ACM 104 **October 16, 2023**

Problem Set 2

Problem 1. (10 points) Subspaces

part (a) W is not a subspace. Let $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Clearly det(A) = 0 and det(B) = 0, therefore $A, B \in W$. Now consider their sum $A + B$:

$$A + B = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix}$$

We have that det(A+B) $= 4 \cdot 3 - 7 \cdot 2 = 12 - 14 = -2$. Hence $A + B \notin W$ as det(A+B) $\neq 0$. Therefore, W is not a subspace as it is not closed under addition.

part (b)
W is a subspace as it has a zero vector, is closed under addition, and is closed under scalar multiplication.

Proof:
Let $A, B \in W$.
We first show W has a zero vector. The zero matrix is the zero vector of W as $A + 0 = A + 0 = A$ and $tr\ 0 = \sum_{i=1}^{n} 0 = 0$.

Because $A, B \in W$ then $trA = trB = 0$. Let a_{ii} and b_{ii} denote the diagonal in the ith row and ith column of matrix A and matrix B respectively. By the definition of matrix addition we have that $(A+B)_{ii} = a_{ii} + b_{ii}$. By the definition of trace we have

$$tr(A + B) = \sum_{i=1}^{n}(a_{ii} + b_{ii}) = \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} = trA + trB = 0 + 0 = 0$$

Therefore we have $(A + B) \in W$. Hence W is closed under addition.

Now let α be a scalar. So we have (using the same notation as before):

$$(\alpha \cdot A)_{ii} = \sum_{i=1}^{n} \alpha \cdot a_{ii} = \alpha \sum_{i=1}^{n} a_{ii} = \alpha \cdot trA = 0$$

Therefore we have $\alpha \cdot A \in W$. Hence W is closed under scalar multiplication.

part (c)
W is not a subspace. Consider $f(x) = 1$ for all x, so f is continuous ($f(x) \in V$). Furthermore, $f(0)f(1) = 1$ so $f(x) \in W$. Let α be a scalar, then we have $\alpha f(0) \cdot \alpha f(1) = \alpha^2$. Clearly $\alpha f(x) \in V$.
However $\alpha f(0) \cdot \alpha f(1) = \alpha^2 \neq 1$ for $|\alpha| \neq 1$. Therefore W is not closed under scalar multiplication, hence it is not a subspace.

part (d)
W is a subspace as it has a zero vector, is closed under addition, and is closed under scalar multiplication.

Proof:
Let $f(x), g(x) \in W$.
We first show W has a zero vector. The zero vector in W is $h(x) = 0$ as $f(x) + h(x) = h(x) + f(x) = f(x)$ and $h(\frac{1}{2}) = \int_0^1 h(t)\, dt = \int_0^1 0\, dt = 0$ (so $h(x) \in W$).

Because $f(x), g(x) \in W$ then we have:

$$f(\frac{1}{2}) + g(\frac{1}{2}) = \int_0^1 f(t)\, dt + \int_0^1 g(t)\, dt = \int_0^1 (f(t) + g(t))\, dt$$

Therefore we have $(f(x) + g(x)) \in W$. Hence W is closed under addition.

Now let α be a scalar. So we have

$$\alpha \cdot f(\frac{1}{2}) = \alpha \int_0^1 f(t)\, dt = \int_0^1 (\alpha \cdot f(t))\, dt$$

Therefore we have $\alpha \cdot f(x) \in W$. Hence W is closed under scalar multiplication.

part (e)
W is a subspace as it has a zero vector, is closed under addition, and is closed under scalar multiplication.

Proof:
Let $f(x, y), g(x, y) \in W$.
We first show W has a zero vector. The zero vector in W is $h(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
as $f(x, y) + h(x, y) = h(x, y) + f(x, y) = f(x, y)$ and $\nabla \cdot h(x, y) = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) = 0 + 0 = 0$; so $h(x, y) \in W$.

Let $f(x, y) = \begin{bmatrix} f_1(x, y) \\ f_2(x, y) \end{bmatrix}$ and $g(x, y) = \begin{bmatrix} g_1(x, y) \\ g_2(x, y) \end{bmatrix}$. Because $f(x, y), g(x, y) \in W$ then we have:

$$\nabla \cdot (f(x, y) + g(x, y)) = \frac{\partial}{\partial x}(f_1(x, y) + g_1(x, y)) + \frac{\partial}{\partial y}(f_2(x, y) + g_2(x, y))$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial g_1}{\partial x} + \frac{\partial g_2}{\partial y} = \nabla \cdot f(x, y) + \nabla \cdot g(x, y) = 0 + 0 = 0$$

Therefore we have $(f(x, y) + g(x, y)) \in W$. Hence W is closed under addition.

