# Conversions Between Cartesian and Classical Orbital Elements States

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**Abstract**

This document describes conversions between Cartesian and classical orbital element state variables. Transformations in from Cartesian to classical orbital elements and associated Jacobians are given. Transformations in the other direction will be added later.

## Contents

# List of Acronyms

**COE** classical orbit elements

# 1 State Representations

## 1.1 Cartesian State

The Cartesian state consists of the position and velocity vector of the spacecraft in an assumed inertial reference frame whose origin is the spacecraft's flyby body:

$$\boldsymbol{x}_c = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix}_{6\times 1} . \tag{1}$$

## 1.2 Classical Orbital Elements State

The classical orbit elements (COE) state is given by the vector:

$$\boldsymbol{x}_k = \begin{pmatrix} a \\ e \\ i \\ \Omega \\ \omega \\ \nu \end{pmatrix}_{6\times 1} . \tag{2}$$

# 2 Cartesian State to COE State Transformation

## 2.1 Semimajor Axis

$$E = \frac{v^2}{2} - \frac{\mu}{r} \tag{3}$$

$$a = -\frac{\mu}{2E} \tag{4}$$

### 2.2 Eccentricity

$$\boldsymbol{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\boldsymbol{r} - \left(\boldsymbol{r}^T\boldsymbol{v}\right)\boldsymbol{v}\right] \tag{5}$$

$$e = ||\boldsymbol{e}|| \tag{6}$$

### 2.3 Inclination

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \tag{7}$$

$$i = \mathrm{acos}\left(\frac{h_z}{h}\right) \tag{8}$$

### 2.4 Right Ascension of the Ascending Node

$\boldsymbol{h}$ is obtained from Eq. (7). Then

$$\boldsymbol{n} = \boldsymbol{k} \times \boldsymbol{h} \tag{9}$$

$$\Omega = \mathrm{acos}\left(\frac{n_x}{n}\right) \tag{10}$$

$$\text{if } n_y < 0: \quad \Omega \leftarrow 2\pi - \Omega. \tag{11}$$

### 2.5 Argument of Periapse

$\boldsymbol{e}$ is obtained from Eq. (5) and $\boldsymbol{n}$ is obtained from Eq. (9). Then

$$\omega = \mathrm{acos}\left(\frac{\boldsymbol{n}^T\boldsymbol{e}}{ne}\right) \tag{12}$$

with the quadrant check:

$$\text{if } e_z < 0: \quad \omega \leftarrow 2\pi - \omega. \tag{13}$$

### 2.6 True anomaly

$\boldsymbol{e}$ is obtained from Eq. (5). Then

$$\nu = \mathrm{acos}\left(\frac{\boldsymbol{e}^T\boldsymbol{r}}{er}\right) \tag{14}$$

with the quadrant check:

$$\text{if } \boldsymbol{r}^T\boldsymbol{v} < 0 : \quad \nu \leftarrow 2\pi - \nu. \tag{15}$$

# 3 Cartesian State to COE State Transformation Jacobian

## 3.1 Derivatives of Position Vector

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

## 3.2 Derivatives of Velocity Vector

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

## 3.3 Derivatives of Vector Magnitude

This relation holds regardless of what $\boldsymbol{x}$ is. It is therefore used for, e.g., $r$, $v$, etc.

$$\frac{\partial x}{\partial \boldsymbol{x}} = \frac{\boldsymbol{x}^T}{x} \tag{18}$$

## 3.4 Derivatives of Inverse Cosine

This relation holds regardless of what $x$ is.

$$\frac{\partial\left[\operatorname{acos}(x)\right]}{\partial x} = \frac{-1}{\sqrt{1-x^2}} \tag{19}$$

## 3.5 Skew-symmetric Cross Vector

This relation holds regardless of what $\boldsymbol{x}$ is. It is therefore used for, e.g., $\boldsymbol{r}$, $\boldsymbol{v}$, etc.

$$\{\boldsymbol{x}\}^\times = \begin{bmatrix} 0 & -x_z & x_y \\ x_z & 0 & -x_x \\ -x_y & x_x & 0 \end{bmatrix} \tag{20}$$

### 3.6 Derivatives of Energy

$$\frac{\partial E}{\partial \boldsymbol{x}_c} = \frac{2v}{\mu}\frac{\partial v}{\partial \boldsymbol{v}}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{\mu}{r^2}\frac{\partial r}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{21}$$

Eqs. (16), (17), and (18) are used to calculate the intermediate quantities.

