Categorical Representation Theory

1. Prologue

Before we get into representation theory, let's *very* briefly review some basic definitions from higher category theory. We will follow [Mal18] and [Str95, §9] as our main references.

Definition 1.1. (Bicategory). *A* bicategory $\mathscr{C}$ *consists of*

- *a class* $\mathrm{Ob}(\mathscr{C})$ *of* objects *(or* 0-cells*);*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, a* Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ *whose objects are called* 1-morphisms *(or* 1-cells*), whose morphisms are called* 2-morphisms *(or* 2-cells*) and where composition of 2-morphisms is known as* vertical composition *and denoted* $\circ_v$*;*
- *for each triple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k} \in \mathrm{Ob}(\mathscr{C})$*, a functor* $\circ_h : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ *known as* horizontal composition*;*
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a distinguished 1-morphism* $\mathrm{id}_{\mathtt{i}} \in \mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{i}))$ *known as the* identity morphism *for* $\mathtt{i}$*;*
- *for each pair of objects* $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, natural isomorphisms* l *and* r *satisfying*
$$\begin{pmatrix} f \mapsto \mathrm{id}_{\mathtt{j}} \circ_h f \\ \alpha \mapsto \mathrm{id}_{\mathrm{id}_{\mathtt{j}}} \circ_h \alpha \end{pmatrix} \overset{l}{\Rightarrow} \begin{pmatrix} f \mapsto f \\ \alpha \mapsto \alpha \end{pmatrix} \overset{r}{\Leftarrow} \begin{pmatrix} f \mapsto f \circ_h \mathrm{id}_{\mathtt{i}} \\ \alpha \mapsto \alpha \circ_h \mathrm{id}_{\mathrm{id}_{\mathtt{i}}} \end{pmatrix},$$
known respectively as a left *and* right unitor *(whose components* l_f *and* r_f *are 2-morphisms);*
- *for each quadruple of objects* $\mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l} \in \mathrm{Ob}(\mathscr{C})$*, a natural isomorphism* a *between the two horizontal composition functors* $\mathscr{C}(\mathtt{k}, \mathtt{l}) \times \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{l})$ *given by*
$$\begin{pmatrix} h \times g \times f \mapsto (h \circ_h g) \circ_h f \\ \gamma \times \beta \times \alpha \mapsto (\gamma \circ_h \beta) \circ_h \alpha \end{pmatrix} \overset{a}{\Rightarrow} \begin{pmatrix} h \times g \times f \mapsto h \circ_h (g \circ_h f) \\ \gamma \times \beta \times \alpha \mapsto \gamma \circ_h (\beta \circ_h \alpha) \end{pmatrix},$$
known as an associator *(whose components* $a_{h,g,f}$ *are 2-morphisms);*

such that the pentagon diagram

$$\begin{tikzcd}
 & ((k \circ_h h) \circ_h g) \circ_h f \arrow[dl, "a_{k,h,g} \circ_h \mathrm{id}_f"'] \arrow[dr, "a_{k \circ_h h, g, f}"] & \\
(k \circ_h (h \circ_h g)) \circ_h f \arrow[d, "a_{k, h \circ_h g, f}"'] & & (k \circ_h h) \circ_h (g \circ_h f) \arrow[d, "a_{k,h,g \circ_h f}"] \\
k \circ_h ((h \circ_h g) \circ_h f) \arrow[rr, "\mathrm{id}_k \circ_h a_{h,g,f}"'] & & k \circ_h (h \circ_h (g \circ_h f))
\end{tikzcd}$$

and the triangle diagram

$$\begin{tikzcd}
(g \circ_h \mathrm{id}_{\mathtt{j}}) \circ_h f \arrow[rr, "a_{g, \mathrm{id}_{\mathtt{j}}, f}"] \arrow[dr, "r_g \circ_h \mathrm{id}_f"'] & & g \circ_h (\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[dl, "\mathrm{id}_g \circ_h l_f"] \\
 & g \circ_h f &
\end{tikzcd}$$

commute, for all 1-morphisms $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*,* $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k}, \mathtt{l}))$*,* $k \in \mathrm{Ob}(\mathscr{C}(\mathtt{l}, \mathtt{m}))$*.*

A 2-category is a *strict* bicategory; that is, a bicategory whose unitors and associators are all identities. In this case the pentagon and triangle diagrams hold automatically. Observe that (strict) bicategories $\mathscr{C}$ with a single object $\bullet$ are in bijection with (strict) monoidal categories under taking the monoidal delooping; in particular, our monoidal category is nothing but the End-category $\mathscr{C}(\bullet,\bullet)$, where the monoidal product is given by horizontal composition.

We shall henceforth adopt the notation $\mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j}) \coloneqq \mathrm{Ob}(\mathscr{C}(\mathtt{i},\mathtt{j}))$ and $\mathrm{Mor}^2_{\mathscr{C}}(f,g) \coloneqq \mathrm{Mor}_{\mathscr{C}(\mathtt{i},\mathtt{j})}(f,g)$, for $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ and $f,g \in \mathrm{Mor}(\mathscr{C}(\mathtt{i},\mathtt{j}))$. Unfortunately, we will frequently change our notation for 1-morphisms and 2-morphisms depending on what makes sense contextually. For instance, in light of the above remark, 1-morphisms will often be objects of a monoidal category, whence we will write them as X, Y, Z and so on; meanwhile, sometimes they will be realized as functors, in which case we will use F, G, H and so on. The same goes for 2-morphisms. We hope this will not cause any unnecessary confusion!

Definition 1.2. (Pseudofunctor). *A pseudofunctor* F *between bicategories* $\mathscr{C}$ *and* $\mathscr{D}$ *consists of*

- *a map* $\mathrm{F} : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C})$*;*
- *for each pair of objects* $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$*, a functor* $\mathrm{F} : \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}),\mathrm{F}(\mathtt{j}))$*;*
- *for each triple of objects* $\mathtt{i},\mathtt{j},\mathtt{k} \in \mathrm{Ob}(\mathscr{C})$*, a natural isomorphism* m *between the two "composition by* F*" functors* $\mathscr{C}(\mathtt{j},\mathtt{k}) \times \mathscr{C}(\mathtt{i},\mathtt{j}) \to \mathscr{D}(\mathrm{F}(\mathtt{i}),\mathrm{F}(\mathtt{k}))$ *given by*
$$\begin{pmatrix} g \times f \mapsto \mathrm{F}(g) \circ_h \mathrm{F}(f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta) \circ_h \mathrm{F}(\alpha) \end{pmatrix} \xRightarrow{m} \begin{pmatrix} g \times f \mapsto \mathrm{F}(g \circ_h f) \\ \beta \times \alpha \mapsto \mathrm{F}(\beta \circ_h \alpha) \end{pmatrix},$$
whose components $m_{g,f}$ *are 2-morphisms;*
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, an isomorphism* $i : \mathrm{id}_{\mathrm{F}(\mathtt{i})} \to \mathrm{F}(\mathrm{id}_{\mathtt{i}})$ *in* $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}),\mathrm{F}(\mathtt{i})))$*;*

such that the hexagon diagram
$$\begin{tikzcd}
 & (\mathrm{F}(h) \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) \arrow[dl, "m_{h,g}\circ_h \mathrm{id}_{\mathrm{F}(f)}"'] \arrow[dr, "a_{\mathrm{F}(h),\mathrm{F}(g),\mathrm{F}(f)}"] & \\
\mathrm{F}(h \circ_h g) \circ_h \mathrm{F}(f) \arrow[d, "m_{h\circ_h g,f}"'] & & \mathrm{F}(h) \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[d, "\mathrm{id}_{\mathrm{F}(h)}\circ_h m_{g,f}"] \\
\mathrm{F}((h \circ_h g) \circ_h f) \arrow[dr, "\mathrm{F}(a_{h,g,f})"'] & & \mathrm{F}(h) \circ_h \mathrm{F}(g \circ_h f) \arrow[dl, "m_{h,g\circ_h f}"] \\
 & \mathrm{F}(h \circ_h (g \circ_h f)) &
\end{tikzcd}$$
and the squares
$$\begin{tikzcd}
\mathrm{id}_{\mathrm{F}(\mathtt{j})} \circ_h \mathrm{F}(f) \arrow[r, "l_{\mathrm{F}(f)}"] \arrow[d, "i\circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & \mathrm{F}(f) \\
\mathrm{F}(\mathrm{id}_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[r, "m_{\mathrm{id}_{\mathtt{j}},f}"'] & \mathrm{F}(\mathrm{id}_{\mathtt{j}} \circ_h f) \arrow[u, "\mathrm{F}(l_f)"']
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\mathrm{F}(f) \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[r, "r_{\mathrm{F}(f)}"] \arrow[d, "\mathrm{id}_{\mathrm{F}(f)}\circ_h i"'] & \mathrm{F}(f) \\
\mathrm{F}(f) \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}}) \arrow[r, "m_{f,\mathrm{id}_{\mathtt{i}}}"'] & \mathrm{F}(f \circ_h \mathrm{id}_{\mathtt{i}}) \arrow[u, "\mathrm{F}(r_f)"']
\end{tikzcd}$$
commute, for all 1-morphisms $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i},\mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j},\mathtt{k}))$*,* $h \in \mathrm{Ob}(\mathscr{C}(\mathtt{k},\mathtt{l}))$*.*

As before, a 2-functor is a pseudofunctor where m and i are identity. In the same way that bicategories generalize monoidal categories, pseudofunctors generalize strong monoidal functors, preserving both vertical composition (strictly) and horizontal composition (up to isomorphism).

