CDS110 **October 22, 2024**

Pset 03

Problem 1

Part A

Consider the counterexample below:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = T\Lambda T^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Because A is diagonalizable as shown above then we have:

$$e^A = e^{T\Lambda T^{-1}} = Te^{\Lambda}T^{-1}$$

Thus we have:

$$e^{\Lambda} = \begin{bmatrix} e^1 & 0 \\ 0 & e^{-1} \end{bmatrix} = \begin{bmatrix} e & 0 \\ 0 & \frac{1}{e} \end{bmatrix}$$

$$e^A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & \frac{1}{e} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{e^2+1}{2e} & \frac{e^2-1}{2e} \\ \frac{e^2-1}{2e} & \frac{e^2+1}{2e} \end{bmatrix} = \begin{bmatrix} 1.543 & 1.175 \\ 1.175 & 1.543 \end{bmatrix}$$

Thus the example above is a counterexample to the equation given in the question as $e^0 \neq 1.543$ and $e^1 \neq 1.175$.

Part B

First $(TAT^{-1})^n = TA^nT^{-1}$ will be proven by induction. We first show that it hold for the base cases:

$$(TAT^{-1})^0 = TA^0T^{-1} \implies I = TIT^{-1} = I$$

$$(TAT^{-1})^1 = TAT^{-1}$$

Clearly the above base cases hold. Now we assume the following holds $(TAT^{-1})^n = TA^nT^{-1}$ (inductive hypothesis) for some $n = k$. Now for $k+1$ as the exponent we want to prove the following:

$$(TAT^{-1})^{k+1} = TA^{k+1}T^{-1}$$

Using the inductive hypothesis we have:

$$(TAT^{-1})^{k+1} = (TAT^{-1})^k(TAT^{-1}) = TA^kT^{-1}TAT^{-1} = TA^kIAT^{-1} = TA^kAT^{-1} = TA^{k+1}T^{-1}$$

Therefore, by induction, we have shown that $(TAT^{-1})^n = TA^nT^{-1}$ is true for any non negative n

Using the result above

$$e^{TAT^{-1}} = \sum_{k=0}^{\infty} \frac{(TAT^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{TA^kT^{-1}}{k!} = T\left(\sum_{k=0}^{\infty} \frac{A^k}{k!}\right)T^{-1} = Te^AT^{-1}$$

Thus we have shown that $e^{TAT^{-1}} = Te^AT^{-1}$

Part C
First we will show that $det(e^B) = e^{tr(B)}$ is true for any upper triangular matrix $B \in R^{n\times n}$.
First note, via cofactor expansion, we have:

$$det(B) = det\left(\begin{bmatrix} B_{11} & * & * & \cdots & * \\ 0 & B_{22} & * & \cdots & * \\ 0 & 0 & \ddots & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & \cdots & 0 & B_{nn} \end{bmatrix}\right) = B_{11} det\left(\begin{bmatrix} B_{22} & * & \cdots & * \\ 0 & \ddots & \cdots & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & B_{nn} \end{bmatrix}\right)$$

This is because for $B_{r1} = 0$ for $r > 1$. Iterating again we have:

$$B_{11} det\left(\begin{bmatrix} B_{22} & * & \cdots & * \\ 0 & \ddots & \cdots & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & B_{nn} \end{bmatrix}\right) = B_{11}B_{22} det\left(\begin{bmatrix} B_{33} & * & \cdots & * \\ 0 & \ddots & \cdots & * \\ \vdots & \ddots & \ddots & * \\ 0 & \cdots & 0 & B_{nn} \end{bmatrix}\right)$$

If we keep iterating we have that $det(B) = \prod_{i=1}^{n} B_{ii}$ with $B \in R^{n\times n}$ and with B as an upper triangular matrix.
Now by definition of a matrix exponential we have:

