Basics of Module Categories

1. Definition of a Module Category

Over the course of this seminar, we have seen various examples of tensor categories, which categorify the notion of rings. Continuing down this path, we will begin to look at module categories over monoidal categories, which categorify the notion of modules over rings. As it turns out, module categories happen to be an incredibly natural and convenient language, and in fact many of the categories we have seen in earlier weeks admit the structure of a module category. They also appear readily in the background of subfactor theory, conformal field theory and weak Hopf algebras. It is our goal to introduce them while keeping this motivating context in mind; towards the end, we will aim to provide some "exotic" examples of module categories coming from the world of Jones mathematics.

Definition 1.1. (Module Category). *Let* $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ *be a monoidal category. A* left module category over $\mathcal{C}$ *is a category* $\mathcal{M}$ *endowed with*

- *a* left action *(or* module product*)* bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$*;*
- *a natural isomorphism with components of the form* $m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$*, for all* $X, Y \in \mathcal{C}$ *and* $M \in \mathcal{M}$*, called the* module associativity constraint*;*
- *a natural isomorphism with components of the form* $l_M : \mathbb{1} \otimes M \xrightarrow{\sim} M$*, for all* $M \in \mathcal{M}$*, called the* unit constraint*;*

such that the pentagon diagram

$$\begin{tikzcd}
 & ((X \otimes Y) \otimes Z) \otimes M \arrow[dl, "a_{X,Y,Z} \otimes \mathrm{id}_M"'] \arrow[dr, "m_{X \otimes Y, Z, M}"] & \\
(X \otimes (Y \otimes Z)) \otimes M \arrow[d, "m_{X, Y \otimes Z, M}"'] & & (X \otimes Y) \otimes (Z \otimes M) \arrow[d, "m_{X,Y,Z \otimes M}"] \\
X \otimes ((Y \otimes Z) \otimes M) \arrow[rr, "\mathrm{id}_X \otimes m_{Y,Z,M}"'] & & X \otimes (Y \otimes (Z \otimes M))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes M \arrow[rr, "m_{X,\mathbb{1},M}"] \arrow[dr, "r_X \otimes \mathrm{id}_M"'] & & X \otimes (\mathbb{1} \otimes M) \arrow[dl, "\mathrm{id}_X \otimes l_M"] \\
 & X \otimes M &
\end{tikzcd}$$

commute for all $X, Y \in \mathcal{C}$ *and* $M \in \mathcal{M}$*.*

We define right module categories similarly, except that our monoidal category $\mathcal{C}$ now acts from the right, and the natural isomorphisms and coherence diagrams are modified accordingly. In particular, a right $\mathcal{C}$-module category is nothing but a left $\mathcal{C}^{\mathrm{op}}$-module category, where $\mathcal{C}^{\mathrm{op}}$ is the monoidal category opposite to $\mathcal{C}$ ([EGNO16, Definition 2.1.5]).

Just like we have bimodules, we may also define bimodule categories. As one might guess, a $(\mathcal{C},\mathcal{D})$-bimodule category is nothing but a category that is both a left $\mathcal{C}$-module category and a right $\mathcal{D}$-module category, along with some additional compatibility constraints.

Definition 1.2. (Bimodule Category). *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A* $(\mathcal{C},\mathcal{D})$-bimodule category *is a category $\mathcal{M}$ that has both a left $\mathcal{C}$-module category structure and a right $\mathcal{D}$-module category structure, with module associativity constraints $m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M)$ and $n_{M,W,Z} : M \otimes (W \otimes Z) \xrightarrow{\sim} (M \otimes W) \otimes Z$ respectively, together with a natural isomorphism with components $b_{X,M,Z}(X \otimes M) \otimes Z \xrightarrow{\sim} X \otimes (M \otimes Z)$ for all $X, Z \in \mathcal{C}$, $M \in \mathcal{M}$ called the* middle associativity constraint*, such that the diagrams*

$$\begin{tikzcd}
& ((X \otimes Y) \otimes M) \otimes Z \arrow[dl, "m_{X,Y,M} \otimes \mathrm{id}_Z"'] \arrow[dr, "b_{X \otimes Y,M,Z}"] & \\
(X \otimes (Y \otimes M)) \otimes Z \arrow[d, "b_{X,Y \otimes M,Z}"'] & & (X \otimes Y) \otimes (M \otimes Z) \arrow[d, "m_{X,Y,M \otimes Z}"] \\
X \otimes ((Y \otimes M) \otimes Z) \arrow[rr, "\mathrm{id}_X \otimes b_{Y,M,Z}"'] & & X \otimes (Y \otimes (M \otimes Z))
\end{tikzcd}$$

