Classification of Fusion Categories

1. Prologue

What are fusion categories? What are near-groups, Haagerup–Izumi categories and quadratic categories? What is modular data? What are $6j$ symbols? What is the even part of a subfactor?

Let $\mathcal{C}$ be a fusion category over $\Bbbk$ with representatives $\{X_i\}_{i\in\Gamma}$ of isomorphism classes of simple objects, and choose bases for each multiplicity space $H^l_{i,j} := \mathrm{Hom}_{\mathcal{C}}(X_l, X_i \otimes X_j)$. The *(quantum) 6j-symbols of $\mathcal{C}$* are the matrix blocks

$$\Phi^{i_4}_{i_1,i_2,i_3} : \bigoplus_{j\in\Gamma}(H^{i_4}_{j,i_3} \otimes_{\Bbbk} H^{j}_{i_1,i_2}) \to \bigoplus_{l\in\Gamma}(H^{i_4}_{i_1,l} \otimes_{\Bbbk} H^{l}_{i_2,i_3})$$

of the change-of-basis matrices $\Phi_{i_1,i_2,i_3} := \bigoplus_{i_4\in\Gamma} \Phi^{i_4}_{i_1,i_2,i_3}$, and are given by

$$\Phi^{i_4}_{i_1,i_2,i_3} := v^{-1}(X_{i_4}, X_{i_1}, X_{i_2} \otimes X_{i_3}) \circ (\text{よ}(\alpha_{X_{i_1},X_{i_2},X_{i_3}}))_{X_{i_4}} \circ u(X_{i_4}, X_{i_1} \otimes X_{i_2}, X_{i_3}),$$

where

$$u(U,V,W) : \bigoplus_{j\in\Gamma}(\mathrm{Hom}_{\mathcal{C}}(U, X_j \otimes W) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X_j, V)) \to \mathrm{Hom}(U, V \otimes W),$$
$$v(U,V,W) : \bigoplus_{l\in\Gamma}(\mathrm{Hom}_{\mathcal{C}}(U, V \otimes X_l) \otimes_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(X_l, W)) \to \mathrm{Hom}(U, V \otimes W)$$

are the maps defined by

$$u(U,V,W) := \bigoplus_{j\in\Gamma} u_j(U,V,W) \quad \text{and} \quad v(U,V,W) := \bigoplus_{l\in\Gamma} v_l(U,V,W),$$

with

$$[u_j(U,V,W)](f \otimes_{\Bbbk} g) := (g \otimes \mathrm{id}_W) \circ f \quad \text{and} \quad [v_l(U,V,W)](f' \otimes_{\Bbbk} g') := (\mathrm{id}_V \otimes g') \circ f'$$

for all $f \in \mathrm{Hom}_{\mathcal{C}}(U, X_j \otimes W)$, $g \in \mathrm{Hom}_{\mathcal{C}}(X_j, V)$, $f' \in \mathrm{Hom}_{\mathcal{C}}(U, V \otimes X_l)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X_l, W))$. Recall now the Yoneda embedding

$$\text{よ}_*(X) := \mathrm{Hom}_{\mathcal{C}}(-, X),$$
$$\text{よ}_*(f : X \to Y) := (\mathrm{Hom}_{\mathcal{C}}(-, X) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, Y)).$$

Taking the Yoneda embedding of a component $\alpha_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$ of the associativity natural isomorphism α, we obtain a natural isomorphism

$$\text{よ}_*(\alpha_{X,Y,Z}) = (\mathrm{Hom}_{\mathcal{C}}(-, (X \otimes Y) \otimes Z) \Rightarrow \mathrm{Hom}_{\mathcal{C}}(-, X \otimes (Y \otimes Z))).$$

Thus we have an isomorphism of vector spaces

$$[\text{よ}_*(\alpha_{X,Y,Z})](W) : \mathrm{Hom}_{\mathcal{C}}(W, (X \otimes Y) \otimes Z) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(W, X \otimes (Y \otimes Z));$$

that is, an invertible matrix. In other words, the associativity is given by matrices indexed by X, Y, Z, W. But why do we call these $6j$ symbols? Well, we can simplify our picture further. Suppose we have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X_4, (X_1 \otimes X_2) \otimes X_3) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X_4, X_1 \otimes (X_2 \otimes X_3)).$$

