CDS110 **November 26, 2024**

Pset 08

Problem 1

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u = Ax + Bu$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = Cx$$

Part A

Computing the Reachability Matrix W_C:

$$W_C = \begin{bmatrix}B & AB\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

Computing the Observability Matrix W_O:

$$W_O = \begin{bmatrix}C\\CA\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

Because $rank(W_C) = rank(W_O) = 2$ (i.e both matrices are full rank), then the system is both reachable and observable.

Part B

Let $u = -K\hat{x} + k_r r$ where $\hat{x}$ is the output of the observer of the state such that $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$. Now let $e = x - \hat{x}$. Thus we have:

$$\dot{x} = Ax + B(-K\hat{x} + k_r r) = Ax - BK\hat{x} + Bk_r r = Ax - BK(x - e) + Bk_r r$$

$$\dot{x} = (A - BK)x + Bke + Bk_r r$$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$$

Letting $z = \begin{bmatrix}x\\e\end{bmatrix}$, then we have

$$\dot{z} = \begin{bmatrix}A - BK & BK\\0 & A - LC\end{bmatrix}z + \begin{bmatrix}Bk_r r\\0\end{bmatrix}$$

Thus we have that the characteristic polynomial of the closed system is
$\lambda(s) = det(sI - A + BK)det(sI - A + LC)$. Now, let $L = \begin{bmatrix}\ell_1\\\ell_2\end{bmatrix}$ and let $K = \begin{bmatrix}k_1 & k_2\end{bmatrix}$. This gives us the following:

$$sI - A + BK = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix} = \begin{bmatrix}s & -1\\k_1 - 1 & s + k_2\end{bmatrix}$$

$$det(sI - A + BK) = s^2 + k_2 s + (k_1 - 1) = s^2 + a_1 s + a_2$$

$$sI - A + LC = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \ell_1 \\ 0 & \ell_2 \end{bmatrix} = \begin{bmatrix} s & \ell_1 - 1 \\ -1 & s + \ell_2 \end{bmatrix}$$

$$det(sI - A + LC) = s^2 + \ell_2 s + (\ell_1 - 1) = s^2 + b_1 s + b_2$$

Thus, we have $L = \begin{bmatrix} b_2 + 1 \\ b_1 \end{bmatrix}$ and $K = \begin{bmatrix} a_2 + 1 & a_1 \end{bmatrix}$.

Part C

As described in part B we have $u = -K\hat{x} + k_r r$ and $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$. By substitution we have:

$$\dot{\hat{x}} = A\hat{x} + B(-K\hat{x} + k_r r) + L(y - C\hat{x})$$

$$\dot{\hat{x}} = (A - BK - LC)\hat{x} + Ly + Bk_r r$$

By substituting we have:

$$A - BK - LC = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \begin{bmatrix} 0 & \ell_1 \\ 0 & \ell_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 - \ell_1 \\ 1 - k_1 & -k_2 - \ell_2 \end{bmatrix}$$

Thus, the state space representation of the full controller is:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 - \ell_1 \\ 1 - k_1 & -k_2 - \ell_2 \end{bmatrix} \hat{x} + \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} y + \begin{bmatrix} 0 \\ k_r \end{bmatrix} r$$

$$u = -\begin{bmatrix} k_1 & k_2 \end{bmatrix} \hat{x} + k_r r$$

Substituting $k_1 = a_2 + 1$, $k_2 = a_1$, $\ell_1 = b_2 + 1$ and $\ell_2 = b_1$ we get:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & -b_2 \\ -a_1 & -a_1 - b_1 \end{bmatrix} \hat{x} + \begin{bmatrix} b_2 + 1 \\ b_1 \end{bmatrix} y + \begin{bmatrix} 0 \\ k_r \end{bmatrix} r$$

$$u = -\begin{bmatrix} a_2 + 1 & a_1 \end{bmatrix} \hat{x} + k_r r$$

Part D

For the open loop controller we have $r = y = 0$. Thus the state space representation becomes:

$$\dot{\hat{x}} = \begin{bmatrix} 0 & 1 - \ell_1 \\ 1 - k_1 & -k_2 - \ell_2 \end{bmatrix} \hat{x}$$

First note:

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 - \ell_1 \\ 1 - k_1 & -k_2 - \ell_2 \end{bmatrix} = \begin{bmatrix} s & \ell_1 - 1 \\ k_1 - 1 & s + k_2 + \ell_2 \end{bmatrix}$$

Thus, the characteristic polynomial of the system is:

$$\lambda(s) = det\left(\begin{bmatrix} s & \ell_1 - 1 \\ k_1 - 1 & s + k_2 + \ell_2 \end{bmatrix}\right) = s(s + k_2 + \ell_2) - (\ell_1 - 1)(k_1 - 1) = s^2 + (k_2 + \ell_2)s - (\ell_1 - 1)(k_1 - 1)$$

