CDS110 **October 15, 2024**

Pset 02

Problem 1

$$Y[k] = C[k] + I[k] + G[k]$$

From the given equation above we have:

$$Y[k+1] = C[k+1] + I[k+1] + G[k+1]$$

Substituting for $C[k+1]$:

$$Y[k+1] = aY[k] + I[k+1] + G[k+1]$$

At equilibrium we have $Y[k+1] = Y[k] = Y_e$. Thus we have:

$$Y_e = aY_e + I[k+1] + G[k+1]$$

At equilibrium, the consumption value C would also reach an equilibrium value such that $C_e = C[k+1] = C[k]$. This is because $C[k+1] = aY[k]$. In other words, since the consumption C increases with GNP (Y), and at equilibrium the GNP stabilizes to some value Y_e, then consumption C would also stabilize to some value $C_e = C[k+1] = C[k]$. Therefore we have:

$$I[k+1] = b(C[k+1] - C[k]) = 0$$

Hence

$$Y_e = aY_e + G_e \implies Y_e(1-a) = G_e \implies Y_e = \frac{1}{1-a}G_e$$

As indicated in the problem with $a = 0.25$ we would have $Y_e = 4G_e$

Now we rewrite $C[k+1]$:

$$C[k+1] = aY[k] = a(C[k] + I[k] + G[k]) = aC[k] + aI[k] + aG[k]$$

Now we rewrite $I[k+1]$ by substituting $C[k+1]$ as written above:

$$I[k+1] = b(C[k+1] - C[k]) = b(aC[k] + aI[k] + aG[k] - C[k])$$

$$I[k+1] = (ab-b)C[k] + abI[k] + abG[k]$$

Thus the model can be written as the following discrete-time state model:

$$\begin{bmatrix} C[k+1] \\ I[k+1] \end{bmatrix} = \begin{bmatrix} a & a \\ ab-b & ab \end{bmatrix} \begin{bmatrix} C[k] \\ I[k] \end{bmatrix} + \begin{bmatrix} a \\ ab \end{bmatrix} G[k]$$

$$Y[k] = C[k] + I[k] + G[k]$$

Problem 2

Part A Rewriting the equation given:

$$m\ddot{q} = -k(q - aq^3) - c\dot{q}$$

$$\ddot{q} = -\frac{k}{m}(q - aq^3) - c\dot{q}$$

To write the system in state space form, let $x_1 = q$ and $x_2 = \dot{q}$. Thus the equation above becomes:

$$\dot{x}_2 = -\frac{k}{m}(x_1 - ax_1^3) - \frac{c}{m}x_2$$

Note that $\dot{x}_1 = \dot{q} = x_2$. Therefore we have the following state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{k}{m}(x_1 - ax_1^3) - \frac{c}{m}x_2 \end{bmatrix}$$

Part B
To find the equilibrium points, first set both derivatives to zero:

$$\dot{x}_1 = 0$$

$$\dot{x}_2 = -\frac{k}{m}(x_1 - ax_1^3) - \frac{c}{m}x_2 = 0$$

From the first equation we see that $\dot{x}_1 = x_2 = 0$. Thus the equilibrium value for x_2 is 0. Substituting 0 for x_2 in $\dot{x}_2$ yields the following:

$$\dot{x}_2 = -\frac{k}{m}(x_1 - ax_1^3) = 0$$

$$\dot{x}_2 = -\frac{k}{m}x_1(1 - ax_1^2) = 0$$

From the equation above, the equilibrium values for x_1 are $0, \sqrt{\frac{1}{a}}, -\sqrt{\frac{1}{a}}$ Therefore the equilibrium points are $(0, 0)$, $(\sqrt{\frac{1}{a}}, 0)$, and $(-\sqrt{\frac{1}{a}}, 0)$.
Part C
The Jacobian J of the system is:

$$J = \begin{bmatrix} \frac{\partial \dot{x}_1}{\partial x_1} & \frac{\partial \dot{x}_1}{\partial x_2} \\ \frac{\partial \dot{x}_2}{\partial x_1} & \frac{\partial \dot{x}_2}{\partial x_2} \end{bmatrix}$$