Definition 1.3. (Pseudonatural Transformation). *A* pseudonatural transformation Φ *between pseudofunctors* F *to* G *consists of*

- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, a* 1*-morphism* $\Phi_{\mathtt{i}} \in \mathrm{Ob}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{i})))$*;*
- *for each* 1*-morphism* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, a* 2*-morphism* $\Phi_f : \Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \to \mathrm{G}(f) \circ_h \Phi_{\mathtt{i}}$ *in* $\mathrm{Mor}(\mathscr{D}(\mathrm{F}(\mathtt{i}), \mathrm{G}(\mathtt{j})))$*;*

such that

- *for each* 2*-morphism* $\alpha : f \to g$ *in* $\mathrm{Mor}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*, the square*

$$\begin{tikzcd}
\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f) \arrow[r, "\mathrm{id}_{\Phi_{\mathtt{j}}} \circ_h \mathrm{F}(\alpha)"] \arrow[d, "\Phi_f"'] & \Phi_{\mathtt{j}} \circ_h \mathrm{F}(g) \arrow[d, "\Phi_g"] \\
\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}} \arrow[r, "\mathrm{G}(\alpha) \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"'] & \mathrm{G}(g) \circ_h \Phi_{\mathtt{i}}
\end{tikzcd}$$

  commutes (that is, the 2*-morphisms* Φ_f *are the components of a natural transformation);*
- *for each pair of* 1*-morphisms* $f \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j}))$*,* $g \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k}))$*, the octagon diagram*

$$\begin{tikzcd}
 & (\Phi_{\mathtt{k}} \circ_h \mathrm{F}(g)) \circ_h \mathrm{F}(f) \arrow[r, "a_{\Phi_{\mathtt{k}}, \mathrm{F}(g), \mathrm{F}(f)}"] \arrow[dl, "\Phi_g \circ_h \mathrm{id}_{\mathrm{F}(f)}"'] & \Phi_{\mathtt{k}} \circ_h (\mathrm{F}(g) \circ_h \mathrm{F}(f)) \arrow[dr, "\mathrm{id}_{\Phi_{\mathtt{k}}} \circ_h m_{g,f}"] & \\
(\mathrm{G}(g) \circ_h \Phi_{\mathtt{j}}) \circ_h \mathrm{F}(f) \arrow[d, "a_{\mathrm{G}(g), \Phi_{\mathtt{j}}, \mathrm{F}(f)}"'] & & & \Phi_{\mathtt{k}} \circ_h \mathrm{F}(g \circ_h f) \arrow[d, "\Phi_{g \circ_h f}"] \\
\mathrm{G}(g) \circ_h (\Phi_{\mathtt{j}} \circ_h \mathrm{F}(f)) \arrow[dr, "\mathrm{id}_{\mathrm{G}(g)} \circ_h \Phi_f"'] & & & \mathrm{G}(g \circ_h f) \circ_h \Phi_{\mathtt{i}} \\
 & \mathrm{G}(g) \circ_h (\mathrm{G}(f) \circ_h \Phi_{\mathtt{i}}) & (\mathrm{G}(g) \circ_h \mathrm{G}(f)) \circ_h \Phi_{\mathtt{i}} \arrow[l, "a_{\mathrm{G}(g), \mathrm{G}(f), \Phi_{\mathtt{i}}}"] \arrow[ur, "m_{g,f} \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"'] &
\end{tikzcd}$$

  commutes;
- *for each object* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*, the pentagon diagram*

$$\begin{tikzcd}
\Phi_{\mathtt{i}} \circ_h \mathrm{id}_{\mathrm{F}(\mathtt{i})} \arrow[r, "r_{\Phi_{\mathtt{i}}}"] \arrow[d, "\mathrm{id}_{\Phi_{\mathtt{i}}} \circ_h i"'] & \Phi_{\mathtt{i}} & \mathrm{id}_{\mathrm{G}(\mathtt{i})} \circ_h \Phi_{\mathtt{i}} \arrow[l, "l_{\Phi_{\mathtt{i}}}"'] \arrow[d, "i \circ_h \mathrm{id}_{\Phi_{\mathtt{i}}}"] \\
\Phi_{\mathtt{i}} \circ_h \mathrm{F}(\mathrm{id}_{\mathtt{i}}) \arrow[rr, "\Phi_{\mathrm{id}_{\mathtt{i}}}"'] & & \mathrm{G}(\mathrm{id}_{\mathtt{i}}) \circ_h \Phi_{\mathtt{i}}
\end{tikzcd}$$

  commutes.

If each Φ_f *is invertible (they form a natural isomorphism), we call* Φ *a* pseudonatural isomorphism*.*

We say that two bicategories are *biequivalent* if there exists an invertible pseudofunctor between them. For the sections that follow, we will assume that all categories are essentially small, that all bicategories are essentially small and that all fields are algebraically closed.

Note that given a bicategory $\mathscr{C}$ and objects $X, Y \in \mathrm{Ob}(\mathscr{C})$, we have that $\mathscr{C}(X, Y)$ is a $(\mathscr{C}(X, X), \mathscr{C}(Y, Y))$-bimodule category. Check bicategory of bifinite bimodules as in [DGG14].

Examples of bicategories.

Given an algebra (or more generally a ring), one can build a 2-category whose objects are algebras, 1-morphisms are modules (A, B)-bimodules and 2-morphisms are bimodule maps. In subfactor theory, we do this for a unital inclusion $N \subseteq M$ of type II_1 subfactors; we set $\mathrm{Ob}(\mathscr{C}) := \{M, N\}$ and $\mathscr{C}(X, Y) := \mathsf{Bimod}(X, Y)$ ("bimodule summands of basic constructions" / bifinite (X, Y)-bimodules). This produces a 2-category.

Given a shaded planar algebra P, take $\mathrm{Ob}(\mathscr{C}) := \{-, +\}$ and let $\mathscr{C}(\varepsilon, \eta) := \mathsf{Rect}_P(\varepsilon, \eta)$ be the subcategory of the rectangular category consisting of those morphisms whose domain shading is ε and whose codomain shading is η, with composition running from bottom to top. In other words,

$$\mathrm{Mor}^1_{\mathscr{C}}(\varepsilon, \eta) := \{2k : k \in \mathbb{N} \text{ is even if } \varepsilon = \eta \text{ and odd otherwise}\}$$

and $\mathrm{Mor}^2_{\mathscr{C}}(m, n) := P_{m+n,\varepsilon}$, for $m, n \in \mathrm{Mor}^1_{\mathscr{C}}(\varepsilon, \eta)$ ([DGG14]). This gives a $\mathbb{C}$-linear (but unfortunately not additive) 2-category. Look into "The Temperley–Lieb algebra at roots of unity".

Explicit example of multifinitary bicategory: group planar algebra? Maybe we can generalize it, see https://scholars.unh.edu/cgi/viewcontent.cgi?article=1338&context=dissertation.

2. Finitary Birepresentation Theory

Definition 2.1. (Idempotent Complete). *Let $\mathcal{C}$ be a category. An* idempotent *is an endomorphism $p : A \to A$ in $\mathcal{C}$ such that $p \circ p = p$. An idempotent is said to* split *if there is an object B and morphisms $\pi : A \to B$, $\iota : B \to A$ in $\mathcal{C}$ such that $p = \iota \circ \pi$ and* $\mathrm{id}_B = \pi \circ \iota$*. A category is said to be* idempotent complete *(or* idempotent split*) if every idempotent splits.*

Note that the condition $\mathrm{id}_B = \pi \circ \iota$ implies that π is an epimorphism and ι is a monomorphism. To see why, suppose we have morphisms $h, k : B \to C$ with $h \circ \pi = k \circ \pi$. Then $h = h \circ \pi \circ \iota = k \circ \pi \circ \iota = k$, whence π is an epimorphism. Similarly, given morphisms $h, k : C \to B$ with $\iota \circ h = \iota \circ k$, we have that $h = \pi \circ \iota \circ h = \pi \circ \iota \circ k = k$, whence ι is a monomorphism. Thus, because ι is a monomorphism, B is by definition a subobject of A. In other words, a category being idempotent complete means that every idempotent $p : A \to A$ can be seen as a projection onto some subobject B followed by an inclusion back into A. Moreover, in the additive setting we have the following result.

