Indecomposable Soergel Bimodules of Types A_1 and A_2

The goal of these notes is to explicitly step through the classification of the indecomposable Soergel bimodules of type A_2, while also covering the simpler A_1 case as a stepping stone. Throughout these notes, $\Bbbk$ will always be taken to be an algebraically closed field of characteristic zero.

Definition 1. (Geometric Representation). *Let (W, S) be a Coxeter system and V the $\Bbbk$-vector space with formal basis $\{\alpha_s : s \in S\}$. Define a symmetric, bilinear form on V by linearly extending*

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right), & m_{st} \neq \infty; \\ -1, & m_{st} = \infty. \end{cases}$$

From this, we define an action of $s \in S$ on the basis elements $\alpha_t \in V$ by the linear automorphism

$$s(\alpha_t) \coloneqq \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s,$$

which reflects α_t across α_s. The geometric representation *of (W, S) is the representation induced by linearly extending this reflection action to an action of W on all of V.*

Recall that by definition, $R \coloneqq \bigoplus_{i=0}^{\infty} \operatorname{Sym}^i(V)$, where $\operatorname{Sym}^i(V)$ is the quotient of the ith tensor power $V^{\otimes i}$ by the action of the symmetric group S_i, viewed as a $\Bbbk$-module that is $\mathbb{Z}$-graded in degree 2. Our reflection action also extends to an action on R by taking $s(fg) \coloneqq s(f)s(g)$, for $f, g \in R$. It is not too difficult to show that $\operatorname{Sym}^i(V)$ is isomorphic to the additive subgroup of the polynomial ring $\Bbbk[\alpha_s : s \in S]$ consisting only of homogeneous polynomials of degree i, meaning we may identify $R = \Bbbk[\alpha_s : s \in S]$. With this in mind, we have a crucial lemma that we will invoke regularly.

Lemma 2. *Suppose that M is a graded (R, R)-bimodule that is generated by a homogeneous element $m \in M$, in the sense that $M = RmR$. Then M is indecomposable.*

Proof. Let d denote the degree of m. Because M is generated by m, and because $R^0 = \Bbbk$, we have that $M^d = R^0 m R^0 = \Bbbk m$. In other words, M^d is a one-dimensional vector space. Suppose that $M \cong L \oplus N$. In this case, M^d is isomorphic as a $\Bbbk$-vector space to $L^d \oplus N^d$. Assume without loss of generality that $m \in L^d$. This forces $N^d = 0$, as M^d is one-dimensional, whence we have that $M = RmR \subseteq L$, forcing $N = 0$. Thus M is indecomposable. This completes the proof. ■

From this lemma, it follows that R itself is indecomposable as an (R, R)-bimodule, as it is generated by $1 \in \Bbbk$. Moreover, suppose we define

$$B_s \coloneqq R \otimes_{R^s} R(1)$$

for any $s \in S$, where $R^s \coloneqq \{f \in R : s(f) = f\}$. This is also clearly indecomposable by our lemma, as it is generated as an (R, R)-bimodule by $1 \otimes_{R^s} 1$ (where we note that, since $1 \in R^0 = R(1)^{-1}$, we have $1 \otimes_{R^s} 1 \in B_s^{0-1} = B_s^{-1}$, meaning it is homogeneous of degree -1). In particular, it is easy to see that grading shifts of R and B_s are all indecomposable too.

Definition 3. (Soergel Bimodule). *Let (W, S) be a Coxeter system and $\underline{w} \coloneqq (s_1, \ldots, s_k)$ an expression. The* Bott–Samelson bimodule corresponding to $\underline{w}$ *is the graded (R, R)-bimodule*

$$\begin{aligned} BS(\underline{w}) &\coloneqq B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \\ &= (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R(k). \end{aligned}$$

A Soergel bimodule *is any graded (R, R)-bimodule that is isomorphic to a finite direct sum of grading shifs of direct summands of Bott–Samelson bimodules.*

Another useful lemma is that any polynomial in R can be split into the sum of an s-invariant component and an s-antiinvariant component, for any $s \in S$.

