Categorical Representation Theory

1. Prologue

Before we get into representation theory, let's *very* briefly review some basic definitions from higher category theory. We will follow [Mal18] and [Str95, §9] as our main references.

Definition 1.1. (Bicategory). *A* bicategory $\mathscr{C}$ *consists of*

- *a class* $\mathrm{Ob}(\mathscr{C})$ *of* objects *(or* 0-cells*);*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, a* Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ *whose objects are called* 1-morphisms *(or* 1-cells*), whose morphisms are called* 2-morphisms *(or* 2-cells*) and where composition of* 2*-morphisms is known as* vertical composition *and denoted* $\circ_v$*;*
- *for each triple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathrm{Ob}(\mathscr{C})$*, a functor* $\circ_h : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ *known as* horizontal composition*;*
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a distinguished* 1*-morphism* $\mathrm{id}_{\mathtt{i}} \in \mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{i}))$ *known as the* identity morphism *for* $\mathtt{i}$ *(sometimes denoted* $\mathbb{1}_{\mathtt{i}}$*, alluding to the monoidality of* $\mathscr{C}(\mathtt{i}, \mathtt{i})$*);*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, natural isomorphisms* l *and* r *satisfying*
$$\begin{pmatrix} f \mapsto \mathrm{id}_{\mathtt{j}} \circ_h f \\ \alpha \mapsto \mathrm{id}_{\mathrm{id}_{\mathtt{j}}} \circ_h \alpha \end{pmatrix} \overset{l}{\Rightarrow} \begin{pmatrix} f \mapsto f \\ \alpha \mapsto \alpha \end{pmatrix} \overset{r}{\Leftarrow} \begin{pmatrix} f \mapsto f \circ_h \mathrm{id}_{\mathtt{i}} \\ \alpha \mapsto \alpha \circ_h \mathrm{id}_{\mathrm{id}_{\mathtt{i}}} \end{pmatrix},$$
known respectively as a left *and* right unitor *(whose components* l_f *and* r_f *are* 2*-morphisms);*
- *for each quadruple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} \in \mathrm{Ob}(\mathscr{C})$*, a natural isomorphism* a *between the two horizontal composition functors* $\mathscr{C}(\mathtt{k}, \mathtt{l}) \times \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{l})$ *given by*
$$\begin{pmatrix} h \times g \times f \mapsto (h \circ_h g) \circ_h f \\ \gamma \times \beta \times \alpha \mapsto (\gamma \circ_h \beta) \circ_h \alpha \end{pmatrix} \overset{a}{\Rightarrow} \begin{pmatrix} h \times g \times f \mapsto h \circ_h (g \circ_h f) \\ \gamma \times \beta \times \alpha \mapsto \gamma \circ_h (\beta \circ_h \alpha) \end{pmatrix},$$
known as an associator *(whose components* $a_{h,g,f}$ *are* 2*-morphisms);*

such that the pentagon diagram

$$\begin{tikzcd}
 & ((k \circ_h h) \circ_h g) \circ_h f \arrow[dl, "a_{k,h,g} \circ_h \mathrm{id}_f"'] \arrow[dr, "a_{k \circ_h h, g, f}"] & \\
(k \circ_h (h \circ_h g)) \circ_h f \arrow[d, "a_{k, h \circ_h g, f}"'] & & (k \circ_h h) \circ_h (g \circ_h f) \arrow[d, "a_{k,h,g \circ_h f}"] \\
k \circ_h ((h \circ_h g) \circ_h f) \arrow[rr, "\mathrm{id}_k \circ_h a_{h,g,f}"'] & & k \circ_h (h \circ_h (g \circ_h f))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(g \circ_h \mathrm{id}_{\mathtt{j}}) \circ_h f \arrow[rr, "a_{g, \mathrm{id}_{\mathtt{j}}, f}"] \arrow[dr, "r_g \circ_h \mathrm{id}_f"'] & & g \circ_h (\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[dl, "\mathrm{id}_g \circ_h l_f"] \\
 & g \circ_h f &
\end{tikzcd}$$

commute, for all 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*,* $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k}, \mathtt{l}))$*,* $k \in \mathrm{Ob}(\mathscr{C}(\mathtt{l}, \mathtt{m}))$*.*

A 2-category is a *strict* bicategory; that is, a bicategory whose unitors and associators are all identities. In this case the pentagon and triangle diagrams hold automatically. Observe that (strict) bicategories $\mathscr{C}$ with a single object $\bullet$ are in bijection with (strict) monoidal categories under taking the monoidal delooping; in particular, our monoidal category is nothing but the End-category $\mathscr{C}(\bullet, \bullet)$, where the monoidal product is given by horizontal composition.

We shall henceforth adopt the notation $\mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j}) \coloneqq \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$ and $\mathrm{Mor}^2_{\mathscr{C}}(f, g) \coloneqq \mathrm{Mor}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(f, g)$, for $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ and $f, g \in \mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$. Unfortunately, we will frequently change our notation for 1-morphisms and 2-morphisms depending on what makes sense contextually. For instance, in light of the above remark, 1-morphisms will often be objects of a monoidal category, whence we will write them as X, Y, Z and so on; meanwhile, sometimes they will be realized as functors, in which case we will use F, G, H and so on. The same goes for 2-morphisms. We hope this will not cause any unnecessary confusion!

Definition 1.2. (Lax Functor). *A* lax functor F *between bicategories $\mathscr{C}$ and $\mathscr{D}$ consists of*

- *a map $\mathrm{F} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$;*
- *for each pair of objects $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, a functor $\mathrm{F} : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{j}))$;*
- *for each triple of objects $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathrm{Ob}(\mathscr{C})$, a natural transformation m between the two "composition by F" functors $\mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{k}))$ given by*
$$\begin{pmatrix} g \times f \mapsto \mathrm{F}(g) \circ_h \mathrm{F}(f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta) \circ_h \mathrm{F}(\alpha) \end{pmatrix} \overset{m}{\Longrightarrow} \begin{pmatrix} g \times f \mapsto \mathrm{F}(g \circ_h f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta \circ_h \alpha) \end{pmatrix},$$
whose components $m_{g,f}$ are 2-morphisms;
- *for each object $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, a 2-morphism $i : \mathrm{id}_{\mathrm{F}(\mathtt{i})} \Rightarrow \mathrm{F}(\mathrm{id}_{\mathtt{i}})$ in $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{F}(\mathtt{i})))$;*

such that the hexagon diagram

$$\begin{tikzcd}
& (\mathrm{F}(h) \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) \arrow[dl, "m_{h,g} \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] \arrow[dr, "a_{\mathrm{F}(h),\mathrm{F}(g),\mathrm{F}(f)}"] & \\
\mathrm{F}(h \circ_h g) \circ_h \mathrm{F}(f) \arrow[d, "m_{h \circ_h g, f}"'] & & \mathrm{F}(h) \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[d, "\mathrm{id}_{\mathrm{F}(h)} \circ_h m_{g,f}"] \\
\mathrm{F}((h \circ_h g) \circ_h f) \arrow[dr, "\mathrm{F}(a_{h,g,f})"'] & & \mathrm{F}(h) \circ_h \mathrm{F}(g \circ_h f) \arrow[dl, "m_{h, g \circ_h f}"] \\
& \mathrm{F}(h \circ_h (g \circ_h f)) &
\end{tikzcd}$$

and the squares

$$\begin{tikzcd}
\mathrm{id}_{\mathrm{F}(\mathtt{j})} \circ_h \mathrm{F}(f) \arrow[r, "l_{\mathrm{F}(f)}"] \arrow[d, "i \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & \mathrm{F}(f) \\
\mathrm{F}(\mathrm{id}_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[r, "m_{\mathrm{id}_{\mathtt{j}}, f}"'] & \mathrm{F}(\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[u, "\mathrm{F}(l_f)"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathrm{F}(f) \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[r, "r_{\mathrm{F}(f)}"] \arrow[d, "\mathrm{id}_{\mathrm{F}(f)} \circ_h i"'] & \mathrm{F}(f) \\
\mathrm{F}(f) \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}}) \arrow[r, "m_{f, \mathrm{id}_{\mathtt{i}}}"'] & \mathrm{F}(f \circ_h \mathrm{id}_{\mathtt{i}}) \arrow[u, "\mathrm{F}(r_f)"']
\end{tikzcd}$$

commute, for all 1-morphisms $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$, $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$, $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k}, \mathtt{l}))$. If m and i are invertible, we call F a pseudofunctor*. If they are identity, we call it a* 2-functor*.*

In the same way that bicategories generalize monoidal categories, pseudofunctors generalize strong monoidal functors, preserving both vertical composition (strictly) and horizontal composition (up to isomorphism).

Definition 1.3. (Lax Natural Transformation). *A* lax natural transformation Φ *from the lax functor* $\mathrm{F} : \mathscr{C} \to \mathscr{D}$ *to the lax functor* $\mathrm{G} : \mathscr{C} \to \mathscr{D}$ *consists of*

- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a* 1*-morphism* $\Phi_{\mathtt{i}} : \mathrm{F}(\mathtt{i}) \to \mathrm{G}(\mathtt{i})$ *in* $\mathrm{Ob}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{i})))$*;*
- *for each* 1*-morphism* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, a* 2*-morphism* $\Phi_f : G(f) \circ_h \Phi_{\mathtt{i}} \Rightarrow \Phi_{\mathtt{j}} \circ_h \mathrm{F}(f)$ *in* $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{j})))$*;*

such that

- *for each* 2*-morphism* $\alpha : f \Rightarrow g$ *in* $\mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, the square*

$$\begin{tikzcd}
\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}} \arrow[r, "\mathrm{G}(\alpha)\circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"] \arrow[d, "\Phi_f"'] & \mathrm{G}(g) \circ_h \Phi_{\mathtt{i}} \arrow[d, "\Phi_g"] \\
\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \arrow[r, "\mathrm{id}_{\Phi_{\mathtt{j}}} \circ_h \mathrm{F}(\alpha)"'] & \Phi_{\mathtt{j}} \circ_h \mathrm{F}(g)
\end{tikzcd}$$

  commutes (that is, the 2*-morphisms* Φ_f *are the components of a natural transformation);*
- *for each pair of* 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*, the octagon diagram*

$$\begin{tikzcd}
& \mathrm{G}(g) \circ_h (\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}}) \arrow[rr, "a_{\mathrm{G}(g),\mathrm{G}(f),\Phi_{\mathtt{i}}}"] \arrow[dl, "\mathrm{id}_{\mathrm{G}(g)} \circ_h \Phi_f"'] & & (\mathrm{G}(g) \circ_h \mathrm{G}(f)) \circ_h \Phi_{\mathtt{i}} \arrow[dr, "m_{g,f} \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"] & \\
\mathrm{G}(g) \circ_h (\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f)) \arrow[d, "a_{\mathrm{G}(g),\Phi_{\mathtt{j}},\mathrm{F}(f)}"'] & & & & \mathrm{G}(g \circ_h f) \circ_h \Phi_{\mathtt{i}} \arrow[d, "\Phi_{g \circ_h f}"] \\
(\mathrm{G}(g) \circ_h \Phi_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[dr, "\Phi_g \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & & & & \Phi_{\mathtt{k}} \circ_h \mathrm{F}(g \circ_h f) \\
& (\Phi_{\mathtt{k}} \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) & & \Phi_{\mathtt{k}} \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[ll, "a_{\Phi_{\mathtt{k}},\mathrm{F}(g),\mathrm{F}(f)}"] \arrow[ur, "\mathrm{id}_{\Phi_{\mathtt{k}}} \circ_h m_{g,f}"'] &
\end{tikzcd}$$

  commutes;
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, the pentagon diagram*

$$\begin{tikzcd}
\mathrm{id}_{\mathrm{G}(\mathtt{i})} \circ_h \Phi_{\mathtt{i}} \arrow[r, "l_{\Phi_{\mathtt{i}}}"] \arrow[d, "i \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"'] & \Phi_{\mathtt{i}} & \Phi_{\mathtt{i}} \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[l, "r_{\Phi_{\mathtt{i}}}"'] \arrow[d, "\mathrm{id}_{\Phi_{\mathtt{i}}} \circ_h i"] \\
\mathrm{G}(\mathrm{id}_{\mathtt{i}}) \circ_h \Phi_{\mathtt{i}} \arrow[rr, "\Phi_{\mathrm{id}_{\mathtt{i}}}"'] & & \Phi_{\mathtt{i}} \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}})
\end{tikzcd}$$

  commutes.

If each Φ_f *is invertible (they form a natural isomorphism), we call* Φ *a* pseudonatural transformation *or* strong transformation*. If they are identity, we call it a* strict 2-natural transformation*.*

For the following definitions, we refer to [Lei98].

Definition 1.4. (Modification). *Let Φ and Ψ be lax natural transformations from $F : \mathscr{C} \to \mathscr{D}$ to $G : \mathscr{C} \to \mathscr{D}$. A* modification $\Gamma : \Phi \to \Psi$ *consists of*

- *for each object $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, a 2-morphism $\Gamma_{\mathtt{i}} : \Phi_{\mathtt{i}} \Rightarrow \Psi_{\mathtt{i}}$*

such that

- *for each 1-morphism $f : \mathtt{i} \to \mathtt{j}$, the square*

$$\begin{tikzcd} \mathrm{G}(f) \circ_h \Phi_{\mathtt{i}} \arrow[r, "\mathrm{id}_{G(f)} \circ_h \Gamma_{\mathtt{i}}"] \arrow[d, "\Phi_f"'] & \mathrm{G}(f) \circ_h \Psi_{\mathtt{i}} \arrow[d, "\Psi_f"] \\ \Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \arrow[r, "\Gamma_{\mathtt{j}} \circ_h \mathrm{id}_{F(f)}"'] & \Psi_{\mathtt{j}} \circ_h \mathrm{F}(f) \end{tikzcd}$$

  commutes.

Definition 1.5. (Internal Equivalence). *An* internal equivalence *in a bicategory $\mathscr{C}$ consists of a pair of 1-morphisms $f : \mathtt{i} \to \mathtt{j}$ and $g : \mathtt{j} \to \mathtt{i}$ together with an isomorphism $g \circ f \to \mathrm{id}_{\mathtt{i}}$ in $\mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{i}))$ and an isomorphism $f \circ g \to \mathrm{id}_{\mathtt{j}}$ in $\mathrm{Mor}(\mathscr{C}(\mathtt{j}, \mathtt{j}))$. In this case, $\mathtt{i}$ and $\mathtt{j}$ are said to be* equivalent.

We denote by $\mathsf{Lax}(\mathscr{C}, \mathscr{D})$ the "functor bicategory" whose objects are lax functors from $\mathscr{C}$ to $\mathscr{D}$, whose 1-morphisms are lax natural transformations and whose 2-morphisms are modifications. Although this is not in general a 2-category, it is if $\mathscr{D}$ is. We also define the sub-bicategory $[\mathscr{C}, \mathscr{D}]$ of $\mathsf{Lax}(\mathscr{C}, \mathscr{D})$ consisting of pseudofunctors, pseudonatural transformations and modifications.

Definition 1.6. (Biequivalence). *A* biequivalence *between two bicategories $\mathscr{C}$ and $\mathscr{D}$ consists of pseudofunctors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ together with pseudonatural transformations $G \circ F \Rightarrow \mathrm{id}_{\mathscr{C}}$ and $F \circ G \Rightarrow \mathrm{id}_{\mathscr{D}}$ that are also internal equivalences in the bicategories $[\mathscr{C}, \mathscr{C}]$ and $[\mathscr{D}, \mathscr{D}]$. In this case, we write $\mathscr{C} \simeq \mathscr{D}$.*

Finally, let $\mathscr{C}^{\mathrm{op}}$, $\mathscr{C}^{\mathrm{co}}$ and $\mathscr{C}^{\mathrm{co,op}}$ denote the bicategories obtained by reversing only horizontal composition, only vertical composition and both compositions, respectively.

For the sections that follow, we will assume – unless stated otherwise – that all categories are essentially small, that all bicategories are essentially small and that all fields are algebraically closed.

Note that given a bicategory $\mathscr{C}$ and objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, we have that $\mathscr{C}(X,Y)$ is a $(\mathscr{C}(X,X), \mathscr{C}(Y,Y))$-bimodule category. Check bicategory of bifinite bimodules as in [DGG14].

Examples of bicategories.

Given an algebra (or more generally a ring), one can build a 2-category whose objects are algebras, 1-morphisms are modules (A,B)-bimodules and 2-morphisms are bimodule maps. In subfactor theory, we do this for a unital inclusion $N \subseteq M$ of type II_1 subfactors; we set $\mathrm{Ob}(\mathscr{C}) \coloneqq \{M, N\}$ and $\mathscr{C}(X,Y) \coloneqq \mathsf{Bimod}(X,Y)$ ("bimodule summands of basic constructions" / bifinite (X,Y)-bimodules). This produces a 2-category.

Given a shaded planar algebra P, take $\mathrm{Ob}(\mathscr{C}) \coloneqq \{-,+\}$ and let $\mathscr{C}(\varepsilon,\eta) \coloneqq \mathsf{Rect}_P(\varepsilon,\eta)$ be the subcategory of the rectangular category consisting of those morphisms whose source shading is ε and whose target shading is η, with composition running from bottom to top. In other words,

$$\mathrm{Mor}^1_{\mathscr{C}}(\varepsilon,\eta) \coloneqq \{2k : k \in \mathbb{N} \text{ is even if } \varepsilon = \eta \text{ and odd otherwise}\}$$

and $\mathrm{Mor}^2_{\mathscr{C}}(m,n) \coloneqq P_{m+n,\varepsilon}$, for $m,n \in \mathrm{Mor}^1_{\mathscr{C}}(\varepsilon,\eta)$ ([DGG14]). This gives a $\mathbb{C}$-linear (but unfortunately not additive) 2-category. Look into "The Temperley–Lieb algebra at roots of unity".

Explicit example of multifinitary bicategory: group planar algebra? Maybe we can generalize it, see https://scholars.unh.edu/cgi/viewcontent.cgi?article=1338&context=dissertation.