and

$$\begin{tikzcd}
& X \otimes (M \otimes (W \otimes Z)) \arrow[dl, "\mathrm{id}_X \otimes n_{M,W,Z}"'] & \\
X \otimes ((M \otimes W) \otimes Z) & & (X \otimes M) \otimes (W \otimes Z) \arrow[ul, "b_{X,M,W \otimes Z}"'] \arrow[d, "n_{X \otimes M,W,Z}"] \\
(X \otimes (M \otimes W)) \otimes Z \arrow[u, "b_{X,M \otimes W,Z}"] & & ((X \otimes M) \otimes W) \otimes Z \arrow[ll, "b_{X,M,W} \otimes \mathrm{id}_Z"]
\end{tikzcd}$$

commute for all $X, Y \in \mathcal{C}$, $Z, W \in \mathcal{D}$ and $M \in \mathcal{M}$.

Currently we have defined module categories over general monoidal categories. There is, however, a particularly rich theory for module categories over tensor categories. As is the norm, we should first introduce some additional "compatibility" adjectives.

Definition 1.3. (Module Category over Tensor Category). *Let $\mathcal{C}$ be a multitensor category. A* left module category over $\mathcal{C}$ *is a locally finite, Abelian $\mathcal{C}$-module category $\mathcal{M}$ such that the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear on morphisms and exact in the first variable.*

Remark 1.4. Recall that all small limits can be written as an equalizer of a pair of maps between products, and all small colimits can be written as a coequalizer of a pair of maps between coproducts. Moreover, in an additive category, all products and coproducts correspond to direct sums, while in an Abelian category all equalizers and coequalizers correspond to kernels and cokernels, respectively. In other words, the requirement that the module product be exact in the first variable is equivalent to saying it preserves both kernels and cokernels, whence we get for free that it preserves direct sums.

2. Recurring Examples

Example 2.1. (Regular Representation) Every multitensor category $\mathcal{C}$ is trivially a $(\mathcal{C}, \mathcal{C})$-bimodule category, as the tensor product is bilinear on morphisms by definition and biexact by [EGNO16, Proposition 4.2.1] (which we saw in lecture 2). More generally, $\mathcal{C}$ is a $(\mathcal{D}_1, \mathcal{D}_2)$-bimodule category for all multitensor subcategories $\mathcal{D}_1, \mathcal{D}_2 \subseteq \mathcal{C}$. Okay, let's maybe see some more substantial examples.

Example 2.2. The simplest example of a tensor category is $\mathcal{C} \coloneqq \mathsf{Vec}$, the category of finite-dimensional $\Bbbk$-vector spaces, so let's start here. Obviously, for any $X \in \mathrm{Ob}(\mathcal{C})$, we have $X \cong \Bbbk^n$ for some $n \in \mathbb{N}$. Thus any locally finite, Abelian category over $\Bbbk$ admits a unique (up to equivalence) $\mathcal{C}$-module structure given by $X \otimes M \coloneqq M^{\oplus n}$, for $X \in [\Bbbk^n]$ and $M \in \mathrm{Ob}(\mathcal{M})$.

Example 2.3. Let G be a group and $\mathcal{C} \coloneqq \mathsf{Vec}_G$ the category of G-graded finite-dimensional $\Bbbk$-vector spaces, whose tensor product is given by $(V \otimes W)_g \coloneqq \bigoplus_{hk=g} V_h \otimes W_k$. A module category over $\mathcal{C}$ is an Abelian category $\mathcal{M}$ endowed with an *action of G on $\mathcal{M}$*; that is, a monoidal functor $F : \mathsf{Cat}(G) \to \mathsf{Aut}_{\otimes}(\mathcal{M})$. Here $\mathsf{Cat}(G)$ is the monoidal category whose objects are elements of G, whose morphisms are only identity morphisms and whose monoidal product is multiplication in G, while $\mathsf{Aut}_{\otimes}(\mathcal{M})$ is the monoidal category whose objects are tensor autoequivalences of $\mathcal{M}$, whose morphisms are natural isomorphisms of tensor functors and whose monoidal product is functor composition. Note that we can always construct an action of the form $F(g) : M \mapsto \delta_g \otimes M$, where δ_g is the object with $\Bbbk$ for its gth graded piece and $\{0\}$ for all others.

Example 2.4. If $F : \mathcal{C} \to \mathcal{M}$ is a tensor functor, then $\mathcal{M}$ is a module category over $\mathcal{C}$ with $X \otimes Y \coloneqq F(X) \otimes Y$. In particular, if G is a finite group and $L \subseteq G$ is a subgroup, the tensor functor $\mathrm{Res} : \mathsf{Rep}(G) \to \mathsf{Rep}(L)$ given by restriction induces a $\mathsf{Rep}(G)$-module structure on $\mathsf{Rep}(L)$.

