Higher Representation Theory

1. Prologue

Throughout these notes, we will often talk about both weak and strict 2-categories. We will use "bicategory" to mean a weak 2-category, which roughly consists of a class of objects, a class of 1-morphisms between objects and a class of 2-morphisms between 1-morphisms, together with unitors, associators and a notion of horizontal composition. Meanwhile, a "2-category" will be taken to mean a strict 2-category, which is roughly a bicategory whose unitors and associators are equalities rather than mere natural transformations. Note that there is a bijection between one-object bicategories (respectively, 2-categories) and monoidal (respectively, strict monoidal) categories. We will also assume that $\mathbb{k}$ is an algebraically closed field.

Should be more precise and put basic higher category theoretic definitions here, as well as assumptions: our bicategories are essentially small, pseudofunctors are what [Lei98] calls homomorphisms, our natural transformations between pseudofunctors are strong transformations, fields are algebraically closed, etc.

2. Finitary Birepresentation Theory

Definition 2.1. (Idempotent Complete). *Let $\mathcal{C}$ be a category. An* idempotent *is an endomorphism $e : A \to A$ in $\mathcal{C}$ such that $e \circ e = e$. An idempotent is said to* split *if there is an object B and morphisms $\pi : A \to B$, $\iota : B \to A$ in $\mathcal{C}$ such that $e = \iota \circ \pi$ and $\mathrm{id}_B = \pi \circ \iota$. A category is said to be* idempotent complete *(or* idempotent split*) if every idempotent splits.*

Note that the condition $\mathrm{id}_B = \pi \circ \iota$ implies that π is an epimorphism and ι is a monomorphism. To see why, suppose we have morphisms $h, k : B \to C$ with $h \circ \pi = k \circ \pi$. Then $h = h \circ \pi \circ \iota = k \circ \pi \circ \iota = k$, whence π is an epimorphism. Similarly, given morphisms $h, k : C \to B$ with $\iota \circ h = \iota \circ k$, we have that $h = \pi \circ \iota \circ h = \pi \circ \iota \circ k = k$, whence ι is a monomorphism. Thus, because ι is a monomorphism, B is by definition a subobject of A. In other words, a category being idempotent complete means that every idempotent $e : A \to A$ can be seen as a projection onto some subobject B followed by an inclusion back into A. Moreover, in the additive setting we have the following result.

Proposition 2.2. *An idempotent $e : A \to A$ belonging to a preadditive category splits if and only if $A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$.*

Proof. Suppose $e : A \to A$ is an idempotent that splits. Then by definition we have a subobject I of A together with an epimorphism $\pi_I : A \to I$ and a monomorphism $\iota_I : I \to A$ satisfying $e = \iota_I \circ \pi_I$ and $\mathrm{id}_I = \pi_I \circ \iota_I$. Moreover, because $\mathrm{id}_A - e$ is also idempotent, there similarly exists a subobject K of A together with an epimorphism $\pi_K : A \to K$ and a monomorphism $\iota_K : K \to A$ satisfying $\mathrm{id}_A - e = \iota_K \circ \pi_K$ and $\mathrm{id}_K = \pi_K \circ \iota_K$. Because $\mathrm{id}_A = \iota_I \circ \pi_I + \iota_K \circ \pi_K$, we have the biproduct diagram

$$I \underset{\iota_I}{\overset{\pi_I}{\leftrightarrows}} A \underset{\iota_K}{\overset{\pi_K}{\rightleftarrows}} K.$$

By [Mac13, Theorem VIII.2.2], it follows that $A = I \oplus K$. We claim now that $\mathrm{Im}(e) = I$; we shall prove this by showing that e admits the canonical decomposition

$$K \xrightarrow{\iota_K} A \xrightarrow{\pi_I} I \xrightarrow{\iota_I} A \xrightarrow{\pi_K} K.$$

In particular, we claim that $\mathrm{Ker}(e) = (K, \iota_K)$, that $\mathrm{Coker}(e) = (K, \pi_K)$, that $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and that $\mathrm{Ker}(\pi_K) = (I, \iota_I)$. We show that the first two hold and remark that showing the remaining two is essentially the same. First, observe that

$$e \circ \iota_K = \iota_I \circ \pi_I \circ \iota_K = (\mathrm{id}_A - \iota_K \circ \pi_K) \circ \iota_K = \iota_K - \iota_K \circ \pi_K \circ \iota_K = \iota_K - \iota_K = 0.$$

Moreover, given an object K' together with a morphism $k' : K' \to A$ for which $e \circ k' = 0$, we see that by taking $\ell := \pi_K \circ k'$, we have that

$$\iota_K \circ \ell = \iota_K \circ \pi_K \circ k' = (\mathrm{id}_A - e) \circ k' = k'.$$

Thus $\mathrm{Ker}(e) = (K, \iota_K)$. As for $\mathrm{Coker}(e)$, we observe that

$$\pi_K \circ e = \pi_K \circ \iota_I \circ \pi_I = \pi_K \circ (\mathrm{id}_A - \iota_K \circ \pi_K) = \pi_K - \pi_K \circ \iota_K \circ \pi_K = \pi_K - \pi_K = 0,$$

and that for any object C' together with a morphism $c' : A \to C'$ for which $c' \circ e = 0$, taking $\ell := c' \circ \iota_K$ gives us

$$\ell \circ \pi_K = c' \circ \iota_K \circ \pi_K = c' \circ (\mathrm{id}_A - e) = c'.$$

That $\mathrm{Coker}(\iota_K) = (I, \pi_I)$ and $\mathrm{Ker}(\pi_K) = (I, \iota_I)$ follow similarly, whence $A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$.

