Math Specification for EMTG-182: Constraint in Two-Body-Rotating Reference Frame

Noble Hatten

June 15, 2023

Abstract

This document is a mathematical specification for ticket EMTG-182: a boundary state constraint expressed in a two-body-rotating reference frame (e.g., the restricted three-body problem rotating frame).

Contents

1 Scenario Statement

The scenario under consideration consists of two EMTG Universe bodies B_1 and B_2, and a spacecraft s. B_1 and B_2 can be either the central body of the propagation or any body defined in the Universe of the central body. The geometry of the scenario is described as follows:

- $\boldsymbol{r}$ is the position vector of s with respect to the central body in the inertial frame. This is part of the state vector.

- $\boldsymbol{v}$ is the inertial velocity vector of s with respect to the central body in the inertial frame. This is part of the state vector.

- $\boldsymbol{r}_1$ is the position vector of B_1 with respect to the central body in theinertial frame. This is an explicit function of time only.

- $\boldsymbol{v}_1$ is the inertial velocity vector of B_1 with respect to the central body in the inertial frame. This is an explicit function of time only.

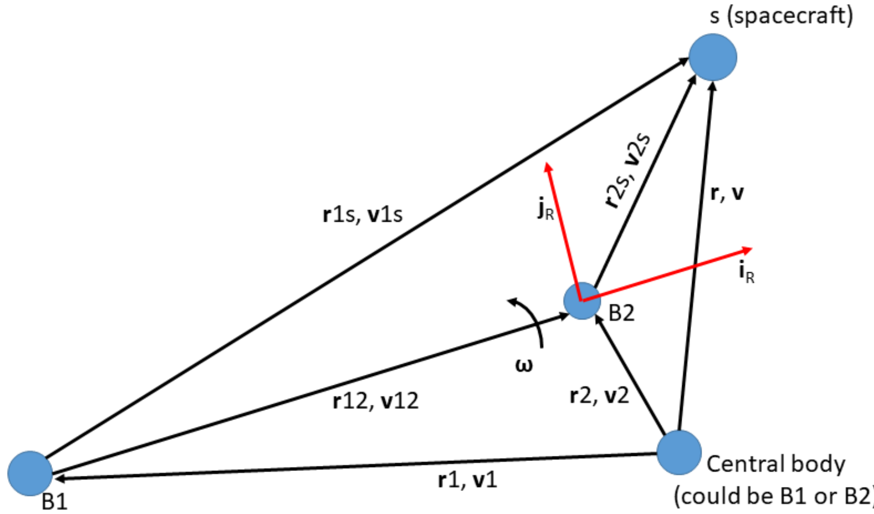

Figure 1: Geometry of the scenario.

- $\boldsymbol{r}_2$ is the position vector of B_2 with respect to the central body in the inertial frame. This is an explicit function of time only.
- $\boldsymbol{v}_2$ is the inertial velocity vector of B_2 with respect to the central body in the inertial frame. This is an explicit function of time only.
- $\boldsymbol{r}_{12} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ is the inertial position vector from B_1 to B_2 in the inertial frame. This is an explicit function of time only.
- $\boldsymbol{v}_{12} = \boldsymbol{v}_2 - \boldsymbol{v}_1$ is the inertial velocity vector of B_2 with respect to B_1 in the inertial frame. This is an explicit function of time only.
- $\boldsymbol{r}_{1s} = \boldsymbol{r}_s - \boldsymbol{r}_1$ is the inertial position vector from B_1 to s in the inertial frame.
- $\boldsymbol{v}_{1s} = \boldsymbol{v}_s - \boldsymbol{v}_1$ is the inertial velocity vector of s with respect to B_1 in the inertial frame.
- $\boldsymbol{r}_{2s} = \boldsymbol{r}_s - \boldsymbol{r}_2$ is the inertial position vector from B_2 to s in the inertial frame.
- $\boldsymbol{v}_{2s} = \boldsymbol{v}_s - \boldsymbol{v}_2$ is the inertial velocity vector of s with respect to B_2 in the inertial frame.
- $\boldsymbol{\omega}$ is the instantaneous angular velocity vector of B_2 with respect to B_1. This is an explicit function of time only.

In practice, the inertial frame is the ICRF.
The geometry is displayed graphically in Figure 1.

