From Subfactors to Richard Thompson's Groups and their Generalizations

Daniel Dunmore
Supervised by Dr. Arnaud Brothier

School of Mathematics and Statistics
UNSW Sydney

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A Brief History

There are three main periods that will be especially relevant to this talk.

- 1930s – 1940s: Murray and von Neumann introduce subfactors in the context of their “rings of operators”.
- 1980s – 1990s: the tour de force of Jones.
- 2010s – 2020s: a “happy accident” produces a new machine for the study of certain classes of particularly stubborn groups.

What is a Subfactor?

A *von Neumann algebra* (VNA) is a self-adjoint subalgebra M of $\mathcal{B}(H)$ equal to its *von Neumann closure* M'' [von Neumann 30]. *Very* heuristically, VNAs are like a "non-commutative" or "quantum" analogue of measure theory.

Definition (Subfactor)

A *factor* is a von Neumann algebra M with trivial centre.
A *subfactor* is a unital inclusion of factors $N \subseteq M$.

Every von Neumann algebra can be decomposed uniquely as a "direct integral" (a kind of measure theoretic generalization of the direct sum) of its factors [Murray-von Neumann 49].

Classification of Factors

We classify factors by their orthogonal projections ($p \in M : p = p^2 = p^*$).

- Type I: contains at least one minimal projection.
- Type II: not type I yet contains at least one non-zero finite projection.
- Type III: neither type I nor II.

Type I factors are trivial: they are all $\mathcal{B}(H)$ for some Hilbert space H.

Two kinds of type II factors: type II_1 (1_M is finite, or equivalently M admits a unique, faithful, tracial state $\tau : M \to \mathbb{C}$) and type II_∞ (tensor product of type II_1 and type I_∞).

Type III are tricky, originally considered to be pathological.

Examples of Subfactors

Example (Group von Neumann Algebra)

The *group von Neumann algebra* $\mathcal{L}G := (\lambda_G(G))''$ of a discrete group G is not only a factor, but a type II_1 factor, if and only if G is an ICC group; for instance F_n, S_∞, etc.

Example (Crossed Products)

Crossed products, while slightly technical, are a great way of producing subfactors. For instance, all type III_1 factors are subalgebras of $M \otimes \mathcal{B}(L^2(\mathbb{R}))$, for M a factor of type II_∞.

Invariants of Subfactors – Index

From now on, we will assume all subfactors $N \subseteq M$ are type II_1.

The index $[M : N]$ is a measure of the "relative dimension" of a subfactor; it is finite if and only if M is a finitely-generated N-module [Pimsner-Popa 86].

The index was *completely categorized* by Jones in the 1980s [Jones 83]:

Theorem (Jones Index Theorem)

$[M : N] \in \{4\cos^2(\pi/n) : n \geq 3, n \in \mathbb{N}\} \cup [4, \infty]$

Invariants of Subfactors – Basic Construction

Let e_N be the orthogonal projection of $L^2(M)$ onto $L^2(N)$ taking M to N. The GNS construction gives us $\pi_\tau : M \to \mathcal{B}(L^2(M))$ in a *canonical way*. When $[M : N] < \infty$, we obtain a tower of type II_1 subfactors

$$M_{-1} := \pi_\tau(N) \subseteq \pi_\tau(M) \subseteq (\pi_\tau(M) \cup \{e_N\})'' =: M_1.$$

Invariants of Subfactors – Standard Invariant

From a finite index subfactor $N \subseteq M$, we can construct an infinite grid of finite-dimensional von Neumann algebras called the *standard invariant*:

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subseteq & N' \cap M & \subseteq & N' \cap M_1 & \subseteq & N' \cap M_2 & \subseteq & \cdots \\ & & \cup\!| & & \cup\!| & & \cup\!| & & \\ & \mathbb{C} = & M' \cap M & \subseteq & M' \cap M_1 & \subseteq & M' \cap M_2 & \subseteq & \cdots \end{array}$$

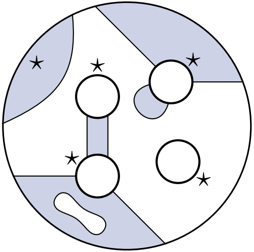
This data is highly admissible to a variety of interesting axiomatizations!

