

# CLASSIFICATION OF FUSION CATEGORIES

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## 1. PROLOGUE

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What are fusion categories? What are near-groups, Haagerup–Izumi categories and quadratic categories? What is modular data? What are  $6j$  symbols? What is the even part of a subfactor?

Let  $\mathcal{C}$  be a fusion category over  $\mathbb{k}$ . Recall the Yoneda embedding

$$\mathfrak{Y}_*(X) := \text{Hom}_{\mathcal{C}}(-, X) \quad \text{and} \quad \mathfrak{Y}_*(f : X \rightarrow Y) := (\text{Hom}_{\mathcal{C}}(-, X) \Rightarrow \text{Hom}_{\mathcal{C}}(-, Y)).$$

Taking the Yoneda embedding of a component  $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$  of the associativity natural isomorphism  $\alpha$ , we obtain a natural isomorphism

$$\mathfrak{Y}_*(\alpha_{X,Y,Z}) = (\text{Hom}_{\mathcal{C}}(-, (X \otimes Y) \otimes Z) \Rightarrow \text{Hom}_{\mathcal{C}}(-, X \otimes (Y \otimes Z))).$$

Thus we have an isomorphism of vector spaces

$$[\mathfrak{Y}_*(\alpha_{X,Y,Z})](W) : \text{Hom}_{\mathcal{C}}(W, (X \otimes Y) \otimes Z) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}}(W, X \otimes (Y \otimes Z));$$

that is, an invertible matrix. In other words, the associativity is given by matrices indexed by  $X, Y, Z, W$ . But we can simplify things further! Suppose we now choose representatives  $\{X_i\}_{i \in \Gamma}$  of isomorphism classes of simple objects and bases for each *multiplicity space*  $H_{i,j}^h := \text{Hom}_{\mathcal{C}}(X_h, X_i \otimes X_j)$ . Then by semisimplicity,

$$X_i \otimes X_j \cong \bigoplus_{h \in \Gamma} X_h^{\dim_{\mathbb{k}}(H_{i,j}^h)},$$

whence we have both

$$\begin{aligned} \bigoplus_{m \in \Gamma} H_{m,i_3}^{i_0} \otimes_{\mathbb{k}} H_{i_1,i_2}^m &= \bigoplus_{m \in \Gamma} \text{Hom}_{\mathcal{C}}(X_{i_0}, X_m \otimes X_{i_3}) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(X_m, X_{i_1} \otimes X_{i_2}) \\ &\cong \text{Hom}_{\mathcal{C}}(X_{i_0}, (X_{i_1} \otimes X_{i_2}) \otimes X_{i_3}) \end{aligned}$$

and

$$\begin{aligned} \bigoplus_{n \in \Gamma} H_{i_1,n}^{i_0} \otimes_{\mathbb{k}} H_{i_2,i_3}^n &= \bigoplus_{n \in \Gamma} \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes X_n) \otimes_{\mathbb{k}} \text{Hom}_{\mathcal{C}}(X_n, X_{i_2} \otimes X_{i_3}) \\ &\cong \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes (X_{i_2} \otimes X_{i_3})). \end{aligned}$$

We thus obtain canonical bases for the above Hom-spaces. The (*quantum*) *6j-symbols* of  $\mathcal{C}$  are then defined to be the matrix blocks of the form

$$\Phi_{i_1,i_2,i_3}^{i_0,m,n} : H_{m,i_3}^{i_0} \otimes_{\mathbb{k}} H_{i_1,i_2}^m \rightarrow H_{i_1,n}^{i_0} \otimes_{\mathbb{k}} H_{i_2,i_3}^n$$

such that the block-diagonal matrices  $\Phi_{i_1,i_2,i_3}^{i_0} := \bigoplus_{m \in \Gamma} \bigoplus_{n \in \Gamma} \Phi_{i_1,i_2,i_3}^{i_0,m,n}$  give a change-of-basis

$$\Phi_{i_1,i_2,i_3}^{i_0} : \text{Hom}_{\mathcal{C}}(X_{i_0}, (X_{i_1} \otimes X_{i_2}) \otimes X_{i_3}) \cong \text{Hom}_{\mathcal{C}}(X_{i_0}, X_{i_1} \otimes (X_{i_2} \otimes X_{i_3}))$$

between the canonical bases. These matrices  $\Phi_{i_1,i_2,i_3}^{i_0}$  themselves form the blocks of a block-diagonal matrix  $\Phi_{i_1,i_2,i_3} := \bigoplus_{i_0 \in \Gamma} \Phi_{i_1,i_2,i_3}^{i_0}$ , which gives us a change-of-basis  $(X_{i_1} \otimes X_{i_2}) \otimes X_{i_3} \cong X_{i_1} \otimes (X_{i_2} \otimes X_{i_3})$  describing how the associator acts on each summand  $X_{i_0}$ . As it happens, we in fact have that

