ACM 104 **November 19, 2023**

Problem Set 6

Problem 1. (10 points) The Power Method (aka von Mises Iteration)

See code attached.

Problem 2. (10 points) Ranking US Airports using PageRank

See code attached.

Problem 3. (10 points) Matrix Diagonalization

$$F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Finding the eigenvalues of F :

$$det(F - \lambda I) = det\left(\begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix}\right) = -\lambda(1-\lambda) - 1 = 0$$

$$-\lambda(1-\lambda) - 1 = \lambda^2 - \lambda - 1 = 0$$

Using the quadratic formula, the two eigenvalues are :

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \qquad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

Finding eigenvector $v = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^T$ corresponding to λ_1:

$$Fv = \lambda_1 v \Rightarrow (F - \lambda_1 I)v = 0$$

$$\begin{aligned} (1-\lambda_1)v_1 + v_2 &= 0 \quad (1) \\ v_1 - \lambda_1 v_2 &= 0 \quad (2) \end{aligned}$$

From (2) we have $v_1 = \lambda_1 v_2$. Substituting this in (1) we have that $v_2(\lambda_1 - \lambda_1^2 + 1) = 0$. Because $\lambda_1 - \lambda_1^2 + 1 = -(\lambda_1^2 - \lambda_1 - 1) = 0$ then this means that v_2 is our free variable. Thus for $v_2 = 1$ we have:

$$v = \begin{bmatrix} \lambda_1 v_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

By the same logic, we have the eigenvector $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ corresponding to λ_2 (with $u_2 = 1$):

$$u = \begin{bmatrix} \lambda_2 u_2 \\ u_2 \end{bmatrix} = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$$

Let V be defined as:

$$V = \begin{bmatrix} v & u \end{bmatrix} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$

$$det(V) = \frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} = \sqrt{5}$$

So we have:

$$V^{-1} = \frac{1}{det(V)} \begin{bmatrix} 1 & -\frac{1-\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$$

Putting this together we have the diagonalization of F:

$$F = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1} = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1-\sqrt{5}}{2\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & \frac{1+\sqrt{5}}{2\sqrt{5}} \end{bmatrix}$$

Problem 4. (10 points) Principal Component Analysis

See code attached.

Problem 5. (10 points) Spectral Method for Graph Partitioning

See code attached.

ACM/IDS 104 - Problem Set 6 - MATLAB Problems

Before writing your MATLAB code, it is always good practice to get rid of any leftover variables and figures from previous scripts.

```
clc; clear; close all;
```

NOTE: Start with Problem 1, at the *bottom of this livescript*. (It's at the bottom because it's a MATLAB live function)

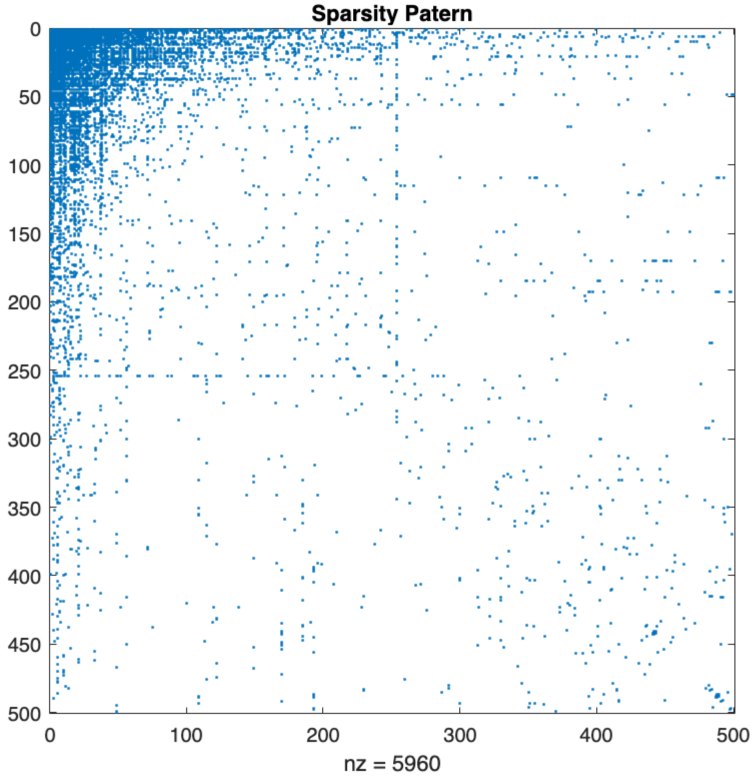
Problem 2 (10 points) Ranking US Airports using PageRank