2. Finitary Birepresentation Theory

Definition 2.1. (Idempotent Complete). *Let $\mathcal{C}$ be a category. An* idempotent *is an endomorphism $p : A \to A$ in $\mathcal{C}$ such that $p \circ p = p$. An idempotent is said to* split *if there is an object B and morphisms $\pi : A \to B$, $\iota : B \to A$ in $\mathcal{C}$ such that $p = \iota \circ \pi$ and $\mathrm{id}_B = \pi \circ \iota$. A category is said to be* idempotent complete *(or* idempotent split*) if every idempotent splits.*

Note that the condition $\mathrm{id}_B = \pi \circ \iota$ implies that π is an epimorphism and ι is a monomorphism. To see why, suppose we have morphisms $h, k : B \to C$ with $h \circ \pi = k \circ \pi$. Then $h = h \circ \pi \circ \iota = k \circ \pi \circ \iota = k$, whence π is an epimorphism. Similarly, given morphisms $h, k : C \to B$ with $\iota \circ h = \iota \circ k$, we have that $h = \pi \circ \iota \circ h = \pi \circ \iota \circ k = k$, whence ι is a monomorphism. Thus, because ι is a monomorphism, B is by definition a subobject of A. In other words, a category being idempotent complete means that every idempotent $p : A \to A$ can be seen as a projection onto some subobject B followed by an inclusion back into A. Moreover, in the additive setting we have the following result.

Proposition 2.2. *An idempotent $p : A \to A$ belonging to a preadditive category splits if and only if $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.*

Proof. Suppose $p : A \to A$ is an idempotent that splits. Then by definition we have a subobject I of A together with an epimorphism $\pi_I : A \to I$ and a monomorphism $\iota_I : I \to A$ satisfying $p = \iota_I \circ \pi_I$ and $\mathrm{id}_I = \pi_I \circ \iota_I$. Moreover, because $\mathrm{id}_A - p$ is also idempotent, there similarly exists a subobject K of A together with an epimorphism $\pi_K : A \to K$ and a monomorphism $\iota_K : K \to A$ satisfying $\mathrm{id}_A - p = \iota_K \circ \pi_K$ and $\mathrm{id}_K = \pi_K \circ \iota_K$. Because $\mathrm{id}_A = \iota_I \circ \pi_I + \iota_K \circ \pi_K$, we have the biproduct diagram

$$I \underset{\iota_I}{\overset{\pi_I}{\leftrightarrows}} A \underset{\iota_K}{\overset{\pi_K}{\rightleftarrows}} K.$$

By [Mac13, Theorem VIII.2.2], it follows that $A = I \oplus K$. We claim now that $\mathrm{Im}(p) = I$; we shall prove this by showing that p admits the canonical decomposition

$$K \xrightarrow{\iota_K} A \xrightarrow{\pi_I} I \xrightarrow{\iota_I} A \xrightarrow{\pi_K} K.$$

In particular, we claim that $\mathrm{Ker}(p) = (K, \iota_K)$, that $\mathrm{Coker}(p) = (K, \pi_K)$, that $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and that $\mathrm{Ker}(\pi_K) = (I, \iota_I)$. We show that the first two hold and remark that showing the remaining two is essentially the same. First, observe that

$$p \circ \iota_K = \iota_I \circ \pi_I \circ \iota_K = (\mathrm{id}_A - \iota_K \circ \pi_K) \circ \iota_K = \iota_K - \iota_K \circ \pi_K \circ \iota_K = \iota_K - \iota_K = 0.$$

Moreover, given an object K' together with a morphism $k' : K' \to A$ for which $p \circ k' = 0$, we see that by taking $\ell \coloneqq \pi_K \circ k'$, we have that

$$\iota_K \circ \ell = \iota_K \circ \pi_K \circ k' = (\mathrm{id}_A - p) \circ k' = k'.$$

Thus $\mathrm{Ker}(p) = (K, \iota_K)$. As for $\mathrm{Coker}(p)$, we observe that

$$\pi_K \circ p = \pi_K \circ \iota_I \circ \pi_I = \pi_K \circ (\mathrm{id}_A - \iota_K \circ \pi_K) = \pi_K - \pi_K \circ \iota_K \circ \pi_K = \pi_K - \pi_K = 0,$$

and that for any object C' together with a morphism $c' : A \to C'$ for which $c' \circ p = 0$, taking $\ell \coloneqq c' \circ \iota_K$ gives us

$$\ell \circ \pi_K = c' \circ \iota_K \circ \pi_K = c' \circ (\mathrm{id}_A - p) = c'.$$

That $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and $\mathrm{Ker}(\pi_K) = (I, \iota_I)$ follow similarly, whence $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.

Conversely, suppose that $p : A \to A$ is an idempotent for which $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$. Then we have the canonical decomposition

$$\mathrm{Ker}(p) \xrightarrow{k} A \xrightarrow{\pi} \mathrm{Im}(p) \xrightarrow{\iota} A \xrightarrow{c} \mathrm{Coker}(p).$$

By definition this means that $p = \iota \circ \pi$, so we need only show that $\mathrm{id}_{\mathrm{Im}(p)} = \pi \circ \iota$. But note that π is a cokernel and ι is a kernel, hence they are an epimorphism and a monomorphism, respectively. Thus by the definition of epimorphisms and monomorphisms, we may cancel $p \circ p = p$ on the right by ι and on the left by π, whence we obtain nothing but

$$p \circ p = p \implies \iota \circ \pi \circ \iota \circ \pi = \iota \circ \pi \implies \pi \circ \iota = \mathrm{id}_{\mathrm{Im}(p)}$$

as desired. Thus p splits. This completes the proof. ■

This result is not only important in its own right, but psychologically helpful: it tells us that split idempotents categorify in some heuristic sense the notion of projections from linear algebra, which always split. Moreover, recall that a preadditive category is said to be *Karoubian* (or *pseudo-Abelian*) if every idempotent admits a kernel (or, equivalently, if every idempotent admits an image, as we may obtain the image by considering $\mathrm{Ker}(\mathrm{id}_A - p)$). We therefore have the following corollary.

Corollary 2.3. *A preadditive category is Karoubian if and only if it is idempotent complete.*

Example 2.4. The category of projective modules over a ring is the Karoubi envelope of its full subcategory of free modules, as a module is projective if and only if it is a direct summand of a free module. In other words, categories of projective modules are idempotent complete in a universal way.

Remark 2.5. Let $\mathcal{C}$ be an additive category. Then by [Mac13, §VIII.2], its morphisms form a matrix calculus; that is, for any $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $X \cong \bigoplus_{i=1}^m X_i$ and $Y \cong \bigoplus_{j=1}^n Y_j$, we have that

$$f = \sum_{j=1}^{n} \sum_{i=1}^{m} (\iota_{Y_j} \circ f_{i,j} \circ \pi_{X_i})$$

for $f_{i,j} := \pi_{Y_j} \circ f \circ \iota_{X_i}$, where $\pi_{X_i} : X \to X_i$ and $\pi_{Y_j} : Y \to Y_j$ are epimorphisms while $\iota_{X_i} : X_i \to X$ and $\iota_{Y_j} : Y_j \to Y$ are monomorphisms for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

We also have the following folklorish theorem, which has some nice exactness implications in the module category setting. Due to its very technical nature we will only offer a (fallible) sketch of the proof, but the full proof should follow similarly to the proof of the embedding theorem for Abelian categories. Please see the wonderful write-up given in [Jun19] for more details on these subtleties.

Theorem 2.6. (Freyd–Mitchell Embedding Theorem). *A small category is multifinitary if and only if there is an exact, full embedding into the exact category* $\mathsf{Mod}_p(A)$ *of finitely generated, projective modules over some finite-dimensional, associative* $\Bbbk$*-algebra* A.

Sketch. First, $\mathsf{Mod}_p(A)$ is certainly Karoubi (in fact, the category of projective modules over any ring is the Karoubi envelope of its full subcategory of free modules), and it also has both finitely many isomorphism classes of indecomposable objects and finite-dimensional $\Bbbk$-vector spaces of morphisms.

Conversely, let $\mathcal{C}$ be multifinitary. We wish to find a full, exact embedding $\mathcal{C} \to \mathsf{Mod}_p(A)$ for some finite-dimensional, associative $\Bbbk$-algebra A. Denote by $\mathcal{L} := \mathsf{Fun}_l(\mathcal{C}, \mathsf{Vect}_{\Bbbk}^{\text{f.d.}})$ the category of left exact, $\Bbbk$-linear functors from $\mathcal{C}$ to $\mathsf{Vect}_{\Bbbk}^{\text{f.d.}}$ (which we note are also automatically additive). The contravariant Yoneda embedding $X \mapsto \mathcal{C}(X, -)$ gives us a full, exact embedding よ $: \mathcal{C}^{\text{op}} \to \mathcal{L}$, as $\mathcal{C}(X, -)$ is left exact, which corresponds to a full, exact, covariant embedding よ$^{\text{op}} : \mathcal{C} \to \mathcal{L}^{\text{op}}$ by duality†. Because $\mathcal{L}$ is complete with injective cogenerator given by the direct product $\prod_{X \in \text{Ob}(\mathcal{C})} \mathcal{C}(X, -) \in \text{Ob}(\mathcal{L})$, it follows that $\mathcal{L}^{\text{op}}$ is cocomplete and admits a corresponding projective generator $P' \in \text{Ob}(\mathcal{L}^{\text{op}})$. Let $\mathcal{L}'$ be the small, exact, full subcategory of $\mathcal{L}^{\text{op}}$ generated by the image of よ, which we note is itself multifinitary by the Yoneda lemma; the final step is to embed $\mathcal{L}'$ into the category of projective modules. Well, suppose we write $I := \coprod_{F \in \text{Ob}(\mathcal{L}')} \text{Mor}_{\mathcal{L}'}(P', F)$ and define $P := \bigoplus_{i \in I} P'$. Of course $A := \text{End}_{\mathcal{L}'}(P)$ is a finite-dimensional, associative $\Bbbk$-algebra. Moreover, for any $F \in \text{Ob}(\mathcal{L}')$, we may endow $\text{Mor}_{\mathcal{L}'}(P, F)$ with the structure of a finitely-generated, projective A-module in a canonical way by taking $a \cdot x := x \circ a$ for all $a \in A$ and $x \in \text{Mor}_{\mathcal{L}'}(P, F)$. Since P is also a projective generator, we have a full, exact embedding $\mathscr{F} : \mathcal{L}' \to \mathsf{Mod}_p(A)$ sending $F \mapsto \text{Mor}_{\mathcal{L}'}(P, F)$. Thus $\mathscr{F} \circ$ よ$^{\text{op}} : \mathcal{C} \to \mathsf{Mod}_p(A)$ is itself a full, exact embedding, giving us the desired result. □

Remark 2.7. This result tells us that a small category is multifinitary if and only if it is equivalent to a full subcategory of the category of finitely generated, projective modules over some finite-dimensional, associative $\Bbbk$-algebra ([MMM+23]). It may also be worth noting that we can replace "projective" with "injective" in this alternative definition, as these two notions are dual in the sense that being projective is of course the same as being injective in the opposite category.

Theorem 2.8. ([EMTW20, Theorem 11.53]). *An additive category is Krull–Schmidt if and only if it is idempotent complete and all of its endomorphism rings are semiperfect.*

By (ii), we mean that for all $X \in \text{Ob}(\mathcal{C})$, the ring $R := \text{End}_{\mathcal{C}}(X)$ admits a complete, orthogonal set of idempotents $p_1, \ldots, p_n$ such that each $p_i R p_i$ is a local ring. This in fact implies that X is indecomposable if and only if its endomorphism ring is local. In the setting where $\mathcal{C}$ is $\Bbbk$-linear with finite-dimensional $\Bbbk$-vector spaces of morphisms, its endomorphism rings will just be matrix rings, which are always semiperfect (as is any finite-dimensional, associative $\Bbbk$-algebra); for such categories, being Krull–Schmidt is equivalent to being idempotent complete.

If representation theory can naïvely be described as "group theory in linear sets", birepresentation theory can be described as "group theory in linear categories". We will now make precise the notion of "linear categories" that we will find ourselves working with.

Definition 2.9. (Finitary Category). *An additive, $\Bbbk$-linear category $\mathcal{C}$ is called* multifinitary *if it is idempotent complete, has finitely many isomorphism classes of indecomposable objects and has finite-dimensional $\Bbbk$-vector spaces of morphisms. If $\mathcal{C}$ is monoidal, this is equivalent to being* finitary. *Otherwise, we ask that the monoidal product be $\Bbbk$-bilinear, and say that it is* finitary *only if its unit object is indecomposable.*

† Since this will inevitably be someone's first time seeing it, よ is the hiragana for "yo" (as in "Yoneda").

This definition may seem a bit scary at first. However, observe that a (multi)fusion category is exactly a rigid, semisimple (multi)finitary monoidal category whose objects have finite length. In other words, we can think of finitary monoidal categories as categories that are not necessarily rigid or Abelian, but are otherwise "as fusion as possible".

Historically, categorical representation theory can be traced back to Jones' discovery of a remarkable polynomial invariant for knots originating from subfactor theory ([Jon85]), showing that the Alexander polynomial is not the only invariant of its kind and sparking major breakthroughs in knot theory. This was quickly generalized, and a variety of new knot polynomials were discovered. Reshetikhin and Turaev later explained how these invariants could be obtained from the representation theory of quantum groups – in more modern terms, via certain kinds of fusion categories called *modular tensor categories* ([RT91]). Following this was an influential categorification of the Jones polynomial due to Khovanov ([Kho00]); this so-called *Khovanov homology* is strictly stronger than its decategorification, being able to detect certain differences between distinct knots that are invisible to the Jones polynomial. This naturally led to the expectation that such categorifications could be found by studying some kind of "2-representation theory" of quantum groups. This is exactly what Webster showed in [Web17], building off the budding program of higher representation theory initiated by Chuang and Rouquier in [CR08] and continued by Rouquier in [Rou08].

Definition 2.10. *Let $\mathfrak{A}_{\Bbbk}^{f}$ denote the 2-category whose objects are multifinitary categories, whose 1-morphisms are $\Bbbk$-linear functors and whose 2-morphisms are natural transformations.*

We briefly remind the reader that any $\Bbbk$-linear functor between additive, $\Bbbk$-linear categories is automatically additive! For the following definition, we interpret the End-categories $\mathscr{C}(\mathtt{i},\mathtt{i})$ as being monoidal categories with respect to the composition of 1-morphisms.

Definition 2.11. (Finitary Bicategory). *A bicategory $\mathscr{C}$ is said to be* (multi)finitary *if*

(i). *it has finitely many objects;*
(ii). *for any pair $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, the Hom-category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is (multi)finitary;*
(iii). *horizontal composition of 2-morphisms is $\Bbbk$-bilinear.*

A monoidal category $\mathcal{C}$ is (multi)finitary if and only if its monoidal delooping $\mathsf{B}\mathcal{C}$ is (multi)finitary.

Definition 2.12. (Birepresentation). *A* birepresentation *of a bicategory $\mathscr{C}$ is a pseudofunctor from $\mathscr{C}$ to Cat, the 2-category of small categories. A* 2-representation *of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to Cat.*

Definition 2.13. (Finitary Birepresentation). *A* (multi)finitary birepresentation *of a (multi)finitary bicategory $\mathscr{C}$ is a covariant, $\Bbbk$-linear pseudofunctor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$. A* (multi)finitary 2-representation *of a (multi)finitary 2-category $\mathscr{C}$ is a covariant, $\Bbbk$-linear 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$.*

Definition 2.14. (Finitary Module Category). *A* (multi)finitary module category *over a (multi)finitary monoidal category $\mathcal{C}$ is a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ for which the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is $\Bbbk$-bilinear.*

Remark 2.15. Recall that a representation of a group G is morally nothing but a functor $F : \mathsf{B}G \to \mathsf{Vect}$, where the action of G on $V := F(\bullet)$ is given by $g \cdot v := [F(g)](v)$ for all $v \in V$ and $g \in \mathrm{Mor}(\mathsf{B}G)$. Analogously, given a 2-representation $\mathrm{M} : \mathscr{C} \to \mathsf{Cat}$, we have a 2-*action* of $\mathscr{C}$ given by $F \cdot X := [\mathrm{M}(F)](X)$ for all $X \in \mathrm{M}(\mathtt{i})$ and $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$, where $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. The upshot here is that we should think of the 1-morphisms in our 2-category as our "group elements", with composition becoming "group multiplication".

Proposition 2.16. *Let $\mathcal{C}$ be a monoidal (respectively, strict monoidal) category. There exists a bijection between $\mathcal{C}$-module categories $\mathcal{M}$ and birepresentations (respectively, 2-representations) M of the delooping category $\mathsf{B}\mathcal{C}$. Moreover, $\mathcal{M}$ is (multi)finitary if and only if M is (multi)finitary.*

Proof. Let $\mathcal{C}$ be a monoidal category and $\mathsf{B}\mathcal{C}$ its delooping category. Recall that by [EGNO16, Proposition 7.1.3] there is a bijection between $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors of the form $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$. Such a functor F induces a canonical birepresentation $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ that takes the single object $\bullet \in \mathrm{Ob}(\mathsf{B}\mathcal{C})$ to $\mathcal{M}$ and otherwise acts on the 1-morphisms and 2-morphisms by F. Conversely, let $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ be a birepresentation and write $\mathcal{M} := \mathrm{M}(\bullet)$. This naturally induces a functor $F : \mathcal{C} \to \mathsf{Cat}(\mathcal{M}, \mathcal{M}) = \mathrm{End}(\mathcal{M})$ that acts on objects and morphisms of $\mathcal{C}$ by M. Clearly these two constructions are inverse to each other and (multi)finitarity of both sides is equivalent. It is easy to see that if $\mathcal{C}$ is strict, the result for 2-representations follows similarly. This completes the proof. ∎

We also note that by Remark 2.7, every multifinitary module category is automatically exact, as equivalences of categories preserve projectivity of objects and hence every object must be projective.

