

CLASSIFICATION OF FUSION CATEGORIES

1. PROLOGUE

What are fusion categories? What are near-groups, Haagerup–Izumi categories and quadratic categories? What is modular data? What are $6j$ symbols? What is the even part of a subfactor?

Let \mathcal{C} be a fusion category over \mathbb{k} . Recall the Yoneda embedding

$$\mathfrak{J}_*(X) := \text{Hom}_{\mathcal{C}}(-, X) \quad \text{and} \quad \mathfrak{J}_*(f : X \rightarrow Y) := (\text{Hom}_{\mathcal{C}}(-, X) \Rightarrow \text{Hom}_{\mathcal{C}}(-, Y)).$$

Taking the Yoneda embedding of a component $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$ of the associativity natural isomorphism α , we obtain a natural isomorphism

$$\mathfrak{J}_*(\alpha_{X,Y,Z}) = (\text{Hom}_{\mathcal{C}}(-, (X \otimes Y) \otimes Z) \Rightarrow \text{Hom}_{\mathcal{C}}(-, X \otimes (Y \otimes Z))).$$

Thus we have an isomorphism of vector spaces

$$[\mathfrak{J}_*(\alpha_{X,Y,Z})](W) : \text{Hom}_{\mathcal{C}}(W, (X \otimes Y) \otimes Z) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(W, X \otimes (Y \otimes Z));$$

that is, an invertible matrix. In other words, the associativity is given by matrices indexed by X, Y, Z, W . But we can simplify things further! Suppose we now choose representatives $\{X_i\}_{i \in \Gamma}$ of isomorphism classes of simple objects and bases for each *multiplicity space* $H_{i,j}^h := \text{Hom}_{\mathcal{C}}(X_h, X_i \otimes X_j)$. Then by semisimplicity,

$$X_i \otimes X_j \cong \bigoplus_{h \in \Gamma} X_h^{\dim_{\mathbb{k}}(H_{i,j}^h)},$$

whence we have both

$$\begin{aligned} \bigoplus_{m \in \Gamma} H_{m,i_3}^{i_0} \otimes_{\mathbb{k}} H_{i_1,i_2}^m &= \bigoplus_{m \in \Gamma} \text{Hom}_{\mathcal{C}}(X_{i_0}, X_m \otimes X_{i_3}) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(X_m, X_{i_1} \otimes X_{i_2}) \\ &\cong \text{Hom}_{\mathcal{C}}(X_{i_0}, (X_{i_1} \otimes X_{i_2}) \otimes X_{i_3}) \end{aligned}$$

and

$$\begin{aligned} \bigoplus_{n \in \Gamma} H_{i_1,n}^{i_0} \otimes_{\mathbb{k}} H_{i_2,i_3}^n &= \bigoplus_{n \in \Gamma} \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes X_n) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(X_n, X_{i_2} \otimes X_{i_3}) \\ &\cong \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes (X_{i_2} \otimes X_{i_3})). \end{aligned}$$

We thus obtain canonical bases for the above Hom-spaces. The (*quantum*) *6j-symbols* of \mathcal{C} are then defined to be the matrix blocks of the form

$$\Phi_{i_1,i_2,i_3}^{i_0,m,n} : H_{m,i_3}^{i_0} \otimes_{\mathbb{k}} H_{i_1,i_2}^m \rightarrow H_{i_1,n}^{i_0} \otimes_{\mathbb{k}} H_{i_2,i_3}^n$$

such that the block-diagonal matrices $\Phi_{i_1,i_2,i_3}^{i_0} := \bigoplus_{m \in \Gamma} \bigoplus_{n \in \Gamma} \Phi_{i_1,i_2,i_3}^{i_0,m,n}$ give a change-of-basis

$$\Phi_{i_1,i_2,i_3}^{i_0} : \text{Hom}_{\mathcal{C}}(X_{i_0}, (X_{i_1} \otimes X_{i_2}) \otimes X_{i_3}) \cong \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes (X_{i_2} \otimes X_{i_3}))$$

between the canonical bases. These matrices $\Phi_{i_1,i_2,i_3}^{i_0}$ themselves form the blocks of a block-diagonal matrix $\Phi_{i_1,i_2,i_3} := \bigoplus_{i_0 \in \Gamma} \Phi_{i_1,i_2,i_3}^{i_0}$, which gives us a change-of-basis $(X_{i_1} \otimes X_{i_2}) \otimes X_{i_3} \cong X_{i_1} \otimes (X_{i_2} \otimes X_{i_3})$ describing how the associator acts on each summand X_{i_0} . As it happens, we in fact have that