Conversely, suppose that $e : A \to A$ is an idempotent for which $A = \mathrm{Im}(e) \oplus \mathrm{Ker}(e)$. Then we have the canonical decomposition

$$\mathrm{Ker}(e) \xrightarrow{k} A \xrightarrow{\pi} \mathrm{Im}(e) \xrightarrow{\iota} A \xrightarrow{c} \mathrm{Coker}(e).$$

By definition this means that $e = \iota \circ \pi$, so we need only show that $\mathrm{id}_{\mathrm{Im}(e)} = \pi \circ \iota$. But note that π is a cokernel and ι is a kernel, hence they are an epimorphism and a monomorphism, respectively. Thus by the definition of epimorphisms and monomorphisms, we may cancel $e \circ e = e$ on the right by ι and on the left by π, whence we obtain nothing but

$$e \circ e = e \implies \iota \circ \pi \circ \iota \circ \pi = \iota \circ \pi \implies \pi \circ \iota = \mathrm{id}_{\mathrm{Im}(e)}$$

as desired. Thus e splits. This completes the proof. ∎

This result is not only important in its own right, but psychologically helpful: it tells us in some heuristic sense that split idempotents categorify the notion of projections from linear algebra, which always split. Moreover, recall that a preadditive category is said to be *Karoubian* (or *pseudo-Abelian*) if every idempotent admits a kernel (or, equivalently, if every idempotent admits an image, as we may obtain the image by considering $\mathrm{Ker}(\mathrm{id}_A - e)$). We therefore have the following corollary.

Corollary 2.3. *A preadditive category is Karoubian if and only if it is idempotent complete.*

Example 2.4. The category of projective modules over a ring is the Karoubi envelope of its full subcategory of free modules, as a module is projective if and only if it is a direct summand of a free module. In other words, categories of projective modules are idempotent complete in a universal way.

Definition 2.5. (Finitary Category). *An additive, $\Bbbk$-linear category $\mathcal{C}$ is called* multifinitary *if it is idempotent complete, it has finitely many isomorphism classes of indecomposable objects and it has finite-dimensional $\Bbbk$-vector spaces of morphisms. If $\mathcal{C}$ is not monoidal, this is equivalent to being* finitary*; otherwise, it is said to be* finitary *if its unit object is indecomposable.*

Some authors write instead that a category is finitary if it is equivalent to the category of finitely generated, projective modules over some finite-dimensional, associative $\Bbbk$-algebra. Such a category is certainly Karoubi by our previous remarks, and it also has both finitely many isomorphism classes of indecomposable objects and finite-dimensional $\Bbbk$-vector spaces of morphisms. The converse direction follows from a finitary modification of the Freyd-Mitchell embedding theorem. It may also be worth noting that we can replace "projective" with "injective" in the previous definition, as these two notions are dual in the sense that being projective is the same as being injective in the opposite category.

Theorem 2.6. ([Sha23, Theorem 6.1]). *Let $\mathcal{C}$ be an additive, $\Bbbk$-linear category with finite-dimensional $\Bbbk$-vector spaces of morphisms. Then the following are equivalent.*

(i). *$\mathcal{C}$ is Krull–Schmidt.*
(ii). *$\mathcal{C}$ is idempotent complete.*
(iii). *An object $X \in \mathrm{Ob}(\mathcal{C})$ is indecomposable if and only if the ring $\mathrm{End}_{\mathcal{C}}(X)$ is local.*

See [Sha23, Theorem 6.1] for the proof. We will only be using the fact that (ii) implies (iii), for which there is alternatively a proof given in [Ela20, Lecture 1, Theorem 2]. This theorem tells us that a multifinitary category is precisely a Krull–Schmidt category with finitely many isomorphism classes of indecomposable objects!

Definition 2.7. *Let $\mathfrak{A}^f_{\Bbbk}$ denote the 2-category whose objects are multifinitary categories, whose 1-morphisms are $\Bbbk$-linear functors and whose 2-morphisms are natural transformations.*

Note that any $\Bbbk$-linear functor between additive, $\Bbbk$-linear categories is automatically additive.

Definition 2.8. (Hom-Category). *Let $\mathscr{C}$ be a bicategory with objects $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. We define the* Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ *to be the category whose objects are the 1-morphisms of $\mathscr{C}$ from $\mathtt{i}$ to $\mathtt{j}$ and whose 1-morphisms are the 2-morphisms of $\mathscr{C}$ that are horizontally between $\mathtt{i}$ and $\mathtt{j}$, where composition is given by vertical composition. We interpret the* End-categories $\mathscr{C}(\mathtt{i}, \mathtt{i})$ *as being monoidal categories with respect to the composition of 1-morphisms.*

Definition 2.9. (Finitary Bicategory). *A bicategory $\mathscr{C}$ is said to be* (multi)finitary *if*

- *it has finitely many objects;*
- *for any pair $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, the Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is (multi)finitary;*
- *horizontal composition of 2-morphisms is both biadditive and $\Bbbk$-bilinear.*

Observe that $\mathscr{C}(\mathtt{i}, \mathtt{i})$ has a canonical monoidal structure given by composition, which is strict if $\mathscr{C}$ is a 2-category. Thus for finitary bicategories, we are subtly asking that the corresponding identity 1-morphism $\mathrm{id}_{\mathtt{i}}$ be indecomposable as an object of $\mathscr{C}(\mathtt{i}, \mathtt{i})$ for every $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$.

In a finitary 2-category, vertical composition will automatically be both biadditive and $\Bbbk$-bilinear as a consequence of the Hom-categories $\mathcal{C}(\mathtt{i}, \mathtt{j})$ being additive and $\Bbbk$-linear. However, we genuinely must ask that horizontal composition be biadditive and $\Bbbk$-bilinear.

