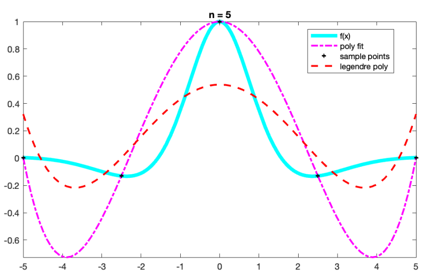
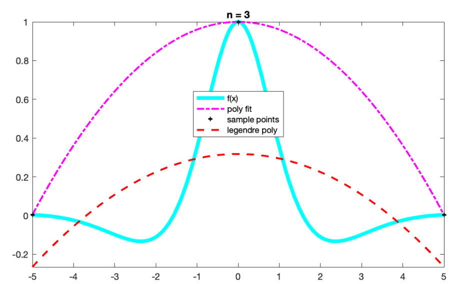
ACM 104 **November 13, 2023**

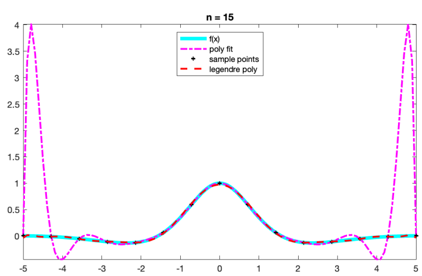
Problem Set 5

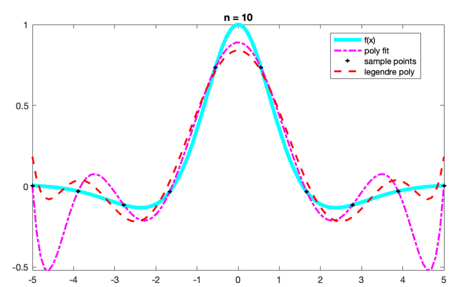
Problem 1.(10 points) Application of Projections to Approximation

See code attached. Plots shown below.









Problem 2. (10 points) Hermite Polynomials

Let $H_0(x), H_1(x), H_2(x), H_3(x), H_4(x)$ denote the first five monic Hermite Polynomials. Applying the Gram-Schmidt process (using the standard basis $\{1, x, x^2, x^3, ...\}$ for polynomials):

$$H_0(x) = 1$$

$$H_1(x) = x - \frac{\langle x, 1\rangle}{\langle 1, 1\rangle}(1)$$

Solving for the inner products:

$$\langle x, 1\rangle = \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\, dx = 0 \qquad \langle 1, 1\rangle = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

So we have:

$$H_1(x) = x$$

Continuing:

$$H_2(x) = x^2 - \frac{\langle x^2, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^2, x\rangle}{\langle x, x\rangle}(x)$$

$$\langle x^2, 1\rangle = \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi} \qquad \langle 1, 1\rangle = \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

$$\langle x^2, x\rangle = \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}}\, dx = 0 \qquad \langle x, x\rangle = \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

So we have:

$$H_2(x) = x^2 - 1$$

Continuing:

$$H_3(x) = x^3 - \frac{\langle x^3, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^3, x\rangle}{\langle x, x\rangle}(x) - \frac{\langle x^3, x^2-1\rangle}{\langle x^2-1, x^2-1\rangle}(x^2-1)$$

$$\langle x^3, 1\rangle = \int_{-\infty}^{\infty} x^3 e^{-\frac{x^2}{2}}\, dx = 0 \qquad \langle x^3, x^2-1\rangle = \int_{-\infty}^{\infty} x^3(x^2-1) e^{-\frac{x^2}{2}}\, dx = 0$$

$$\langle x^3, x\rangle = \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}}\, dx = 3\sqrt{2\pi} \qquad \langle x, x\rangle = \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

