

INDECOMPOSABLE SOERGEL BIMODULES OF TYPES A_1 AND A_2

The goal of these notes is to explicitly step through the classification of the indecomposable Soergel bimodules of type A_2 , while also covering the simpler A_1 case as a stepping stone. Throughout these notes, \mathbb{k} will always be taken to be an algebraically closed field of characteristic zero.

Definition 1. (Geometric Representation). *Let (W, S) be a Coxeter system and V the \mathbb{k} -vector space with formal basis $\{\alpha_s : s \in S\}$. Define a symmetric, bilinear form on V by linearly extending*

$$(\alpha_s, \alpha_t) = \begin{cases} -\cos\left(\frac{\pi}{m_{st}}\right), & m_{st} \neq \infty; \\ -1, & m_{st} = \infty. \end{cases}$$

From this, we define an action of $s \in S$ on the basis elements $\alpha_t \in V$ by the linear automorphism

$$s(\alpha_t) := \alpha_t - 2(\alpha_s, \alpha_t)\alpha_s,$$

which reflects α_t across α_s . The geometric representation of (W, S) is the representation induced by linearly extending this reflection action to an action of W on all of V .

Recall that by definition, $R := \bigoplus_{i=0}^{\infty} \text{Sym}^i(V)$, where $\text{Sym}^i(V)$ is the quotient of the i th tensor power $V^{\otimes i}$ by the action of the symmetric group S_i , viewed as a \mathbb{k} -module that is \mathbb{Z} -graded in degree 2. Our reflection action also extends to an action on R by taking $s(fg) := s(f)s(g)$, for $f, g \in R$. It is not too difficult to show that $\text{Sym}^i(V)$ is isomorphic to the additive subgroup of the polynomial ring $\mathbb{k}[\alpha_s : s \in S]$ consisting only of homogeneous polynomials of degree i , meaning we may identify $R = \mathbb{k}[\alpha_s : s \in S]$. With this in mind, we have a crucial lemma that we will invoke regularly.

Lemma 2. *Suppose that M is a graded (R, R) -bimodule that is generated by a homogeneous element $m \in M$, in the sense that $M = RmR$. Then M is indecomposable.*

Proof. Let d denote the degree of m . Because M is generated by m , and because $R^0 = \mathbb{k}$, we have that $M^d = R^0 m R^0 = \mathbb{k}m$. In other words, M^d is a one-dimensional vector space. Suppose that $M \cong L \oplus N$. In this case, M^d is isomorphic as a \mathbb{k} -vector space to $L^d \oplus N^d$. Assume without loss of generality that $m \in L^d$. This forces $N^d = 0$, as M^d is one-dimensional, whence we have that $M = RmR \subseteq L$, forcing $N = 0$. Thus M is indecomposable. This completes the proof. ■

From this lemma, it follows that R itself is indecomposable as an (R, R) -bimodule, as it is generated by $1 \in \mathbb{k}$. Moreover, suppose we define

$$B_s := R \otimes_{R^s} R(1)$$

for any $s \in S$, where $R^s := \{f \in R : s(f) = f\}$. This is also clearly indecomposable by our lemma, as it is generated as an (R, R) -bimodule by $1 \otimes_{R^s} 1$ (where we note that, since $1 \in R^0 = R(1)^{-1}$, we have $1 \otimes_{R^s} 1 \in B_s^{0-1} = B_s^{-1}$, meaning it is homogeneous of degree -1). In particular, it is easy to see that grading shifts of R and B_s are all indecomposable too.

Definition 3. (Soergel Bimodule). Let (W, S) be a Coxeter system and $\underline{w} := (s_1, \dots, s_k)$ an expression. The Bott–Samelson bimodule corresponding to \underline{w} is the graded (R, R) -bimodule

$$\begin{aligned} BS(\underline{w}) &:= B_{s_1} \otimes_R \cdots \otimes_R B_{s_k} \\ &= (R \otimes_{R^{s_1}} R(1)) \otimes_R \cdots \otimes_R (R \otimes_{R^{s_k}} R(1)) \\ &\cong R \otimes_{R^{s_1}} R \otimes_{R^{s_2}} \cdots \otimes_{R^{s_k}} R(k). \end{aligned}$$

A Soergel bimodule is any graded (R, R) -bimodule that is isomorphic to a finite direct sum of grading shifts of direct summands of Bott–Samelson bimodules.