Proposition 2.2. *An idempotent $p : A \to A$ belonging to a preadditive category splits if and only if $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.*

Proof. Suppose $p : A \to A$ is an idempotent that splits. Then by definition we have a subobject I of A together with an epimorphism $\pi_I : A \to I$ and a monomorphism $\iota_I : I \to A$ satisfying $p = \iota_I \circ \pi_I$ and $\mathrm{id}_I = \pi_I \circ \iota_I$. Moreover, because $\mathrm{id}_A - p$ is also idempotent, there similarly exists a subobject K of A together with an epimorphism $\pi_K : A \to K$ and a monomorphism $\iota_K : K \to A$ satisfying $\mathrm{id}_A - p = \iota_K \circ \pi_K$ and $\mathrm{id}_K = \pi_K \circ \iota_K$. Because $\mathrm{id}_A = \iota_I \circ \pi_I + \iota_K \circ \pi_K$, we have the biproduct diagram

$$\begin{tikzcd} I \arrow[r, shift right, "\iota_I"'] & A \arrow[l, shift right, "\pi_I"'] \arrow[r, shift left, "\pi_K"] & K. \arrow[l, shift left, "\iota_K"] \end{tikzcd}$$

By [Mac13, Theorem VIII.2.2], it follows that $A = I \oplus K$. We claim now that $\mathrm{Im}(p) = I$; we shall prove this by showing that p admits the canonical decomposition

$$K \xrightarrow{\iota_K} A \xrightarrow{\pi_I} I \xrightarrow{\iota_I} A \xrightarrow{\pi_K} K.$$

In particular, we claim that $\mathrm{Ker}(p) = (K, \iota_K)$, that $\mathrm{Coker}(p) = (K, \pi_K)$, that $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and that $\mathrm{Ker}(\pi_K) = (I, \iota_I)$. We show that the first two hold and remark that showing the remaining two is essentially the same. First, observe that

$$p \circ \iota_K = \iota_I \circ \pi_I \circ \iota_K = (\mathrm{id}_A - \iota_K \circ \pi_K) \circ \iota_K = \iota_K - \iota_K \circ \pi_K \circ \iota_K = \iota_K - \iota_K = 0.$$

Moreover, given an object K' together with a morphism $k' : K' \to A$ for which $p \circ k' = 0$, we see that by taking $\ell \coloneqq \pi_K \circ k'$, we have that

$$\iota_K \circ \ell = \iota_K \circ \pi_K \circ k' = (\mathrm{id}_A - p) \circ k' = k'.$$

Thus $\mathrm{Ker}(p) = (K, \iota_K)$. As for $\mathrm{Coker}(p)$, we observe that

$$\pi_K \circ p = \pi_K \circ \iota_I \circ \pi_I = \pi_K \circ (\mathrm{id}_A - \iota_K \circ \pi_K) = \pi_K - \pi_K \circ \iota_K \circ \pi_K = \pi_K - \pi_K = 0,$$

and that for any object C' together with a morphism $c' : A \to C'$ for which $c' \circ p = 0$, taking $\ell \coloneqq c' \circ \iota_K$ gives us

$$\ell \circ \pi_K = c' \circ \iota_K \circ \pi_K = c' \circ (\mathrm{id}_A - p) = c'.$$

That $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and $\mathrm{Ker}(\pi_K) = (I, \iota_I)$ follow similarly, whence $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$.

Conversely, suppose that $p : A \to A$ is an idempotent for which $A = \mathrm{Im}(p) \oplus \mathrm{Ker}(p)$. Then we have the canonical decomposition

$$\mathrm{Ker}(p) \xrightarrow{k} A \xrightarrow{\pi} \mathrm{Im}(p) \xrightarrow{\iota} A \xrightarrow{c} \mathrm{Coker}(p).$$

By definition this means that $p = \iota \circ \pi$, so we need only show that $\mathrm{id}_{\mathrm{Im}(p)} = \pi \circ \iota$. But note that π is a cokernel and ι is a kernel, hence they are an epimorphism and a monomorphism, respectively. Thus by the definition of epimorphisms and monomorphisms, we may cancel $p \circ p = p$ on the right by ι and on the left by π, whence we obtain nothing but

$$p \circ p = p \implies \iota \circ \pi \circ \iota \circ \pi = \iota \circ \pi \implies \pi \circ \iota = \mathrm{id}_{\mathrm{Im}(p)}$$

as desired. Thus p splits. This completes the proof. ■

This result is not only important in its own right, but psychologically helpful: it tells us that split idempotents categorify in some heuristic sense the notion of projections from linear algebra, which always split. Moreover, recall that a preadditive category is said to be *Karoubian* (or *pseudo-Abelian*) if every idempotent admits a kernel (or, equivalently, if every idempotent admits an image, as we may obtain the image by considering $\mathrm{Ker}(\mathrm{id}_A - p)$). We therefore have the following corollary.

Corollary 2.3. *A preadditive category is Karoubian if and only if it is idempotent complete.*

Example 2.4. The category of projective modules over a ring is the Karoubi envelope of its full subcategory of free modules, as a module is projective if and only if it is a direct summand of a free module. In other words, categories of projective modules are idempotent complete in a universal way.

If representation theory can naïvely be described as "group theory in linear sets", birepresentation theory can be described as "group theory in linear categories". We will now make precise the notion of "linear categories" that we will find ourselves working with.

Definition 2.5. (Finitary Category). *An additive, $\Bbbk$-linear category $\mathcal{C}$ is called* multifinitary *if it is idempotent complete, it has finitely many isomorphism classes of indecomposable objects and it has finite-dimensional $\Bbbk$-vector spaces of morphisms. If $\mathcal{C}$ is not monoidal, this is equivalent to being* finitary*; otherwise, it is said to be* finitary *if its unit object is indecomposable.*

Remark 2.6. Let $\mathcal{C}$ be an additive category. Then by [Mac13, §VIII.2], its morphisms form a matrix calculus; that is, for any $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ with $X \cong \bigoplus_{i=1}^{m} X_i$ and $Y \cong \bigoplus_{j=1}^{n} Y_j$, we have that

$$f = \sum_{j=1}^{n} \sum_{i=1}^{m} (\iota_{Y_j} \circ f_{i,j} \circ \pi_{X_i})$$

for $f_{i,j} := \pi_{Y_j} \circ f \circ \iota_{X_i}$, where $\pi_{X_i} : X \to X_i$ and $\pi_{Y_j} : Y \to Y_j$ are epimorphisms while $\iota_{X_i} : X_i \to X$ and $\iota_{Y_j} : Y_j \to Y$ are monomorphisms for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

We also have the following folklorish theorem, which has some nice exactness implications in the module category setting. Due to its very technical nature we will only offer a (fallible) sketch of the proof, but the full proof should follow similarly to the proof of the embedding theorem for Abelian categories. Please see the wonderful write-up given in [Jun19] for more details on these subtleties.

Theorem 2.7. (Freyd–Mitchell Embedding Theorem). *A small category is multifinitary if and only if there is an exact, full embedding into the exact category* $\mathsf{Mod}_p(A)$ *of finitely generated, projective modules over some finite-dimensional, associative* $\Bbbk$*-algebra* A*.*

Sketch. First, $\mathsf{Mod}_p(A)$ is certainly Karoubi (in fact, the category of projective modules over any ring is the Karoubi envelope of its full subcategory of free modules), and it also has both finitely many isomorphism classes of indecomposable objects and finite-dimensional $\Bbbk$-vector spaces of morphisms.

Conversely, let $\mathcal{C}$ be multifinitary. We wish to find a full, exact embedding $\mathcal{C} \to \mathsf{Mod}_p(A)$ for some finite-dimensional, associative $\Bbbk$-algebra A. Denote by $\mathcal{L} := \mathsf{Fun}_l(\mathcal{C}, \mathsf{Vect}_{\Bbbk}^{\text{f.d.}})$ the category of left exact, $\Bbbk$-linear functors from $\mathcal{C}$ to $\mathsf{Vect}_{\Bbbk}^{\text{f.d.}}$ (which we note are also automatically additive). The contravariant Yoneda embedding $X \mapsto \mathcal{C}(X, -)$ gives us a full, exact embedding よ $: \mathcal{C}^{\text{op}} \to \mathcal{L}$, as $\mathcal{C}(X, -)$ is left exact, which corresponds to a full, exact, covariant embedding よ$^{\text{op}} : \mathcal{C} \to \mathcal{L}^{\text{op}}$ by duality[†]. Because $\mathcal{L}$ is complete with injective cogenerator given by the direct product $\prod_{X \in \text{Ob}(\mathcal{C})} \mathcal{C}(X, -) \in \text{Ob}(\mathcal{L})$, it follows that $\mathcal{L}^{\text{op}}$ is cocomplete and admits a corresponding projective generator $P' \in \text{Ob}(\mathcal{L}^{\text{op}})$. Let $\mathcal{L}'$ be the small, exact, full subcategory of $\mathcal{L}^{\text{op}}$ generated by the image of よ, which we note is itself multifinitary by the Yoneda lemma; the final step is to embed $\mathcal{L}'$ into the category of projective modules. Well, suppose we write $I := \bigsqcup_{F \in \text{Ob}(\mathcal{L}')} \text{Mor}_{\mathcal{L}'}(P', F)$ and define $P := \bigoplus_{i \in I} P'$. Of course $A := \text{End}_{\mathcal{L}'}(P)$ is a finite-dimensional, associative $\Bbbk$-algebra. Moreover, for any $F \in \text{Ob}(\mathcal{L}')$, we may endow $\text{Mor}_{\mathcal{L}'}(P, F)$ with the structure of a finitely-generated, projective A-module in a canonical way by taking $a \cdot x := x \circ a$ for all $a \in A$ and $x \in \text{Mor}_{\mathcal{L}'}(P, F)$. Since P is also a projective generator, we have a full, exact embedding $\mathscr{F} : \mathcal{L}' \to \mathsf{Mod}_p(A)$ sending $F \mapsto \text{Mor}_{\mathcal{L}'}(P, F)$. Thus $\mathscr{F} \circ$ よ$^{\text{op}} : \mathcal{C} \to \mathsf{Mod}_p(A)$ is itself a full, exact embedding, giving us the desired result. □

Remark 2.8. This result tells us that a small category is multifinitary if and only if it is equivalent to a full subcategory of the category of finitely generated, projective modules over some finite-dimensional, associative $\Bbbk$-algebra ([MMM+23]). It may also be worth noting that we can replace "projective" with "injective" in this alternative definition, as these two notions are dual in the sense that being projective is of course the same as being injective in the opposite category.