So we have:

$$H_3(x) = x^3 - 3x$$

Continuing:

$$H_4(x) = x^4 - \frac{\langle x^4, 1\rangle}{\langle 1, 1\rangle}(1) - \frac{\langle x^4, x\rangle}{\langle x, x\rangle}(x) - \frac{\langle x^4, x^2-1\rangle}{\langle x^2-1, x^2-1\rangle}(x^2-1) - \frac{\langle x^4, x^3-3x\rangle}{\langle x^3-3x, x^3-3x\rangle}(x^3-3x)$$

$$\langle x^4, 1\rangle = \int_{-\infty}^{\infty} x^4 e^{-\frac{x^2}{2}}\, dx = 3\sqrt{2\pi} \qquad \langle 1, 1\rangle = \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

$$\langle x^4, x\rangle = \int_{-\infty}^{\infty} x^5 e^{-\frac{x^2}{2}}\, dx = 0 \qquad \langle x^4, x^3-3x\rangle = \int_{-\infty}^{\infty} x^4(x^3-3x) e^{-\frac{x^2}{2}}\, dx = 0$$

$$\langle x^4, x^2-1\rangle = \int_{-\infty}^{\infty} x^4(x^2-1) e^{-\frac{x^2}{2}}\, dx = 12\sqrt{2\pi} \qquad \langle x^2-1, x^2-1\rangle = \int_{-\infty}^{\infty} (x^2-1)^2 e^{-\frac{x^2}{2}}\, dx = 2\sqrt{2\pi}$$

So we have:

$$H_4(x) = x^4 - 3 - 6(x^2-1) = x^4 - 6x^2 + 3$$

Problem 3. (10 points) Orthogonal Compliments

Finding the orthogonal compliment $W_1^{\perp}$:
Because $W \subset \mathbb{R}^3$ then $dim(\mathbb{R}^3) = dim(W) + dim(W_1^{\perp}) \Rightarrow dim(W_1^{\perp}) = 3 - 2 = 1$. Therefore the basis of $W_1^{\perp}$ consists of only one vector $\vec{u}$ that is perpendicular to both v_1 and v_2 with respect to the dot product $\langle \cdot, \cdot \rangle_1$. Therefore $\vec{u}$ is simply the cross product of $v_1 = (1,2,3)^T$ and $v_2 = (2,0,1)^T$ (or a multiple of the cross product):

$$\vec{u} = v_1 \times v_2 = (2, 5, -4)^T$$

Verifying:

$$\langle \vec{u}, v_1 \rangle_1 = 2 + 10 - 12 = 0$$

$$\langle \vec{u}, v_2 \rangle_1 = 4 + 0 - 4 = 0$$

Hence $W_1^{\perp} = span(\vec{u}) = span((2,5,-4)^T)$

Finding the orthogonal compliment $W_2^{\perp}$:
Because $W \subset \mathbb{R}^3$ then $dim(\mathbb{R}^3) = dim(W) + dim(W_2^{\perp}) \Rightarrow dim(W_2^{\perp}) = 3 - 2 = 1$. Therefore the basis of $W_2^{\perp}$ consists of only one vector $\vec{w} = (a,b,c)^T$ that is perpendicular to both $v_1 = (1,2,3)^T$ and $v_2 = (2,0,1)^T$ with respect to the dot product $\langle \cdot, \cdot \rangle_2$. So the following must be satisfied:

$$\langle \vec{w}, v_1 \rangle_2 = a + 4b + 9c = 0$$

$$\langle \vec{w}, v_1 \rangle_2 = 2a + 3c = 0$$

So we have $2a + 3c = 0 \Rightarrow c = -\frac{2}{3}a$. Substituting we have:

$$a + 4b + 9(-\frac{2}{3}a) = a + 4b - 6a = 4b - 5a = 0 \Rightarrow a = \frac{4}{5}b \Rightarrow c = -\frac{2}{3}a = -\frac{8}{15}b$$

Therefore $\vec{w} = (\frac{4}{5}b, b, -\frac{8}{15}b)^T$ where $b \in \mathbb{R}$ and $b \neq 0$. Hence, generally, $W_2^{\perp} = span(\vec{w}) = span((\frac{4}{5}b, b, -\frac{8}{15}b)^T)$ where $b \in \mathbb{R}$ and $b \neq 0$.

