CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110

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Problem 1. (25 points) Properties of the Matrix Exponential

In this problem, we will examine the properties of the matrix exponential. Recall that this function is critically important to us as the matrix exponential describes the solution to a given linear system:

$$\dot{x}(t) = Ax(t) \Rightarrow x(t) = \exp(At)x_0, \tag{1}$$

with the dynamics matrix $A \in \mathbb{R}^{n\times n}$, the state $x \in \mathbb{R}$, and the initial condition $x_0 \in \mathbb{R}^n$. Using (1), we will look at some of this function's salient properties!

(a) While the matrix exponential has many intuitive relations with the scalar exponential, there are many intuitive *incorrect* relations as well. For instance, is it true that for $\forall A \in \mathbb{R}^{n\times n}$,

$$\exp(A) = \begin{bmatrix} e^{a_{11}} & \dots & e^{a_{1n}} \\ \vdots & \ddots & \vdots \\ e^{a_{n1}} & \dots & e^{a_{nn}} \end{bmatrix}? \tag{2}$$

Provide a proof or a counterexample.

(b) In the study of stability, and indeed in the next problem, we will see it is often convenient to transform the matrix exponential into different forms. Show that:

$$\exp(TAT^{-1}) = T\exp(A)T^{-1}, \tag{3}$$

where $A \in \mathbb{R}^{n\times n}$ and $T \in \mathbb{R}^{n\times n}$ is an invertible transformation matrix.

(c) Let's say we want to look backwards in time and get our initial condition given our current state. Namely, we want to find:

$$x_0 = \exp(At)^{-1}x(t) \tag{4}$$

In this case, we might want to take the inverse of the matrix exponential, but will it exist? Turns out, it always does! Show that:

$$\det(\exp(A)) = \exp(\mathrm{tr}(A)), \tag{5}$$

where $\mathrm{tr}(A)$ is the trace of A and $\det(A)$ is the determinant of A. Note that the exponentials on either side of the equality are different: while the one on the left is a matrix exponential, the one on the right is a scalar exponential. Why does this result imply invertability of $\exp(A)$?

Problem 2. (25 points) Eigenvalues for Stability

As we consider the stability of our linear system, recall that what will often matter to us is the eigenvalues of the $A \in \mathbb{R}^{n\times n}$ matrix of our system. Let's take a look at what properties of these eigenvalues matter to us.

(a) Consider the following linear system:

$$\dot{x}(t) = Ax(t), \tag{6}$$

with $A \in \mathbb{R}^{2\times 2}$ and $x \in \mathbb{R}^2$. Further, we are given that this system has complex eigenvalues. Note that for a real-valued A, this means that both eigenvectors and eigenvalues must be complex conjugate pairs, i.e.:

$$\lambda_{1,2} = \sigma \pm j\omega, \tag{7}$$

$$v_{1,2} = v_r \pm jv_j. \tag{8}$$

Construct a similarity transformation T such that we can rewrite the system as:

$$\dot{x}(t) = T \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} T^{-1}x(t). \tag{9}$$

(b) Now that we've seen how to decompose our system in this way, let's do a quick change of variables $z = T^{-1}x$ and consider our new system:

$$\dot{z}(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z(t). \tag{10}$$

Show that the solution to this differential equation is the following:

$$z(t) = \exp(\sigma t) \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix} z_0. \tag{11}$$

How do σ and ω each affect the stability of the overall system?

Hint: Decompose the new 'A' matrix into a diagonal part plus a skew symmetric part. Use also the property that $\exp(A + B) = \exp(A)\exp(B)$.

Problem 3. (25 points) Linearization

While linear systems are relatively easy to analyze, most systems in the real world are nonlinear. It is tempting therefore to always linearize systems, and in many practical situations this is perfectly fine! However, there are some large caveats present here, which we will explore in this problem.

(a) Consider the following model of the populations of predator and prey animals in an ecosystem:

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} rx_1\left(1 - \frac{x_1}{k}\right) - \frac{ax_1x_2}{c+x_1} \\ b\frac{ax_1x_2}{c+x_1} - dx_2 \end{bmatrix}. \tag{12}$$

where x_1 is the prey population, x_2 is the predator population, r is the prey population growth rate, b is the predator population growth rate, d is the predator mortality rate, k is the maximum prey population the environment can sustain, and a and c quantify the prey consumption rate. The state is thus $x = (x_1, x_2) \in \mathbb{R}^2$.

Find the equilibrium point(s), and compute the Jacobian linearization around each. What can you conclude about the stability of each of these equilibria? Do they match your intuition? Show stability or instability.

(b) With the following parameter values: a=3.2, b=0.6, c=50, d=0.56, k=125, and r=1.6, plot the dynamics for a few different initial conditions of the full nonlinear system, and the linearizations around each equilibrium point. When do the dynamics look similar to one another? When do they look different? Be sure to test not only different initial conditions but also different timescales.

(c) Why do we linearize around equilibrium points? Can we linearize around other points and get meaningful information about the system behavior? Briefly discuss.

Problem 4. (25 points) Continuous Control with Discrete-time Inputs

While the world is in continuous time [citation needed], computers (and therefore many controllers) operate in discrete time. However, while you can just take approximate discretizations (such as the Euler discretization), you can also rigorously get an exact discretization of your system. While this is often too difficult in practice, it is an interesting theoretical exercise!

Recall that the general solution to the differential equation $\dot{x}(t) = Ax(t) + Bu(t)$ with an initial condition $x(0) = x_0$ is given by the expression:

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau. \tag{13}$$

If we use a sampling time Δt, and $u(t)$ is constant over the sampling interval $[t, t + \Delta t]$, show that we can discretize the system perfectly with no error as:

$$x(k+1) = A_d x(k) + B_d u(k), \tag{14}$$

where $A_d = e^{A\Delta t}$ is the discrete dynamics matrix and $B_d = \int_0^{\Delta t} e^{A(\Delta t - \tau)}Bd\tau$ is the discrete control matrix. This form of discretization, where we hold an input constant over a sampling interval, is known as a zero order hold discretization.

Hint: Examine the general solution for $x(t)$ from $t = 0$ to $t = t + \Delta t$. Then see if you can split the integral into $x(t)$ from $t = 0$ to $t = t$ and from $t = t$ to $t = t + \Delta t$.