal{H})$ is a closed subspace of $\mathcal{H}$ that is left invariant under all the operators in $\pi(G)$. The continuous representation $(\pi, \mathcal{H})$ is said to be* irreducible *if $\mathcal{H} \neq \{0\}$ and there are no proper, non-trivial invariant subspaces.*

Definition 4. (Bounded Equivalence). *Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be continuous representations of a Lie group G. We define the space of* intertwining operators *between $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ to be*

$$\mathrm{Hom}_G(\pi, \pi') \coloneqq \{L : \mathcal{H} \to \mathcal{H}' : L \text{ is continuous, linear and } \pi'(g) \circ L = L \circ \pi(g) \text{ for all } g \in G\}.$$

We say that two continuous representations are boundedly equivalent *if there exists an invertible intertwining operator between them.*

Definition 5. (Dual Object). *Let G be a Lie group that is the direct product of a compact group and an Abelian group, such that every irreducible continuous representation of G is finite-dimensional. We define the* dual object $\widehat{G}$ *of G to be the set of bounded equivalence classes of irreducible continuous representations of G.*

Definition 6. (Admissible Representation). *Let G be a real Lie group with K a maximal compact subgroup. A continuous representation $(\pi, \mathcal{H})$ of G is said to be* K-admissible *if $\mathrm{Hom}_K(V_\delta, \mathcal{H})$ is finite-dimensional for all irreducible continuous representations $(\delta, V_\delta) \in \widehat{K}$ of K.*

Definition 7. ($(\mathfrak{g}, K)$-Module). *Let G be a real Lie group with complexified Lie algebra $\mathfrak{g}$ and maximal compact subgroup K. A* $(\mathfrak{g}, K)$-module *is a complex vector space V together with a map $\pi : \mathfrak{g} \sqcup K \to \mathrm{End}(V)$ that restricts to a Lie algebra representation $\pi|_{\mathfrak{g}}$ (that is, V is a $\mathcal{U}(\mathfrak{g})$-module) and a group representation $\pi|_K$ satisfying certain compatibility conditions. A $(\mathfrak{g}, K)$-module V is said to be* K-admissible *if $\mathrm{Hom}_K(V_\delta, V)$ is finite-dimensional for all irreducible continuous representations $(\delta, V_\delta) \in \widehat{K}$ of K. A K-admissible $(\mathfrak{g}, K)$-module which is finitely-generated over $\mathcal{U}(\mathfrak{g})$ is called a* Harish-Chandra module.

Definition 8. (K-Finiteness). *Let G be a real Lie group with K a maximal compact subgroup and $(\pi, \mathcal{H})$ a continuous representation of G. A vector $v \in \mathcal{H}$ is said to be* K-finite *if $\mathrm{span}\{\pi(k)v : k \in K\}$ is finite-dimensional. We denote $\mathcal{H}_K \coloneqq \{v \in \mathcal{H} : v \text{ is } K\text{-finite}\}$.*

Theorem 9. ([Vog81, Theorem 0.3.5]). *Let G be a real Lie group with Lie algebra $\mathfrak{g}_0$ and maximal compact subgroup K. If $(\pi, \mathcal{H})$ is a K-admissible representation of G, the limit*

$$\widehat{\pi}_0(x)v \coloneqq \lim_{t \to 0} \frac{\pi(\exp(tx))v - v}{t}$$

exists for all $x \in \mathfrak{g}_0$ and $v \in \mathcal{H}_K$, where $\exp : \mathfrak{g}_0 \to G$ is the exponential map. In particular, this defines a Lie algebra representation $\widehat{\pi}_0 : \mathfrak{g}_0 \to \mathrm{End}(\mathcal{H}_K)$. Let $\widehat{\pi}|_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{End}(\mathcal{H}_K)$ be its complexification, which exists since $\mathcal{H}$ is complex. By definition $\mathcal{H}_K$ induces a group representation $\widehat{\pi}|_K : K \to \mathrm{End}(\mathcal{H}_K)$, whence these representations endow $\mathcal{H}_K$ with the structure of a $(\mathfrak{g}, K)$-module.