Setting $b = 1$ we have that $W_2^{\perp} = span(\vec{w}) = span((\frac{4}{5}, 1, -\frac{8}{15})^T)$.

Problem 4. (10 points) Complete Matrices

$$A - \lambda I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix}$$

Solving for eigenvalues ($A - \lambda I$ must be singular so set $det(A - \lambda I) = 0$:

$$det(A - \lambda I) = -\lambda det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\right) - det\left(\begin{bmatrix} 0 & 1-\lambda \\ 1 & 0 \end{bmatrix}\right) = \lambda^2(1-\lambda) + (1-\lambda) = 0$$

Factoring out $(1 - \lambda)$:

$$(1-\lambda)(\lambda^2 + 1) = 0$$

Thus we have the eigenvalues $\lambda_1 = 1, \lambda_2 = i, \lambda_3 = -i$.
Because the eigenvalues are distinct and complex then the corresponding eigenvectors v_1, v_2, v_3 form a basis of $\mathbb{C}^3$. Therefore A only admits an eigenvector basis of $\mathbb{C}^3$ (it does not admit an eigenvector basis of $\mathbb{R}^3$). Hence, A is complete (as A can is complete if and only if it has an eigenbasis).

Problem 5. (10 points) Gerschgorin Theorem

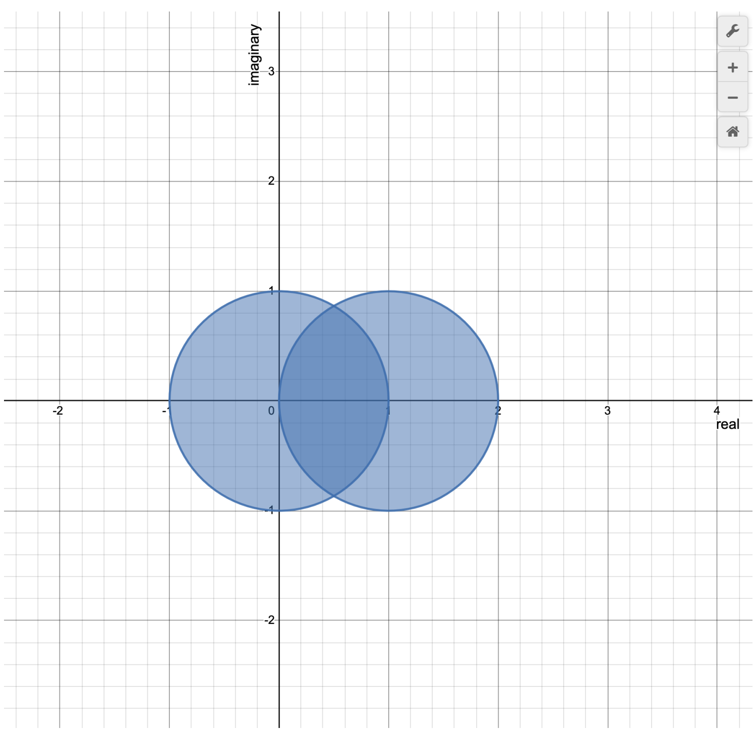
part (a) Gerschgorin disks:

$$D_1 = \{z \in \mathbb{C} : |z - 0| \leq 1\}$$

$$D_2 = \{z \in \mathbb{C} : |z - 1| \leq 1\}$$

$$D_3 = \{z \in \mathbb{C} : |z - 1| \leq 1\}$$

Sketch the Gerschgorin domain in the complex plane:



part (b) Let $spec(A)$ denote the set of eigenvalues of A and $spec(A^T)$ denote the set of eigenvalues of A^T.
Claim: $spec(A) = spec(A^T)$
Proof: By definition, the eigenvalues of A are the values of λ such that $det(A - \lambda I) = 0$. Similarly, the eigenvalues of A^T are the values of λ such that $det(A^T - \lambda I) = 0$. Observe that $A^T - \lambda I = (A - \lambda I)^T$. By the properties of determinants, for any square matrix B we have $det(B) = det(B^T)$. Therefore the eigenvalues of A^T are the values of λ such that $det((A - \lambda I)^T) = det(A - \lambda I) = 0$. Hence, A and A^T have the same eigenvalues. Thus, $spec(A) = spec(A^T)$.