For example...

The Category of Planar Algebras

Planar algebras were devised by Jones in 1999 as a new axiomatization of the standard invariant.

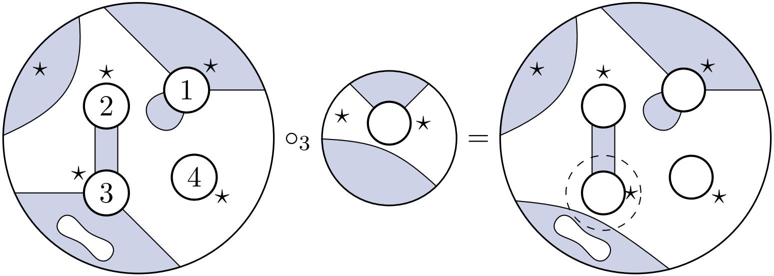
Basic idea: take a family of vector spaces P_ι indexed by $\iota \in \mathbb{N} \times \{-, +\}$. We construct morphisms between them via *shaded planar tangles*.

$T :=$ 

To the left is a *shaded planar* 6*-tangle of kind "$-$"* representing a multilinear map

$$Z_T : P_{4,-} \times P_{2,+} \times P_{4,+} \times P_{0,+} \to P_{6,-}.$$

Composition of planar tangles is defined as follows.
Note that we have explicitly ordered the "input discs".

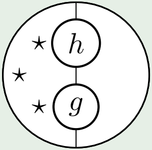


Remark. Thomas will mention in his talk *pivotal categories*, which have some notion of "rotational invariance". There is in fact a "folklore" classification of planar algebras in terms of certain pivotal categories!

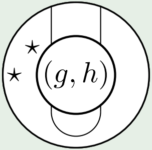
Examples of Planar Algebras

Example (Group Planar Algebra)

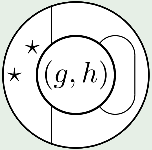
Let $P_{2n} := \mathbb{C}G^n$, and define a planar algebra structure by



$:= gh,$



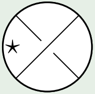
$:= hg,$



$:= g.$

Example (Conway Tangles)

Take $P_{2n} := C_n$, where C_n is the vector space of Conway n-tangles. This gives a planar algebra generated by the crossing



$\longleftrightarrow \chi \in C_2$

under the Reidemeister relations. This setting has been useful for Jensen!

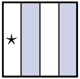
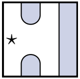
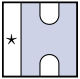
Subfactor Planar Algebras

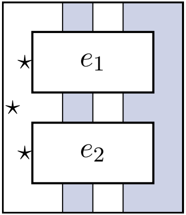
Given a subfactor $N \subseteq M$ (with some additional adjectives...), we obtain a planar algebra by taking $P_{n,+} = N' \cap M_{n-1}$ and $P_{n,-} = M' \cap M_n$.

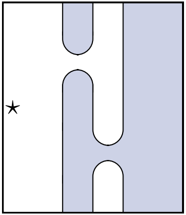
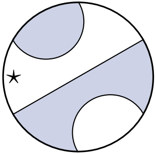
This planar algebra will be shaded with $\dim(P_{0,\pm}) = 1$, and of course each $P_{n,\pm}$ will be a finite-dimensional VNA. If the subfactor is of finite index and the traces on N' and each M_n are compatible, the planar algebra will be spherical and each P_ι will admit a positive-definite inner product [Jones 21].

We call such planar algebras *subfactor planar algebras*. Many subfactor planar algebras admit subfactors, although the correspondence is only bijective among *amenable* subfactors [Popa 94].

