Contextualizing Categorical Representation Theory

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Roadmap

A Brief History

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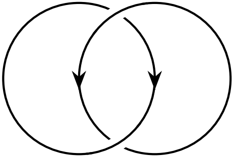
Our story begins in the 1980s with Jones' surprising discovery of a polynomial invariant for oriented links ([Jon85]).



$$\rightsquigarrow \quad -t^{1/2}(1+t^2)$$

This created a link (no pun intended!) between operator algebras and low dimensional topology, initiated the field of quantum topology and lead to an explosion of new knot polynomials.

A Brief History

In the 1990s, Reshetikhin and Turaev explained how these invariants could be obtained from the representation theory of quantum groups ([RT91]).

Following this was a categorification of the Jones polynomial due to Khovanov that is strictly stronger as a knot invariant ([Kho00]).

If the Jones polynomial may be obtained from the representation theory of quantum groups, what about its categorification? Spoiler: [Web13a], [Web13b].

What Is Categorification?

What Is Categorification?

Well... it's complicated. There are many flavours of categorification.

The most precise (and unhelpful) definition of categorification is that it's a right inverse to decategorification.

Roughly speaking, categorification is the process of endowing an algebraic object with *higher algebraic structure*.

What Is Categorification? Examples

The canonical example of categorification is $\mathsf{Vect}_{\Bbbk}^{\text{f.d.}}$, the category of finite-dimensional $\Bbbk$-vector spaces, which categorifies the natural numbers.

We can decategorify by taking the dimension.

The direct sum $\oplus$ categorifies addition:

$$\dim(U \oplus V) = \dim(U) + \dim(V).$$

The tensor product $\otimes_{\Bbbk}$ categorifies multiplication:

$$\dim(U \otimes_{\Bbbk} V) = \dim(U) \cdot \dim(V).$$

What Is Categorification? Examples

The additive and multiplicative identities are categorified by 0 and $\Bbbk$:

$$0 \oplus V \cong V \cong V \oplus 0,$$

$$\Bbbk \otimes_{\Bbbk} V \cong V \cong V \otimes_{\Bbbk} \Bbbk.$$

Injections and surjections categorify the order relation:

$$\exists f : U \hookrightarrow V \iff \dim(U) \leq \dim(V),$$

$$\exists g : U \twoheadrightarrow V \iff \dim(U) \geq \dim(V).$$

The category of finite-dimensional vector spaces contains all the structure of the natural numbers in addition to *higher, linear algebraic structure*!

We can also recover the entirety of $\mathbb{N}$ by looking at isomorphism classes of objects (this is known as the *Grothendieck group* of $\mathrm{Vect}_{\Bbbk}^{\text{f.d.}}$).

What Is Categorification? Examples

In a similar vein, the category of finite-dimensional, $\mathbb{Z}$-graded vector spaces categorifies the quantum numbers $\mathbb{N}[q, q^{-1}]$.

Decategorification is done via the graded dimension:

$$\mathrm{qdim}\left(V = \bigoplus_{j \in \mathbb{Z}} V^j\right) := \sum_{j \in \mathbb{Z}} q^j \dim(V^j).$$

Grading shifts correspond to division by the formal variable q:

$$\mathrm{qdim}(V(n)) = \sum_{j \in \mathbb{Z}} \dim(V^{j+n}) q^j = q^{-n} \mathrm{qdim}(V)\,.$$

Already there are some nice applications of this.

Suppose we have an Abelian category $\mathcal{C}$. In the world of algebraic topology, we often look at $\mathbb{Z}$-graded complexes $C := (C_*(X), \partial_*)$ of the form

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} \cdots,$$

where the C_i are objects in $\mathcal{C}$ and the ∂_i are morphisms.

Given two complexes $A := (A_*, a_*)$ and $B := (B_*, b_*)$ with a morphism $d : A \to B$, we may also define a complex via the mapping cone $\mathrm{Cone}(d)$:

$$\cdots \to A_n \oplus B_{n-1} \xrightarrow{\begin{pmatrix} a_n & 0 \\ d_n & -b_n \end{pmatrix}} A_{n+1} \oplus B_n \to \cdots.$$

What Is Categorification? Examples

This is a bit abstract, so let's be more concrete. Consider Ab, the category of Abelian groups.