By Gerschgorin Theorem, $spec(A) \subset D_A$ and $spec(A^T) \subset D_{A^T}$.
As proven above we have $spec(A) = spec(A^T)$. Therefore, $spec(A) \subset D_A$ and $spec(A) \subset D_{A^T}$. Thus, $spec(A) \subset D_A \cap D_{A^T}$. Hence, the eigenvalues of (any) matrix A must lie in its refined Gerschgorin domain $D_A^* = D_A \cap D_{A^T}$.

part (c): Gerschgorin disk for A:

$$D_1^A = \{z \in \mathbb{C} : |z - 0| \leq 1\}$$

$$D_2^A = \{z \in \mathbb{C} : |z - 1| \leq 1\}$$

$$D_3^A = \{z \in \mathbb{C} : |z - 1| \leq 1\}$$

$$D_A = D_1^A \cup D_2^A \cup D_3^A$$

Gerschgorin disk for A^T:

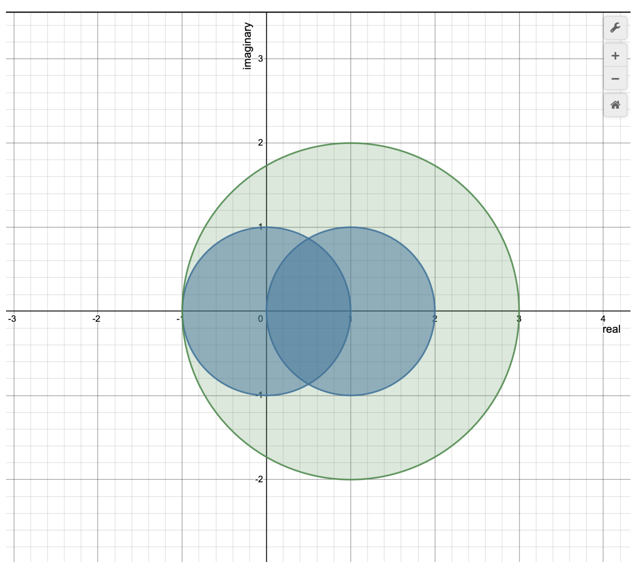
$$D_1^{A^T} = \{z \in \mathbb{C} : |z - 0| \leq 0\}$$

$$D_2^{A^T} = \{z \in \mathbb{C} : |z - 1| \leq 2\}$$

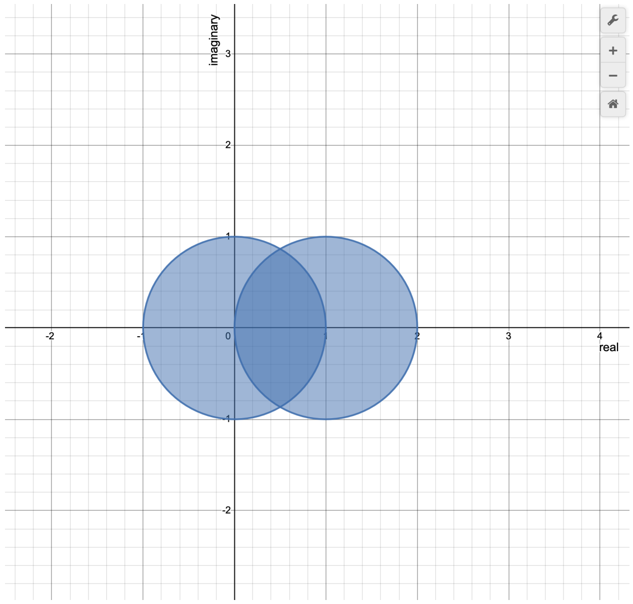
$$D_3^{A^T} = \{z \in \mathbb{C} : |z - 1| \leq 1\}$$

$$D_{A^T} = D_1^{A^T} \cup D_2^{A^T} \cup D_3^{A^T}$$

Sketch of D_A (in blue) and D_{A^T} (in green):