Definition 2.17. (Equivalence of Birepresentations). *We say that two birepresentations M and N of $\mathscr{C}$ are* equivalent *if there exists a pseudonatural transformation $\Phi : \mathrm{M} \to \mathrm{N}$ such that the component $\Phi_{\mathtt{i}} : \mathrm{M}(\mathtt{i}) \to \mathrm{N}(\mathtt{i})$ is an equivalence of categories for each $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$.*

Definition 2.18. (Yoneda Birepresentation). *Let $\mathscr{C}$ be a bicategory and consider the pseudofunctor $\mathscr{C}(\mathtt{i}, -) : \mathscr{C} \to \mathsf{Cat}$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, such that*

- *objects $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ are sent to the Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$;*
- *1-morphisms of the form $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$ are sent to the "horizontal post-composition by F" functor $F_* : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ given by $G \mapsto F \circ G$ and $(\gamma : G \to G') \mapsto \mathrm{id}_F \circ_h \gamma$;*
- *2-morphisms of the form $\alpha \in \mathrm{Mor}^2_{\mathscr{C}}(F, F')$, for $F, F' \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$, are sent to the "horizontal post-composition by α" natural transformation $\alpha_* : F_* \Rightarrow F'_*$, whose components are given by $(\alpha_*)_G := \alpha \circ_h \mathrm{id}_G$ for each $G \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$.*

We call this the Yoneda *(or* principal*)* birepresentation corresponding to $\mathtt{i}$ *and denote it by $\mathbb{P}_{\mathtt{i}}$.*

If $\mathscr{C}$ is (multi)finitary, then its corresponding Yoneda birepresentations are also all (multi)finitary, and if $\mathscr{C}$ is a (multi)finitary 2-category its Yoneda birepresentations are (multi)finitary 2-representations. As an example, let $\mathscr{C}$ be the monoidal delooping of $\mathcal{C}$. Then $\mathbb{P}_\bullet$ maps $\bullet$ to $\mathcal{C}$, maps 1-morphisms $X \in \mathrm{Ob}(\mathcal{C})$ to the left tensor product functor given by $Y \mapsto X \otimes Y$ and $(f : Y \to Y') \mapsto \mathrm{id}_X \otimes f$, and maps 2-morphisms $f : X \Rightarrow X'$ to the natural transformation $Y \mapsto f \otimes \mathrm{id}_Y$. This is nothing but the birepresentation corresponding to the regular $\mathcal{C}$-module category $\mathcal{C}$!

Definition 2.19. (Ideal). *A* left *(respectively* right*)* ideal *of a category $\mathcal{C}$ is a collection $\mathcal{I} \coloneqq \{\mathcal{I}(X,Y) : X, Y \in \mathrm{Ob}(\mathcal{C})\}$, where each $\mathcal{I}(X,Y)$ is a non-empty subclass of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, such that $\mathcal{I}$ is stable under post-composition (respectively pre-composition) with morphisms from $\mathcal{C}$. If $\mathcal{C}$ is preadditive, we additionally ask that $\mathcal{I}(X,Y)$ be an Abelian subgroup of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ for all pairs $X, Y \in \mathrm{Ob}(\mathcal{C})$, and if $\mathcal{C}$ is $\Bbbk$-linear we ask that they be $\Bbbk$-subspaces. This gives us a notion of summing such ideals. We say that $\mathcal{I}$ is a* two-sided *(or* bilateral*)* ideal *if it is both a left ideal and a right ideal, and that it is a subideal of $\mathcal{J}$ if its classes of morphisms are subclasses. An ideal is said to be* proper *if there exists some pair $X, Y \in \mathrm{Ob}(\mathcal{C})$ for which $\mathcal{I}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$, and* maximal *if it is proper and not a subideal of any other proper ideal.*

Let $\mathcal{I}$ be an ideal of a category $\mathcal{C}$. As we have implied previously, if $\mathcal{C}$ is a preadditive, then $\mathrm{End}_{\mathcal{C}}(X)$ is a ring for all $X \in \mathrm{Ob}(\mathcal{C})$, and it follows that the valid choices for $\mathcal{I}(X,X)$ coincide exactly with the ring ideals of $\mathrm{End}_{\mathcal{C}}(X)$. Similarly, if $\mathcal{C}$ is a $\Bbbk$-linear category, then each $\mathrm{End}_{\mathcal{C}}(X)$ is an associative, unital algebra, and the valid choices for $\mathcal{I}(X,X)$ coincide with algebra ideals (that is, a subspace of $\mathrm{End}_{\mathcal{C}}(X)$ that is closed under algebra multiplication).

Example 2.20. Consider a subcategory $\mathcal{T}$ of $\mathsf{Vect}_{\Bbbk}$ whose endomorphisms of $\Bbbk^n$ are the $n \times n$ upper triangular Toeplitz matrices. Then $\mathrm{End}(\Bbbk^2) \cong \Bbbk[x]/\langle x^2 \rangle$. If we consider the sub-semicategory of $\mathcal{T}$ containing only the object $\Bbbk^2$ and the endomorphisms $\Bbbk\{x\}$ (that is, linear scalings of the matrix with 1 in the off-diagonal), we obtain an ideal of $\mathcal{T}$.

Let M be a multifinitary birepresentation of $\mathscr{C}$ for which each $\mathrm{M}(\mathtt{j})$ is additive and idempotent complete, and let $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ for some $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. Consider the *additive closure* (closure under isomorphisms, direct summands and finite direct sums) of the orbit of X under the action of $\mathscr{C}$; that is, the collection of objects

$$\mathscr{C}(\{X\}) \coloneqq \mathsf{add}(\{[\mathrm{M}(F)](X) : \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})\})$$

where the add denotes the aforementioned additive closure. Due to the additivity of the 1-morphisms of $\mathfrak{A}^f_{\Bbbk}$, it follows that $\mathscr{C}(\{X\})$ is itself stable under the action of $\mathscr{C}$. This therefore induces a multifinitary subbirepresentation $\mathrm{G}_{\mathrm{M}}(\{X\})$ of $\mathscr{C}$ by restriction, with each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ sent to

$$\mathscr{C}_{\mathtt{j}}(\{X\}) \coloneqq \mathsf{Add}(\{[\mathrm{M}(F)](X) : F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})\}),$$

the *additive subcategory* (full subcategory that is closed under isomorphisms, direct summands and finite direct sums) of $\mathrm{M}(\mathtt{j})$ generated by the objects of $\mathscr{C}(\{X\})$ that lie in $\mathrm{M}(\mathtt{j})$. Because $\mathrm{M}(\mathtt{j})$ is Karoubian, this is nothing but the Karoubi envelope of the full subcategory generated by the action of $\mathscr{C}$. In principle this process works for any collection $\{X_i : i \in I\}$ with $X_i \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}_i))$, whence

$$\mathscr{C}(\{X_i : i \in I\}) \coloneqq \mathsf{Add}(\{[\mathrm{M}(F)](X_i) : i \in I, \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_i, \mathtt{j})\})$$

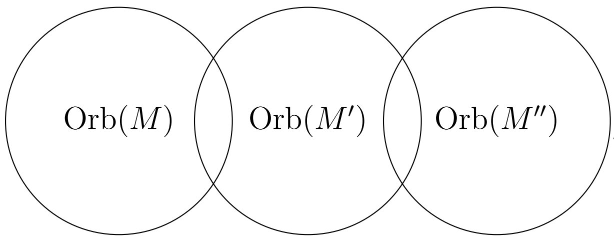
similarly induces a multifinitary subbirepresentation $\mathrm{G}_{\mathrm{M}}(\{X_i : i \in I\})$ of $\mathscr{C}$. In any case, we will only need to consider the single-object situation, as it allows us to make the following evocative definition.

Definition 2.21. (Transitive Birepresentation). *Let M be a multifinitary birepresentation of a multifinitary bicategory $\mathscr{C}$. We say that M is* transitive *if, for every $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ and non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$, the embedding $\mathscr{C}_{\mathtt{j}}(\{X\}) \hookrightarrow \mathrm{M}(\mathtt{j})$ is an equivalence for all $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$.*

Remark 2.22. Recall that a module M is simple if and only if every cyclic submodule generated by a non-zero element of M is equal to M. This is exactly what we're trying to capture with transitivity! In the birepresentation world, however, things are more involved. We will see more on this shortly.

We say that a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ is transitive if, for all $M \in \mathrm{Ob}(\mathcal{M})$, we have that $\mathrm{Orb}(M) := \mathsf{Add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathcal{M}$ by Proposition 2.16. This is equivalent to it having no full Karoubian subcategories, equivalent to its corresponding birepresentation being transitive and equivalent to its split Grothendieck group $\mathrm{Gr}(\mathcal{M})$ being a simple $\mathrm{Gr}(\mathcal{C})$-module.

Being transitive is clearly a much stronger condition for a module category than being indecomposable (in the sense that it is not the direct sum of two non-zero multifinitary module subcategories). Given any $M \in \mathrm{Ob}(\mathcal{M})$ and $M' \in \mathrm{Ob}(\mathrm{Orb}(M))$, it's possible that the orbit of M and the orbit of some $M'' \in \mathrm{Ob}(\mathrm{Orb}(M'))$ may have trivial intersection. For instance,



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This picture becomes much more insightful in light of Lemma 2.29 and Proposition 2.30.

Definition 2.23. ($\mathscr{C}$-Stable Ideal). *Let* M *be a birepresentation of* $\mathscr{C}$*. A* $\mathscr{C}$-stable ideal I *of* M *is a collection* $\mathrm{I} := \{\mathrm{I}(\mathtt{i}) : \mathtt{i} \in \mathrm{Ob}(\mathscr{C})\}$*, where each* $\mathrm{I}(\mathtt{i})$ *is a two-sided ideal of* $\mathrm{M}(\mathtt{i})$ *such that* $[\mathrm{M}(F)](\mathrm{I}(\mathtt{i}))$ *is a subclass of* $\mathrm{I}(\mathtt{j})$ *for all* 1*-morphisms* $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$*. A* $\mathscr{C}$*-stable subideal* I *of a* $\mathscr{C}$*-stable ideal* J *is a* $\mathscr{C}$*-stable ideal for which* $\mathrm{I}(\mathtt{i})$ *is a subideal of* $\mathrm{J}(\mathtt{i})$ *for all* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*. We say that* I *is* proper *if there exists some* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *for which* $\mathrm{I}(\mathtt{i})$ *is proper, and* maximal *if it is proper and not a* $\mathscr{C}$*-stable subideal of any other proper* $\mathscr{C}$*-stable ideal.*

Definition 2.24. (Simple Transitive Birepresentation). *A multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *is said to be* simple transitive *if it admits no proper, non-zero* $\mathscr{C}$*-stable ideals.*

Similarly to before, we say that a $\mathcal{C}$-module category $\mathcal{M}$ is simple transitive if its corresponding birepresentation is simple transitive. In other words, given non-zero $f, g \in \mathrm{Mor}(\mathcal{M})$, we can obtain g by composing f with other morphisms in $\mathcal{M}$ and acting via $\mathcal{C}$ (by taking the left monoidal product).

In light of the module category picture, we see that the "right" way to think about transitivity and simplicity is to observe that transitivity insists that your objects are cyclically generated (that is, $\mathsf{add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathrm{Ob}(\mathcal{M})$ for all non-zero $X \in \mathrm{Ob}(\mathcal{M})$), while simplicity insists that your morphisms are cyclically generated ($\{l \circ (\mathrm{id}_Y \otimes f) \circ r : Y \in \mathrm{Ob}(\mathcal{C}), l, r \in \mathrm{Mor}(\mathcal{M})\} = \mathrm{Mor}(\mathcal{M})$ for all non-zero $f \in \mathrm{Mor}(\mathcal{M})$)! This perspective, in addition to the following result, really elucidates our notion of simplicity for birepresentations.

Proposition 2.25. *Every simple transitive birepresentation is transitive.*

Proof. Let M be a simple transitive birepresentation of a multifinitary bicategory $\mathscr{C}$ and take $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero. Certainly $\mathrm{G}_{\mathrm{M}}(\{X\})$ induces a $\mathscr{C}$-stable ideal of M that is both non-empty and non-zero, as it contains X. Thus by the simplicity of M, it follows that $\mathscr{C}_{\mathtt{j}}(\{X\})$ must be equivalent to $\mathrm{M}(\mathtt{j})$ for each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. Thus M is transitive. This completes the proof. ■

It is not necessarily the case that transitive birepresentations are simple transitive! This will become clear when we look at certain subcategories of the Lusztig–Vogan module categories. However, transitive birepresentations can always be made simple transitive by leaving the objects alone and just throwing away some collection of morphisms. This result will end up being important in formulating the categorical version of the Jordan–Hölder theorem.

Lemma 2.26. *Let* M *be a transitive birepresentation of a multifinitary bicategory* $\mathscr{C}$*. Then* M *admits a unique maximal* $\mathscr{C}$*-stable ideal* I*, and moreover each* $\mathrm{I}(\mathtt{i})$ *contains no identity morphisms apart from the one corresponding to the zero object.*

Proof. Let I be the sum of all $\mathscr{C}$-stable ideals of M that do not contain id_X for any non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and any $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This is certainly itself a $\mathscr{C}$-stable ideal by construction. Moreover, because the sum of any two ideals $\mathcal{I}$ and $\mathcal{J}$ is an ideal containing both $\mathcal{I}$ and $\mathcal{J}$, it follows that I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms. To see that it is genuinely maximal, suppose J is a $\mathscr{C}$-stable ideal containing I. Because I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms, J must contain at least one identity morphism, say id_X for some non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$. Given any non-zero $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, the transitivity of M tells us that Y is either isomorphic to a direct summand of $[\mathrm{M}(F)](X)$, for some $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$, or isomorphic to a direct sum $[\mathrm{M}(F_1)](X) \oplus \cdots \oplus [\mathrm{M}(F_n)](X)$, for some $F_1, \ldots, F_n \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$. We claim that id_Y must lie in $\mathrm{J}(\mathtt{j})$ in both cases.

Let $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$ with $[\mathrm{M}(F)](X) = X_1 \oplus \cdots \oplus X_n$ and suppose that $\varphi : Y \to X_k$ is an isomorphism for some $1 \leq k \leq n$. Then $[\mathrm{M}(F)](\mathrm{id}_X) = \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \in \mathrm{J}(\mathtt{j})$ by $\mathscr{C}$-stability. But by pre-composing with $\iota_{X_k} \circ \varphi^{-1} : Y \to X_k \to X_1 \oplus \cdots \oplus X_n$ and post-composing with $\varphi \circ \pi_{X_k} : X_1 \oplus \cdots \oplus X_n \to X_k \to Y$, we obtain that $\mathrm{id}_Y = (\varphi \circ \pi_{X_k}) \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ (\iota_{X_k} \circ \varphi^{-1}) \in \mathrm{J}(\mathtt{j})$.

Suppose now that $\varphi : Y \to X_1 \oplus \cdots \oplus X_n$ is an isomorphism, where for each $1 \leq k \leq n$ we have $X_k = [\mathrm{M}(F_k)](X)$ for some $F_k \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$. As before, $[\mathrm{M}(F_k)](\mathrm{id}_X) = \mathrm{id}_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$ by $\mathscr{C}$-stability. Moreover, because $\mathrm{J}(\mathtt{j})$ is an ideal, $\iota_{X_k} \circ \mathrm{id}_{X_k} \circ \pi_{X_k} = \iota_{X_k} \circ \pi_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$. Thus by the definition of an additive category, $\mathrm{id}_{X_1 \oplus \cdots \oplus X_n} = \iota_{X_1} \circ \pi_{X_1} + \cdots + \iota_{X_n} \circ \pi_{X_n} \in \mathrm{J}(\mathtt{j})$, whence it follows that $\mathrm{id}_Y = \varphi^{-1} \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ \varphi \in \mathrm{J}(\mathtt{j})$.

We have thus shown that J must contain *all* identity morphisms and therefore cannot be proper, meaning that I must in fact be maximal as claimed. The uniqueness of I follows by construction. This completes the proof. ■

Let $\mathcal{C}$ be a preadditive category admitting a two-sided ideal $\mathcal{I}$. We define a congruence relation $\sim_{\mathcal{I}}$ on each $\mathcal{C}(X,Y)$ by $f \sim_{\mathcal{I}} g$ if and only if $f - g \in \mathcal{I}(X,Y)$. This motivates the following definition.

Definition 2.27. (Quotient Category). *Let $\mathcal{C}$ be a preadditive category admitting a two-sided ideal $\mathcal{I}$. We define the* quotient category $\mathcal{C}/\mathcal{I}$ *to be the subcategory whose classes of morphisms are given by $\mathcal{C}/\mathcal{I}(X,Y) := \mathcal{C}(X,Y)/\sim_{\mathcal{I}}$, for all $X, Y \in \mathrm{Ob}(\mathcal{C})$.*

Theorem-Definition 2.28. (Simple Quotient). *A transitive birepresentation* M *of a multifinitary bicategory $\mathscr{C}$ is simple transitive if and only if the unique maximal $\mathscr{C}$-stable ideal* I *from Lemma 2.26 is the zero ideal. The simple transitive subbirepresentation* $\underline{\mathrm{M}}$ *of* M *that sends each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *to the quotient subcategory* $\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i})$ *is known as the* simple quotient *of* M.