Given any topological space X, we can always define an ($\mathbb{N}$-graded) singular chain complex $S(X)$, where $S_n(X)$ is the free Abelian group with basis given by all singular n-simplices in X and $\partial_n : S_n \to S_{n-1}$ is the boundary homomorphism.

What Is Categorification? Examples

If the (co)homology groups $H_n(X) \coloneqq \mathrm{Ker}(\partial_n)/\mathrm{Im}(\partial_{n-1})$ are finite rank,

$$\mathrm{rk}(H_n(X)) = \dim(H_n(X) \otimes_{\mathbb{Z}} \mathbb{Q}) = b_n.$$

In other words, *finite (co)homology categorifies Betti numbers*.

If there are a finite number of non-zero Betti numbers (e.g., you have a finite C.W. complex), we may define the *graded Euler characteristic*,

$$\widehat{\chi}_q(X) \coloneqq \mathrm{qdim}(S(X)) = \sum_{i \in \mathbb{N}} q^i \dim(H_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}),$$

whence we find that $\widehat{\chi}_{-1}(X) = \chi(X)$. In other words, *bounded (co)chain complexes of finite (co)homology categorify Euler–Poincaré characteristics*.

With this in mind, let's try to understand how exactly Khovanov homology categorifies the Jones polynomial.

What Is Categorification? Examples

Definition (Jones, 1985 and Kauffman, 1987)

The *Kauffman bracket polynomial* $\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$ of an unoriented link diagram D is defined recursively via the local rules

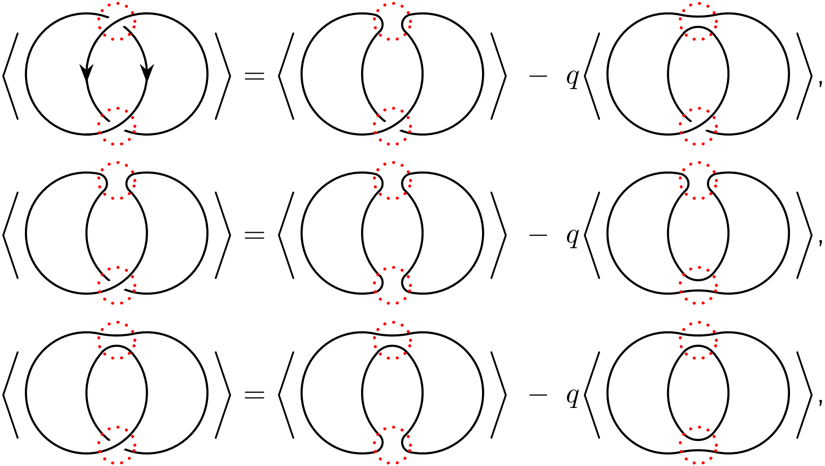
- $\langle \varnothing \rangle = 1$;
- $\langle \times \rangle = \langle)\,(\rangle - q\langle \asymp \rangle$;
- $\langle \bigcirc \sqcup D' \rangle = (q + q^{-1})\langle D' \rangle$, for any unoriented link diagram D'.

The *Jones polynomial* $V_q(L)$ of an oriented link L is then the invariant

$$V_q(L) = \frac{(-1)^{n_-} q^{n_+ - 2n_-}}{q+q^{-1}} \langle D \rangle \in \mathbb{Z}[q, q^{-1}],$$

where D is the associated unoriented diagram, n_+ is the number of positive crossings $\times$ and n_- is the number of negative crossings $\times$.

What Is Categorification? Examples



$$\implies V_q(\mathsf{Hopf}) = \tfrac{q^2}{q+q^{-1}}(q+q^{-1})(q^3+q^{-1}) = q^5+q.$$

What Is Categorification? Examples

Suppose that we associate for each link L with diagram D a $\mathbb{Z}$-graded complex $C(L) \coloneqq (C_*(D), \partial_*)$ of $\mathbb{Z}$-graded Abelian groups. That is,

$$C_i(D) = \bigoplus_{j \in \mathbb{Z}} C_i^j(D).$$

Denoting by $H_i(D)$ the ith cohomology group of $C(D)$ and by $H_i^j(D)$ its jth graded summand, we may define

$$\widehat{\widehat{\chi}}(C(L)) \coloneqq \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \dim(H_i^j(D) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

What Khovanov did was find a way of building complexes $C(L)$ such that $\widehat{\widehat{\chi}}(C(L)) = (q + q^{-1}) V_q(L)$.