Theorem 2.9. ([Sha23, Theorem 6.1]). *Let* $\mathcal{C}$ *be an additive,* $\Bbbk$*-linear category with finite-dimensional* $\Bbbk$*-vector spaces of morphisms. Then the following are equivalent.*

(i). $\mathcal{C}$ *is Krull–Schmidt.*
(ii). $\mathcal{C}$ *is idempotent complete.*
(iii). *An object* $X \in \text{Ob}(\mathcal{C})$ *is indecomposable if and only if the ring* $\text{End}_{\mathcal{C}}(X)$ *is local.*

See [Sha23, Theorem 6.1] for the proof. There is alternatively a proof for (ii) $\implies$ (iii) given in [Ela20, Lecture 1, Theorem 2]. This theorem tells us that a multifinitary category is precisely a Krull–Schmidt category with finitely many isomorphism classes of indecomposable objects!

Definition 2.10. *Let* $\mathfrak{A}_{\Bbbk}^{f}$ *denote the 2-category whose objects are multifinitary categories, whose 1-morphisms are* $\Bbbk$*-linear functors and whose 2-morphisms are natural transformations.*

[†] Since this will inevitably be someone's first time seeing it, よ is the hiragana for "yo" (as in "Yoneda").

We briefly remind the reader that any $\Bbbk$-linear functor between additive, $\Bbbk$-linear categories is automatically additive!

For the following definition, we interpret the End-categories $\mathscr{C}(\mathtt{i},\mathtt{i})$ as being monoidal categories with respect to the composition of 1-morphisms.

Definition 2.11. (Finitary Bicategory). *A bicategory $\mathscr{C}$ is said to be* (multi)finitary *if*

- *it has finitely many objects;*
- *for any pair $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, the Hom-category $\mathscr{C}(\mathtt{i},\mathtt{j})$ is (multi)finitary;*
- *horizontal composition of 2-morphisms is $\Bbbk$-bilinear.*

In a multifinitary bicategory, vertical composition will automatically be both biadditive and $\Bbbk$-bilinear as a consequence of the Hom-categories $\mathcal{C}(\mathtt{i},\mathtt{j})$ being additive and $\Bbbk$-linear. However, we genuinely must ask that horizontal composition be biadditive and $\Bbbk$-bilinear.

Proposition 2.12. *A monoidal category $\mathcal{C}$ is (multi)finitary if and only if its monoidal delooping $\mathsf{B}\mathcal{C}$ is (multi)finitary.*

Proof. If $\mathsf{B}\mathcal{C}$ is (multi)finitary, it is obvious that $\mathcal{C}$ is (multi)finitary, since $\mathcal{C} = \mathsf{B}\mathcal{C}(\bullet,\bullet)$, where $\mathrm{Ob}(\mathsf{B}\mathcal{C}) = \{\bullet\}$. Conversely, suppose $\mathcal{C}$ is (multi)finitary. Clearly $\mathsf{B}\mathcal{C}$ has finitely many objects (in particular, it has only one), and $\mathsf{B}\mathcal{C}(\bullet,\bullet) = \mathcal{C}$ is (multi)finitary. Finally, horizontal composition is given by the monoidal product and is hence biadditive and $\Bbbk$-bilinear. This completes the proof. ■

Definition 2.13. (Birepresentation). *A* birepresentation *of a bicategory $\mathscr{C}$ is a pseudofunctor from $\mathscr{C}$ to Cat, the 2-category of small categories. A* 2-representation *of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to Cat.*

Definition 2.14. (Finitary Birepresentation). *A* (multi)finitary birepresentation *of a (multi)finitary bicategory $\mathscr{C}$ is a covariant, $\Bbbk$-linear pseudofunctor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$. A* (multi)finitary 2-representation *of a (multi)finitary 2-category $\mathscr{C}$ is a covariant, $\Bbbk$-linear 2-functor from $\mathscr{C}$ to $\mathfrak{A}_{\Bbbk}^{f}$.*

Definition 2.15. (Finitary Module Category). *A* (multi)finitary module category *over a (multi)finitary monoidal category $\mathcal{C}$ is a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ for which the module product bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is $\Bbbk$-bilinear.*

Remark 2.16. Recall that a representation of a group G is morally nothing but a functor $F : \mathsf{B}G \to \mathsf{Vect}$, where the action of G on $V := F(\bullet)$ is given by $g \cdot v := [F(g)](v)$ for all $v \in V$ and $g \in \mathrm{Mor}(\mathsf{B}G)$. Analogously, given a 2-representation $\mathrm{M} : \mathscr{C} \to \mathsf{Cat}$, we have a 2-*action* of $\mathscr{C}$ given by $F \cdot X := [\mathrm{M}(F)](X)$ for all $X \in \mathrm{M}(\mathtt{i})$ and $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$, where $\mathtt{i},\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. The upshot here is that we should think of the 1-morphisms in our 2-category as our “group elements”, with composition becoming “group multiplication”.

Proposition 2.17. *Let $\mathcal{C}$ be a monoidal (respectively, strict monoidal) category. There exists a bijection between $\mathcal{C}$-module categories $\mathcal{M}$ and birepresentations (respectively, 2-representations) M of the delooping category $\mathsf{B}\mathcal{C}$. Moreover, $\mathcal{M}$ is (multi)finitary if and only if M is (multi)finitary.*

Proof. Let $\mathcal{C}$ be a monoidal category and $\mathsf{B}\mathcal{C}$ its delooping category. Recall that by [EGNO16, Proposition 7.1.3] there is a bijection between $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors of the form $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$. Such a functor F induces a canonical birepresentation $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ that takes the single object $\bullet \in \mathrm{Ob}(\mathsf{B}\mathcal{C})$ to $\mathcal{M}$ and otherwise acts on the 1-morphisms and 2-morphisms by F. Conversely, let $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ be a birepresentation and write $\mathcal{M} := \mathrm{M}(\bullet)$. This naturally induces a functor $F : \mathcal{C} \to \mathsf{Cat}(\mathcal{M}, \mathcal{M}) = \mathrm{End}(\mathcal{M})$ that acts on objects and morphisms of $\mathcal{C}$ by M. Clearly these two constructions are inverse to each other and (multi)finitarity of both sides is equivalent. It is easy to see that if $\mathcal{C}$ is strict, the result for 2-representations follows similarly. This completes the proof. ■

We also note that by Remark 2.8, every multifinitary module category is automatically exact, as equivalences of categories preserve projectivity of objects and hence every object must be projective.

Definition 2.18. (Equivalence of Birepresentations). *We say that two birepresentations M and N of $\mathscr{C}$ are* equivalent *if there exists a pseudonatural isomorphism $\Phi : \mathrm{M} \to \mathrm{N}$ such that the component $\Phi_{\mathtt{i}} : \mathrm{M}(\mathtt{i}) \to \mathrm{N}(\mathtt{i})$ is an equivalence of categories for each $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$.*

Definition 2.19. (Yoneda Birepresentation). *Let $\mathscr{C}$ be a bicategory and consider the pseudofunctor $\mathscr{C}(\mathtt{i}, -) : \mathscr{C} \to \mathsf{Cat}$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, such that*

- *objects $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ are sent to the Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$;*
- *1-morphisms of the form $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$ are sent to the "horizontal post-composition by F" functor $F_* : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ given by $G \mapsto F \circ G$ and $(\gamma : G \to G') \mapsto \mathrm{id}_F \circ_h \gamma$;*
- *2-morphisms of the form $\alpha \in \mathrm{Mor}^2_{\mathscr{C}}(F, F')$, for $F, F' \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$, are sent to the "horizontal post-composition by α" natural transformation $\alpha_* : F_* \Rightarrow F'_*$, whose components are given by $(\alpha_*)_G := \alpha \circ_h \mathrm{id}_G$ for each $G \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$.*

We call this the Yoneda *(or* principal*)* birepresentation corresponding to $\mathtt{i}$ *and denote it by $\mathbb{P}_{\mathtt{i}}$.*

If $\mathscr{C}$ is (multi)finitary, then its corresponding Yoneda birepresentations are also all (multi)finitary, and if $\mathscr{C}$ is a (multi)finitary 2-category its Yoneda birepresentations are (multi)finitary 2-representations. As an example, let $\mathscr{C}$ be the monoidal delooping of $\mathcal{C}$. Then $\mathbb{P}_\bullet$ maps $\bullet$ to $\mathcal{C}$, maps 1-morphisms $X \in \mathrm{Ob}(\mathcal{C})$ to the left tensor product functor given by $Y \mapsto X \otimes Y$ and $(f : Y \to Y') \mapsto \mathrm{id}_X \otimes f$, and maps 2-morphisms $f : X \to X'$ to the natural transformation $Y \mapsto f \otimes \mathrm{id}_Y$. This is nothing but the birepresentation corresponding to the regular $\mathcal{C}$-module category $\mathcal{C}$!

Definition 2.20. (Ideal). *A* left *(respectively* right*)* ideal *of a category $\mathcal{C}$ is a collection $\mathcal{I} := \{\mathcal{I}(X,Y) : X,Y \in \mathrm{Ob}(\mathcal{C})\}$, where each $\mathcal{I}(X,Y)$ is a non-empty subclass of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, such that $\mathcal{I}$ is stable under post-composition (respectively pre-composition) with morphisms from $\mathcal{C}$. If $\mathcal{C}$ is preadditive, we additionally ask that $\mathcal{I}(X,Y)$ be an Abelian subgroup of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ for all pairs $X,Y \in \mathrm{Ob}(\mathcal{C})$, and if $\mathcal{C}$ is $\Bbbk$-linear we ask that these also be $\Bbbk$-subspaces. We say that $\mathcal{I}$ is a* two-sided *(or* bilateral*)* ideal *if it is both a left ideal and a right ideal, and that it is a subideal of $\mathcal{J}$ if its classes morphisms are subclasses. An ideal is said to be* proper *if there exists some pair $X,Y \in \mathrm{Ob}(\mathcal{C})$ for which $\mathcal{I}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$, and* maximal *if it is proper and not a subideal of any other proper ideal.*

Let $\mathcal{I}$ be an ideal of a category $\mathcal{C}$. As we have implied previously, if $\mathcal{C}$ is a preadditive, then $\mathrm{End}_{\mathcal{C}}(X)$ is a ring for all $X \in \mathrm{Ob}(\mathcal{C})$, and it follows that the valid choices for $\mathcal{I}(X,X)$ coincide exactly with the ring ideals of $\mathrm{End}_{\mathcal{C}}(X)$. Similarly, if $\mathcal{C}$ is a $\Bbbk$-linear category, then each $\mathrm{End}_{\mathcal{C}}(X)$ is an associative, unital algebra, and the valid choices for $\mathcal{I}(X,X)$ coincide with algebra ideals (that is, a subspace of $\mathrm{End}_{\mathcal{C}}(X)$ that is closed under algebra multiplication).