Now let α be a scalar. So we have

$$\nabla \cdot (\alpha \cdot f(x, y)) = \alpha \cdot \frac{\partial f_1}{\partial x} + \alpha \cdot \frac{\partial f_2}{\partial y} = \alpha \cdot (\nabla \cdot f(x, y)) = 0$$

Therefore we have $\alpha \cdot f(x, y) \in W$. Hence W is closed under scalar multiplication.

Problem 2. (10 points) Polynomials

part (a)
Define A to be the matrix (note that each column represents a polynomial given in the problem):

$$A = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Performing row operations to get the row echelon form:

$$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_1+3r_3 \to r_1} \begin{bmatrix} 0 & 2 & 4 \\ 0 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{2r_2+r_3 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

Hence we have that the rank of A is 3. Because A was formed with 3 vector columns (each representing one of the polynomials), then this means that the polynomials p_1, p_2, and p_3 are linearly independent.

part (b)
Because the three polynomials p_1, p_2, p_3 are linearly independent, and since the dimension of $\mathcal{P}^{(2)}$ is 3, then the polynomials span $\mathcal{P}^{(2)}$.

part (c)
The polynomials p_1, p_2, p_3 form a basis of $\mathcal{P}^{(2)}$ because they are linearly independent and span $\mathcal{P}^{(2)}$.

Now Solving for the coordinates of $q(x) = 1$

$$\left[\begin{array}{ccc|c} -3 & 2 & 1 & 1 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1+3r_3 \to r_1} \left[\begin{array}{ccc|c} 0 & 2 & 4 & 1 \\ 0 & -1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & 4 & 1 \end{array}\right] \xrightarrow{2r_2+r_3 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 8 & 1 \end{array}\right]$$

Hence the coordinates of $q(x) = 1$ are $(\frac{-1}{8}, \frac{1}{4}, \frac{1}{8})$

Problem 3. (10 points) Fibonacci Sequences

part (a)
Note that any sequence $f \in \mathcal{F}$ is dictated by only its first two values x_1 and x_2 (in other words x_1 and x_2 are free). Hence to show that $\mathcal{F}$ is a vector space, we only need to show that is is closed under addition, closed under scalar multiplication, and that it has a zero vector as $\mathcal{F}$ inherits the other properties (associativity, commutativity, etc) from the real numbers.

Let $f_1, f_2 \in \mathcal{F}$.

We first show $\mathcal{F}$ has a zero vector. Consider the sequence $h = (h_1, h_2, h_3...)$ where $h_i = 0$ for all i. Clearly $h_n = h_{n-1} + h_{n-2}$ for $n = 3, ...$ Hence $h \in \mathcal{F}$. Because all values in the sequence h are zero then we have that $f_1 + h = h + f_1 = f_1$. Hence $h = (h_1, h_2, h_3...)$ where $h_i = 0$ for all i is the zero vector for $\mathcal{F}$.

Now we have $f_1 = (x_1, x_2, x_3, ...)$ and $f_2 = (y_1, y_2, y_3, ...)$. Let $g = f_1 + f_2 = (z_1, z_2, z_3, ...)$. Because $f_1, f_2 \in \mathcal{F}$, the for $n = 3, ...$ we have:

$$z_n = x_n + y_n = (x_{n-1} + x_{n-2}) + (y_{n-1} + y_{n-2}) = (x_{n-1} + y_{n-1}) + (x_{n-2} + y_{n-2}) = z_{n-1} + z_{n-2}$$

Therefore we have $(f_1 + f_2) \in W$. Hence $\mathcal{F}$ is closed under addition.

Now let α be a scalar. So we have (for n = 3, ...):

$$\alpha f_1 = (\alpha x_1, \alpha x_2, \alpha x_3, ...) = (b_1, b_2, b_3, ...)$$

$$b_n = (\alpha x_n) = \alpha(x_{n-1} + x_{n-2}) = \alpha x_{n-1} + \alpha x_{n-2} = b_{n-1} + b_{n-2}$$

Therefore we have $(\alpha f_1) \in W$. Hence $\mathcal{F}$ is closed under scalar multiplication.

Thus $\mathcal{F}$ as we have shown that is has a zero vector, is closed under scalar addition, and is closed under scalar multiplication. As stated before, these are all the conditions to show that $\mathcal{F}$ is a vector space, hence $\mathcal{F}$ is a vector space.

part (b)
The dimension of $\mathcal{F}$ is two. Notice that the first two values of any sequence $f \in \mathcal{F}$ constitute the values of the rest of the sequence. In other words, $\mathcal{F}$ only has 2 free variables. Hence its dimension is 2.
Therefore a basis for $\mathcal{F}$ is $f_1 = (1, 0, 1, 1, 2, ...)$ and $f_2 = (0, 1, 1, 2, 3, ...)$
part (c)
The coordinates of the original sequence in the basis given in part b is $(1, 1)$ as $f^* = f_1 + f_2$