$$\Phi_{i_1,i_2,i_3}^{i_0,m,n} = v_n^{-1}(X_{i_0}, X_{i_1}, X_{i_2} \otimes X_{i_3}) \circ [\mathfrak{J}_*(\alpha_{X_{i_1},X_{i_2},X_{i_3}})](X_{i_0}) \circ u_m(X_{i_0}, X_{i_1} \otimes X_{i_2}, X_{i_3}),$$

where

$$[u_m(W, U, V)](f \otimes_{\mathbb{k}} g) := (g \otimes \text{id}_V) \circ f \quad \text{and} \quad [v_n(W, U, V)](f' \otimes_{\mathbb{k}} g') := (\text{id}_U \otimes g') \circ f'$$

for all $f \in \text{Hom}_{\mathcal{C}}(W, X_m \otimes V)$, $g \in \text{Hom}_{\mathcal{C}}(X_m, U)$, $f' \in \text{Hom}_{\mathcal{C}}(W, U \otimes X_n)$ and $g' \in \text{Hom}_{\mathcal{C}}(X_n, V)$.

More explicitly, suppose we have an isomorphism

$$\text{Hom}_{\mathcal{C}}(X_4, (X_1 \otimes X_2) \otimes X_3) \cong \text{Hom}_{\mathcal{C}}(X_4, X_1 \otimes (X_2 \otimes X_3)).$$

Let X_5 and X_6 be simple summands of $(X_1 \otimes X_2)$ and $(X_2 \otimes X_3)$, respectively. Then we can determine our isomorphism by determining the block matrices of the form

$$\mathrm{Hom}_{\mathcal{C}}(X_4, X_5 \otimes X_3) \rightarrow \mathrm{Hom}_{\mathcal{C}}(X_4, X_1 \otimes X_6)$$

for all such X_5 and X_6 . These matrices are exactly the $6j$ symbols, where the six simple objects $X_1, X_2, X_3, X_4, X_5, X_6$ play the role of the eponymous “six j ’s”. This is also where u and v come in; given $f \in H_{5,3}^4$ and $g \in H_{1,2}^5$, we have $[u_5(X_4, X_1 \otimes X_2, X_3)](f \otimes_{\mathbb{K}} g) : X_4 \rightarrow (X_1 \otimes X_2) \otimes X_3$, and similarly $[v_6(X_4, X_1, X_2 \otimes X_3)](f' \otimes_{\mathbb{K}} g') : X_4 \rightarrow X_1 \otimes (X_2 \otimes X_3)$ for $f' \in H_{1,6}^4$ and $g' \in H_{2,3}^6$.

Example 1.1. ($\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$). Consider $\mathrm{Vec}_{\mathbb{Z}/2\mathbb{Z}}$, a category with two isomorphism classes of simple objects – $[\mathbb{1}]$ and $[X]$ – satisfying the fusion rule

$$[X] \otimes [X] = [\mathbb{1}].$$

Let’s assume that it’s skeletal – that is, $X \otimes X = \mathbb{1}$ – and start by computing the change-of-basis matrix $\Phi_{X,X,X}$. This matrix can be written in block diagonal form as

$$\Phi_{X,X,X} = \begin{pmatrix} \Phi_{X,X,X}^{\mathbb{1}} & 0 \\ 0 & \Phi_{X,X,X}^X \end{pmatrix}.$$

First, observe that

$$\Phi_{X,X,X}^{\mathbb{1}} : \mathrm{Hom}(\mathbb{1}, (X \otimes X) \otimes X) \rightarrow \mathrm{Hom}(\mathbb{1}, X \otimes (X \otimes X));$$

but $(X \otimes X) \otimes X = X = X \otimes (X \otimes X)$, so $\Phi_{X,X,X}^{\mathbb{1}}$ is just the isomorphism of 0-dimensional vector spaces; that is, $\Phi_{X,X,X}^{\mathbb{1}} = 0$. Let’s now compute $\Phi_{X,X,X}^X$. We shall start by choosing bases for the multiplicity spaces $H_{X,X}^{\mathbb{1}}$, $H_{1,X}^X$ and $H_{X,1}^X$. These spaces are all 1-dimensional, so we will choose basis elements $\iota_{X,X}^{\mathbb{1}}$, λ_X^{-1} and ρ_X^{-1} , respectively. Here we have a canonical choice of basis elements for $H_{1,X}^X$ and $H_{X,1}^X$; namely, the left and right unitors λ and ρ . This culminates in the basis elements