Example 2.21. Consider a subcategory $\mathcal{T}$ of $\mathsf{Vect}_{\Bbbk}$ whose endomorphisms of $\Bbbk^n$ are the $n \times n$ upper triangular Toeplitz matrices. Then $\mathrm{End}(\Bbbk^2) \cong \Bbbk[x]/\langle x^2 \rangle$. If we consider the sub-semicategory of $\mathcal{T}$ containing only the object $\Bbbk^2$ and the endomorphisms $\Bbbk\{x\}$ (that is, linear scalings of the matrix with 1 in the off-diagonal), we obtain an ideal of $\mathcal{T}$.

Let M be a multifinitary birepresentation of $\mathscr{C}$ for which each $\mathrm{M}(\mathtt{j})$ is additive and idempotent complete, and let $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ for some $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. Consider the *additive closure* (closure under isomorphisms, direct summands and finite direct sums) of the orbit of X under the action of $\mathscr{C}$; that is, the collection of objects

$$\mathscr{C}(\{X\}) := \mathsf{add}(\{[\mathrm{M}(F)](X) : \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})\})$$

where the add denotes the aforementioned additive closure. Due to the additivity of the 1-morphisms of $\mathfrak{A}^f_{\Bbbk}$, it follows that $\mathscr{C}(\{X\})$ is itself stable under the action of $\mathscr{C}$. This therefore induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X\})$ of $\mathscr{C}$ by restriction, with each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ sent to

$$\mathscr{C}_{\mathtt{j}}(\{X\}) := \mathsf{Add}(\{[\mathrm{M}(F)](X) : F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})\}),$$

the *additive subcategory* (full subcategory that is closed under isomorphisms, direct summands and finite direct sums) of $\mathrm{M}(\mathtt{j})$ generated by the objects of $\mathscr{C}(\{X\})$ that lie in $\mathrm{M}(\mathtt{j})$. Because $\mathrm{M}(\mathtt{j})$ is Karoubian, this is nothing but the Karoubi envelope of the full subcategory generated by the action of $\mathscr{C}$. In principle this process works for any collection $\{X_i : i \in I\}$ with $X_i \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}_i))$, whence

$$\mathscr{C}(\{X_i : i \in I\}) := \mathsf{Add}(\{[\mathrm{M}(F)](X_i) : i \in I, \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_i,\mathtt{j})\})$$

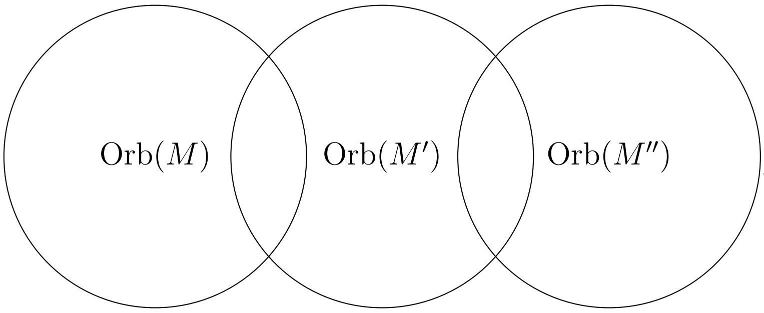
similarly induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X_i : i \in I\})$ of $\mathscr{C}$. In any case, we will only need to consider the single-object situation, as it allows us to make the following evocative definition.

Definition 2.22. (Transitive Birepresentation). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory $\mathscr{C}$. We say that* M *is* transitive *if, for every $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ and non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$, the embedding $\mathscr{C}_{\mathtt{j}}(\{X\}) \hookrightarrow \mathrm{M}(\mathtt{j})$ is an equivalence for all $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$.*

Remark 2.23. Recall that a module M is simple if and only if every cyclic submodule generated by a non-zero element of M is equal to M. This is exactly what we're trying to capture with transitivity! In the birepresentation world, however, things are more involved. We will see more on this shortly.

We say that a multifinitary $\mathcal{C}$-module category $\mathcal{M}$ is transitive if, for all $M \in \mathrm{Ob}(\mathcal{M})$, we have that $\mathrm{Orb}(M) := \mathsf{add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathrm{Ob}(\mathcal{M})$ by Proposition 2.17. This is equivalent to it having no full Karoubian subcategories, equivalent to its corresponding birepresentation being transitive and equivalent to its split Grothendieck group $\mathrm{Gr}(\mathcal{M})$ being a simple $\mathrm{Gr}(\mathcal{C})$-module.

Being transitive is clearly a much stronger condition for a module category than being indecomposable (in the sense that it is not the direct sum of two non-zero multifinitary module subcategories). Given any $M \in \mathrm{Ob}(\mathcal{M})$ and $M' \in \mathrm{Orb}(M)$, it's possible that the orbit of M and the orbit of some $M'' \in \mathrm{Orb}(M')$ may have trivial intersection. For instance,



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This picture becomes much more insightful in light of Lemma 2.29 and Proposition 2.30.

Definition 2.24. ($\mathscr{C}$-Stable Ideal). *Let* M *be a birepresentation of* $\mathscr{C}$*. A* $\mathscr{C}$-stable ideal I *of* M *is a collection* $\mathrm{I} := \{\mathrm{I}(\mathtt{i}) : \mathtt{i} \in \mathrm{Ob}(\mathscr{C})\}$*, where each* $\mathrm{I}(\mathtt{i})$ *is a two-sided ideal of* $\mathrm{M}(\mathtt{i})$ *such that* $[\mathrm{M}(F)](\mathrm{I}(\mathtt{i}))$ *is a subclass of* $\mathrm{I}(\mathtt{j})$ *for all* 1*-morphisms* $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$*. A* $\mathscr{C}$*-stable subideal* I *of a* $\mathscr{C}$*-stable ideal* J *is a* $\mathscr{C}$*-stable ideal for which* $\mathrm{I}(\mathtt{i})$ *is a subideal of* $\mathrm{J}(\mathtt{i})$ *for all* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$*. We say that* I *is* proper *if there exists some* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *for which* $\mathrm{I}(\mathtt{i})$ *is proper, and* maximal *if it is proper and not a* $\mathscr{C}$*-stable subideal of any other proper* $\mathscr{C}$*-stable ideal.*

Definition 2.25. (Simple Birepresentation). *A multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *is said to be* simple *if it admits no proper, non-zero* $\mathscr{C}$*-stable ideals.*

Similarly to before, we say that a $\mathcal{C}$-module category $\mathcal{M}$ is simple if its corresponding birepresentation is simple. In other words, given non-zero $f, g \in \mathrm{Mor}(\mathcal{M})$, we can obtain g by composing f with other morphisms in $\mathcal{M}$ and acting via $\mathcal{C}$ (by taking the left monoidal product with identity morphisms).

In light of the module category picture, we see that the "right" way to think about transitivity and simplicity is to observe that transitivity is really just asking that your birepresentation is cyclically generated by its objects ($\mathsf{add}(\{X \otimes M : X \in \mathrm{Ob}(\mathcal{C})\}) = \mathrm{Ob}(\mathcal{M})$ for all $X \in \mathrm{Ob}(\mathcal{M})$), while simplicity is asking that it is cyclically generated by its morphisms ($\{g \otimes f : g \in \mathrm{Mor}(\mathcal{C})\} = \mathrm{Mor}(\mathcal{M})$ for all $f \in \mathrm{Mor}(\mathcal{M})$)! This perspective, in addition to the following result, really elucidates our notion of simplicity for birepresentations.

Proposition 2.26. *Every simple birepresentation is transitive.*

Proof. Let M be a simple birepresentation of a multifinitary bicategory $\mathscr{C}$ and take $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero. Certainly $\mathrm{G}_{\mathrm{M}}(\{X\})$ is non-zero (as it contains X) and hence induces a non-proper $\mathscr{C}$-stable ideal of M by simplicity. Thus for each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, we know that $\mathrm{Mor}(\mathscr{C}_{\mathtt{j}}(\{X\}))$ cannot be proper and so $\mathscr{C}_{\mathtt{j}}(\{X\})$ must be equivalent to $\mathrm{M}(\mathtt{j})$. In other words, M is transitive. This completes the proof. ■

It is not necessarily the case that transitive birepresentations are simple! Many of the Lusztig-Vogan module categories we will study are not simple, but they are all transitive. We do, however, have the following proposition. This result will end up being important in formulating the categorical version of the Jordan–Hölder theorem.