2 Two-Body Rotating Frame

The two-body rotating frame is defined such that the origin is coincident with B_2, and the axes are:

- $\hat{\boldsymbol{i}}_R = \hat{\boldsymbol{r}}_{12}$
- $\hat{\boldsymbol{k}}_R = \hat{\boldsymbol{\omega}}$

- $\hat{\boldsymbol{j}}_R$ completes the right-handed set. I.e., $\hat{\boldsymbol{j}}_R = \hat{\boldsymbol{k}}_R \times \hat{\boldsymbol{i}}_R$.

A subscript R is used to indicate that a vector is expressed in the two-body rotating frame.

2.1 Angular Velocity of Body 2 with Respect to Body 1

The magnitude of the angular velocity is the time derivative of the angular displacement of B_2 about B_1, and the direction of the angular velocity vector is in the direction of the angular momentum of B_2 about B_1. For brevity, redefine $\boldsymbol{r} \triangleq \boldsymbol{r}_{12}$ and $\boldsymbol{v} \triangleq \boldsymbol{v}_{12}$ in this section only for the purposes of this derivation.

Common knowledge:

$$\dot{\nu} = \frac{h}{r^2} \tag{1}$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \tag{2}$$

So:

$$\begin{aligned}
\boldsymbol{\omega} &= \dot{\nu}\hat{\boldsymbol{h}} && (3)\\
&= \frac{h}{r^2}\frac{\boldsymbol{h}}{h} && (4)\\
&= r^{-2}\boldsymbol{h} && (5)\\
&= r^{-2}\left(\boldsymbol{r} \times \boldsymbol{v}\right) && (6)
\end{aligned}$$

We also require the derivatives of $\boldsymbol{\omega}$. $\boldsymbol{\omega}$ is an explicit function of time only because it only depends on the positions and velocities of B_1 and B_2, which are ephemeris lookups.

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} &= \frac{\mathrm{d}\left(r^{-2}\boldsymbol{h}\right)}{\mathrm{d}t} && (7)\\
&= \frac{\mathrm{d}r^{-2}}{\mathrm{d}t}\boldsymbol{h} + r^{-2}\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} && (8)
\end{aligned}$$

Then:

$$\begin{aligned}
\frac{\mathrm{d}r^{-2}}{\mathrm{d}t} &= -2r^{-3}\frac{\mathrm{d}r}{\mathrm{d}t} && (9)\\
\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{\partial r}{\partial \boldsymbol{r}}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} && (10)\\
&= \frac{\boldsymbol{r}^T}{r}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} && (11)\\
\to \frac{\mathrm{d}r^{-2}}{\mathrm{d}t} &= \frac{-2}{r^4}\boldsymbol{r}^T\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} && (12)
\end{aligned}$$

Important note: $\boldsymbol{v}$ is not substituted for $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ here because these two quantities are not necessarily equal, depending on how the ephemeris is implemented.

We also have

$$\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}}\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \tag{13}$$

where:

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{r}} = -\{\boldsymbol{v}\}^{\times} \tag{14}$$
$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{v}} = \{\boldsymbol{r}\}^{\times} \tag{15}$$
$$\tag{16}$$

Like $\boldsymbol{r}$ and $\boldsymbol{v}$, $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$ and $\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$ are ephemeris lookups. Again, note that $\boldsymbol{v}$ is not substituted for $\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}$.

2.2 Transformation from ICRF to Two-Body Rotating Frame

To express a generic vector given in frame I in the two-body rotating frame R, we have

$$\boldsymbol{r}_R = \boldsymbol{R}^{I\to R}\boldsymbol{r}_I \tag{17}$$

From the dot product definition of the direction cosine matrix, we get

$$\boldsymbol{R}^{I\to R} = \begin{bmatrix}\hat{\boldsymbol{i}}_R & \hat{\boldsymbol{j}}_R & \hat{\boldsymbol{k}}_R\end{bmatrix}^T, \tag{18}$$

where the unit vectors are defined in Section 2 and the quantities expressed in the I frame.

2.2.1 Time Derivative of Transformation Matrix

The time derivative of $\boldsymbol{R}^{I\to R}$ is obtained by differentiating the unit vectors of the rotating frame with respect to time.

$$\dot{\hat{\boldsymbol{i}}}_R = \frac{\partial \hat{\boldsymbol{i}}_R}{\partial \boldsymbol{r}_{12}}\frac{\mathrm{d}\boldsymbol{r}_{12}}{\mathrm{d}t} \tag{19}$$