Proposition 2.10. *A monoidal category $\mathcal{C}$ is (multi)finitary if and only if its monoidal delooping $\mathsf{B}\mathcal{C}$ is (multi)finitary.*

Proof. If $\mathsf{B}\mathcal{C}$ is (multi)finitary, it is obvious that $\mathcal{C}$ is (multi)finitary, since $\mathcal{C} = \mathsf{B}\mathcal{C}(\bullet, \bullet)$, where $\mathrm{Ob}(\mathsf{B}\mathcal{C}) = \{\bullet\}$. Conversely, suppose $\mathcal{C}$ is (multi)finitary. Clearly $\mathsf{B}\mathcal{C}$ has finitely many objects (in particular, it has only one), and $\mathsf{B}\mathcal{C}(\bullet, \bullet) = \mathcal{C}$ is (multi)finitary. Finally, horizontal composition is given by the monoidal product and is hence biadditive and $\Bbbk$-bilinear. This completes the proof. ■

Definition 2.11. (Birepresentation). *A* birepresentation *of a bicategory $\mathscr{C}$ is a pseudofunctor from $\mathscr{C}$ to Cat, the 2-category of small categories. A* 2-representation *of a 2-category $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to Cat.*

Definition 2.12. (Finitary Birepresentation). *A* (multi)finitary birepresentation *of a (multi)finitary bicategory $\mathscr{C}$ is a $\Bbbk$-linear pseudofunctor from $\mathscr{C}$ to $\mathfrak{A}^f_{\Bbbk}$. A* (multi)finitary 2-representation *of a (multi)finitary 2-category $\mathscr{C}$ is a $\Bbbk$-linear 2-functor from $\mathscr{C}$ to $\mathfrak{A}^f_{\Bbbk}$.*

Every 2-representation is a birepresentation. Moreover, if $\mathscr{C}$ is a (multi)finitary 2-category, then any (multi)finitary 2-representation of $\mathscr{C}$ is a (multi)finitary birepresentation.

Intuition 2.13. Recall that a representation of a group G is morally nothing but a functor $F : \mathsf{B}G \to \mathsf{Vect}$, where the action of G on $V := F(\bullet)$ is given by $g \cdot v := [F(g)](v)$ for all $v \in V$ and $g \in \mathrm{Mor}(\mathsf{B}G)$. Analogously, given a 2-representation $\mathrm{M} : \mathscr{C} \to \mathsf{Cat}$, we have a 2-*action* of $\mathscr{C}$ given by $F \cdot X := [\mathrm{M}(F)](X)$ for all $X \in \mathrm{M}(\mathtt{i})$ and $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$, where $\mathtt{i}, \mathtt{j} \in \mathrm{Ob}(\mathscr{C})$. The upshot here is that we should think of the 1-morphisms in our 2-category as our "group elements", with composition becoming "group multiplication". This perspective motivates the following definitions.

Proposition 2.14. *Let $\mathcal{C}$ be a monoidal (respectively, strict monoidal) category. There exists a bijection between $\mathcal{C}$-module categories and birepresentations (respectively, 2-representations) of the delooping category $\mathsf{B}\mathcal{C}$.*

Proof. Let $\mathcal{C}$ be a monoidal category and $\mathsf{B}\mathcal{C}$ its delooping category. Recall that there is a bijection between $\mathcal{C}$-module structures on a category $\mathcal{M}$ and monoidal functors of the form $F : \mathcal{C} \to \mathrm{End}(\mathcal{M})$. Such a functor F induces a canonical birepresentation $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ that takes the single object $\bullet \in \mathrm{Ob}(\mathsf{B}\mathcal{C})$ to $\mathcal{M}$ and otherwise acts on the 1-morphisms and 2-morphisms by F. Conversely, let $\mathrm{M} : \mathsf{B}\mathcal{C} \to \mathsf{Cat}$ be a birepresentation and write $\mathcal{M} := \mathrm{M}(\bullet)$. This naturally induces a functor $F : \mathcal{C} \to \mathsf{Cat}(\mathcal{M}, \mathcal{M}) = \mathrm{End}(\mathcal{M})$ that acts on objects and morphisms of $\mathcal{C}$ by M. Clearly these two constructions are inverse to each other. It is easy to see that if $\mathcal{C}$ is strict, the result for 2-representations follows similarly. This completes the proof. ■

Definition 2.15. (Equivalence of Birepresentations). *We say that two birepresentations* M *and* N *of $\mathscr{C}$ are* equivalent *if there exists a* strong transformation $\Phi : \mathrm{M} \to \mathrm{N}$ *such that the component* $\Phi_{\mathtt{i}} : \mathrm{M}(\mathtt{i}) \to \mathrm{N}(\mathtt{i})$ *is an equivalence of categories for each* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *(see [Lei98] for the definition of a strong transformation).*

Example 2.16. (Yoneda Birepresentation). Let $\mathscr{C}$ be a bicategory and consider the pseudofunctor $\mathscr{C}(\mathtt{i}, -) : \mathscr{C} \to \mathsf{Cat}$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, such that

- objects $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ are sent to the Hom-category $\mathscr{C}(\mathtt{i}, \mathtt{j})$;
- 1-morphisms of the form $F \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k})) = \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$ are sent to the "post-composition by F" functor $F_* : \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$ given by $G \mapsto F \circ G$ and $(\gamma : G \Rightarrow G') \mapsto \mathbb{1}_F \circ_h \gamma$;
- 2-morphisms of the form $\alpha : F \Rightarrow G$, for $F, G \in \mathrm{Ob}(\mathscr{C}(\mathtt{j}, \mathtt{k})) = \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{j}, \mathtt{k})$, are sent to the "horizontal post-composition by α" natural transformation given by $\alpha_* : \beta \mapsto \alpha \circ_h \beta$, where $\beta : F' \Rightarrow G'$ for $F', G' \in \mathrm{Ob}(\mathscr{C}(\mathtt{i}, \mathtt{j})) = \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}, \mathtt{j})$.

We call this the *Yoneda* (or *principal*) *birepresentation corresponding to* $\mathtt{i}$ and denote it by $\mathbb{P}_{\mathtt{i}}$. We can see the Yoneda birepresentation in action by taking $\mathscr{C}$ to be the monoidal delooping of Vect. Then $\mathbb{P}_\bullet$ maps $\bullet$ to Vect, maps 1-morphisms $V \in \mathrm{Ob}(\mathsf{Vect})$ to the left tensor product functor given by $U \mapsto V \otimes U$ and $(f : U \to U') \mapsto \mathrm{id}_V \otimes f$, and maps 2-morphisms $f : U \to V$ to the left tensor product map given by $g \mapsto f \otimes g$. This is in fact a finitary 2-representation; in general, if $\mathscr{C}$ is (multi)finitary, then its corresponding Yoneda birepresentations are also all (multi)finitary, and if $\mathscr{C}$ is a (multi)finitary 2-category its Yoneda birepresentations are (multi)finitary 2-representations.