Another useful lemma is that any polynomial in R can be split into the sum of an s -invariant component and an s -antiinvariant component, for any $s \in S$.

Lemma 4. For any $s \in S$ and $f \in R$, we have that $f + s(f) \in R^s$ and $f - s(f) \in R^s\alpha_s$. In particular, we have a graded (R^s, R^s) -bimodule splitting $R \cong R^s \oplus R^s\alpha_s \cong R^s \oplus R^s(-2)$ for all $s \in S$.

Proof. Let $s \in S$ and $f := \alpha_t$ for any $t \in S$. Observe that $P_s(f) := \frac{1}{2}(f + s(f)) = \alpha_t - (\alpha_s, \alpha_t)\alpha_s$ is in R^s (alternatively, $s(f + s(f)) = s(f) + s^2(f) = s(f) + f$). Similarly, $\partial_s(f)\alpha_s := \frac{1}{2}(f - s(f)) = (\alpha_s, \alpha_t)\alpha_s$ is clearly in $R^s\alpha_s$ (alternatively, $s(f - s(f)) = s(f) - s^2(f) = s(f) - f$). By linearity of the action of S , sums will preserve this splitting, so all that remains is to show that products preserve it too. Suppose $f := f_1 + f_2\alpha_s$ and $g := g_1 + g_2\alpha_s$, for $f_1, f_2, g_1, g_2 \in R^s$, such that $fg = f_1g_1 + f_1g_2\alpha_s + f_2g_1\alpha_s + f_2g_2\alpha_s^2$. We see that $P_s(fg) = g_1g_2 + h_1h_2\alpha_s^2$ and $\partial_s(fg)\alpha_s = g_1h_2\alpha_s + g_2h_1\alpha_s$, whence fg admits a unique decomposition of the form $fg = P_s(fg) + \partial_s(fg)\alpha_s$. Clearly this induces an isomorphism of graded (R^s, R^s) -bimodules, as for any $f \in R$ and $g, h \in R^s$, we have that $hfg \mapsto (hP_s(f)g, h\partial_s(f)\alpha_s g)$. This completes the proof. ■

Note that this is an isomorphism of (R^s, R^s) -bimodules, not (R, R) -bimodules! This lemma allows us to completely classify the indecomposables in the type A_1 case.

Proposition 5. (Indecomposable Soergel Bimodules of Type A_1). The indecomposable Soergel bimodules of type A_1 are, up to grading shift and isomorphism, R and B_s .

Proof. Let $W = S_2 = \langle s \rangle$ and $S = \{s\}$. By our previous observations, we know that grading shifts of R and B_s are indecomposable, as they are generated by the homogeneous unit tensors. The next step is to check the Bott–Samelson $B_s \otimes_R B_s$. But observe that

$$\begin{aligned} B_s \otimes_R B_s &= (R \otimes_{R^s} R(1)) \otimes_R (R \otimes_{R^s} R(1)) \\ &\cong R \otimes_{R^s} R \otimes_{R^s} R(2) \\ &\cong R \otimes_{R^s} (R^s \oplus R^s(-2)) \otimes_{R^s} R(2) \tag{Lemma 4} \\ &\cong (R \otimes_{R^s} R(2)) \oplus (R \otimes_{R^s} R) \\ &= B_s(1) \oplus B_s(-1). \end{aligned}$$

Thus $B_s \otimes_R B_s$ is decomposable, meaning that we have exhausted all candidates for indecomposables. This completes the proof. ■

Remark 6. Observe that the indecomposables are, up to grading shift, in bijection with W , just as we would expect by the categorification theorem.

One last lemma that we will find useful before we attempt to classify the indecomposable Soergel bimodules of type A_2 is the following.