Proof. Naturally if I – the sum of all $\mathscr{C}$-stable ideals without non-zero identity morphisms – is the zero ideal, then M must contain no proper, non-zero $\mathscr{C}$-stable ideals. Conversely, if M is simple transitive, then because I is the sum of proper $\mathscr{C}$-stable ideals, they must all be zero. Finally, because I is maximal, every morphism $f \in \mathrm{Mor}(\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i}))$ must generate either $\{0\}$ or $\mathrm{M}(\mathtt{i})/\mathrm{I}(\mathtt{i})$ under composition with morphisms in $\mathscr{C}$, whence $\underline{\mathrm{M}}$ is simple transitive. This completes the proof. ■

Let M be a multifinitary birepresentation of $\mathscr{C}$. We denote by Ind(M) the set of isomorphism classes of indecomposable objects in every $\mathrm{M}(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$; that is,

$$\mathrm{Ind}(\mathrm{M}) = \bigsqcup_{\mathtt{i} \in \mathrm{Ob}(\mathscr{C})} \{[X] \in \mathrm{M}(\mathtt{i}) : X \text{ is indecomposable}\}.$$

Note that Ind(M) is clearly finite, as $\mathscr{C}$ has finitely many objects and each category $\mathrm{M}(\mathtt{i}) \in \mathrm{Ob}(\mathfrak{A}_{\Bbbk}^{f})$ has finitely many isomorphism classes of indecomposable objects.

For $X, Y \in \mathrm{Ind}(\mathrm{M})$, where for instance $X \in \mathrm{M}(\mathtt{i}_X)$ and $Y \in \mathrm{M}(\mathtt{i}_Y)$, we write $Y \geq X$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_X, \mathtt{i}_Y)$ such that Y is isomorphic to a direct summand of $[\mathrm{M}(F)](X)$.

Lemma 2.29. *Let* M *be a multifinitary birepresentation. The binary relation $\geq$ defined above defines a preorder on* $\mathrm{Ind}(\mathrm{M})$ *known as the* action preorder.

Proof. Clearly $\geq$ is reflexive, as we can just take $F := \mathrm{id}_{\mathtt{i}}$. Moreover, suppose that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$ and Y is isomorphic to a direct summand of $[\mathrm{M}(G)](Z)$; that is,

$$[\mathrm{M}(F)](Y) \cong X \oplus X_1 \oplus X_2 \oplus \cdots,$$
$$[\mathrm{M}(G)](Z) \cong Y \oplus Y_1 \oplus Y_2 \oplus \cdots.$$

In order to show transitivity, we would like to show that X is isomorphic to a direct summand of $[\mathrm{M}(FG)](Z)$. Well, because the morphisms of $\mathfrak{A}_{\Bbbk}^{f}$ are additive, we simply observe that

$$\begin{aligned}[\mathrm{M}(FG)](Z) &\cong [\mathrm{M}(F)](Y) \oplus [\mathrm{M}(F)](Y_1) \oplus [M(F)](Y_2) \oplus \cdots \\ &\cong X \oplus X_1 \oplus X_2 \oplus \cdots \oplus [\mathrm{M}(F)](Y_1) \oplus [\mathrm{M}(F)](Y_2) \oplus \cdots.\end{aligned}$$

This completes the proof. ■

Suppose we define an equivalence relation $\sim$ given by $X \sim Y$ if and only if $X \geq Y$ and $Y \geq X$. Obviously $\geq$ extends to a partial order on $\mathrm{Ind}(\mathrm{M})/\sim$. In particular, we have the following result.

Proposition 2.30. *Let* M *be a multifinitary birepresentation. Then* M *is transitive if and only if* $\mathrm{Ind}(\mathrm{M})/\!\sim$ *has only one element.*

Proof. Suppose $\mathrm{Ind}(\mathrm{M})/\!\sim$ is a singleton and take any $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero as a representative. Then for any indecomposable $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, there exists some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$ for which Y is isomorphic to a direct summand of $[M(F)](X)$, since $Y \geq X$. In other words, the additive subcategory $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, as by definition it is closed under direct summands and direct sums. Thus M is transitive.

Conversely, suppose M is transitive, and consider any pair of indecomposables $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$. Because $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, we know by the definition of $\mathscr{C}_{\mathtt{j}}(\{X\})$ that Y is isomorphic to a direct summand of $[M(F)](X)$ for some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$; that is, $Y \geq X$. The same argument applied to $\mathscr{C}_{\mathtt{i}}(\{Y\})$ shows us that $X \geq Y$, whence $\mathrm{Ind}(\mathrm{M})/\!\sim$ has only one element. This completes the proof. ■

Definition 2.31. (Directed Order Ideal). *A* directed order coideal *of a partially ordered set* $(P, \geq)$ *is a non-empty subset* I *such that*

- *for all* $x \in I$ *and* $y \in P$*,* $y \geq x$ *implies that* $y \in I$ *(upper set);*
- *for all* $x, y \in I$*, there is some* $z \in I$ *such that* $x \geq z$ *and* $y \geq z$ *(downward directed set).*

Remark 2.32. This definition is slightly unusual. It is more typical to talk of *directed order ideals* of partially ordered sets $(P, \leq)$, which are non-empty subsets that are lower sets and upward directed sets. We have chosen to use coideals rather than ideals due to how we have defined our partial order; the standard notion of a directed order ideal goes "downwards", while we want to go "upwards". To more explicitly illustrate why we have made this choice, suppose we have some $\mathcal{C}$-module category of R-modules with indecomposables $R, X \in \mathrm{Ob}(\mathcal{C})$. Naturally we would expect $X \geq R$, and indeed with our setup this will be true, as we can always consider the functor $\mathrm{M}(X) = - \otimes_R X$.

Let M be a multifinitary birepresentation of $\mathscr{C}$ and Q a directed order coideal of $\mathrm{Ind}(\mathrm{M})/\!\sim$. For $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, define $\mathrm{M}_Q(\mathtt{i})$ to be the additive subcategory of $\mathrm{M}(\mathtt{i})$ generated by every indecomposable object $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ whose equivalence class lies in Q. Then $\mathrm{M}_Q : \mathtt{i} \mapsto \mathrm{M}_Q(\mathtt{i})$ induces a multifinitary subbirepresentation of M, known as the subbirepresentation of M associated to Q.

Let $Q \subset R$ be a pair of directed order coideals in $\mathrm{Ind}(\mathrm{M})/\!\sim$ and let $\mathrm{I}_Q(\mathtt{i})$ denote the ideal in $\mathrm{M}_R(\mathtt{i})$ generated by the identity morphisms in $\mathrm{M}_Q(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This collection of ideals is $\mathscr{C}$-stable, whence the multifinitary birepresentation M_R induces a multifinitary birepresentation $\mathrm{M}_{R/Q} : \mathtt{i} \mapsto \mathrm{M}_R(\mathtt{i})/\mathrm{I}_Q(\mathtt{i})$. This is known as the quotient of M associated to $Q \subset R$. Note that if $|R \setminus Q| = 1$, then $|\mathrm{Ind}(\mathrm{M}_{R/Q})/\!\sim| = 1$, so $\mathrm{M}_{R/Q}$ will be transitive by Proposition 2.30.

Choose $r \in \mathrm{Ind}(\mathrm{M})/\!\sim$ and let X_r be the maximal directed order coideal in $\mathrm{Ind}(\mathrm{M})/\!\sim$ that does not contain r. In other words, $(\mathrm{Ind}(\mathrm{M})/\!\sim) \setminus X_r$ – the complement of X_r – has maximal element r. Thus we also obtain a directed order coideal $Y_r := X_r \cup \{r\}$, as r being maximal in the complement means Y_r will be an upper set, whence the associated quotient M_{Y_r/X_r} is transitive by Proposition 2.30. We henceforth let $\underline{\mathrm{M}}_r$ denote the simple quotient $\underline{\mathrm{M}}_{Y_r/X_r}$.

Consider a filtration of directed order coideals

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim$$

such that $|Q_i \backslash Q_{i-1}| = 1$ for all $i \in \{1, \dots, n\}$. We call this a *complete filtration.* As shown previously, from such a filtration we have a corresponding *weak Jordan–Hölder series*

$$\{0\} = \mathrm{M}_{Q_0} \subset \mathrm{M}_{Q_1} \subset \cdots \subset \mathrm{M}_{Q_n} = \mathrm{M}$$

consisting of subbirepresentations whose *weak composition quotients* $\mathrm{L}_i := \underline{\mathrm{M}}_{Q_i/Q_{i-1}}$ are simple transitive birepresentations for all $1 \leq i \leq n$. We are now ready to state [MM16, Theorem 8].

Theorem 2.33. (Weak Jordan–Hölder Theorem). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *admitting the two complete filtrations*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim,$$

$$\varnothing = Q'_0 \subset Q'_1 \subset \cdots \subset Q'_m = \mathrm{Ind}(\mathrm{M})/\sim,$$

with weak composition quotients $\{\mathrm{L}_i\}_{i=1}^{n}$ *and* $\{\mathrm{L}'_j\}_{j=1}^{m}$ *respectively. Then* $m = n$*, and moreover there exists a permutation* $\sigma \in S_n$ *such that* L_i *and* $\mathrm{L}'_{\sigma(i)}$ *are equivalent for all* $i \in \{1, \dots, n\}$*.*

Proof. We clearly have $m = n = |\mathrm{Ind}(\mathrm{M})/\sim|$ by the definition of a complete filtration. Suppose now that $r \in \mathrm{Ind}(\mathrm{M})/\sim$; then there exist unique $i, j \in \{1, 2, \dots, n\}$ for which $Q_i \backslash Q_{i-1} = Q'_j \backslash Q'_{j-1} = \{r\}$. If we can show that the birepresentations L_i and L'_j are both equivalent to $\underline{\mathrm{M}}_r$, then we are done. In particular, by symmetry it is enough to show that L_i is equivalent to $\underline{\mathrm{M}}_r$.

Let I_{X_r} be the $\mathscr{C}$-stable ideal in M_{Y_r} for which $\mathrm{M}_{Y_r/X_r} = \mathrm{M}_{Y_r}/\mathrm{I}_{X_r}$ and $\mathrm{I}_{Q_{i-1}}$ the $\mathscr{C}$-stable ideal in M_{Q_i} for which $\mathrm{M}_{Q_i/Q_{i-1}} = \mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$. Since $\{r\} = Q_i \setminus Q_{i-1}$, we know by construction that $Q_{i-1} \subseteq X_r$, as X_r is by definition the maximal directed order coideal not containing r; similarly, $Q_i \subseteq Y_r$. This second inclusion induces a pseudonatural transformation from M_{Q_i} to M_{Y_r} (a collection of natural isomorphisms from functors between Hom-categories to functors between Hom-subcategories), whence the first inclusion induces a pseudonatural transformation $\sigma : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Y_r/X_r}$ by taking the quotient. Now, $\mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$ contains only the objects generated by indecomposables in the equivalence class r. But for any such pair of indecomposable objects $X, Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$ lying in the equivalence class r, we have that $\mathrm{I}_{X_r}(X, Y) \subseteq \mathrm{I}_{Q_{i-1}}(X, Y)$ by the aforementioned inequalities. Thus the pseudonatural transformation σ factors through $\mathrm{M}_{Q_i/Q_{i-1}}$, in the sense that there exist lax natural transformations $\sigma_1 : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Q_i/Q_{i-1}}$ and $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ such that $\sigma = \sigma_2 \circ \sigma_1$. In particular, this gives us a lax natural transformation $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ that is obviously surjective on morphisms by fullness; therefore, because $\mathrm{M}_{Q_i/Q_{i-1}}$ and M_{Y_r/X_r} are both transitive, taking their simple quotients via Theorem 2.28 induces an equivalence between L_i and $\underline{\mathrm{M}}_r$ as desired, whence the result follows. This completes the proof. ■

I'm not happy with this proof, it's handwavey and I think I've got it wrong. Also, in what sense is this weak Jordan–Hölder theorem "weak"? The decategorifications of these simple quotients are "transitive $\mathbb{N}$-modules" and usually not simple.

3. Categories of Soergel Bimodules

Throughout this chapter and the next, we will explore one of the main examples that has motivated the theory from the previous chapter: namely, categories of Soergel bimodules. Given a Coxeter system (W, S), we may define a corresponding Iwahori–Hecke algebra. These algebras appear all over mathematics, playing important roles in areas such as the representation theory of Lie groups and quantum groups, knot theory and statistical mechanics, among others. From this same Coxeter system, we may also construct a category of so-called Soergel bimodules, which is an algebraic categorification of the corresponding Iwahori–Hecke algebra. These categories were fundamental to the recent, purely algebraic proofs of the Kazhdan–Lusztig Conjecture ([KL79, Conjecture 1.5]) and Kazhdan–Lusztig Positivity Conjecture ([KL79, p. 166]) by Elias and Williamson ([EW14, Theorem 1.1 and Corollary 1.2, respectively]), and have since become an indispensable tool in Lie theory.

Before introducing categories of Soergel bimodules, we will briefly give definitions of Coxeter systems and Iwahori–Hecke algebras. We will not give much insight into these objects; for a more detailed survey, please see the wonderful book of Elias, Makisumi, Thiel and Williamson ([EMTW20]).

Definition 3.1. (Coxeter System). *Let S be a finite set and $(m_{st})_{s,t\in S}$ a matrix satisfying*

- $m_{ss} = 1$, *for each* $s \in S$;
- $m_{st} = m_{ts} \in \{2, 3, \dots\} \sqcup \{\infty\}$, *for* $s \neq t \in S$.

Consider now the subgroup W of F_S, the free group over S, with presentation

$$W = \langle s \in S : (st)^{m_{st}} = 1 \text{ for all } s, t \in S \text{ with } m_{st} < \infty \rangle.$$

The pair (W, S) is known as a Coxeter system*, while W is known as a* Coxeter group.

We call the generating set S the set of *simple reflections* and $(m_{st})_{s,t\in S}$ a *Coxeter matrix.* Because $s^2 = 1$ for all $s \in S$, the relation $(st)^{m_{st}} = 1$ is equivalent to the braid relation $sts \cdots = tst \cdots$ for all $s, t \in S$ with $m_{st} < \infty$, where both sides are the product of m_{st} simple reflections.

Definition 3.2. (Expression). *Let (W, S) be a Coxeter system with $w \in W$. An* expression for w of length k *is any sequence $\underline{w} := (s_1, \dots, s_k)$, for some not necessarily unique choice of $s_1, \dots, s_k \in S$, such that $w = s_1 s_2 \cdots s_k$. The* length *of w, denoted $\ell(w)$, is the length of its shortest expression, and any expression for w of length $\ell(w)$ is said to be* reduced*. Moreover, $\ell(w) = 0$ if and only if $w = 1$.*

Definition 3.3. (Bruhat Order). *Let (W, S) be a Coxeter system and $\Phi \coloneqq \{wsw^{-1} : s \in S, w \in W\}$ the set of conjugates of simple reflections in W. For $x, y \in W$, we write $x \leq y$ if and only if there exists a chain $x = x_0, x_1, \dots, x_k = y$ such that $\ell(x_i) < \ell(x_{i+1})$ and $x_i^{-1}x_{i+1} \in \Phi$ for all $0 \leq i < k$. This defines a partial order on W, known as the* Bruhat order.

Theorem 3.4. (Matsumoto's Theorem). *For any two reduced expressions of an element of a Coxeter group, the first can always be transformed into the second by repeatedly applying the braid relation.*

This result was first shown in [Mat64]. A sketch is given in [EMTW20, Theorem 2.20], and the full proof can be found in [GP00, Theorem 1.2.2].

Definition 3.5. (Iwahori–Hecke Algebra). *Let (W,S) be a Coxeter system and v a formal variable. The* (one-parameter) Iwahori–Hecke algebra *corresponding to (W,S) is the unital, associative $\mathbb{Z}[v,v^{-1}]$-algebra $\mathscr{H}(W,S)$ with generators $\{\delta_s : s \in S\}$ and relations*

- *(braid relation) $\delta_s\delta_t\delta_s \cdots = \delta_t\delta_s\delta_t \cdots$ for all $s,t \in S$ with $m_{st} < \infty$, where both sides are the product of m_{st} generators;*
- *(quadratic relation) $(\delta_s - v^{-1})(\delta_s + v) = 0$, for all $s \in S$.*

Note that we can expand and rearrange the quadratic relation as $1 = \delta_s^2 + (v - v^{-1})\delta_s$, whence multiplying through by δ_s^{-1} gives us $\delta_s^{-1} = \delta_s + (v - v^{-1})$.

Remark 3.6. If we identify our parameter v with 1, the Iwahori–Hecke algebra reduces to the group algebra $\mathbb{Z}W$. In other words, it is a deformation of the group algebra of its associated Coxeter group.

Remark 3.7. We have defined above the *one-parameter* Iwahori–Hecke algebra, as opposed to the more general *multiparameter* Iwahori–Hecke algebra, where instead of taking $\mathscr{H}(W,S)$ to be over the ring of one-parameter Laurent polynomials $\mathbb{Z}[v,v^{-1}]$ we consider a family of units $\{v_s : s \in S\}$ and take $\mathscr{H}(W,S)$ to be over the ring $\mathbb{Z}[v_s^{\pm 1} : s \in S]$.

Let (W,S) be a Coxeter system, and take $(s_1,\ldots,s_\ell)$ and $(t_1,\ldots,t_\ell)$ to be two reduced expressions for $w \in W$. Then $\delta_{s_1}\delta_{s_2}\cdots\delta_{s_\ell} = \delta_{t_1}\delta_{t_2}\cdots\delta_{t_\ell} =: \delta_w$ by Matsumoto's Theorem and the braid relation. In particular, by [EMTW20, Theorem 3.5], the set $\{\delta_w : w \in W\}$ forms a $\mathbb{Z}[v,v^{-1}]$-basis for $\mathscr{H}(W,S)$ with $\delta_{\mathrm{id}} = 1$, known as the *standard basis.* Another important basis is the Kazhdan–Lusztig basis.

Definition 3.8. (Kazhdan–Lusztig Involution). *Let $\mathscr{H}(W,S)$ be an Iwahori–Hecke algebra. The* Kazhdan–Lusztig involution *is the $\mathbb{Z}$-linear automorphism $h \mapsto \overline{h}$ on $\mathscr{H}(W,S)$, defined by*

$$\overline{\delta_s} := \delta_s^{-1} = \delta_s + (v - v^{-1})$$

on generators and by $\overline{v} = v^{-1}$ on Laurent polynomials. The Kazhdan–Lusztig anti-involution *is the $\mathbb{Z}$-linear anti-automorphism $h \mapsto \omega(h)$ defined similarly on generators and Laurent polynomials.*

Given $w \in W$ admitting a reduced expression $(s_1,\ldots,s_\ell)$, we find that

$$\begin{aligned}\overline{\delta_w} &= \overline{\delta_{s_1}}\cdots\overline{\delta_{s_\ell}} = \delta_{s_1}^{-1}\cdots\delta_{s_\ell}^{-1} = (\delta_{w^{-1}})^{-1},\\ \omega(\delta_w) &= \overline{\delta_{s_\ell}}\cdots\overline{\delta_{s_1}} = \delta_{s_\ell}^{-1}\cdots\delta_{s_1}^{-1} = \delta_w^{-1}.\end{aligned}$$

With this, we are ready to define the Kazhdan–Lusztig basis.