Let X_5 and X_6 be simple summands of $(X_1\otimes X_2)$ and $(X_2\otimes X_3)$, respectively. Then we can determine our isomorphism by determining the matrices of the form

$$\mathrm{Hom}_{\mathcal{C}}(X_4, X_5 \otimes X_3) \to \mathrm{Hom}_{\mathcal{C}}(X_4, X_1 \otimes X_6)$$

for all such X_5 and X_6. These matrices are exactly the $6j$ symbols, where the six simple objects $X_1, X_2, X_3, X_4, X_5, X_6$ play the role of the eponymous "six j's". This is also where u and v come in; given $f \in H^4_{5,3}$ and $g \in H^5_{1,2}$, we have $[u_5(X_4, X_1 \otimes X_2, X_3)](f \otimes_{\Bbbk} g) : X_4 \to (X_1 \otimes X_2) \otimes X_3$, and similarly $[v_6(X_4, X_1, X_2 \otimes X_3)](f' \otimes_{\Bbbk} g') : X_4 \to X_1 \otimes (X_2 \otimes X_3)$ for $f' \in H^4_{1,6}$ and $g' \in H^6_{2,3}$.

Note that these matrices are indeed parameterized by all "six j's", as there could be many invertible matrices that give us a maps of the aforementioned form. We need X_2 in order to use the pentagon diagram for the associativity constraint and hence determine the specific invertible matrix corresponding to the associator. Moreover, X_2 tells us how the blocks

$$\mathrm{Hom}_{\mathcal{C}}(X_4, X_5 \otimes X_3) \to \mathrm{Hom}_{\mathcal{C}}(X_4, X_1 \otimes X_6)$$

fit together into the block diagonal matrix

$$\mathrm{Hom}_{\mathcal{C}}(X_4, (X_1 \otimes X_2) \otimes X_3) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(X_4, X_1 \otimes (X_2 \otimes X_3)).$$

Example 1.1. ($\mathsf{Vec}_{\mathbb{Z}/2\mathbb{Z}}$). Consider $\mathsf{Vec}_{\mathbb{Z}/2\mathbb{Z}}$, a category with two isomorphism classes of simple objects – $[\mathbb{1}]$ and $[X]$ – satisfying the fusion rule

$$X \otimes X \cong \mathbb{1}.$$

Let's start by computing the change-of-basis matrix $\Phi_{X,X,X}$. This matrix can be written in block diagonal form as

$$\Phi_{X,X,X} = \begin{pmatrix} \Phi^{\mathbb{1}}_{X,X,X} & 0 \\ 0 & \Phi^{X}_{X,X,X} \end{pmatrix}.$$

First, observe that

$$\Phi^{\mathbb{1}}_{X,X,X} : \mathrm{Hom}(\mathbb{1}, (X \otimes X) \otimes X) \to \mathrm{Hom}(\mathbb{1}, X \otimes (X \otimes X));$$

but $(X \otimes X) \otimes X \cong X \cong X \otimes (X \otimes X)$, so $\Phi^{\mathbb{1}}_{X,X,X}$ is just the isomorphism of 0-dimensional vector spaces; that is, $\Phi^{\mathbb{1}}_{X,X,X} = 0$. Let's now compute $\Phi^{X}_{X,X,X}$. We shall start by choosing bases for the multiplicity spaces $H^{\mathbb{1}}_{X,X}$, $H^{X}_{\mathbb{1},X}$ and $H^{X}_{X,\mathbb{1}}$. These spaces are all 1-dimensional, so we will choose basis elements $\iota^{\mathbb{1}}_{X,X}$, λ_X^{-1} and ρ_X^{-1}, respectively. Here we have a canonical choice of basis elements for $H^{X}_{\mathbb{1},X}$ and $H^{X}_{X,\mathbb{1}}$; namely, the left and right unitors λ and ρ. This culminates in the basis elements

$$\iota^{X}_{(X\otimes X)\otimes X} \coloneqq (\iota^{\mathbb{1}}_{X,X} \otimes \mathrm{id}_X) \circ \lambda_X^{-1} = [u_{\mathbb{1}}(X, X \otimes X, X)](\lambda_X^{-1} \otimes_{\Bbbk} \iota^{\mathbb{1}}_{X,X})$$

and

$$\iota^{X}_{X\otimes(X\otimes X)} \coloneqq (\mathrm{id}_X \otimes \iota^{\mathbb{1}}_{X,X}) \circ \rho_X^{-1} = [v_{\mathbb{1}}(X, X, X \otimes X)](\rho_X^{-1} \otimes_{\Bbbk} \iota^{\mathbb{1}}_{X,X})$$

for $\mathrm{Hom}(X, (X \otimes X) \otimes X)$ and $\mathrm{Hom}(X, X \otimes (X \otimes X))$, respectively. These are once again both 1-dimensional spaces, so let's define a constant $\mu_1 \in \Bbbk$ for which

$$\Phi^{X}_{X,X,X}(\iota^{X}_{(X\otimes X)\otimes X}) \eqqcolon \mu_1 \cdot \iota^{X}_{X\otimes(X\otimes X)}.$$