The PageRank algorithm can be used for ranking not only web pages, but any "entities" organized into a network. The `USAirTransportation.mat` file contains the adjacency matrix A for the network of (anonymized) US airports: nodes represent airports and links represent air travel connections among them. That is, airports i and j are connected $(A_{ij} = 1)$ if there are direct flights between them. The network in `USAirTransportation.mat` is obtained by considering the 500 US airports with the largest amount of traffic from publicly available data. Implement the PageRank algorithm (see Lecture 12) for ranking the airports. Use your function from Problem 1 for finding the importance score vector. What are the ID numbers of 10 most important airports for $\alpha = 0.10, 0.15, 0.20$?

Functions that may be useful for this problem:

`repmat(), sort(), disp(), strcat(), num2str()`

```
load USAirTransportation.mat % dataset
%{
Let us see how matrix A looks like
%}
figure;
spy(A);
title('Sparsity Patern');
```




```
alpha = [0.10 0.15 0.20]; % values of alpha
[~,n]=size(A); % size of A
%{
Implement the algorithm below
You can use bar() to visualize the scores
Report the top 10 airports (in descending order) for each alpha
%}

% form matrix L that is column stochastic
L = zeros(n,n);
for i = 1 : n
    ni = sum(A(:,i));
    L(:,i) = A(:,i) * (1/ni);
end

S = ones(n,n) * (1/n);

disp("The ids are given in descending order, so first id is the most important.");
```

```
The ids are given in descending order, so first id is the most important.
```

```
for a = alpha
    L_tilde = (1-a) * L  + a * S;
    [lambda, x] = power_method(L_tilde);
```

```
    x_sorted = sort(x, "descend");
    %x_top_ten = x_sorted(1:10, 1);

    % get id_nums
    id_nums = zeros(1,10);
    for i = 1:10
        id_nums(1, i) = find(x == x_sorted(i,1));
    end
    disp(['For alpha =' num2str(a) ' the 10 most important airports ids: [' num2str(id_

end
```

```
For alpha =0.1 the 10 most important airports ids: [6   1   7   3   2  21  11   8  18  10]
For alpha =0.15 the 10 most important airports ids: [6   7   3   1   2  21  11   8  10  18]
For alpha =0.2 the 10 most important airports ids: [6   7   3   1   2  21  11  10   8  18]
```

Problem 4 (10 points) Principal Component Analysis

In this problem, you will find the principal components of the dataset **pca_data.mat**, which contains the $m \times n$ matrix X. Columns of X correspond to $n = 10^4$ measurements and rows of X correspond to $m = 2$ quantities (features). Let us see what the data looks like:

```
load pca_data.mat
figure;
plot(X(1,:),X(2,:),'.m','MarkerSize',10);
title("PCA raw data");
axis equal
```

Part (a)

Find the $m \times m$ covariance matrix of X. Use `disp()` to display it.

```
n = 10^4;
X_raw = X;
% preprocess X so that it has rows with zero mean
X(1, :) = X(1,:) - (1/n) * sum(X(1,:));
X(2, :) = X(2,:) - (1/n) * sum(X(2,:));
Cx = (1/n) * (X * X');
disp(Cx);
```

```
    0.9829   -0.6808
   -0.6808    1.9508
```

Part (b)