Lemma 4. *For any $s \in S$ and $f \in R$, we have that $f + s(f) \in R^s$ and $f - s(f) \in R^s\alpha_s$. In particular, we have a graded (R^s, R^s)-bimodule splitting $R \cong R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$ for all $s \in S$.*

Proof. Let $s \in S$ and $f \coloneqq \alpha_t$ for any $t \in S$. Observe that $P_s(f) \coloneqq \frac{1}{2}(f+s(f)) = \alpha_t - (\alpha_s, \alpha_t)\alpha_s$ is in R^s (alternatively, $s(f+s(f)) = s(f)+s^2(f) = s(f)+f$). Similarly, $\partial_s(f)\alpha_s \coloneqq \frac{1}{2}(f - s(f)) = (\alpha_s, \alpha_t)\alpha_s$ is clearly in $R^s\alpha_s$ (alternatively, $s(f-s(f)) = s(f)-s^2(f) = s(f)-f$). By linearity of the action of S, sums will preserve this splitting, so all that remains is to show that products preserve it too. Suppose $f \coloneqq f_1+f_2\alpha_s$ and $g \coloneqq g_1+g_2\alpha_s$, for $f_1, f_2, g_1, g_2 \in R^s$, such that $fg = f_1g_1 + f_1g_2\alpha_s + f_2g_1\alpha_s + f_2g_2\alpha_s^2$. We see that $P_s(fg) = g_1g_2 + h_1h_2\alpha_s^2$ and $\partial_s(fg)\alpha_s = g_1h_2\alpha_s + g_2h_1\alpha_s$, whence fg admits a unique decomposition of the form $fg = P_s(fg) + \partial_s(fg)\alpha_s$. Clearly this induces an isomorphism of graded (R^s, R^s)-bimodules, as for any $f \in R$ and $g, h \in R^s$, we have that $hfg \mapsto (hP_s(f)g, h\partial_s(f)\alpha_s g)$. This completes the proof. ■

Note that this is an isomorphism of (R^s, R^s)-bimodules, *not* (R, R)-bimodules! This lemma allows us to completely classify the indecomposables in the type A_1 case.

Proposition 5. (Indecomposable Soergel Bimodules of Type A_1). *The indecomposable Soergel bimodules of type A_1 are, up to grading shift and isomorphism, R and B_s.*

Proof. Let $W = S_2 = \langle s \rangle$ and $S = \{s\}$. By our previous observations, we know that grading shifts of R and B_s are indecomposable, as they are generated by the homogeneous unit tensors. The next step is to check the Bott–Samelson $B_s \otimes_R B_s$. But observe that

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) && \text{(Lemma 4)} \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

Thus $B_s \otimes_R B_s$ is decomposable, meaning that we have exhausted all candidates for indecomposables. This completes the proof. ■

Remark 6. Observe that the indecomposables are, up to grading shift, in bijection with W, just as we would expect by the categorification theorem.

One last lemma that we will find useful before we attempt to classify the indecomposable Soergel bimodules of type A_2 is the following.

Lemma 7. *Let $s, t \in S$ with $s \neq t$. Then R^s and R^t generate R as a ring if and only if $m_{st} \neq \infty$.*

Proof. Recall that $R \coloneqq \mathrm{Sym}(V)$. As a polynomial ring, R is generated by its linear terms – those being the vectors in V. The linear terms in R^s are precisely the vectors fixed by s. This "intersection of R^s with V" is nothing but

$$H_s \coloneqq \{v \in V : s(v) = v\} = \{v \in V : (v, \alpha_s) = 0\}.$$

Note that the map $\varphi_s : V \to \Bbbk$ given by $v \mapsto (v, \alpha_s)$ is obviously a surjective linear map, as $\varphi_s(\lambda\alpha_s)$ always maps to $\lambda \in \Bbbk$. We therefore have that

$$\dim(H_s) = \dim(\ker(\varphi_s)) = \dim(V) - \dim(\mathrm{rank}(\varphi_s)) = \dim(V) - 1$$

by the rank-nullity theorem. In other words, H_s is a hyperplane. When $m_{st} \neq \infty$, we have that H_s and H_t are distinct. Because they are hyperplanes, this means that they must span V, whence R^s and R^t must generate R. This completes the proof. ∎