In order to determine μ_1, we need to use the pentagon equation. To this end, we first choose bases for the Hom-spaces

$$\begin{gathered} \mathrm{Hom}(\mathbb{1}, (\mathbb{1} \otimes X) \otimes X), \\ \mathrm{Hom}(\mathbb{1}, \mathbb{1} \otimes (X \otimes X)), \\ \mathrm{Hom}(\mathbb{1}, (X \otimes \mathbb{1}) \otimes X), \\ \mathrm{Hom}(\mathbb{1}, X \otimes (\mathbb{1} \otimes X)), \\ \mathrm{Hom}(\mathbb{1}, (X \otimes X) \otimes \mathbb{1}), \\ \mathrm{Hom}(\mathbb{1}, X \otimes (X \otimes \mathbb{1})). \end{gathered}$$

We can write bases for these Hom-spaces in terms of the bases we chose previously. That is,

$$\begin{aligned}
\iota^{\mathbb{1}}_{(\mathbb{1}\otimes X)\otimes X} &\coloneqq (\lambda_X^{-1} \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{X,X},\\
\iota^{\mathbb{1}}_{\mathbb{1}\otimes(X\otimes X)} &\coloneqq (\mathrm{id}_{\mathbb{1}} \otimes \iota^{\mathbb{1}}_{X,X}) \circ \lambda_{\mathbb{1}}^{-1},\\
\iota^{\mathbb{1}}_{(X\otimes\mathbb{1})\otimes X} &\coloneqq (\rho_X^{-1} \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{X,X},\\
\iota^{\mathbb{1}}_{X\otimes(\mathbb{1}\otimes X)} &\coloneqq (\mathrm{id}_X \otimes \lambda_X^{-1}) \circ \iota^{\mathbb{1}}_{X,X},\\
\iota^{\mathbb{1}}_{(X\otimes X)\otimes\mathbb{1}} &\coloneqq (\iota^{\mathbb{1}}_{X,X} \otimes \mathrm{id}_{\mathbb{1}}) \circ \lambda_{\mathbb{1}}^{-1},\\
\iota^{\mathbb{1}}_{X\otimes(X\otimes\mathbb{1})} &\coloneqq (\mathrm{id}_X \otimes \rho_X^{-1}) \circ \iota^{\mathbb{1}}_{X,X}.
\end{aligned}$$

This culminates in the three new constants given by

$$\begin{aligned}
\Phi^{\mathbb{1}}_{\mathbb{1},X,X}(\iota^{\mathbb{1}}_{(\mathbb{1}\otimes X)\otimes X}) &\eqqcolon \mu_2 \cdot \iota^{\mathbb{1}}_{\mathbb{1}\otimes(X\otimes X)},\\
\Phi^{\mathbb{1}}_{X,\mathbb{1},X}(\iota^{\mathbb{1}}_{(X\otimes\mathbb{1})\otimes X}) &\eqqcolon \mu_3 \cdot \iota^{\mathbb{1}}_{X\otimes(\mathbb{1}\otimes X)},\\
\Phi^{\mathbb{1}}_{X,X,\mathbb{1}}(\iota^{\mathbb{1}}_{(X\otimes X)\otimes\mathbb{1}}) &\eqqcolon \mu_4 \cdot \iota^{\mathbb{1}}_{X\otimes(X\otimes\mathbb{1})}.
\end{aligned}$$

The final bases we need are for the Hom-spaces

$$\begin{aligned}
&\mathrm{Hom}(\mathbb{1}, ((X\otimes X)\otimes X)\otimes X),\\
&\mathrm{Hom}(\mathbb{1}, (X\otimes (X\otimes X))\otimes X),\\
&\mathrm{Hom}(\mathbb{1}, (X\otimes X)\otimes (X\otimes X)),\\
&\mathrm{Hom}(\mathbb{1}, X\otimes ((X\otimes X)\otimes X)),\\
&\mathrm{Hom}(\mathbb{1}, X\otimes (X\otimes (X\otimes X))).
\end{aligned}$$