$$\Phi_{i_1,i_2,i_3}^{i_0,m,n} = v_n^{-1}(X_{i_0}, X_{i_1}, X_{i_2} \otimes X_{i_3}) \circ [\mathfrak{Y}_*(\alpha_{X_{i_1}, X_{i_2}, X_{i_3}})](X_{i_0}) \circ u_m(X_{i_0}, X_{i_1} \otimes X_{i_2}, X_{i_3}),$$

where

$$[u_m(W, U, V)](f \otimes_{\mathbb{k}} g) := (g \otimes \text{id}_V) \circ f \quad \text{and} \quad [v_n(W, U, V)](f' \otimes_{\mathbb{k}} g') := (\text{id}_U \otimes g') \circ f'$$

for all  $f \in \text{Hom}_{\mathcal{C}}(W, X_m \otimes V)$ ,  $g \in \text{Hom}_{\mathcal{C}}(X_m, U)$ ,  $f' \in \text{Hom}_{\mathcal{C}}(W, U \otimes X_n)$  and  $g' \in \text{Hom}_{\mathcal{C}}(X_n, V)$ .

More explicitly, suppose we have an isomorphism

$$\text{Hom}_{\mathcal{C}}(X_4, (X_1 \otimes X_2) \otimes X_3) \cong \text{Hom}_{\mathcal{C}}(X_4, X_1 \otimes (X_2 \otimes X_3)).$$

Let  $X_5$  and  $X_6$  be simple summands of  $(X_1 \otimes X_2)$  and  $(X_2 \otimes X_3)$ , respectively. Then we can determine our isomorphism by determining the block matrices of the form

$$\text{Hom}_{\mathcal{C}}(X_4, X_5 \otimes X_3) \rightarrow \text{Hom}_{\mathcal{C}}(X_4, X_1 \otimes X_6)$$

for all such  $X_5$  and  $X_6$ . These matrices are exactly the  $6j$  symbols, where the six simple objects  $X_1, X_2, X_3, X_4, X_5, X_6$  play the role of the eponymous “six  $j$ ’s”. This is also where  $u$  and  $v$  come in; given  $f \in H_{5,3}^4$  and  $g \in H_{1,2}^5$ , we have  $[u_5(X_4, X_1 \otimes X_2, X_3)](f \otimes_{\mathbb{k}} g) : X_4 \rightarrow (X_1 \otimes X_2) \otimes X_3$ , and similarly  $[v_6(X_4, X_1, X_2 \otimes X_3)](f' \otimes_{\mathbb{k}} g') : X_4 \rightarrow X_1 \otimes (X_2 \otimes X_3)$  for  $f' \in H_{1,6}^4$  and  $g' \in H_{2,3}^6$ .

**Example 1.1.** ( $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ ). Consider  $\text{Vec}_{\mathbb{Z}/2\mathbb{Z}}$ , a category with two isomorphism classes of simple objects –  $[\mathbb{1}]$  and  $[X]$  – satisfying the fusion rule

$$[X] \otimes [X] = [\mathbb{1}].$$

Let’s assume that it’s skeletal – that is,  $X \otimes X = \mathbb{1}$  – and start by computing the change-of-basis matrix  $\Phi_{X,X,X}$ . This matrix can be written in block diagonal form as

$$\Phi_{X,X,X} = \begin{pmatrix} \Phi_{X,X,X}^{\mathbb{1}} & 0 \\ 0 & \Phi_{X,X,X}^X \end{pmatrix}.$$

First, observe that

$$\Phi_{X,X,X}^{\mathbb{1}} : \text{Hom}(\mathbb{1}, (X \otimes X) \otimes X) \rightarrow \text{Hom}(\mathbb{1}, X \otimes (X \otimes X));$$

but  $(X \otimes X) \otimes X = X = X \otimes (X \otimes X)$ , so  $\Phi_{X,X,X}^{\mathbb{1}}$  is just the isomorphism of 0-dimensional vector spaces; that is,  $\Phi_{X,X,X}^{\mathbb{1}} = 0$ . Let’s now compute  $\Phi_{X,X,X}^X$ . We shall start by choosing bases for the multiplicity spaces  $H_{X,X}^{\mathbb{1}}, H_{\mathbb{1},X}^X$  and  $H_{X,\mathbb{1}}^X$ . These spaces are all 1-dimensional, so we will choose basis elements  $\iota_{X,X}^{\mathbb{1}}, \lambda_X^{-1}$  and  $\rho_X^{-1}$ , respectively. Here we have a canonical choice of basis elements for  $H_{\mathbb{1},X}^X$  and  $H_{X,\mathbb{1}}^X$ ; namely, the left and right unitors  $\lambda$  and  $\rho$ . This culminates in the basis elements