Solving for the partial derivatives:

$$\frac{\partial \dot{x}_1}{\partial x_1} = \frac{\partial}{\partial x_1}(x_2) = 0$$

$$\frac{\partial \dot{x}_1}{\partial x_2} = \frac{\partial}{\partial x_2}(x_2) = 1$$

$$\frac{\partial \dot{x}_2}{\partial x_1} = \frac{\partial}{\partial x_1}(-\frac{k}{m}(x_1 - ax_1^3) - \frac{c}{m}x_2) = -\frac{k}{m}(1 - 3ax_1^2)$$

$$\frac{\partial \dot{x}_2}{\partial x_2} = \frac{\partial}{\partial x_2}(-\frac{k}{m}(x_1 - ax_1^3) - \frac{c}{m}x_2) = -\frac{c}{m}$$

By substitution:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m}(1 - 3ax_1^2) & -\frac{c}{m} \end{bmatrix}$$

Substituting in the values m=1000,k=250,a=0.01,and c=100 we have:

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{250}{1000}(1 - 3(0.01)x_1^2) & -\frac{100}{1000} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4}(1 - 0.03x_1^2) & -\frac{1}{10} \end{bmatrix}$$

Now linearizing the system at equilibrium point (0,0) we have

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4}(1 - 0.03 \cdot 0^2) & -\frac{1}{10} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = J \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{1}{4}x_1 - \frac{1}{10}x_2 \end{bmatrix}$$

Now linearizing the system at equilibrium point $(\sqrt{\frac{1}{a}}, 0) = (\sqrt{\frac{1}{0.01}}, 0) = (10, 0)$ we have

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4}(1 - 0.03 \cdot 10^2) & -\frac{1}{10} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = J \begin{bmatrix} x_1 - 10 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{2}(x_1 - 10) - \frac{1}{10}x_2 \end{bmatrix}$$

Now linearizing the system at equilibrium point $(-\sqrt{\frac{1}{a}}, 0) = (-\sqrt{\frac{1}{0.01}}, 0) = (-10, 0)$ we have

$$J = \begin{bmatrix} 0 & 1 \\ -\frac{1}{4}(1 - 0.03 \cdot (-10)^2) & -\frac{1}{10} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{10} \end{bmatrix}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = J \begin{bmatrix} x_1 + 10 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{1}{2}(x_1 + 10) - \frac{1}{10}x_2 \end{bmatrix}$$

To determining stability at equilibrium point (0,0), we calculate the eigenvalues of its Jacobian:

$$det(J - \lambda I) = det\left(\begin{bmatrix} -\lambda & 1 \\ -\frac{1}{4} & -\frac{1}{10} - \lambda \end{bmatrix}\right) = 0$$

$$\lambda^2 + \frac{1}{10}\lambda + \frac{1}{4} = \lambda^2 + 0.1\lambda + 0.25 = 0$$

$$\lambda = \frac{-0.1 \pm \sqrt{0.1^2 - 4(0.25)}}{2} = \frac{-0.1 \pm \sqrt{-0.99}}{2}$$

From above we see that the real part for both eigenvalues will be negative. Therefore, the linearization of the system at point (0,0) is asymptotically stable at the equilibrium point (0,0).

Note that the linearization at (10,0) and (-10,0) have the same Jacobian. To determining their stability we calculate the eigenvalues of its Jacobian:

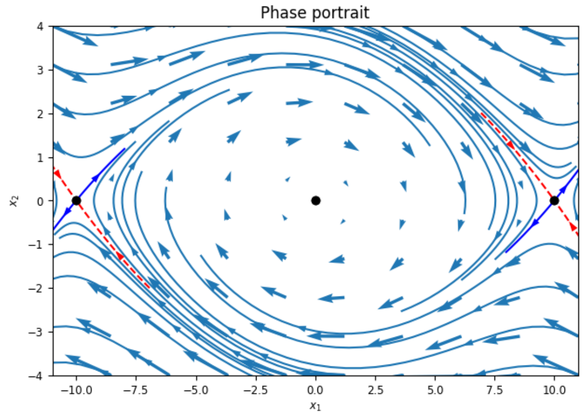
$$det(J - \lambda I) = det\left(\begin{bmatrix} -\lambda & 1 \\ \frac{1}{2} & -\frac{1}{10} - \lambda \end{bmatrix}\right) = 0$$