Following the same procedure as before, we have

$$\begin{aligned}
\iota^{\mathbb{1}}_{((X\otimes X)\otimes X)\otimes X} &\coloneqq (\iota^{X}_{(X\otimes X)\otimes X} \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{X,X} = ((\iota^{\mathbb{1}}_{X,X} \otimes \mathrm{id}_X) \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{(\mathbb{1}\otimes X)\otimes X},\\
\iota^{\mathbb{1}}_{(X\otimes(X\otimes X))\otimes X} &\coloneqq (\iota^{X}_{X\otimes(X\otimes X)} \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{X,X} = ((\mathrm{id}_X \otimes \iota^{\mathbb{1}}_{X,X}) \otimes \mathrm{id}_X) \circ \iota^{\mathbb{1}}_{(X\otimes\mathbb{1})\otimes X},\\
\iota^{\mathbb{1}}_{(X\otimes X)\otimes(X\otimes X)} &\coloneqq (\iota^{\mathbb{1}}_{X,X} \otimes (\mathrm{id}_X \otimes \mathrm{id}_X)) \circ \iota^{\mathbb{1}}_{\mathbb{1}\otimes(X\otimes X)} = ((\mathrm{id}_X \otimes \mathrm{id}_X) \otimes \iota^{\mathbb{1}}_{X,X}) \circ \iota^{\mathbb{1}}_{(X\otimes X)\otimes\mathbb{1}},\\
\iota^{\mathbb{1}}_{X\otimes((X\otimes X)\otimes X)} &\coloneqq (\mathrm{id}_X \otimes \iota^{X}_{(X\otimes X)\otimes X}) \circ \iota^{\mathbb{1}}_{X,X} = (\mathrm{id}_X \otimes (\iota^{\mathbb{1}}_{X,X} \otimes \mathrm{id}_X)) \circ \iota^{\mathbb{1}}_{X\otimes(\mathbb{1}\otimes X)},\\
\iota^{\mathbb{1}}_{X\otimes(X\otimes(X\otimes X))} &\coloneqq (\mathrm{id}_X \otimes \iota^{X}_{X\otimes(X\otimes X)}) \circ \iota^{\mathbb{1}}_{X,X} = (\mathrm{id}_X \otimes (\mathrm{id}_X \otimes \iota^{\mathbb{1}}_{X,X})) \circ \iota^{\mathbb{1}}_{X\otimes(X\otimes\mathbb{1})}.
\end{aligned}$$

The pentagon diagram thus gives us

$$\begin{tikzcd}
 & \iota^{\mathbb{1}}_{((X\otimes X)\otimes X)\otimes X} \arrow[dl, mapsto, "\alpha_{X,X,X}\otimes\mathrm{id}_X"'] \arrow[dr, mapsto, "\alpha_{X\otimes X,X,X}"] & \\
\mu_1 \cdot \iota^{\mathbb{1}}_{(X\otimes(X\otimes X))\otimes X} \arrow[d, mapsto, "\alpha_{X,X\otimes X,X}"'] & & \mu_2 \cdot \iota^{\mathbb{1}}_{(X\otimes X)\otimes(X\otimes X)} \arrow[d, mapsto, "\alpha_{X\otimes X,X,X}"] \\
\mu_1\mu_3 \cdot \iota^{\mathbb{1}}_{X\otimes((X\otimes X)\otimes X)} \arrow[r, mapsto, "\alpha_{X,X,X}\otimes\mathrm{id}_X"'] & \mu_1^2\mu_3 \cdot \iota^{\mathbb{1}}_{X\otimes(X\otimes(X\otimes X))} \arrow[r, equal] & \mu_2\mu_4 \cdot \iota^{\mathbb{1}}_{X\otimes(X\otimes(X\otimes X))}
\end{tikzcd}.$$

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2. The Cuntz Algebra Approach of Izumi

Take Vec_G to be skeletal. Consider an associativity constraint $a_{ghk} : ghk - - > ghk$. Since ghk is a simple object, $\mathrm{Hom}(ghk, ghk) \cong \Bbbk$, whence $a_{ghk} = \lambda_{ghk}\mathrm{id}_{ghk}$ for some $\lambda_{ghk} \in \Bbbk^\times$. Note that the pentagon diagram enforces certain conditions on our choice of λ_{ghk}; in particular, if we look at this diagram, we'll see that $\lambda_{ghk} = \omega(g, h, k)$ for some 3-cocycle ω. By this, we mean a map $\omega : G \times G \times G \to \Bbbk^\times$ satisfying

$$\omega(x, y, zw)\omega(xy, z, w)\omega(y, z, w)^{-1}\omega(x, yz, w)^{-1}\omega(x, y, z)^{-1} = 1$$

for all $x, y, z, w \in G$. We will henceforth denote by Vec_G^ω the category of G-graded vector spaces with associativity constraint $a_{ghk} = \omega(g, h, k)\mathrm{id}_{ghk}$, for all $g, h, k \in G$, and Vec_G the category of G-graded vector spaces with trivial associativity.