$$\iota_{(X \otimes X) \otimes X}^X := (\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_X) \circ \lambda_X^{-1} = [u_{\mathbb{1}}(X, X \otimes X, X)](\lambda_X^{-1} \otimes_{\mathbb{k}} \iota_{X,X}^{\mathbb{1}})$$

and

$$\iota_{X \otimes (X \otimes X)}^X := (\text{id}_X \otimes \iota_{X,X}^{\mathbb{1}}) \circ \rho_X^{-1} = [v_{\mathbb{1}}(X, X, X \otimes X)](\rho_X^{-1} \otimes_{\mathbb{k}} \iota_{X,X}^{\mathbb{1}})$$

for  $\text{Hom}(X, (X \otimes X) \otimes X)$  and  $\text{Hom}(X, X \otimes (X \otimes X))$ , respectively. These are once again both 1-dimensional spaces, so let’s define a constant  $\mu_1 \in \mathbb{k}^\times$  for which

$$\Phi_{X,X,X}^X(\iota_{(X \otimes X) \otimes X}^X) =: \mu_1 \cdot \iota_{X \otimes (X \otimes X)}^X.$$

In order to gain constraints on  $\mu_1$ , we need to use the pentagon equation. To this end, we first choose bases for the Hom-spaces

$$\begin{aligned} &\text{Hom}(\mathbb{1}, (\mathbb{1} \otimes X) \otimes X), \\ &\text{Hom}(\mathbb{1}, \mathbb{1} \otimes (X \otimes X)), \\ &\text{Hom}(\mathbb{1}, (X \otimes \mathbb{1}) \otimes X), \\ &\text{Hom}(\mathbb{1}, X \otimes (\mathbb{1} \otimes X)), \\ &\text{Hom}(\mathbb{1}, (X \otimes X) \otimes \mathbb{1}), \\ &\text{Hom}(\mathbb{1}, X \otimes (X \otimes \mathbb{1})). \end{aligned}$$

We can write bases for these Hom-spaces in terms of the bases we chose previously. That is,

$$\begin{aligned}\iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}} &:= (\lambda_X^{-1} \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}}, \\ \iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}} &:= (\text{id}_{\mathbb{1}} \otimes \iota_{X,X}^{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1}, \\ \iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}} &:= (\rho_X^{-1} \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}}, \\ \iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}} &:= (\text{id}_X \otimes \lambda_X^{-1}) \circ \iota_{X,X}^{\mathbb{1}}, \\ \iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}} &:= (\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1}, \\ \iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}} &:= (\text{id}_X \otimes \rho_X^{-1}) \circ \iota_{X,X}^{\mathbb{1}}.\end{aligned}$$

This culminates in the three new constants given by

$$\begin{aligned}\Phi_{\mathbb{1}, X, X}^{\mathbb{1}}(\iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}}) &=: \mu_2 \cdot \iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}}, \\ \Phi_{X, \mathbb{1}, X}^{\mathbb{1}}(\iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}}) &=: \mu_3 \cdot \iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}}, \\ \Phi_{X, X, \mathbb{1}}^{\mathbb{1}}(\iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}}) &=: \mu_4 \cdot \iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}}.\end{aligned}$$

The final bases we need are for the Hom-spaces

$$\begin{aligned}\text{Hom}(\mathbb{1}, ((X \otimes X) \otimes X)), \\ \text{Hom}(\mathbb{1}, (X \otimes (X \otimes X)) \otimes X), \\ \text{Hom}(\mathbb{1}, (X \otimes X) \otimes (X \otimes X)), \\ \text{Hom}(\mathbb{1}, X \otimes ((X \otimes X) \otimes X)), \\ \text{Hom}(\mathbb{1}, X \otimes (X \otimes (X \otimes X))).\end{aligned}$$