Lemma 7. *Let $s, t \in S$ with $s \neq t$. Then R^s and R^t generate R as a ring if and only if $m_{st} \neq \infty$.*

Proof. Recall that $R := \text{Sym}(V)$. As a polynomial ring, R is generated by its linear terms – those being the vectors in V . The linear terms in R^s are precisely the vectors fixed by s . This “intersection of R^s with V ” is nothing but

$$H_s := \{v \in V : s(v) = v\} = \{v \in V : (v, \alpha_s) = 0\}.$$

Note that the map $\varphi_s : V \rightarrow \mathbb{k}$ given by $v \mapsto (v, \alpha_s)$ is obviously a surjective linear map, as $\varphi_s(\lambda \alpha_s)$ always maps to $\lambda \in \mathbb{k}$. We therefore have that

$$\dim(H_s) = \dim(\ker(\varphi_s)) = \dim(V) - \dim(\text{rank}(\varphi_s)) = \dim(V) - 1$$

by the rank-nullity theorem. In other words, H_s is a hyperplane. When $m_{st} \neq \infty$, we have that H_s and H_t are distinct. Because they are hyperplanes, this means that they must span V , whence R^s and R^t must generate R . This completes the proof. ■

Remark 8. This result becomes trivial when considering that, for any $s \in S$, the ring R^s is generated by the unit, α_s^2 and elements of the form $\alpha_t - (\alpha_s, \alpha_t)\alpha_s$, for all $t \in S \setminus \{s\}$. It is easy to see that, given fixed $s, t \in S$ with $s \neq t$, we have

$$\alpha_s = \frac{1}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_s - (\alpha_s, \alpha_t)\alpha_t) + \frac{(\alpha_s, \alpha_t)}{(\alpha_s, \alpha_t)^2 + 1}(\alpha_t - (\alpha_s, \alpha_t)\alpha_s)$$

if and only if $m_{st} \neq \infty$, and similarly for α_t . However, showing that these polynomials generate R^s is non-trivial, following from the Chevalley–Shephard–Todd theorem.

Let $s, t \in S$ with $s \neq t$ and $m_{st} \neq \infty$. Observe that

$$B_s \otimes_R B_t \cong R \otimes_{R^s} R \otimes_{R^t} R(2) \quad \text{and} \quad B_t \otimes_R B_s \cong R \otimes_{R^t} R \otimes_{R^s} R(2).$$

It follows from Lemma 2 that these are both indecomposable, as by Lemma 7 they are generated by the degree -2 elements $1 \otimes_{R^s} 1 \otimes_{R^t} 1$ and $1 \otimes_{R^t} 1 \otimes_{R^s} 1$, respectively. We shall therefore write $B_{st} := B_s \otimes_R B_t$ and $B_{ts} := B_t \otimes_R B_s$ from this point onwards.

Remark 9. As it happens, $B_s \otimes_R B_t$ and $B_t \otimes_R B_s$ remain indecomposable when $m_{st} = \infty$, but proving this is somewhat more involved. In addition to our previous lemma no longer holding (since $H_s = H_t$ end up being the same 1-dimensional space, generated by $\alpha_s + \alpha_t$), these bimodules are also no longer cyclic, meaning we cannot use Lemma 2!

The remainder of these notes will be in proving the following result, where $B_{sts} := R \otimes_{R^{s,t}} R(3)$. This notation will be justified by the following proposition. However, we note that – remarkably – a similar definition can be made for any Soergel bimodule B_{w_0} corresponding to the longest element w_0 of its associated Coxeter group.

Proposition 10. (Indecomposable Soergel Bimodules of Type A_2). *The indecomposable Soergel bimodules of type A_2 are, up to grading shift and isomorphism, R , B_s , B_t , B_{st} , B_{ts} and B_{sts} .*

Proof. Let $W = S_3 = \{1, s, t, st, ts, sts\}$ and $S = \{s, t\}$. We have already seen that R , B_s , B_t , B_{st} and B_{ts} are indecomposable Soergel bimodules. Certainly B_{sts} is indecomposable as an (R, R) -bimodule by Lemma 2, as it is generated by the degree -3 element $1 \otimes_{R^{s,t}} 1$, but it remains to be shown that it is indeed a Soergel bimodule. We will begin by showing that it appears as a direct summand of both $B_s \otimes_R B_t \otimes_R B_s$ and $B_t \otimes_R B_s \otimes_R B_t$. Once we've done this, we will show that these indecomposables are exhaustive.