Consider the category $\mathrm{End}(M)$, for M a hyperfinite type III factor. This category is strict, as $\rho \otimes \sigma \coloneqq \rho \circ \sigma$ by definition. Every near-group category with group G contains some copy of Vec_G^ω corresponding to the group-like part. Because every unitary near-group category is a subcategory of $\mathrm{End}(M)$ and is hence itself strict, we know that it will actually contain the "strictification" of some Vec_G^ω. However, Izumi shows that if $\mathcal{C}$ is any fusion category containing a simple object that is fixed under tensor products with invertibles (that is, there exists some simple object X such that $X \otimes g \cong X$ for all invertible g), then it contains a copy of Vec_G, for G the group of isomorphism classes of invertible objects. He shows in addition that if the fusion category is also unitary, then $g \otimes X = X$ (but we may not necessarily have that $X \otimes g = X$). The upshot is that we almost know how objects are tensored, since the group-like part will have trivial associativity (that is, $g \otimes h = gh$). We just need to understand $X \otimes g$ and $X \otimes X$, as well as the morphisms.

In [Izu17], Izumi showed that every unitary near-group category $\mathcal{C}$ with multiplicity m is equivalent to a subcategory of $\mathrm{End}(M)$, where M is the hyperfinite type III_1 factor. In particular, it is generated by a single irreducible endomorphism $\rho \in \mathrm{End}_0(M)$ satisfying the fusion rules

$$\begin{aligned}
[\rho] \otimes [\rho] &= \bigoplus_{g \in G} [\alpha_g] \oplus [\rho]^{\oplus m}, \\
[\alpha_g] \otimes [\alpha_h] &= [\alpha_{gh}], \\
[\alpha_g] \otimes [\rho] = [\rho] \otimes [\alpha_g] &= [\rho],
\end{aligned}$$

where the map $\alpha : G \to \mathrm{Aut}(M)$ induces an injective homomorphism from G into $\mathrm{Out}(M)$.

The main result of [Izu17] is [Izu17, Theorem 4.9]. Essentially, there is a bijective correspondence between the set of equivalence classes of unitary near-group categories with finite group G and multiplicity parameter m and the set of equivalence classes of admissible tuples $(\mathcal{K}, j_1, j_2, V, U_{\mathcal{K}}, \chi, l)$ (see [Izu17, Definition 4.8]). Here $\mathcal{K}$ is the finite-dimensional Hilbert space $\mathrm{Hom}(\rho, \rho^2)$, j_1 and j_2 are two antilinear isometries of $\mathcal{K}$, V and $U_{\mathcal{K}}$ are unitary representations of G on $\mathcal{K}$, $\{\chi_g\}_{g \in G}$ are characters of G and l is a linear map from $\mathcal{K}$ to the set $\mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{K})$ of bounded operators $\mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$.

By [Izu17, Theorem 9.1], the unitary near-group categories with finite Abelian group G and $m = |G|$ are completely classified tuples of the form $(\langle \cdot, \cdot \rangle, a, b, c)$, where $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{T}$ is a non-degenerate symmetric bicharacter and where $a : G \to \mathbb{T}$, $b : G \to \mathbb{T}$ and $c \in \mathbb{T}$ satisfy various conditions. When we say that $\langle \cdot, \cdot \rangle$ is a bicharacter, we mean that

$$\langle xy, z \rangle = \langle x, z \rangle \langle y, z \rangle \qquad \text{and} \qquad \langle x, yz \rangle = \langle x, y \rangle \langle x, z \rangle$$

for all $x, y, z \in G$. By non-degenerate, we mean that

$$\langle x, \cdot \rangle = \langle y, \cdot \rangle$$

if and only if $x = y$. This is equivalent to the map $\varphi : G \to \mathrm{Hom}(G, \mathbb{T})$ given by $x \mapsto \langle x, \cdot \rangle$ being an isomorphism.

Definition 2.1. (Cuntz Algebra). *Let $\{S_i\}_{i=1}^n$ be a set of isometries on an infinite-dimensional Hilbert space $\mathscr{H}$. Suppose moreover that these isometries satisfy the* Cuntz relation

$$\sum_{k=1}^{n} S_k S_k^* = 1.$$

The Cuntz algebra *$\mathcal{O}_n$ is the universal C^*-algebra $C^*(S_1, \ldots, S_n)$.*

Remark 2.2. Note that, as isometries, $S_i^* S_i = 1$. In particular, we must have that $S_i^* S_j = \delta_{i,j}$ for all $i, j \in \{1, \ldots, n\}$. This follows from the fact that a sum of projections is itself a projection if and only if the projections in the sum are pairwise orthogonal. The Cuntz relation is essentially ensuring that the sum of the projections $S_i S_i^*$ is the trivial projection.