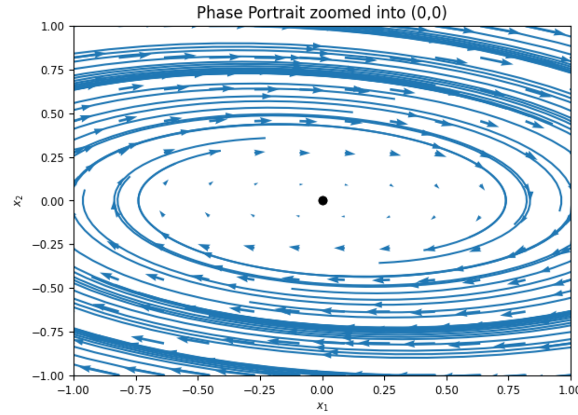
$$\lambda^2 + \frac{1}{10}\lambda - \frac{1}{2} = \lambda^2 + 0.1\lambda - 0.50 = 0$$

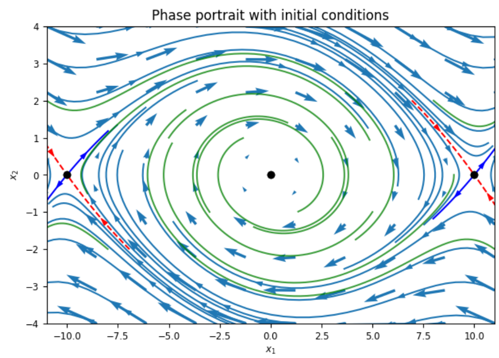
$$\lambda = \frac{-0.1 \pm \sqrt{0.1^2 + 4(0.5)}}{2} = \frac{-0.1 \pm \sqrt{2.01}}{2}$$

From above we see that the real part of one eigenvalues will be positive ($\lambda = \frac{-0.1+\sqrt{2.01}}{2}$). Therefore, the linearization of the system at point (10,0) is unstable at the equilibrium point (10,0) and the linearization of the system at point (-10,0) is unstable at the equilibrium point (-10,0).

