Applied 354 Lecture Notes

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1 Stokes and Divergance Integrals

Calculate Divergance Theorem Given the vector field, where $z = 0$:

$$\underline{v} = 3y^2z\underline{i} + 2y^2\underline{j} + yz^2\underline{k} \tag{1}$$

$$\underline{v} = 2y^2\underline{j} \tag{2}$$

The divergance theorem is given by:

$$\iint_A \nabla \cdot \underline{v} dA = \oint \underline{n} \cdot \underline{v} dr$$

Calculating the inside of the LHS:

$$\nabla \cdot \underline{v} = \underline{j}\frac{\partial}{\partial y} \cdot 2y^2\underline{j} = 4y$$

Therefor calculating LHS:

$$\iint_A \nabla \cdot \underline{v}\ dA = \int_2^3 \int_2^4 4y\ dxdy$$

$$\iint_A \nabla \cdot \underline{v}\ dA = \left| 8\frac{y^2}{2} \right|_2^3 = 20$$

Calculating the RHS, using the graphic finding the normal vectors to each side. Only two terms are present as i dot j will be zero, so the verticals are zero.

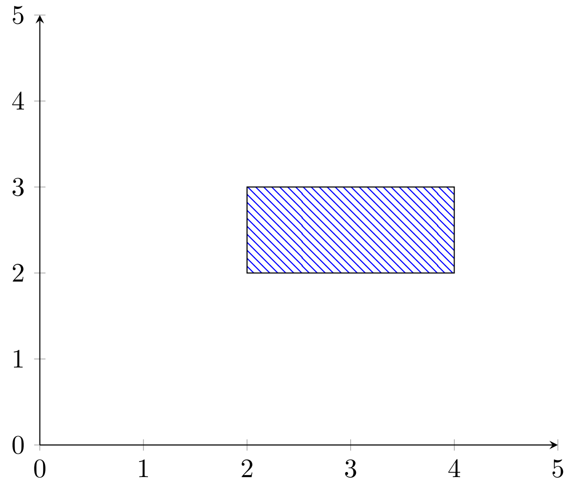
$$\oint \underline{n} \cdot \underline{v} dr = \int_2^4 -\underline{j} \cdot 2y^2\underline{j} dx + \int_2^4 \underline{j} \cdot 2y^2\underline{j} dx$$

Using Figure:1 replacing the y values from the graphic, 2 and 3 respectivly

$$= \int_2^4 -8dx + \int_2^4 18dx = 20$$

Therefor RHS = LHS

Figure 1: Vector Field Figure



Calculate Stokes Integral Theorem for the velocity field $v = x\underline{j}$.
Stokes integral is given by:

$$\oint_{\partial S} \underline{v} \cdot dr = \iint_S \underline{n} \cdot \nabla \times \underline{v} \, dS$$

Calculating the LHS:

$$\begin{aligned}\oint_{\partial S} \underline{v} \cdot dr &= \int (x\underline{j}) \cdot (dx \cdot \underline{i} + dy \cdot \underline{j}) \\ &= \int xdy\end{aligned}$$

The only non zero term is in the vertical because the dot product is 0 for non-equal unit vectors [1]
Based on Figure: 2 respect the rotations when picking the bounds for the integrals:

$$\begin{aligned}LHS &= \int_2^3 xdy + \int_3^2 xdy \\ &= \int_2^3 4dy + \int_3^2 2dy \\ &= 2\end{aligned}$$

Calculating the RHS: The normal vector using the right hand rule, based on

[1]Split integrals if more than one direction in velocity field

the rotation is $\underline{k}$

$$\iint_S \underline{n} \cdot \nabla \times \underline{v} \, dS = \iint \underline{k} \cdot \underline{k} \, dS$$
$$= \int_2^3 \int_2^4 dxdy$$
$$= 2$$

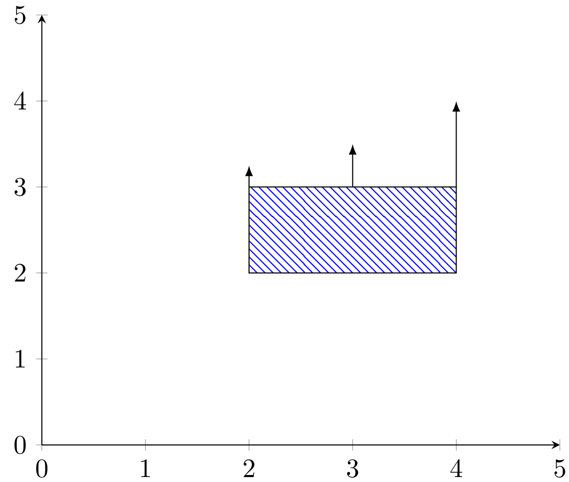
The inside cross product between the Del and V:

$$\nabla \times \underline{v} = \underline{i}\frac{\partial}{\partial x} \times (x\underline{j}) = \underline{k}$$