Putting this all together we have: $D_A^* = D_A \cap D_{A^T} = D_A$
Sketch of D_A^*:



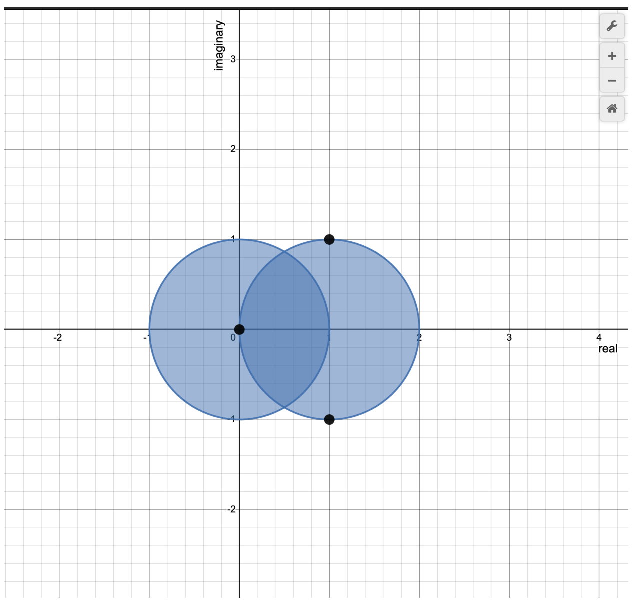
part (d):

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

Solving for eigenvalues ($A - \lambda I$ must be singular so set $det(A - \lambda I) = 0$:

$$det(A - \lambda I) = -\lambda det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) - det\left(\begin{bmatrix} 0 & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = -\lambda((1-\lambda)^2 + 1) = 0$$

Thus we have the eigenvalues $\lambda_1 = 0, \lambda_2 = 1 - i, \lambda_3 = 1 + i$.
They do belong to D_A^* (shown below, eigenvalues are shown in black):



ACM/IDS 104 - Problem Set 5 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

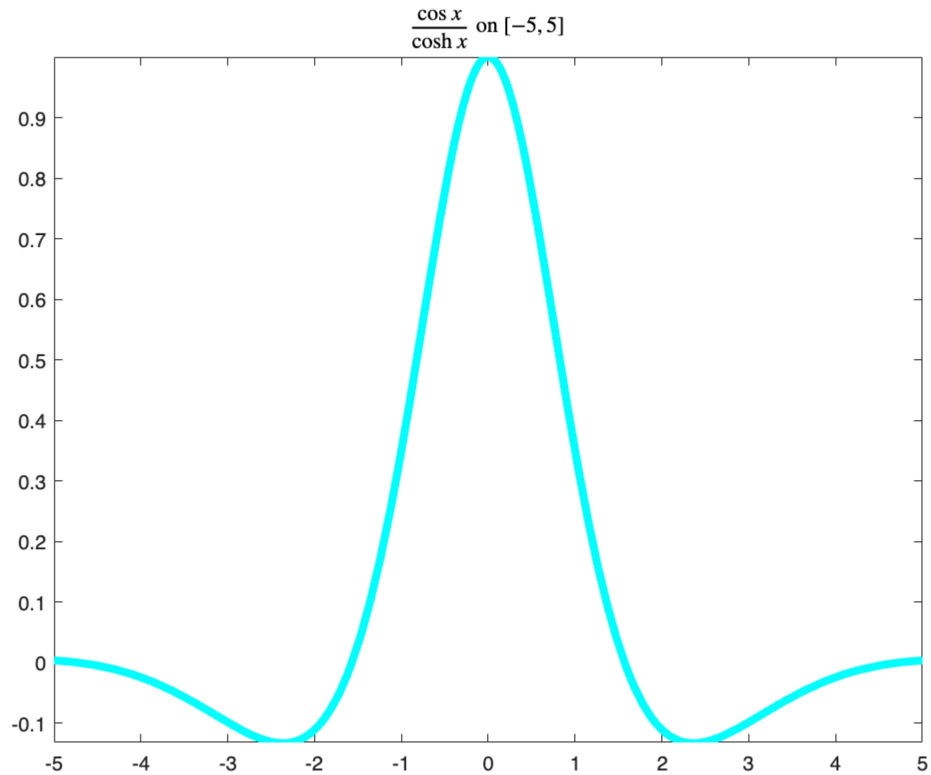
Problem 1 (10 points) Application of Projections to Approximation