Theorem-Definition 3.9. (Kazhdan–Lusztig Basis). *Let $\mathscr{H}(W,S)$ be an Iwahori–Hecke algebra. Then it admits a unique $\mathbb{Z}[v,v^{-1}]$-basis $\{b_w : w \in W\}$ such that each b_w satisfies*

- *(self-duality) $\overline{b_w} = b_w$,*
- *(degree bound) $b_w = \sum_{y \in W} h_{y,w}\delta_y$,*

for some $h_{y,w} \in v\mathbb{Z}_{\geq 0}[v]$ with $h_{w,w} := 1$ and $h_{y,w} := 0$ whenever $y \not\leq w$ under the Bruhat order. This basis is known as the Kazhdan–Lusztig basis*, and the coefficients $h_{y,w}$ are called* Kazhdan–Lusztig polynomials.

A proof for existence can be found in [EMTW20, Theorem 3.25], while a proof for uniqueness can be found in [EMTW20, Lemma 3.10]. A helpful example for $W = S_3$ is given in [EMTW20, §3.3.1]. Note that all coefficients of Kazhdan–Lusztig polynomials are non-negative; this is the aforementioned *Kazhdan–Lusztig Positivity Conjecture*, which was first proven for general Coxeter systems by Elias and Williamson in [EW14, Corollary 1.2]. Observe also that *any* set $\{b_w : w \in W\}$ satisfying the degree bound condition will be a basis; in particular, the Kazhdan–Lusztig polynomials induce a triangular change of basis matrix with 1's along the diagonal that maps the standard basis to $\{b_w : w \in W\}$, giving an isomorphism between the two sets.

Definition 3.10. (Standard Form). *Let $\mathscr{H}(W,S)$ be an Iwahori–Hecke algebra. The* standard trace $\tau : \mathscr{H}(W,S) \to \mathbb{Z}[v, v^{-1}]$ *is the map that extracts the coefficient of δ_{id}; that is, it is the $\mathbb{Z}[v, v^{-1}]$-linear map for which $\tau(\delta_{\mathrm{id}}) = 1$ and $\tau(\delta_w) = 0$ for all $w \neq \mathrm{id}$. We then define the* standard form *on $\mathscr{H}(W,S)$ to be the sesquilinear form given by $(a,b) \coloneqq \tau(\omega(a)b)$, for all $a, b \in \mathscr{H}(W,S)$*

Definition 3.11. (Geometric Representation). *Let (W,S) be a Coxeter system and V the $\Bbbk$-vector space with basis $\{\alpha_s : s \in S\}$, where $\mathrm{char}(\Bbbk) = 0$. Define a symmetric, bilinear form on V by*

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right), & m_{st} \neq \infty; \\ -1, & m_{st} = \infty. \end{cases}$$

From this, we define an action of $s \in S$ on the basis elements $\alpha_t \in V$ by the linear automorphism

$$s(\alpha_t) \coloneqq \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s,$$

which reflects α_t across α_s. The geometric representation *of (W,S) is the representation induced by linearly extending this reflection action to an action of W on all of V.*

Let (W,S) be a Coxeter system with geometric representation $V \coloneqq \mathrm{span}_\Bbbk\{\alpha_s : s \in S\}$, and let $R \coloneqq \mathrm{Sym}(V) = \bigoplus_{i=0}^{\infty} \mathrm{Sym}^i(V)$ be the symmetric algebra of V, where $\mathrm{Sym}^i(V)$ is the *ith symmetric power of V*. We define this to be the quotient of the ith tensor power $V^{\otimes i}$ by the action of the symmetric group S_i, viewed as a $\Bbbk$-module. For technical reasons we will take R to be *graded in degree* 2; that is, we will take all odd graded pieces to be $\{0\}$ and label $\mathrm{Sym}^i(V)$ as its $2i$th graded piece rather than its ith graded piece. We will comment more on why we do this in Remark 3.28 at the end of this chapter.

For the time being, let's try to understand what $\mathrm{Sym}^i(V)$ actually looks like. Let $S = \{s_1, s_2, \dots\}$, and observe that $V^{\otimes i}$ admits the basis $\{\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}} : s_{j_1}, \dots, s_{j_i} \in S\}$. We define an action of the symmetric group S_i on basis elements in $V^{\otimes i}$ by $\sigma \cdot (\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}}) = (\alpha_{s_{\sigma(j_1)}} \otimes \cdots \otimes \alpha_{s_{\sigma(j_i)}})$, for $\sigma \in S_i$. This extends linearly to an action of S_i on all of $V^{\otimes i}$. Thus quotienting by the action of the symmetric group has the effect of making $\otimes$ commutative; this can be seen clearly by considering finite S and small i, for instance. Therefore, by linearly extending the map that sends basis elements $(\alpha_{s_{j_1}} \otimes \cdots \otimes \alpha_{s_{j_i}})$ to degree i monomials $\alpha_{s_{j_1}} \cdots \alpha_{s_{j_i}}$, we obtain a $\Bbbk$-module isomorphism from $\mathrm{Sym}^i(V)$ to the additive subgroup of the polynomial ring $\Bbbk[\alpha_s : s \in S]$ consisting only of homogeneous degree i polynomials. The upshot is that we may identify R with the $\mathbb{Z}$-graded polynomial ring $\Bbbk[\alpha_s : s \in S]$, whose $2i$th graded piece is the additive subgroup of homogeneous polynomials of degree i for all non-negative i, with all other graded pieces being $\{0\}$.

Remark 3.12. Note that the reflection action of W on V given in Definition 3.11 induces an action of W on R. This is given on monomials by $w(\alpha_{s_{j_1}} \cdots \alpha_{s_{j_i}}) = w(\alpha_{s_{j_1}}) \cdots w(\alpha_{s_{j_i}})$ and extended linearly to an action on all of R. Given a Coxeter group (W, S) and some $I \subseteq S$, we denote by W_I the subgroup of W generated by I, known as the *(standard) parabolic subgroup generated by* I. We say that I is *finitary* if W_I is a finite group, and write $R^I := \{f \in R : w(f) = f \text{ for all } w \in W_I\}$ for the set of W_I-invariant polynomials in R. Given $s \in S$, we will typically write R^s rather than $R^{\{s\}}$; naturally, since $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$ for any element g of an arbitrary group G, we have that $W_{\{s\}} = \{1, s\}$. Thus $R^s = \{f \in R : s(f) = f\}$, justifying our simplified notation.

Before defining Soergel bimodules, we will introduce Bott–Samelson bimodules. As it happens, the category of Soergel bimodules is the graded closure of the Karoubian envelope of the category of Bott–Samelson bimodules. The easiest non-trivial example of a Bott–Samelson bimodule (and hence of a Soergel bimodule) is, given a Coxeter system (W, S) and any $s \in S$, the graded (R, R)-bimodule

$$B_s := R \otimes_{R^s} R(1),$$

where (1) denotes a grading shift by 1. By this we mean that $R(1)$ is a copy of R whose ith graded piece is the $(i+1)$th graded piece of R. More generally, letting R^i denote the ith graded piece of R, we have that $R(k)^i := R^{i+k}$. Note that grading shifts commute with tensor products, and hence

$$B_s = R \otimes_{R^s} R(1) = (R \otimes_{R^s} R)(1) = R(1) \otimes_{R^s} R.$$

Definition 3.13. (Bott–Samelson Bimodule). *Let* (W, S) *be a Coxeter system and* $\underline{w} := (s_1, \ldots, s_k)$ *an expression. The* Bott–Samelson bimodule corresponding to $\underline{w}$ *is the graded* (R, R)*-bimodule*

$$\begin{aligned} BS(\underline{w}) &:= B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \\ &= (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R(k). \end{aligned}$$

We take the convention that if $\varnothing$ *is the empty expression, then* $BS(\varnothing) := R$*. Denote by* $\mathbb{BS}\mathsf{Bim}(W, S)$ *the* category of Bott–Samelson bimodules corresponding to (W, S)*, whose objects are direct sums of Bott–Samelson bimodules corresponding to expressions in* (W, S) *and whose morphisms are graded* (R, R)*-bimodule homomorphisms; that is,* (R, R)*-bimodule homomorphisms* $\varphi : X \to Y$ *that respect gradings, in the sense that* $\varphi(X^i) \subseteq Y^i$ *for all* $i \in \mathbb{Z}$.

Before we get too ahead of ourselves, a natural question we may ask is where these B_s bimodules come from. Well, for any $s \in S$, one can verify that R^s is generated by α_s^2 and elements of the form $\alpha_t - (\alpha_s, \alpha_t)\alpha_s$, for $t \in S \setminus \{s\}$. This gives us a decomposition of R into s-invariants and s-antiinvariants; that is, $R \cong R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$ *as an* (R^s, R^s)*-bimodule*[†]. Thus we have

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

[†] We will show this more explicitly later on, in Lemma 3.22.

If we now look at the Kazhdan–Lusztig basis element b_s, it follows from uniqueness that $b_s = \delta_s + v$, as this is certainly a self-dual element satisfying the degree bound conditions for s. Thus

$$\begin{aligned} b_s^2 &= \delta_s^2 + 2v\delta_s + v^2 \\ &= (1 + (v^{-1} - v)\delta_s) + 2v\delta_s + v^2 \\ &= 1 + v^{-1}\delta_s + v\delta_s + v^2 \\ &= v(\delta_s + v) + v^{-1}(\delta_s + v) \\ &= vb_s + v^{-1}b_s. \end{aligned}$$

Comparing the expressions for B_s and b_s, we see some striking similarities! This Bott–Samelson bimodule arising from the generator s happens to behave just like an element of the Kazhdan–Lusztig basis, with positive (negative) grading shifts corresponding to multiplication (division) by our formal variable v. But what about the basis elements corresponding $w \in W \setminus S$? For these, Bott–Samelsons alone are insufficient. We therefore introduce Soergel bimodules.

Definition 3.14. (Soergel Bimodule). *Let (W, S) be a Coxeter system. A* Soergel bimodule *is a direct summand of a finite direct sum of grading shifts of Bott–Samelson bimodules corresponding to expressions in (W, S). Denote by $\mathbb{S}\mathsf{Bim}(W, S)$ the* category of Soergel bimodules corresponding to *(W, S), which we define as the "graded closure" of the Karoubian envelope of $\mathbb{BS}\mathsf{Bim}(W, S)$; in other words, it is the closure of the category of Bott–Samelson bimodules corresponding to (W, S) under isomorphisms, direct summands, finite direct sums and grading shifts.*

Remark 3.15. When we constructed R, we allowed $\Bbbk$ to be any field of characteristic zero. In principle, this choice is completely arbitrary for $\mathbb{S}\mathsf{Bim}$; this is one of the powers of Soergel bimodules! The field we choose to work over can be decided based on the context. In [EMTW20], this field is taken to be $\mathbb{R}$, as every symmetric $\mathbb{R}$-bilinear form is determined by its signature, a fundamental fact for doing Hodge theory. Alternatively, we will often be in the setting where we have some complex, semisimple Lie algebra $\mathfrak{g}$, along with a choice of Cartan subalgebra $\mathfrak{h}$ that determines a Weyl group W and Borel subalgebra $\mathfrak{b}$ that determines a set of simple reflections S. In this setting, we will take R to be the ring of regular functions on $\mathfrak{h}^*$, whence $R = \mathrm{Sym}(\mathfrak{h}) = \mathbb{C}[\alpha_s^\vee : s \in S]$ for coroots $\alpha_s^\vee \in \mathfrak{h}$ and $\mathfrak{h}^* = \mathrm{spec}(R)$. In this case, it is a result of Serre's theorem for quasicoherent sheaves that we can view $\mathbb{S}\mathsf{Bim}(W, S)$ as a subcategory of the category of $\mathbb{C}^\times$-equivariant coherent sheaves of $\mathscr{O}_X$-modules on X, where $\mathscr{O}_X$ is the sheaf of regular functions on the pullback $X := \mathfrak{h}^* \times_{\mathfrak{h}^*/W} \mathfrak{h}^* = \mathrm{spec}(R \otimes_{R^W} R)$.

Theorem 3.16. (Soergel's Categorification Theorem I). *There is a unique $\mathbb{Z}[v, v^{-1}]$-algebra homomorphism $c : \mathscr{H}(W, S) \to \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W, S))$ sending $v^k b_s$ to the isomorphism class $[B_s(k)]$, for each $s \in S$. Moreover, the Δ-character function $\mathrm{ch}_\Delta : \mathbb{S}\mathsf{Bim}(W, S) \to \mathscr{H}(W, S)$ descends to a homomorphism $\mathrm{ch} : \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W, S)) \to \mathscr{H}(W, S)$ that is inverse to c, making these isomorphisms.*

Theorem 3.17. (Soergel's Categorification Theorem II). *Let (W, S) be a Coxeter system and $\underline{w}$ a reduced expression for $w \in W$. Then $BS(\underline{w})$ contains, up to isomorphism, a unique indecomposable summand B_w, which does not appear in $BS(\underline{x})$ for any $x < w$ and depends only on w, not the reduced expression. Moreover, all other direct summands are grading shifts of B_x, for $x < w$.*

A statement of and reference for these results can be found in [EMTW20, Theorem 5.24], as well as an algorithm for finding the indecomposables B_w. The precise definition of the Δ-character function is $\mathrm{ch}_\Delta : B \mapsto \sum_{y \in W} v^{\ell(y)} h_y(B)\delta_y$, where $h_y(B) \in v\mathbb{Z}_{\geq 0}[v, v^{-1}]$ is some kind of graded multiplicity (this is slightly technical, see [EMTW20, Theorem 5.10] for details). For any indecomposable Soergel bimodule B_w, the graded multiplicities satisfy $h_y(B_w(k)) = v^k h_{y,w}$ by Soergel's Conjecture.

Corollary 3.18. *The isomorphism classes of indecomposables of* $\mathbb{S}\mathsf{Bim}(W, S)$ *are in bijection with* $W \times \mathbb{Z}$.

Proof. This follows from Theorem 3.17, together with the fact that grading shifts preserve indecomposability. In particular, the indecomposable objects are, up to isomorphism, grading shifts of B_w, for $w \in W$. This completes the proof. ■

For any pre-additive category $\mathcal{C}$ with a *shift functor* – that is, an additive, invertible endofunctor $(1) : \mathcal{C} \to \mathcal{C}$ – we have that $\mathrm{Mor}_{\mathcal{C}}(X(j), Y(i)) = \mathrm{Mor}_{\mathcal{C}}(X, Y(i-j))$, where $X(k) \coloneqq [(1)^k](X)$. We define the corresponding *graded Hom-space* by $\mathrm{Mor}^\bullet_{\mathcal{C}}(X, Y) \coloneqq \bigoplus_{i \in \mathbb{Z}} \mathrm{Mor}^i_{\mathcal{C}}(X, Y)$, where the graded pieces are given by $\mathrm{Mor}^i_{\mathcal{C}}(X, Y) \coloneqq \mathrm{Mor}_{\mathcal{C}}(X, Y(i))$. In other words, this process endows $\mathcal{C}$ with the structure of a *graded category* – a category whose Hom-spaces are $\mathbb{Z}$-graded Abelian groups. Similarly, we can turn any graded category into a category with a shift functor, and in fact by [EMTW20, Proposition 11.9] we have a pair of mutually adjoint pseudofunctors

$$\begin{tikzcd} \left\{\begin{matrix}\text{categories with} \\ \text{a shift functor}\end{matrix}\right\} \arrow[rr, bend left, "(-)^{\mathrm{gr}}"] & & \left\{\begin{matrix}\text{graded} \\ \text{categories}\end{matrix}\right\} \arrow[ll, bend left, "(-)^{\mathrm{sh}}"] \end{tikzcd}$$

In other words, we can always think of the Hom-spaces of $\mathbb{S}\mathsf{Bim}$ as being graded!

Now, given a finite-dimensional $\mathbb{Z}$-graded $\Bbbk$-vector space $V = \oplus V^i$, we define its *graded dimension* to be the Hilbert–Poincaré series $\mathrm{gdim}(V) \coloneqq \sum_{i \in \mathbb{Z}} \dim(V^i) v^i$. Then given a free, finitely-generated left (respectively, right) graded R-module M, we define its *graded rank* to be $\mathrm{grk}(M) \coloneqq \mathrm{gdim}(\Bbbk \otimes_R M)$ (respectively, $\mathrm{grk}(M) \coloneqq \mathrm{gdim}(M \otimes_R \Bbbk)$). Note that the purpose of tensoring with the underlying field here is to promote M to a finite-dimensional $\Bbbk$-vector space; since M is finitely-generated, it admits a finite generating set $\{m_1, \ldots, m_n\}$, whence $\{1 \otimes m_1, \ldots, 1 \otimes m_n\}$ is a $\Bbbk$-basis for $\Bbbk \otimes_R M$.

Theorem 3.19. (Soergel Hom Formula). *For any two Soergel bimodules* $B, B' \in \mathbb{S}\mathsf{Bim}(W, S)$, $\mathrm{Mor}^\bullet_{\mathbb{S}\mathsf{Bim}}(B, B')$ *is free as a left graded* R*-module and as a right graded* R*-module, both with* $\mathrm{grk}(\mathrm{Mor}^\bullet_{\mathbb{S}\mathsf{Bim}}(B, B')) = (\mathrm{ch}_\Delta(B), \mathrm{ch}_\Delta(B'))$, *where* $(-,-)$ *denotes the standard form on* $\mathscr{H}(W, S)$. *In particular, the dimension of* $\mathrm{Mor}_{\mathbb{S}\mathsf{Bim}}(B, B'(i))$ *is given by the coefficient of* v^i *in* $(\mathrm{ch}_\Delta(B), \mathrm{ch}_\Delta(B'))$.

