Conversions Between Cartesian and B-Plane States

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Abstract

This document describes conversions between Cartesian and B-Plane state variables. Transformations in both directions and associated Jacobians are given. Primary emphasis is given to transformations for the "incoming" velocity case. Later, the changes required to adapt the equations to the "outgoing" velocity case are given.

Contents

1 State Representations

1.1 Cartesian State

The Cartesian state consists of the position and velocity vector of the spacecraft in an assumed inertial reference frame whose origin is the spacecraft's flyby body:

$$\boldsymbol{x}_c = \begin{pmatrix} \boldsymbol{r} \\ \boldsymbol{v} \end{pmatrix}_{6\times 1}. \tag{1}$$

1.2 B-Plane State

The primary B-Plane state is given by the vector

$$\boldsymbol{x}_b = \begin{pmatrix} v_\infty \\ \alpha \\ \delta \\ b \\ \theta \\ \nu \end{pmatrix}_{6\times 1}. \tag{2}$$

An alternate B-Plane state substitutes the radius of periapse for b:

$$\boldsymbol{x}_{br_p} = \begin{pmatrix} v_\infty \\ \alpha \\ \delta \\ r_p \\ \theta \\ \nu \end{pmatrix}_{6\times 1}. \tag{3}$$

Full definition of the B-Plane state requires setting a reference vector (frequently some inertial $\boldsymbol{k}$). In this document, the reference vector is denoted $\boldsymbol{\phi}$ and left undefined further.

2 Cartesian State to B-Plane State Transformation

First, define standard convenience variables:

$$\boldsymbol{e} = \frac{1}{\mu}\left[\left(v^2 - \frac{\mu}{r}\right)\boldsymbol{r} - \left(\boldsymbol{r}^T\boldsymbol{v}\right)\boldsymbol{v}\right] \tag{4}$$

$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \tag{5}$$

$$\boldsymbol{P} = \boldsymbol{h} \times \boldsymbol{e}. \tag{6}$$

Then, $\hat{\boldsymbol{S}}$, in the direction of the incoming asymptote (i.e., $\boldsymbol{v}_{\infty,in}$) is given by

$$\hat{\boldsymbol{S}} = \frac{1}{e}\hat{\boldsymbol{e}} + \sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{P}}. \tag{7}$$

Additional variables are defined by

$$\hat{\boldsymbol{T}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{\phi}}{||\hat{\boldsymbol{S}} \times \boldsymbol{\phi}||} \tag{8}$$

$$\hat{\boldsymbol{R}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{T}}{||\hat{\boldsymbol{S}} \times \boldsymbol{T}||} \tag{9}$$

$$\hat{\boldsymbol{B}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{h}}{||\hat{\boldsymbol{S}} \times \boldsymbol{h}||} \tag{10}$$

$$\boldsymbol{B} = b\left[\sqrt{1 - \frac{1}{e^2}}\hat{\boldsymbol{e}} - \frac{1}{e}\hat{\boldsymbol{P}}\right] \tag{11}$$

$$b = \frac{h^2}{\mu\sqrt{e^2 - 1}}. \tag{12}$$

Then, the B-Plane dot products are

$$B_T = \boldsymbol{B}^T\hat{\boldsymbol{T}} \tag{13}$$

$$B_R = \boldsymbol{B}^T\hat{\boldsymbol{R}}. \tag{14}$$

The B-Plane clock angle is given by

$$\theta = \mathrm{atan2}\left(B_R, B_T\right). \tag{15}$$

The magnitude of the velocity at infinity is

$$v_\infty = \sqrt{v^2 - \frac{2\mu}{r}}. \tag{16}$$

The radius of periapsis is

$$r_p = \frac{\mu\left(e - 1\right)}{v_\infty^2}. \tag{17}$$

The right ascension and declination of the incoming asymptote are given by

$$\alpha = \mathrm{atan2}\left(S_y, S_x\right) \tag{18}$$

$$\delta = \mathrm{asin}\left(\frac{S_z}{S}\right) \tag{19}$$

$$= \mathrm{asin}\left(S_z\right). \tag{20}$$

True anomaly is given by the angle between e and r:

$$\nu = \mathrm{atan2}\left(||\boldsymbol{e} \times \boldsymbol{r}||, \boldsymbol{e}^T \boldsymbol{r}\right), \tag{21}$$

with the quadrant check:

$$\text{if } \boldsymbol{r}^T \boldsymbol{v} < 0: \quad \nu \leftarrow 2\pi - \nu. \tag{22}$$