Part D

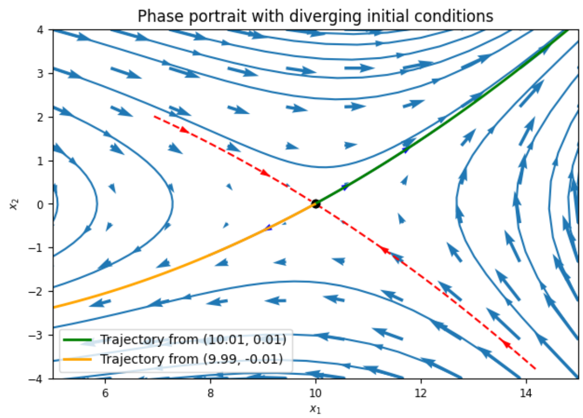


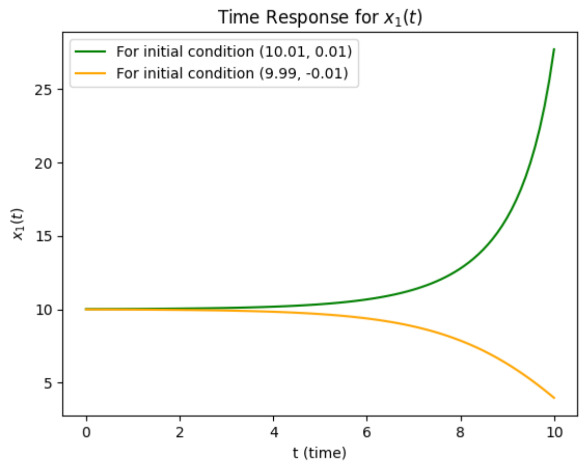


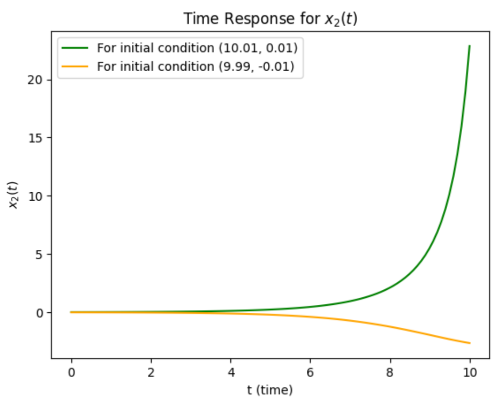


Above are three plots. The one titled "Phase portrait" is the phase Portrait that includes the equilibrium points (black points) separatrices (red dashed line and the the darker blue lines) and streamlines (the lighter blue lines). The one titled a "Phase Portrait zoomed into (0,0)" is the same as the one "Phase portrait" but just zoomed into the equilibrium point (0,0). The plot titled "Phase Portrait with initial conditions" contains the same as the one tilted "Phase portrait" in addition to also including the trajectories from initial conditions (lines are green).

Part E Above are the plots for the time responses and trajectories for both initial conditions (10.01, 0.01)







and (9.99, -0.01). Note that the norm of the difference of the initial conditions is 0.028 which is less than 0.1

Problem 3

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$

Finding the equilibrium points, we set the derivative to zero and solve for x:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) = 0$$

The equation above is true when $x = 0$ or when $1 - \frac{x}{k} = 0 \implies x = k$. Therefore the two equilibrium points are $x = 0$ and $x = k$ assuming both k and r are nonzero.

Now we linearize around the equilibrium point $x = 0$

$$\frac{dx}{dt} = f = rx\left(1 - \frac{x}{k}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(rx\left(1 - \frac{x}{k}\right)\right) = r - \frac{2rx}{k}$$

$$\dot{x} = f(0) + \frac{\partial f}{\partial x}|_{x=0} \cdot (x - 0) = rx$$

Thus the linearizaiton at equilibrium $x = 0$ is $\dot{x} = rx$. Now using the linearization, we see that for $r > 0$, the equilibrium $x = 0$ is unstable as x would grow/increase away from a population of zero whenever x is near 0. For $r < 0$, the equilibrium $x = 0$ is stable as x would decrease towards a population of zero whenever x is near 0.

Now we linearize around the equilibrium point $x = k$

$$\frac{dx}{dt} = f = rx\left(1 - \frac{x}{k}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(rx\left(1 - \frac{x}{k}\right)\right) = r - \frac{2rx}{k}$$

$$\dot{x} = f(k) + \frac{\partial f}{\partial x}|_{x=k} \cdot (x - k) = -r(x - k)$$

Thus the linearizaiton at equilibrium $x = k$ is $\dot{x} = -r(x - k)$. Now using the linearization, we see that for $r > 0$, the equilibrium $x = k$ is stable as x would grow/increase towards a population of k when $x < k$ such that x is near k and x would decrease towards a population of k when $x > k$ such that x is near k. For $r < 0$, the equilibrium $x = k$ is unstable as x would decrease away from population of k when $x < k$ such that x is near k and x would increase/grow away from a population of k when $x > k$ such that x is near k.

Problem 4

Part A
Assume the eigenvectors $v_1, v_2, ..., v_n$ are linearly dependent. So we have that v_{k+1}, where $k+1$ is the vector with the lowest index that can be expressed as a linear combination of vectors $v_1, v_2, ..., v_k$ (that is these vectors from index 1 to k are linearly independent) and where $2 \leq k+1 \leq n$. So we have a solution to the following equation:

$$c_1 v_1 + c_2 v_2 + ... + c_k v_k = v_{k+1} \quad (1)$$

where $c_1, c_2, ..., c_k$ are all constants. Now, applying A to both sides of the equation yields:

$$A(c_1 v_1 + c_2 v_2 + ... + c_k v_k) = A v_{k+1}$$

By definition of eigenvectors and eigenvalues we have $Av_i = \lambda_i v_i$ for all $i \in [1, n]$. Thus the above equation becomes:

$$c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + ... + c_k \lambda_k v_k = \lambda_{k+1} v_{k+1} \quad (2)$$