Proposition 2.5. ([EGNO16, Proposition 7.1.3]). *Let $\mathcal{C}$ be a monoidal category and $\mathsf{End}(\mathcal{M})$ the category of endofunctors over a category $\mathcal{M}$. We have that the $\mathcal{C}$-module structures on $\mathcal{M}$ are in bijection with monoidal functors of the form $F : \mathcal{C} \to \mathsf{End}(\mathcal{M})$.*

This proposition categorifies the fact that a module over a ring is nothing but a representation! We have also a similar result for the more restrictive module structure given in Definition 1.3.

Proposition 2.6. ([EGNO16, Proposition 7.3.3]). *Let $\mathcal{C}$ be a multitensor category and $\mathsf{End}_l(\mathcal{M})$ the category of left exact endofunctors over a category $\mathcal{M}$. We have that the $\mathcal{C}$-module structures on $\mathcal{M}$ are in bijection with exact monoidal functors of the form $F : \mathcal{C} \to \mathsf{End}_l(\mathcal{M})$.*

Example 2.7. Suppose we consider the category $\mathcal{M} \coloneqq \mathsf{Vec}$ of finite-dimensional $\Bbbk$-vector spaces. Over what tensor categories is $\mathcal{M}$ a module category? Well, the covariant Yoneda embedding よ $: V \mapsto \mathrm{Hom}_{\mathcal{M}}(V, -)$ (here よ is the Hiragana for "yo") induces a tensor embedding of $\mathcal{M}$ into $\mathsf{End}(\mathcal{M})$, the category of endofunctors over $\mathcal{M}$. In particular, the essential image of this embedding is $\mathsf{End}_l(\mathcal{M})$, the category of left exact endofunctors over $\mathcal{M}$. Thus Proposition 2.6 tells us that, given a tensor category $\mathcal{C}$, we have a bijective correspondence between $\mathcal{C}$-module structures on Vec and fibre functors of the form $F : \mathcal{C} \to \mathsf{Vec}$.

3. Back to Definition Land

We have seen the initial definition of a module category, along with some simple examples. Building on this, we briefly introduce some more basic definitions, beginning with module subcategories and module functors. These obviously categorify submodules and module homomorphismsm, respectively.

Definition 3.1. (Module Subcategory). *Let $\mathcal{M}$ be a $\mathcal{C}$-module category. A* module subcategory $\mathcal{N}$ of $\mathcal{M}$ *is a full subcategory $\mathcal{N} \subseteq \mathcal{M}$ that is closed under the module product bifunctor.*

Definition 3.2. (Module Functor). *Let $\mathcal{M}$ and $\mathcal{N}$ be module categories over $\mathcal{C}$ with associativity constraints m and n, respectively. A* $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{N}$ *is a functor $F : \mathcal{M} \to \mathcal{N}$, together with a natural isomorphism having components*

$$s_{X,M} : F(X \otimes M) \to X \otimes F(M),$$

for all $X \in \mathcal{C}$ and $M \in \mathcal{M}$, such that the diagrams

$$\begin{tikzcd}
 & F((X \otimes Y) \otimes M) \arrow[dl, "F(m_{X,Y,M})"'] \arrow[dr, "s_{X \otimes Y, M}"] & \\
F(X \otimes (Y \otimes M)) \arrow[d, "s_{X, Y \otimes M}"'] & & (X \otimes Y) \otimes F(M) \arrow[d, "n_{X,Y,F(M)}"] \\
X \otimes F(Y \otimes M) \arrow[rr, "\mathrm{id}_X \otimes s_{Y,M}"'] & & X \otimes (Y \otimes F(M))
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(\mathbb{1} \otimes M) \arrow[rr, "s_{\mathbb{1},M}"] \arrow[dr, "F(l_M)"'] & & \mathbb{1} \otimes F(M) \arrow[dl, "l_{F(M)}"] \\
 & F(M) &
\end{tikzcd}$$

commute for all $X, Y \in \mathcal{C}$ and $M \in \mathcal{M}$.

Definition 3.3. (Equivalence of Module Categories). *Let $\mathcal{M}$ and $\mathcal{N}$ be module categories over $\mathcal{C}$. A* $\mathcal{C}$-module equivalence *is a module functor $F : \mathcal{M} \to \mathcal{N}$ that is also an equivalence of categories.*

Remark 3.4. A version of Mac Lane's strictness theorem exists for module categories; for any module category, one can show that there is a module equivalence between it and some strict module category.