Definition 10. (Infinitesimal Equivalence). *Let (π, V) and (π', V') be $(\mathfrak{g}, K)$-modules. We define the space of* intertwining operators *between (π, V) and (π', V') to be*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(\pi, \pi') \coloneqq \{L : V \to V' : L \text{ is complex, linear and } \pi'(x) \circ L = L \circ \pi(x) \text{ for all } x \in \mathfrak{g} \sqcup K\}.$$

We say that two $(\mathfrak{g}, K)$-modules are equivalent if there exists an invertible intertwining operator between them. Two continuous representations are said to be infinitesimally equivalent *if their corresponding Harish-Chandra modules are equivalent.*

Theorem 11. ([Vog81, Theorem 0.3.10]). *Let G be a real Lie group with complexified Lie algebra $\mathfrak{g}$ and maximal compact subgroup K. Then every irreducible $(\mathfrak{g}, K)$-module is the Harish-Chandra module of an irreducible K-admissible representation of G. In particular, every irreducible $(\mathfrak{g}, K)$-module is automatically K-admissible, and we have a bijective correspondence*

$$\frac{\{\text{irreducible } K\text{-admissible representations of } G\}}{\text{infinitesimal equivalence}} \longleftrightarrow \frac{\{\text{irreducible Harish-Chandra modules}\}}{\text{equivalence}}$$
$$\longleftrightarrow \frac{\{\text{irreducible } (\mathfrak{g}, K)\text{-modules}\}}{\text{equivalence}}$$

via Theorem 9.

From now on, we shall assume the following notation. Let $\mathbb{G}$ be a connected, complex, reductive algebraic group defined over $\mathbb{R}$ and G its group of $\mathbb{R}$-rational points. Let θ be the Cartan involution corresponding to G, and write $\mathbb{K} := \mathbb{G}^\theta$ and $K := G^\theta$ for the corresponding fixed-point subgroups, where we recall that K is necessarily maximal compact (see [AC09, §3] or my notes on categorical representation theory). Denote by $\mathfrak{g}$ the complexification of the Lie algebra of G and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Finally, let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$ ([Vog81, Definition 0.2.5]).

If V is an irreducible $(\mathfrak{g}, K)$-module, Dixmier's generalization of Schur's lemma ([Kna13, Proposition 5.19]) tells us that every endomorphism of an irreducible $(\mathfrak{g}, K)$-module V is a scalar. In particular, by treating V as a $\mathcal{U}(\mathfrak{g})$-module, the center $Z(\mathcal{U}(\mathfrak{g}))$ acts on V by

$$z \cdot v = \chi_V(z)v$$

for all $z \in Z(\mathcal{U}(\mathfrak{g}))$ and $v \in V$, where $\chi_V(z) \in \mathbb{C}$. The resulting homomorphism $\chi_V : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ is known as the *infinitesimal character of* V. Two equivalent $(\mathfrak{g}, K)$-modules will always share the same infinitesimal character.

By [Vog81, Theorem 0.2.8], we have an algebra isomorphism $\xi : Z(\mathcal{U}(\mathfrak{g})) \to \mathcal{U}(\mathfrak{h})^W$ known as the *Harish-Chandra isomorphism.* Suppose we let $\lambda \in \mathfrak{h}^*$. This corresponds to an algebra homomorphism $\lambda : \mathfrak{h} \to \mathbb{C}$ and hence lifts to an algebra homomorphism $\lambda : \mathcal{U}(\mathfrak{h}) \to \mathbb{C}$. Composing the latter map with the Harish-Chandra isomorphism, we obtain a map $\xi_\lambda : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$. In fact, we have the following surprising result.

Theorem 12. ([Vog81, Corollary 0.2.10]). *Every homomorphism from $Z(\mathcal{U}(\mathfrak{g}))$ to $\mathbb{C}$ is of the form ξ_λ, for some $\lambda \in \mathfrak{h}^*$. Moreover, $\xi_\lambda = \xi_{\lambda'}$ if and only if there exists some $w \in W$ for which $\lambda' = w\lambda$.*

In other words, if we have a map $\chi : \mathsf{Adm}_K(G) \to \mathfrak{h}^*/W$ that takes equivalence classes of irreducible K-admissible representations of G to W-conjugacy classes of elements in $\mathfrak{h}^*$. Given some $\lambda \in \mathfrak{h}^*/W$, we will write

$$\mathsf{Adm}_K(G, \lambda) := \{\pi \in \mathsf{Adm}_K(G) : \chi(\pi) = \lambda\}$$

for the set of equivalence classes of irreducible K-admissible representations of G with infinitesimal character λ. By [Vog81, Corollary 5.4.17], this set is finite.

