1 Some theory

Transformation A and after that B, with initial coordinates as input and the final coordinates as output.

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = \begin{pmatrix} B \end{pmatrix} \begin{pmatrix} A \end{pmatrix} \begin{pmatrix} x_{initial} \\ y_{initial} \\ z_{initial} \\ 1 \end{pmatrix} \tag{1.1}$$

Move the point:

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{initial} \\ y_{initial} \\ z_{initial} \\ 1 \end{pmatrix} \tag{1.2}$$

To move the coordinate system, not the points, just take negative value. Rotate the point for α around x axis:

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{initial} \\ y_{initial} \\ z_{initial} \\ 1 \end{pmatrix} \tag{1.3}$$

Rotate the point for α around y axis:

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & 1 & 0 & 0 \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{initial} \\ y_{initial} \\ z_{initial} \\ 1 \end{pmatrix} \tag{1.4}$$

Rotate the point for α around z axis:

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{initial} \\ y_{initial} \\ z_{initial} \\ 1 \end{pmatrix} \tag{1.5}$$

To rotate the coordinate system, not the points, just transpose the matrix.

2 Input data

Let the inputs be $X, Y, Z, phi_x, phi_y, phi_z$

The point X, Y, Z we are inputing is not essentially the middle of the triangle, but rather the instrument tip. The tip offset related to the center of the triangle is equal to $X_{instrument}$, $Y_{instrument}$ and $Z_{instrument}$.

a	b	h
6.3	31	15

Table 1: Linear dimensions

	1	2	3	4	5	6
x	2.6	0.5	-3.1	-3.1	0.5	2.6
y	2.1	3.3	1.2	-1.2	-3.3	-2.1
z	0	0	0	0	0	0

Table 2: Triangle dimensions

	16	23	45
x	-7.9	3.95	3.95
y	0	6.84	-6.84
z	-3.14	-3.14	-3.14

Table 3: The "intersection origin" points

3 Main part

The first step is to determine the coordinates of the points 1 through 6. For that, we first move the triangle points so that the instrument tip coincides with the origin [A], then we rotate all the points as desired [B] and move them so the tip gets to the desired point [C]. Therefore:

$$\begin{pmatrix} x_{final} \\ y_{final} \\ z_{final} \\ 1 \end{pmatrix} = (C)\,(B)\,(A) \begin{pmatrix} x_{triangle} \\ y_{triangle} \\ z_{triangle} \\ 1 \end{pmatrix} \tag{3.1}$$

$$(A) = \begin{pmatrix} 1 & 0 & 0 & -X_{instrument} \\ 0 & 1 & 0 & -Y_{instrument} \\ 0 & 0 & 1 & -Z_{instrument} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

$$(B) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_x & -\sin\phi_x & 0 \\ 0 & \sin\phi_x & \cos\phi_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi_y & 0 & -\sin\phi_y & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi_y & 0 & \cos\phi_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi_z & -\sin\phi_z & 0 & 0 \\ \sin\phi_z & \cos\phi_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.3}$$

$$(C) = \begin{pmatrix} 1 & 0 & 0 & X \\ 0 & 1 & 0 & Y \\ 0 & 0 & 1 & Z \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.4}$$

Now we need to find the middle points between 1 and 6, 2 and 3, 4 and 5.

$$x_{middle} = \frac{x_1 + x_2}{2} \tag{3.5}$$

Let (l, m, n) be the coordinates for each of the found points. To use them further in the calculation, we need to move to the new "intersection point" coordinate system. For that we just need to do a move of origin [D]

$$(D) = \begin{pmatrix} 1 & 0 & 0 & -x_{intersection} \\ 0 & 1 & 0 & -y_{intersection} \\ 0 & 0 & 1 & -z_{intersection} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.6}$$

Let c be close to b*.

$$x^2 + y^2 + z^2 = h^2 \tag{3.7}$$

$$(x - l)^2 + (y - m)^2 + (z - n)^2 = c^2 \tag{3.8}$$

$$y = 0 \tag{3.9}$$

Substitute y = 0

$$x^2 + z^2 = h^2 \tag{3.10}$$

$$(x - l)^2 + m^2 + (z - n)^2 = c^2 \tag{3.11}$$

Isolate the constants

$$z^2 = -x^2 + h^2 \tag{3.12}$$

$$x^2 - 2xl + z^2 - 2nz = c^2 - l^2 - m^2 - n^2 \tag{3.13}$$

Substitute z^2

$$-x^2 - 2zn + x^2 - 2xl = c^2 - l^2 - m^2 - n^2 - h^2 \tag{3.14}$$

$$-2zn - 2xl = c^2 - l^2 - m^2 - n^2 - h^2 \tag{3.15}$$

Substitute x

$$-l\sqrt{h^2 - x^2} - xl = \frac{c^2 - l^2 - m^2 - n^2 - h^2}{2} = K \tag{3.16}$$