In order to show that B_{sts} is a Soergel bimodule, we first claim that

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \quad \text{and} \quad B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t.$$

To this end, let's define an (R, R) -bimodule homomorphism $\iota_{sts} : B_{sts} \rightarrow B_s \otimes_R B_t \otimes_R B_s$ by extending

$$\iota_{sts} : 1 \otimes_{R^{s,t}} 1 \mapsto 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

This is clearly a well-defined homomorphism of graded (R, R) -bimodules, as $f \otimes_{R^{s,t}} 1 = 1 \otimes_{R^{s,t}} f$ if and only if $f \in R^{s,t}$ if and only if $f \in R^s, R^t$. In fact, this reasoning also shows that it is injective. Suppose we define another (R, R) -bimodule homomorphism $\iota_s : B_s \rightarrow B_s \otimes_R B_t \otimes_R B_s$ by extending

$$\iota_s : 1 \otimes_{R^s} 1 \mapsto \frac{1}{2}(1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1).$$

In order to show that this is well-defined, we need to show that

$$f \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1 = 1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} f$$

for all $f \in R^s$. By Lemma 4, for any $f \in R$ we have

$$f = P_t(f) + \partial_t(f)\alpha_t,$$

where $P_t(f), \partial_t(f) \in R^t$. If $f \in R^s$, then

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 &= 1 \otimes_{R^s} f \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} (P_t(f) + \partial_t(f)\alpha_t) \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 \\ &= (1 \otimes_{R^s} P_t(f) \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1) + (1 \otimes_{R^s} \partial_t(f) \alpha_t^2 \otimes_{R^t} 1 \otimes_{R^s} 1) \\ &= (1 \otimes_{R^s} \alpha_t \otimes_{R^t} P_t(f) \otimes_{R^s} 1) + (1 \otimes_{R^s} 1 \otimes_{R^t} \partial_t(f) \alpha_t^2 \otimes_{R^s} 1), \end{aligned}$$

and similarly

$$f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 = (1 \otimes_{R^s} 1 \otimes_{R^t} P_t(f) \alpha_t \otimes_{R^s} 1) + (1 \otimes_{R^s} \alpha_t \otimes_{R^t} \partial_t(f) \alpha_t \otimes_{R^s} 1).$$

Summing these and combining the “even” terms $P_t(f)$ with the “odd” terms $\partial_t(f)\alpha_t$, it follows that

$$\begin{aligned} f \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} 1 + f \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} f \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} f \alpha_t \otimes_{R^s} 1 \\ &= 1 \otimes_{R^s} \alpha_t \otimes_{R^t} 1 \otimes_{R^s} f + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} f. \end{aligned}$$

Thus ι_s is well-defined, and like ι_{sts} it is a monomorphism. Because $\langle 1 \otimes_{R^t} 1 \rangle$ and $\langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle$ clearly generate disjoint (R^s, R^s) -bimodules, ι_{sts} and ι_s are themselves disjoint. If we can show

$$R \otimes_{R^t} R \cong \langle 1 \otimes_{R^t} 1 \rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle,$$

then $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$, with a similar computation yielding $B_t \otimes_R B_s \otimes_R B_t \cong B_{sts} \oplus B_t$.

While we could just figure out how to write $x \otimes_{R^t} y$ in $\langle 1 \otimes_{R^t} 1 \rangle \oplus \langle \alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t \rangle$ for all $x, y \in \{1, \alpha_s, \alpha_t\}$, there is perhaps a more informative way to prove this result. This is to find explicit projections $\pi_s : B_s \otimes_R B_t \otimes_R B_s \rightarrow B_s$ and $\pi_{sts} : B_s \otimes_R B_t \otimes_R B_s \rightarrow B_{sts}$ for which

$$\begin{aligned}\pi_{sts} \circ \iota_{sts} &= \text{id}_{B_{sts}}, \\ \pi_s \circ \iota_s &= \text{id}_{B_s}, \\ \iota_{sts} \circ \pi_{sts} + \iota_s \circ \pi_s &= \text{id}_{B_s \otimes_R B_t \otimes_R B_s}.\end{aligned}$$