3 Cartesian State to B-Plane State Transformation Jacobian

3.1 Derivatives of Position Vector

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

3.2 Derivatives of Velocity Vector

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{24}$$

3.3 Derivatives of Right Ascension of Velocity at Infinity

$$\frac{\partial \alpha}{\partial \hat{\boldsymbol{S}}} = \begin{bmatrix} -\frac{\hat{S}_y}{\hat{S}_x^2 + \hat{S}_y^2} & \frac{\hat{S}_x}{\hat{S}_x^2 + \hat{S}_y^2} & 0 \end{bmatrix} \tag{25}$$

$$\frac{\partial \alpha}{\partial \boldsymbol{x}_c} = \frac{\partial \alpha}{\partial \hat{\boldsymbol{S}}} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} \tag{26}$$

3.4 Derivatives of Declination of Velocity at Infinity

$$\frac{\partial \delta}{\partial \hat{\boldsymbol{S}}} = \begin{bmatrix} 0 & 0 & \dfrac{1}{\sqrt{1 - \hat{S}_z^2}} \end{bmatrix} \tag{27}$$

$$\frac{\partial \delta}{\partial \boldsymbol{x}_c} = \frac{\partial \delta}{\partial \hat{\boldsymbol{S}}} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} \tag{28}$$

3.5 Derivatives of Velocity at Infinity Magnitude

$$\frac{\partial v_\infty}{\partial \boldsymbol{x}_c} = \frac{1}{2} \left(v^2 - \frac{2\mu}{r} \right)^{-\frac{1}{2}} \left(2\boldsymbol{v}^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{2\mu}{r^3} \boldsymbol{r}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \right) \tag{29}$$

3.6 Derivatives of Periapsis Radius

$$\frac{\partial r_p}{\partial \boldsymbol{x}_c} = \mu \left[\hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} v_\infty^{-2} - 2\left(e - 1\right) v_\infty^{-3} \frac{\partial v_\infty}{\partial \boldsymbol{x}_c} \right] \tag{30}$$

3.7 Derivatives of B-Plane Clock Angle

$$\frac{\partial \theta}{\partial \boldsymbol{x}_c} = \frac{B_T}{B_R^2 + B_T^2} \frac{\partial B_R}{\partial \boldsymbol{x}_c} - \frac{B_R}{B_R^2 + B_T^2} \frac{\partial B_T}{\partial \boldsymbol{x}_c} \tag{31}$$

3.8 Derivatives of B_T

$$\frac{\partial B_T}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{T}}^T \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} + \boldsymbol{B}^T \frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_c} \tag{32}$$

3.9 Derivatives of B_R

$$\frac{\partial B_R}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{R}}^T \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} + \boldsymbol{B}^T \frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_c} \tag{33}$$

3.10 Derivatives of B Vector Magnitude

$$\boldsymbol{\xi}_1 \triangleq -\left(e^2 - 1\right)^{-\frac{3}{2}} \boldsymbol{e}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{34}$$

$$\boldsymbol{\xi}_2 \triangleq 2\boldsymbol{h}^T \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} \tag{35}$$

$$\frac{\partial b}{\partial \boldsymbol{x}_c} = \frac{1}{\mu}\left[h^2 \boldsymbol{\xi}_1 + \left(e^2 - 1\right)^{-\frac{1}{2}} \boldsymbol{\xi}_2\right] \tag{36}$$

3.11 Derivatives of B Vector

$$\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_c} = \hat{\boldsymbol{B}} \frac{\partial b}{\partial \boldsymbol{x}_c} + b \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} \tag{37}$$

3.12 Derivatives of B Unit Vector

$$\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} = \frac{1}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} \tag{38}$$

$$\frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} = \frac{1}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right)\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{39}$$

$$\zeta_1 \triangleq \sqrt{1 - \frac{1}{e^2}} \tag{40}$$

$$\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} = \frac{1}{e^3}\left(1 - \frac{1}{e^2}\right)^{-\frac{1}{2}} \hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{41}$$

$$\boldsymbol{\zeta}_2 \triangleq -\frac{1}{e^2}\hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{42}$$

$$\boldsymbol{\xi}_1 \triangleq \zeta_1 \frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} \tag{43}$$

$$\boldsymbol{\xi}_2 \triangleq \hat{\boldsymbol{e}} \frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} \tag{44}$$

$$\boldsymbol{\xi}_3 \triangleq -\frac{1}{e}\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} \tag{45}$$

$$\boldsymbol{\xi}_4 \triangleq -\hat{\boldsymbol{P}}\boldsymbol{\zeta}_2 \tag{46}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{47}$$

