STAT 5361 Statistical Computing Solution

2018-10-16

Contents

Chapter 1

Exercise 3.1

1.1 Get Fisher Information

$$
\begin{aligned}
f(x;\theta) &= \frac{1}{\pi(1+(x-\theta)^2)} \\
\Rightarrow l(\theta) &= \sum_{i=1}^{n} \ln(f(X_i;\theta)) = -n\ln(\pi) - \sum_{i=1}^{n} \ln(1+(X_i-\theta)^2) \\
\Rightarrow l'(\theta) &= -2\sum_{i=1}^{n} \frac{\theta - X_i}{1+(\theta-X_i)^2} \\
\Rightarrow l''(\theta) &= -2\sum_{i=1}^{n} [\frac{1}{1+(\theta-X_i)^2} - \frac{2(\theta-X_i)^2}{[1+(\theta-X_i)^2]^2}] = -2\sum_{i=1}^{n} \frac{1-(\theta-X_1)^2}{[1+(\theta-X_i)^2]^2} \\
\Rightarrow I_n(\theta) &= -El''(\theta) \\
&= 2nE\frac{1-(\theta-X)^2}{[1+(\theta-X)^2]^2} \\
&= \frac{2n}{\pi}\int_R \frac{1-(x-\theta)^2}{(1+(x-\theta)^2)^3}dx \\
&= \frac{2n}{\pi}\int_R \frac{1-x^2}{(1+x^2)^3}dx \\
&= \frac{2n}{\pi}\int_R \frac{-1}{(1+x^2)^2} + 2\frac{2}{(1+x^2)^3}dx
\end{aligned}
$$

Also:

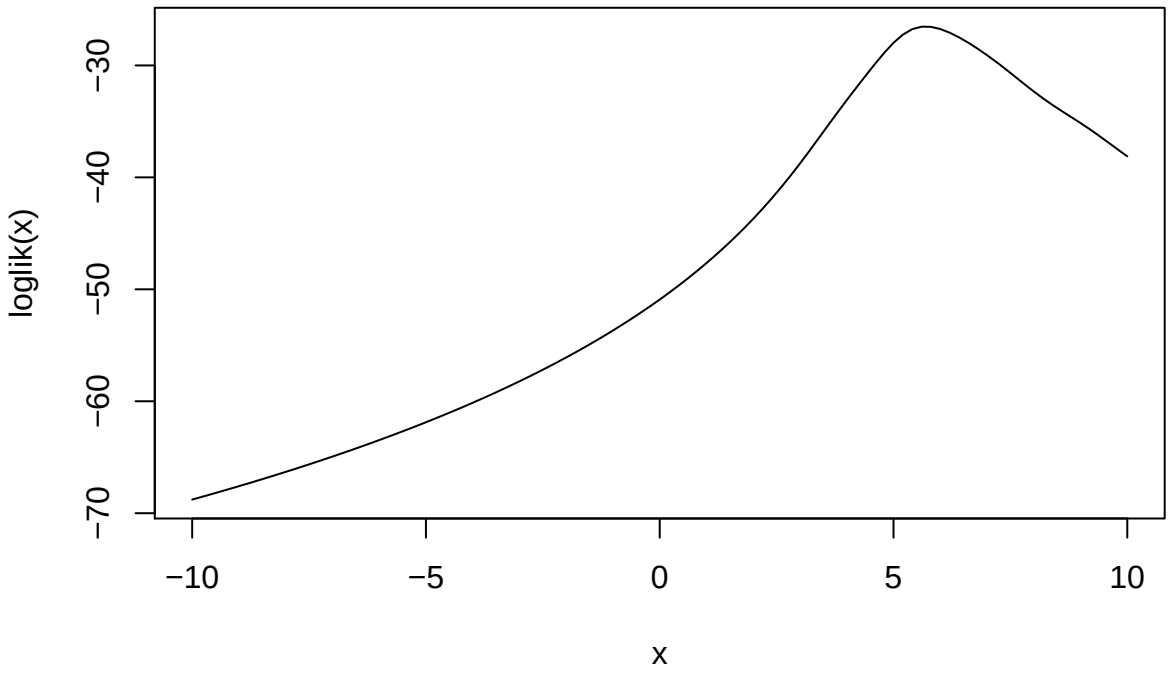
$$
\begin{aligned}
M_k &= \int_R \frac{1}{(1+x^2)^k} dx \\
&= \int_R \frac{1+x^2}{(1+x^2)^{k+1}} dx \\
&= M_{k+1} + \int_R \frac{x^2}{(1+x^2)^{k+1}} dx \\
&= M_{k+1} + \int_R \frac{2kx}{(1+x^2)^{k+1}} \frac{x}{2k} dx = M_{k+1} + \frac{1}{2k} M_k
\end{aligned}
$$

Since $M_1 = \pi$, we have $M_2 = \pi/2, M_3 = 3\pi/8$, then $I_n(\theta) = n/2$.