$$\iota_{(X \otimes X) \otimes X}^X := (\iota_{X,X}^{\mathbb{1}} \otimes \mathrm{id}_X) \circ \lambda_X^{-1} = [u_1(X, X \otimes X, X)](\lambda_X^{-1} \otimes_{\mathbb{K}} \iota_{X,X}^{\mathbb{1}})$$

and

$$\iota_{X \otimes (X \otimes X)}^X := (\mathrm{id}_X \otimes \iota_{X,X}^{\mathbb{1}}) \circ \rho_X^{-1} = [v_1(X, X, X \otimes X)](\rho_X^{-1} \otimes_{\mathbb{K}} \iota_{X,X}^{\mathbb{1}})$$

for $\mathrm{Hom}(X, (X \otimes X) \otimes X)$ and $\mathrm{Hom}(X, X \otimes (X \otimes X))$, respectively. These are once again both 1-dimensional spaces, so let’s define a constant $\mu_1 \in \mathbb{K}^\times$ for which

$$\Phi_{X,X,X}^X(\iota_{(X \otimes X) \otimes X}^X) =: \mu_1 \cdot \iota_{X \otimes (X \otimes X)}^X.$$

In order to gain constraints on μ_1 , we need to use the pentagon equation. To this end, we first choose bases for the Hom-spaces

$$\begin{aligned} & \mathrm{Hom}(\mathbb{1}, (\mathbb{1} \otimes X) \otimes X), \\ & \mathrm{Hom}(\mathbb{1}, \mathbb{1} \otimes (X \otimes X)), \\ & \mathrm{Hom}(\mathbb{1}, (X \otimes \mathbb{1}) \otimes X), \\ & \mathrm{Hom}(\mathbb{1}, X \otimes (\mathbb{1} \otimes X)), \\ & \mathrm{Hom}(\mathbb{1}, (X \otimes X) \otimes \mathbb{1}), \\ & \mathrm{Hom}(\mathbb{1}, X \otimes (X \otimes \mathbb{1})). \end{aligned}$$

We can write bases for these Hom-spaces in terms of the bases we chose previously. That is,

$$\begin{aligned}
\iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}} &:= (\lambda_X^{-1} \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}}, \\
\iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}} &:= (\text{id}_{\mathbb{1}} \otimes \iota_{X,X}^{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1}, \\
\iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}} &:= (\rho_X^{-1} \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}}, \\
\iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}} &:= (\text{id}_X \otimes \lambda_X^{-1}) \circ \iota_{X,X}^{\mathbb{1}}, \\
\iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}} &:= (\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1}, \\
\iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}} &:= (\text{id}_X \otimes \rho_X^{-1}) \circ \iota_{X,X}^{\mathbb{1}}.
\end{aligned}$$

This culminates in the three new constants given by

$$\begin{aligned}
\Phi_{\mathbb{1},X,X}^{\mathbb{1}}(\iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}}) &:= \mu_2 \cdot \iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}}, \\
\Phi_{X,\mathbb{1},X}^{\mathbb{1}}(\iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}}) &:= \mu_3 \cdot \iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}}, \\
\Phi_{X,X,\mathbb{1}}^{\mathbb{1}}(\iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}}) &:= \mu_4 \cdot \iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}}.
\end{aligned}$$

The final bases we need are for the Hom-spaces

$$\begin{aligned}
&\text{Hom}(\mathbb{1}, ((X \otimes X) \otimes X) \otimes X), \\
&\text{Hom}(\mathbb{1}, (X \otimes (X \otimes X)) \otimes X), \\
&\text{Hom}(\mathbb{1}, (X \otimes X) \otimes (X \otimes X)), \\
&\text{Hom}(\mathbb{1}, X \otimes ((X \otimes X) \otimes X)), \\
&\text{Hom}(\mathbb{1}, X \otimes (X \otimes (X \otimes X))).
\end{aligned}$$