3.13 Derivatives of R Unit Vector

$$\boldsymbol{R} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{T} \tag{48}$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_c} = -\left\{\boldsymbol{T}\right\}^{\times} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} + \left\{\hat{\boldsymbol{S}}\right\}^{\times} \frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} \tag{49}$$

$$\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_c} = \frac{1}{R}\left(\boldsymbol{I} - \frac{1}{R^2}\boldsymbol{R}\boldsymbol{R}^T\right)\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_c} \tag{50}$$

3.14 Derivatives of T Unit Vector

$$\boldsymbol{T} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{\phi} \tag{51}$$

$$\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} = -\left\{\boldsymbol{\phi}\right\}^{\times} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} + \left\{\hat{\boldsymbol{S}}\right\}^{\times} \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_c} \tag{52}$$

$$\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_c} = \frac{1}{T}\left(\boldsymbol{I} - \frac{1}{T^2}\boldsymbol{T}\boldsymbol{T}^T\right)\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_c} \tag{53}$$

3.15 Derivatives of S Unit Vector

$$\frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} = \frac{1}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} \tag{54}$$

$$\zeta_1 \triangleq \sqrt{1 - \frac{1}{e^2}} \tag{55}$$

$$\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} = \frac{1}{e^3}\left(1 - \frac{1}{e^2}\right)^{-\frac{1}{2}} \hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{56}$$

$$\boldsymbol{\xi}_1 \triangleq -\frac{1}{e^2}\hat{\boldsymbol{e}}\hat{\boldsymbol{e}}^T \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{57}$$

$$\boldsymbol{\xi}_2 \triangleq \frac{1}{e}\frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_c} \tag{58}$$

$$\boldsymbol{\xi}_3 \triangleq \hat{\boldsymbol{P}}\frac{\partial \zeta_1}{\partial \boldsymbol{x}_c} \tag{59}$$

$$\boldsymbol{\xi}_4 \triangleq \zeta_1 \frac{\partial \hat{\boldsymbol{P}}}{\partial \boldsymbol{x}_c} \tag{60}$$

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{61}$$

3.16 Derivatives of Angular Momentum Vector

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} = \begin{bmatrix} -\{\boldsymbol{v}\}^\times & \{\boldsymbol{r}\}^\times \end{bmatrix} \tag{62}$$

3.17 Derivatives of P Vector

$$\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{x}_c} = -\{\boldsymbol{e}\}^\times \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_c} + \{\boldsymbol{h}\}^\times \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} \tag{63}$$

3.18 Derivatives of Eccentricity Vector

$$\boldsymbol{\zeta}_1 \triangleq \boldsymbol{r}^T \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \boldsymbol{v}^T \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{64}$$

$$\frac{\partial r}{\partial \boldsymbol{r}} = \frac{\boldsymbol{r}^T}{r} \tag{65}$$

$$\frac{\partial v}{\partial \boldsymbol{v}} = \frac{\boldsymbol{v}^T}{v} \tag{66}$$

$$\boldsymbol{\xi}_1 \triangleq \boldsymbol{r}\left(2v\frac{\partial v}{\partial \boldsymbol{v}}\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} + \frac{\mu}{r^2}\frac{\partial r}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c}\right) + \left(v^2 - \frac{\mu}{r}\right)\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_c} \tag{67}$$

$$\boldsymbol{\xi}_2 \triangleq \boldsymbol{v}\boldsymbol{\zeta}_1 + \left(\boldsymbol{r}^T\boldsymbol{v}\right)\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_c} \tag{68}$$

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_c} = \frac{1}{\mu}\left(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\right) \tag{69}$$

3.19 Derivatives of True Anomaly

The derivatives of true anomaly with respect to the Cartesian state are obtained by differentiating Eq. (21).

$$\frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{e}} = -\{\boldsymbol{r}\}^{\times} \tag{70}$$

$$\frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{r}} = \{\boldsymbol{e}\}^{\times} \tag{71}$$

$$\frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{x}_c} = \frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{e}}\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{x}_c} + \frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{r}}\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{x}_c} \tag{72}$$

$$\xi_1 \triangleq ||\boldsymbol{e}\times\boldsymbol{r}|| \tag{73}$$

$$\xi_2 \triangleq \boldsymbol{e}^T\boldsymbol{r} \tag{74}$$

$$\frac{\partial\xi_1}{\partial\boldsymbol{x}_c} = \frac{1}{\xi_1}(\boldsymbol{e}\times\boldsymbol{r})^T\frac{\partial(\boldsymbol{e}\times\boldsymbol{r})}{\partial\boldsymbol{x}_c} \tag{75}$$

$$\frac{\partial\xi_2}{\partial\boldsymbol{x}_c} = \boldsymbol{e}^T\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{x}_c} + \boldsymbol{r}^T\frac{\partial\boldsymbol{e}}{\partial\boldsymbol{x}_c} \tag{76}$$

$$\frac{\partial\nu}{\partial\xi_1} = \frac{\xi_2}{\xi_1^2+\xi_2^2} \tag{77}$$

$$\frac{\partial\nu}{\partial\xi_2} = -\frac{\xi_1}{\xi_1^2+\xi_2^2} \tag{78}$$

$$\frac{\partial\nu}{\partial\boldsymbol{x}_c} = \frac{\partial\nu}{\partial\xi_1}\frac{\partial\xi_1}{\partial\boldsymbol{x}_c} + \frac{\partial\nu}{\partial\xi_2}\frac{\partial\xi_2}{\partial\boldsymbol{x}_c} \tag{79}$$