Remark 8. This result becomes trivial when considering that, for any $s \in S$, the ring R^s is generated by the unit, α_s^2 and elements of the form $\alpha_t - (\alpha_s, \alpha_t)\alpha_s$, for all $t \in S \setminus \{s\}$. It is easy to see that, given fixed $s, t \in S$ with $s \neq t$, we have

$$\alpha_s = \frac{1}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_s - (\alpha_s, \alpha_t)\alpha_t) + \frac{(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_t - (\alpha_s, \alpha_t)\alpha_s)$$

if and only if $m_{st} \neq \infty$, and similarly for α_t. However, showing that these polynomials generate R^s is non-trivial, following from the Chevalley–Shephard–Todd theorem.

Let $s, t \in S$ with $s \neq t$ and $m_{st} \neq \infty$. Observe that

$$B_s \otimes_R B_t \cong R \otimes_{R^s} R \otimes_{R^t} R(2) \qquad \text{and} \qquad B_t \otimes_R B_s \cong R \otimes_{R^t} R \otimes_{R^s} R(2).$$

It follows from Lemma 2 that these are both indecomposable, as by Lemma 7 they are generated by the degree -2 elements $1 \otimes_{R^s} 1 \otimes_{R^t} 1$ and $1 \otimes_{R^t} 1 \otimes_{R^s} 1$, respectively. We shall therefore write $B_{st} \coloneqq B_s \otimes_R B_t$ and $B_{ts} \coloneqq B_t \otimes_R B_s$ from this point onwards.

Remark 9. As it happens, $B_s \otimes_R B_t$ and $B_t \otimes_R B_s$ remain indecomposable when $m_{st} = \infty$, but proving this is somewhat more involved. In addition to our previous lemma no longer holding (since $H_s = H_t$ end up being the same 1-dimensional space, generated by $\alpha_s + \alpha_t$), these bimodules are also no longer cyclic, meaning we cannot use Lemma 2!

The remainder of these notes will be in proving the following result, where $B_{sts} \coloneqq R \otimes_{R^{s,t}} R(3)$. This notation will be justified by the following proposition. However, we note that – remarkably – a similar definition can be made for any Soergel bimodule B_{w_0} corresponding to the longest element w_0 of its associated Coxeter group.

Proposition 10. (Indecomposable Soergel Bimodules of Type A_2). *The indecomposable Soergel bimodules of type A_2 are, up to grading shift and isomorphism, R, B_s, B_t, B_{st}, B_{ts} and B_{sts}.*

Proof. Let $W = S_3 = \{1, s, t, st, ts, sts\}$ and $S = \{s, t\}$. We have already seen that R, B_s, B_t, B_{st} and B_{ts} are indecomposable Soergel bimodules. Certainly B_{sts} is indecomposable as an (R, R)-bimodule by Lemma 2, as it is generated by the degree -3 element $1 \otimes_{R^{s,t}} 1$, but it remains to be shown that it is indeed a Soergel bimodule. We will begin by showing that it appears as a direct summand of both $B_s \otimes_R B_t \otimes_R B_s$ and $B_t \otimes_R B_s \otimes_R B_t$. Once we've done this, we will show that these indecomposables are exhaustive.

In order to show that B_{sts} is a Soergel bimodule, we first claim that

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \qquad \text{and} \qquad B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t.$$

To this end, let's define an (R, R)-bimodule homomorphism $\iota_{sts} : B_{sts} \to B_s \otimes_R B_t \otimes_R B_s$ by extending

$$\iota_{sts} : 1 \otimes_{R^{s,t}} 1 \mapsto 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

This is clearly a well-defined homomorphism of graded (R, R)-bimodules, as $f \otimes_{R^{s,t}} 1 = 1 \otimes_{R^{s,t}} f$ if and only if $f \in R^{s,t}$ if and only if $f \in R^s, R^t$. In fact, this reasoning also shows that it is injective. Suppose we define another (R, R)-bimodule homomorphism $\iota_s : B_s \to B_s \otimes_R B_t \otimes_R B_s$ by extending

$$\iota_s : 1 \otimes_{R^s} 1 \mapsto \frac{1}{2}(1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1).$$

