

# Categorification and the Lusztig–Vogan Module

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# Prologue

In this talk, I would like to motivate categorical representation theory from the perspective of real Lie groups.

I'll start by introducing Kazhdan–Lusztig theory and describing the success of categorification in giving us a new perspective in this realm.

I'll then mention Lusztig and Vogan's real group analogue and where I fit into understanding its categorification.

In particular, I would like to walk through some explicit examples of categorical Jordan–Hölder decompositions.

# Kazhdan–Lusztig Theory

In the late 1970s, Kazhdan and Lusztig introduced a family of polynomials now known as **Kazhdan–Lusztig polynomials**, defined as change-of-basis coefficients between the standard basis and the following self-dual basis for the Iwahori–Hecke algebra  $\mathcal{H}(W, S)$ .

$$b_x = \sum_{y \in W} h_{y,x}(v) \delta_y$$

**Theorem (Kazhdan–Lusztig Conjecture ([KL79]: [BB81], [BK81]))**

*For all  $x, y \in W$ , the Jordan–Hölder multiplicity of the composition factor  $L_y$  inside the Verma module  $M_x$  is given by*

$$[M_x : L_y] = h_{y,x}(1).$$

Let  $R := \mathbb{R}[\alpha_s : s \in S]$  and define an  $(R, R)$ -bimodule

$$B_s := R \otimes_{R^s} R(1)$$

for each  $s \in S$ . Given  $w \in W$  with expression  $\underline{w} = (s_1, \dots, s_k)$ , we define the **Bott–Samelson bimodule**

$$BS(w) := B_{s_1} \otimes_R \cdots \otimes_R B_{s_k}.$$

The Karoubian envelope of the category generated by these  $(R, R)$ -bimodules is known as the **category of Soergel bimodules** and denoted  $\mathbb{S}\mathrm{Bim}(W, S)$ .

The indecomposables of  $\mathbb{S}\mathrm{Bim}(W, S)$  are, up to grading shifts, indexed by  $W$ , with  $B_w$  denoting the unique indecomposable summand of  $BS(w)$  that does not appear in  $BS(x)$  for any  $x < w$ .

For all  $x \in W$ , we define the **standard bimodule**  $R_x$  to be the regular  $(R, R)$ -bimodule  $R$  whose right action is twisted by reflection by  $x$ .

For any Soergel bimodule  $B \in \text{Ob}(\mathcal{SBim}(W, S))$ , we have a so-called  **$\Delta$ -filtration** of  $(R, R)$ -bimodules

$$0 = B^k \subset B^{k-1} \subset \dots \subset B^1 \subset B^0 = B$$

with subquotients  $B^i/B^{i+1} \cong R_{y_i}^{\oplus h_{y_i}(B)}$ , where  $h_{y_i}(B) \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ .

By definition,  $R_y^v := R_y(1)$  and  $R_y^{v^{-1}} := R_y(-1)$ .

## Theorem (Soergel's Conjecture ([Soe92]: [EW14]))

*For all  $x, y \in W$ , the graded multiplicity of the composition factor  $R_y$  in the  $\Delta$ -filtration of the Soergel bimodule  $B_x$  is given by*

$$h_y(B_x) = h_{y,x}.$$

*Equivalently, the indecomposables of  $\mathbb{S}\text{Bim}(W, S)$  categorify the canonical basis for  $\mathcal{H}(W, S)$ .*

On top of this, Soergel bimodules carry also a good deal of homological and geometric structure that isn't seen in the decategorified world.

# Real Reductive Groups

The objects we are interested in studying are real reductive groups.

Their full representation theory is totally unmanageable.

Harish-Chandra's approach: study a nice subclass that *is* manageable.

The “right” class of representations are [admissible representations](#).

Irreducible  $K$ -admissible representations are in bijection with irreducible  $(\mathfrak{g}, K)$ -modules, up to some notion of equivalence.