Find the m principal components of X and show them on top of the original data. That is, if v is a principal component, then plot the line $l(t) = \mu + vt$ over the scatter plot shown above, where $\mu \in \mathbb{R}^2$ is the centroid of the data.

```
%{
Find the eigenvectors
Normalize
Obtain the centroids
```

```
Create the lines
Plot
%}

[V, D] = eig(Cx);
V(:, 1) = V(:,1) / norm( V(:,1));
V(:, 2) = V(:,2) / norm( V(:,2));
x1_centroid = (1/n) * sum(X_raw(1,:));
x2_centroid = (1/n) * sum(X_raw(2,:));

figure;
hold on
plot(X_raw(1,:),X_raw(2,:),'.m','MarkerSize',10);
xt = @(t) x1_centroid + V(1, 1) * t ;
yt = @(t) x2_centroid + V(2,1) * t;
fplot(xt,yt, "-b");

xt = @(t) x1_centroid + V(1, 2) * t ;
yt = @(t) x2_centroid + V(2,2) * t;
fplot(xt,yt, "-b");

hold off
title("PCA raw data with principal components");
axis equal
```

Part (c)

If we change the basis of $\mathbb{R}^m$ from the standard basis to the basis of the principal components $[v_1, \cdots, v_m] = V$ of X, the data matrix will transform to $Y = V^{-1}X$. Find the covariance of the transformed data Y. Use **`disp()`** to display it.

```
Cy = D;
disp(Cy);
```

```
    0.6316         0
         0    2.3021
```

Part (d)

Principal component analysis is a very popular data analysis technique and it is often used in various applications. As such, it is implemented in most numerical software packages. In MATLAB, this is a built-in function **`pca()`**. In the simplest form, **`V = pca(X)`** returns the principal components of the data matrix X, where rows of X correspond to the *measurements* and columns correspond to the *features*. Find the principal components of X using **`pca()`** and compare them with the ones found in part (b). Are they the same?

```
V_mat = pca(X');

fprintf("Matlab's result show on next disp output\n");
```

```
Matlab's result show on next disp output
```

```
disp(V_mat);
```

```
   -0.4586    0.8886
    0.8886    0.4586
```

```
fprintf("Results from part b shown on next disp output");
```

```
Results from part b shown on next disp output
```

```
disp(V);
```

```
   -0.8886   -0.4586
   -0.4586    0.8886
```

```
% Yes, if you were plot the lines corresponding to the principal
% compenents, then they would be the same.
```

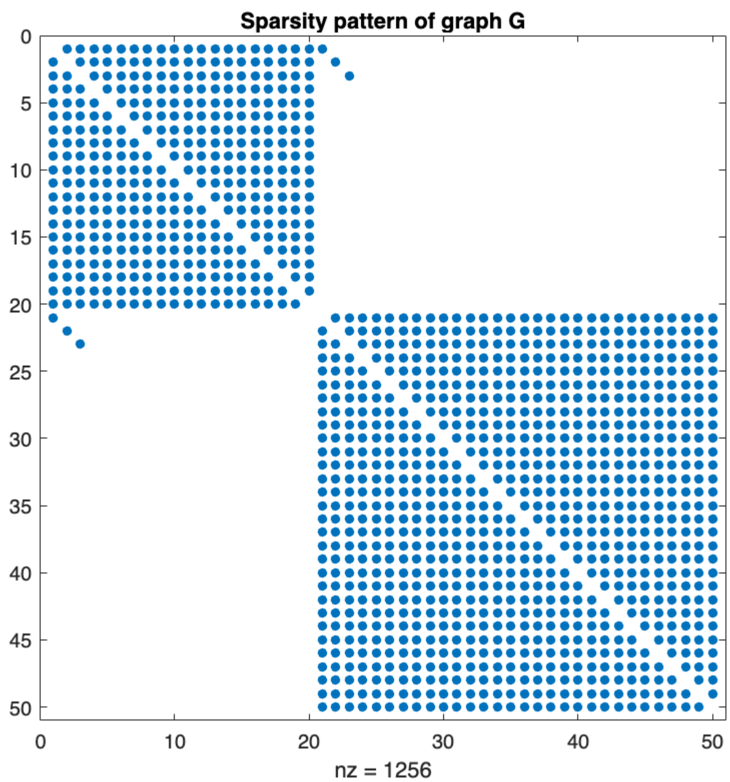
Problem 5 (10 points) Spectral Method for Graph Partitioning

A complete graph K_n with n vertices is a graph in which each pair of graph vertices is connected by an edge. Let G be a barbell-like graph obtained by connecting K_{n_1} and K_{n_2} with $m = 3$ bridges, where $n_1 = 20$ and $n_2 = 30$.