Following the same procedure as before, we have

$$\begin{aligned}\iota_{((X \otimes X) \otimes X) \otimes X}^{\mathbb{1}} &:= (\iota_{(X \otimes X) \otimes X}^X \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}} = ((\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_X) \otimes \text{id}_X) \circ \iota_{(\mathbb{1} \otimes X) \otimes X}^{\mathbb{1}}, \\ \iota_{(X \otimes (X \otimes X)) \otimes X}^{\mathbb{1}} &:= (\iota_{X \otimes (X \otimes X)}^X \otimes \text{id}_X) \circ \iota_{X,X}^{\mathbb{1}} = ((\text{id}_X \otimes \iota_{X,X}^{\mathbb{1}}) \otimes \text{id}_X) \circ \iota_{(X \otimes \mathbb{1}) \otimes X}^{\mathbb{1}}, \\ \iota_{(X \otimes X) \otimes (X \otimes X)}^{\mathbb{1}} &:= (\iota_{X,X}^{\mathbb{1}} \otimes (\text{id}_X \otimes \text{id}_X)) \circ \iota_{\mathbb{1} \otimes (X \otimes X)}^{\mathbb{1}} = ((\text{id}_X \otimes \text{id}_X) \otimes \iota_{X,X}^{\mathbb{1}}) \circ \iota_{(X \otimes X) \otimes \mathbb{1}}^{\mathbb{1}}, \\ \iota_{X \otimes ((X \otimes X) \otimes X)}^{\mathbb{1}} &:= (\text{id}_X \otimes \iota_{(X \otimes X) \otimes X}^X) \circ \iota_{X,X}^{\mathbb{1}} = (\text{id}_X \otimes (\iota_{X,X}^{\mathbb{1}} \otimes \text{id}_X)) \circ \iota_{X \otimes (\mathbb{1} \otimes X)}^{\mathbb{1}}, \\ \iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}} &:= (\text{id}_X \otimes \iota_{X \otimes (X \otimes X)}^X) \circ \iota_{X,X}^{\mathbb{1}} = (\text{id}_X \otimes (\text{id}_X \otimes \iota_{X,X}^{\mathbb{1}})) \circ \iota_{X \otimes (X \otimes \mathbb{1})}^{\mathbb{1}}.\end{aligned}$$

The pentagon diagram thus gives us

$$\begin{array}{ccccc} & & \iota_{((X \otimes X) \otimes X) \otimes X}^{\mathbb{1}} & & \\ & \swarrow \alpha_{X,X,X \otimes \text{id}_X} & & \searrow \alpha_{X \otimes X,X,X} & \\ \mu_1 \cdot \iota_{(X \otimes (X \otimes X)) \otimes X}^{\mathbb{1}} & & & & \mu_2 \cdot \iota_{(X \otimes X) \otimes (X \otimes X)}^{\mathbb{1}} \\ \downarrow \alpha_{X,X \otimes X,X} & & & & \downarrow \alpha_{X,X,X \otimes X} \\ \mu_1 \mu_3 \cdot \iota_{X \otimes ((X \otimes X) \otimes X)}^{\mathbb{1}} & \xrightarrow[\text{id}_X \otimes \alpha_{X,X,X}]{} & \mu_1^2 \mu_3 \cdot \iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}} & \xlongequal{\quad} & \mu_2 \mu_4 \cdot \iota_{X \otimes (X \otimes (X \otimes X))}^{\mathbb{1}} \end{array}.$$

This gives us the relation  $\mu_1^2\mu_3 = \mu_2\mu_4$ . However, there could be more constraints on these constants! As it turns out, though, there is a simple way of determining these. Suppose we use the more insightful labeling convention

$$\begin{aligned}\omega(X, X, X) &:= \mu_1, \\ \omega(\mathbb{1}, X, X) &:= \mu_2, \\ \omega(X, \mathbb{1}, X) &:= \mu_3, \\ \omega(X, X, \mathbb{1}) &:= \mu_4.\end{aligned}$$

The reason why we have chosen this relabeling is because now acting by  $\alpha_{A,B,C}$  in the pentagon diagram introduces a factor of  $\omega(A, B, C)$ . If we apply this to the general pentagon diagram

$$\begin{array}{ccccc} & & ((A \otimes B) \otimes C) \otimes D & & \\ & \swarrow \alpha_{A,B,C} \otimes \text{id}_D & & \searrow \alpha_{A \otimes B, C, D} & \\ (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D), \\ \downarrow \alpha_{A,B \otimes C, D} & & & & \downarrow \alpha_{A,B,C \otimes D} \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\text{id}_A \otimes \alpha_{B,C,D}} & & & A \otimes (B \otimes (C \otimes D)) \end{array}$$

we obtain the constraints

$$\omega(A, B, C)\omega(A, BC, D)\omega(B, C, D) = \omega(AB, C, D)\omega(A, B, CD)$$

for all  $A, B, C, D \in \text{Ob}(\text{Vec}_{\mathbb{Z}/2\mathbb{Z}})$ . This is exactly the definition for a 3-cocycle  $\omega$ ! In general, the possible  $6j$ -symbols – and hence the possible associators – for  $\text{Vec}_G$  are exactly given by 3-cocycles. We will henceforth denote by  $\text{Vec}_G^\omega$  the category of  $G$ -graded vector spaces with associativity constraint  $\alpha_{ghk} = \omega(g, h, k)\text{id}_{ghk}$ , for all  $g, h, k \in G$ , and  $\text{Vec}_G$  the category with trivial associativity.