The partial cancels the x term and i cross j by convention is k. Only the x partial is included as all the other partials would be zero.
Therefor LHS = RHS

Figure 2, has arrows indicating the force and magnitude of different positions, they indicate that the rotation is anti-clockwise

Figure 2: Stokes Integral Figure



2 Streakline example

Given the velocity field: $\underline{v} = xt\underline{i} + \underline{j}$. Determine:

1. The stream lines
2. The path line (Parametric and explicit)
3. The streak lines (Parametric and explicit)

2.1 Stream lines

$$\frac{dx}{u} = \frac{dy}{v} => \frac{dx}{xt} = \frac{dy}{1}$$

$$\int \frac{dx}{x} = \int tdy$$
$$ln(x) = ty + c_1$$
$$x = e^{ty+c_1} = C_2e^{ty}$$

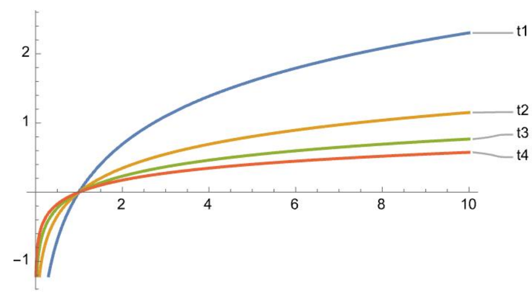


Figure 3: Stream Lines at different times

2.2 Path Lines

Parametric form:

$$\frac{dX}{dt} = Xt$$
$$\int \frac{dX}{x} = \int tdt$$
$$ln(X) = \frac{t^2}{2} + C_3$$
$$X = C_4e^{\frac{t^2}{2}}$$

At $t = 0$, $X = X_0$. Therefor $C_4 = X_0$

$$X = X_0e^{\frac{t^2}{2}} \tag{3}$$

Calculating the Y component:

$$\frac{dY}{dt} = 1$$
$$\int dY = \int dt$$
$$Y = t + C_5$$

At $t = 0$, $Y = Y_0$. Therefor $C_5 = Y_0$

$$Y = t + Y_0 \tag{4}$$

Explicit Form: Eliminate t, $t = Y - Y_0$

$$X = X_0 e^{0.5(Y-Y_0)^2}$$

$$\ln(\frac{X}{X_0}) = \frac{1}{2}(Y - Y_0)^2$$

$$Y = Y_0 + \sqrt{2\ln(\frac{X}{X_0})} \tag{5}$$

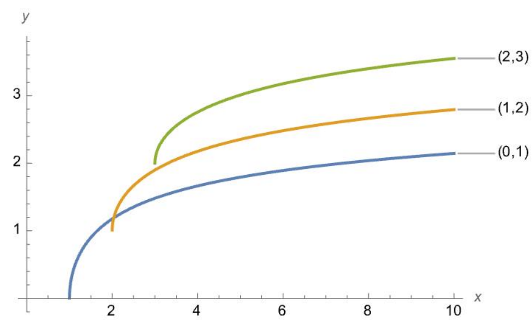


Figure 4: Path lines for different starting points

2.3 Streak lines

Parametric Form: Calculating the X component:

$$X = X_0 e^{0.5t^2} \tag{6}$$

$$X^* = X_0 e^{0.5\tau^2} \tag{7}$$

$$X_0 = X^* e^{-0.5\tau^2} \tag{8}$$

Substitute (8) into (6). Resulting in:

$$X = X^* e^{0.5(t^2-\tau^2)} \tag{9}$$

Calculating the Y component:

$$Y = Y_0 + t \tag{10}$$

$$Y^* = Y_0 + \tau \tag{11}$$

$$Y_0 = Y^* - \tau \tag{12}$$

Substitute (10) into (12). Resulting in:

$$Y = Y^* + t - \tau \tag{13}$$

Explicit Form: Using (9) to eliminate τ:

$$\ln(\frac{X}{X^*}) = \frac{1}{2}(t^2 - \tau^2)$$

$$\tau = \sqrt{t^2 - 2\ln(\frac{X}{X^*})} \tag{14}$$

Substitute (14) into (13), which results in the explicit form:

$$Y = Y^* + t - \sqrt{t^2 - 2\ln(\frac{X}{X^*})} \tag{15}$$