Example 3.20. Consider the graded Hom-space $\mathrm{Mor}^{\bullet}_{\mathbb{S}\mathsf{Bim}}(B_s, B_s \otimes_R B_s)$. Since $\mathrm{ch}_\Delta(B_s) = b_s = \delta_s + v$ and $\mathrm{ch}_\Delta(B_s \otimes_R B_s) = b_s^2 = v^2 + v\delta_s + v^{-1}\delta_s + 1$, the standard form gives us

$$\begin{aligned}(\mathrm{ch}_\Delta(B_s), \mathrm{ch}_\Delta(B_s \otimes_R B_s)) &= \tau(\omega(b_s)b_s^2) = \tau(b_s^3) = \tau((v+v^{-1})b_s^2) = \tau((v+v^{-1})^2 b_s) \\ &= \tau((v+v^{-1})^2(\delta_s + v)) = (v+v^{-1})^2 v \\ &= v^3 + 2v + v^{-1}.\end{aligned}$$

In other words, this tells us that as $\Bbbk$-vector spaces, $\mathrm{Mor}_{\mathbb{S}\mathsf{Bim}}(B_s, B_s \otimes_R B_s(3))$ has dimension 1, $\mathrm{Mor}_{\mathbb{S}\mathsf{Bim}}(B_s, B_s \otimes_R B_s(1))$ has dimension 2 and $\mathrm{Mor}_{\mathbb{S}\mathsf{Bim}}(B_s, B_s \otimes_R B_s(-1))$ has dimension 1, with all other graded pieces having dimension 0. This is an extremely powerful result!

Any category of Soergel bimodules is naturally a strict monoidal category. Although they are not Abelian, by the Soergel Hom Formula we can view their classes of morphisms as finite-dimensional $\Bbbk$-vector spaces by tensoring with $\Bbbk$, and in fact one can show using Theorem 2.8 that they are Krull–Schmidt. By Corollary 3.18 it will only be *graded multifinitary*, as we only have finitely many indecomposable objects under both isomorphisms and grading shifts. Fortunately, however, the the only time having finitely many isomorphism classes of indecomposable objects comes into play is with the weak Jordan–Hölder Theorem, where we only need $\mathrm{Ind}(\mathrm{M})/\sim$ to be finite.

Theorem 3.21. (Soergel's Conjecture). *The isomorphism* $\mathrm{ch} : \mathrm{Gr}(\mathbb{S}\mathsf{Bim}(W,S)) \to \mathscr{H}(W,S)$ *from Theorem 3.16 sends* $B_w(k)$ *to the isomorphism class* $v^k b_w$*, for each* $w \in W$*.*

This result, which was first proven by Elias and Williamson ([EW14, Theorem 1.1]), generalizes the Kazhdan–Lusztig Conjecture and finally gives us an algebraic categorification of the Iwahori–Hecke algebra. This also means that our categories of Soergel bimodules are actually deceptively easy to work with; up to isomorphism, we can manipulate the objects by viewing them not as bimodules but as elements in the corresponding Iwahori–Hecke algebra. That said, Soergel bimodules also give us a collection of morphisms to look at. In fact, from the categorical perspective, the morphisms are really the items of interest – compared to the Iwahori–Hecke algebra, they provide us with genuinely new information! As it turns out, even these morphisms are nice to work with, thanks to Theorem 3.19 and the fact that they admit a graphical calculus.

At this point, there are just a few loose ends that I'd like to tie up, which will hopefully become useful later on. Earlier we claimed that there exists an (R^s, R^s)-bimodule isomorphism $R \cong R^s \oplus R^s(-2)$ (which we note is certainly *not* an isomorphism of (R,R)-bimodules, as R^s is itself not an R-module). I would like to show this more explicitly.

Lemma 3.22. *For any* $s \in S$ *and* $f \in R$*, we have that* $f + s(f) \in R^s$ *and* $f - s(f) \in R^s\alpha_s$*. In particular, we have an* (R^s, R^s)*-bimodule splitting* $R \cong R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$ *for every* $s \in S$*.*

Proof. Let $s \in S$ and $f := \alpha_t$ for any $t \in S$. Observe that $g := \frac{1}{2}(f + s(f)) = \alpha_t - (\alpha_s, \alpha_t)\alpha_s \in R^s$, while $h\alpha_s := \frac{1}{2}(f - s(f)) = (\alpha_s, \alpha_t)\alpha_s \in R^s\alpha_s$. All that remains is to show that this holds for products. Suppose $f_1 := g_1 + h_1\alpha_s$ and $f_2 := g_2 + h_2\alpha_s$; then $f := f_1 f_2 = g_1g_2 + g_1h_2\alpha_s + g_2h_1\alpha_s + h_1h_2\alpha_s^2$ admits a unique decomposition of the form $g + h\alpha_s$, for $g := \frac{1}{2}(f + s(f)) = g_1g_2 + h_1h_2\alpha_s^2$ and $h\alpha_s := \frac{1}{2}(f - s(f)) = g_1h_2\alpha_s + g_2h_1\alpha_s$. This completes the proof. ■

Definition 3.23. (Demazure Operator). *For each $s \in S$, we define the corresponding* Demazure operator *to be the graded bimodule homomorphism $\partial_s : R \to R^s(-2)$ given by*

$$\partial_s : f \mapsto \frac{f - s(f)}{\alpha_s},$$

for all $f \in R$.

Clearly this map is well-defined by Lemma 3.22. Essentially, the Demazure operators encapsulate the aforementioned splitting of R into an s-invariant part and an s-antiinvariant part (in the sense that $s(f) = -f$). In particular, it is easy to see that for any $f \in R$, we have that $\partial_s(f\alpha_s) = f + s(f)$ while $\alpha_s\partial_s(f) = f - s(f)$. Thus we can rephrase our splitting in terms of Demazure operators as

$$f = \partial_s \left(f\frac{\alpha_s}{2} \right) + \frac{\alpha_s}{2}\partial_s(f).$$

As it happens, the Demazure operators have many nice properties.

Lemma 3.24. ([EMTW20, Lemma 4.15]). *Let $s \in S$. Then*

(i). *∂_s is an (R^s, R^s)-bimodule map;*
(ii). *$s \circ \partial_s = \partial_s$ and $\partial_s \circ s = -\partial_s$;*
(iii). *$\partial_s \circ \partial_s = 0$;*
(iv). *there exists a short exact sequence*

$$0 \to R^s \to R \xrightarrow{\partial_s} R^s(-2) \to 0;$$

(v). *(twisted Leibniz rule) $\partial_s(fg) = \partial_s(f)g + s(f)\partial_s(g)$, for all $f, g \in R$;*
(vi). *(braid relation) for distinct simple reflections $s, t \in S$ with $m_{st} < \infty$,*

$$\partial_s \circ \partial_t \circ \partial_s \circ \cdots = \partial_t \circ \partial_s \circ \partial_t \circ \cdots,$$

where both sides are the composition of m_{st} Demazure operators.

Definition 3.25. *The Demazure operator corresponding to $w \in W$ is given by*

$$\partial_w := \partial_{s_1} \circ \cdots \circ \partial_{s_k},$$

for any reduced expression $\underline{w} := (s_1, \ldots, s_k)$ for w.

By (vi) above and Matsumoto's Theorem, this Demazure operator will not depend on the choice of reduced expression.

In the next chapter, we will be often find it helpful to view the dummy object in the monoidal delooping of $\mathbb{S}\mathsf{Bim}$ as a category of modules over the *covariant algebra*. Let's briefly make some definitions for the purpose of preparing for this.

Definition 3.26. (Reflection Faithful Representation). *Let (W, S) be a Coxeter system. A faithful representation $\rho : W \to \mathrm{GL}(V)$ is further said to be* reflection faithful *if, for each $x \in W$, the corresponding fixed-point set $\{v \in V : [\rho(x)](v) = v\}$ has codimension 1 if and only if x is a* reflection *in W (that is, x is conjugate to an element of S).*

In other words, we are asking that elements of W act as reflections on V if and only if they are reflections in W. Previously, our ring R was defined in terms of the geometric representation, which is in general faithful but may fail to be reflection faithful (such as when W is an affine Weyl group, for instance). For what follows, we will work with $R := \mathrm{Sym}(V)$ for an arbitrary reflection faithful representation (ρ, V).

Definition 3.27. (Coinvariant Algebra). *Let W be finite and denote by R_+^W the graded subspace of R^W consisting of everything in strictly positive degrees; that is,*

$$R_+^W := \bigoplus_{i=1}^{\infty} R^i.$$

Letting I_W be the homogeneous ideal in R generated by the elements in R_+^W, we define the coinvariant algebra of W *to be the graded algebra*

$$C := R/I_W.$$

While this definition may seem a bit obtuse, the point is that we want to quotient out by the ideal generated by all of the *non-constant* W-invariant elements.

Remark 3.28. In the following chapters, we will typically have a sequence of subgroups $T \subset B \subset G$, where G is a connected, complex, reductive algebraic group, B is a Borel subgroup and T is a maximal torus. These inclusions induce a Weyl group $W \coloneqq N_G(T)/T$, where

$$N_G(T) \coloneqq \{g \in G : gtg^{-1} \in T, \text{ for all } t \in T\};$$

a set $\Phi = \Phi(G,T)$ of roots of G relative to T; a set S of simple roots of W corresponding to B, such that (W,S) is a Coxeter system (it is a fact that the set of Borel subgroups of G containing T are in bijection with the set of bases for Φ); and a complex flag manifold (or generalized flag variety) G/B. Suppose now that we choose, for each $w \in W$, a representative $g_w \in N_G(T)$ for which $w = g_wT$. Letting $C_w \coloneqq Bg_wB$, which is well-defined since T is a subgroup of B, we have the following stratification of G into a disjoint union of (B,B)-double cosets:

$$G = \bigsqcup_{w \in W} C_w.$$

This is known as a *Bruhat decomposition* of G, and the double cosets C_w are known as *Bruhat cells.* Importantly, this descends to a stratification of the flag variety G/B in terms of *Schubert cells* of the form $X_w \coloneqq C_w/B$:

$$G/B = \bigsqcup_{w \in W} X_w.$$

The cells X_w are locally closed subvarieties of G/B. We refer to the Zariski closure $\overline{X_w}$ as the *Schubert variety of* w, and remark that $\overline{X_y} \subseteq \overline{X_w}$ if and only if $y \leq w$. As an aside, we have that

$$v^{\ell(w)-\ell(y)} h_{y,w}(v^{-1}) = \sum_{i \geq 0} v^i \dim_{\mathbb{C}}(IH^{2i}_{X_y}(\overline{X_w}, \mathbb{C})),$$

where $h_{y,w}$ is a Kazhdan–Lusztig polynomial for the Kazhdan–Lusztig basis of $\mathscr{H}(W,S)$. The term $IH^{2i}_{X_y}(X_w)$ is the stalk of the ith hypercohomology of IC_w at a (equivalently, any) point of X_y, where IC_w denotes the intersection cohomology sheaves of $\overline{X_w}$ ([EMTW20, Theorem 13.13]). This result was first shown in the appendix to [KL79] (see Remark A11). One thing we may observe here is that we only sum over even graded pieces of the local intersection cohomology; the reason for this is that the odd graded pieces are all zero. In fact, because we have a left action of B on G/B, we can consider the B-equivariant cohomology $H^*_B(G/B)$, which we will find is a ring that is also graded in degree 2. As it turns out, $H^*_B(G/B) \cong R \otimes_{R^W} R$, which is why we grade R the way that we do!

Go through the SL_2 and SL_3 examples in detail to put everything together, it'd be good to see an example of the Kazhdan–Lusztig basis of a Iwahori–Hecke algebra and compare it to the indecomposables in the corresponding category of Soergel bimodules. I should also start typing up Lie theory notes.

4. Simple Module Categories over Soergel Bimodules

Before we talk about the Lusztig–Vogan module categories, it might be helpful to understand the classification result of Mackaay, Mazorchuk, Miemietz, Tubbenhauer and Zhang ([MMM+23]). A family of birepresentations that play a particularly important role are cell birepresentations. In the context of Soergel bimodules, these module categories categorify Kazhdan–Lusztig cell representations of the Iwahori–Hecke algebra as a consequence of the categorification theorem.

Let $\mathscr{C}$ be a multifinitary bicategory with indecomposables $F \in \mathrm{Ob}(\mathscr{C}(\mathtt{i},\mathtt{j}))$ and $G \in \mathrm{Ob}(\mathscr{C}(\mathtt{k},\mathtt{l}))$. We write $G \geq_L F$ if there exists some $H \in \mathscr{C}(\mathtt{j},\mathtt{l})$ for which G is isomorphic to a direct summand of $H \circ F$ in the Hom-category $\mathscr{C}(\mathtt{k},\mathtt{l})$ (note that this forces $\mathtt{k} = \mathtt{i}$). This is known as the *left preorder* (c.f. the action preorder). The set of equivalence classes of an indecomposable object F is called the *left cell of* F and denoted $\mathcal{L}_F$. Similarly, we have a *right preorder* defined by $G \geq_R F$ if there exists some $H \in \mathrm{Ob}(\mathscr{C}(\mathtt{k},\mathtt{i}))$ for which G is isomorphic to a direct summand of $F \circ H$, with equivalence classes known as *right cells* and denoted $\mathcal{R}_F$. Finally, we also write $G \geq_J F$ if there exist $H_l \in \mathrm{Ob}(\mathscr{C}(\mathtt{j},\mathtt{l}))$ and $H_r \in \mathrm{Ob}(\mathscr{C}(\mathtt{k},\mathtt{i}))$ for which G is isomorphic to a direct summand of $H_l \circ F \circ H_r$, with equivalence classes known as *two-sided cells* and denoted $\mathcal{J}_F$. Such a cell is said to be *idempotent* if there exist $G, H, K \in \mathcal{J}_F$ such that K is isomorphic to a direct summand of $G \circ H$.

Let $\mathcal{L}_F$ be the left cell of $F \in \mathrm{Ob}(\mathscr{C}(\mathtt{s},\mathtt{t}))$ and define a multifinitary subbirepresentation N of the Yoneda birepresentation $\mathbb{P}_{\mathtt{s}}$ given by restricting each $\mathbb{P}_{\mathtt{s}}(\mathtt{i}) = \mathscr{C}(\mathtt{s},\mathtt{i})$ to the additive subcategory

$$\mathrm{N}(\mathtt{i}) := \mathsf{Add}(\{G \in \mathrm{Ob}(\mathscr{C}(\mathtt{s},\mathtt{i})) : G \geq_L \mathcal{L}_F\}).$$

We will refer to this as the *pre-cell birepresentation of* $\mathscr{C}$ *associated to* $\mathcal{L}_F$.

Proposition 4.1. *Let* N *be the pre-cell birepresentation of a multifinitary bicategory* $\mathscr{C}$ *associated to a left cell* $\mathcal{L}$*. Then* N *admits a unique maximal* $\mathscr{C}$*-stable ideal* I*, and moreover each* $\mathrm{I}(\mathtt{i})$ *contains no identity morphisms corresponding to objects in* $\mathcal{L}$*.*

Proof. The proof follows similarly to Lemma 2.26, with I given by the sum of all $\mathscr{C}$-stable ideals of N that do not contain id_F for any $F \in \mathcal{L}$. Note that transitivity is no longer needed, as containing one such id_F means that you must contain id_G for all $G \in \mathcal{L}$ by $\mathscr{C}$-stability and the definition of left cells. Maximality also tells us that we must in fact include all identity morphisms corresponding to objects F for which $F >_L \mathcal{L}$. This completes the proof. ■

Definition 4.2. (Cell Birepresentation). *Let* N *be the pre-cell birepresentation of a multifinitary bicategory* $\mathscr{C}$ *associated to a left cell* $\mathcal{L}$*. The simple transitive subbirepresentation* $\mathscr{C}_{\mathcal{L}}$ *of* N *that sends each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *to the quotient subcategory* $\mathrm{N}(\mathtt{i})/\mathrm{I}(\mathtt{i})$ *is known as the* cell birepresentation *of* $\mathscr{C}$ *associated to* $\mathcal{L}$*.*

Unlike when going from transitive to simple transitive, going from pre-cell to cell typically annihilates objects. In particular, we will annihilate every object F for which $F >_L \mathcal{L}$.

Let M be a birepresentation of $\mathscr{C}$. For any $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$, we define

$$\mathrm{Ann}_{\mathscr{C}}(X) := \{F \in \mathrm{Ob}(\mathscr{C}(\mathtt{i},\mathtt{j})) : \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), [\mathrm{M}(F)](X) = 0\}$$

to be the *annihilator of* X. We define the *annihilator of* M to be

$$\mathrm{Ann}_{\mathscr{C}}(\mathrm{M}) := \bigcap_{X} \mathrm{Ann}_{\mathscr{C}}(X),$$

the collection of 1-morphisms that are annihilated by M.

Definition 4.3. (Biideal). *A* left *(respectively* right*)* biideal $\mathscr{I}$ *of a bicategory* $\mathscr{C}$ *is a collection* $\mathscr{I} := \{\mathscr{I}(\mathtt{i},\mathtt{j}) : \mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})\}$*, where each* $\mathscr{I}(\mathtt{i},\mathtt{j})$ *is a left (respectively right) ideal of the Hom-category* $\mathscr{C}(\mathtt{i},\mathtt{j})$ *that is, in addition, stable under horizontal post-composition (respectively pre-composition). We say that* $\mathscr{I}$ *is a* two-sided biideal *if it is both a left biideal and a right biideal.*

The annihilator $\mathrm{Ann}_{\mathscr{C}}(\mathrm{M})$ gives us a two-sided biideal by taking all the 2-morphisms between its 1-morphisms. This in turn gives us a multifinitary bicategory $\mathscr{C}_{\mathrm{M}} := \mathscr{C}/\mathrm{Ann}_{\mathscr{C}}(\mathrm{M})$ by quotienting each Hom-category. It is easy to see that the two-sided cells of $\mathscr{C}_{\mathrm{M}}$ are also two-sided cells of $\mathscr{C}$, although some two-sided cells of $\mathscr{C}$ may disappear from $\mathscr{C}_{\mathrm{M}}$ if they are annihilated by M.