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + ...$$

Now because B is an upper triangular matrix then e^B is also an upper triangular matrix since the product of any two upper triangular matrices is also an upper triangular matrix. Therefore, using the result for the determinant of an upper triangular matrix we have:

$$det(e^B) = \prod_{i=1}^{n} (e^B)_{ii}$$

Note that for any upper triangular matrix $B \in R^{n\times n}$ we have $(B^k)_{ii} = (B_{ii})^k$. To see this, first notice that $(B^{k+1})_{ii} = \sum_{j=1}^{n} (B^k)_{ij} B_{ji}$ with $k > 1$. Since B is an upper triangular matrix, then B^k is also an upper triangular matrix. Therefore $B_{ij} = 0$ for $i > j$ and $(B^k)_{ji} = 0$ for $j > i$. Therefore, the summation reduces to $(B^{k+1})_{ii} = (B^k)_{ii} B_{ii}$. With this in mind, we will prove $(B^k)_{ii} = (B_{ii})^k$ via induction. For the base case we have that $(B^1)_{ii} = (B_{ii})^1$ holds. Now assume the statement $(B^k)_{ii} = (B_{ii})^k$ holds for some $k = m$. Now for $k = m+1$ we want to prove $(B^{m+1})_{ii} = (B_{ii})^{m+1}$. Using the property $(B^{k+1})_{ii} = (B^k)_{ii} B_{ii}$, we have $(B^{m+1})_{ii} = (B^m)_{ii} B_{ii}$. Now using the inductive hypothesis we have that $(B^{m+1})_{ii} = (B_{ii})^m B_{ii} = (B_{ii})^{m+1}$. Thus we have proven that $(B^k)_{ii} = (B_{ii})^k$ for any upper triangular matrix $B \in R^{n\times n}$.
Thus we have:

$$(e^B)_{ii} = I + B_{ii} + \frac{1}{2!}(B^2)_{ii} + \frac{1}{3!}(B^3)_{ii} + ... = I + B_{ii} + \frac{1}{2!}(B_{ii})^2 + \frac{1}{3!}(B_{ii})^3 + ... = e^{B_{ii}}$$

Therefore for any upper triangular matrix $B \in R^{n\times n}$ we have:

$$det(e^B) = \prod_{i=1}^{n} e^{B_{ii}} = e^{\sum_{i=1}^{n} B_{ii}} = e^{tr(B)}$$

Now let A be any square matrix such that $A \in R^{n\times n}$. Now Let A be written in its Jordan canonical form $A = PJP^{-1}$. From part B we have:

$$e^A = e^{PJP^{-1}} = Pe^J P^{-1}$$

Now note that by the properties of determinants we have

$$det(Pe^J P^{-1}) = det(P) \cdot det(e^J) \cdot det(P^{-1}) = det(P) \cdot det(e^J) \cdot \frac{1}{det(P)} = det(e^J)$$

Thus we have that $det(e^A) = det(e^J)$. Now because J is an upper triangular matrix then using the result had on the previous page we have:

$$det(e^A) = det(e^J) = e^{tr(J)}$$

By the properties of the trace we have:

$$tr(A) = tr(PJP^{-1}) = tr(JP^{-1}P) = tr(J)$$

Therefore for any square matrix $A \in R^{n\times n}$ we have:

$$det(e^A) = e^{tr(A)}$$

Since $e^{tr(A)} \neq 0$ for any $A \in R^{n\times n}$ then this means that the determinant of e^A will always be nonzero. Because the determinant of e^A will always be nonzero, then e^A is invertible.