1.2 Implement loglikelihood with a random sample and plot against θ

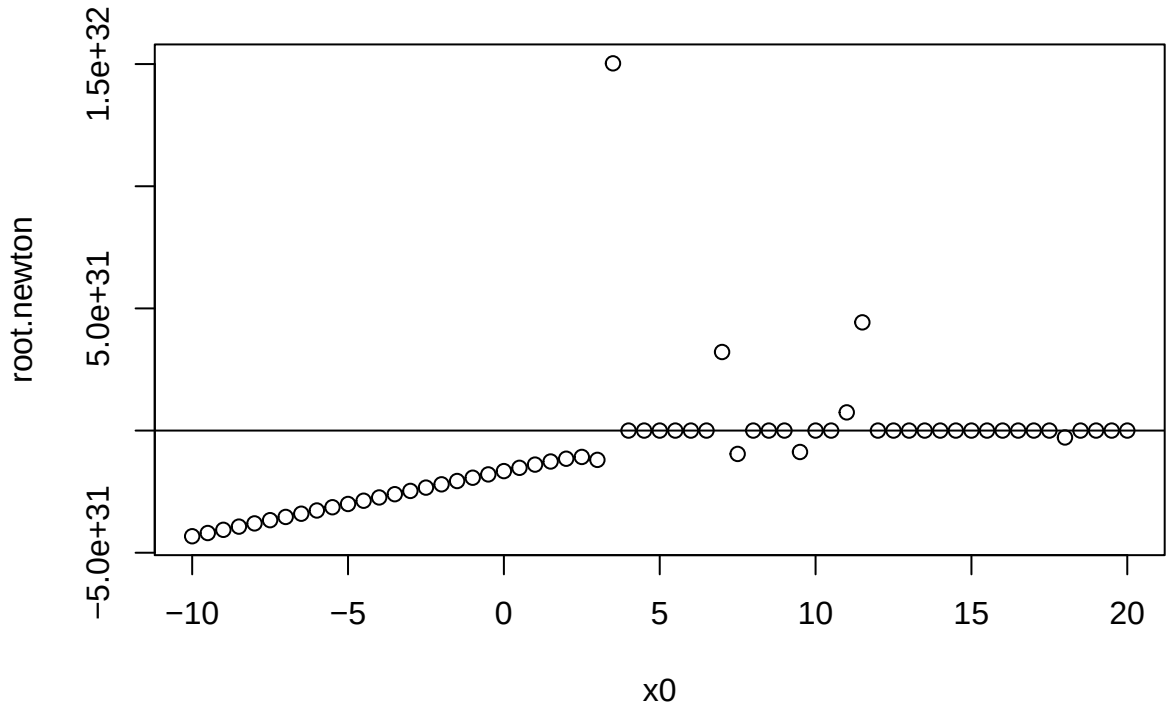
Use the loglikelihood function we got from above, set n = 10 and plug in the generated sample value X_i, we can get the loglikelihood function. When generating sample, the location parameter was set to be $\theta = 5$. The loglikelihood function curve against θ are shown in Figure **??**:

```
set.seed(20180909)
n <- 10
X <- rcauchy(n, location = 5, scale = 1)
loglik.0 <- function(theta) {
  l <- sum(dcauchy(X, location = theta, log = TRUE))
  l
}
loglik <- function(theta) {
  l <- sapply(theta, FUN = loglik.0)
  l
}
curve(loglik, -10, 10)
```



1.3 Newton-Raphson method

```
library(pracma)
## define the derivitive function
dev.loglik <- function(theta) {
  dev.l <- -2 * sum((theta-X)/(1+(theta-X)^2))
  dev.l
}
## define the hessian function
hessian.loglik <- function(theta) {
  h <- -2 * sum((1-(theta-X)^2) * (1+(theta-X)^2)^(-2))
  h
}
x0 <- seq(-10, 20, by = 0.5)
root.newton <- rep(0, length(x0))
for (i in 1:length(x0)) {
  root.newton[i] <- newtonRaphson(dev.loglik, x0 = x0[i], dfun = hessian.loglik)$root
}
plot(x0, root.newton)
abline(h = 5.442)
```




```
root.newton
```

