CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110

Problem Set #8

Eric Mazumdar
Fall 2024

Issued: 19 Nov 2024
Due: 26 Nov 2024

Problem 1. (30 points) Consider a normalized inverted pendulum with a rate sensor described by

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u = Ax + Bu, \qquad y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = Cx.$$

(a) Is this system reachable? Is it observable?

(b) Design a controller based on state feedback and an observer such that the matrices $A - BK$ and $A - LC$ have the characteristic polynomials $s^2 + a_1 s + a_2$ and $s^2 + b_1 s + b_2$ with all coefficients positive.

(c) Construct a state space representation of the full controller (estimator + state feedback) that takes r and y as inputs and outputs u.

(d) Show that the open loop controller (with inputs r and y set to zero) has an eigenvalue in the right half plane.

Problem 2. (40 points) The lateral dynamics of a planar vectored thrust aircraft described by

$$\begin{aligned} m\ddot{x} &= F_1\cos\theta - F_2\sin\theta - c\dot{x} \\ m\ddot{y} &= F_1\sin\theta + F_1\cos\theta - mg - c\dot{y} \\ J\ddot{\theta} &= rF_1 \end{aligned} \tag{1}$$

can be obtained by considering the motion described by the states $z = (x, \theta, \dot{x}, \dot{\theta})$. Assume that we can only measure the position of the aircraft x, corrupted by white noise with intensity $R_w = 10^{-4}$. For the following problem, please refer to the accompanying skeleton notebook.

(a) Construct an estimator for these dynamics by setting the eigenvalues of the observer into a *Butterworth pattern*[1] with $\lambda_{\text{bw}} = -3.83 \pm 9.24i$, $-9.24 \pm 3.83i$. Show the response of the estimator starting from $\hat{z}(0) = (0.2, 0, 0, 0)$, assuming that the system remains at the origin but with noisy measurement of the x position.

(b) Construct an optimal estimator for the system assuming that the system is subject to white noise disturbances v with intensity $R_v = 0.1$ applied at the system input. Compute the response of the estimator as in part (a) and compare the initial condition response to that obtained in part (a).

[1] The Butterworth estimator is the "maximally flat" filter - i.e., it is designed to have the maximally sharp rolloff, so that desirable frequencies are minimally impacted by the filter.

(c) Design a state-feedback LQR controller of the form $u = -K\hat{x} + k_f r$. For each of the estimators above, combine them with the controller and plot the step response of the closed loop system in the presence of noisy measurements.

Problem 3. (30 points)

Consider measuring the velocity of an object moving in one dimension. The object is subject to random accelerations, and the goal is to measure the velocity x from noisy measurements. The system and measurement equations are given by

$$\begin{aligned} \dot{x} &= w \\ y &= x + v \\ w &\sim (0, Q) \\ v &\sim (0, R) \end{aligned} \tag{2}$$

(a) Show that the continuous time covariance update equation

$$\dot{P} = -PC^T R^{-1} CP + AP + PA^T + Q \tag{3}$$

reduces to

$$\dot{P} = -\frac{P^2}{R} + Q \tag{4}$$

(b) Show that $\lim_{t\to\infty} P = \sqrt{QR}$. Give intuition to this result.

(c) Show that $\lim_{t\to\infty} K = \sqrt{Q/R}$, where K is the Kalman gain. How does K change as a function of the process and measurement noise? Why does this make sense?