Proposition-Definition 3.5. ([EGNO16, Proposition 7.3.4]). *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be module categories over $\mathcal{C}$. The* direct sum $\mathcal{M} \coloneqq \mathcal{M}_1 \oplus \mathcal{M}_2$, *whose module product, associativity constraints and unit constraints are sums of those of $\mathcal{M}_1$ and $\mathcal{M}_2$, is also a module category over $\mathcal{C}$.*

Definition 3.6. (Simple Module Category). *A module category is said to be* indecomposable *if there is no module equivalence between it and a non-trivial direct sum of module categories (that is, a direct sum of non-zero module categories). It is called* simple *if it has no non-trivial module subcategories.*

4. Exact Module Categories

Earlier, we mentioned that module categories categorify modules over rings. Indeed, if $\mathcal{M}$ is a module category over a monoidal category $\mathcal{C}$, then the Grothendieck group $\mathrm{Gr}(\mathcal{M})$ forms a module over the ring $\mathrm{Gr}(\mathcal{C})$ (in fact, it forms a $\mathbb{Z}_+$-module over the $\mathbb{Z}_+$-ring $\mathrm{Gr}(\mathcal{C})$ – see [Ost03]). There's a problem, however; because module functors as we defined them need not be exact, they will not necessarily descend to morphisms of Grothendieck group modules. Another issue we currently have is that even in the simplest example of a tensor category, with $\mathcal{C} := \mathsf{Vec}$, there is no hope in classifying its module categories (as this is equivalent to understanding every locally finite Abelian category)! We will therefore find it helpful to look instead at a smaller, nicer class of module categories.

Definition 4.1. (Exact Module Category). *Let $\mathcal{C}$ be a multitensor category with "enough projectives", in the sense that every simple object has a projective cover. A module category over $\mathcal{C}$ is said to be* exact *if, for any projective $P \in \mathrm{Ob}(\mathcal{C})$ and any $M \in \mathrm{Ob}(\mathcal{M})$, the object $P \otimes M$ is projective in $\mathcal{M}$.*

In a technical sense, we may think of exact module categories as somehow playing a role analogous to the role of projective modules.

Example 4.2. Any multitensor category with enough projectives is an exact module category over itself. This is a simple consequence of [EGNO16, Proposition 4.2.12].

Example 4.3. Let $\mathcal{M}$ be a module category over $\mathcal{C} := \mathsf{Vec}$. Because the unit object of Vec is projective, it follows that $\mathcal{M}$ is exact if and only if every object is projective, as we have that $M \cong \mathbb{1} \otimes M$ for any object $M \in \mathrm{Ob}(\mathcal{M})$. Moreover, it is a fact that a locally finite Abelian category $\mathcal{M}$ is semisimple if and only if all of its objects are projective; thus the exact module categories over Vec are precisely the semisimple ones. It follows that the exact module categories over Vec are classified by the cardinality of their set of isomorphism classes of simple objects.

Example 4.4. By [EGNO16, Corollary 4.2.13], a rigid monoidal category with enough projectives is semisimple if and only if its unit is projective. Thus our argument from Example 4.3 generalizes; if $\mathcal{C}$ is semisimple, then any exact $\mathcal{C}$-module category will also be semisimple.

Lemma 4.5. ([EGNO16, Lemma 7.6.1]) *Exact module categories have enough projectives.*

Lemma 4.6. ([EGNO16, Corollary 7.6.4]) *An object in an exact module category is projective if and only if it is injective.*

Remark 4.7. It is an immediate corollary of Lemma 4.5 that all exact module categories with finitely many isomorphism classes of simple objects are finite (as being finite means being locally finite, having enough projectives and having finitely many isomorphism classes of simple objects), whence Lemma 4.6 will tell us that our module category is *quasi-Frobenius* (finite Abelian category where objects are projective if and only if they are injective). This is actually a nice situation to be in, as such categories are amenable to homological techniques; in particular, any quasi-Frobenius category is either semisimple or of infinite homological dimension ([EGNO16, Remark 6.1.4]).

It is also worth commenting on our remark from the beginning of this section, regarding how module categories decategorify. If $\mathcal{M}$ is an indecomposable exact module category, then the module it decategorifies to is not only indecomposable, but irreducible.

Proposition 4.8. ([EGNO16, Proposition 7.7.2]) *Let $\mathcal{M}$ be an indecomposable exact module category over $\mathcal{C}$. Then $\mathrm{Gr}(\mathcal{M})$ is a simple $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$.*

Remark 4.9. In the case of a finite multitensor category $\mathcal{C}$, there are only finitely many $\mathbb{Z}_+$-modules over $\mathrm{Gr}(\mathcal{C})$ that are of the form $\mathrm{Gr}(\mathcal{M})$, for some indecomposable exact $\mathcal{C}$-module category $\mathcal{M}$. This is a consequence of [EGNO16, Proposition 3.4.6].

To conclude this section, we have the following cute characterization of exact module categories in terms of exact module functors.