Problem 4. (10 points) Kernel, Image, Cokernel, Coimage

Basis for Image of A:
Let a_j be the the jth column of A. Observe that any column in A is the linear combination of two of its column vectors. More specifically we have that

$$a_j = \begin{bmatrix} 1 \\ n+1 \\ \vdots \\ n^2-n+1 \end{bmatrix} + (j-1)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ n+1 \\ \vdots \\ n^2-n+1 \end{bmatrix} + (j-1)\left(\begin{bmatrix} 2 \\ n+2 \\ \vdots \\ n^2-n+2 \end{bmatrix} - \begin{bmatrix} 1 \\ n+1 \\ \vdots \\ n^2-n+1 \end{bmatrix}\right) = a_1 + (j-1)(a_2 - a_1)$$

Therefore, $Im\ A = span(a_1, a_2, ..., a_n) = span(a_1, a_2)$. Note that a_1 and a_2 are linearly independent as the only solution to $\alpha_1 a_1 + \alpha_2 a_2 = 0$ is $\alpha_1 = \alpha_2 = 0$. Hence, because a_1 and a_2 are linearly independent and span $Im\ A$ then a basis for $Im\ A$ is $\{a_1, a_2\}$.

Basis for Coimage of A:
Let a^i be the the ith row of A. Observe that any row in A is the linear combination of two of its row vectors. More specifically we have that

$$a^i = \begin{bmatrix} 1 & 2 & \dots & n \end{bmatrix} + (i-1)\begin{bmatrix} n & n & \dots & n \end{bmatrix} = a^1 + (i-1)(a^2 - a^1)$$

Therefore, $coIm\ A = span(a^1, a^2, ..., a^n) = span(a^1, a^2)$. Note that a^1 and a^2 are linearly independent as the only solution to $\alpha_1 a^1 + \alpha_2 a^2 = 0$ is $\alpha_1 = \alpha_2 = 0$. Hence, because a^1 and a^2 are linearly independent and span $coIm\ A$ then a basis for $coIm\ A$ is $\{a^1, a^2\}$.

Basis for Kernel of A:
We first perform row operations on A.

$$\begin{bmatrix} 1 & 2 & \dots & n \\ n+1 & n+2 & \dots & 2n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix} \xrightarrow{-r_1+r_2 \to r_2} \begin{bmatrix} 1 & 2 & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & & \vdots \\ n^2-n+1 & n^2-n+2 & \dots & n^2 \end{bmatrix}$$

As stated, for the ith row of A we have $a^i = a^1 + (i-1)\begin{bmatrix} n & n & \dots & n \end{bmatrix}$. So when we apply the row operation $r_i - r_1 \to r_i$, followed by $r_i - (i-1)r_2 \to r_i$ for every $i > 2$ then we have the matrix shown below

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ n & n & n & \dots & n \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Performing the operation $-nr_1 + r_2 \to r_2$ gives us the row echelon form of A. Afterwards, another two operations can be performed to get the reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 0 & -n & -2n & \dots & n-n^2 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \xhookrightarrow{-\frac{1}{n}r_2 \to r_2} \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \xhookrightarrow{-2r_2+r_1 \to r_1} \begin{bmatrix} 1 & 0 & -1 & \dots & 2-n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Let the basic variables (which correspond to the pivots) be denoted as x_1 and x_2. Let all the other free variables be denoted $x_3, x_4, ..., x_n$. We have that the solution to $Ax = 0$ is given by:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \sum\limits_{i=3}^{n} (i-2)x_i \\ \sum\limits_{i=3}^{n} (1-i)x_i \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

For $k > 2$ let the vectors v_k be defined such that the first entry of v_k is $(k-2)x_k$, the second entry of v_k is $(1-k)x_k$, the kth entry of v_k is 1 and all other entries are 0. So we have that $span(v_3, v_4, ..., v_n) = ker\ A$. By construction of v_k, all of these vectors are linearly independent as well. Therefore $v_3, v_4, ..., v_n$ form a basis for the kernel of A.