Problem 2

Part A
By the definition of an eigenvector and eigenvalue we have:

$$Av_1 = \lambda_1 v_1$$

In this case $v_1 = v_r + jv_j$ and $\lambda_1 = \sigma + j\omega$
Thus by substitution we have:

$$A(v_r + jv_j) = (\sigma + j\omega)(v_r + jv_j)$$

$$A(v_r + jv_j) = \sigma v_r + \sigma jv_j + j\omega v_r + j^2\omega v_j = \sigma v_r + \sigma jv_j + j\omega v_r - \omega v_j$$

$$Av_r + j(Av_j) = (\sigma v_r - \omega v_j) + j(\sigma v_j + \omega v_r)$$

Hence we have:

$$Av_r = \sigma v_r - \omega v_j$$

$$Av_j = \sigma v_j + \omega v_r$$

In matrix form this becomes:

$$A \begin{bmatrix} v_r & v_j \end{bmatrix} = \begin{bmatrix} v_r & v_j \end{bmatrix} \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Let $T = \begin{bmatrix} v_r & v_j \end{bmatrix}$ so we have:

$$AT = T \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \implies A = T \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} T^{-1}$$

Thus $T = \begin{bmatrix} v_r & v_j \end{bmatrix}$ is the transformation matrix.

Part B
Given the system below:

$$\dot{z}(t) = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} z(t)$$

The solution is:

$$z(t) = exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t \right) z_0$$

Rewriting the new 'A' matrix we have:

$$\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} + \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$$

Thus using the property $exp(A + B) = exp(A)exp(B)$. We have

$$exp\left(\begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} t \right) = exp\left(\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} t \right) exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t \right)$$

Now solving the first matrix exponential (Note $I^n = I$ for any non negative n):

$$exp\left(\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} t \right) = exp(I\sigma t) = \sum_{k=0}^{\infty} \frac{(I\sigma t)^k}{k!} = I \sum_{k=0}^{\infty} \frac{(\sigma t)^k}{k!} = Ie^{\sigma t} = e^{\sigma t} I$$

Now we solve the next matrix exponential (first we let B equal the following matrix):

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t\right) = exp\left(\omega Bt\right) = \sum_{k=0}^{\infty} \frac{(\omega Bt)^k}{k!} = I + \omega Bt + \frac{1}{2!}(\omega Bt)^2 + ...$$

First notice that $B^2 = -I$, $B^3 = -B$, $B^4 = I$, $B^5 = B$, ... and so on. In other words the powers of B repeat in form $B, -I, -B, I$. Using this we simply the series:

$$exp\left(\omega Bt\right) = \sum_{k=0}^{\infty} \frac{(\omega Bt)^k}{k!} = I + (\omega t)B - \frac{(\omega t)^2 I}{2!} - \frac{(\omega t)^3 B}{3!} + \frac{(\omega t)^4 I}{4!} + \frac{(\omega t)^5 B}{5!} + ...$$

$$exp\left(\omega Bt\right) = \sum_{k=0}^{\infty} \frac{(\omega Bt)^k}{k!} = \left(1 - \frac{(\omega t)^2}{2!} + \frac{(\omega t)^4}{4!} + ...\right) I + \left(\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} + ...\right) B$$

Thus we have:

$$exp\left(\omega Bt\right) = \sum_{k=0}^{\infty} \frac{(\omega Bt)^k}{k!} = cos(\omega t)I + sin(\omega t)B = \begin{bmatrix} cos(\omega t) & 0 \\ 0 & cos(\omega t) \end{bmatrix} + \begin{bmatrix} 0 & sin(\omega t) \\ -sin(\omega t) & 0 \end{bmatrix} = \begin{bmatrix} cos(\omega t) & sin(\omega t) \\ -sin(\omega t) & cos(\omega t) \end{bmatrix}$$

Therefore by substitution we have:

$$z(t) = exp\left(\begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix} t\right) exp\left(\begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} t\right) z_0 = e^{\sigma t} \begin{bmatrix} cos(\omega t) & sin(\omega t) \\ -sin(\omega t) & cos(\omega t) \end{bmatrix} z_0$$

$$z(t) = e^{\sigma t} \begin{bmatrix} cos(\omega t) & sin(\omega t) \\ -sin(\omega t) & cos(\omega t) \end{bmatrix} z_0$$