Like with the calculation of true anomaly itself, a quadrant check is required at the end of the derivatives calculations:

$$\text{if } \boldsymbol{r}^T\boldsymbol{v} < 0: \quad \frac{\partial\nu}{\partial\boldsymbol{x}_c} \leftarrow -\frac{\partial\nu}{\partial\boldsymbol{x}_c} \tag{80}$$

4 B-Plane State to Cartesian State Transformation

The transformation from B-Plane state to Cartesian state is accomplished by expressing the Cartesian state as a function of $\boldsymbol{e}$, $\boldsymbol{h}$, and ν:

$$\boldsymbol{r} = \frac{h^2}{\mu(1+e\cos\nu)}\left[\hat{\boldsymbol{e}}\cos\nu + \hat{\boldsymbol{P}}\sin\nu\right] \tag{81}$$

$$\boldsymbol{v} = -\frac{\mu}{h}\left[\hat{\boldsymbol{e}}\sin\nu - (e+\cos\nu)\hat{\boldsymbol{P}}\right]. \tag{82}$$

True anomaly ν is known because it is a member of $\boldsymbol{x}_b$. The rest of the elements needed to calculate the Cartesian state are given by:

$$e = \sqrt{1 + \frac{v_\infty^4 b^2}{\mu^2}} \tag{83}$$

$$e = \frac{r_p v_\infty^2}{\mu} + 1 \tag{84}$$

$$h = v_\infty b \tag{85}$$

$$h = r_p v_p \tag{86}$$

$$v_p = \sqrt{v_\infty^2 + \frac{2\mu}{r_p}} \tag{87}$$

$$\boldsymbol{v}_\infty = v_\infty \begin{bmatrix} \cos\delta\cos\alpha \\ \cos\delta\sin\alpha \\ \sin\delta \end{bmatrix} \tag{88}$$

$$B_R = \boldsymbol{B}^T \hat{\boldsymbol{R}} = b \sin\theta \tag{89}$$

$$B_T = \boldsymbol{B}^T \hat{\boldsymbol{T}} = b \cos\theta \tag{90}$$

$$\hat{\boldsymbol{S}} = \hat{\boldsymbol{v}}_\infty \tag{91}$$

$$\hat{\boldsymbol{T}} = \frac{\hat{\boldsymbol{S}} \times \boldsymbol{\phi}}{||\hat{\boldsymbol{S}} \times \boldsymbol{\phi}||} \tag{92}$$

$$\hat{\boldsymbol{R}} = \frac{\hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}}}{||\hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}}||} \tag{93}$$

$$\boldsymbol{B} = B_R \hat{\boldsymbol{R}} + B_T \hat{\boldsymbol{T}} \tag{94}$$

$$\hat{\boldsymbol{h}} = \frac{\boldsymbol{B} \times \hat{\boldsymbol{S}}}{||\boldsymbol{B} \times \hat{\boldsymbol{S}}||} \tag{95}$$

$$\boldsymbol{h} = h \hat{\boldsymbol{h}} \tag{96}$$

$$\nu_{\infty,in} = -\mathrm{acos}\left(-\frac{1}{e}\right) \tag{97}$$

$$\hat{\boldsymbol{e}} = \frac{\hat{\boldsymbol{S}} \cos(\pi - \nu_{\infty,in}) - \hat{\boldsymbol{B}} \sin(\pi - \nu_{\infty,in})}{||\hat{\boldsymbol{S}} \cos(\pi - \nu_{\infty,in}) - \hat{\boldsymbol{B}} \sin(\pi - \nu_{\infty,in})||} \tag{98}$$

$$\boldsymbol{e} = e \hat{\boldsymbol{e}} \tag{99}$$

5 B-Plane State to Cartesian State Transformation Jacobian

5.1 Derivatives of Magnitude of B Vector

$$\frac{\partial b}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0] \tag{100}$$

$$\frac{\partial b}{\partial \boldsymbol{x}_{br_p}} = \left[\left(\frac{r_p}{v_\infty} \left(\frac{\partial v_p}{\partial v_\infty} - \frac{v_p}{v_\infty} \right) \right) \quad 0 \quad 0 \quad \left(\frac{v_p}{v_\infty} + \frac{r_p}{v_\infty} \frac{\partial v_p}{\partial r_p} \right) \quad 0 \quad 0 \right] \tag{101}$$