### 3.7 Derivatives of Semimajor Axis

$$\frac{\partial a}{\partial \boldsymbol{x}_c} = \frac{\mu}{2E^2}\frac{\partial E}{\partial \boldsymbol{x}_c} \tag{22}$$

Eq. (21) is used to calculate the intermediate quantities.

### 3.8 Derivatives of Angular Momentum Vector

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} -\{\boldsymbol{v}\}^\times & \{\boldsymbol{r}\}^\times \end{bmatrix} \tag{23}$$

Eq. (20) is used to calculate the intermediate quantities.

### 3.9 Derivatives of Node Vector

The node vector derivatives are simplified by noting that $\frac{\partial \boldsymbol{k}}{\partial \boldsymbol{x}_c} = \boldsymbol{0}$.

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{h}} = \{k\}^\times \tag{24}$$

$$\frac{\partial \boldsymbol{n}}{\partial \boldsymbol{x}_c} = \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{h}}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} \tag{25}$$

$$\tag{26}$$

Eqs. (20) and (23) are used to calculate the intermediate quantities.

### 3.10 Derivatives of Eccentricity Vector

$$\boldsymbol{\zeta}_1 \triangleq \boldsymbol{r}^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \boldsymbol{v}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{27}$$

$$\frac{\partial r}{\partial \boldsymbol{r}} = \frac{\boldsymbol{r}^T}{r} \tag{28}$$

$$\frac{\partial v}{\partial \boldsymbol{v}} = \frac{\boldsymbol{v}^T}{v} \tag{29}$$

$$\boldsymbol{\xi}_1 \triangleq \boldsymbol{r} \left( 2v \frac{\partial v}{\partial \boldsymbol{v}} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{\mu}{r^2} \frac{\partial r}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \right) + \left( v^2 - \frac{\mu}{r} \right) \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{30}$$

$$\boldsymbol{\xi}_2 \triangleq \boldsymbol{v} \boldsymbol{\zeta}_1 + \left( \boldsymbol{r}^T \boldsymbol{v} \right) \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} \tag{31}$$

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} = \frac{1}{\mu} \left( \boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 \right) \tag{32}$$

### 3.11 Derivatives of Eccentricity

Eqs. (18) and (32) are used to calculate the derivatives of $e$:

$$\frac{\partial e}{\partial \boldsymbol{x}_c} = \frac{\partial e}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{33}$$

### 3.12 Derivatives of Inclination

$$\frac{\partial i}{\partial \boldsymbol{x}_c} = \frac{\partial i}{\partial \boldsymbol{h}} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} \tag{34}$$

where

$$\frac{\partial i}{\partial \boldsymbol{h}} = \frac{\partial \mathrm{acos}\left(\frac{h_z}{h}\right)}{\partial \left(\frac{h_z}{h}\right)} \frac{\partial \left(\frac{h_z}{h}\right)}{\partial \boldsymbol{h}} \tag{35}$$

$$\frac{\partial \mathrm{acos}\left(\frac{h_z}{h}\right)}{\partial \left(\frac{h_z}{h}\right)} = \frac{-1}{\sqrt{1 - \left(\frac{h_z}{h}\right)^2}} \tag{36}$$

$$\frac{\partial \left(\frac{h_z}{h}\right)}{\partial \boldsymbol{h}} = \frac{1}{h} \frac{\partial h_z}{\partial \boldsymbol{h}} - \frac{h_z}{h^3} \boldsymbol{h}^T \tag{37}$$

$$\frac{\partial h_z}{\partial \boldsymbol{h}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \tag{38}$$

Eq. (23) is also used to calculate the intermediate quantities.

### 3.13 Derivatives of Right Ascension of the Ascending Node

$$\frac{\partial \Omega}{\partial \boldsymbol{x}_c} = \frac{\partial \Omega}{\partial \boldsymbol{n}} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{x}_c} \tag{39}$$

where

$$\frac{\partial \Omega}{\partial \boldsymbol{n}} = \frac{\partial \mathrm{acos}\left(\frac{n_x}{n}\right)}{\partial \left(\frac{n_x}{n}\right)} \frac{\partial \left(\frac{n_x}{n}\right)}{\partial \boldsymbol{n}} \tag{40}$$

$$\frac{\partial \mathrm{acos}\left(\frac{n_x}{n}\right)}{\partial \left(\frac{n_x}{n}\right)} = \frac{-1}{\sqrt{1 - \left(\frac{n_x}{n}\right)^2}} \tag{41}$$

$$\frac{\partial \left(\frac{n_x}{n}\right)}{\partial \boldsymbol{n}} = \frac{1}{n} \frac{\partial n_x}{\partial \boldsymbol{n}} - \frac{n_x}{n^3} \boldsymbol{n}^T \tag{42}$$

$$\frac{\partial n_x}{\partial \boldsymbol{n}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{43}$$

Eq. (24) is also used to calculate the intermediate quantities.