Hence we have shown that the solution of the differential equation is as shown above. With that in mind, we see that σ affects the stability of the system while ω affects the oscillatory behavior of the system. If $\sigma > 0$ this would mean that the magnitude of the system's trajectories would increase over time via exponential growth making it unstable. If $\sigma = 0$, the magnitude of the system's trajectories would not increase nor decrease, which would mean that the system is stable. The magnitude of the system's trajectories would decrease for $\sigma \leq 0$ via exponential decay leading to the system being asymptotically stable. Lastly, ω controls the frequency of the oscillations exhibited by the system (i.e if $\omega = 0$ then there will be no oscillatory behavior of the system).

Problem 3

Part A

$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} rx_1(1-\frac{x_1}{k}) - \frac{ax_1x_2}{c+x_1} \\ b\frac{ax_1x_2}{c+x_1} - dx_2 \end{bmatrix}$$

For system above, when $x_2 = 0$ we also have $\dot{x}_2 = 0$. With $x_2 = 0$, $\dot{x}_1 = 0$ when $x_1 = 0$ and when $x_1 = k$. Therefore, $(0,0)$ and $(k,0)$ are equilibrium points.
Now if $x_2 \neq 0$, then $\dot{x}_2 = 0$ when $b\frac{ax_1}{c+x_1} = d$. Rearranging the equation we get $bax_1 = d(c+x_1) = dc + dx_1$. Solving for x_1 yields $x_1 = \frac{dc}{ba-d}$. Now setting $\dot{x}_1 = 0$ we get $r(c+x_1)x_1(1-\frac{x_1}{k}) = ax_1x_2$. This yields $(x_2)_0 = \frac{r(c+(x_1)_0)(x_1)_0(1-\frac{(x_1)_0}{k})}{ax_1}$ where $(x_1)_0 = \frac{dc}{ba-d}$. Thus the point $((x_1)_0, (x_2)_0)$, where $(x_1)_0, (x_2)_0$ are defined as in the previous sentence, is also an equilibrium point assuming $\frac{dc}{ba-d} \neq 0$ and $ba - d \neq 0$.

To test stability, we first calculate the Jacobian. In general:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} r - 2r\frac{x_1}{k} - \left(\frac{ax_2}{c+x_1} - \frac{ax_1x_2}{(c+x_1)^2}\right) & -\frac{ax_1}{c+x_1} \\ b\left(\frac{ax_2}{c+x_1} - \frac{ax_1x_2}{(c+x_1)^2}\right) & b\frac{ax_1}{c+x_1} - d \end{bmatrix} = \begin{bmatrix} r(1-2\frac{x_1}{k}) - \frac{ax_2c}{(c+x_1)^2} & -\frac{ax_1}{c+x_1} \\ b\frac{ax_2c}{(c+x_1)^2} & b\frac{ax_1}{c+x_1} - d \end{bmatrix}$$

Linearizing at the equilibrium point $x_e = (0,0)$ we have the Jacobian and the linearized dynamics around $(0,0)$ written as an equation:

$$J = \begin{bmatrix} r & 0 \\ 0 & -d \end{bmatrix}$$

$$\dot{x}(t) = J\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

For this linearized dynamics, the eigenvalues are $\lambda_1 = r$ and $\lambda_2 = -d$. Because r is the prey population growth, then we would typically have $r > 0$ which would mean that one of the eigenvalues is positive and hence the dynamics around (0,0) is unstable. For $r > 0$ with $d > 0$ we would have the prey population growing while the predator population decreasing for the linearized dynamics. or $r > 0$ with $d < 0$ we would have both the the prey population and predator population increasing for the linearized dynamics. Both of these situations lead to an unstable behavior around (0,0). In the event that $r < 0$ and $d < 0$ then we would have the prey population decreasing while the predator population increasing which would lead to unstable behavior around (0,0). Lastly for $r < 0$ and $d > 0$ we would have a stable equilibrium where both the prey population and predator population are decreasing.
Linearizing at the equilibrium point $x_e = (k,0)$ we have the Jacobian and the linearized dynamics around $(k,0)$ written as an equation:

$$J = \begin{bmatrix} -r & -\frac{ak}{c+k} \\ 0 & b\frac{ak}{c+k} - d \end{bmatrix}$$

$$\dot{x}(t) = J\begin{bmatrix} x_1 - k \\ x_2 \end{bmatrix}$$

For this linearized dynamics, the eigenvalues are $\lambda_1 = -r$ and $\lambda_2 = b\frac{ak}{c+k} - d$. Because r is the prey population growth, then we would typically have $r > 0$ which would mean that $\lambda_1 < 0$. Now because $\lambda_1 < 0$, then the dynamics around (k,0) would be stable if $b\frac{ak}{c+k} - d < 0$ ($\lambda_2 < 0$), otherwise if $b\frac{ak}{c+k} - d > 0$ ($\lambda_2 > 0$), then the dynamics around (k,0) would be unstable.

Linearizing at the equilibrium point $((x_1)_0, (x_2)_0)$ where $(x_2)_0 = \frac{r(c+(x_1)_0)(x_1)_0(1-\frac{(x_1)_0}{k})}{ax_1}$ and $(x_1)_0 = \frac{dc}{ba-d}$ we have the Jacobian and the linearized dynamics around this point written as an equation:

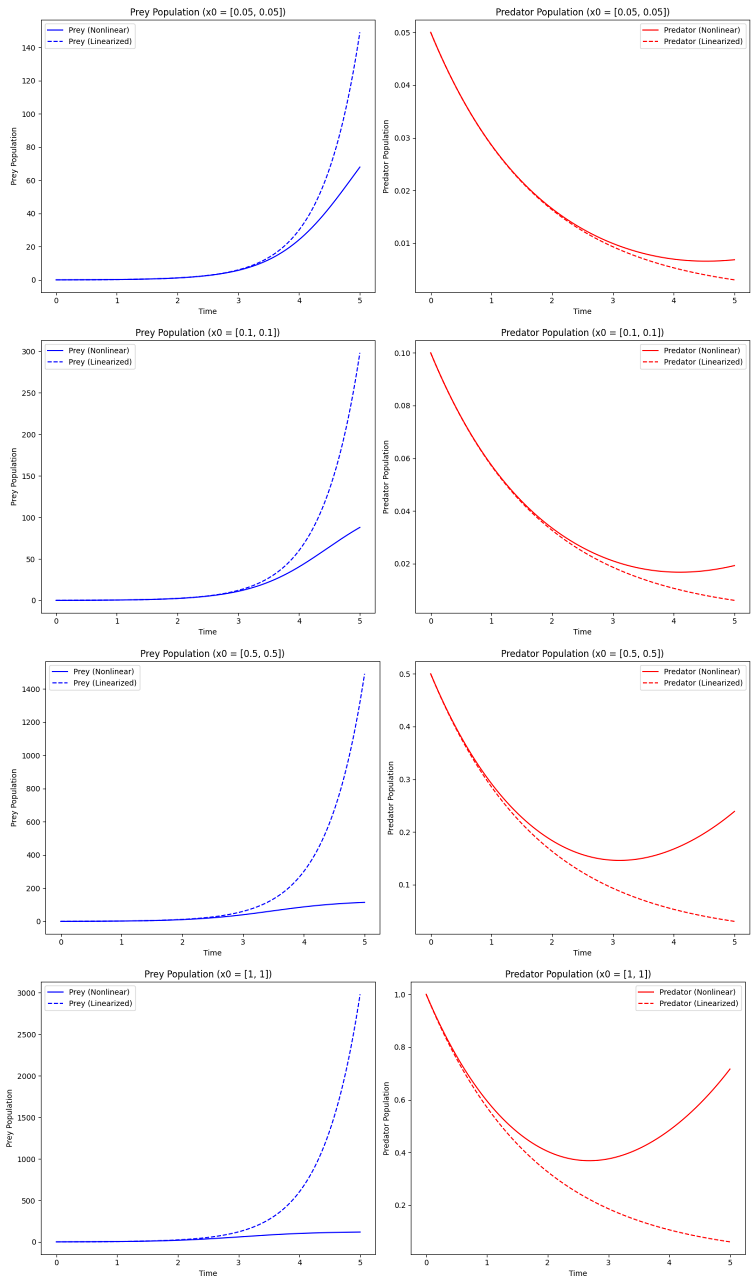
$$J = \begin{bmatrix} r(1-2\frac{(x_1)_0}{k}) - \frac{a(x_2)_0 c}{(c+(x_1)_0)^2} & -\frac{a(x_1)_0}{c+(x_1)_0} \\ b\frac{a(x_2)_0 c}{(c+(x_1)_0)^2} & b\frac{a(x_1)_0}{c+(x_1)_0} - d \end{bmatrix}$$