In order to show that this is well-defined, we need to show that

$$f \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1 = 1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} f$$

for all $f \in R^s$. By Lemma 4, for any $f \in R$ we have

$$f = P_t(f) + \partial_t(f)\alpha_t,$$

where $P_t(f), \partial_t(f) \in R^t$. If $f \in R^s$, then

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 &= 1 \otimes_{R^s} f\alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} (P_t(f) + \partial_t(f)\alpha_t)\alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= (1 \otimes_{R^s} P_t(f)\alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1) + (1 \otimes_{R^s} \partial_t(f)\alpha_t^2 \otimes_{R^t} 1 \otimes_{R^s} 1) \\ &= (1 \otimes_{R^s} \alpha_t \otimes_{R^t} P_t(f) \otimes_{R^s} 1) + (1 \otimes_{R^s} 1 \otimes_{R^t} \partial_t(f)\alpha_t^2 \otimes_{R^s} 1), \end{aligned}$$

and similarly

$$f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 = (1 \otimes_{R^s} 1 \otimes_{R^t} P_t(f)\alpha_t \otimes_{R^s} 1) + (1 \otimes_{R^s} \alpha_t \otimes_{R^t} \partial_t(f)\alpha_t \otimes_{R^s} 1).$$

Summing these and combining the "even" terms $P_t(f)$ with the "odd" terms $\partial_t(f)\alpha_t$, it follows that

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 + f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} f \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} f\alpha_t \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} f + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} f. \end{aligned}$$

Thus ι_s is well-defined, and like ι_{sts} it is a monomorphism. Because $\langle 1 \otimes_{R^t} 1 \rangle$ and $\langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle$ clearly generate disjoint (R^s, R^s)-bimodules, ι_{sts} and ι_s are themselves disjoint. If we can show

$$R \otimes_{R^t} R \cong \langle 1 \otimes_{R^t} 1 \rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle,$$

then $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$, with a similar computation yielding $B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t$.

While we could figure out how to write $f \otimes_{R^t} g \in \langle 1 \otimes_{R^t} 1\rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t\rangle$ for all $f, g \in R$, there is perhaps a more informative way to prove this result. This is to find explicit projections $\pi_s : B_s \otimes_R B_t \otimes_R B_s \to B_s$ and $\pi_{sts} : B_s \otimes_R B_t \otimes_R B_s \to B_{sts}$ for which

$$\begin{aligned} \pi_{sts} \circ \iota_{sts} &= \mathrm{id}_{B_{sts}}, \\ \pi_s \circ \iota_s &= \mathrm{id}_{B_s}, \\ \iota_{sts} \circ \pi_{sts} + \iota_s \circ \pi_s &= \mathrm{id}_{B_s \otimes_R B_t \otimes_R B_s}. \end{aligned}$$

Note that this is precisely what $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$ means by definition. It is not too hard to find π_s. One can easily show that it is nothing but

$$\pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -2\partial_s(fg) \otimes_{R^s} 1,$$

where we remind the reader that we are using the relatively non-standard normalization $\delta_s : f \mapsto (f - s(f))/2\alpha_s$ of the Demazure operator. But how can we find π_{sts}? Well, let

$$e_s := \iota_s \circ \pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -\delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

As we would expect, e_s is an idempotent, as

$$e_s^2 = \iota_s \circ \pi_s \circ \iota_s \circ \pi_s = \iota_s \circ \mathrm{id}_{B_s} \circ \pi_s = e_s.$$

As mentioned above, we want to find $e_{sts} = \iota_{sts} \circ \pi_{sts}$ satisfying

$$e_{sts} + e_s = \mathrm{id}_{B_s \otimes_R B_t \otimes_R B_s}.$$

This gives us an explicit formula for e_{sts}; namely, $e_{sts} = \mathrm{id}_{B_s \otimes_R B_t \otimes_R B_s} - e_s$, whence

$$e_{sts} = 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 + \delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

We can use this to determine π_{sts}. First, we know that π_{sts} will be some map of the form

$$\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto \sum_i h_i \otimes_{R^{s,t}} k_i.$$