## 2. THE CUNTZ ALGEBRA APPROACH OF IZUMI

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Consider the category  $\text{End}(M)$ , for  $M$  a hyperfinite type III factor. This category is strict, as  $\rho \otimes \sigma := \rho \circ \sigma$  by definition. Every near-group category with group  $G$  contains some copy of  $\text{Vec}_G^\omega$  corresponding to the group-like part. Because every unitary near-group category is a subcategory of  $\text{End}(M)$  and is hence itself strict, we know that it will actually contain the “strictification” of some  $\text{Vec}_G^\omega$ . However, Izumi shows that if  $\mathcal{C}$  is any fusion category containing a simple object that is fixed under tensor products with invertibles (that is, there exists some simple object  $X$  such that  $X \otimes g \cong X$  for all invertible  $g$ ), then it contains a copy of  $\text{Vec}_G$ , for  $G$  the group of isomorphism classes of invertible objects. He shows in addition that if the fusion category is also unitary, then  $g \otimes X = X$  (but we may not necessarily have that  $X \otimes g = X$ ). The upshot is that we almost know how objects are tensored, since the group-like part will have trivial associativity (that is,  $g \otimes h = gh$ ). We just need to understand  $X \otimes g$  and  $X \otimes X$ , as well as the morphisms.

In [Izu17], Izumi showed that every unitary near-group category  $\mathcal{C}$  with multiplicity  $m$  is equivalent to a subcategory of  $\text{End}(M)$ , where  $M$  is the hyperfinite type III<sub>1</sub> factor. In particular, it is generated by a single irreducible endomorphism  $\rho \in \text{End}_0(M)$  satisfying the fusion rules

$$\begin{aligned} [\rho] \otimes [\rho] &= \bigoplus_{g \in G} [\alpha_g] \oplus [\rho]^{\oplus m}, \\ [\alpha_g] \otimes [\alpha_h] &= [\alpha_{gh}], \\ [\alpha_g] \otimes [\rho] &= [\rho] \otimes [\alpha_g] = [\rho], \end{aligned}$$

where the map  $\alpha : G \rightarrow \text{Aut}(M)$  induces an injective homomorphism from  $G$  into  $\text{Out}(M)$ .

The main result of [Izu17] is [Izu17, Theorem 4.9]. Essentially, there is a bijective correspondence between the set of equivalence classes of unitary near-group categories with finite group  $G$  and multiplicity parameter  $m$  and the set of equivalence classes of admissible tuples  $(\mathcal{K}, j_1, j_2, V, U_\mathcal{K}, \chi, l)$  (see [Izu17, Definition 4.8]). Here  $\mathcal{K}$  is the finite-dimensional Hilbert space  $\text{Hom}(\rho, \rho^2)$ ,  $j_1$  and  $j_2$  are two antilinear isometries of  $\mathcal{K}$ ,  $V$  and  $U_\mathcal{K}$  are unitary representations of  $G$  on  $\mathcal{K}$ ,  $\{\chi_g\}_{g \in G}$  are characters of  $G$  and  $l$  is a linear map from  $\mathcal{K}$  to the set  $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{K})$  of bounded operators  $\mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{K}$ .

By [Izu17, Theorem 9.1], the unitary near-group categories with finite Abelian group  $G$  and  $m = |G|$  are completely classified tuples of the form  $(\langle \cdot, \cdot \rangle, a, b, c)$ , where  $\langle \cdot, \cdot \rangle : G \times G \rightarrow \mathbb{T}$  is a non-degenerate symmetric bicharacter and where  $a : G \rightarrow \mathbb{T}$ ,  $b : G \rightarrow \mathbb{T}$  and  $c \in \mathbb{T}$  satisfy various conditions. When we say that  $\langle \cdot, \cdot \rangle$  is a bicharacter, we mean that

$$\langle xy, z \rangle = \langle x, z \rangle \langle y, z \rangle \quad \text{and} \quad \langle x, yz \rangle = \langle x, y \rangle \langle x, z \rangle$$

for all  $x, y, z \in G$ . By non-degenerate, we mean that

$$\langle x, \cdot \rangle = \langle y, \cdot \rangle$$

if and only if  $x = y$ . This is equivalent to the map  $\varphi : G \rightarrow \text{Hom}(G, \mathbb{T})$  given by  $x \mapsto \langle x, \cdot \rangle$  being an isomorphism.

**Definition 2.1.** (Cuntz Algebra). *Let  $\{S_i\}_{i=1}^n$  be a set of isometries on an infinite-dimensional Hilbert space  $\mathcal{H}$ . Suppose moreover that these isometries satisfy the Cuntz relation*

$$\sum_{k=1}^n S_k S_k^* = 1.$$

*The Cuntz algebra  $\mathcal{O}_n$  is the universal  $C^*$ -algebra  $C^*(S_1, \dots, S_n)$ .*

**Remark 2.2.** Note that, as isometries,  $S_i^* S_i = 1$ . In particular, we must have that  $S_i^* S_j = \delta_{i,j}$  for all  $i, j \in \{1, \dots, n\}$ . This follows from the fact that a sum of projections is itself a projection if and only if the projections in the sum are pairwise orthogonal. The Cuntz relation is essentially ensuring that the sum of the projections  $S_i S_i^*$  is the trivial projection.

