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C^*-Algebras of Discrete Groups

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Outline

The topics we would like to talk about today are related to the C^*-algebras of discrete groups and the theory of amenability. In particular, we would like to introduce

- the definition of a (discrete) group C^*-algebra;
- the classical characterization of amenability;
- the connection between amenability and group C^*-algebras;
- alternative characterizations of amenability.

Motivation

can also mention use in proofs wrt non-simply-connected manifolds

Applications of group C^*-algebras:

- representation theory – we will see this shortly
- topology – all compact spaces admit C^*-algebras

Applications of amenability:

- ergodic theory – involves the action of a group on something, measurable actions of amenable groups are unique

Unitary Representations

complex Hilbert space

A *unitary representation* is a homomorphism $\pi : G \to \mathcal{U}(\underline{H})$.

Left regular representation: unitary rep λ taking g to λ_g, where

$$\lambda_g : \ell^2(G) \to \ell^2(G)$$
$$\xi \mapsto \underline{g \cdot \xi}.$$

$(g \cdot \xi)(h) = \xi(g^{-1}h), \ \forall g, h \in G$

Group C^*-Algebras [Murray-von Neumann 36-43; Segal 47]

First define the *group algebra*, $\mathbb{C}G$: elements are $\sum_{g\in G} a_g g$, for finitely many non-zero $a_g \in \mathbb{C}$.

Unitary reps of G extend bijectively to unital reps of $\mathbb{C}G$.
(unit-preserving) and to " reps of C^*

Two most significant group C^*-algebras:

- *Reduced*, $C_r^*(G)$: norm completion wrt $\|\cdot\|_r := \|\lambda(\cdot)\|$.
- *Universal*, $C^*(G)$: $\|\cdot\|_U := \sup\{\|\pi(\cdot)\| : \text{unitary rep } \pi\}$.

Hierarchy: $G \subset \mathbb{C}G \subseteq C_r^*(G) \subseteq \underline{C^*(G)}$.

Larger completion!

Finite: $C^*(\mathbb{Z}_n) \cong C(\mathbb{Z}_n)$,

$C^*(S_3) \cong M_2(\mathbb{C}) \oplus \mathbb{C} \oplus \mathbb{C}$

Abelian: $C^*(\mathbb{Z}) \cong C(\mathbb{T})$

In general, $C_r^*(G) = C^*(G)$ for finite or Abelian groups.

Free group $F_2 := \langle a, b \rangle$, $C_r^*(G) \not\cong C^*(G)$!

The reduced group C^*-algebra is easy, while the universal one can be quite unfriendly. Can we study $C^*(G)$ by means of $C_r^*(G)$?

Amenability [von Neumann 29]

A discrete group G is amenable if it admits a G-*invariant mean*; a linear functional $m : \ell^\infty(G) \to \mathbb{C}$ satisfying

- $m(\chi_G) = 1$; $\chi_G(g) = 1, \forall g \in G$
- $m(f) \geq 0$ for all $f \geq 0$;
- $m(g \cdot f) = m(f)$.

Amenability is inherited by subgroups and preserved under group homomorphisms.

G-invariant mean for finite group (it really is just a mean):

$$m : f \mapsto \frac{1}{|G|} \sum_{g \in G} f(g)$$

More examples: Abelian groups, infinite dihedral group, etc.

sub-exponential growth.

Non-examples: braid groups, free groups, etc.

contains free subgroup (ping-pong lemma)

Amenability via Weak Containment [Hulanicki-Reiter, 66]

Let G be a discrete group. Then the following are equivalent:

- G is amenable;
- $\|\lambda(a)\| \geq \|\chi_G(a)\|$, $\forall a \in \mathbb{C}G$;
- $\|\lambda(a)\| \geq \|\pi(a)\|$, $\forall$ unitary reps π of G and $a \in \mathbb{C}G$.

Immediate corollary: G is amenable iff $\|\cdot\|_r = \|\cdot\|_U$. That is, amenability is equivalent to saying that $C^*(G) = C^*_r(G)$!

Recall that $\|\cdot\|_r = \|\lambda(\cdot)\|$, $\|\cdot\|_U = \sup \|\pi(\cdot)\|$

Banach-Tarski Paradox

G acts on S^2 is the same as G acting on G.

Banach-Tarski paradox: a 2-sphere S^2 can be decomposed into a finite number of disjoint subsets, which can be rearranged by rotation only (actions of $SO(3)$) to form two exact copies of S^2.

A discrete group admits such a paradoxical decomposition iff it is not amenable. [Neumann 29]

Rotation group SO(3) has free subgroup.

Do we observe this paradox on the circle?

SO(2) is obviously Abelian, so no.

This concludes the presentation. I hope you found this brief look into the theory of group C^*-algebras as enjoyable and insightful as I have.

Thank you very much for your attention!

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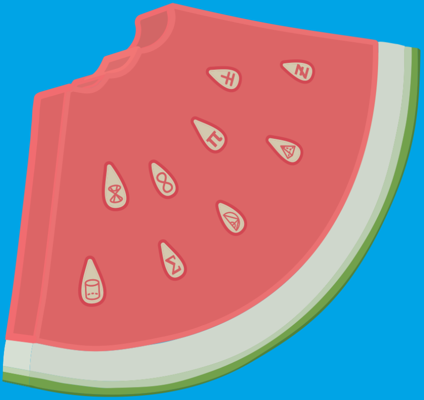

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