We shall henceforth fix $\lambda \in \mathfrak{h}^*$ non-singular and integral ($\langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_+$ for all simple coroots $\check{\alpha} \in \check{\Delta}$).

Definition 13. (Block). *The smallest equivalence relation generated by*

$$V \sim V' \iff \begin{array}{c}\text{there exists an indecomposable } (\mathfrak{g},K)\text{-module } V_B \text{ such that we} \\ \text{have a non-split short exact sequence } 0 \to V \to V_B \to V' \to 0\end{array},$$

where V and V' are irreducible $(\mathfrak{g},K)$-modules, is known as block equivalence. *The corresponding equivalence classes of irreducible $(\mathfrak{g},K)$-modules are known as* blocks.

By [Vog81, Lemma 9.2.3], every $(\mathfrak{g},K)$-module V of finite length (that is, every $(\mathfrak{g},K)$-module admitting a notion of a Jordan–Hölder series) can be written as a direct sum

$$V = \bigoplus_{\text{blocks } B} V_B,$$

where each V_B is some $(\mathfrak{g},K)$-module whose irreducible sub-$(\mathfrak{g},K)$-modules all belong to the block B. Moreover, blocks of irreducible $(\mathfrak{g},K)$-modules are somehow determined by their infinitesimal characters ([Vog81, Theorem 9.2.11]).

The original definition of a block from [Vog81, Definition 9.2.1] is that block equivalence is generated by two irreducible $(\mathfrak{g},K)$-modules having a non-zero first cohomology group $\mathrm{Ext}^1_{\mathfrak{g},K}(V,V')$. Blocks give us a way of computing the composition series of certain standard representations, generalizing the algorithm given by Kazhdan and Lusztig for Verma modules in [KL79].

Remark 14. We have been primarily living in the Harish-Chandra world. Atlas, on the other hand, uses a more geometric language, owing to the Langlands classification. Here, every irreducible admissible representation corresponds to a pair (x,y), where x is a $\mathbb{K}$-orbit in $\mathbb{G}/\mathbb{B}$ and y is a $\mathbb{K}^\vee$-orbit in $\mathbb{G}^\vee/\mathbb{B}^\vee$ (see [AC09, §10] and [Ada08, §8]). Here $\mathbb{B}$ is some Borel subgroup of $\mathbb{G}$, $\mathbb{G}^\vee$ is the Langlands dual of $\mathbb{G}$, $\mathbb{B}^\vee$ is a Borel subgroup of $\mathbb{G}^\vee$ and $\mathbb{K}^\vee$ is the complexification of a maximal compact subgroup $K^\vee$ of a real form $G^\vee$ of $\mathbb{G}^\vee$. In this language, a block is a set of the form

$$B(G^\vee) \coloneqq \{(x,y) \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B} \times \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee : \theta^t_{x,H} = -\theta^\vee_{y,H}\},$$

arising from a real form $G^\vee$. Here $\theta^t_{x,H} = -\theta^\vee_{y,H}$ is a technical compatibility condition. Moreover,

$$\mathsf{Adm}_K(G,\lambda) \longleftrightarrow \bigsqcup_{\text{dual real forms } G^\vee} B(G^\vee).$$

Similarly, the irreducible admissible representations of a real form $G^\vee$ can be broken up into blocks corresponding to real forms of $\mathbb{G}$; Vogan duality tells us that if $(x,y) \in \mathbb{K}\backslash\mathbb{G}/\mathbb{B} \times \mathbb{K}^\vee\backslash\mathbb{G}^\vee/\mathbb{B}^\vee$ corresponds to an admissible representation of G, then (y,x) will correspond to an admissible representation of the real form $G^\vee$ of $\mathbb{G}^\vee$ corresponding to $\mathbb{K}^\vee$.