Like with the calculation of $\Omega$ itself, a quadrant check is required at the end of the derivatives calculations:

$$\text{if } n_y < 0: \quad \frac{\partial \Omega}{\partial \boldsymbol{x}_c} \leftarrow -\frac{\partial \Omega}{\partial \boldsymbol{x}_c} \tag{44}$$

### 3.14 Derivatives of Argument of Periapse

$$\frac{\partial \omega}{\partial \boldsymbol{x}_c} = \frac{\partial \omega}{\partial \alpha} \frac{\partial \alpha}{\partial \boldsymbol{x}_c} \tag{45}$$

where

$$\alpha = \frac{\boldsymbol{n}^T \boldsymbol{e}}{ne} \tag{46}$$

$$\frac{\partial \omega}{\partial \alpha} = \frac{-1}{\sqrt{1 - \left(\frac{\boldsymbol{n}^T \boldsymbol{e}}{ne}\right)^2}} \tag{47}$$

$$\frac{\partial \alpha}{\partial \boldsymbol{x}_c} = \frac{\partial \alpha}{\partial \boldsymbol{n}} \frac{\partial \boldsymbol{n}}{\partial \boldsymbol{x}_c} + \frac{\partial \alpha}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{48}$$

$$\frac{\partial \alpha}{\partial \boldsymbol{n}} = \frac{\boldsymbol{e}^T}{e} \left[ \frac{1}{n} \boldsymbol{I}_{3\times3} - \frac{1}{n^3} \boldsymbol{n}\boldsymbol{n}^T \right] \tag{49}$$

$$\frac{\partial \alpha}{\partial \boldsymbol{e}} = \frac{\boldsymbol{n}^T}{n} \left[ \frac{1}{e} \boldsymbol{I}_{3\times3} - \frac{1}{e^3} \boldsymbol{e}\boldsymbol{e}^T \right] \tag{50}$$

Eqs. (24) and (32) are also used to calculate the intermediate quantities.

Like with the calculation of $\omega$ itself, a quadrant check is required at the end of the derivatives calculations:

$$\text{if } e_z < 0: \quad \frac{\partial \omega}{\partial \boldsymbol{x}_c} \leftarrow -\frac{\partial \omega}{\partial \boldsymbol{x}_c} \tag{51}$$

### 3.15 Derivatives of True Anomaly

$$\frac{\partial \nu}{\partial \boldsymbol{x}_c} = \frac{\partial \omega}{\partial \beta} \frac{\partial \beta}{\partial \boldsymbol{x}_c} \tag{52}$$

where

$$\beta = \frac{\boldsymbol{r}^T \boldsymbol{e}}{re} \tag{53}$$

$$\frac{\partial \omega}{\partial \beta} = \frac{-1}{\sqrt{1 - \left(\frac{\boldsymbol{r}^T \boldsymbol{e}}{re}\right)^2}} \tag{54}$$

$$\frac{\partial \beta}{\partial \boldsymbol{x}_c} = \frac{\partial \beta}{\partial \boldsymbol{r}} \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} + \frac{\partial \alpha}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{55}$$

$$\frac{\partial \beta}{\partial \boldsymbol{r}} = \frac{\boldsymbol{e}^T}{e} \left[\frac{1}{r} \boldsymbol{I}_{3\times3} - \frac{1}{r^3} \boldsymbol{r} \boldsymbol{r}^T\right] \tag{56}$$

$$\frac{\partial \beta}{\partial \boldsymbol{e}} = \frac{\boldsymbol{r}^T}{r} \left[\frac{1}{e} \boldsymbol{I}_{3\times3} - \frac{1}{e^3} \boldsymbol{e} \boldsymbol{e}^T\right] \tag{57}$$

Eqs. (16) and (32) are also used to calculate the intermediate quantities.

Like with the calculation of true anomaly itself, a quadrant check is required at the end of the derivatives calculations:

# 4 COE State to Cartesian State Transformation

# 5 COE State to Cartesian State Transformation Jacobian