Proposition 2.27. *Let* M *be a transitive birepresentation of a multifinitary bicategory* $\mathscr{C}$*. Then* M *admits a unique maximal* $\mathscr{C}$*-stable ideal* I*, and moreover each* $\mathrm{I}(\mathtt{i})$ *contains no identity morphisms apart from the one corresponding to the zero object.*

Proof. Let I be the sum (as vector spaces) of all $\mathscr{C}$-stable ideals of M that do not contain id_X for any non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and any $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This is certainly itself $\mathscr{C}$-stable ideal by construction. Moreover, because $U, V \subseteq U + V$ for vector spaces U, V, it follows that a sum of ideals is itself an ideal containing each ideal being summed, whence I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms. To see that it is genuinely maximal, suppose J is a $\mathscr{C}$-stable ideal containing I. Because I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms, J must contain at least one identity morphism, say id_X for some non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$. Given any non-zero $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, the transitivity of M tells us that Y is either isomorphic to a direct summand of $[\mathrm{M}(F)](X)$, for some $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$, or isomorphic to a direct sum $[\mathrm{M}(F_1)](X) \oplus \cdots \oplus [\mathrm{M}(F_n)](X)$, for some $F_1, \ldots, F_n \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$. We claim that id_Y must lie in $\mathrm{J}(\mathtt{j})$ in both cases.

Let $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$ with $[\mathrm{M}(F)](X) = X_1 \oplus \cdots \oplus X_n$ and suppose that $\varphi : Y \to X_k$ is an isomorphism for some $1 \leq k \leq n$. Then $[\mathrm{M}(F)](\mathrm{id}_X) = \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \in \mathrm{J}(\mathtt{j})$ by $\mathscr{C}$-stability. But by pre-composing with $\iota_{X_k} \circ \varphi^{-1} : Y \to X_k \to X_1 \oplus \cdots \oplus X_n$ and post-composing with $\varphi \circ \pi_{X_k} : X_1 \oplus \cdots \oplus X_n \to X_k \to Y$, we obtain that $\mathrm{id}_Y = (\varphi \circ \pi_{X_k}) \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ (\iota_{X_k} \circ \varphi^{-1}) \in \mathrm{J}(\mathtt{j})$.

Suppose now that $\varphi : Y \to X_1 \oplus \cdots \oplus X_n$ is an isomorphism, where for each $1 \leq k \leq n$ we have $X_k = [\mathrm{M}(F_k)](X)$ for some $F_k \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i},\mathtt{j})$. As before, $[\mathrm{M}(F_k)](\mathrm{id}_X) = \mathrm{id}_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$ by $\mathscr{C}$-stability. Moreover, because $\mathrm{J}(\mathtt{j})$ is an ideal, $\iota_{X_k} \circ \mathrm{id}_{X_k} \circ \pi_{X_k} = \iota_{X_k} \circ \pi_{X_k} \in \mathrm{J}(\mathtt{j})$ for all $1 \leq k \leq n$. Thus by the definition of an additive category, $\mathrm{id}_{X_1 \oplus \cdots \oplus X_n} = \iota_{X_1} \circ \pi_{X_1} + \cdots + \iota_{X_n} \circ \pi_{X_n} \in \mathrm{J}(\mathtt{j})$, whence it follows that $\mathrm{id}_Y = \varphi^{-1} \circ \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \circ \varphi \in \mathrm{J}(\mathtt{j})$.

We have thus shown that J must contain *all* identity morphisms and therefore cannot be proper, meaning that I must in fact be maximal as claimed. The uniqueness of I follows by construction. This completes the proof. ■

Proposition-Definition 2.28. (Simple Quotient). *A transitive birepresentation* M *of a multifinitary bicategory* $\mathscr{C}$ *is simple if and only if the unique maximal* $\mathscr{C}$*-stable ideal* I *from Proposition 2.27 is the zero ideal. The simple birepresentation* $\underline{\mathrm{M}}$ *given by the quotient* M/I *is known as the* simple quotient *of* M.

Proof. Naturally if I – the sum of all $\mathscr{C}$-stable ideals without non-zero identity morphisms – is the zero ideal, then M must contain no proper, non-zero $\mathscr{C}$-stable ideals. Conversely, if M is simple, then because I is the sum of proper $\mathscr{C}$-stable ideals, they must all be zero. Because I is maximal, every morphism $f \in \mathrm{Mor}([\mathrm{M}/\mathrm{I}](\mathtt{i}))$ must generate either $\{0\}$ or $[\mathrm{M}/\mathrm{I}](\mathtt{i})$ under multiplication by $\mathscr{C}$, whence it follows that M/I is simple. This completes the proof. ■

Let M be a multifinitary birepresentation of $\mathscr{C}$. We denote by $\mathrm{Ind}(\mathrm{M})$ the set of isomorphism classes of indecomposable objects in every $\mathrm{M}(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$; that is,

$$\mathrm{Ind}(\mathrm{M}) = \bigsqcup_{\mathtt{i} \in \mathrm{Ob}(\mathscr{C})} \{[X] \in \mathrm{M}(\mathtt{i}) : X \text{ is indecomposable}\}.$$

Note that $\mathrm{Ind}(\mathrm{M})$ is clearly finite, as $\mathscr{C}$ has finitely many objects and each category $\mathrm{M}(\mathtt{i}) \in \mathrm{Ob}(\mathfrak{A}^f_{\Bbbk})$ has finitely many isomorphism classes of indecomposable objects.

For $X, Y \in \mathrm{Ind}(\mathrm{M})$, where for instance $X \in \mathrm{M}(\mathtt{i}_X)$ and $Y \in \mathrm{M}(\mathtt{i}_Y)$, we write $Y \leq X$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_X, \mathtt{i}_Y)$ such that Y is isomorphic to a direct summand of $[\mathrm{M}(F)](X)$.

Lemma 2.29. *Let* M *be a multifinitary birepresentation. The binary relation* $\leq$ *defined above defines a preorder on* $\mathrm{Ind}(\mathrm{M})$ *known as the* action preorder.

Proof. Clearly $\geq$ is reflexive, as we can just take $F \coloneqq \mathrm{id}_{\mathtt{i}}$. Moreover, suppose that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$ and Y is isomorphic to a direct summand of $[\mathrm{M}(G)](Z)$; that is,

$$[\mathrm{M}(F)](Y) \cong X \oplus X_1 \oplus X_2 \oplus \cdots,$$
$$[\mathrm{M}(G)](Z) \cong Y \oplus Y_1 \oplus Y_2 \oplus \cdots.$$

In order to show transitivity, we would like to show that X is isomorphic to a direct summand of $[\mathrm{M}(FG)](Z)$. Well, because the morphisms of $\mathfrak{A}^f_{\Bbbk}$ are additive, we simply observe that

$$\begin{aligned}[\mathrm{M}(FG)](Z) &\cong [\mathrm{M}(F)](Y) \oplus [\mathrm{M}(F)](Y_1) \oplus [M(F)](Y_2) \oplus \cdots \\ &\cong X \oplus X_1 \oplus X_2 \oplus \cdots \oplus [\mathrm{M}(F)](Y_1) \oplus [\mathrm{M}(F)](Y_2) \oplus \cdots.\end{aligned}$$

This completes the proof. ■

Suppose we define an equivalence relation $\sim$ given by $X \sim Y$ if and only if $X \leq Y$ and $Y \leq X$. Obviously $\geq$ extends to a partial order on $\mathrm{Ind}(\mathrm{M})/\sim$. In particular, we have the following result.

Proposition 2.30. *Let* M *be a multifinitary birepresentation. Then* M *is transitive if and only if* $\mathrm{Ind}(\mathrm{M})/\sim$ *has only one element.*

Proof. Suppose $\mathrm{Ind}(\mathrm{M})/\sim$ is a singleton and take any $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero as a representative. Then for any indecomposable $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, there exists some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$ for which Y is isomorphic to a direct summand of $[M(F)](X)$, since $Y \leq X$. In other words, the additive subcategory $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, as by definition it is closed under direct summands. Thus M is transitive.

Conversely, suppose M is transitive, and consider any pair of indecomposables $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$. Because $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, we know by the definition of $\mathscr{C}_{\mathtt{j}}(\{X\})$ that Y is isomorphic to a direct summand of $[M(F)](X)$ for some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$; that is, $Y \leq X$. The same argument applied to $\mathscr{C}_{\mathtt{i}}(\{Y\})$ shows us that $X \leq Y$, whence $\mathrm{Ind}(\mathrm{M})/\sim$ has only one element. This completes the proof. ■

Definition 2.31. (Directed Order Ideal). *A* directed order ideal *of a partially ordered set* $(P, \leq)$ *is a non-empty subset* I *such that*

- *for all* $x \in I$ *and* $y \in P$, $y \leq x$ *implies that* $y \in I$ *(lower set);*
- *for all* $x, y \in I$, *there is some* $z \in I$ *such that* $x \leq z$ *and* $y \leq z$ *(upward directed set).*

Remark 2.32. In the literature, this partial order is often written as $X \geq Y$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_Y, \mathtt{i}_X)$ such that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$; that is, the inequality symbol is the other way around. As a result, they consider *coideals* (non-empty subsets that are upper sets and downward directed sets) rather than ideals in what follows. This is because we want to go "downwards", in the sense that we collect all direct summands. Why do they define the order "backwards"? Is there some (cell?) interpretation for which that is more natural?

Let M be a multifinitary birepresentation of $\mathscr{C}$ and Q a directed order ideal of $\mathrm{Ind}(\mathrm{M})/\sim$. For $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, define $\mathrm{M}_Q(\mathtt{i})$ to be the additive subcategory of $\mathrm{M}(\mathtt{i})$ generated by the indecomposable objects $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ whose equivalence class lies in Q. Then $\mathrm{M}_Q : \mathtt{i} \mapsto \mathrm{M}_Q(\mathtt{i})$ induces a multifinitary sub-birepresentation of M, known as the sub-birepresentation of M associated to Q.