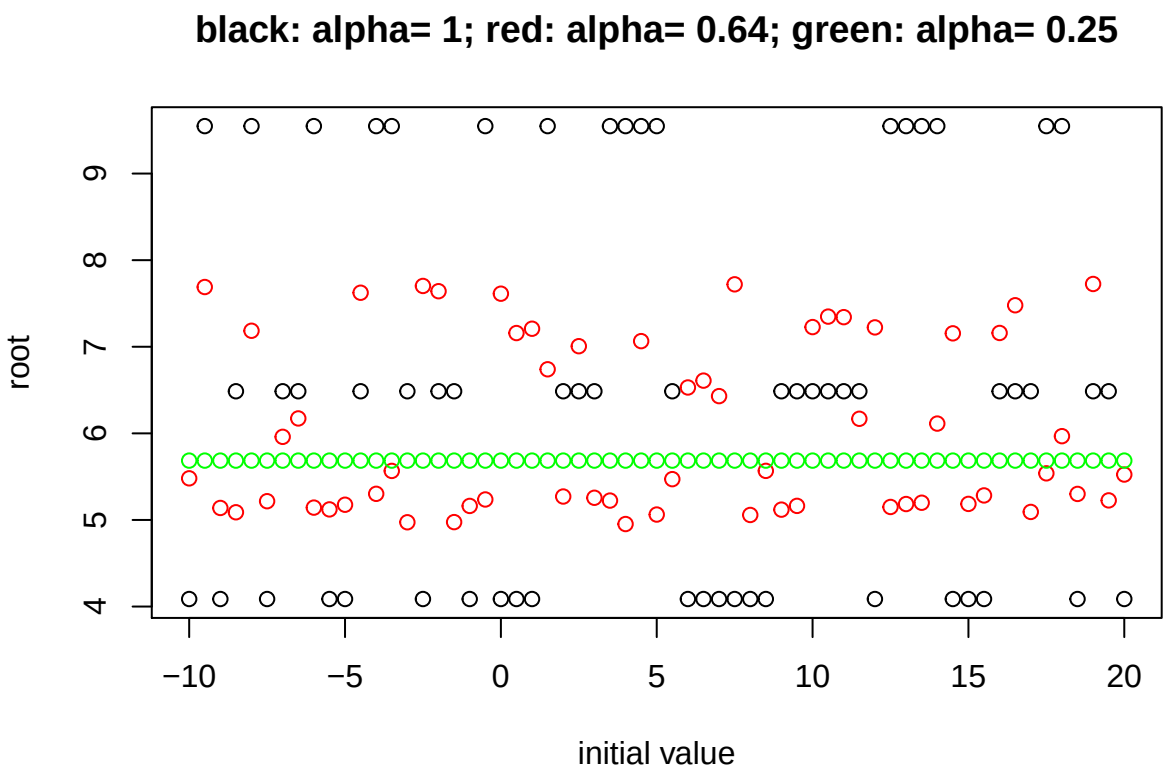
```
##  [1] -4.324741e+31 -4.193577e+31 -4.062249e+31 -3.930748e+31 -3.799064e+31
##  [6] -3.667185e+31 -3.535100e+31 -3.402796e+31 -3.270261e+31 -3.137479e+31
## [11] -3.004436e+31 -2.871118e+31 -2.737510e+31 -2.603599e+31 -2.469374e+31
## [16] -2.334832e+31 -2.199981e+31 -2.064850e+31 -1.929508e+31 -1.794100e+31
## [21] -1.658922e+31 -1.524582e+31 -1.392396e+31 -1.265439e+31 -1.151924e+31
## [26] -1.079358e+31 -1.199750e+31  1.502957e+32  2.056366e+01  2.108229e+01
## [31]  5.685422e+00  5.685422e+00  5.685422e+00  5.685422e+00  3.215974e+31
## [36] -9.558888e+30  1.937744e+01  2.108229e+01  5.685422e+00 -8.759488e+30
## [41]  2.108229e+01  5.685422e+00  7.439560e+30  4.429077e+31  2.056366e+01
## [46]  2.056366e+01  2.056366e+01  2.108229e+01  2.108229e+01  2.108230e+01
## [51]  1.937743e+01  2.056366e+01  2.056366e+01  1.937744e+01  1.937743e+01
## [56]  1.937744e+01 -2.825479e+30  2.056366e+01  2.056366e+01  2.056366e+01
## [61]  2.056366e+01
```

We can see that Newton method doesn't converge when initial value is not close to the real root.

1.4 Fixed point method

```
## self-defined fixed point methods to find mle
## input gradiant of loglikelihood function, x0 is initial value
FixPoint.mle <- function(dev.loglik, alpha, x0, maxiter = 100,
                         tol = .Machine$double.eps^0.5){
  x <- x0
  for (i in 1:maxiter) {
   x.new <- alpha * dev.loglik(x) + x
   if (abs(x.new - x) < tol) break
   x <- x.new
  }
```

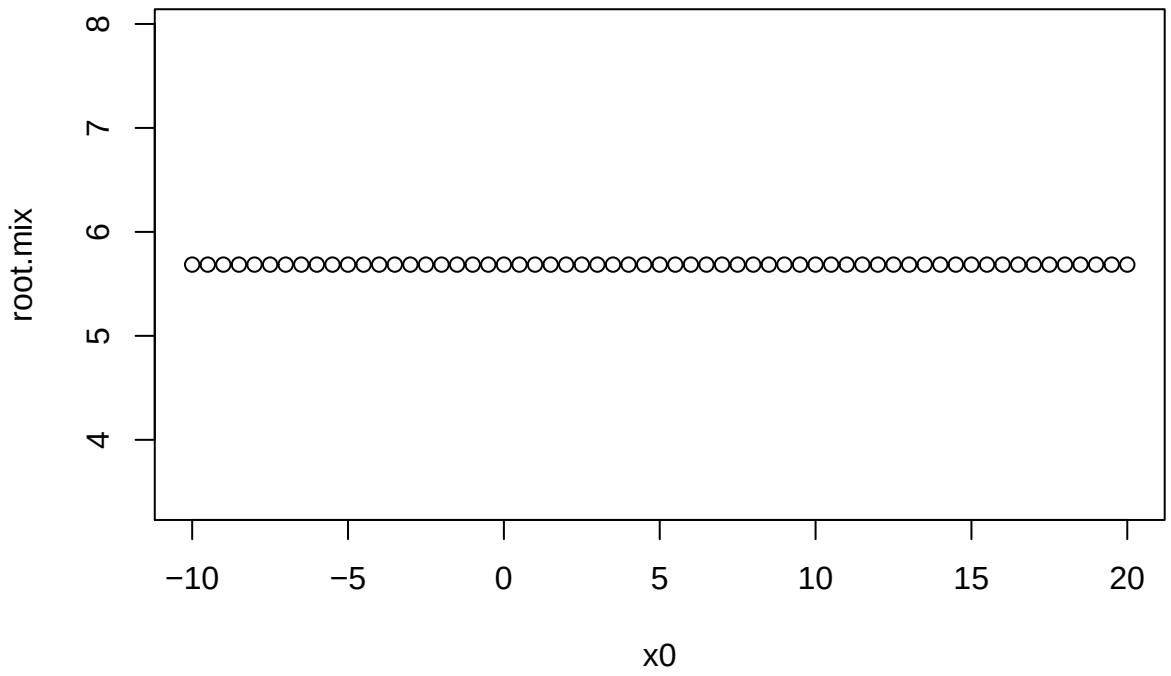
```
  if (i == maxiter) warning("maximum iteration has reached")
  return(list(root = x, niter = i))
}
alpha <- c(1, 0.64, 0.25)
root.fixpoint <- matrix(0, ncol = length(alpha), nrow = length(x0))
for (i in 1:length(alpha)) {
  for (j in 1:length(x0)) {
    root.fixpoint[j, i] <- FixPoint.mle(dev.loglik = dev.loglik, alpha = alpha[i],
                                        x0 = x0[j])$root
  }
}
plot(x0, root.fixpoint[, 1], ylim = c(min(root.fixpoint), max(root.fixpoint)),
     ylab = "root", xlab = "initial value",
     main = paste0("black: ", expression(alpha), "= 1; red: ", expression(alpha),
     "= 0.64; green: ", expression(alpha), "= 0.25"))
points(x0, root.fixpoint[, 2], col = "red")
points(x0, root.fixpoint[, 3], col = "green")
```



