The Russell Transformation

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Abstract

This document describes the "Russell" transformation, which is used to produce a fictitious independent variable for step-size control in N-body gravitational environments.

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1 The Russell Transformation

The transformation takes the form

$$\mathrm{d}t = g\mathrm{d}s, \tag{1}$$

where

$$g = \prod_{i=1}^{N} \rho_i^{\alpha}, \tag{2}$$

where

$$\rho_i = \frac{A^2 + A}{A + \left(\frac{A(1-C)}{C+A}\right)^{\frac{r_i}{B r_{H,i}}}} - A. \tag{3}$$

Then, the equations of motion are written as

$$\begin{aligned} \boldsymbol{f}_s &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}s} && (4)\\ &= \boldsymbol{f}_t \frac{\mathrm{d}t}{\mathrm{d}s} && (5)\\ &= \boldsymbol{f}_t g && (6) \end{aligned}$$

2 Jacobian of the Russell Transformation

We require the Jacobians $\frac{\partial \boldsymbol{f}_s}{\partial \boldsymbol{x}}$ and $\frac{\partial \boldsymbol{f}_s}{\partial s}$.

$$\begin{aligned} \frac{\partial \boldsymbol{f}_s}{\partial \boldsymbol{x}} &= \frac{\partial}{\partial \boldsymbol{x}}\left[\boldsymbol{f}_t g\right] && (7)\\ &= \frac{\partial \boldsymbol{f}_t}{\partial \boldsymbol{x}} g + \boldsymbol{f}_t \frac{\partial g}{\partial \boldsymbol{x}} && (8)\\ \frac{\partial \boldsymbol{f}_s}{\partial s} &= \frac{\partial \boldsymbol{f}_s}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s} && (9)\\ &= \frac{\partial}{\partial t}\left[\boldsymbol{f}_t g\right] g && (10)\\ &= \frac{\partial \boldsymbol{f}_t}{\partial t} g^2 + \boldsymbol{f}_t \frac{\partial g}{\partial t} g && (11) \end{aligned}$$

$\frac{\partial \boldsymbol{f}_t}{\partial \boldsymbol{x}}$ and $\frac{\partial \boldsymbol{f}_t}{\partial t}$ are the usual Jacobians for equations of motion when time is the independent variable, and $\boldsymbol{f}_t$ are the time equations of motion. Thus, the only terms that need to be derived are $\frac{\partial g}{\partial \boldsymbol{x}}$ and $\frac{\partial g}{\partial t}$.

3 Jacobian of ρ_i

In g, the only non-constant terms are the ρ_i. Each ρ_i is solely dependent on r_i. Using MATLAB symbolic math, we arrive at

$$\begin{aligned} \frac{\partial \rho_i}{\partial r_i} &= -\frac{A\gamma^{\frac{r_i}{B r_{H,i}}}\log\left(\gamma\right)\left(A+1\right)}{B r_{H,i}\left(A+\gamma^{\frac{r_i}{B r_{H,i}}}\right)^2} && (12)\\ \gamma &\triangleq \frac{A\left(1-C\right)}{C+A}, && (13) \end{aligned}$$

where γ is a convenience function. Then, r_i is only directly dependent on $\boldsymbol{r}_i$:

$$\begin{aligned} \frac{\partial r_i}{\partial \boldsymbol{x}} &= \frac{\partial r_i}{\partial \boldsymbol{r}_i}\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{x}} && (14)\\ \frac{\partial r_i}{\partial \boldsymbol{r}_i} &= \frac{\boldsymbol{r}_i^T}{r_i}. && (15) \end{aligned}$$

If we call $\boldsymbol{r}_i$ the vector from the spacecraft to body i and $\boldsymbol{r}_{c,i}$ the vector from the central body of integration to body i, then

$$\boldsymbol{r}_i = \boldsymbol{r}_{c,i} - \boldsymbol{r}. \tag{16}$$

The derivatives are

$$\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{x}} = \frac{\partial}{\partial \boldsymbol{x}} \left(\boldsymbol{r}_{c,i} - \boldsymbol{r} \right). \tag{17}$$

In Eq. (17), $\boldsymbol{r}_{c,i}$ depends only on time, and $\boldsymbol{r}$ depends only on the position state of the spacecraft. So,

$$\frac{\partial \boldsymbol{r}_i}{\partial t} = \boldsymbol{v}_{c,i} \tag{18}$$

$$\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{r}} = -\boldsymbol{I} \tag{19}$$

and all other derivatives are zero.

So,

$$\frac{\partial \rho_i}{\partial \boldsymbol{x}} = \frac{\partial \rho_i}{\partial r_i} \frac{\partial r_i}{\partial \boldsymbol{x}} \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{x}}, \tag{20}$$

where we may assume that t is part of the augmented state vector.

4 Jacobian of g

The derivative of g with respect to each of the ρ_i is

$$\frac{\partial g}{\partial \rho_i} = \alpha \rho_i^{\alpha-1} \prod_{\substack{j=1 \\ j \neq i}}^{N} \rho_j^{\alpha}. \tag{21}$$

Then, if we concatenate all ρ_i into the vector $\boldsymbol{\rho}$, we finally get

$$\frac{\partial g}{\partial \boldsymbol{x}} = \frac{\partial g}{\partial \boldsymbol{\rho}} \frac{\partial \boldsymbol{\rho}}{\partial \boldsymbol{x}}, \tag{22}$$

where we may assume that t is part of the augmented state vector.