Note that this is precisely what $B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s$ means by definition. It is not too hard to find π_s . One can easily show that it is nothing but

$$\pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -2\delta_s(fg) \otimes_{R^s} 1,$$

where we remind the reader that we are using the relatively non-standard normalization $\delta_s : f \mapsto (f - s(f))/2\alpha_s$ of the Demazure operator. But how can we find π_{sts} ? Well, let

$$e := \iota_s \circ \pi_s : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto -\delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

As we would expect, e_s is an idempotent, as

$$e_s^2 = \iota_s \circ \pi_s \circ \iota_s \circ \pi_s = \iota_s \circ \text{id}_{B_s} \circ \pi_s = e_s.$$

As mentioned above, we want to find $e_{sts} = \iota_{sts} \circ \pi_{sts}$ satisfying

$$e_{sts} + e_s = \text{id}_{B_s \otimes_R B_t \otimes_R B_s}.$$

This gives us an explicit formula for e_{sts} ; namely, $e_{sts} = \text{id}_{B_s \otimes_R B_t \otimes_R B_s} - e_s$, whence

$$e_{sts} = 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 + \delta_s(fg) \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1.$$

We can use this to determine π_{sts} . First, we know that π_{sts} will be some map of the form

$$\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 \mapsto \sum_i h_i \otimes_{R^{s,t}} k_i.$$

Thus we must rearrange the image of e_{sts} such that the inner tensors are 1. Let's write

$$\begin{aligned}1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 &= 1 \otimes_{R^s} (P_s(f) + \delta_s(f)\alpha_s) \otimes_{R^t} (P_s(g) + \delta_s(g)\alpha_s) \otimes_{R^s} 1 \\ &= P_s(f) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} P_s(g) + \\ &\quad P_s(f) \otimes_{R^s} 1 \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g) + \\ &\quad \delta_s(f) \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} \delta_s(g).\end{aligned}$$

This greatly simplifies our work, as now we need only compute how e_{sts} acts on these four tensors. The first one is simple, since $\delta_s(1) = 0$:

$$e_{sts}(1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1) = 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.$$

As for the next two, observe that

$$(\alpha_s + \alpha_t) - \frac{2}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) = \frac{2}{3} \left(\alpha_s + \frac{1}{2} \alpha_t \right).$$

Hence

$$\begin{aligned}
e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} 1) &= 1 \otimes_{R^s} \alpha_s \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} (\alpha_t \otimes_{R^t} 1 + 1 \otimes_{R^t} \alpha_t) \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} (\alpha_s + \alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} \frac{1}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 + 1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_t \otimes_{R^s} 1 + \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{1}{3}(2\alpha_s + 4\alpha_t) \otimes_{R^s} 1 + 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}(\alpha_s + 2\alpha_t) + \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1.
\end{aligned}$$

The result for $1 \otimes_{R^s} 1 \otimes_{R^t} \alpha_s \otimes_{R^s} 1$ follows by symmetry. We can then play a similar game for the remaining tensor. We begin by observing that

$$\alpha_s + \frac{2}{3} \left(\frac{1}{2}\alpha_s + \alpha_t \right) = \frac{4}{3} \left(\alpha_s + \frac{1}{2}\alpha_t \right).$$

Thus

$$\begin{aligned}
e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1) &= 1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^t} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} \frac{2}{3}(2\alpha_s + \alpha_t) \otimes_{R^s} 1 + \\
&\quad 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^t} \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(8\alpha_s^2 + 8\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \\
&= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(8\alpha_s^2 + 8\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^s} 1 - \\
&\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\
&\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t).
\end{aligned}$$

