CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 110

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Problem 1. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}u = Ax + Bu, \qquad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = Cx \tag{1}$$

(a) Determine the stability of the open loop system, and provide your reasoning.

(b) Compute the reachability matrix W_r and justify why the closed loop system can be made asymptotically stable.

We aim to create a stable closed loop system with the characteristic polynomial $\lambda(s) = s^2+2\zeta_0\omega_0 s+\omega_0^2$ and with unit static gain (steady-state output $y = r$). We will achieve this by applying a control law $u = -Kx + k_\mathrm{f} r$, consisting of state feedback matrix $K = [k_1, k_2]$ and feedforward gain k_f.

(c) In order to do so, first determine the characteristic polynomial of the closed loop system and find the values of k_1 and k_2 that give the desired characteristic polynomial.

(d) Next, compute the equilibrium point x_e and the steady-state output y_e to determine the value of k_f that gives a closed loop system with unit static gain.

(e) Determine the values of ζ_0 and ω_0 for which the closed loop system is asymptotically stable.

Problem 2. Two-Link robot control. Consider a robot (Figure 1) consisting of two rigid links, with the anchor joint free and the elbow joint actuated. Let $\theta = (\theta_1, \theta_2)$ be the vector of joint angles. The equations of motion are then given below:

$$M(\theta)\ddot{\theta} + C(\theta,\dot{\theta})\dot{\theta} + G(\theta) = \begin{bmatrix}0\\\tau\end{bmatrix} \tag{2}$$

where

$$M(\theta,\dot{\theta}) = \begin{bmatrix}a+b+2c\cos(\theta_2) & b+c\cos(\theta_2)\\ b+c\cos(\theta_2) & b\end{bmatrix} \tag{3}$$

$$C(\theta,\dot{\theta}) = \begin{bmatrix}-c\sin(\theta_2)\dot{\theta}_2 & -c\sin(\theta_2)(\dot{\theta}_1+\dot{\theta}_2)\\ c\sin(\theta_2)\dot{\theta}_1 & 0\end{bmatrix} \tag{4}$$

$$G(\theta) = \begin{bmatrix}-d\sin(\theta_1) - e\sin(\theta_1+\theta_2)\\ -e\sin(\theta_1+\theta_2)\end{bmatrix} \tag{5}$$

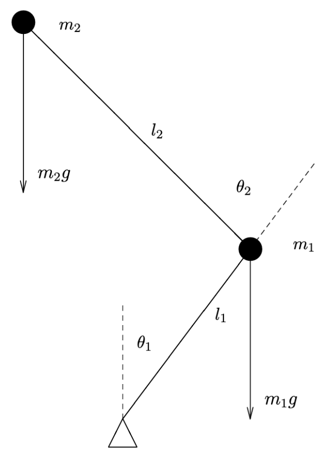

Figure 1: Two-link robot. Note that the actuator is present in the elbow joint.

where

$$\begin{aligned}
a &= m_1 l_1^2 + m_2 l_1^2 \\
b &= m_2 l_2^2 \\
c &= m_2 l_1 l_2 \\
d &= g m_1 l_1 + g m_2 l_1 \\
e &= g m_2 l_2
\end{aligned} \tag{6}$$

Let the system parameters be given as

m_1	m_2	l_1	l_2	g
0.5	1	8	8	10

(a) Let the state vector be

$$x := \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

. Let the control input $u := \tau$ Show that the system can be written in *control-affine form*:

$$\dot{x} = f(x) + g(x)u \tag{7}$$

(b) Given a constant bias torque input $\tau = \tau_0$, determine the set of equilibrium points. Plot these equilibrium points in the θ phase space.

(c) Linearize the system about the vertical equilibrium point ($\theta_1 = \theta_2 = 0$). Is this equilibrium point stable? Is the linearized system controllable?

(d) Now consider the set of equilibrium points determined in part (b). Is the linearized system always controllable? (Hint: plot the determinant of the controllability matrix as a function of θ_2^*). Give intuition to this result - why does this make sense for the physical system?

(e) Design a stabilizing controller of the form $\tau = -Kx$. The linearized system should meet the following specifications: From an initial condition of $(\theta_1 = 0.1, \theta_2 = 0.0)$,

- the 5% settling time for θ_1 should be less than 2 seconds.
- the maximum angle θ_1 before settling should not exceed 1 rad.

(f) Now apply your controller to the nonlinear system, and plot the result alongside the linear solution. Does the system still meet the specifications?

(g) If you were to implement this procedure on a real robot, you might run into several practical issues. For each of the following, briefly describe their effect on the system and how you might modify the control synthesis procedure to account for them.

- Your actuator would have a maximum torque, limiting the control authority.
- Your control system would be implemented digitally, and must be recomputed at discrete intervals.
- Your actuator might have an unknown bias torque (i.e., $u = \tau + d$).

Problem 3. Consider the system

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

with the control law

$$u = -k_1 x_1 - k_2 x_2 + k_\mathrm{f} r.$$

(a) Compute the rank of the reachability matrix for the system and show that it is not reachable.

(b) Compute the characteristic polynomial of the closed loop system and show that the eigenvalues of the system cannot be assigned to arbitrary values.

(Make sure to show your work to receive full credit.)