Following the same procedure as before, we have

$$\begin{aligned}
\iota_{((X \otimes X) \otimes X) \otimes X}^{\mathbb{1}} &:= (\iota_{(X \otimes X) \otimes X}^X \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}} = ((\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_X) \otimes \text{id}_X) \circ \iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}}, \\
\iota_{(X \otimes (X \otimes X)) \otimes X}^{\mathbb{1}} &:= (\iota_{X \otimes (X \otimes X)}^X \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}} = ((\text{id}_X \otimes \iota_{X,X}^{\mathbb{1}}) \otimes \text{id}_X) \circ \iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}}, \\
\iota_{(X \otimes X) \otimes (X \otimes X)}^{\mathbb{1}} &:= (\iota_{X,X}^{\mathbb{1}} \otimes (\text{id}_X \otimes \text{id}_X)) \circ \iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}} = ((\text{id}_X \otimes \text{id}_X) \otimes \iota_{X,X}^{\mathbb{1}}) \circ \iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}}, \\
\iota_{X \otimes ((X \otimes X) \otimes X)}^{\mathbb{1}} &:= (\text{id}_X \otimes \iota_{(X \otimes X) \otimes X}^X) \circ \iota_{X,X}^{\mathbb{1}} = (\text{id}_X \otimes (\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_X)) \circ \iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}}, \\
\iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}} &:= (\text{id}_X \otimes \iota_{X \otimes (X \otimes X)}^X) \circ \iota_{X,X}^{\mathbb{1}} = (\text{id}_X \otimes (\text{id}_X \otimes \iota_{X,X}^{\mathbb{1}})) \circ \iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}}.
\end{aligned}$$

The pentagon diagram thus gives us

$$\begin{array}{ccccc}
& & \iota_{((X \otimes X) \otimes X) \otimes X}^{\mathbb{1}} & & \\
& \swarrow \alpha_{X,X,X} \otimes \text{id}_X & & \searrow \alpha_{X \otimes X, X, X} & \\
\mu_1 \cdot \iota_{(X \otimes (X \otimes X)) \otimes X}^{\mathbb{1}} & & & & \mu_2 \cdot \iota_{(X \otimes X) \otimes (X \otimes X)}^{\mathbb{1}} \\
\downarrow \alpha_{X, X \otimes X, X} & & & & \downarrow \alpha_{X, X, X \otimes X} \\
\mu_1 \mu_3 \cdot \iota_{X \otimes ((X \otimes X) \otimes X)}^{\mathbb{1}} & \xrightarrow{\text{id}_X \otimes \alpha_{X, X, X}} & \mu_1^2 \mu_3 \cdot \iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}} & \xlongequal{\quad} & \mu_2 \mu_4 \cdot \iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}}
\end{array}$$

This gives us the relation $\mu_1^2\mu_3 = \mu_2\mu_4$. However, there could be more constraints on these constants! As it turns out, though, there is a simple way of determining these. Suppose we use the more insightful labeling convention

$$\begin{aligned}\omega(X, X, X) &:= \mu_1, \\ \omega(\mathbb{1}, X, X) &:= \mu_2, \\ \omega(X, \mathbb{1}, X) &:= \mu_3, \\ \omega(X, X, \mathbb{1}) &:= \mu_4.\end{aligned}$$

The reason why we have chosen this relabeling is because now acting by $\alpha_{A,B,C}$ in the pentagon diagram introduces a factor of $\omega(A, B, C)$. If we apply this to the general pentagon diagram

$$\begin{array}{ccc} & ((A \otimes B) \otimes C) \otimes D & \\ \swarrow \alpha_{A,B,C} \otimes \text{id}_D & & \searrow \alpha_{A \otimes B, C, D} \\ (A \otimes (B \otimes C)) \otimes D & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow \alpha_{A, B \otimes C, D} & & \downarrow \alpha_{A, B, C \otimes D} \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\text{id}_A \otimes \alpha_{B, C, D}} & A \otimes (B \otimes (C \otimes D)) \end{array}$$

we obtain the constraints

$$\omega(A, B, C)\omega(A, BC, D)\omega(B, C, D) = \omega(AB, C, D)\omega(A, B, CD)$$

for all $A, B, C, D \in \text{Ob}(\text{Vec}_{\mathbb{Z}/2\mathbb{Z}})$. This is exactly the definition for a 3-cocycle ω ! In general, the possible $6j$ -symbols – and hence the possible associators – for Vec_G are exactly given by 3-cocycles. We will henceforth denote by Vec_G^ω the category of G -graded vector spaces with associativity constraint $\alpha_{ghk} = \omega(g, h, k)\text{id}_{ghk}$, for all $g, h, k \in G$, and Vec_G the category with trivial associativity.