1.5 Fisher scoring and Newton-Raphson

```
## Self-defined fisher scoring method to find mle.
## input gradiant of loglikelihood and sample fisher information.
FisherScore.mle <- function(dev.loglik, information, x0, maxiter = 100,
                            tol = .Machine$double.eps^0.5) {
  x <- x0
  for (i in 1:maxiter) {
    x.new <- x + dev.loglik(x) / information(x)
```

```
    if (abs(x.new - x) < tol) break
    x <- x.new
   }
   if (i == maxiter) warning("maximum iteration has reached")
   return(list(root = x, niter = i))
}
FisherNewton.mle <- function(dev.loglik, information, dfun, x0, maxiter = 100,
                             tol = .Machine$double.eps^0.5) {
  method.fisher <- FisherScore.mle(dev.loglik = dev.loglik, information = information,
                                x0 = x0, maxiter = maxiter, tol = tol)
  x.fisher <- method.fisher$root
  niter.fisher <- method.fisher$niter
  method.newton <- newtonRaphson(fun = dev.loglik, x0 = x.fisher, dfun = dfun, maxiter = maxiter,
                         tol = tol)
  return(list(root = method.newton$root, niter.fisher = niter.fisher,
              niter.newton = method.newton$niter))
}
inf.cauchy <- function(x) n/2

root.mix <- rep(0, length(x0))
for (i in 1:length(x0)) {
  root.mix[i] <- FisherNewton.mle(dev.loglik, inf.cauchy, dfun = hessian.loglik,
                                  x0 = x0[i])$root
}
plot(x0, root.mix)
```



1.6 comparing the different methods

```
library(microbenchmark)
## comparing the speed of different methods
```

```
## use starting point 5, alpha = 0.25 for fixed point method
newton.method <- newtonRaphson(fun = dev.loglik, x0 = 5, dfun = hessian.loglik)
fixpoint.method <- FixPoint.mle(dev.loglik = dev.loglik, alpha = 0.25, x0 = 5)
fishernewton.method <- FisherNewton.mle(dev.loglik = dev.loglik, information = inf.cauchy,
                                dfun = hessian.loglik, x0 = 5)
list(newton.niter = newton.method$niter, fixpoint.niter = fixpoint.method$niter,
     fishernewton.niter = c(fishernewton.method$niter.fisher,
                            fishernewton.method$niter.newton))
```

```
## $newton.niter
## [1] 6
##
## $fixpoint.niter
## [1] 17
##
## $fishernewton.niter
## [1] 8 1
```

Fixed point method is most stable but converges slowly compare to the other two methods. Newton-Raphson methods converges fastest but is the most unstably one. Fisher-Scoring converges slower than Newton, but is very stable and accuracy, after refining with Newton-Raphson methods. Also we can see that if we use fisher scoring root to be the initial value of Newton-Raphson method, it will converge very fast.

