ACM/IDS 104 APPLIED LINEAR ALGEBRA
PROBLEM SET 1

Please submit your solution as a single PDF file, that contains both the written-up and published code parts, via Gradescope by **9pm Tuesday, October 10**. An example of the submission process is shown here: `https://www.gradescope.com/get_started#student-submission`

- For theoretical problems, please use a pen, not a pencil: it is hard to read scanned submission written by a pencil.
- For coding problems, please convert your MATLAB livescripts (`.mlx`) to PDF by selecting `Live Editor` → `Save` → `Export to PDF` and merge them with the rest of your solution.
- After uploading your submission to Gradescope, please label all pages.

Problem 1. (10 POINTS) MATRIX MULTIPLICATION
Let $A \in \mathbb{M}_{m\times p}$ and $B \in \mathbb{M}_{p\times n}$ be $m \times p$ and $p \times n$ matrices respectively. Then, by definition, the (i, j) entry of $C = AB$ is the *dot product* of the i^{th} row of A and the j^{th} column of B:

$$c_{ij} = a^i \cdot b_j = \sum_{k=1}^{p} a_{ik} b_{kj}. \tag{1}$$

Interestingly, we can also compute AB by multiplying columns of A by rows of B. Show that

$$AB = \sum_{k=1}^{p} a_k b^k, \tag{2}$$

where each term $a_k b^k$ is a matrix of size $m \times n$. In particular, if B is a column vector $B = x \in \mathbb{R}^p$, then

$$Ax = \sum_{k=1}^{p} x_k a_k. \tag{3}$$

is a linear combination of columns of A. We will make use of this interpretation of matrix multiplication in the lectures.

Remark: Decomposition (2) is related to the notion of *outer product* (also called *tensor product*) of two vectors. If $u, v \in \mathbb{R}^p$ are two (column) vectors, then their outer product is defined as

$$u \otimes v = uv^T \in \mathbb{M}_{p\times p}. \tag{4}$$

Problem 2. (10 POINTS) NILPOTENT MATRICES
In lecture 1, we saw that $A^2 = 0$ does not imply $A = 0$. This motivates the following definition: a square matrix $A \in \mathbb{M}_{n\times n}$ is called *nilpotent* if $A^k = 0$ for some $k \geq 1$. Show that any strictly upper triangular matrix (that is upper triangular with zero diagonal) is nilpotent.

Problem 3. (10 POINTS) PERMUTED LU FACTORIZATION
Let A_n be the $n \times n$ tridiagonal matrix with all 2's along the main diagonal, and all -1's along the sub- and super-diagonals[1]. Find the factors P_n, L_n, and U_n in the permuted LU factorization of A_n:

$$P_n A_n = L_n U_n. \tag{5}$$

Hint: You can compute the PLU factorization of A_n in MATLAB for different (large) values of n and check how P_n, L_n, and U_n look like. Formulate a hypothesis (a "guess") for P_n, L_n, and U_n from your observations, and then prove it.

[1]Recall that A_n is the coefficient matrix of a linear system which is obtained from discretization of the Poisson equation in 1D, see lecture 1.

Problem 4. (10 POINTS) ORTHOGONAL MATRICES
The Gaussian elimination method results into the permuted LU decomposition, $PA = LU$, which is fundamental for solving linear systems. Another important factorization of a nonsingular matrix is the so-called QR factorization (will discuss in lectures), which can also be used for solving square linear systems (this approach is more computationally expensive, but less prone to inaccuracies and more numerically stable). The QR factorization is based on the notion of *orthogonal* matrix. A square matrix A is called orthogonal if its inverse and transpose are coincide: $A^{-1} = A^T$.

(a) (5 points) Show that any permutation matrix P is orthogonal.
(b) (5 points) Is an orthogonal matrix necessarily a permutation matrix?

Problem 5. (10 POINTS) SKEW-SYMMETRIC MATRICES
In lecture 2, we discussed symmetric matrices and their properties. Another important class of matrices is *skew-symmetric* matrices. A matrix A is skew-symmetric if $A^T = -A$. Show that every square matrix A can be written as a sum of a symmetric matrix S and a skew-symmetric matrix J:

$$A = S + J, \quad S^T = S, \quad J^T = -J. \tag{6}$$

Problem 6. (10 POINTS) DETERMINANTS ARE EXPENSIVE TO COMPUTE.
Complete Problem 6 in `PS1.mlx`.

Problem 7. (10 POINTS) SOLVING LINEAR SYSTEMS
Let B be the following $n \times n$ matrix:

$$B = \begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix} \tag{7}$$

(a) (5 points) Find the rank of B and complete Problem 7a in `PS1.mlx`.
<u>Remark:</u> You don't need MATLAB to find the answer, but you can use it for making the right guess. We have provided a way of checking your answer in Problem 7a of `PS1.mlx`. In your solutions, please explain why your answer is correct for any n, and not only the values explored in MATLAB.
(b) (5 points) Complete Problem 7b in `PS1.mlx`. Remeber to report the non-zero components in your solutions.