Substituting $k_1 = a_2 + 1$, $k_2 = a_1$, $\ell_1 = b_2 + 1$ and $\ell_2 = b_1$ we get:

$$\lambda(s) = s^2 + (a_1 + b_1)s - (b_2)(a_2)$$

This gives us the eigenvalues:

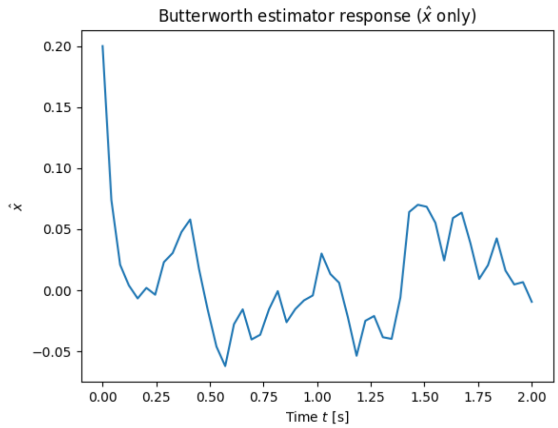
$$\lambda = \frac{-(a_1 + b_1) \pm \sqrt{(a_1 + b_1)^2 + 4b_2 a_2}}{2}$$

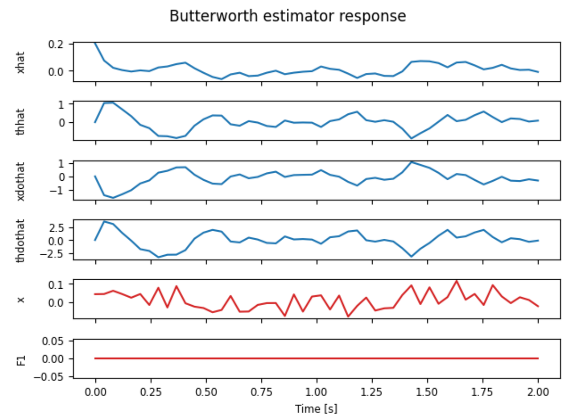
Because $a_1, a_2, b_1, b_2 > 0$, then one of the eigenvalues $\lambda_1 = \frac{-(a_1+b_2)+\sqrt{(a_1+b_1)^2+4b_2 a_2}}{2} > 0$. Thus, there is an eigenvalue that is in the right half plane.

Problem 2

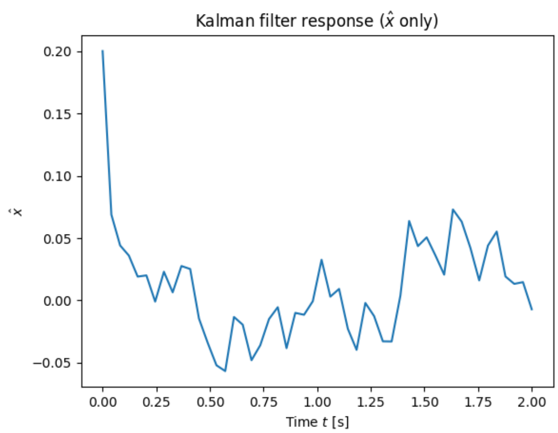
$$m\ddot{x} = F_1 cos(\theta) - F_2 sin(\theta) - c\dot{x}$$
$$m\ddot{y} = F_1 sin(\theta) + F_1 cos(\theta) - mg - c\dot{y}$$
$$J\ddot{\theta} = rF_1$$