Chapter 2

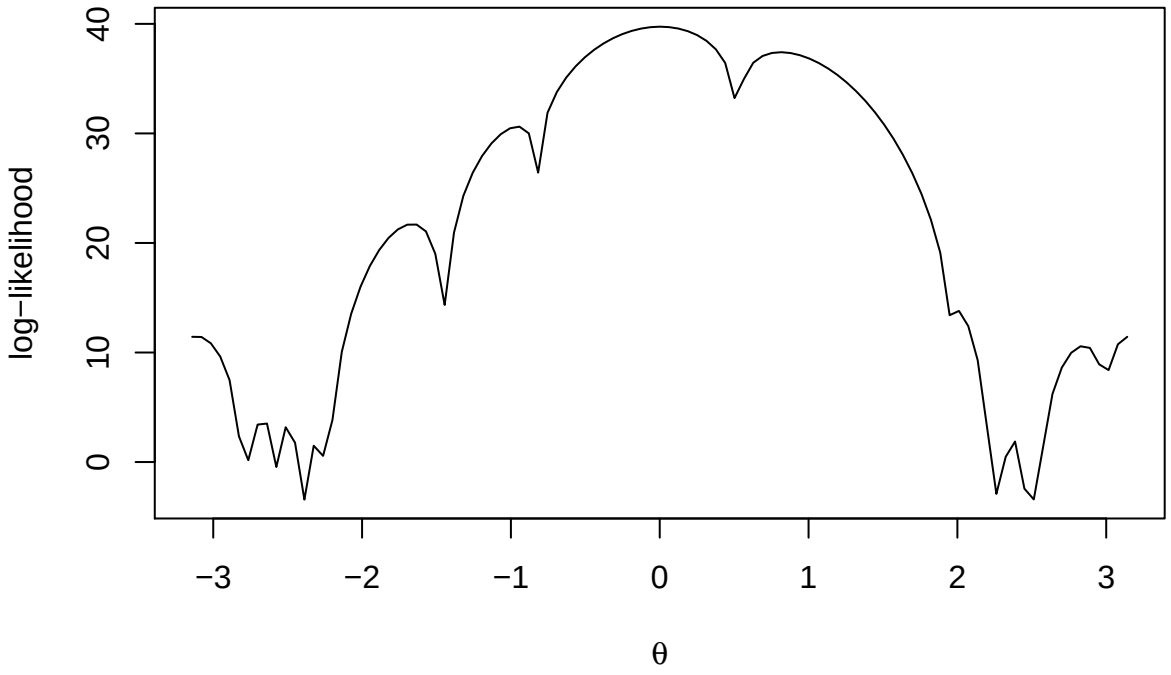
Exercise 3.2

2.1 Find the the log-likelihood function

The log-likelihood function of this distribution is

$$\ell(\mathbf{x}, \theta) = \sum_{i=1}^{n} \log\{1 - \cos(x_i - \theta)\} - n \log 2\pi$$

```
x <- c(3.91, 4.85, 2.28, 4.06, 3.70, 4.04, 5.46, 3.53, 2.28, 1.96,
       2.53, 3.88, 2.22, 3.47, 4.82, 2.46, 2.99, 2.54, 0.52)

loglikelihood <- function(theta){
  n <- length(x)
  s <- sum(log(1 - cos(x - theta))) + n * log(2 * pi)
  return(s)
}
loglikelihood <- Vectorize(loglikelihood)
curve(loglikelihood, -pi, pi, xlab = expression(theta), ylab = "log-likelihood")
```



2.2 Find the method-of-moments estimator

The expectation of $\mathbf{x}|\theta$ is

$$\begin{aligned}\mathbb{E}(x|\theta) &= \int_0^{2\pi} x\frac{1-\cos(x-\theta)}{2\pi}dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} x - x\cos(x-\theta)dx \\ &= \pi + \sin(\theta) \\ &= \bar{X_n}\end{aligned}$$

Thus,

```
theta_tilde <- asin(mean(x) - pi)
theta_tilde
```

```
## [1] 0.09539407
```

2.3 Find the MLE

Since

$$\frac{\partial \ell(\mathbf{x};\theta)}{\partial \theta} = \sum_{i=1}^{n} \frac{-\sin(x_i-\theta)}{1-\cos(x_i-\theta)}$$

$$\frac{\partial^2 \ell(\mathbf{x};\theta)}{\partial \theta^2} = \sum_{i=1}^{n} \frac{\cos(x_i-\theta)-\cos^2(x_i-\theta)-\sin^2(x_i-\theta)}{(1-\cos(x_i-\theta))^2}$$

The Newton-Raphson method is

$$\hat{\theta}^{(t+1)} = \hat{\theta}^{(t)} - \left\{ \frac{\partial^2 \ell(\mathbf{x}; \hat{\theta}^{(t)})}{\partial \theta^2} \right\}^{-1} \frac{\partial \ell(\mathbf{x}; \hat{\theta}^{(t)})}{\partial \theta}$$

```
lfd <- function(theta){
  sum(-sin(x-theta)/(1-cos(x-theta)))
}

lsd <- function(theta){
  sum((cos(x-theta) - (cos(x-theta))^2 - (sin(x-theta))^2)/(1-cos(x-theta))^2)
}

Newton <- function(init){
  theta0 <- init
  i <- 0
  diff <- 1
  msg <- "converge"
  while(abs(diff) > 0.0000001){
    lfd <- lfd(theta0)
    lsd <- lsd(theta0)
    diff <- (lfd/lsd)
    theta1 <- theta0 - diff
    theta0 <- theta1
    i <- i+1
    #cat(i)
    if(i >= 150){
      msg <- "Not converge"
      theta0 <- Inf
      break
    }
  }
  return(list(theta = theta0, itr = i, msg = msg))
}
Newton(theta_tilde)
```

```
## $theta
## [1] 0.003118157
##
## $itr
## [1] 4
##
## $msg
## [1] "converge"
```

2.4 $\theta_0 = 2.7$ or $\theta_0 = -2.7$

```
Newton(-2.7)
```

```
## $theta
## [1] -2.668857
##
```

```
## $itr
## [1] 4
##
## $msg
## [1] "converge"
```

```
Newton(2.7)
```

```
## $theta
## [1] 2.848415
##
## $itr
## [1] 5
##
## $msg
## [1] "converge"
```

The $\hat{\theta}$ we got is different.