$$\dot{x}(t) = J \begin{bmatrix} x_1 - (x_1)_0 \\ x_2 - (x_2)_0 \end{bmatrix}$$

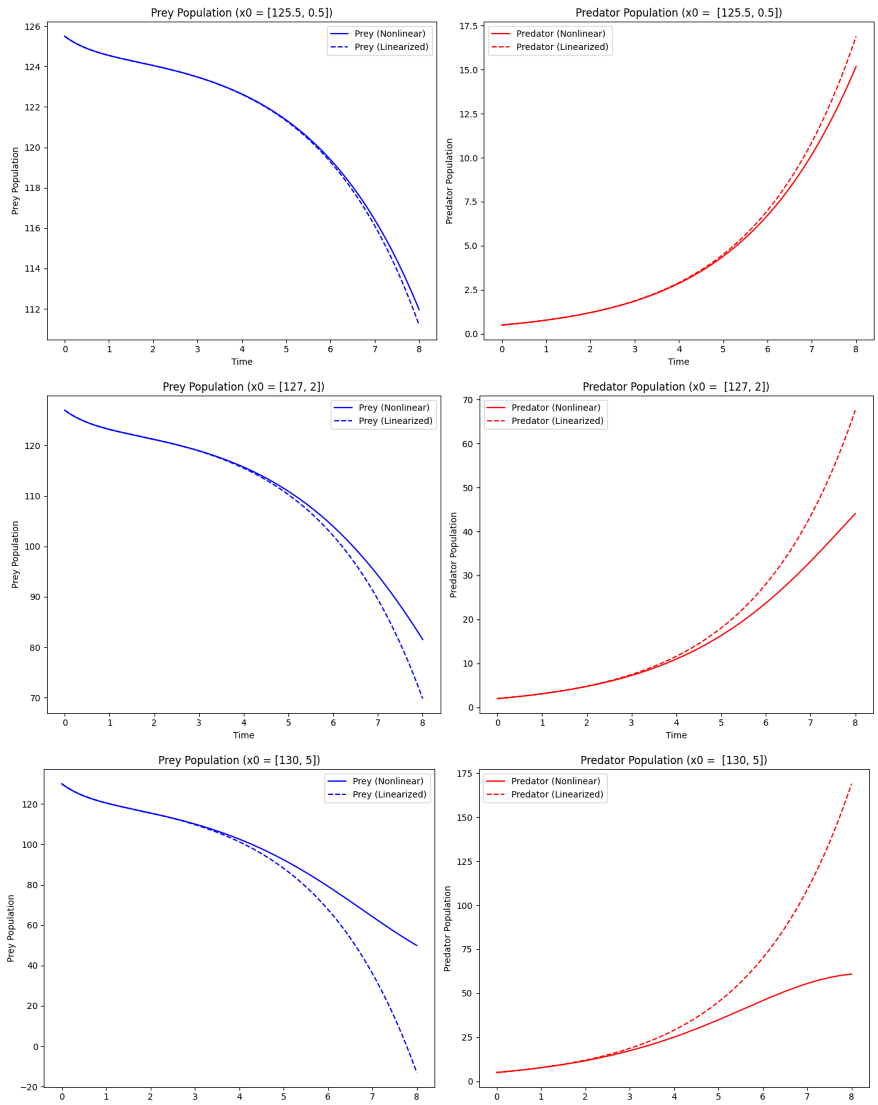
Without the values for the parameters we cannot precisely say whether this equilibrium point is stable or unstable as the eigenvalues would depend on the values of the parameters. If the real part of the eigenvalues of J are all negative then we can say the equilibrium is asymptotically stable, otherwise if any of them are positive then we can say the equilibrium is unstable. If neither one of these is the case, then we can use other methods (e.e Lyapunov functions).

