ACM/IDS 104 APPLIED LINEAR ALGEBRA
PROBLEM SET 5

Please submit your solution as a single PDF file, that contains both the written-up and published code parts, via Gradescope by **9pm Tuesday, November 14**. An example of the submission process is shown here: `https://www.gradescope.com/get_started#student-submission`

- For theoretical problems, please use a pen, not a pencil: it is hard to read scanned submission written by a pencil.
- For coding problems, please convert your MATLAB livescripts (`.mlx`) to PDF by selecting `Live Editor` → `Save` → `Export to PDF` and merge them with the rest of your solution.
- After uploading your submission to Gradescope, please label all pages.

Problem 1.(10 POINTS) APPLICATION OF PROJECTIONS TO APPROXIMATION
Complete Problem 1 in `PS5.mlx`.

Problem 2. (10 POINTS) HERMITE POLYNOMIALS
The Hermite[1] polynomials are orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)e^{-\frac{x^2}{2}}\,dx. \tag{1}$$

They are extensively used in probability theory. Recall that the standard normal (with zero mean and unit variance) pdf is $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \propto e^{-\frac{x^2}{2}}$. Find the first five monic Hermite polynomials.

Problem 3. (10 POINTS) ORTHOGONAL COMPLIMENTS
Let $W \subset \mathbb{R}^3$ be a subspace spanned by $v_1 = (1,2,3)^T$ and $v_2 = (2,0,1)^T$. Find the orthogonal compliments $W_1^\perp$ and $W_2^\perp$ to W with respect to the dot product $\langle x, y\rangle_1 = x_1y_1 + x_2y_2 + x_3y_3$ and the weighted inner product $\langle x, y\rangle_2 = x_1y_1 + 2x_2y_2 + 3x_3y_3$.

Problem 4. (10 POINTS) COMPLETE MATRICES
Consider the following matrix:

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2}$$

Is this matrix complete? Does it admit eigenvector basis of $\mathbb{R}^3$, $\mathbb{C}^3$?

Problem 5. (10 POINTS) GERSCHGORIN THEOREM
Consider the following matrix:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \tag{3}$$

(a) (3 points) Find all Gerschgorin disks and sketch the Gerschgorin domain in the complex plane.
(b) (3 points) Show that the eigenvalues of (any) matrix A must lie in its *refined* Gerschgorin domain $D_A^* = D_A \cap D_{A^T}$.
(c) (3 points) Find and sketch the refined Gerschgorin domain for A in (3).
(d) (1 point) Find the eigenvalues of A and check that they indeed belong to D_A^*.

[1]Originally defined by Laplace and studied in detail by Chebyshov. One more manifestation of the Arnold principle: if a notion bears a personal name, then this name is not the name of the discoverer, `https://www.math.fsu.edu/~wxm/Arnold.htm`.