Definition 2.17. (Ideal). *A* left *(respectively* right*)* ideal *of a category $\mathcal{C}$ is a collection $\mathcal{I} \coloneqq \{\mathcal{I}(X,Y) : X,Y \in \mathrm{Ob}(\mathcal{C})\}$, where each $\mathcal{I}(X,Y)$ is a non-empty subclass of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, such that $\mathcal{I}$ is stable under post-composition (respectively pre-composition) with morphisms from $\mathcal{C}$. If $\mathcal{C}$ is preadditive, we additionally ask that $\mathcal{I}(X,Y)$ be an Abelian subgroup of $\mathrm{Mor}_{\mathcal{C}}(X,Y)$ for all pairs $X,Y \in \mathrm{Ob}(\mathcal{C})$, and if $\mathcal{C}$ is $\Bbbk$-linear we ask that these also be $\Bbbk$-subspaces. We say that $\mathcal{I}$ is a* two-sided *(or* bilateral*)* ideal *if it is both a left ideal and a right ideal, and that it is a subideal of $\mathcal{J}$ if its classes morphisms are subclasses. An ideal is said to be* proper *if there exists some pair $X,Y \in \mathrm{Ob}(\mathcal{C})$ for which $\mathcal{I}(X,Y) \subset \mathrm{Mor}_{\mathcal{C}}(X,Y)$, and* maximal *if it is proper and not a subideal of any other proper ideal.*

Let $\mathcal{I}$ be an ideal of a category $\mathcal{C}$. As we have implied previously, if $\mathcal{C}$ is a preadditive, then $\mathrm{End}_{\mathcal{C}}(X)$ is a ring for all $X \in \mathrm{Ob}(\mathcal{C})$, and it follows that the valid choices for $\mathcal{I}(X,X)$ coincide exactly with the ring ideals of $\mathrm{End}_{\mathcal{C}}(X)$. Similarly, if $\mathcal{C}$ is a $\Bbbk$-linear category, then each $\mathrm{End}_{\mathcal{C}}(X)$ is an associative, unital algebra, and the valid choices for $\mathcal{I}(X,X)$ coincide with algebra ideals (that is, a subspace of $\mathrm{End}_{\mathcal{C}}(X)$ that is closed under algebra multiplication).

Example 2.18. Consider a subcategory $\mathcal{T}$ of $\mathsf{Vect}_{\Bbbk}$ whose endomorphisms of $\Bbbk^n$ are the $n \times n$ upper triangular Toeplitz matrices. Then $\mathrm{End}(\Bbbk^2) \cong \Bbbk[x]/\langle x^2 \rangle$. If we consider the sub-semicategory of $\mathcal{T}$ containing only the object $\Bbbk^2$ and the endomorphisms $\Bbbk\{x\}$ (that is, linear scalings of the matrix with 1 in the off-diagonal), we obtain an ideal of $\mathcal{T}$.

Let M be a birepresentation of a multifinitary bicategory $\mathscr{C}$ for which each $\mathrm{M}(\mathtt{j})$ is additive and idempotent complete, and let $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ for some $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. Consider the *additive closure* (closure under isomorphisms, direct summands and finite direct sums) of the orbit of X under the action of $\mathscr{C}$; that is, the collection

$$\mathscr{C}(\{X\}) \coloneqq \mathsf{add}(\{[\mathrm{M}(F)](X) : \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})\})$$

where the add denotes the aforementioned additive closure. Due to the additivity of the 1-morphisms of $\mathfrak{A}^f_{\Bbbk}$, it follows that $\mathscr{C}(\{X\})$ is itself stable under the action of $\mathscr{C}$. This therefore induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X\})$ of $\mathscr{C}$ by restriction, with each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$ sent to

$$\mathscr{C}_{\mathtt{j}}(\{X\}) \coloneqq \mathsf{add}(\{[\mathrm{M}(F)](X) : F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})\}),$$

the *additive subcategory* (full subcategory that is closed under isomorphisms, direct summands and finite direct sums) of $\mathrm{M}(\mathtt{j})$ generated by the objects of $\mathscr{C}(\{X\})$ that lie in $\mathrm{M}(\mathtt{j})$. In principle this process works for any collection of objects $\{X_i : i \in I\}$ with $X_i \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}_i))$, whence

$$\mathscr{C}(\{X_i : i \in I\}) \coloneqq \mathsf{add}(\{[\mathrm{M}(F)](X_i) : i \in I, \mathtt{j} \in \mathrm{Ob}(\mathscr{C}), F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_i,\mathtt{j})\})$$

similarly induces a finitary sub-birepresentation $\mathrm{G}_{\mathrm{M}}(\{X_i : i \in I\})$ of $\mathscr{C}$. In any case, we will only need to consider the single-object situation, as it allows us to make the following evocative definition.

Definition 2.19. (Transitive Birepresentation). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory $\mathscr{C}$. We say that* M *is* transitive *if, for every $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ and non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$, the embedding $\mathscr{C}_{\mathtt{j}}(\{X\}) \hookrightarrow \mathrm{M}(\mathtt{j})$ is an equivalence for all $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$.*

Remark 2.20. Recall that a module M is simple if and only if every cyclic submodule generated by a non-zero element of M is equal to M. This is exactly what we're trying to capture with transitivity! In the 2-representation world, however, things are slightly more involved, as this does not imply that there are no non-trivial ideals (a natural property we would like to ask of simple 2-representations).