Part B

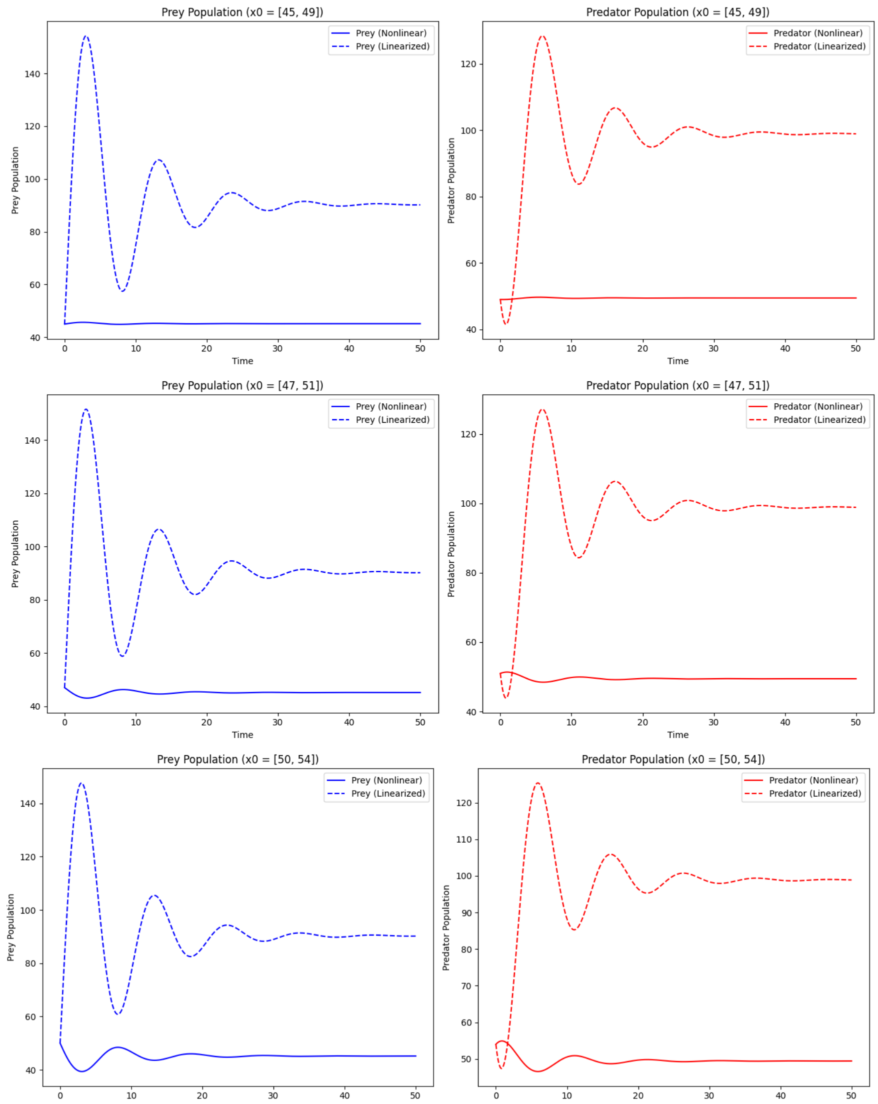
Plots for nonlinear dynamics and linearized dynamics around point (0,0) starting at different initial conditions.