What Is Categorification? Examples

For suitable morphisms d_* (the *differentials*), you can in fact construct a *Khovanov bracket* $[\![D]\!]$ as a higher analogue to the Kauffman bracket, giving a similar recursive process for computing the Khovanov homology.

$$
\begin{array}{lcr}
\langle \varnothing \rangle = 1 & \rightsquigarrow & [\![\varnothing]\!] = 0 \to \mathbb{Q} \to 0, \\
\langle \times \rangle = \langle)\,(\rangle - q\langle \asymp \rangle & \rightsquigarrow & [\![\times]\!] = \mathrm{Cone}(d_* : [\![ )\,( ]\!] \to [\![\asymp]\!](1)), \\
\langle \bigcirc \sqcup D' \rangle = (q + q^{-1})\langle D' \rangle & \rightsquigarrow & [\![\bigcirc \sqcup D']\!] = A \otimes [\![D']\!], \\
(-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle & \rightsquigarrow & [\![D]\!]\{-n_-\}(n_+ - 2n_-).
\end{array}
$$

Note that $\{-\}$ denotes grading shifts on the level of the complex, while $(-)$ denotes grading shifts of the Abelian groups. Here $A \coloneqq \mathbb{Q}[x]/\langle x^2 \rangle$ are the dual numbers, with $\mathrm{qdim}(A) = q + q^{-1}$.

What Is Categorification? Examples

One more nice example is $\mathbb{S}\mathsf{Bim}$, the $\mathbb{Z}$-graded category of Soergel bimodules. It categorifies the Iwahori–Hecke $\mathbb{Z}[v, v^{-1}]$-algebra.

Irreducible Soergel bimodules correspond bijectively to elements of the so-called *Kazhdan–Lusztig basis* of the Iwahori–Hecke algebra.

Direct sums and tensor products of Soergel bimodules correspond to sums and products of elements of the Iwahori–Hecke algebra:

$$B \oplus B' \rightsquigarrow b + b' \quad \text{and} \quad B \otimes B' \rightsquigarrow bb'.$$

Grading shifts correspond to multiplication by the formal variable v:

$$B(n) \rightsquigarrow v^n b.$$

The graded rank of Hom-spaces corresponds to the standard form:

$$\mathrm{grk}(\mathrm{Hom}^{\bullet}_{\mathbb{S}\mathsf{Bim}}(B, B')) = (b, b').$$

What Is Categorification? Examples

That Soergel bimodules categorify the Iwahori–Hecke algebra was only recently proven by Elias and Williamson in 2014 ([EW14]).

This concluded with fully algebraic proofs of an important pair of conjectures in Lie theory posed by Kazhdan and Lusztig ([KL79]), which had been highly sought after for a few decades.

Classical Representation Theory

Classical Representation Theory

First of all, what are representations? Well, we can model them nicely using the categorical language.

1. Start with a finite group G.
2. Deloop it: $\mathrm{Ob}(\mathsf{B}G) := \{\bullet\}$, $\mathrm{Hom}(\mathsf{B}G) := G$.
3. Take a functor from $\mathsf{B}G$ to $\mathsf{Vect}_{\Bbbk}^{\mathsf{f.d.}}$.
4. This is exactly a finite-dimensional representation of G over $\Bbbk$ (or, if you prefer, a finitely-generated $\Bbbk[G]$-module)!

In general, we can think of representations as functors from one category to another.

Classical Representation Theory

Representations enjoy a very rich and beautiful theory.

Let $\Bbbk$ be an algebraically closed field with $\mathrm{char}(\Bbbk) \nmid |G|$.

- All finitely-generated $\Bbbk[G]$-modules are built from extensions of their simple submodules.
- Two simple $\Bbbk[G]$-modules are isomorphic if and only if they have the same character.
- There is a bijective correspondence between isomorphism classes of simple $\Bbbk[G]$-modules and conjugacy classes of G.
- The regular $\Bbbk[G]$-module contains every simple $\Bbbk[G]$-module as a summand.

We want to categorify this theory!

Representation Theory: Categorified

Representation Theory: Categorified

Before, we started with a finite group G. Now let's start with a category.

