The Banach–Tarski Paradox and Amenability

Daniel Dunmore

University of New South Wales

d.dunmore@unsw.edu.au

April 7, 2025

Today's Story

The main goal of my talk today will be to explain the famous Banach–Tarski paradox, first described in 1924.

This "veridical paradox" (to use the terminology of Quine) is in fact the middle child in a triplet of "paradoxes" from the early 1900s, with its elder and younger siblings being the Hausdorff paradox (1914) and the von Neumann paradox (1929), respectively.

If there's enough time, I would then like to give an overview of the surprising impact it has had via the notion of amenability.

Roadmap

1. Paradoxical Groups
2. The Hausdorff Paradox
3. The Banach–Tarski Paradox
4. Enter: Amenability

Paradoxical Decompositions

Definition (Paradoxical Decomposition)

Let G be a group and X a G-set. We say that the action of G on X is ***G*-paradoxical** if there exist pairwise disjoint $A_1, \dots, A_n, B_1, \dots, B_m \subseteq X$ and elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$X = \bigcup_{i=1}^{n} g_i \cdot A_i = \bigcup_{j=1}^{m} h_j \cdot B_j.$$

In this case, X is called a ***G*-paradoxical set**, and the collection (A_i, B_j, g_i, h_j) is called a ***G*-paradoxical decomposition** of X.

Any group G is itself a G-set with respect to, say, left multiplication, so we have a natural notion of groups themselves being paradoxical.

Paradoxical Decompositions

Example (Finite Groups)

A finite group G is never paradoxical! This is because we would require

$$|A_1| + \cdots + |A_n| = |G| = |B_1| + \cdots + |B_m|,$$

which means these subsets cannot be disjoint.

Example (Abelian Groups)

Abelian groups are also never paradoxical (more on this later).

Example (Free Groups)

Let F_2 be the non-Abelian free group with free generating set $\{a, b\}$. Let $\omega(x)$ be the set of all reduced words in F_2 starting with x. Then F_2 admits the paradoxical decomposition $F_2 = \omega(a) \cup a \cdot \omega(a^{-1}) = \omega(b) \cup b \cdot \omega(b^{-1})$.

Paradoxical Decompositions

We can in fact push this property of the group onto a set on which it acts using the following theorem, which makes use of the Axiom of Choice.

Theorem

If G is a paradoxical group, then any set X that it acts freely on is G-paradoxical.

Moreover, because non-identity subgroups always act freely via left multiplication, we obtain immediately the following result (where the forward direction is of course tautological).

Corollary

A group G is paradoxical if and only if it contains a paradoxical subgroup.

Paradoxical Decompositions

Looking for free subgroups is one of the most convenient ways of checking for paradoxical decompositions. For example...

Example (Braid Groups)

Any braid group on three or more strands contains F_2 as a subgroup, and is hence paradoxical.

Example (3D Rotation Group)

The 3D rotation group $\mathrm{SO}(3)$ contains F_2 as a subgroup, and is hence paradoxical.

Another nice application of our theorem is the paradox of Hausdorff.

Theorem (Hausdorff Paradox, 1914)

There exists a countable subset of the sphere $M \subset S^2$ such that its complement is paradoxical with respect to the canonical action of $\mathsf{SO}(3)$.

Although one can certainly come up with an explicit $\mathsf{SO}(3)$-paradoxical decomposition, the fastest way to show this is to find a free action of a free subgroup $F \subset \mathsf{SO}(3)$.

Proof. Fix a free subgroup F and let M be the set of all points in S^2 that are fixed by F. Because F is countable and every non-trivial rotation of S^2 fixes two points, M itself is countable. Since $S^2 \setminus M$ is invariant under the (free) action of F, the result follows from our previous theorem. ∎

Let's take this a step further.

Definition (Equidecomposability)

Let G be a group and X a G-set. We say that two subsets $A, B \subseteq X$ are ***G*-equidecomposable** if there exists a partition $A_1, \ldots, A_n$ of A, a partition $B_1, \ldots, B_n$ of B and $g_1, \ldots, g_n \in G$ such that $g_i \cdot A_i = B_i$ for all i. This defines an equivalence relation of sets, denoted by $A \sim_G B$.

Example ("Reverse Hilbert's Hotel")

Suppose $A = S^1 \setminus \{(1, 0)\}$ and $B = S^1$. Let g_1 be the clockwise rotation by 1 radian, g_2 the identity, $A_1 = \{(\cos(n), \sin(n)) : n \in \mathbb{Z}_+\}$ and $A_2 = A \setminus A_1$. Observe that $g_1 \cdot A_1 = \{(\cos(n), \sin(n)) : n \in \mathbb{N}\} =: B_1$ is countably infinite. Thus by taking $B_2 = A_2$, we find that $A \sim_{\mathsf{SO}(2)} B$.

The Banach–Tarski Paradox

Proposition

If A is G-paradoxical and B is G-equidecomposable to A, then B is G-paradoxical.

The proof of this is straightforward and not particularly interesting. We do, however, have the following results.

Proposition

Let M be a countable subset of S^2. Then $S^2 \setminus M$ is $\mathrm{SO}(3)$-equidecomposable to S^2.

Corollary

The sphere S^2 is $\mathrm{SO}(3)$-paradoxical.