$$\sqrt{h^2 - x^2} = -\frac{K + xl}{l} \tag{3.17}$$

$$l^2h^2 - l^2x^2 = K^2 + 2Kxl + n^2x^2 \tag{3.18}$$

$$l^2h^2 - K^2 = 2Kxl + n^2x^2 + l^2x^2 \tag{3.19}$$

$$(n^2 + l^2)x^2 + 2Klx - l^2h^2 + K^2 = 0 \tag{3.20}$$

Discriminant

$$D = 4K^2n^2 - 4(n^2 + l^2)(K^2 - l^2h^2) \tag{3.21}$$

$$D' = K^2n^2 - (n^2 + l^2)(K^2 - l^2h^2) \tag{3.22}$$

We need the outer solution, therefore the bigger x is our aim:

$$x = \frac{-Kl + \sqrt{D'}}{(n^2 + l^2)} \tag{3.23}$$

Substitute that for z:

$$z = \sqrt{h^2 - x^2} \tag{3.24}$$

Yay :)
To find out an angle, we will use two vectors: a vector $\vec{A}$ with coordinates $1, 0, 0$ and vector $\vec{B}$ with newly found coordinates.

$$\vec{A} \cdot \vec{B} = |A||B| \cos\theta \tag{3.25}$$

$$\theta = \arccos(\frac{x_A x_B + y_A y_B + z_A z_B}{\sqrt{x_A^2 + y_A^2 + z_A^2}\sqrt{x_B^2 + y_B^2 + z_B^2}}) \tag{3.26}$$

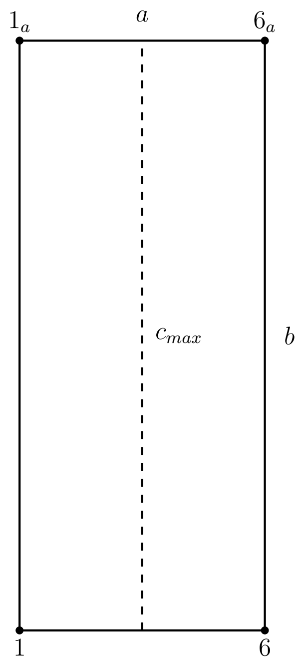
Simplify that a bit

$$\theta = \arccos(\frac{x_B}{\sqrt{x_B^2 + z_B^2}}) \tag{3.27}$$

We use the right-hand rule to determine angles, so add a minus sign if $z_B < 0$!

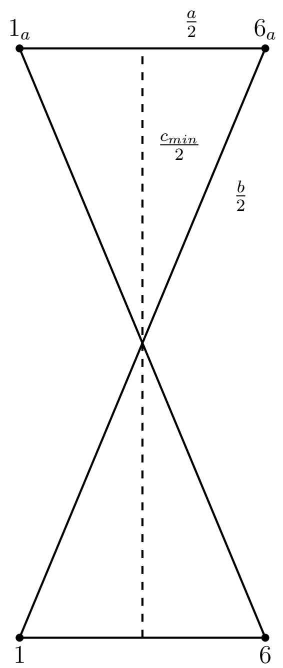
4 Comments

* What is c, exactly? Take a quadrilateral 1_a6_a61, for example. c is the distance between two midpoints: between 1_a and 6_a and between 6 and 1 The quadrilateral we are facing is a skewed one, therefore its distance between midpoints is not constant and mostly hard to evaluate, but we can look at the extreme cases:



In this case it is just a flat parallelogram, therefore the distance between midpoints is equal to b. Therefore,

$$c_{max} = b \tag{4.1}$$



In this case $\frac{c_{min}}{2}$, $\frac{b}{2}$ and $\frac{a}{2}$ make up a right-angled triangle, so

$$c_{min} = \sqrt{b^2 - a^2} \tag{4.2}$$

How big is the difference between c_{min} and c_{max}? We can get a feeling for that looking at $\left|\frac{c_{max}-c_{min}}{c_{min}}\right|$:

$$\left|\frac{c_{max} - c_{min}}{c_{min}}\right| = \left|\frac{b - \sqrt{b^2 - a^2}}{b}\right| = \left|\frac{b - b\sqrt{1^2 - \frac{a}{b}^2}}{b}\right| = \left|1 - \sqrt{1 - (\frac{a}{b})^2}\right| \tag{4.3}$$