Real Groups and Categorical Representation Theory

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Overview

The theme of this talk will be using categorification to study the representation theory of real Lie groups.

The material is quite dense; as a result, I would like to take a "hands-on" approach during this talk. I will be leaving out precise definitions of Soergel bimodules and birepresentations, and instead focusing on telling a coherent story that conveys where I fit into the picture, as well as the flavour of what I'm doing.

Roadmap

Real Groups and Lusztig–Vogan Modules

Motivation I

Physics is, (very) roughly speaking, the study of symmetry and symmetry breaking.

For our purposes, this is captured by the study of Lie groups and their representations.

The primary motivation for this talk will be the classification of unitary representations of real Lie groups.

Understanding the entire class is hopeless for now, so we instead look at the more manageable class of admissible representations of real reductive groups.

Motivation II

For real reductive groups in the so-called Harish–Chandra class, the irreducible admissible representations are classified by a certain set of parameters due to the Langlands classification.

This classification scheme is similar to other classification schemes in representation theory! That is, we have some standard principal series module, with every irreducible module contained uniquely within it.

Given such a scheme, we're often interested in the relationship between the principal series and irreducibles.

Motivation III

To this end, Lusztig and Vogan created a family of modules over the Iwahori–Hecke algebra that are built from connected real reductive groups ([Vog82], [LV83]).

These modules admit a standard basis and a canonical basis, with the change-of-basis coefficients giving rise to a collection of polynomials.

The values of these Kazhdan–Lusztig–Vogan polynomials at 1 determine the Jordan–Hölder multiplicities of irreducible submodules of the standard module for the underlying reductive group.

The philosophy of categorification has been fruitful in similar contexts. What can it tell us here?

Lusztig–Vogan Module Categories

The General Setup

The Lusztig–Vogan (LV) module categories take as ingredients

- a connected, complex, reductive algebraic group G;
- a Borel subgroup $B \subseteq G$;
- a holomorphic involution $\theta : G \to G$;
- a finite index subgroup $K \subseteq G^\theta$ *(the identity component of* G^θ*)*;
- a maximal torus $T_K \subseteq B \cap K$.

$$\begin{aligned} \theta &\rightsquigarrow \text{real form of } G \\ T_K \subseteq G &\rightsquigarrow \text{Weyl group} \\ B &\rightsquigarrow \text{choice of simple roots} \end{aligned}$$

The Equal Rank Case

1. Let W be a Weyl group with $S \subseteq W$ a choice of simple roots and $W_K \subseteq W$ a finite index subgroup.
2. Define a polynomial algebra $R := \Bbbk[\alpha_s : s \in S]$ graded in degree 2.
3. We have a *geometric action* of W on R given by
$$s(\alpha_t) = \alpha_t + 2\cos\left(\frac{\pi}{m_{st}}\right)\alpha_s.$$
4. For any right coset $w \in W_K \backslash W$, let R_w be the *w-standard (R^{W_K}, R)-bimodule* given by R as a vector space with the right action given by the w-twisting $p \cdot_w f := pw(f)$, for $p \in R_w$, $f \in R$.

Definition (Larson and Romanov, 2022)

The corresponding *(equal rank) Lusztig–Vogan module category* is
$$\mathcal{N}^0_{LV} = \langle R_w \otimes_R X : w \in W_K \backslash W,\ X \in \mathrm{Ob}(\mathbb{S}\mathrm{Bim}(W,S)) \rangle_{\oplus,\ominus,(1)}.$$

Example: $\mathsf{SL}(2, \mathbb{R})$

Possibly the simplest LV category is the one corresponding to $\mathsf{SL}(2, \mathbb{R})$.

Our ingredients are

- $G = \mathsf{SL}(2, \mathbb{C})$,
- $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C} \right\}$,
- $\theta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$,
- $T_K = K = G^\theta = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{C}^\times \right\}$.

This corresponds to $W = S_2 = \{1, s\}$, $S = \{s\}$ and $W_K = \{1\}$.

The geometric action is given by $1(\alpha_t) = \alpha_t$ and $s(\alpha_t) = -\alpha_t$.

Example: SL$(2, \mathbb{R})$

Let's compute the indecomposables of this category!

We'll start by computing products of standard bimodules and indecomposables from $\mathbb{S}$Bim.

$$R_1 \otimes_R B_1 \cong R, \qquad R_1 \otimes_R B_s \cong B_s,$$
$$R_s \otimes_R B_1 \cong R_s, \qquad R_s \otimes_R B_s \cong B_s.$$

As it turns out, all of these are indecomposable in $\mathcal{N}^0_{LV}$, so luckily we don't need to look at direct summands!

Example: SL$(2, \mathbb{R})$

Here's something interesting we can observe.