5.2 Derivatives of Radius of Periapse

Important: This relationship is useful on when $\boldsymbol{x}_{br_p}$ is used and *not* when $\boldsymbol{x}_b$ is used.

$$\frac{\partial r_p}{\partial \boldsymbol{x}_{br_p}} = [0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0] \tag{102}$$

5.3 Derivatives of Velocity at Periapse

$$\frac{\partial v_p}{\partial \boldsymbol{x}_{br_p}} = \left[\frac{v_\infty}{v_p} \quad 0 \quad 0 \quad \left(-\frac{\mu}{v_p r_p^2} \right) \quad 0 \quad 0 \right] \tag{103}$$

5.4 Derivatives of B-Plane Clock Angle

$$\frac{\partial \theta}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0] \tag{104}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.5 Derivatives of True Anomaly

With $\boldsymbol{x}_b$ defined as in Eq. (2), the derivatives of true anomaly are

$$\frac{\partial \nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1] \tag{105}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.6 Derivatives of Eccentricity Vector

$$\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} = \hat{\boldsymbol{e}} \frac{\partial e}{\partial \boldsymbol{x}_b} + e \frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_b} \tag{106}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.7 Derivatives of Eccentricity Magnitude

$$\frac{\partial e}{\partial \boldsymbol{x}_b} = \frac{1}{2}\left(1 + \frac{v_\infty^4 b^2}{\mu^2}\right)^{-\frac{1}{2}} \begin{bmatrix} \frac{4v_\infty^3 b^2}{\mu^2} & 0 & 0 & \frac{2v_\infty^4 b}{\mu^2} & 0 & 0 \end{bmatrix} \tag{107}$$

$$\frac{\partial e}{\partial \boldsymbol{x}_{br_p}} = \frac{v_\infty}{\mu} \begin{bmatrix} 2r_p & 0 & 0 & v_\infty & 0 & 0 \end{bmatrix} \tag{108}$$

5.8 Derivatives of Angular Momentum Vector

$$\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} = \hat{\boldsymbol{h}} \frac{\partial h}{\partial \boldsymbol{x}_b} + h \frac{\partial \hat{\boldsymbol{h}}}{\partial \boldsymbol{x}_b} \tag{109}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.9 Derivatives of Angular Momentum Magnitude

$$\frac{\partial h}{\partial \boldsymbol{x}_b} = \begin{bmatrix} b & 0 & 0 & v_\infty & 0 & 0 \end{bmatrix} \tag{110}$$

$$\frac{\partial h}{\partial \boldsymbol{x}_{br_p}} = \begin{bmatrix} \frac{r_p v_\infty}{v_p} & 0 & 0 & \left(v_p - \frac{\mu}{r_p v_p}\right) & 0 & 0 \end{bmatrix} \tag{111}$$

5.10 Derivatives of Angular Momentum Unit Vector

$$\boldsymbol{\gamma} \triangleq \boldsymbol{B} \times \hat{\boldsymbol{S}} \tag{112}$$

$$\frac{\partial \hat{\boldsymbol{h}}}{\partial \boldsymbol{x}_b} = \left(-\frac{1}{\gamma^3}\boldsymbol{\gamma}\boldsymbol{\gamma}^T + \frac{1}{\gamma}\boldsymbol{I}\right)\left(-\left\{\hat{\boldsymbol{S}}\right\}^\times \frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} + \{\boldsymbol{B}\}^\times \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b}\right) \tag{113}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.11 Derivatives of S Unit Vector

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} = \begin{bmatrix} 0 & -\cos\delta\sin\alpha & -\sin\delta\cos\alpha & 0 & 0 & 0 \\ 0 & \cos\delta\cos\alpha & -\sin\delta\sin\alpha & 0 & 0 & 0 \\ 0 & 0 & \cos\delta & 0 & 0 & 0 \end{bmatrix} \tag{114}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.12 Derivatives of B Vector

$$\frac{\partial \sin\nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad \cos\theta \quad 0] \tag{115}$$

$$\frac{\partial \cos\nu}{\partial \boldsymbol{x}_b} = [0 \quad 0 \quad 0 \quad 0 \quad 0 - \sin\theta \quad 0] \tag{116}$$

$$\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} = \sin\theta\hat{\boldsymbol{R}}\frac{\partial b}{\partial \boldsymbol{x}_b} + b\hat{\boldsymbol{R}}\frac{\partial \sin\theta}{\partial \boldsymbol{x}_b} + b\sin\theta\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_b} + \cos\theta\hat{\boldsymbol{T}}\frac{\partial b}{\partial \boldsymbol{x}_b} + b\hat{\boldsymbol{T}}\frac{\partial \cos\theta}{\partial \boldsymbol{x}_b} + b\cos\theta\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} \tag{117}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.13 Derivatives of T Unit Vector