**Example 2.3.** (Fibonacci Category). Let's look at the Fibonacci category. This is the near-group with  $G = \{0\}$  and  $m = 1$ . Our choice for  $\langle \cdot, \cdot \rangle$  is obvious, and [Izu17, Lemma 7.1] tells us that

$$c^3 a(0) = \sqrt{n} = 1 \implies a(0) = c^{-3}.$$

Moreover, [Izu17, Theorem 9.1] tells us that  $b$  is defined by  $b : 0 \mapsto -1/d$ , where  $d$  corresponds to the dimension of our irreducible generator  $\rho$ . Let's determine  $c$  and  $d$ . Because  $b$  is equal to its own Fourier transform, [Izu17, Theorem 9.1] tells us that

$$b(0) = ca(0)b(0) \implies a(0) = c^{-1}.$$

In order for  $c^{-1} = c^{-3}$ , we require  $c = \pm 1$ . Finally, [Izu17, Equation 9.5] tells us that

$$\begin{aligned} b(0)b(0)b(0) &= b(0)b(0) \mp \frac{1}{d}, \\ &\implies -\frac{1}{d^3} = \frac{1}{d^2} \mp \frac{1}{d}, \\ &\implies \pm d^2 - d - 1 = 0. \end{aligned}$$

This only has a real solution when  $c = 1$ , whence  $d$  is nothing but the golden ratio (as it cannot be negative in the unitary case - this alternative solution, known as the Galois dual, corresponds to a non-unitary near-group in this case and many others). This is exactly what we would expect, as  $d$  is the dimension of  $X$  (where  $d^2 = 1 + d$  comes from the fusion rule  $X^2 = \mathbb{1} \oplus X$ ).

**Example 2.4.** ( $G = \mathbb{Z}/2\mathbb{Z}$ ). Let's look at the case where  $G = \mathbb{Z}/2\mathbb{Z}$  and  $m = 2$ . This near-group corresponds to the even part of the type  $A_4$  subfactor. We know the dimension is

$$d_{\pm} := \frac{m \pm \sqrt{m^2 + 4n}}{2} = 1 \pm \sqrt{3}.$$

In the unitary setting, we of course ask that  $d$  be positive, and hence we choose  $d = d_+$ . The only possibility for a non-degenerate bicharacter is

$$\langle 0, 0 \rangle = 1, \quad \langle 0, 1 \rangle = \langle 1, 0 \rangle = 1 \quad \text{and} \quad \langle 1, 1 \rangle = -1.$$

From [Izu17, Equation 7.8], it follows that

$$a(0) = 1 \quad \text{and} \quad a(1) = \pm i.$$

Meanwhile, [Izu17, Equation 9.4] tells us that

$$\overline{b(1)} = \pm ib(1) \implies \Re(b(1)) = \mp \Im(b(1)),$$

whence [Izu17, Equation 9.3] gives us

$$\Re(b(1))^2 + \Im(b(1))^2 = (b(1)\overline{b(1)})^2 = \frac{1}{2} \implies b(1) = \frac{1 - a(1)}{2}.$$

It then follows from evaluating [Izu17, Equation 9.1] with  $g = 0$  and rearranging for  $c$  that

$$c = \frac{1 - \sqrt{3} + a(1)(1 + \sqrt{3})}{2\sqrt{2}}.$$

Note that we may choose either  $a(1) = i$  or  $a(1) = -i$ ; both of these lead to solutions. Moreover, in the non-unitary setting, we may take the Galois conjugate of  $d$ .

**Example 2.5.** ( $G = \mathbb{Z}/2\mathbb{Z}$ ). Let's determine the Haagerup–Izumi categories with  $G = \mathbb{Z}/2\mathbb{Z}$ . Let

$$d_{\pm} := \frac{n \pm \sqrt{n^2 + 4}}{2},$$

where in this example  $d := 1 + \sqrt{2}$ . Izumi's classification involves a triplet  $(\epsilon_h(g), \omega(g), A_{h,k}(g))$ , where  $\epsilon_h(g) \in \{-1, 1\}$ ,  $\omega(g) \in \mathbb{T}$  and  $A_{h,k}(g) \in \mathbb{C}$  satisfy [Izu18, Equations 4.1–4.9]. Well, we know

$$\epsilon_0(0) = \epsilon_1(0) = 1 \quad \text{and} \quad \epsilon_0(1) = \epsilon_0(1)\epsilon_0(1) \implies \epsilon_0(1) = 1.$$