Irreducible  $(\mathfrak{g}, K)$ -modules can then be realized geometrically as  $K$ -equivariant local systems on  $K$ -orbits of the flag variety via Beilinson–Bernstein localization.

# The Lusztig–Vogan Module

Let  $G$  be a connected complex reductive group with holomorphic involution  $\theta$ . This data is equivalent to choosing a real form  $G_{\mathbb{R}}$  of  $G$ .

Fix a finite index subgroup  $K \subseteq G^{\theta}$ , a Borel subgroup  $B \subset G$ , a maximal torus  $T \subset K \cap B$  and a non-singular integral infinitesimal character  $\lambda$ .

$$\left\{ \begin{array}{l} \text{irreducible } (\mathfrak{g}, K)\text{-modules} \\ \text{with character } \lambda \end{array} \right\} \longleftrightarrow \mathcal{D} := \left\{ \begin{array}{l} \text{irreducible } K\text{-equivariant} \\ \text{local systems on } K\text{-orbits} \\ \text{of the flag variety } G/B \end{array} \right\}$$

The free  $\mathbb{Z}[v^{\pm 1}]$ -module with basis  $\mathcal{D}$  is known as the **Lusztig–Vogan module** and denoted  $\mathcal{M}_{LV}$ . It admits an action of  $\mathcal{H}(W, S)$ .

The Lusztig–Vogan module can be thought of as a **real group analogue to Kazhdan–Lusztig theory**.



# The Lusztig–Vogan Module

For each  $\delta \in \mathscr{D}$ , there exists a “self-dual” element

$$C_\delta = \sum_{\gamma \in \mathscr{D}} P_{\gamma, \delta}(v) \gamma$$

in  $\mathcal{M}_{LV}$ . The  $P_{\gamma, \delta}$  are known as [Kazhdan–Lusztig–Vogan polynomials](#).

## Theorem ([LV83])

*For all  $\delta, \gamma \in W$ , the Jordan–Hölder multiplicity of the composition factor  $L_\gamma$  inside the  $(\mathfrak{g}, K)$ -module  $M_\delta$  is given by*

$$[M_\delta : L_\gamma] = P_{\gamma, \delta}(1).$$

Question: can we learn more about real groups via categorification?

# The Lusztig–Vogan Category

In 2022, Larson and Romanov came up with a categorification of the principle block of the Lusztig–Vogan module as a module category over the category of Soergel bimodules.

For the sake of simplicity, we will restrict to the case where  $K$  is the connected component containing the identity and  $T$  is Abelian.

## Theorem ([LR22])

*The Grothendieck group of*

$$\mathcal{N}_{LV}^0 := \langle R_w \otimes_R X : [w] \in W_K \setminus W, X \in \text{Ob}(\mathcal{SBim}(W, S)) \rangle_{\oplus, \ominus, (1)}$$

*is the block of  $\mathcal{M}_{LV}$  containing the trivial representation.*

# The Lusztig–Vogan Category

Victor showed us a diagrammatic calculus for the morphisms.  
This is already some new categorical structure.

I'm interested in understanding its (weak) Jordan–Hölder decompositions.

I'd like to use the remainder of the talk to walk through a couple of concrete examples.

## Example – Type $A_1$

Let  $G := \mathrm{SL}(2, \mathbb{C})$ . This group has two real forms: a compact form  $\mathrm{SU}(2)$  and a split form  $\mathrm{SU}(1, 1)$ . Let's look at the latter.

The corresponding Weyl groups are  $W = S_2 = \{1, s\}$  and  $W_K = \{1\}$ , so  $W_K \backslash W = \{[1], [s]\}$ .

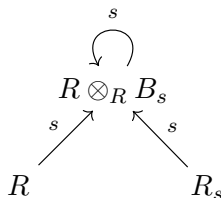
Tensoring standard bimodules with indecomposable Soergel bimodules gives us four initial candidates for indecomposables (up to grading shifts).

$$\begin{aligned} R \otimes_R R &\cong R, & R \otimes_R B_s &\cong B_s, \\ R_s \otimes_R R &\cong R_s, & R_s \otimes_R B_s &\cong B_s. \end{aligned}$$

All of these are indecomposable, so we're done.