2.5 Repeat the above using 200 equally spaced starting values

```
init <- seq(-pi, pi, length.out=200)
result <- NULL
for(initi in init){
  result <- rbind(result, c(initi, Newton(initi)$theta))
}
colnames(result) <- c("Initial_value", "theta_hat")
split(result, result[,2])
```

```
## $`-3.11247050669846`
##  [1] -3.141593 -3.110019 -3.078445 -3.046871 -3.015297 -2.983724 -2.952150
##  [8] -2.920576 -2.889002 -2.857428 -2.825855 -3.112471 -3.112471 -3.112471
## [15] -3.112471 -3.112471 -3.112471 -3.112471 -3.112471 -3.112471 -3.112471
## [22] -3.112471
##
## $`-2.78655685241805`
## [1] -2.794281 -2.786557
##
## $`-2.78655685241804`
## [1] -2.762707 -2.786557
##
## $`-2.66885745902142`
##  [1] -2.731133 -2.699560 -2.667986 -2.636412 -2.604838 -2.668857 -2.668857
##  [8] -2.668857 -2.668857 -2.668857
##
## $`-2.50935603320277`
##  [1] -2.573264 -2.541691 -2.510117 -2.478543 -2.446969 -2.415395 -2.509356
##  [8] -2.509356 -2.509356 -2.509356 -2.509356
##
## $`-2.38826662826452`
## [1] -2.383822 -2.388267
##
## $`-2.29792596896698`
```

```
## [1] -2.352248 -2.297926
##
## $`-2.29792596896697`
## [1] -2.320674 -2.289100 -2.257526 -2.297926 -2.297926 -2.297926
##
## $`-2.23219189887219`
## [1] -2.225953 -2.232192
##
## $`-1.66271239546243`
##  [1] -2.194379 -2.162805 -2.131231 -2.099657 -2.068084 -2.036510 -2.004936
##  [8] -1.973362 -1.941788 -1.910215 -1.878641 -1.847067 -1.815493 -1.783919
## [15] -1.752346 -1.720772 -1.689198 -1.594477 -1.531329 -1.499755 -1.468181
## [22] -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712
## [29] -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712
## [36] -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712 -1.662712
##
## $`-1.66271239546242`
## [1] -1.657624 -1.626050 -1.562903 -1.662712 -1.662712 -1.662712
##
## $`-1.44750255268373`
## [1] -1.436608 -1.447503
##
## $`-0.95440583712848`
##  [1] -1.4050339 -1.3103125 -1.2787387 -1.2155911 -1.1208697 -1.0577221
##  [7] -1.0261484 -0.9945746 -0.9544058 -0.9544058 -0.9544058 -0.9544058
## [13] -0.9544058 -0.9544058 -0.9544058 -0.9544058
##
## $`-0.954405837128479`
##  [1] -1.3734601 -1.3418863 -1.1524435 -1.0892959 -0.9630008 -0.8998532
##  [7] -0.8682794 -0.8367056 -0.9544058 -0.9544058 -0.9544058 -0.9544058
## [13] -0.9544058 -0.9544058 -0.9544058 -0.9544058
##
## $`-0.954405837128476`
## [1] -0.9314270 -0.9544058
##
## $`-0.95440583712847`
## [1] -1.2471649 -0.9544058
##
## $`-0.954405837128466`
## [1] -1.1840173 -0.9544058
##
## $`0.00311815708656577`
## [1] -0.489393830  0.003118157
##
## $`0.0031181570865658`
## [1] -0.078934489  0.003118157
##
## $`0.00311815708656581`
## [1] 0.110508284 0.003118157
##
## $`0.00311815708656585`
## [1] -0.236803466  0.003118157
##
## $`0.00311815708656587`
```

```
## [1] -0.142082080 -0.047360693  0.003118157  0.003118157
##
## $`0.00311815708656589`
## [1] -0.678836604 -0.584115217  0.003118157  0.003118157
##
## $`0.00311815708656591`
## [1] -0.110508284  0.015786898  0.003118157  0.003118157
##
## $`0.00311815708656593`
## [1] -0.710410399  0.142082080  0.299951057  0.003118157  0.003118157
## [6]  0.003118157
##
## $`0.00311815708656597`
## [1] -0.457820035  0.003118157
##
## $`0.00311815708656598`
## [1] -0.015786898  0.236803466  0.268377262  0.003118157  0.003118157
## [6]  0.003118157
##
## $`0.00311815708656599`
## [1] -0.805131786  0.003118157