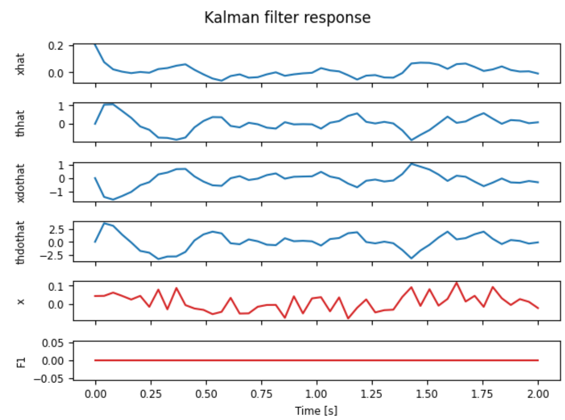
Part A





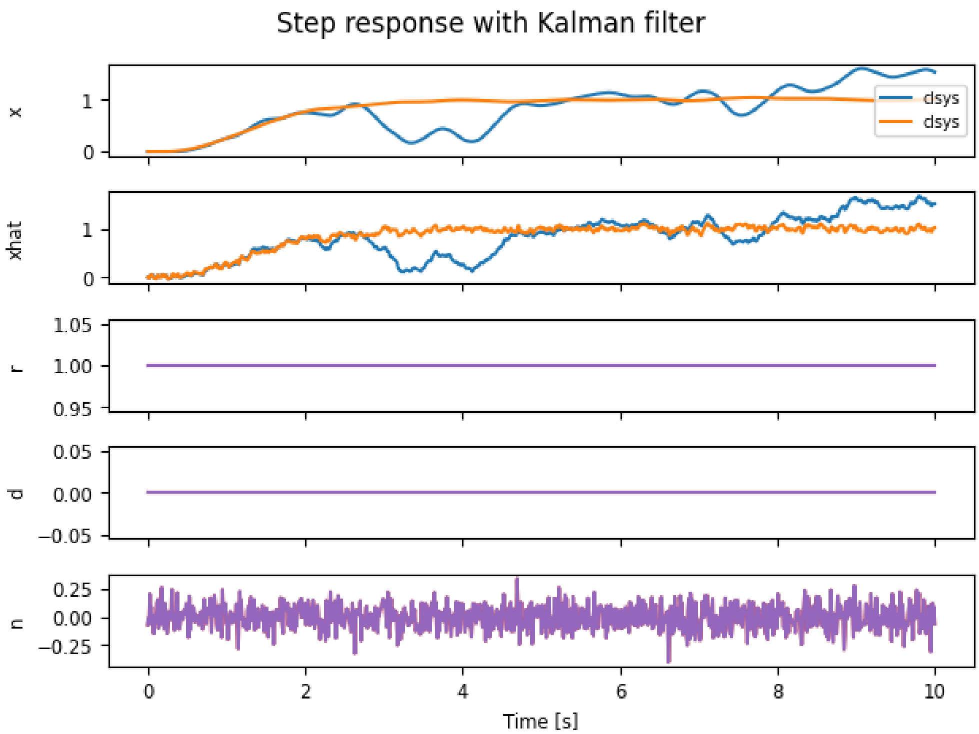
Part B





Comparing the plots from part B with those in part A, we see that the plots are very similar. This is not much distinction between using the given Butterworth Estimator and a Kalman Filter in this case.

Part C



Step response with Kalman filter
x
xhat
r
d
n
dsys
dsys
0
1
1.05
1.00
0.95
0.05
0.00
−0.05
0.25
0.00
−0.25
0
2
4
6
8
10
Time [s]

Problem 3

$$\dot{x} = w$$
$$\dot{y} = x + v$$
$$w \sim (0, Q)$$
$$v \sim (0, R)$$

Part A

As given we have the continuous time covariance update equation:

$$\dot{P} = -PC^T R^{-1} CP + AP + PA^T + Q$$

From the equations given in the problem set we have $A = 0$, thus $\dot{P}$ reduces to:

$$\dot{P} = -PC^T R^{-1} CP + Q$$

Since we are working in one dimension, R is just a scalar, hence we have $R^{-1} = \frac{1}{R}$. Furthermore, we also have C as a scalar $C = 1 \implies C^T = 1$. Therefore, we have:

$$\dot{P} = \frac{1}{R}(-P(1)(1)P) + Q = -\frac{P^2}{R} + Q$$

Part B

As time progresses, eventually $P(t)$ will converge. Hence as t approaches infinity we would have $\dot{P} = 0$ ($\lim_{t\to\infty} \dot{P} = 0$). Using the equation from Part A we have:

$$0 = -\frac{P^2}{R} + Q \implies P^2 = QR \implies P = \sqrt{QR}$$

This means that as t approaches infinity we have $P = \sqrt{QR}$ ($\lim_{t\to\infty} P = \sqrt{QR}$).
Intuitively, P is the variance of the error of the state estimate. So if either the variance of the process noise or measurement noise is large (either Q or R is large) then it would make sense if the variance of the error of the state estimate is also large (P is large). Similarly, if either the variance of the process noise or measurement noise is small (either Q or R is small) then it would make sense if the variance of the error of the state estimate is also small (P is small).

Part C

In this case we have $K = PC^T R^{-1} = \frac{P}{R}$. Hence we have:

$$\lim_{t\to\infty} K = \frac{\lim_{t\to\infty} P}{R} = \frac{\sqrt{QR}}{R} = \sqrt{\frac{Q}{R}}$$

As the variance of the measurement noise increases, the value of the kalman gain decreases. As the variance of the process noise increases, the value of the kalman gain increases. In other words, the kalman gain is proportional to the process noise but inversely proportional to the measurement noise.
Intuitively, the Kalman Gain indicates how much our estimate should be changed with a given prior. If there is low variance in our measurement noise (R is low) then our confidence in our measurement is high (so the Kalman Gain should be higher, i.e. we would want to change our estimate more). Similarly, if the is high variance measurement noise (R is high) then we are less confidence in our measurements (so the Kalman Gain should be lower). Now if there is low variance in our process noise (Q is low) then this suggests that our state would not vary much (i.e the state dynamics are more predictable), so the Kalman Gain should be low as well. If there is high variance in our process noise (Q is high) then this suggests that our state would vary more (i.e the state dynamics are less predictable), so the Kalman Gain should be high to account for that variance.
With this intuition, the Kalman Gain is essentially a compromise on the response given the variances of the two noises (the process noise and the measurement noise).