Definition 2.21. ($\mathscr{C}$-Stable Ideal). *Let* M *be a birepresentation of* $\mathscr{C}$. *A* $\mathscr{C}$-stable ideal I *of* M *is a collection* $\mathrm{I} := \{\mathrm{I}(\mathtt{i}) : \mathtt{i} \in \mathrm{Ob}(\mathcal{C})\}$, *where each* $\mathrm{I}(\mathtt{i})$ *is a two-sided ideal of* $\mathrm{M}(\mathtt{i})$ *such that* $[\mathrm{M}(F)](\mathrm{I}(\mathtt{i}))$ *is a subclass of* $\mathrm{I}(\mathtt{j})$ *for all* 1*-morphisms* $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$. *A* $\mathscr{C}$*-stable subideal* I *of a* $\mathscr{C}$*-stable ideal* J *is a* $\mathscr{C}$*-stable ideal for which* $\mathrm{I}(\mathtt{i})$ *is a subideal of* $\mathrm{J}(\mathtt{i})$ *for all* $\mathtt{i} \in \mathrm{Ob}(\mathcal{C})$. *We say that* I *is* proper *if there exists some* $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$ *for which* $\mathrm{I}(\mathtt{i})$ *is proper, and* maximal *if it is proper and not a* $\mathscr{C}$*-stable subideal of any other proper* $\mathscr{C}$*-stable ideal.*

What is the advantage of $\mathscr{C}$-stability when quotienting?

Definition 2.22. (Simple 2-Representation). *A multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *is said to be* simple *if it admits no non-zero, proper* $\mathscr{C}$*-stable ideals.*

Proposition 2.23. *Every simple birepresentation is transitive.*

Proof. Let M be a simple birepresentation of a multifinitary bicategory $\mathscr{C}$ and take $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero. Certainly $\mathrm{G}_{\mathrm{M}}(\{X\})$ induces a $\mathscr{C}$-stable ideal of M, which cannot be proper by simplicity. Thus for each $\mathtt{j} \in \mathrm{Ob}(\mathscr{C})$, we know that $\mathrm{Mor}(\mathscr{C}_{\mathtt{j}}(\{X\}))$ cannot be proper and hence $\mathscr{C}_{\mathtt{j}}(\{X\})$ must be equivalent to $\mathrm{M}(\mathtt{j})$. In other words, M is transitive. This completes the proof. ∎

Example 2.24. Transitive does not imply simple?

Remark 2.25. Let $\mathcal{C}$ be an additive category. Then by [Mac13, §VIII.2], its morphisms form a matrix calculus; that is, for any $f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$ with $X \cong \bigoplus_{i=1}^m X_i$ and $Y \cong \bigoplus_{j=1}^n Y_j$, we have that

$$f = \sum_{j=1}^{n} \sum_{i=1}^{m} (\iota_{Y_j} \circ f_{i,j} \circ \pi_{X_i})$$

for $f_{i,j} := \pi_{Y_j} \circ f \circ \iota_{X_i}$, where $\pi_{X_i} : X \to X_i$ and $\pi_{Y_j} : Y \to Y_j$ are epimorphisms while $\iota_{X_i} : X_i \to X$ and $\iota_{Y_j} : Y_j \to Y$ are monomorphisms for all $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$.

Proposition 2.26. *Let* M *be a transitive birepresentation of a multifinitary bicategory* $\mathscr{C}$*. Then* M *admits a unique maximal* $\mathscr{C}$*-stable ideal* I*, and moreover each* $\mathrm{I}(\mathtt{i})$ *contains no identity morphisms apart from the one corresponding to the zero object.*

Proof. Let I be the sum (as vector spaces) of all $\mathscr{C}$-stable ideals of M that do not contain id_X for any non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and any $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This is certainly itself $\mathscr{C}$-stable ideal by construction. Moreover, because $U, V \subseteq U + V \subseteq W$ for vector subspaces $U, V \subseteq W$, it follows that I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms. Because I is maximal with respect to $\mathscr{C}$-stable ideals not containing identity morphisms, J must contain at least one identity morphism, say id_X for some non-zero $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$. Given any non-zero $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, the transitivity of M tells us that there exists some $F \in \mathrm{Mor}^1_{\mathcal{C}}(\mathtt{i}, \mathtt{j})$ with $[\mathrm{M}(F)](X) = X_1 \oplus \cdots \oplus X_n$ such that $X_k \cong Y$ for some $1 \le k \le n$. This means that

$$[\mathrm{M}(F)](\mathrm{id}_X) = \mathrm{id}_{X_1 \oplus \cdots \oplus X_n} \in \mathrm{J}(\mathtt{j})$$

by $\mathscr{C}$-stability. But by pre-composing with $\iota_{X_k} \circ \varphi^{-1} : Y \to X_k \to X_1 \oplus \cdots \oplus X_n$ and post-composing with $\varphi \circ \pi_{X_k} : X_1 \oplus \cdots \oplus X_n \to X_k \to Y$, where $\varphi : X_k \xrightarrow{\sim} Y$ is our isomorphism, we obtain that $\mathrm{id}_Y \in [\mathrm{J}(\mathtt{j})](Y, Y)$, and hence that $\mathrm{id}_Y \in \mathrm{J}(\mathtt{j})$. Thus J must contain *all* identity morphisms and therefore cannot be proper, meaning that I must in fact be maximal as claimed. The uniqueness of I follows by construction. This completes the proof. ■

Proposition-Definition 2.27. (Simple Quotient). *A transitive birepresentation* M *of a multifinitary bicategory* $\mathscr{C}$ *is simple if and only if the unique maximal* $\mathscr{C}$*-stable ideal* I *from Proposition 2.26 is the zero ideal. The simple birepresentation* $\underline{\mathrm{M}}$ *given by the quotient* M/I *is known as the* simple quotient *of* M*.*

Proof. Naturally if I – the sum of all $\mathscr{C}$-stable ideals without non-zero identity morphisms – is the zero ideal, then M must contain no non-zero, proper $\mathscr{C}$-stable ideals. Conversely, if M is simple, then because I is the sum of proper $\mathscr{C}$-stable ideals, they must all be zero. Because I is maximal, every morphism $f \in \mathrm{Mor}([\mathrm{M}/\mathrm{I}](\mathtt{i}))$ must generate either $\{0\}$ or $[\mathrm{M}/\mathrm{I}](\mathtt{i})$ under multiplication by $\mathscr{C}$, whence it follows that M/I is simple. This completes the proof. ■

Let M be a multifinitary birepresentation of $\mathscr{C}$. We denote by Ind(M) the set of isomorphism classes of indecomposable objects in every $\mathrm{M}(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$; that is,

$$\mathrm{Ind}(\mathrm{M}) = \bigsqcup_{\mathtt{i} \in \mathrm{Ob}(\mathscr{C})} \{[X] \in \mathrm{M}(\mathtt{i}) : X \text{ is indecomposable}\}.$$

Note that Ind(M) is clearly finite, as $\mathscr{C}$ has finitely many objects and each category $\mathrm{M}(\mathtt{i}) \in \mathrm{Ob}(\mathfrak{A}^f_{\Bbbk})$ has finitely many isomorphism classes of indecomposable objects.