##
## $`0.003118157086566`
## [1] -0.741984195  0.003118157
##
## $`0.00311815708656601`
## [1] -0.268377262  0.394672444  0.003118157  0.003118157
##
## $`0.00311815708656602`
## [1] -0.647262808 -0.552541421  0.078934489  0.003118157  0.003118157
## [6]  0.003118157
##
## $`0.00311815708656603`
## [1] -0.773557990 -0.615689013  0.047360693  0.489393830  0.003118157
## [6]  0.003118157  0.003118157  0.003118157
##
## $`0.00311815708656604`
## [1] -0.363098648  0.003118157
##
## $`0.00311815708656606`
## [1] 0.457820035 0.003118157
##
## $`0.00311815708656607`
## [1] -0.426246239  0.003118157
##
## $`0.00311815708656609`
## [1] -0.173655875  0.003118157
##
## $`0.00311815708656611`
## [1] -0.205229671  0.003118157
##
## $`0.00311815708656612`
## [1] -0.394672444  0.363098648  0.003118157  0.003118157
##
```

```
## $`0.00311815708656613`
## [1] -0.299951057  0.003118157
##
## $`0.00311815708656615`
## [1] 0.205229671 0.003118157
##
## $`0.00311815708656793`
## [1] 0.426246239 0.003118157
##
## $`0.00311815708656861`
## [1] -0.520967626  0.003118157
##
## $`0.00311815708656864`
## [1] 0.331524853 0.003118157
##
## $`0.00311815708656926`
## [1] -0.331524853  0.003118157
##
## $`0.00311815708656987`
## [1] 0.173655875 0.003118157
##
## $`0.812637416717926`
## [1] 1.2787387 0.8126374
##
## $`0.812637416717938`
## [1] 0.8051318 1.4050339 0.8126374 0.8126374
##
## $`0.812637416717939`
## [1] 0.6788366 0.8126374
##
## $`0.81263741671794`
##  [1] 0.5209676 0.5525414 0.5841152 0.6156890 0.6472628 0.7104104 0.7419842
##  [8] 0.7735580 0.8367056 0.8682794 0.8998532 0.9314270 0.9630008 0.9945746
## [15] 1.0261484 1.0577221 1.0892959 1.1208697 1.1524435 1.1840173 1.2155911
## [22] 1.2471649 1.3103125 1.3418863 1.3734601 1.4366077 1.4681815 1.4997553
## [29] 1.5313291 1.5629029 1.5944767 1.6260505 1.6576243 1.6891981 1.7207719
## [36] 1.7523457 1.7839194 1.8154932 1.8470670 1.8786408 1.9102146 1.9417884
## [43] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
## [50] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
## [57] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
## [64] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
## [71] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
## [78] 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374 0.8126374
##
## $`2.00722323801594`
##  [1] 1.973362 2.004936 2.068084 2.099657 2.131231 2.162805 2.194379
##  [8] 2.007223 2.007223 2.007223 2.007223 2.007223 2.007223 2.007223
##
## $`2.00722323801595`
## [1] 2.036510 2.007223
##
## $`2.23701292270577`
## [1] 2.225953 2.257526 2.237013 2.237013
##
```

```
## $`2.37471166606864`
##  [1] 2.289100 2.320674 2.352248 2.383822 2.415395 2.446969 2.374712
##  [8] 2.374712 2.374712 2.374712 2.374712 2.374712
##
## $`2.48844965088485`
## [1] 2.478543 2.488450
##
## $`2.48844965088489`
## [1] 2.510117 2.488450
##
## $`2.84841532545741`
##  [1] 2.541691 2.573264 2.604838 2.636412 2.667986 2.699560 2.731133
##  [8] 2.762707 2.794281 2.825855 2.857428 2.920576 2.952150 2.983724
## [15] 2.848415 2.848415 2.848415 2.848415 2.848415 2.848415 2.848415
## [22] 2.848415 2.848415 2.848415 2.848415 2.848415 2.848415 2.848415
##
## $`2.84841532545742`
## [1] 2.889002 2.848415
##
## $`3.17071480048113`
##  [1] 3.015297 3.046871 3.078445 3.110019 3.141593 3.170715 3.170715
##  [8] 3.170715 3.170715 3.170715
```