For $P, Q \in \text{Ind}(\mathcal{N}^0_{LV})/\sim$, write $Q \geq P$ if there exists $X \in \mathrm{Ob}(\mathbb{S}\text{Bim})$ for which Q is isomorphic to a direct summand of $P \otimes_R B$.

Stepping back to our example, this gives us the cell structure

$$\begin{tikzcd} \boxed{R} \arrow[dr, dash] & \\ & \boxed{B_s} \\ \boxed{R_s} \arrow[ur, dash] & \end{tikzcd} .$$

Example: $\mathsf{SU}(2,1)$

Let's look at a more involved example.

Our ingredients are

- $G = \mathsf{SL}(3,\mathbb{C})$,
- $\theta : \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \begin{pmatrix} a & b & -c \\ d & e & -f \\ -g & -h & i \end{pmatrix}$,
- $K = G^\theta = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} : a,b,c,d,e \in \mathbb{C},\ (ad-bc)e = 1 \right\}$,
- $T_K = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & \frac{1}{ab} \end{pmatrix} : a, b \in \mathbb{C}^\times \right\}$.

Example: $\mathrm{SU}(2,1)$

The indecomposables of this category are much more difficult to compute. With some work, however, it can be shown that they are

$$R, \quad R_t, \quad R_{ts}, \quad B_t, \quad R_t \otimes_R B_s, \quad R^s \otimes_{R^W} R.$$

Products of standard bimodules and indecomposables are already no longer necessarily indecomposable!

The cell structure is given by

$$\begin{tikzcd}
 & \boxed{R_t} \arrow[dr, dash] & \\
\boxed{R, B_t} \arrow[ur, dash] \arrow[dr, dash] & & \boxed{R_t \otimes_R B_s, R_{ts}} \\
 & \boxed{R^s \otimes_{R^W} R} \arrow[ur, dash] &
\end{tikzcd}.$$

Weak Jordan–Hölder Series

Weak Jordan–Hölder Series

In [MM16, Theorem 8], Mazorchuk and Miemietz developed a higher categorical analogue to the Jordan–Hölder theorem.

This is central to higher category theory, as it shows how different birepresentations decompose into simple (transitive) birepresentations.

It's quite technical, so let's apply it to one of our examples.

Example: SL$(2, \mathbb{R})$

Our filtrations consist of directed order ideals of Ind$(\mathcal{N}^0_{LV})/\sim$.
Each ideal has one less object than the next.

In the SL$(2, \mathbb{R})$ case, one such filtration is

$$Q_1 \coloneqq \{[B_s]\} \subset Q_2 \coloneqq \{[R_s], [B_s]\} \subset Q_3 \coloneqq \{[R], [R_s], [B_s]\}.$$

These give rise to the additive $\mathbb{S}$Bim-module subcategories

$$\mathcal{M}_{Q_1} = \langle B_s \rangle_{\oplus,(1)}, \quad \mathcal{M}_{Q_2} = \langle R_s, B_s \rangle_{\oplus,(1)}, \quad \mathcal{M}_{Q_3} = \langle R, R_s, B_s \rangle_{\oplus,(1)}.$$

Denote by $\mathcal{M}_{Q_i/Q_{i-1}}$ the quotient of $\mathcal{M}_{Q_i}$ by the two-sided ideal generated by the identity morphisms in Q_{i-1}.

Example: SL$(2, \mathbb{R})$

What are the simple module categories?

Well, $\mathcal{M}_{Q_1}$ doesn't change.
The quotient functor from $\mathcal{M}_{Q_2}$ to $\mathcal{M}_{Q_2/Q_1}$ takes B_s to 0.
The quotient functor from $\mathcal{M}_{Q_3}$ to $\mathcal{M}_{Q_3/Q_2}$ takes R_s and B_s to 0.

As it happens, the functor sending R_s to R defines an equivalence of module categories $\mathcal{M}_{Q_2/Q_1} \simeq \mathcal{M}_{Q_3/Q_2}$.

Don't be fooled by "$R_s \otimes_R R_s \sim R = 0$"!

These are transitive module categories, but there is a unique way of making them simple by throwing away non-invertible morphisms.

What's Next?

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We would like to eventually look at these simple decompositions outside of the type A case.

The final goal is to consider a notion of extensions in order to build new module categories from such simple constituents.

The hope is that these new module categories, may contain their own interesting representation theoretic data.

References

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[MM16] Mazorchuk, V. and Miemietz, V., *Transitive 2-representations of finitary 2-categories*, Trans. Amer. Math. Soc. 368.**11** (2016), pp. 7623–7644.

[Vog82] Vogan Jr., D. A., *Irreducible characters of semisimple Lie groups IV. Character-multiplicity duality*, Duke Math. J. 49.**4** (1982), pp. 943–1073.

Thank you very much for your attention!