$\frac{\mathrm{d}\boldsymbol{r}_{12}}{\mathrm{d}t}$ is an ephemeris lookup (but not $\boldsymbol{v}_{12}$!). Since $\hat{\boldsymbol{i}}_R = \hat{\boldsymbol{r}}_{12}$, we use the well-known derivative of a unit vector with respect to its non-unitized vector:

$$\frac{\partial \hat{\boldsymbol{r}}_{12}}{\partial \boldsymbol{r}_{12}} = \frac{1}{r_{12}} \left(\boldsymbol{I} - \frac{1}{r_{12}^2} \boldsymbol{r}_{12} \boldsymbol{r}_{12}^T \right) \tag{20}$$

$\dot{\hat{\boldsymbol{k}}}_R$ is similar.

$$\dot{\hat{\boldsymbol{k}}}_R = \frac{\partial \hat{\boldsymbol{k}}_R}{\partial \boldsymbol{\omega}} \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{21}$$

$$= \frac{1}{\omega} \left(\boldsymbol{I} - \frac{1}{\omega^2} \boldsymbol{\omega} \boldsymbol{\omega}^T \right) \frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{22}$$

$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}$ is obtained from the equations in Section 2.1.

For $\dot{\hat{\boldsymbol{j}}}_R$, we can use

$$\dot{\hat{\boldsymbol{j}}}_R = \frac{\partial \hat{\boldsymbol{j}}_R}{\partial \hat{\boldsymbol{i}}_R} \dot{\hat{\boldsymbol{i}}}_R + \frac{\partial \hat{\boldsymbol{j}}_R}{\partial \hat{\boldsymbol{k}}_R} \dot{\hat{\boldsymbol{k}}}_R \tag{23}$$

Since $\hat{\boldsymbol{j}}_R = \hat{\boldsymbol{k}}_R \times \hat{\boldsymbol{i}}_R$,

$$\frac{\partial \hat{\boldsymbol{j}}_R}{\partial \hat{\boldsymbol{i}}_R} = \left\{ \hat{\boldsymbol{k}}_R \right\}^\times \tag{24}$$

$$\frac{\partial \hat{\boldsymbol{j}}_R}{\partial \hat{\boldsymbol{k}}_R} = -\left\{ \hat{\boldsymbol{i}}_R \right\}^\times \tag{25}$$

2.2.2 Position with Respect to Body 2

The specific position quantity we wish to constrain is the position of the spacecraft relative to Body 2, expressed in the rotating frame: $\boldsymbol{r}_{2s,R}$. Using Eq. (17), we have

$$\boldsymbol{r}_{2s,R} = \boldsymbol{R}^{I \to R} \boldsymbol{r}_{2s,I} \tag{26}$$

$$= \boldsymbol{R}^{I \to R} \left(\boldsymbol{r}_{s,I} - \boldsymbol{r}_{2,I} \right) \tag{27}$$

$\boldsymbol{r}_{s,I}$ is made up only of elements of the state vector (i.e., it is not an explicit function of time), while $\boldsymbol{r}_{2,I}$ is *only* a function of time. We require the derivatives of $\boldsymbol{r}_{2s,R}$ with respect of our independent variables. The relevant independent variables are the position state $\boldsymbol{r}_{s,I}$ and time t. Differentiating with respect to $\boldsymbol{r}_{s,I}$ gives

$$\frac{\partial \boldsymbol{r}_{2s,R}}{\partial \boldsymbol{r}_{s,I}} = \boldsymbol{R}^{I \to R} \tag{28}$$

Differentiating with respect to t gives

$$\frac{\partial \boldsymbol{r}_{2s,R}}{\partial t} = \dot{\boldsymbol{R}}^{I\to R}\boldsymbol{r}_{s,I} - \left[\dot{\boldsymbol{R}}^{I\to R}\boldsymbol{r}_{2,I} + \boldsymbol{R}^{I\to R}\frac{\mathrm{d}\boldsymbol{r}_{2,I}}{\mathrm{d}t}\right] \tag{29}$$

All elements of Eq. (29) except $\boldsymbol{r}_{s,I}$ are explicit functions of time only. $\frac{\mathrm{d}\boldsymbol{r}_{2,I}}{\mathrm{d}t}$ is obtained from ephemeris lookups. $\dot{\boldsymbol{R}}^{I\to R}$ is obtained by differentiating the unit vectors of the rotating frame with respect to time.

2.2.3 Velocity with Respect to Body 2

For the velocity vector, we must differentiate, in the colloquial sense, between velocities relative to the inertial frame and the rotating frame. This difference is denoted with a preceding superscript: I indicates the derivative is taken with respect to the inertial frame, and R indicates that the derivative is taken with respect to the rotating frame.