Basis for CoKernel of A:
We first perform row operations on A^T.

$$\begin{bmatrix} 1 & n+1 & \dots & n^2-n+1 \\ 2 & n+2 & \dots & n^2-n+2 \\ \vdots & \vdots & & \vdots \\ n & 2n & \dots & n^2 \end{bmatrix} \xrightarrow{-r_1+r_2\to r_2} \begin{bmatrix} 1 & n+1 & \dots & n^2-n+1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & & \vdots \\ n & 2n & \dots & n^2 \end{bmatrix}$$

Note that for the ith row of A^T we have $b^i = b^1 + (i-1)\begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$ where b_i is the ith row of A^T. So when we apply the row operation $r_i - r_1 \to r_i$, followed by $r_i - (i-1)r_2 \to r_i$ for every $i > 2$ then we have the matrix shown below

$$\begin{bmatrix} 1 & n+1 & 2n+1 & \dots & n^2-n+1 \\ 1 & 1 & 1 & \dots & 1 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Performing the operation $-r_1 + r_2 \to r_2$ gives us the row echelon form of A^T. Afterwards, another two operations can be performed to get the reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 0 & -n & -2n & \dots & n-n^2 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \xhookrightarrow{-\frac{1}{n}r_2\to r_2} \begin{bmatrix} 1 & 2 & 3 & \dots & n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \xhookrightarrow{-2r_2+r_1\to r_1} \begin{bmatrix} 1 & 0 & -1 & \dots & 2-n \\ 0 & 1 & 2 & \dots & n-1 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Notice that the reduced row echelon form of A^T is the same as the reduced row echelon form of A. Therefore the basis for the cokernel of A is the same the basis for the kernal of A (described on previous page).

Problem 5. (10 points) Fundamental Matrix Subspaces.

See code attached.

ACM/IDS 104 - Problem Set 2 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

Problem 5 (10 points) Fundamental Matrix Subspaces

Your task for this problem is to write a function that takes a matrix `A` as its argument, and outputs four matrices: `K, I, cK` and `cI` where:

- Columns of `K` form a basis of the kernel of `A`. If $\ker A = \{0\}$, then `K` must be a zero vector of the appropriate dimension.
- Columns of `I` form a basis of the image of `A`. If $\operatorname{im} A = \{0\}$, then `I` must be a zero vector of the appropriate dimension.
- Columns of `cK` form a basis of the cokernel of `A`. If $\operatorname{coker} A = \{0\}$, then `cK` must be a zero vector of the appropriate dimension.
- Columns of `cI` form a basis of the coimage of `A`. If $\operatorname{coim} A = \{0\}$, then `cI` must be a zero vector of the appropriate dimension.

Move to the bottom of this livescript to write the function.

Now, let us test our function:

```
A = magic(6); % feel free to define A as you like
[K, I, cK, cI] = subspacer(A); % this is how you call a MATLAB function
disp(K);
```

```
   -0.4714
   -0.4714
    0.2357
    0.4714
    0.4714
   -0.2357
```

```
disp(I);
```

```
   -0.4082    0.5574    0.0456   -0.4182    0.3092
   -0.4082   -0.2312    0.6301   -0.2571   -0.5627
   -0.4082    0.4362    0.2696    0.5391    0.1725
   -0.4082   -0.3954   -0.2422   -0.4590    0.3971
   -0.4082    0.1496   -0.6849    0.0969   -0.5766
   -0.4082   -0.5166   -0.0182    0.4983    0.2604
```

```
disp(cK);
```

```
    0.5000
    0.0000
   -0.5000
   -0.5000
    0.0000
    0.5000
```

```
disp(cI);
```

```
   -0.4082    0.6234   -0.3116    0.2495   -0.2511
   -0.4082   -0.6282    0.3425    0.1753   -0.2617
   -0.4082   -0.4014   -0.7732   -0.0621    0.1225
   -0.4082    0.1498    0.2262   -0.4510   -0.5780
   -0.4082    0.1163    0.2996    0.6340    0.3255
   -0.4082    0.1401    0.2166   -0.5457    0.6430
```

START HERE by writing the function:

```
function [K, I, cK, cI] = subspacer(A)
%{
This is the MATLAB function syntax.
-> [K, I, cK, cI] are the outputs of the function.
-> "subspacer" is the name of the function. (you can change that if
                            you wish but make sure you change
                            every function call as well!)
-> A is the argument of the function.
%}
[m, n] = size(A);
r = rank (A);
%{
We start by finding out the dimensions and rank of A.
Let us consider the matrix K. There exist 2 cases:
1) The kernel is trivial i.e. kerA = {0}
2) The kernel is not trivial -> Hint: use null()
Complete the following if/else statement.
%}
if r == n % this condition is done for you
    K = zeros([n 1]);
else
    K = null(A);
end

%{
Now, let us consider the matrix cK.
As above, there exist 2 cases. Remember, you can use ' to
transpose a matrix.
Write a similar if/else statement to produce cK.
%}

if r == m % this condition is done for you
    cK = zeros([m 1]);
else
    cK = null(A');
end

%{
For the image I and coimage cI, there exists only 1 condition
we must test, and that is if rankA = 0. With this in mind,
```

```
complete the following if/else statement.
-> Hint: orth() is useful here.
%}
if r == 0
    I = zeros([n 1]);
    cI = zeros([m 1]);
else
    I = orth(A);
    cI = orth(A');


end
end
```