By [Izu18, Equation 4.7],

$$A_{0,0}(g) = A_{0,0}(g)\omega(g),$$

which tells us that either  $\omega(g) = 1$  or  $A_{0,0}(g) = 0$  for each  $g \in G$ . Let's fix any  $g \in G$  and consider the case when  $A_{0,0}(g) = 0$ . In this case, however, [Izu18, Equations 4.3 and 4.4] give us

$$A_{1,0}(g)\overline{A_{\delta_{g,0}-g,0}(g)} = 1 - \frac{|\omega(g)|}{d} \implies \left| \frac{1}{d} \right| = 1 - \frac{1}{d}.$$

This “equality” is nonsense; we must therefore have  $\omega(g) = 1$  for all  $g \in G$ . Suppose now that  $\epsilon_1(1) = 1$ . Then [Izu18, Equation 4.7] gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \quad \text{and} \quad A_{0,1}(1) = A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us  $A_{1,1}(0) = A_{1,1}(1)$ . Now, [Izu18, Equations 4.4 and 4.6] tell us

$$A_{0,1}(0)A_{1,1}(1) + A_{1,1}(0)A_{1,0}(1) = 0.$$

Thus  $A_{0,1}(g) = A_{1,1}(g) = A_{1,0}(g) = 0$  and hence  $A_{0,0}(g) = -1/d$  by [Izu18, Equation 4.3]. However, in this case we cannot satisfy [Izu18, Equation 4.9]. Suppose instead that  $\epsilon_1(1) = -1$ . With this new 2-cocycle, [Izu18, Equation 4.7] now gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \quad \text{and} \quad A_{0,1}(1) = -A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us  $A_{1,1}(1) = -A_{1,1}(0)$ . We then see by [Izu18, Equation 4.4] that

$$A_{0,1}(0)A_{1,0}(0) + A_{1,1}(0)A_{1,1}(0) = 1 \implies A_{1,0}(0) = \pm \frac{1}{\sqrt{2}} = \pm \frac{1}{d-1},$$

and by [Izu18, Equation 4.9] that

$$A_{0,0}(0)A_{1,0}(0)^2 = A_{1,0}(0)^2 + A_{1,0}(0)^3 \implies A_{0,0}(0) = 1 + A_{1,0}(0) = \frac{d-1 \pm 1}{d-1}.$$

Finally, [Izu18, Equation 4.3] allows us to deduce

$$A_{1,0}(0) = -\frac{1}{d-1},$$

whence

$$A(0) = \frac{1}{d-1} \begin{pmatrix} d-2 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad A(1) = \frac{1}{d-1} \begin{pmatrix} d-2 & -1 \\ -1 & 1 \end{pmatrix}.$$

This category is nothing but the even part of the type  $A_7$  subfactor.

**Remark 2.6.** Suppose that  $|G|$  is odd. Then [Izu18, Equation 4.1] tells us that  $\epsilon_h(g) = 1$ , while [Izu18, Equation 4.2] tells us that  $\omega(g)$  does not depend on  $g$ . Moreover,  $A_{h,k}(g)$  cannot depend on  $g$  by [Izu18, Equation 4.5], and either  $\omega = 1$  or  $A_{0,0} = 0$  by [Izu18, Equation 4.7]. In this case, [Izu18, Equations 4.1–4.9] reduce to the following four equations.

$$\begin{aligned} A_{h,k} &= A_{-k,h-k}\omega = A_{k-h,-h}\bar{\omega}, \\ \sum_{h \in G} A_{h,0} &= -\frac{\bar{\omega}}{d_{\pm}}, \\ \sum_{h \in G} A_{h-g,k} A_{k,h-g'} &= \delta_{g,g'} - \frac{\delta_{k,0}}{d_{\pm}}, \\ \sum_{l \in G} A_{x+y,l} A_{-x,l+p} A_{-y,l+q} &= A_{p+x,q+x+y} A_{q+y,p+x+y} - \frac{\delta_{x,0}\delta_{y,0}}{d_{\pm}}. \end{aligned}$$

The first three equations above are precisely [EG17, Equations 4.7, 4.8 and 4.9]! In particular, to see that our third equation is equivalent to [EG17, Equation 4.9], we simply make the change of variables  $\hat{g} := g' - g$  and  $\hat{h} := h - g'$ , whence we obtain

$$\sum_{\hat{h} \in G} A_{\hat{h}+\hat{g},k} A_{k,\hat{h}} = \delta_{\hat{g},0} - \frac{\delta_{k,0}}{d_{\pm}}.$$

Similarly, using our first equation while making the change of variables  $\hat{l} := l - x - y$ ,  $\hat{p} := p + x + y$ ,  $\hat{q} := q + x + y$ ,  $\hat{x} := -x$  and  $\hat{y} := -y$ , our fourth equation becomes

$$\bar{\omega} \sum_{\hat{l} \in G} A_{\hat{l},\hat{x}+\hat{y}} A_{\hat{x},\hat{l}+\hat{p}} A_{\hat{y},\hat{l}+\hat{q}} = A_{\hat{y}+\hat{p},\hat{q}} A_{\hat{x}+\hat{q},\hat{p}} - \frac{\delta_{\hat{x},0}\delta_{\hat{y},0}}{d_{\pm}},$$

showing that it is equivalent to [EG17, Equation 4.11].