The velocity of the spacecraft relative to the rotating frame centered at B_2, expressed in the inertial frame, is

$$^{R}\boldsymbol{v}_{2s,I} = {}^{I}\boldsymbol{v}_{2s,I} - \boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I} \tag{30}$$

We wish to express this velocity in the rotating frame, so we use our transformation matrix:

$$^{R}\boldsymbol{v}_{2s,R} = \boldsymbol{R}^{I\to R}\left({}^{I}\boldsymbol{v}_{2s,I} - \boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right) \tag{31}$$

We also require the derivatives of ${}^{R}\boldsymbol{v}_{2s}$. ${}^{R}\boldsymbol{v}_{2s}$ depends on both time and on the spacecraft state.

The derivative of ${}^{R}\boldsymbol{v}_{2s,R}$ with respect to an arbitrary variable x is

$$\frac{\partial {}^{R}\boldsymbol{v}_{2s,R}}{\partial x} = \frac{\partial \boldsymbol{R}^{I\to R}}{\partial x}\left({}^{I}\boldsymbol{v}_{2s,I} - \boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right) + \boldsymbol{R}^{I\to R}\frac{\partial\left({}^{I}\boldsymbol{v}_{2s,I} - \boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial x} \tag{32}$$

As before, the derivatives of $\boldsymbol{R}^{I\to R}$ is obtained by differentiating the unit vectors of the rotating frame with respect to time. $\boldsymbol{R}^{I\to R}$ depends only on the location of B_2 with respect to B_1 and is therefore a function of time only. As a result, $\frac{\partial \boldsymbol{R}^{I\to R}}{\partial x}$ is obtained from ephemeris lookups if $x = t$ and 0 otherwise.

For the second term on the right-hand side of Eq. 32, we obtain

$$\frac{\partial\left({}^{I}\boldsymbol{v}_{2s,I} - \boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial x} = \frac{\partial\left({}^{I}\boldsymbol{v}_{2s,I}\right)}{\partial x} - \frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial x} \tag{33}$$

where

$$\frac{\partial\left({}^{I}\boldsymbol{v}_{2s,I}\right)}{\partial x} = \frac{\partial\left({}^{I}\boldsymbol{v}_{s,I}\right)}{\partial x} - \frac{\partial\left({}^{I}\boldsymbol{v}_{2,I}\right)}{\partial x} \tag{34}$$

in which $\frac{\partial\left({}^{I}\boldsymbol{v}_{s,I}\right)}{\partial x} = \boldsymbol{I}$ if $x = {}^{I}\boldsymbol{v}_{s,I}$ and is 0 otherwise. $\frac{\partial\left({}^{I}\boldsymbol{v}_{2,s}\right)}{\partial x}$ is an ephemeris lookup ($\dot{\boldsymbol{v}}_2$) if $x = t$ and is 0 otherwise.

Finally,

$$\frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial x} = \frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial \boldsymbol{\omega}_I}\frac{\partial \boldsymbol{\omega}_I}{\partial x} + \frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial \boldsymbol{r}_{2s,I}}\frac{\partial \boldsymbol{r}_{2s,I}}{\partial x} \tag{35}$$

$$\frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial \boldsymbol{\omega}_I} = -\left\{\boldsymbol{r}_{2s,I}\right\}^{\times} \tag{36}$$

$$\frac{\partial\left(\boldsymbol{\omega}_I \times \boldsymbol{r}_{2s,I}\right)}{\partial \boldsymbol{r}_{2s,I}} = \left\{\boldsymbol{\omega}_I\right\}^{\times} \tag{37}$$

$$\frac{\partial \boldsymbol{r}_{2s,I}}{\partial x} = \frac{\partial \boldsymbol{r}_{s,I}}{\partial x} - \frac{\partial \boldsymbol{r}_{2,I}}{\partial x} \tag{38}$$

$\frac{\partial \boldsymbol{\omega}_I}{\partial x}$ is given in Section 2.1 if $x = t$ and is 0 otherwise. $\frac{\partial \boldsymbol{r}_{s,I}}{\partial x} = \boldsymbol{I}$ if $x = \boldsymbol{r}_{s,I}$ (part of the state vector) and 0 otherwise. $\frac{\partial \boldsymbol{r}_{2,I}}{\partial x}$ is an ephemeris lookup ($\frac{\mathrm{d}\boldsymbol{r}_{2,I}}{\mathrm{d}t}$) if $x = t$ (an ephemeris lookup) and 0 otherwise.