Lemma-Definition 4.4. (Apex). *Let* $\mathscr{C}$ *be a multifinitary bicategory and* M *a transitive birepresentation of* $\mathscr{C}$*. Then* $\mathscr{C}_{\mathrm{M}}$ *admits a unique maximal two-sided cell known as the* apex of M*. Moreover, this two-sided cell is idempotent.*

Proof. For this proof, we will assume that $\mathscr{C}$ has only one object, $\bullet$. The more general proof follows similarly. We shall first proceed by contradiction. Let $\mathcal{J}_F$ and $\mathcal{J}_G$ be two different two-sided cells of $\mathscr{C}_{\mathrm{M}}$ that are maximal with respect to $\geq_J$, in the sense that $H \geq_J \mathcal{J}_F$ implies $\mathcal{J}_F \geq_J H$, and similarly for $\mathcal{J}_G$. Let A (respectively B) be a multiplicity-free direct sum of representatives of 2-isomorphism classes of indecomposable 1-morphisms in $\mathcal{J}_F$ (respectively $\mathcal{J}_G$). In other words, $A = A_1 \oplus \cdots \oplus A_m$ and $B = B_1 \oplus \cdots \oplus B_n$ are direct sums of indecomposable 1-morphisms with each 2-isomorphism class appearing exactly once. Note that we must have that $A \circ B = 0 = B \circ A$, as otherwise maximality would imply that $A = B$ (since $A \circ B \geq_J \mathcal{J}_F, \mathcal{J}_G$, for instance, where two cells with a non-empty intersection must be equal by transitivity of the preorder).

Let $X := X_1 \oplus \cdots \oplus X_k$ now be a multiplicity-free direct sum of representatives of isomorphism classes of indecomposables in $\mathrm{M}(\bullet)$, whence $\mathsf{add}(\{X\}) = \mathrm{Ob}(\mathrm{M}(\bullet))$. Naturally $[\mathrm{M}(A)](X) \neq 0$ by assumption, since $\mathcal{J}_F$ is a two-sided cell of $\mathscr{C}_{\mathrm{M}}$. Consider now any indecomposable summand X' of $[\mathrm{M}(A)](X)$. Then because M is transitive, Proposition 2.30 tells us that there exists, for all $1 \leq i \leq k$, some $H_i \in \mathrm{Ind}(\mathscr{C})$ for which X_i is isomorphic to a direct summand of $[\mathrm{M}(H_i \circ A)](X)$. Moreover, it follows from maximality that $H_i \circ A$ is a direct sum of indecomposables from $\mathcal{J}_F$, since it is clear that $F' \geq_J \mathcal{J}_F$ for any indecomposable summand F' of some $H_i \circ A_j$. In other words,

$$\mathsf{add}(\{[\mathrm{M}(A)](X)\}) = \mathsf{add}(\{[\mathrm{M}(H_1 \circ A)](X), \ldots, [\mathrm{M}(H_k \circ A)](X)\}) = \mathsf{add}(\{X\}),$$

and similarly $\mathsf{add}(\{[\mathrm{M}(B)](X)\}) = \mathsf{add}(\{X\})$. These imply that $\mathsf{add}(\{[\mathrm{M}(A \circ B)](X)\}) = \mathsf{add}(\{X\})$, contradicting $A \circ B = 0$, whence the maximal two-sided cell must be unique. Finally, we see that $\mathsf{add}(\{[\mathrm{M}(F \circ F)](X)\}) = \mathsf{add}(\{[\mathrm{M}(F)](X)\}) = \mathsf{add}(\{X\})$, meaning $F \circ F \neq 0$ and hence that $\mathcal{J}_F$ is idempotent. This completes the proof. ■

Lifting it back up to $\mathscr{C}$, Lemma 4.4 tells us that $\mathscr{C}$ admits a unique two-sided cell that is maximal among cells not annihilated by M. Given a multifinitary bicategory $\mathscr{C}$ and a two-sided cell $\mathcal{J}$, let $\mathscr{C}\text{-stmod}_{\mathcal{J}}$ denote the category of simple transitive birepresentations of $\mathscr{C}$ with apex $\mathcal{J}$.

In the context of [MMM+23], we can essentially perform the classification of simple transitive 2-representations by looking at the apexes. In fact, we can take this reduction one step further via a technique known as *$\mathcal{H}$-reduction.*

Definition 4.5. (Fiab Category). *A (multi)finitary bicategory $\mathscr{C}$ is said to be* quasi (multi)fiab *if it has adjunctions. In other words, if we have an object-preserving $\Bbbk$-linear biequivalence ${}^* : \mathscr{C} \to \mathscr{C}^{\mathrm{co,op}}$ such that, for every $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i},\mathtt{j}))$, there exist adjunction 2-morphisms $\mathrm{ev}_f : f \circ_h f^* \to \mathbb{1}_{\mathtt{j}}$ and $\mathrm{coev}_f : \mathbb{1}_{\mathtt{i}} \to f^* \circ_h f$ for which the diagrams*

$$
\begin{tikzcd}
 & f \arrow[dl, "r_f^{-1}"'] & \\
f \circ_h \mathbb{1}_{\mathtt{i}} \arrow[d, "\mathrm{id}_f \circ_h \mathrm{coev}_f"'] & & \mathbb{1}_{\mathtt{j}} \circ_h f \arrow[ul, "l_f"'] \\
f \circ_h (f^* \circ_h f) \arrow[rr, "a^{-1}_{f,f^*,f}"'] & & (f \circ_h f^*) \circ_h f \arrow[u, "\mathrm{ev}_f \circ_h \mathrm{id}_f"']
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & f^* \arrow[dl, "l_{f^*}^{-1}"'] & \\
\mathbb{1}_{\mathtt{i}} \circ_h f^* \arrow[d, "\mathrm{coev}_f \circ_h \mathrm{id}_{f^*}"'] & & f^* \circ_h \mathbb{1}_{\mathtt{j}} \arrow[ul, "r_{f^*}"'] \\
(f^* \circ_h f) \circ_h f^* \arrow[rr, "a_{f^*,f,f^*}"'] & & f^* \circ_h (f \circ_h f^*) \arrow[u, "\mathrm{id}_{f^*} \circ_h \mathrm{ev}_f"']
\end{tikzcd}
$$

commute. If $\mathrm{id}_{\mathscr{C}}$ and $({}^*)^2$ are internally equivalent in $[\mathscr{C},\mathscr{C}]$, then $\mathscr{C}$ is said to be (multi)fiab. *The strict version of a (quasi) (multi)fiab bicategory is known as a* (quasi) (multi)fiat 2-category.

Definition 4.6. ($\mathcal{J}$-Simplicity). *Let $\mathscr{C}$ be a multifinitary bicategory with $\mathcal{J}$ a two-sided cell. We say that $\mathscr{C}$ is* $\mathcal{J}$-simple *if any non-zero biideal of $\mathscr{C}$ contains the identity 2-morphisms of all 1-morphisms in $\mathcal{J}$.*

Theorem-Definition 4.7. ($\mathcal{J}$-Simple Quotient). *Let $\mathscr{C}$ be a multifinitary bicategory and $\mathcal{J}$ a non-zero two-sided cell. Then there exists a unique biideal $\mathscr{J}$ such that $\mathscr{C}/\mathscr{J}$ is $\mathcal{J}$-simple. We call $\mathscr{C}_{\leq \mathcal{J}} \coloneqq \mathscr{C}/\mathscr{J}$ the* $\mathcal{J}$-simple quotient *of $\mathscr{C}$.*

For a proof of this in the fiat case, see [MM14, Theorem 15]. According to [MMM+21], this also holds in the more general case of multifinitary bicategories, with the $\mathcal{J}$-simple quotient $\mathscr{C}_{\leq \mathcal{J}}$ being unique up to biequivalence. Moreover, if $\mathscr{C}$ is (quasi) multifiab, then the $\mathcal{J}$-simple quotient $\mathscr{C}_{\leq \mathcal{J}}$ is also (quasi) multifiab.

Let $\mathscr{C}_{\mathcal{J}}$ be the 2-full subbicategory of $\mathscr{C}_{\leq \mathcal{J}}$ whose objects consist only of the sources and targets of 1-morphisms in $\mathcal{J}$, and whose Hom-categories are given by

$$
\begin{aligned}
\mathscr{C}_{\mathcal{J}}(\mathtt{i},\mathtt{i}) &\coloneqq \mathsf{Add}((\mathrm{Mor}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathtt{i},\mathtt{i}) \cap \mathcal{J}) \cup \{\mathbb{1}_{\mathtt{i}}\}),\\
\mathscr{C}_{\mathcal{J}}(\mathtt{i},\mathtt{j}) &\coloneqq \mathsf{Add}(\mathrm{Mor}_{\mathscr{C}_{\leq \mathcal{J}}}(\mathtt{i},\mathtt{j}) \cap \mathcal{J}).
\end{aligned}
$$

That is, the Hom-categories are the additive subcategories of $\mathscr{C}_{\leq \mathcal{J}}$ generated by those 1-morphisms that either are in $\mathcal{J}$ or are identity 1-morphisms. If $\mathscr{C}$ is (quasi) multifiab, then so is $\mathscr{C}_{\mathcal{J}}$.

Let $\mathcal{J}$ be a two-sided cell. A *$\mathcal{H}$-cell* is any intersection of the form $\mathcal{L} \cap \mathcal{R}$, for $\mathcal{L} \subseteq \mathcal{J}$ a left cell and $\mathcal{R} \subseteq \mathcal{J}$ a right cell. Observe that if $\mathscr{C}$ is quasi multifiab, * swaps left cell structures with right cell structures. We therefore define, for any left cell $\mathcal{L} \subseteq \mathcal{J}$, the intersection

$$\mathcal{H}(\mathcal{L}) \coloneqq \mathcal{L} \cap \mathcal{L}^* \subseteq \mathcal{J}.$$

This is known as the *$\mathcal{H}$-cell associated to $\mathcal{L}$*. If $\mathscr{C}$ is multifiab, then every $\mathcal{H}(\mathcal{L})$ is stable under * and called a *diagonal $\mathcal{H}$-cell*. We are now ready to state [MMM+21, Theorem 4.32].

Theorem 4.8. (Strong $\mathcal{H}$-Reduction). *Let $\mathscr{C}$ be a fiab bicategory with a two-sided cell $\mathcal{J}$ and a diagonal $\mathcal{H}$-cell $\mathcal{H} \subset \mathcal{J}$. Then there is a biequivalence*

$$\mathscr{C}\text{-stmod}_{\mathcal{J}} \simeq \mathscr{C}_{\mathcal{H}}\text{-stmod}_{\mathcal{H}}.$$

The proof is ultimately not too important, so we will defer the reader to [MMM+21].

From now on, for the sake of convenience, let $\mathscr{S} \coloneqq \mathbb{S}\mathsf{Bim}$. We will also denote by $\mathscr{S}\text{-(g)stmod}_{\mathcal{J}}$ the 2-category of (graded) simple transitive 2-representations of $\mathscr{S}$ with apex $\mathcal{J}$. For any diagonal $\mathcal{H}$-cell $\mathcal{H} \subseteq \mathcal{J}$, strong $\mathcal{H}$-reduction allows us to write

$$\mathscr{S}\text{-(g)stmod}_{\mathcal{J}} \simeq \mathscr{S}_{\mathcal{H}}\text{-(g)stmod}_{\mathcal{H}}.$$

This makes the classification significantly easier, as $\mathscr{S}_{\mathcal{H}}$ turns out to be a great deal smaller than $\mathscr{S}$.

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5. Lusztig–Vogan Module Categories

This chapter will aim to summarize the recent work of Larson and Romanov in their Soergel bimodule approach for algebraically categorifying the trivial block of the Lusztig–Vogan module ([LR22]), which provides interesting examples of module categories over the category of Soergel bimodules. In the most general setting, the construction of a Lusztig–Vogan module category takes as ingredients a connected, complex, reductive algebraic group G, a Borel subgroup B of G, a holomorphic involution θ of G and a finite index subgroup K of the fixed-point subgroup $G^\theta \coloneqq \{g \in G : \theta(g) = g\}$. We will also make the additional assumption that K is the identity component of G^θ (see [LR22] for details).

Let $P \coloneqq \mathrm{Sym}(\mathrm{span}_{\Bbbk}(X(T_K)))$ be the symmetric algebra on the $\Bbbk$-span of the character lattice $X(T_K) \coloneqq \mathrm{Hom}(T_K, \mathbb{C}^\times)$ of a maximal torus T_K of K contained in $B_K \coloneqq B \cap K$, which we grade in degree 2, and similarly let $R \coloneqq \mathrm{Sym}(\mathrm{span}_{\Bbbk}(X(T)))$ be the symmetric algebra on the $\Bbbk$-span of the character lattice of the unique maximal, θ-stable torus $T \coloneqq Z_G(T_K)$ of G containing T_K (see [LR22, Lemma 6.1.2]), which we once again grade in degree 2. Denote by $W \coloneqq N_G(T)/T$ the Weyl group of G corresponding to T, by W^θ the set of elements of W fixed under the involution induced by θ and by $W_K \coloneqq N_K(T_K)/T_K$ the Weyl group of K corresponding to T_K, where we write $P^{W_K} \coloneqq \{p \in P : wp = p \text{ for all } w \in W_K\}$. Note that $W_K \subseteq W^\theta \subseteq W$, and by choosing the set of simple roots in W corresponding to B we obtain a Coxeter system (W, S). Finally, let $\phi : R \to P$ be the algebra homomorphism extending the restriction map $X(T) \to X(T_K)$. For each $w \in W$, we define the *w-standard bimodule* P_w to be the (P^{W_K}, R)-bimodule given by P as a vector space with left action given by left multiplication and right action given by $p \cdot_w f \coloneqq p\phi(w(f))$ for all $p \in P_w$ and $f \in R$, where $w(f)$ denotes w acting on f via the reflection action given in Remark 3.12. Letting ${}^K W^\theta$ be any set of right coset representatives for $W_K \backslash W^\theta$ (which, by [LR22, Corollary 6.1.6], is in bijection with the set of closed K-orbits of the flag variety G/B), we define

$$\mathcal{N}^0_{LV} \coloneqq \langle P_w \otimes_R X : w \in {}^K W^\theta, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W, S)) \rangle_{\oplus, \ominus, (1)}$$

to be the category of (P^{W_K}, R)-bimodules generated by standard bimodules under the right action of Soergel bimodules and closed under direct sums, direct summands and gradning shifts. By [LR22, Theorem 1.3.1], this categorifies the trivial block of the associated module of Lusztig and Vogan.

To study this in full generality involves some deep results from Lie theory; thus for the time being we will restrict our attention to the case where the tori T_K and T are of equal rank (that is, where $T = T_K$ and hence $R = P$). In this situation the picture is much simpler. Let W be a Weyl group together with a choice S of simple roots and finite index subgroup $W_K \subseteq W$. We define a polynomial algebra $R \coloneqq \Bbbk[\alpha_s : s \in S]$, which we recall is isomorphic to the symmetric algebra of the vector space associated with the geometric representation of (W, S) given in Definition 3.11. We therefore have an action of W on R, given on generators $s \in S$ and formal variables $\alpha_t \in R$ by

$$s(\alpha_t) = \alpha_t + 2\cos\left(\frac{\pi}{m_{st}}\right)\alpha_s.$$

We define R^{W_K} to be the polynomials in R that are invariant under action by W_K. By [LR22, Remark 6.2.4], we have $W = W^\theta$ in this case. Letting ${}^K W$ be right coset representatives for $W_K \backslash W$,

$$\mathcal{N}^0_{LV} = \langle R_w \otimes_R X : w \in {}^K W, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W, S)) \rangle_{\oplus, \ominus, (1)},$$

where R_w is the w-standard (R^{W_K}, R)-bimdoule given by R as a vector space with the left action given by left multiplication and the right action given by $p \cdot_w f \coloneqq pw(f)$, for all $p \in R_w$ and $f \in R$.

Remark 5.1. Suppose G is a connected, complex, reductive algebraic group and let $\sigma : G \to G$ be an antiholomorphic involution of G. Then G^σ, the fixed-point subgroup of σ, has the structure of a real Lie group whose complexification is G. Moreover, σ' is G-conjugate to σ (that is, there exists some $h \in G$ for which $\sigma' = \mathrm{int}_h \circ \sigma \circ \mathrm{int}_h^{-1}$, where $\mathrm{int}_h : g \mapsto hgh^{-1}$ is known as the *inner automorphism associated to* h) if and only if $G^{\sigma'}$ and G^σ are isomorphic as real Lie groups. We will call such an isomorphism class a *real form* of G. Now, a classical result of Cartan tells us the following. First, a connected, complex algebraic group is reductive if and only if it is the complexification of a unique connected, compact, real Lie group ([Kam11, p. 34]); thus there is, up to G-conjugation, only one antiholomorphic involution σ_c whose fixed-point subgroup is compact. This is known as the *compact form* of G. Second, let Θ be any G-conjugacy class of holomorphic involutions of G. Then there exists some holomorphic involution $\theta \in \Theta$ that commutes with σ_c, and furthermore each G-conjugacy class of antiholomorphic involutions of G contains $\sigma = \theta \circ \sigma_c$ for some unique choice of initial Θ ([Ada14]). Putting everything together, we have bijections

$$\{\text{real forms of } G\} \longleftrightarrow \left\{\begin{matrix}\text{antiholomorphic} \\ \text{involutions of } G\end{matrix}\right\}/\!\sim \;\longleftrightarrow\; \left\{\begin{matrix}\text{holomorphic} \\ \text{involutions of } G\end{matrix}\right\}/\!\sim,$$

where the equivalence relations are given by conjugation by G. The holomorphic involution corresponding to a real form is known as the *Cartan involution* of that real form. An immediate corollary is that a real form is compact if and only if its Cartan involution is the identity. For any real form, we also have a diamond

$$\begin{tikzcd} & G \arrow[dl, dash, "\sigma"'] \arrow[dr, dash, "\theta"] & \\ G_{\mathbb{R}} \coloneqq G^\sigma \arrow[dr, dash, "\theta"'] & & K \coloneqq G^\theta \arrow[dl, dash, "\sigma"] \\ & K_{\mathbb{R}} \coloneqq G^{\sigma_c} & \end{tikzcd}$$

where $G_{\mathbb{R}}$ is a real Lie group, K is a complex Lie group and $K_{\mathbb{R}}$ is the maximal compact subgroup of $G_{\mathbb{R}}$. Note that K is not in general compact, although it *is* the complexification of a compact group.