### 3. THE LEAVITT ALGEBRA APPROACH OF EVANS–GANNON

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The important result is [EG17, Theorem 2].

**Definition 3.1.** (Leavitt Algebra). *Let  $X := (x_{ij})$  and  $Y := (y_{ij})$  be  $m \times n$  and  $n \times m$  matrices of symbols, respectively. The Leavitt  $K$ -algebra of type  $(m, n)$  is the free associative unital  $K$ -algebra*

$$\mathcal{L}_K(m, n) := \frac{K[x_{ij}, y_{ij}]}{\langle XY = I_m, YX = I_n \rangle}.$$

In other words, it is the universal  $K$ -algebra with generators

$$\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \sqcup \{y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and Leavitt–Cuntz relations

$$\sum_{k=1}^m y_{ik}x_{kj} = \delta_{i,j} \quad \text{and} \quad \sum_{k=1}^n x_{ik}y_{kj} = \delta_{i,j},$$

for all suitable  $i, j$ .

Consider the Leavitt  $\mathbb{C}$ -algebra of type  $(1, n)$ , which we shall henceforth denote by  $\mathcal{L}_n := \mathcal{L}_{\mathbb{C}}(1, n)$ . We have that  $\mathcal{O}_n = C^*(\mathcal{L}_n)$ , where  $x_i = S_i$  and  $y_i = S_i^*$ . Let's think about what this means precisely. The Leavitt–Cuntz relations for  $m = 1$  become

$$y_i x_j = \delta_{i,j} \quad \text{and} \quad \sum_{k=1}^n x_k y_k = 1.$$

We may endow  $\mathcal{L}_n$  with the structure of a  $*$ -algebra by defining a conjugate homogeneous antihomomorphism that sends  $x_i \mapsto y_i$  and  $y_i \mapsto x_i$ . We further define

$$\|a\| := \sup\{p(a) : p \text{ is a } C^*\text{-seminorm on } \mathcal{L}_n\}$$

for all  $a \in \mathcal{L}_n$ , where a  $C^*$ -seminorm is just a seminorm for which  $p(a^*a) = p(a)^2$  and  $p(ab) \leq p(a)p(b)$ . Note that  $0 \leq \|a\| \leq 1$ , as  $1 = \|y_i x_i\| = \|x_i^2\|$  and hence  $\|x_i\| = \|y_i\| = 1$  for all  $i$ . The condition  $p(ab) \leq p(a)p(b)$  ensures that  $\mathcal{I} := \{a \in \mathcal{L}_n : \|a\| = 0\}$  is an ideal in  $\mathcal{L}_n$ . Our  $C^*$ -seminorm then descends to a  $C^*$ -norm on the quotient  $\mathcal{L}_n/\mathcal{I}$ . The completion of  $\mathcal{L}_n/\mathcal{I}$  with respect to this  $C^*$ -norm is known as the universal  $C^*$ -algebra of  $\mathcal{L}_n$ , denoted by  $C^*(\mathcal{L}_n)$ . This is precisely  $\mathcal{O}_n$  by definition. We may therefore view  $\mathcal{L}_n$  as the polynomial part of  $\mathcal{O}_n$ .

**Example 3.2.** (Yang–Lee Category). Let  $G = \{0\}$ . Then [EG17, Equation 4.7] demands that

$$A_{0,0} = \omega A_{0,0} = \bar{\omega} A_{0,0} \implies \omega = 1,$$

whence [EG17, Equation 4.8] tells us that

$$A_{0,0} = -\frac{1}{d_{\pm}}.$$

The rest of [EG17, Equations 4.7–4.10] are satisfied by these choices. Hence by [EG17, Theorem 2], we have two fusion categories for  $G = \{0\}$ ; a unitary one with  $\pm = +$  (the Fibonacci category) and a non-unitary one with  $\pm = -$  (the Yang–Lee category).

**Example 3.3.** ( $G = \mathbb{Z}/2\mathbb{Z}$ ). The equations we must satisfy for  $|G|$  even are given in [Izu18] (is this true?). Adapting our argument from Example 2.5, we see that there is exactly one non-unitary Haagerup–Izumi category with  $G = \mathbb{Z}/2\mathbb{Z}$ . This corresponds to  $\epsilon_h(g) = (-1)^{gh}$ ,  $\omega(g) = 1$ ,

$$A(0) = \frac{1}{d-1} \begin{pmatrix} d & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad A(1) = \frac{1}{d-1} \begin{pmatrix} d & 1 \\ 1 & -1 \end{pmatrix}.$$

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