Proposition 4.10. ([EGNO16, Propositions 7.6.9 and 7.9.7]) *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two module categories over $\mathcal{C}$. If $\mathcal{M}_1$ is exact, then every additive module functor $F : \mathcal{M}_1 \to \mathcal{M}_2$ is exact. Conversely, if every module functor of the form $F : \mathcal{M}_1 \to \mathcal{M}_2$ is exact, then $\mathcal{M}_1$ is exact.*

The proof of the converse here relies on a tool we haven't seen yet, known as the internal Hom. In the sequel, we will introduce these along with another characterization of module categories.

5. Characterizing Module Categories

In this semifinal section, we will introduce some technology that plays an important role in the characterization of module categories. We begin by defining algebra objects and module objects over them. By the end, we will have shown that these algebras also characterize certain module categories.

Definition 5.1. (Algebra Object). *Let $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$ be a monoidal category. An* algebra in $\mathcal{C}$ *is an object $A \in \mathrm{Ob}(\mathcal{C})$, together with a* unit morphism $u : \mathbb{1} \to A$ *and a* multiplication morphism $m : A \otimes A \to A$*, such that the diagrams*

$$\begin{tikzcd} & A & \\ \mathbb{1} \otimes A \arrow[ur, "l_A"] \arrow[rr, "u \otimes \mathrm{id}_A"'] & & A \otimes A \arrow[ul, "m"'] \end{tikzcd} , \quad \begin{tikzcd} & A & \\ A \otimes \mathbb{1} \arrow[ur, "r_A"] \arrow[rr, "\mathrm{id}_A \otimes u"'] & & A \otimes A \arrow[ul, "m"'] \end{tikzcd}$$

and

$$\begin{tikzcd} (A \otimes A) \otimes A \arrow[rr, "a_{A,A,A}"] \arrow[d, "m \otimes \mathrm{id}_A"'] & & A \otimes (A \otimes A) \arrow[d, "\mathrm{id}_A \otimes m"] \\ A \otimes A \arrow[dr, "m"'] & & A \otimes A \arrow[dl, "m"] \\ & A & \end{tikzcd}$$

commute.

Example 5.2. If $\mathcal{C}$ is a rigid monoidal category, then $\mathbb{1}$ is an algebra. For any $X \in \mathrm{Ob}(\mathcal{C})$, the object $A \coloneqq X \otimes X^*$ is also an algebra, with unit $u = \mathrm{coev}_X$ and multiplication $m = \mathrm{id}_X \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^*}$.

Example 5.3. An algebra object in Vec is a finite-dimensional, associative, unital $\Bbbk$-algebra.

Example 5.4. An algebra object in Vec_G is a G-graded algebra. Moreover, if L is a subgroup of G, then the group algebra $\Bbbk L$ is an algebra in Vec_G.

Example 5.5. Let G be a finite group and $\mathsf{Rep}(G)$ the category of finite-dimensional representations of G over $\Bbbk$. The algebra $\mathrm{Fun}(G)$ of functionals from G to $\Bbbk$, where G acts by right translation, is an algebra in $\mathsf{Rep}(G)$.

Definition 5.6. (Module Object). *Let (A, m, u) be an algebra in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, a, l, r)$. A* right module over A *is an object $M \in \mathrm{Ob}(\mathcal{C})$, together with a morphism $p : M \otimes A \to M$, such that the diagrams*

$$\begin{tikzcd}
 & M & \\
M \otimes \mathbb{1} \arrow[ur, "r_M"] \arrow[rr, "\mathrm{id}_M \otimes u"'] & & M \otimes A \arrow[ul, "p"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
(M \otimes A) \otimes A \arrow[rr, "a_{M,A,A}"] \arrow[d, "p \otimes \mathrm{id}_A"'] & & M \otimes (A \otimes A) \arrow[d, "\mathrm{id}_M \otimes m"] \\
M \otimes A \arrow[dr, "p"'] & & M \otimes A \arrow[dl, "p"] \\
 & M &
\end{tikzcd}$$

commute.

We define left A-modules and bimodules similarly. We can also define submodules, subalgebras, ideals and homomorphisms of both algebras and modules in this setting. We won't give explicit definitions for all of these, but we will at least give the definition of module homomorphisms, since we will actually need to use them. The definition of an algebra homomorphism is essentially the same idea, but we would need an additional coherence diagram for the unit morphisms.

Definition 5.7. (Module Homomorphism). *Let (M_1, p_1) and (M_2, p_2) be right modules over an algebra A. A morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}}(M_1, M_2)$ is said to be a* module homomorphism *if the diagram*

$$\begin{tikzcd}
M_1 \otimes A \arrow[r, "\varphi \otimes \mathrm{id}_A"] \arrow[d, "p_1"'] & M_2 \otimes A \arrow[d, "p_2"] \\
M_1 \arrow[r, "\varphi"'] & M_2
\end{tikzcd}$$

commutes. We write $\mathrm{Mor}_A(M_1, M_2)$ for the class of right A-module homomorphisms from M_1 to M_2.