The graph G and the sparsity pattern of its adjacency matrix are shown:

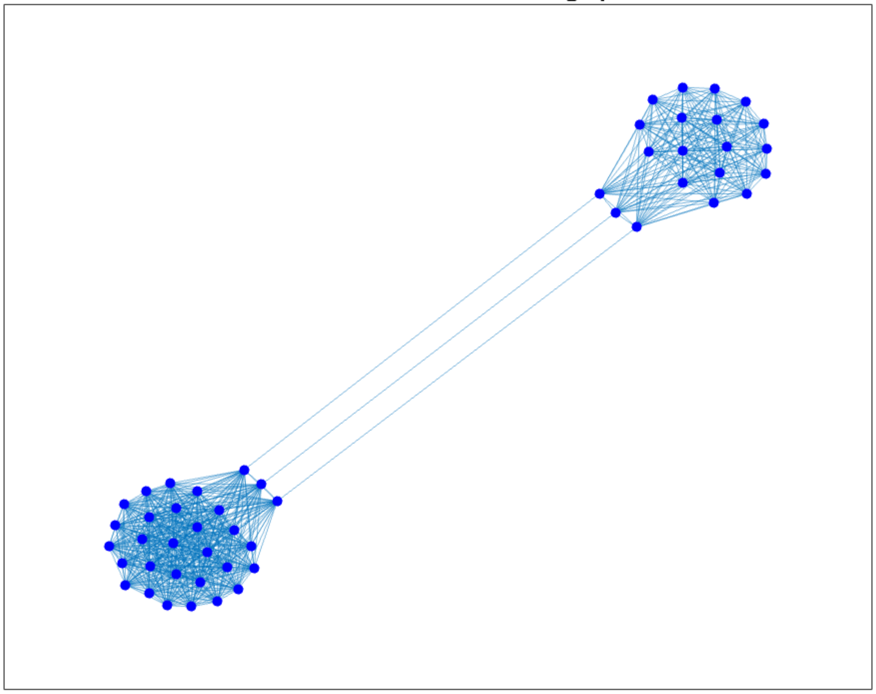
```
%{
Adjacency matrix
%}
n1=20;
n2=30;
```

```
B1=ones(n1,n1)-diag(ones(1,n1));
B2=ones(n2,n2)-diag(ones(1,n2));
B=[B1, zeros(n1,n2);zeros(n2,n1), B2]; % adjacency matrix B of graph G
for i=1:3
    B(i,n1+i)=1; B(n1+i,i)=1; % bridges
end
%{
Visualization
%}
figure;
spy(B);
title("Sparsity pattern of graph G");
```




```
figure;
G = graph(B);
plot(G,'NodeLabel',{},'NodeColor','b');
title("Visualization of the 'barbell-like' graph G");
```