## Example – Type $A_1$

We can graph the relationships between the indecomposables under actions by  $\mathbb{S}\text{Bim}$  as follows.



The simple transitive module subcategories are generated by the strongly connected components of the graph.

In this instance, we have an equivalence of categories  $\mathcal{M}_R \simeq \mathcal{M}_{R_s}$ . Thus our composition quotients are  $\mathcal{M}_{B_s}$  with multiplicity 1 and  $\mathcal{M}_R$  with multiplicity 2. Note that  $\mathcal{H}(W, S)$  has two two-sided cells.

## Example – Type $A_2$

Let  $G := \mathrm{SL}(3, \mathbb{C})$ . This group has three real forms: a compact form  $\mathrm{SU}(3)$ , a split form  $\mathrm{SL}(3, \mathbb{R})$  and a quasi-split form  $\mathrm{SU}(2, 1)$ . We'll look at  $\mathrm{SU}(2, 1)$ .

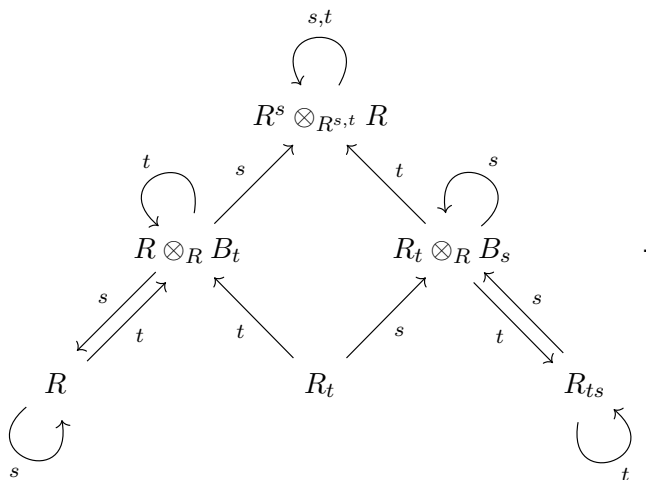
The corresponding Weyl groups are  $W = S_3 = \langle s, t : s^2 = t^2 = (st)^3 = 1 \rangle$  and  $W_K = S_2 = \{1, s\}$ , so  $W_K \backslash W = \{[1], [t], [ts]\}$ .

Not every product of a standard bimodule with an indecomposable Soergel bimodule is indecomposable, so we need to look for direct summands. Our indecomposables are, up to grading shifts,

$$R, \quad R_t, \quad R_{ts}, \quad R \otimes_R B_t, \quad R_t \otimes_R B_s \quad \text{and} \quad R^s \otimes_{R^{s,t}} R.$$

## Example – Type $A_2$

The graph of the action preorder is



We have three simple quotients, where  $\mathcal{M}_R \simeq \mathcal{M}_{R_{ts}}$  has multiplicity 2.

We have just seen two examples for type  $A$  Coxeter systems.

In this case, the action preorder is given by the  $W$ -graph (boring).

For some Coxeter system  $(W, S)$  of type  $A_n$ , there is one equivalence class of simple transitive 2-representations of  $\mathbb{S}\text{Bim}(W, S)$  for each of the  $n + 1$  two-sided cells, and these are all [cell 2-representations](#) ([MMM+23]).



There are already lots of questions!

- Are there Coxeter systems for which the the action preorder for the Lusztig–Vogan category is not predicted by the  $W$ -graph?
- Can we build any interesting module categories from “extensions” of Lusztig–Vogan categories?
- Is there a real group analogue to Soergel’s conjecture?
- What genuinely new information can we learn about real groups?

# References

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Thank you for your attention!