Plots for nonlinear dynamics and linearized dynamics around point (k,0) starting at different initial conditions.



Plots for nonlinear dynamics and linearized dynamics around point $((x_1)_0, (x_2)_0)$ where $(x_2)_0 = \frac{r(c+(x_1)_0)(x_1)_0(1-\frac{(x_1)_0}{k})}{ax_1}$ and $(x_1)_0 = \frac{dc}{ba-d}$ starting at different initial conditions.



Looking at all the plots on the previous 2 pages, we see that the dynamics of the nonlinear system and the linear systems diverge as time progresses. This suggest that the first two points are unstable. From the previous 2 pages, we also see that there is greater divergence as our initial point goes further away from our equilibrium point. The plots on this page, however, indicate that this equilibrium point (where x_1 and x_2 are both nonzero) is stable. As time progresses there seems to be a constant difference between the nonlinear and linearized dynamics (for the nonzero equilibrium point).

Part C

We linearize around equilibrium points to determine the stability of the system around those equilibrium points. This allows us to understand the effects certain parameters may have on our system. With this knowledge, stable systems can be created such that we are able to reach a steady state with parameters set to certain values.

Yes, it is possible to linearize around other points. However, unlike linearizing around equilibrium points, we may only find information for short term time periods whereas linearizing around equilibrium points can tell us the dynamics of a longer time period for some initial condition (near the equilibrium point) (e.g. if a point reaches steady state).

Problem 4

As given we have

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

For $t + \Delta t$ we have:

$$x(t+\Delta t) = e^{A(t+\Delta t)}x_0 + \int_0^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau$$

$$x(t+\Delta t) = e^{A(t+\Delta t)}x_0 + \int_0^t e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau + \int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau$$

$$x(t+\Delta t) = e^{A\Delta t}\left(e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right) + \int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau$$

By substitution we have:

$$x(t+\Delta t) = e^{A\Delta t}x(t) + \int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau$$

Since $u(t)$ is constant over the interval $[t, t+\Delta t]$

$$\int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bu(\tau)d\tau = u(t)\int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bd\tau$$

Rewriting the integral, we have:

$$u(t)\int_t^{t+\Delta t} e^{A(t+\Delta t-\tau)}Bd\tau = u(t)\int_0^{\Delta t} e^{A(\Delta t-\tau)}Bd\tau$$

By substitution we have:

$$x(t+\Delta t) = e^{A\Delta t}x(t) + u(t)\int_0^{\Delta t} e^{A(\Delta t-\tau)}Bd\tau$$

Because we are sampling every Δt then we can denote $x(t) = x(k)$ and $x(t+\Delta t) = x(k+1)$ and since $u(t)$ is constant over the sampling interval $[t, t+\Delta t]$ as stated before we can denote it as $u(t) = u(k)$. Thus the above equation becomes:

$$x(k+1) = e^{A\Delta t}x(k) + u(k)\int_0^{\Delta t} e^{A(\Delta t-\tau)}Bd\tau$$

Let $A_d = e^{A\Delta t}$ and $B_d = \int_0^{\Delta t} e^{A(\Delta t-\tau)}Bd\tau$
Hence we have:

$$x(k+1) = A_d x(k) + B_d u(k)$$

Thus we have shown that the equation above is the discretization of the continuous system where we hold an input constant over a sampling interval. Note that there is no error as no approximations were made and since the control input was constant over the sampling interval, the integral was able to be simplified perfectly without error.