In Problem 4 of PS4, we saw that even higher degree interpolating polynomials may not be accurate approximations to complex functions. We have the function:

$$f(x) = \frac{\cos x}{\cosh x}, \quad \text{on}[-a, a], \quad a = 5$$

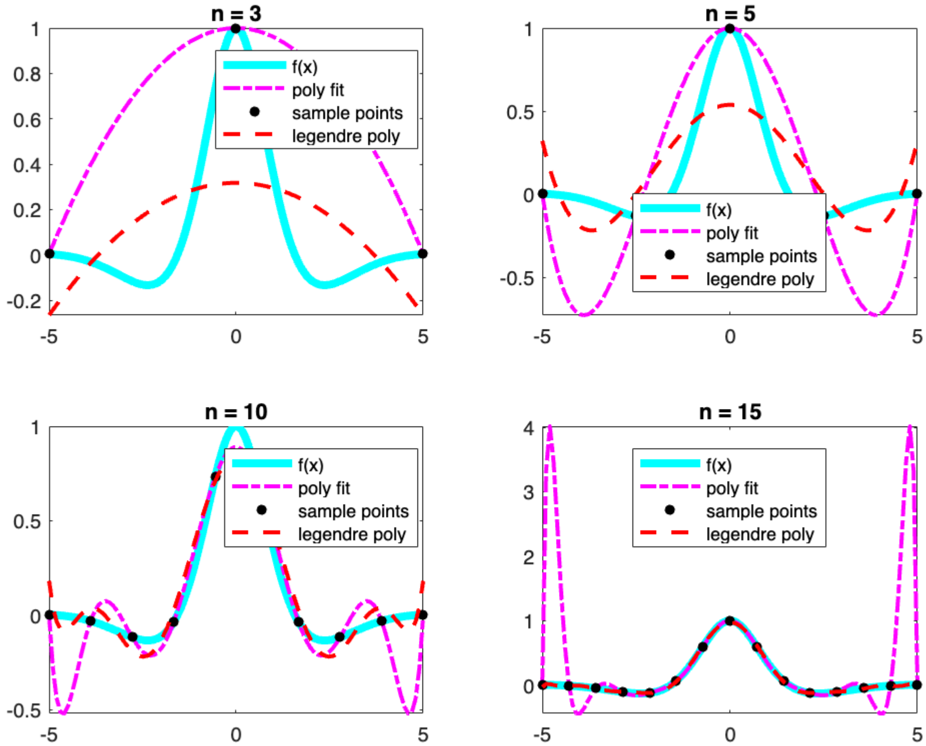
Let us recall how this function looks like and how its interpolating polynomials of degree $(n-1)$ for $n = 3, 5, 10, 15$ behave:

```
%{
Setup
%}
f = @(x) cos(x)./cosh(x); % our function
a = 5; % setting the value of a
n = [3 5 10 15]; % setting the number of points
sub = 1; % subplot index

%{
How f(x) looks like on [-5, 5]
%}
figure;
fplot(f, [-a, a], "-c", "lineWidth", 4);
title("$\frac{\cos{x}}{\cosh{x}}$ on $[-5, 5]$","Interpreter","latex");
```




```
%{
Read the discussion below and complete the code
%}
figure;
for ival = a
    for degree = n-1
        %{
        INTERPOLATING POLYNOMIALS -- no changes needed
        -> Select degree+1 points in the interval
        -> Evaluate f(x) on these points
        -> Find the polynomial coefficients
        %}
        pts = ones(degree+1, 2); % initializing the points
        pts(:, 1) = linspace(-ival, ival, degree+1); % setting the x-values
        for i = 1 : degree+1
            pts(i, 2) = f(pts(i, 1)); % evaluating cos(x) / cosh(x)
        end
        coeffs = polyfit(pts(:, 1), pts(:, 2), degree); % coefficients
        %{
        ORTHOGONAL PROJECTIONS -- TODO
        -> Get transformed Legendre polynomials
        -> Find alpha_k using L^2 inner product
        -> Evaluate alpha_k*Q_k
        %}
```

```
        a0 = -ival;
        b = ival;
        p_star = @(x) 0;
        for i = 0:degree
            leg_poly = @(y) legendreP(i, (2*y - b - a0)/(b-a0));
            leg_poly_norm = integral(@(y) leg_poly(y) .* leg_poly(y), a0, b);
            inner_prod = integral(@(y) f(y) .* leg_poly(y), a0, b);
            alpha_i = inner_prod/leg_poly_norm;
            p_star = @(x) p_star(x) + alpha_i .* leg_poly(x);
        end
        %-> Evaluate alpha_k*Q_k for all points

        points = linspace(-ival, ival, 100);
        p_star_eval = p_star(points);



        %{
        PLOTTING
        Plot f(x), the sampled points, interpolating and approximating
        polynomials
        Please use different colors and linestyles
        %}
        subplot(2, 2, sub);
        fplot(f, [-ival, ival], "-c", "lineWidth", 4);
        hold on
        interpoints = linspace(-ival, ival);
        p = polyval(coeffs, interpoints); % evaluating coeffs in interval
        plot(interpoints, p, "-.m", "lineWidth", 2);
        plot(pts(:, 1), pts(:, 2), "ok", "MarkerSize", 2, "lineWidth", 3);

        % added this
        plot(points, p_star_eval, "--", "lineWidth", 2, 'Color', 'r');

        title(strcat("n = ", int2str(degree+1)));
        sub = sub + 1; % increase subplot index
        legend('f(x)', 'poly fit', 'sample points', 'legendre poly', "location", "best'
    end
end
```