2. THE CUNTZ ALGEBRA APPROACH OF IZUMI

Consider the category $\text{End}(M)$, for M a hyperfinite type III factor. This category is strict, as $\rho \otimes \sigma := \rho \circ \sigma$ by definition. Every near-group category with group G contains some copy of Vec_G^ω corresponding to the group-like part. Because every unitary near-group category is a subcategory of $\text{End}(M)$ and is hence itself strict, we know that it will actually contain the “strictification” of some Vec_G^ω . However, Izumi shows that if \mathcal{C} is any fusion category containing a simple object that is fixed under tensor products with invertibles (that is, there exists some simple object X such that $X \otimes g \cong X$ for all invertible g), then it contains a copy of Vec_G , for G the group of isomorphism classes of invertible objects. He shows in addition that if the fusion category is also unitary, then $g \otimes X = X$ (but we may not necessarily have that $X \otimes g = X$). The upshot is that we almost know how objects are tensored, since the group-like part will have trivial associativity (that is, $g \otimes h = gh$). We just need to understand $X \otimes g$ and $X \otimes X$, as well as the morphisms.

In [Izu17], Izumi showed that every unitary near-group category \mathcal{C} with multiplicity m is equivalent to a subcategory of $\text{End}(M)$, where M is the hyperfinite type III₁ factor. In particular, it is generated by a single irreducible endomorphism $\rho \in \text{End}_0(M)$ satisfying the fusion rules

$$\begin{aligned} [\rho] \otimes [\rho] &= \bigoplus_{g \in G} [\alpha_g] \oplus [\rho]^{\oplus m}, \\ [\alpha_g] \otimes [\alpha_h] &= [\alpha_{gh}], \\ [\alpha_g] \otimes [\rho] &= [\rho] \otimes [\alpha_g] = [\rho], \end{aligned}$$

where the map $\alpha : G \rightarrow \text{Aut}(M)$ induces an injective homomorphism from G into $\text{Out}(M)$.

The main result of [Izu17] is [Izu17, Theorem 4.9]. Essentially, there is a bijective correspondence between the set of equivalence classes of unitary near-group categories with finite group G and multiplicity parameter m and the set of equivalence classes of admissible tuples $(\mathcal{K}, j_1, j_2, V, U_{\mathcal{K}}, \chi, l)$ (see [Izu17, Definition 4.8]). Here \mathcal{K} is the finite-dimensional Hilbert space $\text{Hom}(\rho, \rho^2)$, j_1 and j_2 are two antilinear isometries of \mathcal{K} , V and $U_{\mathcal{K}}$ are unitary representations of G on \mathcal{K} , $\{\chi_g\}_{g \in G}$ are characters of G and l is a linear map from \mathcal{K} to the set $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{K})$ of bounded operators $\mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K}$.

By [Izu17, Theorem 9.1], the unitary near-group categories with finite Abelian group G and $m = |G|$ are completely classified tuples of the form $(\langle \cdot, \cdot \rangle, a, b, c)$, where $\langle \cdot, \cdot \rangle : G \times G \rightarrow \mathbb{T}$ is a non-degenerate symmetric bicharacter and where $a : G \rightarrow \mathbb{T}$, $b : G \rightarrow \mathbb{T}$ and $c \in \mathbb{T}$ satisfy various conditions. When we say that $\langle \cdot, \cdot \rangle$ is a bicharacter, we mean that

$$\langle xy, z \rangle = \langle x, z \rangle \langle y, z \rangle \quad \text{and} \quad \langle x, yz \rangle = \langle x, y \rangle \langle x, z \rangle$$

for all $x, y, z \in G$. By non-degenerate, we mean that

$$\langle x, \cdot \rangle = \langle y, \cdot \rangle$$

if and only if $x = y$. This is equivalent to the map $\varphi : G \rightarrow \text{Hom}(G, \mathbb{T})$ given by $x \mapsto \langle x, \cdot \rangle$ being an isomorphism.

Definition 2.1. (Cuntz Algebra). Let $\{S_i\}_{i=1}^n$ be a set of isometries on an infinite-dimensional Hilbert space \mathcal{H} . Suppose moreover that these isometries satisfy the Cuntz relation

$$\sum_{k=1}^n S_k S_k^* = 1.$$

The Cuntz algebra \mathcal{O}_n is the universal C^* -algebra $C^*(S_1, \dots, S_n)$.

Remark 2.2. Note that, as isometries, $S_i^* S_i = 1$. In particular, we must have that $S_i^* S_j = \delta_{i,j}$ for all $i, j \in \{1, \dots, n\}$. This follows from the fact that a sum of projections is itself a projection if and only if the projections in the sum are pairwise orthogonal. The Cuntz relation is essentially ensuring that the sum of the projections $S_i S_i^*$ is the trivial projection.