The prototypical example of a subfactor planar algebra is the Temperley-Lieb-Jones planar algebra for some parameter δ.

$P_{6,+} :=$ Span$_{\mathbb{C}}\Big\{ 1 :=$ , $e_1 :=$ , $e_2 :=$ , ,  $\Big\}$,

$e_2 e_1 :=$ 

$=$  $=$  .

We replace any closed loops by multiplying outside by a factor of δ.

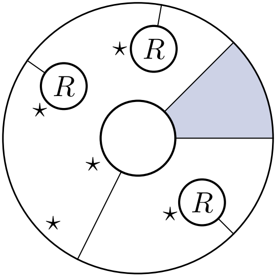
The Dream of Vaughan Jones

Roughly speaking, a conformal field theory (CFT) in our setting consists of a collection of VNAs localized on intervals of the circle, acted on by the group of orientation-preserving diffeomorphisms, and subject to certain physical axioms. Here the circle plays the role of a one-dimensional component of spacetime subject to conformal and chiral symmetry.

As it turns out, these conformal nets automatically give subfactors (as these VNAs are necessarily type III_1 factors). A question that Jones dedicated much time towards answering was: do all type III_1 subfactors come from CFTs in this way?

Jones' Attempt – Naive Approach [Jones 17]

Start with a subfactor planar algebra P and some isometric $R \in P_1$, and let $\mathcal{F}$ be the set of *finite* subsets of S^1, directed by inclusion.
Define morphisms $F_m \to F_n$ for $F_m \subseteq F_n$ as follows.

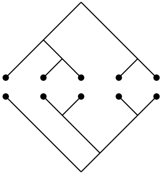
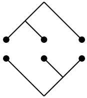


The affine tangle to the left represents a morphism of the form $F_3 \to F_6$.

In the direct limit we obtain an action of $\mathrm{Diff}^+(S^1)$ on a Hilbert space, but the whole thing is very artificial; the Hilbert space is non-separable and the action of diffeomorphisms is hopelessly discontinuous.

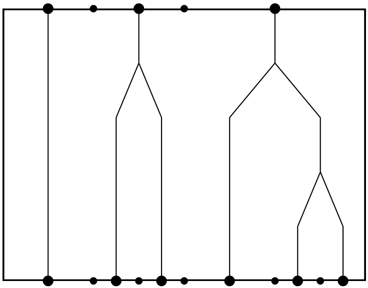
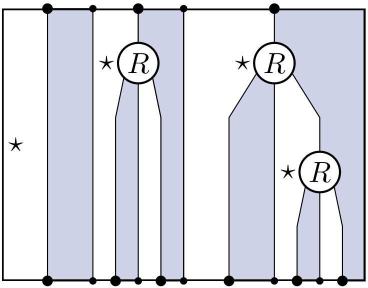
A Quick Detour – Thompson's Groups

Consider the set X consisting of pairs (s, t) of bifurcating trees with $\text{Leaf}(s) = \text{Leaf}(t)$. We draw these by placing s on top and t beneath, and define an equivalence by adding or collapsing opposing pairs of carets:

$\frac{s}{t} :=$  $\sim$  .

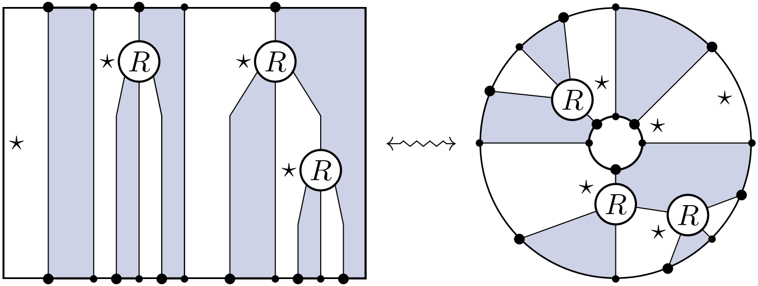
The set $X/\sim$ in fact forms a group; given s/t and t/u, we define their product to be s/u, and write $(s/t)^{-1} := t/s$. This group is known as *Thompson's group* F. If we allow cyclic permutations (resp. any permutation) between leaves, we get *Thompson's group* T (resp. V).