Note that the A-modules of $\mathcal{C}$ together with the A-module homomorphisms form a category $\mathsf{Mod}_{\mathcal{C}}(A)$. Moreover, if $\mathcal{C}$ is a multitensor category, then we have that $\mathrm{Hom}_A(M_1, M_2)$ is a vector subspace of $\mathrm{Hom}_{\mathcal{C}}(M_1, M_2)$, whence $\mathsf{Mod}_{\mathcal{C}}(A)$ is a linear Abelian caegory. We thus have the following proposition.

Proposition 5.8. ([EGNO16, Proposition 7.8.10]). *Let $A \in \mathrm{Ob}(\mathcal{C})$ be an algebra. Then the category $\mathsf{Mod}_{\mathcal{C}}(A)$ defined above is a left $\mathcal{C}$-module category in the appropriate sense.*

We will leave verifying this, along with other properties – such as that $\mathcal{C}$ having enough projectives or being finite implies that $\mathsf{Mod}_{\mathcal{C}}(A)$ also has enough projectives or is finite – as an exercise. We also note that the dual of this proposition also holds just as we would expect; that is, the category A-$\mathsf{Mod}_{\mathcal{C}}$ of *left* A-modules is a *right* $\mathcal{C}$-module category.

Remark 5.9. We should carefully note that while these algebras give us a machine for generating module categories, they will not necessarily give us every module category. Consider, for instance, the following pathological example. If $\mathcal{C} := \mathsf{Vec}$, then the category $\mathbf{Vec}$ of all (possibly infinite-dimensional) vector spaces forms a module category over it in the obvious way. But it is a fact that if $\mathcal{C}$ is finite, then so is $\mathsf{Mod}_{\mathcal{C}}(A)$; thus $\mathbf{Vec}$ cannot appear as a category of modules over an algebra in $\mathcal{C}$.

Example 5.10. Let $\mathcal{C}$ be a rigid monoidal category and $A := X \otimes X^*$ an algebra as in Example 5.2. For every $Y \in \mathrm{Ob}(\mathcal{C})$, the object $M_Y := Y \otimes X^*$ is a right A-module with $p := \mathrm{id}_Y \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^*}$. Moreover, the functor $Y \mapsto M_Y$ is a $\mathcal{C}$-module functor from the regular $\mathcal{C}$-module category $\mathcal{C}$ to $\mathsf{Mod}_{\mathcal{C}}(A)$. We will see soon that by Theorem 5.16, this functor is actually a module equivalence!

Definition 5.11. (Morita Equivalence). *We say that two algebras $A, B \in \mathrm{Ob}(\mathcal{C})$ are* Morita equivalent *if $\mathsf{Mod}_{\mathcal{C}}(A)$ and $\mathsf{Mod}_{\mathcal{C}}(B)$ are equivalent as $\mathcal{C}$-module categories.*

As we mentioned at the beginning of the section, there really is quite a lot you can say about algebra objects, and we cannot possibly hope to cover everything! Instead, we will transition towards stating the characterization of module categories that was promised. To this end, we introduce the notion of internal Hom, which is another surprisingly nice piece of technology for studying module categories.

Let $\mathcal{C}$ be a finite multitensor category and $\mathcal{M}$ a $\mathcal{C}$-module category, with $M_1, M_2 \in \mathrm{Ob}(\mathcal{M})$. Suppose we consider the functor $\mathrm{Hom}_{\mathcal{M}}(- \otimes M_1, M_2) : \mathcal{C} \to \mathsf{Vec}$ for which $X \mapsto \mathrm{Hom}_{\mathcal{M}}(X \otimes M_1, M_2)$. This functor is left exact and hence representable; that is, there exists an object $\underline{\mathrm{Hom}}(M_1, M_2) \in \mathrm{Ob}(\mathcal{C})$ and a natural isomorphism between the functors $\mathrm{Hom}_{\mathcal{M}}(- \otimes M_1, M_2) \cong \mathrm{Hom}_{\mathcal{C}}(-, \underline{\mathrm{Hom}}(M_1, M_2))$.

Definition 5.12. (Internal Hom). *The object $\underline{\mathrm{Hom}}(M_1, M_2)$ representing $\mathrm{Hom}_{\mathcal{M}}(- \otimes M_1, M_2)$ in the previous paragraph is called the* internal Hom from M_1 to M_2*. By the Yoneda lemma, we also have an* internal Hom bifunctor $\underline{\mathrm{Hom}}(-, -) : \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$ *defined by* $(M_1, M_2) \mapsto \underline{\mathrm{Hom}}(M_1, M_2)$.