The derivatives of $\hat{\boldsymbol{T}}$ cannot be fully defined until the reference vector $\boldsymbol{\phi}$ is chosen. In this section, the derivatives are left in terms of the derivatives of $\boldsymbol{\phi}$.

$$\boldsymbol{T} \triangleq \hat{\boldsymbol{S}} \times \boldsymbol{\phi} \tag{118}$$

$$\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b} = -\{\boldsymbol{\phi}\}^\times \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} + \left\{\hat{\boldsymbol{S}}\right\}^\times \frac{\partial \boldsymbol{\phi}}{\partial \boldsymbol{x}_b} \tag{119}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{T^3}\left(\boldsymbol{T}^T\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b}\right)^T \tag{120}$$

$$\frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} = \frac{1}{T}\frac{\partial \boldsymbol{T}}{\partial \boldsymbol{x}_b} + \boldsymbol{T}\boldsymbol{\xi}_2^T \tag{121}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.14 Derivatives of R Unit Vector

$$\boldsymbol{R} \triangleq \hat{\boldsymbol{S}} \times \hat{\boldsymbol{T}} \tag{122}$$

$$\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b} = -\left\{\hat{\boldsymbol{T}}\right\}^{\times} \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} + \left\{\hat{\boldsymbol{S}}\right\}^{\times} \frac{\partial \hat{\boldsymbol{T}}}{\partial \boldsymbol{x}_b} \tag{123}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{R^3}\left(\boldsymbol{R}^T \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b}\right)^T \tag{124}$$

$$\frac{\partial \hat{\boldsymbol{R}}}{\partial \boldsymbol{x}_b} = \frac{1}{R}\frac{\partial \boldsymbol{R}}{\partial \boldsymbol{x}_b} + \boldsymbol{R}\boldsymbol{\xi}_2^T \tag{125}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.15 Derivatives of Eccentricity Unit Vector

$$\beta \triangleq \pi - \nu_{\infty,in} \tag{126}$$

$$c_\beta \triangleq \cos\beta \tag{127}$$

$$s_\beta \triangleq \sin\beta \tag{128}$$

$$\frac{\partial c_\beta}{\partial \boldsymbol{x}_b} = s_\beta \frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} \tag{129}$$

$$\frac{\partial s_\beta}{\partial \boldsymbol{x}_b} = -c_\beta \frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} \tag{130}$$

$$\boldsymbol{\xi}_1 \triangleq c_\beta \hat{\boldsymbol{S}} - s_\beta \hat{\boldsymbol{B}} \tag{131}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{B}}\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{x}_b} \tag{132}$$

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{B}} = \frac{1}{B}\left(\boldsymbol{I} - \frac{1}{B^2}\boldsymbol{B}\boldsymbol{B}^T\right) \tag{133}$$

$$\frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} c_\beta + \hat{\boldsymbol{S}}\frac{\partial c_\beta}{\partial \boldsymbol{x}_b} - \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b} s_\beta - \hat{\boldsymbol{B}}\frac{\partial s_\beta}{\partial \boldsymbol{x}_b} \tag{134}$$

$$\boldsymbol{\xi}_2 \triangleq -\frac{1}{\xi_1^3}\boldsymbol{x}_1^T \frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} \tag{135}$$

$$\frac{\partial \hat{e}}{\partial \boldsymbol{x}_b} = \frac{1}{\xi_1}\frac{\partial \boldsymbol{\xi}_1}{\partial \boldsymbol{x}_b} + \boldsymbol{\xi}_1\boldsymbol{\xi}_2 \tag{136}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.16 Derivatives of Incoming True Anomaly at Infinity

$$\frac{\partial \nu_{\infty,in}}{\partial \boldsymbol{x}_b} = \frac{1}{e\sqrt{e^2-1}}\frac{\partial e}{\partial \boldsymbol{x}_b} \tag{137}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.17 Derivatives of Position Vector

The final derivatives of the position vector utilize the derivatives of $\boldsymbol{h}$, $\boldsymbol{e}$, and ν:

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}_b} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{h}}\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}}\frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{r}}{\partial \nu}\frac{\partial \nu}{\partial \boldsymbol{x}_b} \tag{138}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.17.1 Derivatives of Position Vector with Respect to Angular Momentum Vector

$$\xi_1 \triangleq 2\cos\nu\,\hat{\boldsymbol{e}}\boldsymbol{h}^T \tag{139}$$

$$\xi_2 \triangleq \frac{\sin\nu}{P}\left[2\boldsymbol{P}\boldsymbol{h}^T + h^2\left(-\boldsymbol{I} + \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\{\boldsymbol{e}\}^\times\right] \tag{140}$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{h}} = \frac{1}{\mu\left(1 + e\cos\nu\right)}\left(\xi_1 + \xi_2\right) \tag{141}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.17.2 Derivatives of Position Vector with Respect to Eccentricity Vector

$$\xi_1 \triangleq \left[\hat{\boldsymbol{e}}\cos\nu + \hat{\boldsymbol{P}}\sin\nu\right]\left[\hat{\boldsymbol{e}}^T\frac{-\cos\nu}{\left(1 + e\cos\nu\right)^2}\right] \tag{142}$$

$$\xi_2 \triangleq \frac{1}{1 + e\cos\nu}\left[\frac{\cos\nu}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right) + \frac{\sin\nu}{P}\left(\boldsymbol{I} - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\right)\{\boldsymbol{h}\}^\times\right] \tag{143}$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{e}} = \frac{h^2}{\mu}\left(\xi_1 + \xi_2\right) \tag{144}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.17.3 Derivatives of Position Vector with Respect to True Anomaly

$$\xi_1 \triangleq \frac{e \sin \nu}{(1 + e \cos \nu)^2} \left(\hat{\boldsymbol{e}} \cos \nu + \hat{\boldsymbol{P}} \sin \nu \right) \tag{145}$$

$$\xi_2 \triangleq \frac{1}{1 + e \cos \nu} \left(-\hat{\boldsymbol{e}} \sin \nu + \hat{\boldsymbol{P}} \cos \nu \right) \tag{146}$$

$$\frac{\partial \boldsymbol{r}}{\partial \nu} = \frac{h^2}{\mu} \left(\xi_1 + \xi_2 \right) \tag{147}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.18 Derivatives of Velocity Vector

The final derivatives of the velocity vector utilize the derivatives of $\boldsymbol{h}$, $\boldsymbol{e}$, and ν:

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}_b} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{h}} \frac{\partial \boldsymbol{h}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{e}} \frac{\partial \boldsymbol{e}}{\partial \boldsymbol{x}_b} + \frac{\partial \boldsymbol{v}}{\partial \nu} \frac{\partial \nu}{\partial \boldsymbol{x}_b} \tag{148}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.18.1 Derivatives of Velocity Vector with Respect to Angular Momentum Vector

$$\xi_1 \triangleq -\frac{1}{h^3} \left[\hat{\boldsymbol{e}} \sin \nu - (e + \cos \nu) \hat{\boldsymbol{P}} \right] \boldsymbol{h}^T \tag{149}$$

$$\xi_2 \triangleq -\frac{e + \cos \nu}{hP} \left[-\{\boldsymbol{e}\}^\times + \frac{1}{P^2} \boldsymbol{P} \boldsymbol{P}^T \{\boldsymbol{e}\}^\times \right] \tag{150}$$

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{h}} = -\mu \left(\xi_1 + \xi_2 \right) \tag{151}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.18.2 Derivatives of Velocity Vector with Respect to Eccentricity Vector

$$\xi_1 \triangleq \frac{\sin\nu}{e}\left(\boldsymbol{I} - \frac{1}{e^2}\boldsymbol{e}\boldsymbol{e}^T\right) \tag{152}$$

$$\xi_{21} \triangleq \hat{\boldsymbol{P}}\hat{\boldsymbol{e}}^T \tag{153}$$

$$\xi_{22} \triangleq (e+\cos\nu)\left(\frac{1}{P}\right)\left[\{\boldsymbol{h}\}^\times - \frac{1}{P^2}\boldsymbol{P}\boldsymbol{P}^T\{\boldsymbol{h}\}^\times\right] \tag{154}$$

$$\xi_2 \triangleq -(\xi_{21}+\xi_{22}) \tag{155}$$

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{e}} = -\frac{\mu}{h}(\xi_1+\xi_2) \tag{156}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

5.18.3 Derivatives of Velocity Vector with Respect to True Anomaly

$$\xi_1 \triangleq \cos\nu\,\hat{\boldsymbol{e}} \tag{157}$$

$$\xi_2 \triangleq \sin\nu\,\hat{\boldsymbol{P}} \tag{158}$$

$$\frac{\partial \boldsymbol{v}}{\partial \nu} = -\frac{\mu}{h}(\xi_1+\xi_2) \tag{159}$$

The equation is analogous if $\boldsymbol{x}_{br_p}$ is used; $\boldsymbol{x}_{br_p}$ is substituted for $\boldsymbol{x}_b$.