The Banach–Tarski Paradox

What happens if we try to extend this result to the closed ball B^3? Certainly $B^3 \setminus \{0,0,0\}$ is SO(3)-paradoxical, but we run into issues when including the center, as it is fixed by SO(3).

Letting SE(3) be the special Euclidean group (consisting of SO(3) and the translation group T(3)), however, we quickly obtain the following result.

Theorem (Banach–Tarski Paradox, 1924)

The closed ball at the origin, $B^3 \subset \mathbb{R}^3$, is SE(3)*-paradoxical.*

The idea here is that $B^3 \setminus \{0,0,0\}$ is translation equidecomposable to B^3.

Question. Does any of this work in two dimensions?

Enter: Amenability

In an attempt to understand the paradoxes of Hausdorff, Banach and Tarski, von Neumann defined the following notion of amenability for discrete groups. The locally compact description, as well as the term "amenable" itself, were later given by Day in 1949.

Definition (Amenability)

A discrete group G is said to be **amenable** if it admits a $\boldsymbol{G}$**-invariant mean**; that is, a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ such that

- $m(\chi_G) = 1$ (normalization);
- $m(f) \geq 0$, for all $f \in \ell^\infty(G)$ with $f(g) \geq 0$ for all $g \in G$ (non-negativity);
- $m(g \cdot f) = m(f)$, for all $g \in G$ and $f \in \ell^\infty(G)$ (left-invariance),

where $\chi_G : g \mapsto 1$ is the indicator function and $g \cdot f : h \mapsto f(g^{-1}h)$.

Enter: Amenability

Example (Finite Groups)

If G is a finite group, then the map

$$f \mapsto \frac{1}{|G|} \sum_{g \in G} f(g)$$

for all $f \in \ell^\infty(G)$ defines a G-invariant mean, whence G is amenable.

Example (Abelian Groups)

Discrete Abelian groups are always amenable. This is a consequence of the Markov–Kakutani Fixed Point Theorem.

Example (Free Groups)

The non-Abelian free group F_2 is not amenable.

An Interesting Detour

Last month, Ryan told us the story of Vaughan Jones' dream: given any subfactor, can we build a CFT? In Jones' effort to answer this via a planar algebraic construction, Thompson's group T miraculously popped out.

This group belongs to a family of groups $F \subset T \subset V$, collectively referred to as Thompson's groups. These groups were originally introduced by Thompson in some unpublished handwritten notes in 1965 as potential counterexamples to the following conjecture, first recorded by Day in 1957.

Conjecture (Von Neumann Conjecture)

A group is non-amenable if and only if it contains F_2 as a subgroup.

While this was proven false in 1980 by Olshansky with his so-called Tarski monster groups, the question of whether or not Thompson's groups are amenable remains an open problem to this day!

The Many Faces of Amenability

A surprising property of amenability is that it admits a variety of vastly different characterizations, ranging from algebraic to analytic in nature.

One we're already familiar with, which was spoiled in the introduction, is the following.

Theorem (Tarski's Alternative, 1949)

A discrete group is amenable if and only if it does not admit a paradoxical decomposition.

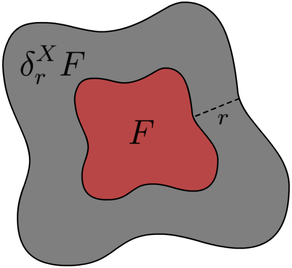
This is only scratching the surface. There are in fact deep connections to harmonic analysis, ergodic theory, asymptotic analysis, operator algebras, homological algebra, automata theory and more. Let's briefly see some!

Theorem (Følner's Property, 1955)

A group G with finite generating set S is amenable if and only if (G, d_S) admits a **Følner sequence**; *a sequence $(F_n)_{n\in\mathbb{N}}$ of finite subsets of G with*

$$\lim_{n\to\infty} \frac{|\delta_r^X F_n|}{|F_n|} = 0,$$

for all $r \in \mathbb{N}$.



Corollary

Let G be a group with finite generating set S. If (G, d_S) has **subexponential growth** *– that is, the size of closed balls of radius n grows no faster than 2^n – then it admits a Følner sequence and is hence amenable.*

The Many Faces of Amenability

Theorem (Reiter's Property (P_p), 1965)

Let $p \in [1, \infty)$. A discrete group G is amenable if and only if, for every finite subset $Q \subset G$ and all $\varepsilon > 0$, there exists some $s \in \ell^p(G)$ such that

- $s \geq 0$;
- $\|s\|_p = 1$;
- $\|q \cdot s - s\|_p < \varepsilon$, *for all* $q \in Q$.

Theorem (Hulanicki 1965, Reiter 1966)

A discrete group G is amenable if and only if its universal C^*-algebra is equal to its reduced C^*-algebra.

The Many Faces of Amenability

Theorem (Johnson, 1972)

A discrete group G is amenable if and only if $\mathcal{H}^1(\ell^1(G), \chi^*) = \{0\}$ for every Banach $\ell^1(G)$-bimodule χ, where $\mathcal{H}^1$ is the Hochschild cohomology.

Theorem (Bartholdi, 2010)

A group G is amenable if and only if all cellular automata on G that admit mutually erasable patterns also admit a Garden of Eden.

Thank you for your attention!