Example 2.3. (Fibonacci Category). Let's look at the Fibonacci category. This is the near-group with $G = \{0\}$ and $m = 1$. Our choice for $\langle \cdot, \cdot \rangle$ is obvious, and [Izu17, Lemma 7.1] tells us that

$$c^3 a(0) = \sqrt{n} = 1 \implies a(0) = c^{-3}.$$

Moreover, [Izu17, Theorem 9.1] tells us that b is defined by $b : 0 \mapsto -1/d$, where d corresponds to the dimension of our irreducible generator ρ. Let's determine c and d. Because b is equal to its own Fourier transform, [Izu17, Theorem 9.1] tells us that

$$b(0) = ca(0)b(0) \implies a(0) = c^{-1}.$$

In order for $c^{-1} = c^{-3}$, we require $c = \pm 1$. Finally, [Izu17, Equation 9.5] tells us that

$$b(0)b(0)b(0) = b(0)b(0) \mp \frac{1}{d},$$

$$\implies -\frac{1}{d^3} = \frac{1}{d^2} \mp \frac{1}{d},$$
$$\implies \pm d^2 - d - 1 = 0.$$

This only has a real solution when $c = 1$, whence d is nothing but the golden ratio (as it cannot be negative in the unitary case - this alternative solution, known as the Galois dual, corresponds to a non-unitary near-group in this case and many others). This is exactly what we would expect, as d is the dimension of X (where $d^2 = 1 + d$ comes from the fusion rule $X^2 = \mathbb{1} \oplus X$).

Example 2.4. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's look at the case where $G = \mathbb{Z}/2\mathbb{Z}$ and $m = 2$. This near-group corresponds to the even part of the type A_4 subfactor. We know the dimension is

$$d_\pm := \frac{m \pm \sqrt{m^2 + 4n}}{2} = 1 \pm \sqrt{3}.$$

In the unitary setting, we of course ask that d be positive, and hence we choose $d = d_+$. The only possibility for a non-degenerate bicharacter is

$$\langle 0, 0 \rangle = 1, \qquad \langle 0, 1 \rangle = \langle 1, 0 \rangle = 1 \qquad \text{and} \qquad \langle 1, 1 \rangle = -1.$$

From [Izu17, Equation 7.8], it follows that

$$a(0) = 1 \qquad \text{and} \qquad a(1) = \pm i.$$

Meanwhile, [Izu17, Equation 9.4] tells us that

$$\overline{b(1)} = \pm i b(1) \implies \Re(b(1)) = \mp \Im(b(1)),$$

whence [Izu17, Equation 9.3] gives us

$$\Re(b(1))^2 + \Im(b(1))^2 = (b(1)\overline{b(1)})^2 = \frac{1}{2} \implies b(1) = \frac{1 - a(1)}{2}.$$

It then follows from evaluating [Izu17, Equation 9.1] with $g = 0$ and rearranging for c that

$$c = \frac{1 - \sqrt{3} + a(1)(1 + \sqrt{3})}{2\sqrt{2}}.$$

Note that we may choose either $a(1) = i$ or $a(1) = -i$; both of these lead to solutions. Moreover, in the non-unitary setting, we may take the Galois conjugate of d.

Example 2.5. ($G = \mathbb{Z}/2\mathbb{Z}$). Let's determine the Haagerup–Izumi categories with $G = \mathbb{Z}/2\mathbb{Z}$. Let

$$d_\pm := \frac{n \pm \sqrt{n^2+4}}{2},$$

where in this example $d := 1 + \sqrt{2}$. Izumi's classification involves a triplet $(\epsilon_h(g), \omega(g), A_{h,k}(g))$, where $\epsilon_h(g) \in \{-1, 1\}$, $\omega(g) \in \mathbb{T}$ and $A_{h,k}(g) \in \mathbb{C}$ satisfy [Izu18, Equations 4.1–4.9]. Well, we know

$$\epsilon_0(0) = \epsilon_1(0) = 1 \qquad \text{and} \qquad \epsilon_0(1) = \epsilon_0(1)\epsilon_0(1) \implies \epsilon_0(1) = 1.$$

By [Izu18, Equation 4.7],

$$A_{0,0}(g) = A_{0,0}(g)\omega(g),$$

which tells us that either $\omega(g) = 1$ or $A_{0,0}(g) = 0$ for each $g \in G$. Let's fix any $g \in G$ and consider the case when $A_{0,0}(g) = 0$. In this case, however, [Izu18, Equations 4.3 and 4.4] give us

$$A_{1,0}(g)\overline{A_{\delta_{g,0}-g,0}(g)} = 1 - \frac{|\omega(g)|}{d} \implies \text{“}\frac{1}{d^2} = 1 - \frac{1}{d}\text{”}.$$