Example 2.3. (Fibonacci Category). Let's look at the Fibonacci category. This is the near-group with $G = \{0\}$ and $m = 1$. Our choice for $\langle \cdot, \cdot \rangle$ is obvious, and [Izu17, Lemma 7.1] tells us that

$$c^3 a(0) = \sqrt{n} = 1 \implies a(0) = c^{-3}.$$

Moreover, [Izu17, Theorem 9.1] tells us that b is defined by $b : 0 \mapsto -1/d$, where d corresponds to the dimension of our irreducible generator ρ . Let's determine c and d . Because b is equal to its own Fourier transform, [Izu17, Theorem 9.1] tells us that

$$b(0) = ca(0)b(0) \implies a(0) = c^{-1}.$$

In order for $c^{-1} = c^{-3}$, we require $c = \pm 1$. Finally, [Izu17, Equation 9.5] tells us that

$$\begin{aligned} b(0)b(0)b(0) &= b(0)b(0) \mp \frac{1}{d}, \\ \implies -\frac{1}{d^3} &= \frac{1}{d^2} \mp \frac{1}{d}, \\ \implies \pm d^2 - d - 1 &= 0. \end{aligned}$$

This only has a real solution when $c = 1$, whence d is nothing but the golden ratio (as it cannot be negative in the unitary case - this alternative solution, known as the Galois dual, corresponds to a non-unitary near-group in this case and many others). This is exactly what we would expect, as d is the dimension of X (where $d^2 = 1 + d$ comes from the fusion rule $X^2 = \mathbb{1} \oplus X$).

Example 2.4. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's look at the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $m = 2$. This near-group corresponds to the even part of the type A_4 subfactor. We know the dimension is

$$d_{\pm} := \frac{m \pm \sqrt{m^2 + 4n}}{2} = 1 \pm \sqrt{3}.$$

In the unitary setting, we of course ask that d be positive, and hence we choose $d = d_+$. The only possibility for a non-degenerate bicharacter is

$$\langle 0, 0 \rangle = 1, \quad \langle 0, 1 \rangle = \langle 1, 0 \rangle = 1 \quad \text{and} \quad \langle 1, 1 \rangle = -1.$$

From [Izu17, Equation 7.8], it follows that

$$a(0) = 1 \quad \text{and} \quad a(1) = \pm i.$$

Meanwhile, [Izu17, Equation 9.4] tells us that

$$\overline{b(1)} = \pm i b(1) \implies \Re(b(1)) = \mp \Im(b(1)),$$

whence [Izu17, Equation 9.3] gives us

$$\Re(b(1))^2 + \Im(b(1))^2 = (b(1)\overline{b(1)})^2 = \frac{1}{2} \implies b(1) = \frac{1 - a(1)}{2}.$$

It then follows from evaluating [Izu17, Equation 9.1] with $g = 0$ and rearranging for c that

$$c = \frac{1 - \sqrt{3} + a(1)(1 + \sqrt{3})}{2\sqrt{2}}.$$

Note that we may choose either $a(1) = i$ or $a(1) = -i$; both of these lead to solutions. Moreover, in the non-unitary setting, we may take the Galois conjugate of d .

Example 2.5. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's determine the Haagerup–Izumi categories with $G = \mathbb{Z}/2\mathbb{Z}$. Let

$$d_{\pm} := \frac{n \pm \sqrt{n^2 + 4}}{2},$$

where in this example $d := 1 + \sqrt{2}$. Izumi's classification involves a triplet $(\epsilon_h(g), \omega(g), A_{h,k}(g))$, where $\epsilon_h(g) \in \{-1, 1\}$, $\omega(g) \in \mathbb{T}$ and $A_{h,k}(g) \in \mathbb{C}$ satisfy [Izu18, Equations 4.1–4.9]. Well, we know

$$\epsilon_0(0) = \epsilon_1(0) = 1 \quad \text{and} \quad \epsilon_0(1) = \epsilon_0(1)\epsilon_0(1) \implies \epsilon_0(1) = 1.$$

By [Izu18, Equation 4.7],

$$A_{0,0}(g) = A_{0,0}(g)\omega(g),$$

which tells us that either $\omega(g) = 1$ or $A_{0,0}(g) = 0$ for each $g \in G$. Let's fix any $g \in G$ and consider the case when $A_{0,0}(g) = 0$. In this case, however, [Izu18, Equations 4.3 and 4.4] give us

$$A_{1,0}(g)\overline{A_{\delta_{g,0}-g,0}(g)} = 1 - \frac{|\omega(g)|}{d} \implies \frac{1}{d^2} = 1 - \frac{1}{d}.$$