For $X, Y \in \mathrm{Ind}(\mathrm{M})$, where for instance $X \in \mathrm{M}(\mathtt{i}_X)$ and $Y \in \mathrm{M}(\mathtt{i}_Y)$, we write $X \le Y$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_Y, \mathtt{i}_X)$ such that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$.

Lemma 2.28. *Let* M *be a multifinitary birepresentation. The binary relation* $\leq$ *defined above defines a preorder on* $\mathrm{Ind}(\mathrm{M})$ *known as the* action preorder*.*

Proof. Clearly $\geq$ is reflexive, as we can just take $F := \mathrm{id}_{\mathtt{i}}$. Moreover, suppose that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$ and Y is isomorphic to a direct summand of $[\mathrm{M}(G)](Z)$; that is,

$$[\mathrm{M}(F)](Y) \cong X \oplus X_1 \oplus X_2 \oplus \cdots ,$$
$$[\mathrm{M}(G)](Z) \cong Y \oplus Y_1 \oplus Y_2 \oplus \cdots .$$

In order to show transitivity, we would like to show that X is isomorphic to a direct summand of $[\mathrm{M}(FG)](Z)$. Well, because the morphisms of $\mathfrak{A}^f_{\Bbbk}$ are additive, we simply observe that

$$\begin{aligned}[\mathrm{M}(FG)](Z) &\cong [\mathrm{M}(F)](Y) \oplus [\mathrm{M}(F)](Y_1) \oplus [M(F)](Y_2) \oplus \cdots \\ &\cong X \oplus X_1 \oplus X_2 \oplus \cdots \oplus [\mathrm{M}(F)](Y_1) \oplus [\mathrm{M}(F)](Y_2) \oplus \cdots .\end{aligned}$$

This completes the proof. ■

Suppose we define an equivalence relation $\sim$ given by $X \sim Y$ if and only if $X \leq Y$ and $Y \leq X$. Obviously $\geq$ extends to a partial order on $\mathrm{Ind}(\mathrm{M})/\sim$. In particular, we have the following result.

Proposition 2.29. *Let* M *be a multifinitary birepresentation. Then* M *is transitive if and only if* $\mathrm{Ind}(\mathrm{M})/\sim$ *has only one element.*

Proof. Suppose $\mathrm{Ind}(\mathrm{M})/\sim$ is a singleton and take any $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ non-zero as a representative. Then for any indecomposable $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$, there exists some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$ for which Y is isomorphic to a direct summand of $[M(F)](X)$, since $Y \leq X$. In other words, the additive subcategory $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, as by definition it is closed under direct summands. Thus M is transitive.

Conversely, suppose M is transitive, and consider any pair of indecomposables $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$. Because $\mathscr{C}_{\mathtt{j}}(\{X\})$ is equivalent to $\mathrm{M}(\mathtt{j})$, we know by the definition of $\mathscr{C}_{\mathtt{j}}(\{X\})$ that Y is isomorphic to a direct summand of $[M(F)](X)$ for some $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i},\mathtt{j})$; that is, $Y \leq X$. The same argument applied to $\mathscr{C}_{\mathtt{i}}(\{Y\})$ shows us that $X \leq Y$, whence $\mathrm{Ind}(\mathrm{M})/\sim$ has only one element. This completes the proof. ■

Definition 2.30. (Directed Order Ideal). *A* directed order ideal *of a partially ordered set* $(P, \leq)$ *is a non-empty subset* I *such that*

- *for all* $x \in I$ *and* $y \in P$*,* $y \leq x$ *implies that* $y \in I$ *(lower set);*
- *for all* $x, y \in I$*, there is some* $z \in I$ *such that* $x \leq z$ *and* $y \leq z$ *(upward directed set).*

Remark 2.31. In the literature, this partial order is often written as $X \geq Y$ if there exists a 1-morphism $F \in \mathrm{Mor}^1_{\mathscr{C}}(\mathtt{i}_Y, \mathtt{i}_X)$ such that X is isomorphic to a direct summand of $[\mathrm{M}(F)](Y)$; that is, the inequality symbol is the other way around. As a result, they consider *coideals* (non-empty subsets that are upper sets and downward directed sets) rather than ideals in what follows. This is because we want to go "downwards", in the sense that we collect all direct summands. Why do they define the order "backwards"? Is there some (cell?) interpretation for which that is more natural?

Let M be a multifinitary birepresentation of $\mathscr{C}$ and Q a directed order ideal of $\mathrm{Ind}(\mathrm{M})/\sim$. For $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$, define $\mathrm{M}_Q(\mathtt{i})$ to be the additive subcategory of $\mathrm{M}(\mathtt{i})$ generated by the indecomposable objects $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ whose equivalence class lies in Q. Then $\mathrm{M}_Q : \mathtt{i} \mapsto \mathrm{M}_Q(\mathtt{i})$ induces a multifinitary sub-birepresentation of M, known as the sub-birepresentation of M associated to Q.