Start once more with a subfactor planar algebra P, but this time choose an isometric $R \in P_4$. We construct a functor Φ sending $n \in \mathbb{N}$ to $P_{n,+}$ and bifurcating forests to (rectangular) planar tangles as follows.

$\Phi :$  $\mapsto$ 

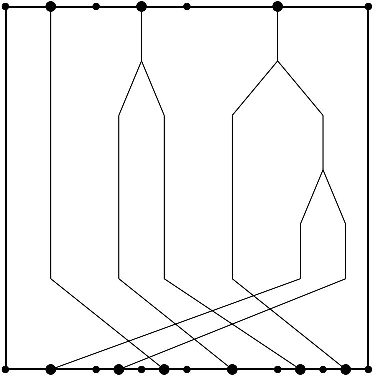
This defines a system directed by growing forests, and in the direct limit we obtain a Hilbert space as well as an action by bifurcating forests. This in fact gives us a unitary representation of F!

If we instead glue the ends of the strip together, we get an affine tangle.

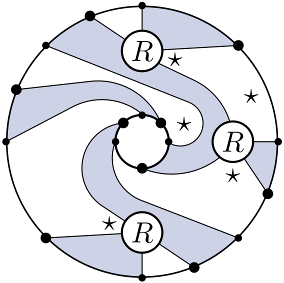


We can also map bifurcating forests with cyclic permutations of their leaves to tangles. Given an *annular representation* of P, where the 2π rotation tangle acts as identity, we obtain a unitary representation of T!

Φ :



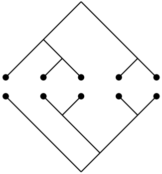
$\mapsto$



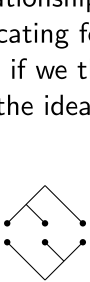
Jones' Technology

The underlying idea here is as follows. The relationship between Thompson's groups and our categories of bifurcating forests is that the former are the *groups of fractions* of the latter; if we think back to our picture of F, the equivalence relation encodes the idea of *formal inverses*.

$$\frac{s}{t} :=$$



$$=$$



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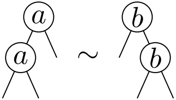
The technology that Jones discovered is a powerful machine that takes in a category $\mathcal{C}$ admitting some group of fractions $G_{\mathcal{C}}$, as well as a functor $\Phi : \mathcal{C} \to \mathcal{D}$ (where $\mathcal{D}$ has sets as objects), and spits out an action of $G_{\mathcal{C}}$ on some direct limit space $\mathscr{X}_\Phi$ that *inherits the structure of the objects of* $\mathcal{D}$.

Forest-Skein Categories

The first step towards capitalizing on the new machinery of Jones is the forest-skein formalism of Brothier. A *forest-skein category* is a category of $\mathcal{S}$-coloured bifurcating trees subject to some set $\mathcal{R}$ of *skein relations*.

$$\mathcal{C}_2 = \mathsf{ForC}^F \left\langle (a), (b) : \cdots \right\rangle.$$



When $\mathcal{R}$ is empty, these categories admit groups of fractions precisely when the trees are monochromatic, whence we recover F, T and V.

It was our hope to perform a similar construction of "discrete CFTs" with the more sophisticated group of symmetries that forest-skein groups afford, but it is challenging to make this work. I believe for the time being a better understanding of when forest-skein categories admit groups of fractions and what their Jones representations look like will be essential.

Thank you very much for your attention!

This was quite a dense talk to begin the seminar with, so please don't hesitate to ask questions. You may also e-mail me if you prefer. Otherwise, let's pass the torch to Thomas!