Let $Q \subset R$ be a pair of directed order ideals in $\mathrm{Ind}(\mathrm{M})/\sim$ and let $\mathrm{I}_Q(\mathtt{i})$ denote the ideal in $\mathrm{M}_R(\mathtt{i})$ generated by the identity morphisms in $\mathrm{M}_Q(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This collection of ideals is $\mathscr{C}$-stable, whence the multifinitary birepresentation M_R induces a multifinitary birepresentation $\mathrm{M}_{R/Q} : \mathtt{i} \mapsto \mathrm{M}_R(\mathtt{i})/\mathrm{I}_Q(\mathtt{i})$. This is known as the quotient of M associated to $Q \subset R$. Note that if $|R \setminus Q| = 1$, then $|\mathrm{Ind}(\mathrm{M}_{R/Q})/\sim| = 1$, so $\mathrm{M}_{R/Q}$ will be transitive by Proposition 2.30.

Choose $r \in \mathrm{Ind}(\mathrm{M})/\sim$ and let X_r be the maximal directed order ideal in $\mathrm{Ind}(\mathrm{M})/\sim$ that does not contain r. In other words, $(\mathrm{Ind}(\mathrm{M})/\sim) \setminus X_r$ – the complement of X_r – has minimal element r. Thus we also obtain a directed order ideal $Y_r := X_r \cup \{r\}$, whence the associated quotient M_{Y_r/X_r} is transitive by Proposition 2.30. We henceforth let $\underline{\mathrm{M}}_r$ denote the simple quotient $\underline{\mathrm{M}}_{Y_r/X_r}$.

Consider a filtration of directed order ideals

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim$$

such that $|Q_i \backslash Q_{i-1}| = 1$ for all $i \in \{1, \ldots, n\}$. We call this a *complete filtration*. As shown previously, from such a filtration we have a corresponding *weak Jordan-Hölder series*

$$\{0\} = \mathrm{M}_{Q_0} \subset \mathrm{M}_{Q_1} \subset \cdots \subset \mathrm{M}_{Q_n} = \mathrm{M}$$

consisting of sub-birepresentations whose *weak composition quotients* $\mathrm{L}_i := \underline{\mathrm{M}}_{Q_i/Q_{i-1}}$ are simple birepresentations for all $i \in \{1, \ldots, n\}$. With this, we have the following result.

Theorem 2.33. (Weak Jordan–Hölder Theorem). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *admitting the two complete filtrations*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim,$$

$$\varnothing = Q'_0 \subset Q'_1 \subset \cdots \subset Q'_m = \mathrm{Ind}(\mathrm{M})/\sim,$$

with weak composition quotients $\{\mathrm{L}_i\}_{i=1}^n$ *and* $\{\mathrm{L}'_j\}_{j=1}^m$ *respectively. Then* $m = n$*, and moreover there exists a permutation* $\sigma \in S_n$ *such that* L_i *and* $\mathrm{L}_{\sigma(i)}$ *are equivalent for all* $i \in \{1, \ldots, n\}$*.*

Proof. We clearly have $m = n = |\mathrm{Ind}(\mathrm{M})/\sim|$ by the definition of a complete filtration. Suppose now that $r \in \mathrm{Ind}(\mathrm{M})/\sim$; then there exist unique $i, j \in \{1, 2, \ldots, n\}$ for which $Q_i \backslash Q_{i-1} = Q'_j \backslash Q_{j-1} = \{r\}$. If we can show that the birepresentations L_i and L'_j are both equivalent to $\underline{\mathrm{M}}_r$, then we are done. In particular, by symmetry it is enough to show that L_i is equivalent to $\underline{\mathrm{M}}_r$.

Let I_{X_r} be the $\mathscr{C}$-stable ideal in M_{Y_r} for which $\mathrm{M}_{Y_r/X_r} = \mathrm{M}_{Y_r}/\mathrm{I}_{X_r}$ and $\mathrm{I}_{Q_{i-1}}$ the $\mathscr{C}$-stable ideal in M_{Q_i} for which $\mathrm{M}_{Q_i/Q_{i-1}} = \mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$. Since $\{r\} = Q_i \setminus Q_{i-1}$, we know by construction that $Q_{i-1} \subseteq X_r$, as X_r is by definition the maximal directed order ideal not containing r; similarly, $Q_i \subseteq Y_r$. This second inclusion induces a pseudonatural isomorphism from M_{Q_i} to M_{Y_r} (a collection of natural isomorphisms from functors between Hom-categories to functors between Hom-subcategories), whence the first inclusion induces a pseudonatural isomorphism $\sigma : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Y_r/X_r}$ by taking the quotient. Now, $\mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$ contains only the objects generated by indecomposables in the equivalence class r. But for any such pair of indecomposable objects $X, Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$ lying in the equivalence class r, we have that $\mathrm{I}_{X_r}(X, Y) \subseteq \mathrm{I}_{Q_{i-1}}(X, Y)$ by the aforementioned inequalities. Thus the pseudonatural isomorphism σ factors through $\mathrm{M}_{Q_i/Q_{i-1}}$, in the sense that there exist pseudonatural transformations $\sigma_1 : \mathrm{M}_{Q_i} \Rightarrow \mathrm{M}_{Q_i/Q_{i-1}}$ and $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ such that $\sigma = \sigma_2 \circ \sigma_1$. In particular, this gives us a pseudonatural transformation $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \Rightarrow \mathrm{M}_{Y_r/X_r}$ that is obviously surjective on morphisms by fullness; therefore, because $\mathrm{M}_{Q_i/Q_{i-1}}$ and M_{Y_r/X_r} are both transitive, taking their simple quotients via proposition 2.28 induces an equivalence between L_i and $\underline{\mathrm{M}}_r$ as desired, whence the result follows. This completes the proof. ■

This proof feels kind of handwavey, need to double-check it slowly. Also, in what sense is this weak Jordan-Hölder theorem "weak"? The decategorifications of these simple quotients are "transitive $\mathbb{N}$-modules" and usually not simple. Should also do some examples here.

3. Category of Soergel Bimodules

Need to transfer my notes into LaTeX later on.

4. Lusztig–Vogan Module Categories

This chapter will aim to summarize the recent work of Larson and Romanov in their Soergel bimodule approach for algebraically categorifying the trivial block of the Lusztig–Vogan module ([LR22]), which provides interesting examples of module categories over the category of Soergel bimodules. In the most general setting, the construction of a Lusztig–Vogan module category takes as ingredients a connected, complex, reductive group G, a Borel subgroup B of G, a holomorphic involution θ of G and a finite-index subgroup K of the fixed-point subgroup $G^\theta := \{g \in G : \theta(g) = g\}$.

With the additional assumption that K is the identity component of G^θ (see [LR22] for details), we can proceed as follows. Let $P := \mathrm{Sym}(\mathrm{span}_{\mathbb{R}}(X_*(T_K)))$ be the symmetric algebra on the $\mathbb{R}$-span of the cocharacter lattice of a maximal torus T_K of K contained in $B_K := B \cap K$, graded such that $\mathrm{span}_{\mathbb{R}}(X_*(T_K))$ has degree 2, and let $R := \mathrm{Sym}(\mathrm{span}_{\mathbb{R}}(X_*(T)))$ be the symmetric algebra on the $\mathbb{R}$-span of the cocharacter lattice of a maximal torus T of G contained in B. Denote by $W := N_G(T)/T$ the Weyl group of G corresponding to T, by W^θ the set of elements of W fixed under the involution induced by θ and by $W_K := N_K(T_K)/T_K$ the Weyl group of K corresponding to T_K, where we write $P^{W_K} := \{p \in P : wp = p \text{ for all } w \in W_K\}$. Note that $W_K \subseteq W^\theta \subseteq W$, and by choosing a set of simple roots in W (whose associated positive roots correspond to B) we obtain a Coxeter system (W, S). Finally, let $\phi : R \to P$ be the algebra homomorphism extending the restriction map $X(T) \to X(T_K)$. For each $w \in W$, we define the *w-standard bimodule* P_w to be the (P_{W_K}, R)-bimodule given by P as a vector space with left action given by left multiplication and right action given by $s \cdot_w r := s\phi(wr)$, for all $s \in P$ and $r \in R$. We define

$$\mathcal{N}^0_{LV} := \langle P_w \otimes_R X : w \in W^\theta, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W,S))\rangle_{\oplus,\ominus,(1)}$$

to be the category of (P^{W_K}, R)-bimodules generated by standard bimodules under the right action of Soergel bimodules and closed under direct sums, direct summands and gradning shifts. By [LR22, Theorem 1.3.1], this categorifies the trivial blocks of the modules of Lusztig and Vogan.