Remark 5.13. Note that we have technically asked that $\mathcal{C}$ be finite. This is not strictly necessary, however: the functor $\mathrm{Hom}_{\mathcal{M}}(- \otimes M_1, M_2)$ is still representable in the non-finite case, only in this case it is represented by an ind-object of $\mathcal{C}$. Much of the following theory extends to this more general case, but we will focus on the finite case for the sake of simplicity.

Proposition 5.14. ([EGNO16, Corollary 7.9.5]). *Let $\mathcal{M}$ be a module category over a multitensor category $\mathcal{C}$ and fix $M \in \mathrm{Ob}(\mathcal{M})$. Then $N \mapsto \underline{\mathrm{Hom}}(M, N)$ is a $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathcal{C}$.*

It follows from Proposition 4.10 and Proposition 5.14 that if $\mathcal{M}$ is an exact module category, then the internal Hom bifunctor from Definition 5.12 is biexact. In fact, the converse is true!

Proposition 5.15. ([EGNO16, Proposition 7.9.7]). *Let $\mathcal{M}$ be a module category over a multitensor category $\mathcal{C}$. If the functor $\underline{\mathrm{Hom}}(M, -) : N \mapsto \underline{\mathrm{Hom}}(M, N)$ is exact for all $M \in \mathcal{M}$, then $\mathcal{M}$ is exact.*

Worth checking is the fact that the object $\underline{\mathrm{Hom}}(M, M)$ is an algebra for every $M \in \mathcal{M}$, while for every $N \in \mathcal{M}$ the object $\underline{\mathrm{Hom}}(M, N)$ is a right $\underline{\mathrm{Hom}}(M, M)$-module. This means that $\mathsf{Mod}_{\mathcal{C}}(\underline{\mathrm{Hom}}(M, M))$ is a left $\mathcal{C}$-module category by Proposition 5.8, whence it follows that $\underline{\mathrm{Hom}}(M, -)$ is a $\mathcal{C}$-module functor from $\mathcal{M}$ to $\mathsf{Mod}_{\mathcal{C}}(\underline{\mathrm{Hom}}(M, M))$. With this, we may state our main result.

Theorem 5.16. ([EGNO16, Theorem 7.10.1]). *Let $\mathcal{M}$ be a module category over a multitensor category $\mathcal{C}$ and $M \in \mathrm{Ob}(\mathcal{M})$ an object for which*

(i). *the functor $\underline{\mathrm{Hom}}(M, -)$ is right exact (it is automatically left exact);*
(ii). *for any $N \in \mathrm{Ob}(\mathcal{M})$, there exists a surjection $X \otimes M \to N$ for some $X \in \mathrm{Ob}(\mathcal{C})$.*

Then the internal Hom functor $\underline{\mathrm{Hom}}(M, -) : \mathcal{M} \to \mathsf{Mod}_{\mathcal{C}}(A)$ is an equivalence of $\mathcal{C}$-module categories, for $A \coloneqq \underline{\mathrm{Hom}}(M, M)$.

Suppose we just look at condition (i) of Theorem 5.16. There are two situations where it is satisfied:

- $\mathcal{M}$ is an arbitrary indecomposable $\mathcal{C}$-module category and $M \in \mathrm{Ob}(\mathcal{M})$ is projective;
- $\mathcal{M}$ is an exact indecomposable $\mathcal{C}$-module category and $M \in \mathrm{Ob}(\mathcal{M})$ is arbitrary.

As for condition (ii), this is equivalent to the fact that $\mathrm{Gr}(\mathcal{M})$ is cyclic as a $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$, generated by the isomorphism class $[M]$.

We conclude this section with a corollary of Theorem 5.16 that more explicitly illustrates the characterization of module categories that we promised at the beginning.

Corollary 5.17. ([EGNO16, Corollary 7.10.5]). *Let $\mathcal{M}$ be a module category over $\mathcal{C}$.*

(i). *If $\mathcal{M}$ is finite, then there exists a module equivalence $\mathcal{M} \cong \mathsf{Mod}_{\mathcal{C}}(A)$ for some algebra $A \in \mathrm{Ob}(\mathcal{C})$.*
(ii). *If $\mathcal{M}$ is exact and $M \in \mathrm{Ob}(\mathcal{M})$ is an object such that the isomorphism class $[M]$ generates $\mathrm{Gr}(\mathcal{M})$ as a $\mathbb{Z}_+$-module over $\mathrm{Gr}(\mathcal{C})$, then there is a module equivalence $\mathcal{M} \cong \mathsf{Mod}_{\mathcal{C}}(A)$ for $A \coloneqq \underline{\mathrm{Hom}}(M, M)$.*

Thus we are able to classify all finite module categories and all exact, "cyclic" module categories in terms of algebra objects. Brilliant!