Let $Q \subset R$ be a pair of directed order ideals in $\mathrm{Ind}(\mathrm{M})/\sim$ and let $\mathrm{I}_Q(\mathtt{i})$ denote the ideal in $\mathrm{M}_R(\mathtt{i})$ generated by the identity morphisms in $\mathrm{M}_Q(\mathtt{i})$, for $\mathtt{i} \in \mathrm{Ob}(\mathscr{C})$. This collection of ideals is $\mathscr{C}$-stable, whence the multifinitary birepresentation M_R induces a multifinitary birepresentation $\mathrm{M}_{R/Q} : \mathtt{i} \mapsto \mathrm{M}_R(\mathtt{i})/\mathrm{I}_Q(\mathtt{i})$. This is known as the quotient of M associated to $Q \subset R$. Note that if $|R \setminus Q| = 1$, then $|\mathrm{Ind}(\mathrm{M}_{R/Q})/\sim| = 1$, so $\mathrm{M}_{R/Q}$ will be transitive by Proposition 2.29.

Choose $r \in \mathrm{Ind}(\mathrm{M})/\sim$ and let X_r be the maximal directed order ideal in $\mathrm{Ind}(\mathrm{M})/\sim$ that does not contain r. In other words, $(\mathrm{Ind}(\mathrm{M})/\sim) \setminus X_r$ – the complement of X_r – has minimal element r. Thus we also obtain a directed order ideal $Y_r \coloneqq X_r \cup \{r\}$, whence the associated quotient M_{Y_r/X_r} is transitive by Proposition 2.29. We henceforth let $\underline{\mathrm{M}}_r$ denote the simple quotient $\underline{\mathrm{M}}_{Y_r/X_r}$.

Before stating the main theorem of this section, we recall the classical Jordan-Hölder theorem for modules over rings. Let M be an R-module admitting the two composition series

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

$$\{0\} = M'_0 \subset M'_1 \subset \cdots \subset M'_m = M,$$

where by *composition series* we mean that these are filtrations of submodules where the *composition factors* $L_i \coloneqq M_i/M_{i-1}$ and $L'_i \coloneqq M'_j/M'_{j-1}$ are simple for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$. The *Jordan-Hölder theorem* tells us that the *composition lengths* are equal – that is, $n = m$ – and moreover that there exists a permutation $\sigma \in S_n$ such that $L_i \cong L'_{\sigma(i)}$ for all $i \in \{1, \ldots, n\}$.

Consider a filtration of directed order ideals

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim$$

such that $|Q_i \setminus Q_{i-1}| = 1$ for all $i \in \{1, \ldots, n\}$. We call this a *complete filtration.* As shown previously, from such a filtration we have a corresponding *weak Jordan-Hölder series*

$$\{0\} = \mathrm{M}_{Q_0} \subset \mathrm{M}_{Q_1} \subset \cdots \subset \mathrm{M}_{Q_n} = \mathrm{M}$$

consisting of sub-birepresentations whose *weak composition quotients* $\mathrm{L}_i \coloneqq \underline{\mathrm{M}}_{Q_i/Q_{i-1}}$ are simple birepresentations for all $i \in \{1, \ldots, n\}$. With this, we have the following result.

Theorem 2.32. (Weak Jordan-Hölder Theorem). *Let* M *be a multifinitary birepresentation of a multifinitary bicategory* $\mathscr{C}$ *admitting the two complete filtrations*

$$\varnothing = Q_0 \subset Q_1 \subset \cdots \subset Q_n = \mathrm{Ind}(\mathrm{M})/\sim,$$

$$\varnothing = Q'_0 \subset Q'_1 \subset \cdots \subset Q'_m = \mathrm{Ind}(\mathrm{M})/\sim,$$

with weak composition quotients $\{\mathrm{L}_i\}_{i=1}^n$ *and* $\{\mathrm{L}'_j\}_{j=1}^m$ *respectively. Then* $m = n$, *and moreover there exists a permutation* $\sigma \in S_n$ *such that* L_i *and* $\mathrm{L}_{\sigma(i)}$ *are equivalent for all* $i \in \{1, \ldots, n\}$.

Proof. We clearly have $m = n = |\mathrm{Ind}(\mathrm{M})/\!\sim|$ by the definition of a complete filtration. Suppose now that $r \in \mathrm{Ind}(\mathrm{M})/\!\sim$; then there exist unique $i, j \in \{1, 2, \ldots, n\}$ for which $Q_i \setminus Q_{i-1} = Q'_j \setminus Q_{j-1} = \{r\}$. If we can show that the birepresentations L_i and L'_j are both equivalent to $\underline{\mathrm{M}}_r$, then we are done. In particular, by symmetry it is enough to show that L_i is equivalent to $\underline{\mathrm{M}}_r$.

Let I_{X_r} be the $\mathscr{C}$-stable ideal in M_{Y_r} for which $\mathrm{M}_{Y_r/X_r} = \mathrm{M}_{Y_r}/\mathrm{I}_{X_r}$ and $\mathrm{I}_{Q_{i-1}}$ the $\mathscr{C}$-stable ideal in M_{Q_i} for which $\mathrm{M}_{Q_i/Q_{i-1}} = \mathrm{M}_{Q_i}/\mathrm{I}_{Q_{i-1}}$. Since $\{r\} = Q_i \setminus Q_{i-1}$, we know by construction that $Q_{i-1} \subseteq X_r$, as X_r is by definition the maximal directed order ideal not containing r. Similarly, $Q_i \subseteq Y_r$; this inclusion induces a transformation from M_{Q_i} to M_{Y_r}, which in turn induces a strong transformation $\sigma : \mathrm{M}_{Q_i} \to \mathrm{M}_{Y_r/X_r}$ by taking the quotient. Double-check this. Meanwhile, for any pair of indecomposable objects $X \in \mathrm{Ob}(\mathrm{M}(\mathtt{i}))$ and $Y \in \mathrm{Ob}(\mathrm{M}(\mathtt{j}))$ lying in the equivalence class r, we have that $\mathrm{I}_{X_r}(X, Y) \subseteq \mathrm{I}_{Q_{i-1}}(X, Y)$ (what does this even mean?). Thus the strong transformation σ factors through $\mathrm{M}_{Q_i/Q_{i-1}}$, in the sense that there exist transformations $\sigma_1 : \mathrm{M}_{Q_i} \to \mathrm{M}_{Q_i/Q_{i-1}}$ and $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \to \mathrm{M}_{Y_r/X_r}$ such that $\sigma = \sigma_2 \circ \sigma_1$. In particular, this gives us a transformation $\sigma_2 : \mathrm{M}_{Q_i/Q_{i-1}} \to \mathrm{M}_{Y_r/X_r}$ that is surjective on morphisms (check how this fits into the equivalence picture). Therefore, because $\mathrm{M}_{Q_i/Q_{i-1}}$ and M_{Y_r/X_r} are both transitive, taking their simple quotients via proposition 2.27 induces an equivalence between L_i and $\underline{\mathrm{M}}_r$ as desired, whence the result follows. This completes the proof. ■