This “equality” is nonsense; we must therefore have $\omega(g) = 1$ for all $g \in G$. Suppose now that $\epsilon_1(1) = 1$. Then [Izu18, Equation 4.7] gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \quad \text{and} \quad A_{0,1}(1) = A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(0) = A_{1,1}(1)$. Now, [Izu18, Equations 4.4 and 4.6] tell us

$$A_{0,1}(0)A_{1,1}(1) + A_{1,1}(0)A_{1,0}(1) = 0.$$

Thus $A_{0,1}(g) = A_{1,1}(g) = A_{1,0}(g) = 0$ and hence $A_{0,0}(g) = -1/d$ by [Izu18, Equation 4.3]. However, in this case we cannot satisfy [Izu18, Equation 4.9]. Suppose instead that $\epsilon_1(1) = -1$. With this new 2-cocycle, [Izu18, Equation 4.7] now gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \quad \text{and} \quad A_{0,1}(1) = -A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(1) = -A_{1,1}(0)$. We then see by [Izu18, Equation 4.4] that

$$A_{0,1}(0)A_{1,0}(0) + A_{1,1}(0)A_{1,1}(0) = 1 \implies A_{1,0}(0) = \pm \frac{1}{\sqrt{2}} = \pm \frac{1}{d-1},$$

and by [Izu18, Equation 4.9] that

$$A_{0,0}(0)A_{1,0}(0)^2 = A_{1,0}(0)^2 + A_{1,0}(0)^3 \implies A_{0,0}(0) = 1 + A_{1,0}(0) = \frac{d-1 \pm 1}{d-1}.$$

Finally, [Izu18, Equation 4.3] allows us to deduce

$$A_{1,0}(0) = -\frac{1}{d-1},$$

whence

$$A(0) = \frac{1}{d-1} \begin{pmatrix} d-2 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad A(1) = \frac{1}{d-1} \begin{pmatrix} d-2 & -1 \\ -1 & 1 \end{pmatrix}.$$

This category is nothing but the even part of the type A_7 subfactor.

Remark 2.6. Suppose that $|G|$ is odd. Then [Izu18, Equation 4.1] tells us that $\epsilon_h(g) = 1$, while [Izu18, Equation 4.2] tells us that $\omega(g)$ does not depend on g . Moreover, $A_{h,k}(g)$ cannot depend on g by [Izu18, Equation 4.5], and either $\omega = 1$ or $A_{0,0} = 0$ by [Izu18, Equation 4.7]. In this case, [Izu18, Equations 4.1–4.9] reduce to the following four equations.

$$A_{h,k} = A_{-k,h-k}\omega = A_{k-h,-h}\bar{\omega},$$

$$\sum_{h \in G} A_{h,0} = -\frac{\bar{\omega}}{d_{\pm}},$$

$$\sum_{h \in G} A_{h-g,k} A_{k,h-g'} = \delta_{g,g'} - \frac{\delta_{k,0}}{d_{\pm}},$$

$$\sum_{l \in G} A_{x+y,l} A_{-x,l+p} A_{-y,l+q} = A_{p+x,q+x+y} A_{q+y,p+x+y} - \frac{\delta_{x,0}\delta_{y,0}}{d_{\pm}}.$$

The first three equations above are precisely [EG17, Equations 4.7, 4.8 and 4.9]! In particular, to see that our third equation is equivalent to [EG17, Equation 4.9], we simply make the change of variables $\hat{g} := g' - g$ and $\hat{h} := h - g'$, whence we obtain

$$\sum_{h \in G} A_{\hat{h}+\hat{g},k} A_{k,\hat{h}} = \delta_{\hat{g},0} - \frac{\delta_{k,0}}{d_{\pm}}.$$

Similarly, using our first equation while making the change of variables $\hat{l} := l - x - y$, $\hat{p} := p + x + y$, $\hat{q} := q + x + y$, $\hat{x} := -x$ and $\hat{y} := -y$, our fourth equation becomes

$$\bar{\omega} \sum_{\hat{l} \in G} A_{\hat{l},\hat{x}+\hat{y}} A_{\hat{x},\hat{l}+\hat{p}} A_{\hat{y},\hat{l}+\hat{q}} = A_{\hat{y}+\hat{p},\hat{q}} A_{\hat{x}+\hat{q},\hat{p}} - \frac{\delta_{\hat{x},0}\delta_{\hat{y},0}}{d_{\pm}},$$

showing that it is equivalent to [EG17, Equation 4.11].