Before we can start building W-graphs from blocks of irreducible Harish-Chandra modules, we need two more ingredients. The first is a notion of cells within these blocks, first appearing in [BV83].

Definition 15. (Harish-Chandra Cell). *Let x,y be two irreducible Harish-Chandra modules with infinitesimal character λ. Write $y \geq x$ if there exists an irreducible finite-dimensional representation f of G such that y is isomorphic to a composition factor of $f \otimes y$. We say that $x \sim y$ if $y \geq x$ and $x \geq y$. The resulting equivalence classes are known as* left Harish-Chandra cells.

Remark 16. Harish-Chandra cells are sometimes called *left W-cells*, since the Harish-Chandra cells of infinitesimal character λ admit an action of the integral Weyl group $W(\lambda)$ and hence correspond to so-called *Harish-Chandra cell representations of* $W(\lambda)$.

Definition 17. (Borho–Jantzen–Duflo τ-Invariant). *Let x be an irreducible Harish-Chandra module with infinitesimal character λ and $R^+(\lambda) := \{\alpha \in \Phi : \langle \lambda, \check{\alpha} \rangle \in \mathbb{Z}_+\}$ the* system of positive integral roots defined by λ. *The* Borho-Jantzen-Duflo τ-invariant of x *is the set $\tau(x) \subseteq R^+(\lambda)$ of simple roots satisfying the equivalent conditions of [Vog81, Corollary 7.2.27].*

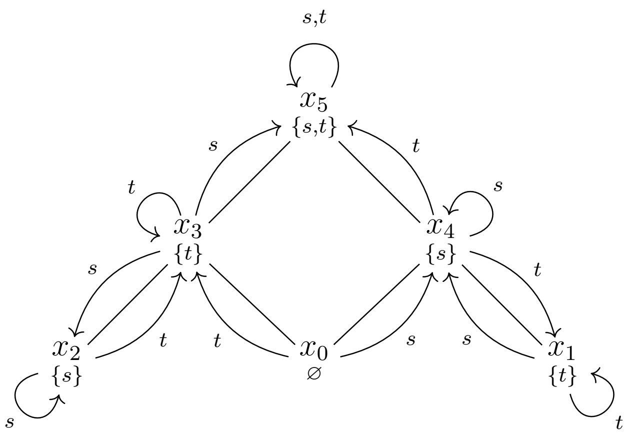
3. The Atlas Construction

Let B be a block of irreducible Harish-Chandra modules of infinitesimal character λ. Using the setup from the previous section, we build a W-graph from this block as follows.

- The vertices are the elements $x \in B$.
- Define $\mu(y, x)$ to be the coefficient of v in the Kazhdan–Lusztig–Vogan polynomial $h_{y,x}(v)$.
- Define I_x to be the Borho–Jantzen–Duflo τ-invariant of x (see [Vog81, Definition 7.3.8]).

Unlike the Kazhdan–Lusztig W-graph, this W-graph is directed. Moreover, the strongly connected components of this graph with respect to the Kazhdan–Lusztig–Vogan basis in fact exhaust the Harish-Chandra cells of B, and the multiplicity $\mu(x, y)$ of an edge $x \to y$ corresponds to the multiplicity with which the representation x appears in $\mathfrak{g} \otimes y$.

Example 18. ($G = \mathrm{SU}(2,1)$). Let's compute the Kazhdan–Lusztig–Vogan W-graph for $\mathrm{SU}(2,1)$. This W-graph only has one block, the trivial block, consisting of 6 irreducible representations. Let's write $B = \{x_0, x_1, x_2, x_3, x_4, x_5\}$. The complexification of the Lie algebra of $\mathrm{SU}(2,1)$ is $\mathfrak{sl}(3, \mathbb{C})$, which has simple roots $\{\alpha, \beta\}$ that we identify with simple reflections $\{s, t\}$. Once again, the multiplicities are all 1 in this example too, and in fact the edges are coincidentally all bidirectional. Recalling the action of b_s on our vertices induced by τ'_s, we have



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We see that the left W-cells of this block are $\{x_0\}$, $\{x_1, x_4\}$, $\{x_2, x_3\}$ and $\{x_5\}$, reflecting once again the left cell structure on W as in our previous example.

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