Example 5.2. Let's work through an example with $G \coloneqq \mathrm{SL}(2, \mathbb{C})$. Recall that G admits two real forms: a compact form $\mathrm{SU}(2)$ and a split form $\mathrm{SL}(2, \mathbb{R})$. The former is the fixed-point subgroup of the antiholomorphic involution $\sigma_c : g \mapsto ((\overline{g})^T)^{-1}$, while the latter is the fixed-point subgroup of the antiholomorphic involution $\sigma_s : g \mapsto \overline{g}$. By Remark 5.1, the Cartan involution of $\mathrm{SU}(2)$ will be trivial; this is a bit boring, so let's look at $\mathrm{SL}(2, \mathbb{R})$ instead. It admits the Cartan involution $\theta' : g \mapsto (g^T)^{-1}$ with fixed-point subgroup

$$G^{\theta'} = \mathrm{SO}(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\}.$$

Note that this is homeomorphic to $\mathbb{C}^\times \coloneqq \mathbb{C}\backslash\{0\}$ and is hence a maximal torus for G as a complex algebraic group (that is, a subgroup that is maximal among subgroups homeomorphic to $(\mathbb{C}^\times)^{\oplus k}$), as there are no groups H for which $G^{\theta'} \subset H \subset G$. In fact, it is the complexification of $\mathrm{SO}(2, \mathbb{R})$, the maximal torus of $\mathrm{SL}(2, \mathbb{R})$ as a real Lie group (that is, the subgroup that is maximal among subgroups homeomorphic to $(S^1)^{\oplus k}$). Since $G^{\theta'}$ is connected, the identity component is the entire group. However, $G^{\theta'}$ is awkward to work with; therefore, instead of using $G^{\theta'}$, consider the following.

Suppose we conjugate θ' by the inner automorphism associated to the matrix
$$h \coloneqq \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} \in \mathrm{SL}(2,\mathbb{C});$$
that is, let $\theta \coloneqq \mathrm{int}_h \circ \theta' \circ \mathrm{int}_h^{-1}$. Realizing θ' as the inner automorphism of the matrix with 1 and -1 on its off-diagonals, we have
$$\begin{aligned}\theta\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right) &\coloneqq \frac{1}{4}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}\\ &= \begin{pmatrix}a & -b\\ -c & d\end{pmatrix}.\end{aligned}$$
This holomorphic involution admits the fixed-point subgroup
$$G^\theta = \left\{\begin{pmatrix}a & 0\\ 0 & a^{-1}\end{pmatrix} : a \in \mathbb{C}^\times\right\},$$
which is of course homeomorphic to $\mathbb{C}^\times$. Once again, since G^θ is connected, the identity component is just the entire group, and hence we will take $K \coloneqq G^\theta$. Once again, G^θ is a maximal torus, and is given by the intersection of the pair of opposite Borel groups
$$B \coloneqq \left\{\begin{pmatrix}a & b\\ 0 & a^{-1}\end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\} \qquad \text{and} \qquad B' \coloneqq \left\{\begin{pmatrix}a & 0\\ b & a^{-1}\end{pmatrix} : a \in \mathbb{C}^\times, b \in \mathbb{C}\right\}.$$
Therefore, let $T_K \coloneqq G^\theta$. The Weyl group corresponding to G is $W \coloneqq N_G(T)/T = S_2 = \{1, s\}$, since
$$N_G(T) = \{g \in G : gtg^{-1} \in T, \text{ for all } t \in T\} = T \sqcup \left\{\begin{pmatrix}0 & b\\ -b^{-1} & 0\end{pmatrix} : b \in \mathbb{C}^\times\right\},$$
whence it follows that the Weyl group corresponding to K is just $W_K \coloneqq N_K(T_K)/T_K = \{1\}$. The only choice of simple roots we can make for W is $S = \{s\}$, whence $R = \mathbb{R}[\alpha_s]$. The action of W on R is given by $1 : \alpha_s \mapsto \alpha_s$ and $s : \alpha_s \mapsto -\alpha_s$, giving us $R^{W_K} = R$ and an easy (R^{W_K}, R)-bimodule structure for R. Note that $W^\theta = W$, since for all $g \in N_G(T)$ we have that $\theta(g) \in gT$. This gives us a very explicit right module category $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ over $\mathbb{S}\mathsf{Bim}$! A standard exercise we can now do is to compute the Jordan–Hölder filtrations of the corresponding birepresentation M that maps $\bullet$ to $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ and $X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim})$ to functors $- \otimes_R X$.

Recall that the indecomposable objects in $\mathbb{S}\mathsf{Bim}$ are, up to grading shifts and isomorphisms, given by Bott–Samelson bimodules of the form $B_w \coloneqq R \otimes_{R^w} R(1)$, for $w \in W$. Because module products distribute over direct sums, the indecomposables of $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ will all be direct summands of $R_{w_1} \otimes_R B_{w_2}$, for $w_1 \in {}^K W$ and $w_2 \in W$; that is, products of generators of $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ with indecomposable Soergel bimodules. In our case with $\mathrm{SL}(2,\mathbb{R})$, we have four candidates:
$$\begin{aligned}R_1 \otimes_R B_1 &\cong (R \otimes_R R) \otimes_R R(1) \cong R, & R_1 \otimes_R B_s &\cong (R \otimes_R R) \otimes_{R^s} R(1) \cong B_s,\\ R_s \otimes_R B_1 &\cong (R_s \otimes_R R) \otimes_R R(1) \cong R_s, & R_s \otimes_R B_s &\cong (R_s \otimes_R R) \otimes_{R^s} R(1) \cong B_s,\end{aligned}$$
where $R_1 = R = B_1$. Note that $R_s \otimes_{R^s} R \cong R \otimes_{R^s} R$, since we can move all elements fixed by s through $\otimes_{R^s}$. Luckily, these are all indecomposable, so we are done. Just like categories of Soergel bimodules, our Lusztig–Vogan module categories are not multifinitary, as they have infinitely many isomorphism classes of indecomposable objects. That said, because module products again distribute over direct sums, we know our module category will be graded multifinitary.

Of course, we also know that $B_s \geq R, R_s$, since B_s is isomorphic to both $[\mathrm{M}(B_s)](R) = R \otimes_R B_s$ and $[\mathrm{M}(B_s)](R_s) = R_s \otimes_R B_s$; thus $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ admits the cell structure

$$\begin{tikzcd}[column sep=small, row sep=tiny] \boxed{R} \arrow[dr, dash] & \\ & \boxed{B_s} \\ \boxed{R_s} \arrow[ur, dash] & \end{tikzcd} .$$

This gives us two filtrations whose corresponding weak composition quotients will of course be equivalent, so we will just compute them for the filtration given by $Q_1 \coloneqq \{[B_s]\}$, $Q_2 \coloneqq \{[R_s], [B_s]\}$ and $Q_3 \coloneqq \{[R], [R_s], [B_s]\}$. These give rise to the additive $\mathbb{S}\mathsf{Bim}$-module subcategories

$$\mathcal{M}_{Q_1} = \langle B_s \rangle_{\oplus,(1)}, \qquad \mathcal{M}_{Q_2} = \langle R_s, B_s \rangle_{\oplus,(1)}, \qquad \mathcal{M}_{Q_3} = \langle R, R_s, B_s \rangle_{\oplus,(1)} \simeq \mathcal{N}^0_{LV},$$

as well as two-sided ideals $\mathcal{I}_{Q_i}$ of $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ generated by the identity morphisms in Q_i. Of course, one of the resulting quotients is trivial, so the subcategory $\mathcal{M}_{Q_1/Q_0} = \mathcal{M}_{Q_1}$ is already transitive. For the remaining transitive quotient subcategories, we observe that $\mathcal{I}_{Q_i}$ consists of all morphisms in $\mathcal{N}^0_{LV}(\mathrm{SL}(2,\mathbb{R}))$ that factor through objects in Q_i (with all other $\mathcal{I}_{Q_i}(X,Y)$ containing only the unique zero morphism). The idea here is that the quotient functor from $\mathcal{M}_{Q_2}$ to $\mathcal{M}_{Q_2/Q_1}$ sends B_s to 0, while the quotient functor from $\mathcal{M}_{Q_3}$ to $\mathcal{M}_{Q_3/Q_2}$ sends both R_s and B_s to 0. As it happens, the functor sending R_s to R defines an equivalence of module categories $\mathcal{M}_{Q_2/Q_1} \simeq \mathcal{M}_{Q_3/Q_2}$. It is easy to be fooled by the fact that $R_s \otimes_R R_s \cong R \sim 0$, but one must remember that our Lusztig–Vogan module categories are *not* monoidal, and thus we cannot tensor by R_s since it doesn't live in $\mathbb{S}\mathsf{Bim}$!

Think more about the morphisms in the weak composition quotients and put a remark here about how our weak composition quotients correspond to indecomposables in the trivial (principal?) block of category $\mathcal{O}$ for $\mathfrak{sl}(2,\mathbb{R})$ (or something like that).

Example 5.3. Let's try looking at $G \coloneqq \mathrm{SL}(3,\mathbb{C})$, a slightly more interesting example. It admits three real forms: a compact form $\mathrm{SU}(3)$, a split form $\mathrm{SL}(3,\mathbb{R})$ and a quasi-split form $\mathrm{SU}(2,1)$. The antiholomorphic involutions whose fixed-point subgroups are $\mathrm{SU}(3)$ and $\mathrm{SL}(3,\mathbb{R})$ are the same as before. However, we now have a third G-conjugacy class of antiholomorphic involutions represented by $\sigma_q \coloneqq \mathrm{int}_{I_{2,1}} \circ \sigma_c$, where $I_{2,1}$ is the signature matrix

$$I_{2,1} \coloneqq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This admits the Cartan involution $\theta \coloneqq \mathrm{int}_{I_{2,1}}$ with fixed-point subgroup

$$G^\theta = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix} : a,b,c,d,e \in \mathbb{C}, (ad-bc)e = 1 \right\} \cong \mathrm{GL}(2,\mathbb{C}).$$

Of course $\mathrm{GL}(2,\mathbb{C})$ is connected, so let's once more take $K \coloneqq G^\theta$. This time G^θ is not itself a maximal torus; however, like before, the subgroup of diagonal matrices with unit determinant is a maximal torus for both G and K, so we still live in the equal rank world. Denote this group by $T = T_K$. In order to compute the Weyl groups, we first observe that

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \frac{1}{ad-bc} \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b & 0 \\ -c & a & 0 \\ 0 & 0 & (ad-bc)^2 \end{pmatrix}.$$

Thus it follows that

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \frac{1}{ad-bc} \end{pmatrix}^{-1} \begin{pmatrix} j & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & \frac{1}{jk} \end{pmatrix} \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & \frac{1}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} adj-bck & bd(j-k) & 0 \\ ac(k-j) & adk-bcj & 0 \\ 0 & 0 & \frac{1}{jk} \end{pmatrix}.$$

In other words,

$$N_K(T_K) = T_K \sqcup \left\{ \begin{pmatrix} 0 & b & 0 \\ c & 0 & 0 \\ 0 & 0 & -\frac{1}{bc} \end{pmatrix} : b, c \in \mathbb{C}^\times \right\},$$

whence $W_K = S_2 = \langle s \rangle$. Meanwhile, $S_3 = \langle s, t : s^2 = t^2 = (st)^3 = 1 \rangle$ is the Weyl group corresponding to G, since the Weyl group for $\mathrm{SL}(n, \mathbb{C})$ is the symmetric group S_n. In particular, one can show that

$$W \coloneqq N_G(T)/T = \left\langle T, \left\{ \begin{pmatrix} 0 & b & 0 \\ c & 0 & 0 \\ 0 & 0 & -\frac{1}{bc} \end{pmatrix} : b, c \in \mathbb{C}^\times \right\}, \left\{ \begin{pmatrix} -\frac{1}{fh} & 0 & 0 \\ 0 & 0 & f \\ 0 & h & 0 \end{pmatrix} : f, h \in \mathbb{C}^\times \right\} \right\rangle.$$

Taking $S = \{s, t\}$, we have $R = \mathbb{R}[\alpha_s, \alpha_t]$. The action of W on R is therefore induced by

$$s : \alpha_s \mapsto -\alpha_s, \qquad s : \alpha_t \mapsto \alpha_s + \alpha_t, \qquad t : \alpha_s \mapsto \alpha_s + \alpha_t, \qquad t : \alpha_t \mapsto -\alpha_t.$$

Note again that $W^\theta = W$, as we would expect, while $R^{W_K} = R^s = \mathbb{R}[\alpha_s^2]$ is the graded subring of R consisting only of polynomials in α_s of even degree. Now that we have explicitly stated the ingredients for this Lusztig–Vogan module category, let's try to compute its Jordan–Hölder filtrations.

As before, $\mathcal{N}^0_{LV}(\mathrm{SU}(2,1))$ will be generated by the standard bimodules R_w, where w is a coset representative for $W_K \backslash W = \{W_K, W_K t, W_K ts\}$. We therefore know automatically that grading shifts of R, R_t and R_{ts} will be indecomposable; in particular, it follows from [LR22, Lemma 6.2.3] that $R_s \cong R_{ss} = R$, $R_{st} \cong R_{sst} = R_t$ and $R_{tst} \cong R_{stst} = R_{ts}$. Unfortunately, for $\mathcal{N}^0_{LV}(\mathrm{SU}(2,1))$, not every product of a generator with an indecomposable from $\mathbb{S}\mathsf{Bim}$ is indecomposable, so we need to look for direct summands. We claim that our indecomposables are, up to grading shifts,

$$R, \quad R_t, \quad R_{ts}, \quad B_t \cong R_t \otimes_R B_t, \quad R_t \otimes_R B_s \cong R_{ts} \otimes_R B_s \quad \text{and} \quad R^s \otimes_{R^W} R.$$

It can be shown that B_t is generated by $1 \otimes_{R^s} 1$, that $R_t \otimes_R B_s$ is generated by $1 \otimes_R 1 \otimes_R 1$ and that $R^s \otimes_{R^W} R$ is obtained from the product $B_s \otimes_R B_t \cong R \oplus (R^s \otimes_{R^W} R)(2)$ and generated by $1 \otimes_{R^W} 1$. To at least see why B_s is decomposable as an (R^s, R)-bimodule, recall that we have an (R^s, R^s)-bimodule isomorphism $R \cong R^s \oplus R^s(-2)$ by Lemma 3.22. By interpreting $B_s \coloneqq R \otimes_{R^s} R(1)$ as an (R^s, R)-bimodule, we can replace the R on the left with the (R^s, R^s)-bimodule $R^s \oplus R^s(-2)$, giving us the (R^s, R)-bimodule isomorphism

$$\begin{aligned} B_s \coloneqq R \otimes_{R^s} R(1) &\cong (R^s \oplus R^s(-2)) \otimes_{R^s} R(1) \cong (R^s \otimes_{R^s} R)(1) \oplus (R^s(-2) \otimes_{R^s} R)(1) \\ &\cong R(1) \oplus R(-1). \end{aligned}$$

Note that although B_s is decomposable in $\mathcal{N}^0_{LV}(\mathrm{SU}(2,1))$, it is still indecomposable in $\mathbb{S}\mathsf{Bim}$, so $R_t \otimes_R B_s$ being indecomposable is not a contradiction! Moving on, the resulting cell structure is

$$\begin{tikzcd}[column sep=small]
& \boxed{R_t} \arrow[dr, dash] & \\
\boxed{R, B_t} \arrow[ur, dash] \arrow[dr, dash] & & \boxed{R_t \otimes B_s, R_{ts}} \\
& \boxed{R^s \otimes_{R^W} R} \arrow[ur, dash] &
\end{tikzcd} .$$

Suppose we consider the filtration given by $Q_1 := \{[R_{ts}]\}$, $Q_2 := \{[R^s \otimes_{R^W} R], [R_{ts}]\}$, $Q_3 := \{[R_t], [R^s \otimes_{R^W} R], [R_{ts}]\}$ and $Q_4 := \{[R], [R_t], [R^s \otimes_{R^W} R], [R_{ts}]\}$. As before, these give us additive $\mathbb{S}\mathsf{Bim}$-module subcategories $\mathcal{M}_i$ generated by the indecomposables in Q_i and two-sided ideals $\mathcal{I}_{Q_i}$ generated by the identity morphisms in Q_i. The resulting transitive module categories are $\mathcal{M}_{Q_1} = \langle R_{ts} \rangle_{\oplus,(1)}$, $\mathcal{M}_{Q_2/Q_1} = \langle R^s \otimes_{R^W} R \rangle_{\oplus,(1)}$, $\mathcal{M}_{Q_3/Q_2} = \langle R_t \rangle_{\oplus,(1)}$ and $\mathcal{M}_{Q_4/Q_3} = \langle R \rangle_{\oplus,(1)}$. In order to understand these as $\mathbb{S}\mathsf{Bim}$-module categories, we need to understand how these generators behave when tensored with the indecomposables R, B_s, B_t, B_{st}, B_{ts} and B_{tst}.

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