To study this in full generality involves some deep results from Lie theory; thus for the time being we will restrict our attention to the case where the tori T_K and T are of equal rank (that is, where $T_K = T$ and hence $P = R$). In this situation the picture is much simpler. Let (W, S) be a Coxeter system with finite index subgroup $W_K \subseteq W$. Given the collection $\{\alpha_s : s \in S\}$ of simple roots, we define a polynomial algebra $R := \mathbb{R}[\alpha_1, \ldots, \alpha_n]$, which we recall is isomorphic to the symmetric algebra of the vector space associated with the geometric representation of (W, S). We therefore have an action of W on R induced by linearly extending the reflection action, given on generators $s \in W$ and simple roots $\alpha_t \in R$ by

$$s(\alpha_t) = \alpha_t + 2\cos\left(\frac{\pi}{m_{st}}\right)\alpha_s.$$

We define R^{W_K} to be the polynomials in R that are invariant under action by W_K. With this, given a generator w representing a coset in W/W_K, we define the w-standard bimodule R_w to be the (R^{W_K}, R)-bimdoule given by R as a vector space with left action given by left multiplication and right action given by $s \cdot_w r := s \cdot w(r)$, for all $s \in R_w$ and $r \in R$. Thus, just as before, the corresponding Lusztig–Vogan module category is

$$\mathcal{M}^0_{LV} := \langle R_w \otimes_R X : [w] \in W/W_K, X \in \mathrm{Ob}(\mathbb{S}\mathsf{Bim}(W,S))\rangle_{\oplus,\ominus,(1)}.$$

Why don't we just do right multiplication for the right action? Why do we look at W/W_K instead of the smaller W^θ? I think $W^\theta \cong W/W_K$ when $G^\theta = K$?

Remark 4.1. A connected, complex, linear algebraic group is reductive if and only if it is the complexification of a unique connected, compact Lie group ([Kam11, p. 34]). Moreover, a classical result of Cartan tells us that, given a connected, complex, reductive group G, we have bijections

$$\{\text{real forms of } G\} \longleftrightarrow \left\{\begin{matrix}\text{antiholomorphic} \\ \text{involutions of } G\end{matrix}\right\}/\sim \;\longleftrightarrow\; \left\{\begin{matrix}\text{holomorphic} \\ \text{involutions of } G\end{matrix}\right\}/\sim,$$

where the equivalence relation is given by conjugation by G (that is, we write $\sigma' \sim \sigma$ if and only if $\sigma' : g \mapsto h\sigma(g)h^{-1}$ for some $h \in G$). The bijection from conjugacy classes of antiholomorphic involutions to real forms is realized by taking fixed-point subgroups, as every real form is the fixed-point subgroup of an antiholomorphic involution. As for the remaining bijection, given an antiholomorphic involution σ of G, we can always choose an antiholomorphic involution σ_c of G that commutes with it, whose fixed-point subgroup G^{σ_c} is compact and for which $\theta := \sigma \circ \sigma_c$ is a holomorphic involution. This involution θ, which is unique up to conjugation, is known as the *Cartan involution* of the real form $G_{\mathbb{R}}$ corresponding to σ ([Ada14]).

Example 4.2. Let's work through an example with $G := \mathrm{SL}(2, \mathbb{C})$. Recall that G admits two real forms: a compact form $\mathrm{SU}(2)$ and a split form $\mathrm{SL}(2, \mathbb{R})$. The former is the fixed-point subgroup of the antiholomorphic involution $\sigma_c : g \mapsto ((\overline{g})^T)^{-1}$, while the latter is the fixed-point subgroup of the antiholomorphic involution $\sigma_s : g \mapsto \overline{g}$. Because $\mathrm{SU}(2)$ is the only compact form of G, it follows that σ_c is (up to conjugation) the only antiholomorphic involution with compact fixed-point subgroup, so the Cartan involution of $\mathrm{SU}(2)$ will just be the trivial involution $\sigma_c^2 = \mathrm{id}$. This is a bit boring, so let's look at $\mathrm{SL}(2, \mathbb{R})$ instead. It admits the Cartan involution $\theta : g \mapsto (g^T)^{-1}$ with fixed-point subgroup

$$G^\theta = \mathrm{SO}(2, \mathbb{C}) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{C}, a^2 + b^2 = 1 \right\}.$$

Since G^θ is connected, the identity component is just $K := G^\theta$. We immediately know that G admits the Weyl group $W = \mathbb{Z}/2\mathbb{Z}$, as it is of type A_1 and hence its root system has only two roots, while K has a trivial Weyl group $W_K = \{0\}$. Alternatively, we can see this by noting that the subgroup $T \subset \mathrm{SU}(2)$ consisting only of diagonal matrices is a maximal torus in G, whence $W := N_G(T)/T = \mathbb{Z}/2\mathbb{Z}$, since

$$N_G(T) = \{g \in G : gtg^{-1} \in T, \text{for all } t \in T\} = T \sqcup \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\};$$

similarly, $T_K := T$ is a maximal torus in K, giving us $W_K := N_K(T_K)/T_K = \{0\}$. The only choice of simple roots we can make is $S = \{1\}$, whence $R = \mathbb{R}[\alpha_1]$. The action of W on R is given by $0 : r \mapsto r$ and $1 : r \mapsto -r$, for $r \in R$, giving us $R^{W_K} = R$ and an easy (R^{W_K}, R)-bimodule structure for R. This gives us a very explicit module category over $\mathbb{S}\mathsf{Bim}(W, S)$! An standard exercise we can now do is to compute the Jordan–Hölder filtrations of $\mathcal{M}^0_{LV}$.

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Motivation

- What is finitary birepresentation theory? My take: a "categorification" of the representation theory of finite-dimensional algebras, arising from the study of knot invariants, tensor categories and operator algebras. In other words, Jones mathematics (lol).
- Stealing some slogans from one of Dani's talks: representation theory is group theory in vector spaces, while birepresentation theory is group theory in linear categories.
- Why is it important? Basically for (somehow) studying the fields I just mentioned. Hopefully by the end of this little series we can figure out the answer to this together.

Plan

- Today I'd like to introduce one of the main theorems of birepresentation theory: the "weak Jordan-Hölder theorem". This theorem is a nice (abstract) motivation for the classification of simple birepresentations.
- In principle I'm not sure how helpful this talk will be. It will mostly be walking you through what I've been thinking about over the past couple of weeks. The main theorem is, I think, quite nice, but getting there will likely be dry. Next time I'd like to focus more on examples (especially Soergel bimodules and their classification) if I'm capable enough.
- Addressing a question: why am I talking about birepresentations? The people were promised 2-representations! In classifying 2-representations of Soergel bimodules, it was quickly found that the language of 2-representations was too restrictive. Later on in this series I will probably transition to 2-representations, but at least in this talk I'd like to state things as generally as possible to avoid issues later on.
- As a reminder to myself, I'd like to maintain a little definition bank so you guys can keep track of definitions.

Multifinitary Categories

- What are multifinitary 1-categories? Add this to the definition bank.
- In linear algebra, all idempotents split. Idempotent complete categories live somewhere between additive categories and Abelian categories. In particular, *all pseudo-Abelian categories are idempotent complete!* These remarks are what "unlocked" idempotent completeness for me.
- Examples of finitary categories (that I can't write down because I haven't prepared any concrete examples): (semi)groups and their representations, quantum groups and their categorifications, tensor categories, fusion categories, modular (tensor) categories, 2-Kac-Moody categories, etc.
- $\mathsf{Vect}_{\Bbbk}$ is a helpful example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$.
- Maybe mention example 2.8? Mazorchuk and Miemietz would have you believe it's a useful example but man is it impenetrable.
- Also, $\mathbb{S}\mathsf{Bim}(W, S)$ lololol
- Define the category of multifinitary categories.
- Finitary bicategories. Add this to the definition bank. We can generate these from finitary monoidal categories: in particular, there is a bijection between monoidal categories and one-object bicategories, with delooping taking us upstairs and Hom-categories taking us downstairs. Mention strict case.

Finitary Birepresentations

- What are finitary birepresentations? There is a bijection between module categories and birepresentations of one-object bicategories. Example: Yoneda birepresentations.
- Equivalence of birepresentations. I don't understand these very well yet.

Simple Birepresentations

- Walk through next block: essentially, I want to define what it means for a birepresentation to be transitive. If you're like me and you keep forgetting what this adjective means, an action of G on a set X is said to be transitive if, for any $x, y \in X$, there is some $g \in G$ for which $g \cdot x = y$.
- Define transitive birepresentations. Add this to the definition bank. What are we trying to capture with transitivity?
- There is a "better" notion of simplicity.
- What are ideals of finitary 1-categories? Mention that I've talked to Dani about this and they gave a somewhat confusing answer, but I think I've figured it out. I'm mentioning this in case I'm actually stupid.
- Give an example – $\mathsf{Vect}_{\Bbbk}$ yippeeee!!
- What are $\mathscr{C}$-stable ideals of birepresentations?
- What are simple birepresentations? Add this to the definition bank. Why are they "better"? They somehow more completely characterize what it means for a representation to be simple. Unfortunately, I don't really have an example of a non-simple transitive birepresentation for you right now asides from Lusztig–Vogan module categories, but I'll hopefully have one next week. Many of the naïve examples of transitive birepresentations end up being simple.
- Result I'll need: transitive birepresentations admit a unique maximal ideal, and quotienting by this ideal gives you a simple birepresentation. We call this the simple quotient of M. Run through a sketch of the proof.

Weak Jordan–Hölder

- Define Ind(M), note that it is finite. Define preorder, outline the proof. Recall: preorder means $X \geq X$ and $X \geq Y, Y \geq Z \implies X \geq Z$. Quotienting by $X \sim Y \iff X \geq Y, Y \geq X$ gives us an honest partial order (pretty much by definition).
- Proposition: M is transitive if and only if $|\mathrm{Ind}(\mathrm{M})/\!\sim| = 1$..
- Define poset ideal and coideal.
- Maybe now would be a good time to remind everyone of the classical Jordan-Hölder theorem. Given a representation M, we would like a sensible notion of composition series.
- How do coideals induce sub-birepresentations?
- How do pairs of coideals induce transitive birepresentations?
- How can we make these birepresentations simple?
- State the theorem and run through a sketch of the proof.