This “equality” is nonsense; we must therefore have $\omega(g) = 1$ for all $g \in G$. Suppose now that $\epsilon_1(1) = 1$. Then [Izu18, Equation 4.7] gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \qquad \text{and} \qquad A_{0,1}(1) = A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(0) = A_{1,1}(1)$. Now, [Izu18, Equations 4.4 and 4.6] tell us

$$A_{0,1}(0)A_{1,1}(1) + A_{1,1}(0)A_{1,0}(1) = 0.$$

Thus $A_{0,1}(g) = A_{1,1}(g) = A_{1,0}(g) = 0$ and hence $A_{0,0}(g) = -1/d$ by [Izu18, Equation 4.3]. However, in this case we cannot satisfy [Izu18, Equation 4.9]. Suppose instead that $\epsilon_1(1) = -1$. With this new 2-cocycle, [Izu18, Equation 4.7] now gives us

$$A_{0,1}(0) = A_{1,1}(0) = A_{1,0}(0) \qquad \text{and} \qquad A_{0,1}(1) = -A_{1,1}(1) = A_{1,0}(1),$$

while [Izu18, Equation 4.8] gives us $A_{1,1}(1) = -A_{1,1}(0)$. We then see by [Izu18, Equation 4.4] that

$$A_{0,1}(0)A_{1,0}(0) + A_{1,1}(0)A_{1,1}(0) = 1 \implies A_{1,0}(0) = \pm\frac{1}{\sqrt{2}} = \pm\frac{1}{d-1},$$

and by [Izu18, Equation 4.9] that

$$A_{0,0}(0)A_{1,0}(0)^2 = A_{1,0}(0)^2 + A_{1,0}(0)^3 \implies A_{0,0}(0) = 1 + A_{1,0}(0) = \frac{d-1\pm 1}{d-1}.$$

Finally, [Izu18, Equation 4.3] allows us to deduce

$$A_{1,0}(0) = -\frac{1}{d-1},$$

whence

$$A(0) = \frac{1}{d-1}\begin{pmatrix} d-2 & -1 \\ -1 & -1 \end{pmatrix} \qquad \text{and} \qquad A(1) = \frac{1}{d-1}\begin{pmatrix} d-2 & -1 \\ -1 & 1 \end{pmatrix}.$$

This category is nothing but the even part of the type A_7 subfactor.

Remark 2.6. Suppose that $|G|$ is odd. Then [Izu18, Equation 4.1] tells us that $\epsilon_h(g) = 1$, while [Izu18, Equation 4.2] tells us that $\omega(g)$ does not depend on g. Moreover, $A_{h,k}(g)$ cannot depend on g by [Izu18, Equation 4.5], and either $\omega = 1$ or $A_{0,0} = 0$ by [Izu18, Equation 4.7]. In this case, [Izu18, Equations 4.1–4.9] reduce to the following four equations.

$$A_{h,k} = A_{-k,h-k}\omega = A_{k-h,-h}\overline{\omega},$$

$$\sum_{h\in G} A_{h,0} = -\frac{\overline{\omega}}{d_\pm},$$

$$\sum_{h\in G} A_{h-g,k}A_{k,h-g'} = \delta_{g,g'} - \frac{\delta_{k,0}}{d_\pm},$$

$$\sum_{l\in G} A_{x+y,l}A_{-x,l+p}A_{-y,l+q} = A_{p+x,q+x+y}A_{q+y,p+x+y} - \frac{\delta_{x,0}\delta_{y,0}}{d_\pm}.$$

The first three equations above are precisely [EG17, Equations 4.7, 4.8 and 4.9]! In particular, to see that our third equation is equivalent to [EG17, Equation 4.9], we simply make the change of variables $\hat{g} \coloneqq g' - g$ and $\hat{h} \coloneqq h - g'$, whence we obtain

$$\sum_{\hat{h}\in G} A_{\hat{h}+\hat{g},k}A_{k,\hat{h}} = \delta_{\hat{g},0} - \frac{\delta_{k,0}}{d_\pm}.$$

Similarly, using our first equation while making the change of variables $\hat{l} \coloneqq l - x - y$, $\hat{p} \coloneqq p + x + y$, $\hat{q} \coloneqq q + x + y$, $\hat{x} \coloneqq -x$ and $\hat{y} \coloneqq -y$, our fourth equation becomes

$$\overline{\omega}\sum_{\hat{l}\in G} A_{\hat{l},\hat{x}+\hat{y}}A_{\hat{x},\hat{l}+\hat{p}}A_{\hat{y},\hat{l}+\hat{q}} = A_{\hat{y}+\hat{p},\hat{q}}A_{\hat{x}+\hat{q},\hat{p}} - \frac{\delta_{\hat{x},0}\delta_{\hat{y},0}}{d_\pm},$$

showing that it is equivalent to [EG17, Equation 4.11].