Thus we must rearrange the image of e_{sts} such that the inner tensors are 1. Let's write

$$\begin{aligned} 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 &= 1 \otimes_{R^s} (P_s(f) + \delta_s(f)\alpha_s) \otimes_{R^t} (P_s(g) + \delta_s(g)\alpha_s) \otimes_{R^s} 1 \\ &= P_s(f) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad P_s(f) \otimes_{R^s} 1 \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g). \end{aligned}$$

This greatly simplifies our work, as now we need only compute how e_{sts} acts on these four tensors. The first one is simple, since $\delta_s(1) = 0$:

$$e_{sts}(1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1) = 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

As for the next two, observe that

$$(\alpha_s + \alpha_t) - \frac{2}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) = \frac{2}{3}\left(\alpha_s + \frac{1}{2}\alpha_t\right).$$

Hence

$$\begin{aligned}
e_{sts}(1\otimes_{R^s}\alpha_s\otimes_{R^t}1\otimes_{R^s}1) &= 1\otimes_{R^s}\alpha_s\otimes_{R^t}1\otimes_{R^s}1 \ + \ 1\otimes_{R^s}(\alpha_t\otimes_{R^t}1+1\otimes_{R^t}\alpha_t)\otimes_{R^s}1\\
&= 1\otimes_{R^s}(\alpha_s+\alpha_t)\otimes_{R^t}1\otimes_{R^s}1 \ + \ 1\otimes_{R^s}1\otimes_{R^t}\alpha_t\otimes_{R^s}1\\
&= 1\otimes_{R^s}\frac{1}{3}(2\alpha_s+\alpha_t)\otimes_{R^t}1\otimes_{R^s}1 \ + \ 1\otimes_{R^s}1\otimes_{R^t}\alpha_t\otimes_{R^s}1 \ +\\
&\quad 1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^t}1\otimes_{R^s}1\\
&= 1\otimes_{R^s}1\otimes_{R^t}\frac{1}{3}(2\alpha_s+4\alpha_t)\otimes_{R^s}1 \ + \ 1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^t}1\otimes_{R^s}1\\
&= 1\otimes_{R^s}1\otimes_{R^t}1\otimes_{R^s}\frac{2}{3}(\alpha_s+2\alpha_t) \ + \ \frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^s}1\otimes_{R^t}1\otimes_{R^s}1.
\end{aligned}$$

The result for $1\otimes_{R^s}1\otimes_{R^t}\alpha_s\otimes_{R^s}1$ follows by symmetry. We can then play a similar game for the remaining tensor. We begin by observing that

$$\alpha_s+\frac{2}{3}\left(\frac{1}{2}\alpha_s+\alpha_t\right)=\frac{4}{3}\left(\alpha_s+\frac{1}{2}\alpha_t\right).$$

Thus

$$\begin{aligned}
e_{sts}(1\otimes_{R^s}\alpha_s\otimes_{R^t}\alpha_s\otimes_{R^s}1) &= 1\otimes_{R^s}\alpha_s\otimes_{R^t}\alpha_s\otimes_{R^s}1\\
&= 1\otimes_{R^s}\frac{2}{3}(2\alpha_s+\alpha_t)\otimes_{R^t}\frac{2}{3}(2\alpha_s+\alpha_t)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}\frac{2}{3}(2\alpha_s+\alpha_t)\otimes_{R^t}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^t}\frac{2}{3}(2\alpha_s+\alpha_t)\otimes_{R^s}1 \ +\\
&\quad 1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^t}\frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^s}1\\
&= 1\otimes_{R^s}1\otimes_{R^t}\frac{2}{9}(8\alpha_s^2+8\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}1\otimes_{R^t}\frac{2}{9}(2\alpha_s^2+5\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}\frac{2}{9}(2\alpha_s^2+5\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^t}1\otimes_{R^s}1 \ +\\
&\quad \frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^s}1\otimes_{R^t}1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t)\\
&= 1\otimes_{R^s}1\otimes_{R^t}\frac{2}{9}(8\alpha_s^2+8\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}1\otimes_{R^t}\frac{2}{9}(2\alpha_s^2+5\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^s}1 \ -\\
&\quad 1\otimes_{R^s}\frac{2}{9}(2\alpha_s^2+5\alpha_s\alpha_t+2\alpha_t^2)\otimes_{R^t}1\otimes_{R^s}1 \ +\\
&\quad \frac{1}{3}(\alpha_s+2\alpha_t)\otimes_{R^s}1\otimes_{R^t}1\otimes_{R^s}\frac{1}{3}(\alpha_s+2\alpha_t).
\end{aligned}$$