From here, we note that

$$R^{s,t} = \mathbb{R}[\alpha_s^2 + \alpha_s\alpha_t + \alpha_t^2];$$

in particular,

$$\alpha_s^2 + \alpha_s\alpha_t \in R^t.$$

We may hence continue with

$$\begin{aligned} e_{sts}(1 \otimes_{R^s} \alpha_s \otimes_{R^t} \alpha_s \otimes_{R^s} 1) &= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(6\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) \\ &= 1 \otimes_{R^s} 1 \otimes_{R^t} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\ &= 1 \otimes_{R^s} \frac{2}{9}(3\alpha_s^2 + 3\alpha_s\alpha_t) \otimes_{R^t} 1 \otimes_{R^s} 1 - \\ &\quad 1 \otimes_{R^s} \frac{2}{9}(2\alpha_s^2 + 5\alpha_s\alpha_t + 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2. \\ &= 1 \otimes_{R^s} \frac{2}{9}(\alpha_s^2 - 2\alpha_s\alpha_t - 2\alpha_t^2) \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 \\ &= \frac{4}{9}(-\alpha_s^2 - \alpha_s\alpha_t - \alpha_t^2) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1 + \\ &\quad \frac{1}{3}(\alpha_s + 2\alpha_t) \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{1}{3}(\alpha_s + 2\alpha_t) + \\ &\quad 1 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} \frac{2}{3}\alpha_s^2 + \\ &\quad \frac{2}{3}\alpha_s^2 \otimes_{R^s} 1 \otimes_{R^t} 1 \otimes_{R^s} 1. \end{aligned}$$

We therefore conclude that

$$\begin{aligned}
\pi_{sts} : 1 \otimes_{R^s} f \otimes_{R^t} g \otimes_{R^s} 1 &\mapsto P_s(f) \otimes_{R^{s,t}} P_s(g) + \\
&\quad \partial_s(f) \frac{2}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} P_s(g) + \\
&\quad \partial_s(f) \otimes_{R^{s,t}} \frac{4}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) P_s(g) + \\
&\quad P_s(f) \otimes_{R^{s,t}} \frac{2}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) \partial_s(g) + \\
&\quad P_s(f) \frac{4}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} \partial_s(g) + \\
&\quad \partial_s(f) \frac{1}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) \otimes_{R^{s,t}} \frac{1}{3} \left(\frac{1}{2} \alpha_s + \alpha_t \right) \partial_s(g) + \\
&\quad \partial_s(f) \otimes_{R^{s,t}} \frac{2}{3} \alpha_s^2 \partial_s(g) + \\
&\quad \partial_s(f) \frac{2}{3} \alpha_s^2 \otimes_{R^{s,t}} \partial_s(g) - \\
&\quad \partial_s(f) \frac{4}{9} (\alpha_s^2 + \alpha_s \alpha_t + \alpha_t^2) \otimes_{R^{s,t}} \partial_s(g).
\end{aligned}$$

Phew!!

All that's left is to clean up a few loose ends. First, we see that B_{st} and B_{ts} are not isomorphic, as

$$B_s \otimes_R B_t \otimes_R B_s \cong B_{sts} \oplus B_s \not\cong B_{st}(-1) \oplus B_{st}(1) \cong B_s \otimes_R B_s \otimes_R B_t.$$

Note that we have used the fact here that direct sum decompositions are unique. Finally, in order to show that all indecomposables have been exhausted, we claim that

$$B_{sts} \otimes_R B_s \cong B_s \otimes_R B_{sts} \cong B_{sts}(1) \oplus B_{sts}(-1) \cong B_t \otimes_R B_{sts} \cong B_{sts} \otimes_R B_t.$$

Recall that in Lemma 4, we found an isomorphism $R \cong R^s \oplus R^s(-2)$ of (R^s, R^s) -bimodules. It is easy to see that this restricts to an isomorphism of $(R^{s,t}, R^s)$ -bimodules, whence

$$\begin{aligned}
B_{sts} \otimes_R B_s &\cong (R \otimes_{R^{s,t}} R(3)) \otimes_R (R \otimes_{R^s} R(1)) \\
&\cong R \otimes_{R^{s,t}} R \otimes_{R^s} R(4) \\
&\cong R \otimes_{R^{s,t}} (R^s \oplus R^s(-2)) \otimes_{R^s} R(4) \tag{Lemma 4} \\
&\cong (R \otimes_{R^{s,t}} R(4)) + (R \otimes_{R^s} R(2)) \\
&\cong B_{sts}(1) \oplus B_{sts}(-1).
\end{aligned}$$

The other isomorphisms follow similarly. We have thus shown that we obtain no new indecomposables by tensoring, meaning we are done with our classification. This completes the proof. \blacksquare