In what sense is this weak Jordan-Hölder theorem "weak"? The decategorifications of these simple quotients are "transitive $\mathbb{N}$-modules" and usually not simple.

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Motivation

- What is finitary birepresentation theory? My take: a "categorification" of the representation theory of finite-dimensional algebras, arising from the study of knot invariants, tensor categories and operator algebras. In other words, Jones mathematics (lol).
- Stealing some slogans from one of Dani's talks: representation theory is group theory in vector spaces, while birepresentation theory is group theory in linear categories.
- Why is it important? Basically for (somehow) studying the fields I just mentioned. Hopefully by the end of this little series we can figure out the answer to this together.

Plan

- Today I'd like to introduce one of the main theorems of birepresentation theory: the "weak Jordan-Hölder theorem". This theorem is a nice (abstract) motivation for the classification of simple birepresentations.
- In principle I'm not sure how helpful this talk will be. It will mostly be walking you through what I've been thinking about over the past couple of weeks. The main theorem is, I think, quite nice, but getting there will likely be dry. Next time I'd like to focus more on examples (especially Soergel bimodules and their classification) if I'm capable enough.
- Addressing a question: why am I talking about birepresentations? The people were promised 2-representations! In classifying 2-representations of Soergel bimodules, it was quickly found that the language of 2-representations was too restrictive. Later on in this series I will probably transition to 2-representations, but at least in this talk I'd like to state things as generally as possible to avoid issues later on.
- As a reminder to myself, I'd like to maintain a little definition bank so you guys can keep track of definitions.

Multifinitary Categories

- What are multifinitary 1-categories? Add this to the definition bank.
- In linear algebra, all idempotents split. Idempotent complete categories live somewhere between additive categories and Abelian categories. In particular, *all pseudo-Abelian categories are idempotent complete!* These remarks are what "unlocked" idempotent completeness for me.
- Examples of finitary categories (that I can't write down because I haven't prepared any concrete examples): (semi)groups and their representations, quantum groups and their categorifications, tensor categories, fusion categories, modular (tensor) categories, 2-Kac-Moody categories, etc.
- $\mathsf{Vec}_{\Bbbk}$ is a helpful example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix}$.
- Maybe mention example 2.8? Mazorchuk and Miemietz would have you believe it's a useful example but man is it impenetrable.
- Also, $\mathbb{S}\mathsf{Bim}(W, S)$ lololol
- Define the category of multifinitary categories.
- Finitary bicategories. Add this to the definition bank. We can generate these from finitary monoidal categories: in particular, there is a bijection between monoidal categories and one-object bicategories, with delooping taking us upstairs and Hom-categories taking us downstairs. Mention strict case.

Finitary Birepresentations

- What are finitary birepresentations? There is a bijection between module categories and birepresentations of one-object bicategories. Example: Yoneda birepresentations.
- Equivalence of birepresentations. I don't understand these very well yet.

Simple Birepresentations

- Walk through next block: essentially, I want to define what it means for a birepresentation to be transitive. If you're like me and you keep forgetting what this adjective means, an action of G on a set X is said to be transitive if, for any $x, y \in X$, there is some $g \in G$ for which $g \cdot x = y$.
- Define transitive birepresentations. Add this to the definition bank. What are we trying to capture with transitivity?
- There is a "better" notion of simplicity.
- What are ideals of finitary 1-categories? Mention that I've talked to Dani about this and they gave a somewhat confusing answer, but I think I've figured it out. I'm mentioning this in case I'm actually stupid.
- Give an example – $\mathsf{Vec}_{\Bbbk}$ yippeeee!!
- What are $\mathscr{C}$-stable ideals of birepresentations?
- What are simple birepresentations? Add this to the definition bank. Why are they "better"? They somehow more completely characterize what it means for a representation to be simple. Unfortunately, I don't really have an example of a non-simple transitive birepresentation for you right now (supposedly they exist), but I'll hopefully have one next week. Many of the naïve examples of transitive birepresentations end up being simple.
- Result I'll need: transitive birepresentations admit a unique maximal ideal, and quotienting by this ideal gives you a simple birepresentation. We call this the simple quotient of M. I haven't finished working through the proof for this yet.

Weak Jordan-Hölder

- Define Ind(M), note that it is finite. Define preorder, outline the proof. Recall: preorder means $X \geq X$ and $X \geq Y, Y \geq Z \implies X \geq Z$. Quotienting by $X \sim Y \iff X \geq Y, Y \geq X$ gives us an honest partial order (pretty much by definition).
- Proposition: M is transitive if and only if $|\mathrm{Ind}(\mathrm{M})/\sim| = 1$. This is yet another folklore result, but I can maybe offer some intuition as to why we might expect this to be true. I still need to prove it rigorously.
- Define poset ideal and coideal.
- Maybe now would be a good time to remind everyone of the classical Jordan-Hölder theorem. Given a representation M, we would like a sensible notion of composition series.
- How do coideals induce sub-birepresentations?
- How do pairs of coideals induce transitive birepresentations?
- How can we make these birepresentations simple?
- State the theorem.
- Maybe go through an example with $\mathsf{Rep}(\mathbb{Z}/2\mathbb{Z})$?