6. Exotic Examples

We have now finally reached the coda. We would like to conclude by showcasing some technical examples. I owe a lot to Pinhas Grossman and Arnaud Brothier for their patience here!

As we mentioned in the introduction, module categories are a nice tool for studying subfactors. Before we give any examples though, we should give a quick background on subfactor theory. A *von Neumann algebra* is a self-adjoint subalgebra $M \subseteq \mathcal{B}(H)$ such that $M = M''$, where $\mathcal{B}(H)$ denotes the set of bounded, linear operators on a (separable) complex Hilbert space H and where we define the *commutant* of $S \subseteq \mathcal{B}(H)$ to be $S' := \{x' \in \mathcal{B}(H) : x'x = xx', \text{ for all } x \in S\}$. A von Neumann algebra is said to be a *factor* if it has trivial centre, and a *subfactor* $N \subseteq M$ is a unital inclusion of factors. The key point here is that every commutative von Neumann algebra is isomorphic to $L^\infty(\Gamma)$ for some measure space Γ ([Sak71, Proposition 1.18.1]); thus the theory of von Neumann algebras can be viewed as a "quantum analogue" to measure theory, where factors are those von Neumann algebras that are "as non-commutative as possible".

Example 6.1. Let $N \subseteq M$ be a finite index, unital inclusion of type II_1 factors. For our purposes, M being a factor of type II_1 means it admits a unique faithful state $\tau : M \to \mathbb{C}$, allowing us to perform a canonical GNS construction. The gist of this construction is to produce $L^2(M)$, the norm-completon of M under $\|x\|_2 := \sqrt{\tau(x^*x)}$, as well as a faithful representation $\pi_l : M \to \mathcal{B}(L^2(M))$ and a faithful antirepresentation $\pi_r : M \to \mathcal{B}(L^2(M))$ given by $\pi_l(x) : \widehat{y} \mapsto \widehat{xy}$ and $\pi_r(x) : \widehat{y} \mapsto \widehat{yx}$, respectively. These representations give left and right actions of both M and N on $L^2(M)$, as well as on the type II_1 factor $M_1 := (\pi_l(M) \cup \{e_N\})''$ given by Jones' basic construction, where $e_N \in \mathcal{B}(L^2(M))$ is the orthogonal projection of $L^2(M)$ onto $L^2(N)$ taking M to N. The category of (N,N)-bimodules generated by M under direct sums, sub-bimodules, conjugates and an appropriately chosen tensor product (namely, *Connes' fusion tensor product*) forms a tensor category ${}_N\mathsf{Mod}_N$. This, together with ${}_M\mathsf{Mod}_M$ generated by M_1, roughly categorifies the "even parts" of the subfactor's standard invariant. A similar process gives us module categories ${}_N\mathsf{Mod}_M$ and ${}_M\mathsf{Mod}_N$ over these tensor categories.

Example 6.2. I am deeply thankful to Pinhas Grossman and Arnaud Brothier for this wonderful example. The Haagerup subfactor is the unique subfactor with index $\frac{5+\sqrt{13}}{4}$ ([AH99]), and is the finite depth subfactor with smallest index greater than 4 ([Haa94]). It is one of the most extensively studied examples of an *exotic* subfactor; that is, a subfactor not arising from groups or quantum groups. The even parts of its standard invariant give rise to two fusion categories: one with three simple objects and one with six. Let's look at the latter, which we shall denote by $\mathscr{H}(\mathbb{Z}/3\mathbb{Z})$. In general, given a finite Abelian group G, we define the *Haagerup-Izumi fusion category* $\mathscr{H}(G)$ to be the fusion category whose set of simple objects $\{g\}_{g\in G} \sqcup \{X_g\}_{g\in G}$ is subject to the fusion rules

$$g \otimes h := gh, \qquad g \otimes X_{\mathbb{1}} := X_g =: X_{\mathbb{1}} \otimes g^{-1}, \qquad X_{\mathbb{1}} \otimes X_{\mathbb{1}} := \mathbb{1} \oplus \Big(\bigoplus_{g\in G} X_g\Big).$$

Naturally for $\mathbb{Z}/3\mathbb{Z}$ our six simple objects are $\{\mathbb{1}, g, g^2, X, gX, g^2X\}$. In [GS12], it was shown that there are three simple (semisimple and indecomposable) module categories over $\mathscr{H}(\mathbb{Z}/3\mathbb{Z})$, which by Corollary 5.17 we may identify with the algebra objects $\mathbb{1}$ (the trivial module category), $\mathbb{1} \oplus X$ (which corresponds to the Haagerup subfactor) and $\mathbb{1} \oplus g \oplus g^2$.

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