From here, we note that
$$R^{s,t} = \mathbb{R}[\alpha_s^2 + \alpha_s\alpha_t + \alpha_t^2];$$
in particular,
$$\alpha_s^2 + \alpha_s\alpha_t \in R^t.$$
We may hence continue with
$$\begin{aligned}
e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1) &= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(6\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\
&= 1 \otimes_{R^s} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\
&= 1 \otimes_{R^s} \frac{2}{9}(\alpha_s^2 - 2\alpha_s\alpha_t - 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 \\
&= \frac{4}{9}(-\alpha_s^2 - \alpha_s\alpha_t - \alpha_t^2) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 + \\
&\quad \frac{2}{3}\alpha_s^2 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.
\end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto\ & P_s(f) \otimes_{R^{s,t}} P_s(g) + \\
& \partial_s(f)\frac{2}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) \otimes_{R^{s,t}} P_s(g) + \\
& \partial_s(f) \otimes_{R^{s,t}} \frac{4}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) P_s(g) + \\
& P_s(f) \otimes_{R^{s,t}} \frac{2}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) \partial_s(g) + \\
& P_s(f)\frac{4}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) \otimes_{R^{s,t}} \partial_s(g) + \\
& \partial_s(f)\frac{1}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) \otimes_{R^{s,t}} \frac{1}{3}\left(\frac{1}{2}\alpha_s + \alpha_t\right) \partial_s(g) + \\
& \partial_s(f) \otimes_{R^{s,t}} \frac{2}{3}\alpha_s^2 \partial_s(g) + \\
& \partial_s(f)\frac{2}{3}\alpha_s^2 \otimes_{R^{s,t}} \partial_s(g) - \\
& \partial_s(f)\frac{4}{9}(\alpha_s^2 + \alpha_s\alpha_t + \alpha_t^2) \otimes_{R^{s,t}} \partial_s(g).
\end{aligned}$$

Phew!!

All that's left is to clean up a few loose ends. First, we see that B_{st} and B_{ts} are not isomorphic, as

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \ncong B_{st}(-1) \oplus B_{st}(1) \cong B_s \otimes_R B_s \otimes_R B_t.$$

Note that we have used the fact here that direct sum decompositions are unique. Finally, in order to show that all indecomposables have been exhausted, we claim that

$$B_{sts} \otimes_R B_s \cong B_s \otimes_R B_{sts} \cong B_{sts}(1) \oplus B_{sts}(-1) \cong B_t \otimes_R B_{sts} \cong B_{sts} \otimes_R B_t.$$

Recall that in Lemma 4, we found an isomorphism $R \cong R^s \oplus R^s(-2)$ of (R^s, R^s)-bimodules. It is easy to see that this restricts to an isomorphism of $(R^{s,t}, R^s)$-bimodules, whence

$$\begin{aligned}
B_{sts} \otimes_R B_s &\cong (R \otimes_{R^{s,t}} R(3)) \otimes_R (R \otimes_{R^s} R(1)) \\
&\cong R \otimes_{R^{s,t}} R \otimes_{R^s} R(4) \\
&\cong R \otimes_{R^{s,t}} (R^s \oplus R^s(-2)) \otimes_{R^s} R(4) && \text{(Lemma 4)} \\
&\cong (R \otimes_{R^{s,t}} R(4)) + (R \otimes_{R^W} R(2)) \\
&\cong B_{sts}(1) \oplus B_{sts}(-1).
\end{aligned}$$

The other isomorphisms follow similarly. We have thus shown that we obtain no new indecomposables by tensoring, meaning we are done with our classification. This completes the proof. ■