6 Outgoing Transformations

6.1 Cartesian State to B-Plane State Transformation

For the Cartesian state to B-Plane transformation, the substantive change is that the vector $\hat{\boldsymbol{S}}$ – which is aligned with the hyperbolic asymptote – changes from being parallel to the incoming asymptote to being parallel to the outgoing asymptote. Consequently, Eq. (7) becomes

$$\hat{\boldsymbol{S}} = -\frac{1}{e}\hat{\boldsymbol{e}} + \sqrt{1-\frac{1}{e^2}}\hat{\boldsymbol{P}} \tag{160}$$

All subsequent calculations proceed as described in Section 2 using the expression for $\hat{\boldsymbol{S}}$ given in Eq. (160) with the exception of Eq. (10), which becomes

$$\hat{\boldsymbol{B}} = \frac{1}{e}\hat{\boldsymbol{P}} + \sqrt{1-\frac{1}{e^2}}\hat{\boldsymbol{e}} \tag{161}$$

6.2 Cartesian State to B-Plane State Transformation Jacobian

Changes to the Cartesian state to B-Plane state transformation Jacobian relative to the expressions presented in Section 3 arise due to the changes presented in Section 6.1. Specifically, Eq. (61) becomes

$$\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_c} = -\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 + \boldsymbol{\xi}_4 \tag{162}$$

using the variable definitions of Section 3.15, overridden where applicable by Section 6.1.

Additionally, Eq. (47) becomes

$$\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_c} = \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 - \boldsymbol{\xi}_3 - \boldsymbol{\xi}_4 \tag{163}$$

using the variable definitions of Section 3.12, overridden where applicable by Section 6.1.

6.3 B-Plane State to Cartesian State Transformation

For the B-Plane state to Cartesian state transformation, the true anomaly at infinity is calculated for the outgoing asymptote rather than for the incoming asymptote:

$$\nu_{\infty,out} = \text{acos}\left(-\frac{1}{e}\right) \tag{164}$$

$\nu_{\infty,out}$ then replaces $\nu_{\infty,in}$ in Eq. (98), which becomes

$$\hat{\boldsymbol{e}} = \hat{\boldsymbol{B}} \sin(\nu_{\infty,out}) + \hat{\boldsymbol{S}} \cos(\nu_{\infty,out}) \tag{165}$$

All other equations of Section 4 still hold with the important note that α and δ must be interpreted as the right ascension and declination, respectively, of the outgoing asymptote. (For the incoming B-Plane transformation, α and δ are the right ascension and declination, respectively, of the incoming asymptote.)

6.4 B-Plane State to Cartesian State Transformation Jacobian

Changes to the Cartesian state to B-Plane state transformation Jacobian relative to the expressions presented in Section 5 arise due to the changes presented in Section 6.3. Specifically, Eq. (137) becomes

$$\frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} = -\frac{1}{e\sqrt{e^2-1}} \frac{\partial e}{\partial \boldsymbol{x}_b} \tag{166}$$

Additionally, Eq. (136) becomes

$$\frac{\partial \hat{\boldsymbol{e}}}{\partial \boldsymbol{x}_b} = \frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b} \cos\left(\nu_{\infty,out}\right) + \hat{\boldsymbol{S}} \frac{\partial \cos\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} + \frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b} \sin\left(\nu_{\infty,out}\right) + \hat{\boldsymbol{B}} \frac{\partial \sin\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} \tag{167}$$

with

$$\frac{\partial \cos\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} = -\sin\left(\nu_{\infty,out}\right) \frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} \tag{168}$$

$$\frac{\partial \sin\left(\nu_{\infty,out}\right)}{\partial \boldsymbol{x}_b} = \cos\left(\nu_{\infty,out}\right) \frac{\partial \nu_{\infty,out}}{\partial \boldsymbol{x}_b} \tag{169}$$

The expressions for $\frac{\partial \hat{\boldsymbol{S}}}{\partial \boldsymbol{x}_b}$ and $\frac{\partial \hat{\boldsymbol{B}}}{\partial \boldsymbol{x}_b}$ do not change from those presented in Section 5.

As with the incoming exressions, the outgoing expressions when $\boldsymbol{x}_{br_p}$ is used instead of $\boldsymbol{x}_b$ are exactly analogous. Derivatives with respect to $\boldsymbol{x}_{br_p}$ are substituted for derivatives with respect to $\boldsymbol{x}_b$ in the chain rule.