Visualization of the 'barbell-like' graph G

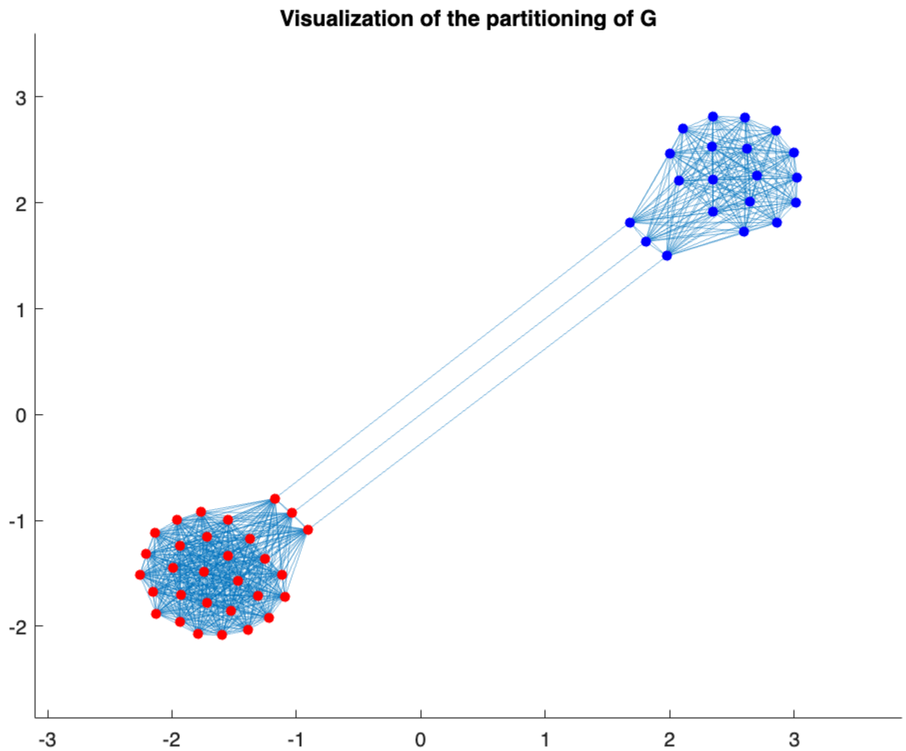


Use the spectral partitioning method to find a division of the graph G into two subgraphs of size $n_1 = 20$ and $n_2 = 30$ such that the number of edges between the subgraphs (the cut size) is minimized. Visualize the obtained partition and find the corresponding cut size. Useful functions for this problem:

`eigs(), find(), highlight(), plot()`

```
%{
-> Find the graph Laplacian
-> Find the Fiedler vector
-> Determine the 2 candidate partitions
-> Visualize
-> Report the cut size using disp()
%}

% LAPLACIAN & FIEDLER VECTOR
[m, n] = size(B);
degrees = zeros(1, m);
for i = 1:m
    degrees(1, i) = sum(B(i, :));
end
L = diag(degrees) - B;
[V,D] = eigs(L,2,'sr');
V2 = V(:,2);

% PARTITIONS
```

```
s_plus = sort(V2, "descend"); %sort in ascending order
s_minus = sort(V2);

n = n1 + n2;
for i = 1:n

    if i <= n1
        s_plus(i,1) = 1;
        s_minus(i,1) = 1;
    else
        s_minus(i,1) = -1;
        s_plus(i,1) = -1;
    end
end

plus_cutsize = (1/4) * (s_plus'* L * s_plus);
minus_cutsize = (1/4) * (s_minus'* L * s_minus);
s = s_plus;
cutsize = plus_cutsize;
if  minus_cutsize < plus_cutsize
    s = s_minus;
    cutsize = minus_cutsize;
end

% VISUALIZATION
group1 = find(s == 1);
group2 = find(s == -1);

figure;
G = graph(B);
hold on;
h = plot(G,'NodeLabel',{},'NodeColor','b');
highlight(h,group1,'NodeColor','b');
highlight(h,group2,'NodeColor','r');
hold off
title("Visualization of the partitioning of G");
```




```
% REPORTING CUT SIZE
disp(cutsize);
```

```
3
```

Problem 1 (10 points) The Power Method (aka Von Mises Iteration)

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_n$ such that:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$$

The eigenvalue λ_1 is called the dominant eigenvalue and the corresponding eigenvector v_1 is called the dominant eigenvector. Write a function that takes A as an argument and uses the Power method (Lecture 12) to compute λ_1 and v_1. We can check that λ_1 is correct by comparing it with the `eig()` function for a random matrix. An example is provided below:

```
E = [9,8,7;6,5,4;3,2,1];
[b, d] = eig(E);
[J, VE] = power_method(E);
disp(d(1));
```

```
16.1168
```

```
disp(J);
```

```
16.1168
```

```
function [lambda, v] = power_method(A)
[n, ~] = size(A); % size of nxn matrix A
v_prev = rand(n, 1); % random starting vector
tol = Inf; % relative error
%{
Implement the algorithm in the while loop
%}
while tol > 1e-15
    v_curr = A * v_prev;
    v_curr = v_curr / norm(v_curr, Inf);
    tol = norm(v_curr - v_prev, Inf);
    v_prev = v_curr;
end
    v = v_prev / norm(v_prev);

    % definition of eigen vector
    % lambda * v = A * v
    lambda_v = A * v;
    % since v is unit vector
    lambda = norm(lambda_v);
end
```