Multiplying eq (1) by λ_{k+1} we have:

$$c_1 \lambda_{k+1} v_1 + c_2 \lambda_{k+1} v_2 + ... + c_k \lambda_{k+1} v_k = \lambda_{k+1} v_{k+1} \quad (3)$$

Subtracting eq (3) from eq (2) we have:

$$c_1(\lambda_1 - \lambda_{k+1})v_1 + c_2(\lambda_2 - \lambda_{k+1})v_2 + ... + c_k(\lambda_k - \lambda_{k+1})v_k = 0$$

As stated before, the vectors $v_1, v_2, ..., v_k$ are linearly independent. Furthermore, these vectors are all nonzero vectors by definition of an eigenvector. Therefore, the equation above can only be true if every coefficient $c_1(\lambda_1 - \lambda_{k+1}), c_2(\lambda_2 - \lambda_{k+1}), ..., c_k(\lambda_k - \lambda_{k+1})$ is zero. As given by the problem, all eigenvalues are distinct, therefore $\lambda_i - \lambda_{k+1} \neq 0$ for every $i \in [1, k]$. Hence, the only way the equation above can be satisfied is if $c_j = 0$ for every $j \in [1, k]$. However, using eq (1), this would make $v_{k+1} = 0$. Thus, we have a contradiction as v_{k+1} is an eigenvector and is not zero by definition. Therefore, the eigenvectors $v_1, v_2, ..., v_n$ are linearly independent.

Since the eigenvectors $v_1, v_2, ..., v_n$ are linearly independent then they are also distinct. If any two vectors were equal to one another $v_i = v_j$ for $i \neq j$, then $v_1, v_2, ..., v_n$ would be linearly dependent. But as shown above we see that there is a contradiction when we make this assumption.Therefore we have shown that $v_i \neq v_j$ for $i \neq j$.

Part B
From the Basis Theorem of Linear Algebra, we know that any set of n linearly independent vectors forms a basis for R^n. From Part A, we have shown that the eigenvectors $v_1, v_2, ..., v_n$ are linearly independent. Therefore the eigenvectors $v_1, v_2, ..., v_n$ form a basis for R^n. Hence, any vector x can be written as a linear combination of the vectors $v_1, v_2, ..., v_n$ such that for any $x \in R^n$ there exists constants α_i for every $i \in [1, n]$ such that $x = \sum_{i=1}^{i=n} \alpha_i v_i$.

Part C
We are given:

$$T = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$$

Multiplying both sides by A we have

$$AT = A \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} Av_1 & Av_2 & \cdots & Av_n \end{bmatrix}$$

By definition of eigenvector, for all i, we have $Av_i = \lambda_i v_i$ where v_i is the corresponding eigenvector to the eigenvalue λ_i. Thus the above reduces to the follwoing:

$$AT = \begin{bmatrix} \lambda_1 v_1 & \lambda_2 v_2 & \cdots & \lambda_n v_n \end{bmatrix}$$

Rewriting the right side of the equation we have:

$$AT = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Multiplying both sides of the equation by T^{-1} we have:

$$T^{-1}AT = T^{-1}T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Simplifying we have:

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Thus we have shown that $T^{-1}AT$ is a diagonal matrix of the form (5.10) in FBS2e.

Part D

$$z = T^{-1}x \implies \dot{z} = T^{-1}\dot{x}$$

Substituting for $\dot{x}$ we have:

$$\dot{z} = T^{-1}\dot{x} = T^{-1}Ax$$

Because $z = T^{-1}x \implies x = Tz$ then we have:

$$\dot{z} = T^{-1}Ax = T^{-1}ATz$$

Using the result from part c we have:

$$\dot{z} = T^{-1}ATz = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} z$$

Simplifying:

$$\dot{z} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_n z_n \end{bmatrix}$$

Thus we have shown that $\dot{z}_i = \lambda_i z_i$ for all $i \in [1, n]$.