3. THE LEAVITT ALGEBRA APPROACH OF EVANS–GANNON

The important result is [EG17, Theorem 2].

Definition 3.1. (Leavitt Algebra). *Let $X := (x_{ij})$ and $Y := (y_{ij})$ be $m \times n$ and $n \times m$ matrices of symbols, respectively. The Leavitt K -algebra of type (m, n) is the free associative unital K -algebra*

$$\mathcal{L}_K(m, n) := \frac{K[x_{ij}, y_{ij}]}{\langle XY = I_m, YX = I_n \rangle}.$$

In other words, it is the universal K -algebra with generators

$$\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \sqcup \{y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and Leavitt–Cuntz relations

$$\sum_{k=1}^m y_{ik}x_{kj} = \delta_{i,j} \quad \text{and} \quad \sum_{k=1}^n x_{ik}y_{kj} = \delta_{i,j},$$

for all suitable i, j .

Consider the Leavitt \mathbb{C} -algebra of type $(1, n)$, which we shall henceforth denote by $\mathcal{L}_n := \mathcal{L}_{\mathbb{C}}(1, n)$. We have that $\mathcal{O}_n = C^*(\mathcal{L}_n)$, where $x_i = S_i$ and $y_i = S_i^*$. Let's think about what this means precisely. The Leavitt–Cuntz relations for $m = 1$ become

$$y_i x_j = \delta_{i,j} \quad \text{and} \quad \sum_{k=1}^n x_k y_k = 1.$$

We may endow \mathcal{L}_n with the structure of a $*$ -algebra by defining a conjugate homogeneous antihomomorphism that sends $x_i \mapsto y_i$ and $y_i \mapsto x_i$. We further define

$$\|a\| := \sup\{p(a) : p \text{ is a } C^*\text{-seminorm on } \mathcal{L}_n\}$$

for all $a \in \mathcal{L}_n$, where a C^* -seminorm is just a seminorm for which $p(a^*a) = p(a)^2$ and $p(ab) \leq p(a)p(b)$. Note that $0 \leq \|a\| \leq 1$, as $1 = \|y_i x_i\| = \|x_i^2\|$ and hence $\|x_i\| = \|y_i\| = 1$ for all i . The condition $p(ab) \leq p(a)p(b)$ ensures that $\mathcal{I} := \{a \in \mathcal{L}_n : \|a\| = 0\}$ is an ideal in \mathcal{L}_n . Our C^* -seminorm then descends to a C^* -norm on the quotient $\mathcal{L}_n/\mathcal{I}$. The completion of $\mathcal{L}_n/\mathcal{I}$ with respect to this C^* -norm is known as the universal C^* -algebra of \mathcal{L}_n , denoted by $C^*(\mathcal{L}_n)$. This is precisely \mathcal{O}_n by definition. We may therefore view \mathcal{L}_n as the polynomial part of \mathcal{O}_n .

Example 3.2. (Yang–Lee Category). Let $G = \{0\}$. Then [EG17, Equation 4.7] demands that

$$A_{0,0} = \omega A_{0,0} = \bar{\omega} A_{0,0} \implies \omega = 1,$$

whence [EG17, Equation 4.8] tells us that

$$A_{0,0} = -\frac{1}{d_{\pm}}.$$

The rest of [EG17, Equations 4.7–4.10] are satisfied by these choices. Hence by [EG17, Theorem 2], we have two fusion categories for $G = \{0\}$; a unitary one with $\pm = +$ (the Fibonacci category) and a non-unitary one with $\pm = -$ (the Yang–Lee category).

Example 3.3. ($G = \mathbb{Z}/2\mathbb{Z}$). The equations we must satisfy for $|G|$ even are given in [Izu18] (is this true?). Adapting our argument from Example 2.5, we see that there is exactly one non-unitary Haagerup–Izumi category with $G = \mathbb{Z}/2\mathbb{Z}$. This corresponds to $\epsilon_h(g) = (-1)^{gh}$, $\omega(g) = 1$,

$$A(0) = \frac{1}{d-1} \begin{pmatrix} d & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A(1) = \frac{1}{d-1} \begin{pmatrix} d & 1 \\ 1 & -1 \end{pmatrix}.$$

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