3. The Leavitt Algebra Approach of Evans–Gannon

The important result is [EG17, Theorem 2].

Definition 3.1. (Leavitt Algebra). *Let* $X \coloneqq (x_{ij})$ *and* $Y \coloneqq (y_{ij})$ *be* $m \times n$ *and* $n \times m$ *matrices of symbols, respectively. The* Leavitt K-algebra of type (m,n) *is the free associative unital* K*-algebra*

$$\mathcal{L}_K(m,n) \coloneqq \frac{K[x_{ij}, y_{ij}]}{\langle XY = I_m, YX = I_n \rangle}.$$

In other words, it is the universal K*-algebra with generators*

$$\{x_{ij} : 1 \leq i \leq m, 1 \leq j \leq n\} \sqcup \{y_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and Leavitt–Cuntz relations

$$\sum_{k=1}^{m} y_{ik} x_{kj} = \delta_{i,j} \qquad \textit{and} \qquad \sum_{k=1}^{n} x_{ik} y_{kj} = \delta_{i,j},$$

for all suitable i, j*.*

Consider the Leavitt $\mathbb{C}$-algebra of type $(1,n)$, which we shall henceforth denote by $\mathcal{L}_n \coloneqq \mathcal{L}_{\mathbb{C}}(1,n)$. We have that $\mathcal{O}_n = C^*(\mathcal{L}_n)$, where $x_i = S_i$ and $y_i = S_i^*$. Let's think about what this means precisely. The Leavitt–Cuntz relations for $m = 1$ become

$$y_i x_j = \delta_{i,j} \qquad \text{and} \qquad \sum_{k=1}^{n} x_k y_k = 1.$$

We may endow $\mathcal{L}_n$ with the structure of a $*$-algebra by defining a conjugate homogeneous antihomomorphism that sends $x_i \mapsto y_i$ and $y_i \mapsto x_i$. We further define

$$\|a\| \coloneqq \sup\{p(a) : p \text{ is a } C^*\text{-seminorm on } \mathcal{L}_n\}$$

for all $a \in \mathcal{L}_n$, where a C^*-seminorm is just a seminorm for which $p(a^*a) = p(a)^2$ and $p(ab) \leq p(a)p(b)$. Note that $0 \leq \|a\| \leq 1$, as $1 = \|y_i x_i\| = \|x_i^2\|$ and hence $\|x_i\| = \|y_i\| = 1$ for all i. The condition $p(ab) \leq p(a)p(b)$ ensures that $\mathcal{I} \coloneqq \{a \in \mathcal{L}_n : \|a\| = 0\}$ is an ideal in $\mathcal{L}_n$. Our C^*-seminorm then descends to a C^*-norm on the quotient $\mathcal{L}_n/\mathcal{I}$. The completion of $\mathcal{L}_n/\mathcal{I}$ with respect to this C^*-norm is known as the universal C^*-algebra of $\mathcal{L}_n$, denoted by $C^*(\mathcal{L}_n)$. This is precisely $\mathcal{O}_n$ by definition. We may therefore view $\mathcal{L}_n$ as the polynomial part of $\mathcal{O}_n$.

Example 3.2. (Yang–Lee Category). Let $G = \{0\}$. Then [EG17, Equation 4.7] demands that

$$A_{0,0} = \omega A_{0,0} = \overline{\omega} A_{0,0} \implies \omega = 1,$$

whence [EG17, Equation 4.8] tells us that

$$A_{0,0} = -\frac{1}{d_\pm}.$$

The rest of [EG17, Equations 4.7–4.10] are satisfied by these choices. Hence by [EG17, Theorem 2], we have two fusion categories for $G = \{0\}$; a unitary one with $\pm = +$ (the Fibonacci category) and a non-unitary one with $\pm = -$ (the Yang–Lee category).

Example 3.3. ($G = \mathbb{Z}/2\mathbb{Z}$). The equations we must satisfy for $|G|$ even are given in [Izu18] (is this true?). Adapting our argument from Example 2.5, we see that there is exactly one non-unitary Haagerup–Izumi category with $G = \mathbb{Z}/2\mathbb{Z}$. This corresponds to $\epsilon_h(g) = (-1)^{gh}$, $\omega(g) = 1$,

$$A(0) = \frac{1}{d-1}\begin{pmatrix} d & 1 \\ 1 & 1 \end{pmatrix} \qquad \text{and} \qquad A(1) = \frac{1}{d-1}\begin{pmatrix} d & 1 \\ 1 & -1 \end{pmatrix}.$$

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