ACM/IDS 104 APPLIED LINEAR ALGEBRA
PROBLEM SET 3

Please submit your solution as a single PDF file, that contains both the written-up and published code parts, via Gradescope by **9pm Tuesday, October 24**. An example of the submission process is shown here: `https://www.gradescope.com/get_started#student-submission`

- For theoretical problems, please use a pen, not a pencil: it is hard to read scanned submission written by a pencil.
- For coding problems, please convert your MATLAB livescripts (`.mlx`) to PDF by selecting `Live Editor` $\rightarrow$ `Save` $\rightarrow$ `Export to PDF` and merge them with the rest of your solution.
- After uploading your submission to Gradescope, please label all pages.

Problem 1. (10 points) Inner Products vs Norms
We know that an inner product $\langle\cdot,\cdot\rangle$ on a vector space V induces a norm on V:

$$\|v\| = \sqrt{\langle v, v\rangle}. \tag{1}$$

(a) (5 points) Suppose we know the norm in (1), i.e. we can compute $\|v\|$ for any $v \in V$. Is it possible to reconstruct the inner product $\langle\cdot,\cdot\rangle$ from the norm? That is, to find $\langle u, v\rangle$ for any $u, v \in V$.
(b) (5 points) Are there two distinct inner products that induce the same norm? In other words, can you find $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$, $\langle\cdot,\cdot\rangle_1 \neq \langle\cdot,\cdot\rangle_2$, such that $\|\cdot\|_1 = \|\cdot\|_2$?

Problem 2. (10 points) Continuously Differentiable Functions
Let $C^1[0,1]$ be a vector space of continuously differentiable functions on the interval $[0,1]$. Let

$$\langle f, g\rangle_1 = \int_0^1 f'(x)g'(x)dx \quad \text{and} \quad \langle f, g\rangle_2 = \int_0^1 (f(x)g(x) + f'(x)g'(x))dx. \tag{2}$$

(a) (3 points) One of the two products in (2) defines an inner product on $C^1[0,1]$. Determine which one and explain why.
(b) (3 points) Denote the inner product from part (a) by $\langle\cdot,\cdot\rangle$. Write down explicitly the Cauchy-Scharz and triangle inequalities based on $\langle\cdot,\cdot\rangle$.
(c) (4 points) Find the angle between $f(x) = 1$ and $g(x) = e^x$.

Remark: The norm induced by $\langle\cdot,\cdot\rangle$ belongs to the family of Sobolev norms. Sobolev spaces are natural homes for solutions of PDEs (more natural that spaces of continuously differentiable functions).

Problem 3. (10 points) The K-means Algorithm for Clustering
Complete Problem 3 in `PS3.mlx`.

Problem 4. (10 points) Gram Matrices
Consider a vector space $C^1[0,1]$ of continuously differentiable functions on $[0,1]$ with the L^2 inner product.

(a) (3 points) Find the Gram matrix G associated with $1, e^x, e^{2x}$.
(b) (3 points) Is G positive definite?
(c) (3 points) Answer (a) and (b) using the inner product from Problem 2.
(d) (1 point) Let $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ be two different inner products on $C^1[0,1]$, and G_1 and G_2 be the Gram matrices associate with $1, e^x, e^{2x}$ computed with respect to $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$. Can you find $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ such that one of the Gram matrices is positive definite and the other one is not?

Problem 5.[1] (10 points) Gram Matrices for Text Classification
Complete Problem 5 in `PS3.mlx`.

[1] Inspired by New York's presidential primary, April 19, 2016.