CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110

Problem Set #1

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Problem 1. (30 pts) Some wisdom: In controls, it's important to build intuition about the system. Oftentimes, the problem becomes easier with the right representation.

The human body is a complex biological machine consisting of many interconnected control systems. One such system is the control of blood glucose level, which is described by Figure 1 and the subsection "Insulin–Glucose Dynamics" , Section 4.6 [FBS2e]. Models of varying complexity have been created, but we will consider a very simplified control scheme shown in the diagram below. The following questions do not require a detailed understanding of the biological mechanisms outlined above but rather a high-level understanding of the underlying control scheme.

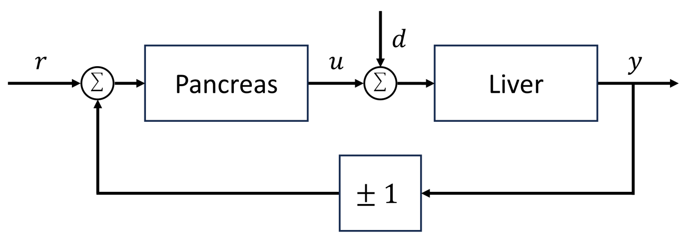

Figure 1: Blood glucose control scheme. Here, r is the desired blood glucose level, u are chemicals secreted into the blood by the pancreas, and y is the output blood glucose level after the liver's response to the secreted chemicals. When the glucose level is high, beta cells in the pancreas secrete insulin, which directs the liver to process excess glucose in the blood supply. When the glucose level is low, alpha cells in the pancreas secrete glucagon, which directs the liver to release its glucose stores. In the shown diagram, d represents disturbances to the secreted chemicals, which may result from causes such as eating, exercise, stress, and more.

(a) Type 1 diabetes is an autoimmune disease where the pancreas's beta cells are destroyed by the immune system. Despite the remaining beta cells working harder to make insulin (to compensate for destroyed cells), the pancreas secretes insufficient insulin. Which of the following functions of the pancreas is *most directly* compromised in this scenario: sensing, computation, or actuation? Provide a short explanation of why.

(b) Type 2 diabetes is a disease where the liver is unable to absorb insulin or use it as effectively, which may be in part due to a reduction in insulin receptors on the surface of cells in the liver. As a result, the liver does not process excess glucose effectively. Which of the following functions of the liver is *most directly* compromised in this scenario: sensing, computation, or actuation? Provide a short explanation of why.

(c) Artificial pancreases are devices that continuously monitor blood glucose levels and secrete insulin and/or glucagon to help regulate glucose in individuals with type 1 diabetes. Consider a hypothetical scenario where one such artificial pancreas is given to an individual, perhaps to someone in the UK (see `https://tinyurl.com/nhs-artificial-pancreas`). Suppose that there is build-up of salts in the tubing that delivers insulin to the individual, and as a result, the individual's blood glucose level is not effectively regulated (thankfully, they receive a cell phone notification and immediately seek medical care). Which of the following functions of the artificial pancreas is *most directly* compromised in this scenario: sensing, computation, or actuation? Provide a short explanation of why.

Problem 2. (30 pts) Let the following differential equation be

$$\ddot{y}(t) + 4\dot{y}(t) + 8y(t) = 2\dot{u}(t) + 3u(t) \tag{1}$$

where $u(t) : \mathbb{R} \to \mathbb{R}$ is a control input and $y(t) : \mathbb{R} \to \mathbb{R}$ is the output function. Define

$$u_\epsilon(t) = \begin{cases} \frac{1}{\epsilon}, & t \in [0, \epsilon] \\ 0, & \text{otherwise} \end{cases}$$

and let $y_\epsilon(t)$ be the solution to the differential equation when $u = u_\epsilon$.

(a) Find

$$\lim_{\epsilon \to 0} y_\epsilon(t).$$

Hint: Don't compute "directly" the solution to the differential equation with $u = u_\epsilon$. Recall the Laplace transform from MA2 (or other similar courses). Assume $y(0) = 0$ and $\dot{y}(0) = 0$.

(b) Now consider the case where $u(t) = 0$ and the output is directly the state ($y = x$), so that the system is now unforced:

$$\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 0 \tag{2}$$

i. Write out the system in (2) in the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with the state $\mathbf{x} = [x, \dot{x}]$.

ii. Compute the eigenvalues for (2). How do the eigenvalues relate to the solution of part a?

Problem 3. (30 pts) More wisdom: In controls, it's important to write high quality code.

In this problem, you will implement a Python simulation of 3 systems using Euler integration. For each system (A, B, and C), you will simulate and plot the time evolution of the state vector $x(t) = [x_1(t), x_2(t)]$, of the control input $u(t)$, and of the output $y(t)$. Note the different initial conditions for each subsystem.

You will observe that all three systems exhibit some form of abnormal behavior. Later in the course, we will revisit this exercise, and try to understand what is the underlying cause of the misbehaving. For now, provide your best interpretation of the plots (qualitative). You can think about how the solutions to the differential equations behave for different initial conditions.

Hint: Use a small integration ΔT, for example 0.001. For System C, you will notice that the selection of ΔT matters in the behaviour of the system.

Deliverable: Google Colab code link and plots in the submission. Qualitative discussion.

Expectations: We will drop points if the code is not cleanly written (i.e., poor formatting, unclear naming, etc.). This will apply throughout the entire class homework when evaluating code.

System A.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

with a unit step input (3) and initial conditions $x_0 = [1, 0]$, as well as another initial condition $x_0 = [0, 0]$.

System B.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

with the unit step input

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

and the initial condition $x = [0, 0]$.

System C.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with the control input

$$u(t) = \begin{cases} e^t, & t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

and initial condition $x_0 = [0, 0]$.