Now, instead of interpolating polynomials, let us approximate $f(x)$ by its orthogonal projection onto the inner space $\mathscr{P}^{(n-1)}_{[-a,a]}$ of polynomials on $[-a, a]$, equipped with the L^2 inner product:

$$f(x) \approx p(x) = \mathrm{pr}_{\mathscr{P}^{(n-1)}_{[-a,a]}} f(x)$$

Recall (Lecture 10) that $p(x)$ is the closest (in the L^2 sense) polynomial to $f(x)$ in $\mathscr{P}^{(n-1)}_{[-a,a]}$, i.e.

$$p(x) = \arg \min_{q \in \mathscr{P}^{(n-1)}_{[-a,a]}} \|f(x) - q(x)\|$$

We know that the transformed Legendre polynomials $\widetilde{Q}_0(x), \cdots, \widetilde{Q}_{n-1}(x)$ form an orthogonal basis of $\mathscr{P}^{(n-1)}_{[-a,a]}$, and, therefore:

$$p(x) = \sum_{k=0}^{n-1} \alpha_k \widetilde{Q}_k(x)$$

where α_k are the coordinates of $p(x)$ in that basis.

Modify the above code to find the approximating polynomials as well. Plot each approximating polynomial on its corresponding subplot. Useful functions for this problem:

```
legendreP(), integral()
```