1. Start with a $\Bbbk$-*finitary* monoidal category $\mathcal{C}$.
2. Deloop it: $\mathrm{Ob}(\mathsf{B}\mathcal{C}) := \{\bullet\}$, $\mathrm{Hom}^1(\mathsf{B}\mathcal{C}) := \mathrm{Ob}(\mathcal{C})$,
   $\mathrm{Hom}^2(\mathsf{B}\mathcal{C}) := \mathrm{Hom}(\mathcal{C})$.
3. Take a pseudofunctor from $\mathsf{B}\mathcal{C}$ to $\mathfrak{A}^f_{\Bbbk}$, the bicategory of $\Bbbk$-finitary categories.
4. This is a $\Bbbk$-finitary birepresentation of $\mathsf{B}\mathcal{C}$ (or, if you prefer, a $\Bbbk$-finitary $\mathcal{C}$-module category)!

To be more precise, birepresentations are horizontal categorifications of module categories, or vertical categorifications of monoidal categories.

In general, we can think of birepresentations as pseudofunctors from a category to Cat, the bicategory of categories (that's a mouthful).

Representation Theory: Categorified

The prototypical example is the *Yoneda birepresentation*.

Let $\mathscr{C}$ be the monoidal delooping of a $\Bbbk$-finitary monoidal category $\mathcal{C}$. We define a pseudofunctor $\mathbb{P} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$ as follows.

1. $\mathbb{P}$ maps the "dummy" object $\bullet$ to $\mathcal{C}$.
2. $\mathbb{P}$ maps 1-morphisms $X \in \mathrm{Ob}(\mathcal{C})$ to functors given by $Y \mapsto X \otimes Y$ and $(f : Y \to Y') \mapsto \mathrm{id}_X \otimes f$.
3. $\mathbb{P}$ maps 2-morphisms $f : X \to X'$ to natural transformations given by $Y \mapsto f \otimes \mathrm{id}_Y$.

This Yoneda birepresentation categorifies the regular representation!

Representation Theory: Categorified

How do we define simplicity for birepresentations? We have two layers: a "discrete" layer of objects and a "continuous" layer of morphisms.

A birepresentation $\mathbf{M}$ of $\mathscr{C}$ is said to be *transitive* if

$$\mathbf{M}(\mathtt{j}) = \mathsf{Add}(\{[\mathbf{M}(F)](X) : F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})\})$$

for all $X \in \mathrm{Ob}(\mathbf{M}(\mathtt{i}))$ and $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ *(recall that M is simple iff every cyclic submodule generated by a non-zero element of M is equal to M)*.

It is said to be *simple* if it admits no proper, non-zero $\mathscr{C}$-stable ideals *(recall that M is simple iff it contains no proper, non-zero ideals)*.

Simple $\implies$ transitive. Every transitive birepresentation admits a unique maximal $\mathscr{C}$-stable ideal that can be quotiented out to make it simple, and a birepresentation is simple if and only if this maximal ideal is zero *(recall that M is simple iff it is isomorphic to a quotient by a maximal ideal)*.

Representation Theory: Categorified

As it turns out, we have (to some degree) a mirroring of some of the results from classical representation theory.

- All "sufficiently nice" birepresentations are built from extensions of their simple sub-representations ([MM16]).
- In this case, simple birepresentations are determined by the matrices describing the actions of the $\mathrm{M}(F)$'s on the Grothendieck group (these matrices play an analogous role to the characters of Frobenius).
- There is a bijective correspondence between equivalence classes of simple birepresentations and Morita equivalence classes of certain (co)algebra 1-morphisms ([MMMT19]).
- We have the Yoneda birepresentation, although even in nice cases there are simple birepresentations that are not contained in it.

There's still a lot of work to do!

Where Do I Fit In?

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Right now, I'm looking at certain module categories over the (monoidal delooping of the) category of Soergel bimodules.

In the type A case, simple module categories over $\mathbb{S}$Bim are identical to the decategorified case, recovering the Kazhdan–Luztig cell structure of the Iwahori–Hecke algebra. For other types, we see non-cell simple birepresentations.

I'm looking at *Lusztig–Vogan module categories*, which categorify the Lusztig–Vogan modules over the Iwahori–Hecke algebra (Victor will tell us more about these tomorrow).

In particular, I'm using the *weak Jordan–Hölder theorem* of Mazorchuk and Miemietz to look at their simple module subcategories in the hopes that they contain new information compared to their decategorifications. I'd also like to see if any interesting new categories can be built via extensions.

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Thank you for your attention!