Chapter 3

Exercise 3.3

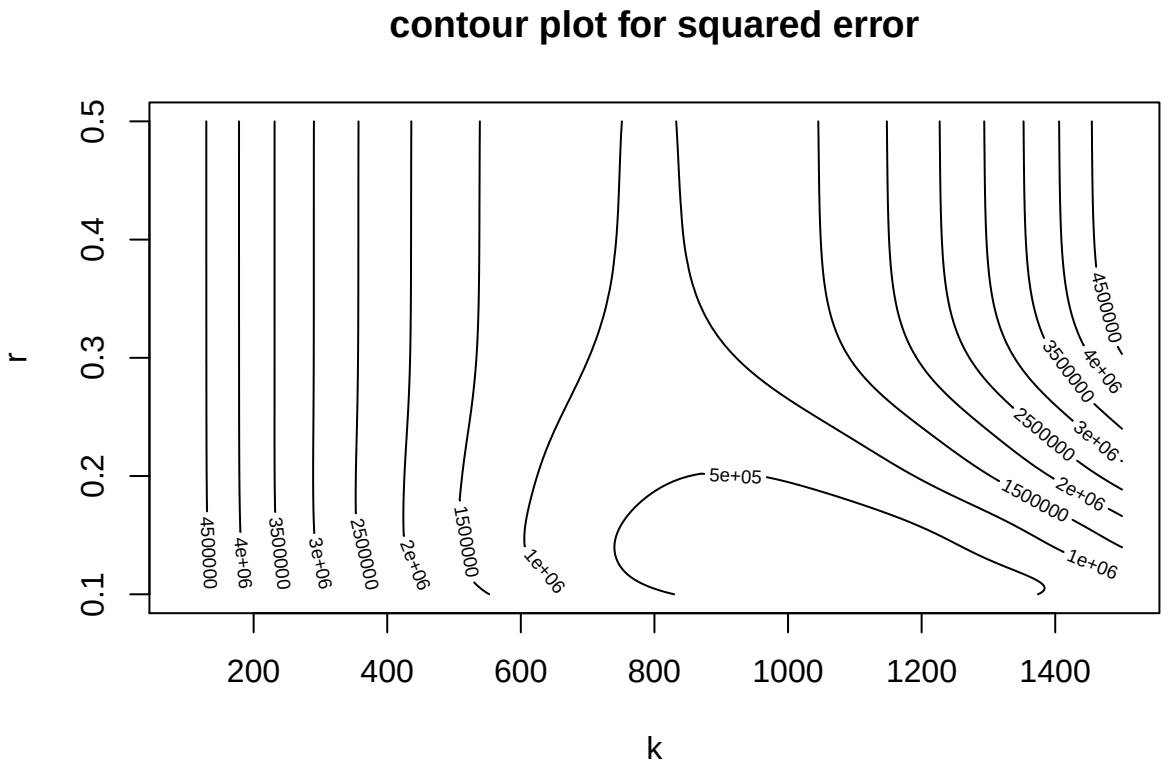
3.1 Fit the population growth model to the beetles data using the Gauss-Newton approach

```
library(graphics)
library(Matrix)
beetles <- data.frame(
    days    = c(0,  8,  28,  41,  63,  69,   97, 117,  135,  154),
    beetles = c(2, 47, 192, 256, 768, 896, 1120, 896, 1184, 1024))

##' define the sum of squared errors function
sqerr <- function(k, r) {
  s <- matrix(0, nrow = length(k), ncol = length(r))
  for (i in 1:length(k)) {
    for (j in 1:length(r)) {
      s[i, j] <- sum((beetles$beetles - k[i] * beetles$beetles[1] /
              (beetles$beetles[1] + (k[i] - beetles$beetles[1]) *
                 exp(-r[j]*beetles$days)))^2)
    }
  }
  s
}

##' define z function
z.vec <- function(k, r) {
  n <- length(beetles$days)
  z <- rep(0, n)
  for (i in 1:n) {
   z[i] <- beetles$beetles[i] - k*beetles$beetles[1] /
     (beetles$beetles[1] + (k - beetles$beetles[1])*exp(-r*beetles$days[i]))
  }
  return(z)
}

##' define A matrix
A.mat <- function(k, r) {
  n <- length(beetles$days)
```

```
  A <- matrix(0, nrow = n, ncol = 2)
  for (i in 1:n) {
    A[i, 1] <- beetles$beetles[1]^2 * (1-exp(-r*beetles$days[i])) /
      (beetles$beetles[1] + (k-beetles$beetles[1])*exp(-r*beetles$days[i]))^2

    A[i, 2] <- beetles$beetles[1]*k*beetles$days[i]*(k-beetles$beetles[1]) *
      exp(-r*beetles$days[i]) / (beetles$beetles[1]+(k-beetles$beetles[1]) *
                                   exp(-r*beetles$days[i]))^2
  }
  return(A)
}

gaussNewton.beetles <- function(para0, z.vec, A.mat, maxiter = 100,
                                tol = .Machine$double.eps^0.5) {
  para <- para0
  for (i in 1:maxiter) {
    Amat <- A.mat(para[1], para[2])
    zvec <- z.vec(para[1], para[2])
    para.new <- para + solve(t(Amat) %*% Amat + 0.0001*diag(nrow = 2)) %*%
      t(Amat) %*% zvec
    if (sum(abs(para.new - para)) < tol) break
    para <- para.new
  }
  if (i == maxiter) warning("maximum iteration has reached")
  return(list(root = para, niter = i))
}

fit <- gaussNewton.beetles(para0 = c(1200, 0.1), z.vec = z.vec, A.mat = A.mat)
fit$root
```

```
##              [,1]
## [1,] 1049.4072443
## [2,]    0.1182684
```

```
k <- seq(100, 1500, by = 10)
r <- seq(0.1, 0.5, by = 0.001)
z <- sqerr(k, r)
contour(k, r, z, xlab = "k", ylab = "r", main = "contour plot for squared error")
```



Estimation using Gauss-Newton method is $\hat{k} = 1049$, $\hat{r} = 0.12$.

3.2 Log normal assumption

```
##' define log-likelihood function
loglikeli <- function(para) {
  k <- para[1]
  r <- para[2]
  sigma2 <- para[3]
  l <- sum(-log(2*pi*sigma2)/2 - (log(beetles$beetles) - log(k) -
                                    log(beetles$beetles[1]) +
                                    log(beetles$beetles[1] +
                                          (k-beetles$beetles[1]) *
                                          exp(-r*beetles$days))^2)/2/sigma2)
  return(l)
}

##' define gradiant of loglikelihood function
grad.my <- function(para) {
  k <- para[1]
  r <- para[2]
  sigma2 <- para[3]
  g <- rep(0, 3)
  g[1] <- sum(-2*(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                    log(beetles$beetles[1]+(k-beetles$beetles[1])*
                          exp(-r*beetles$days)))*
                (-1/k+exp(-r*beetles$days)/
                   (beetles$beetles[1]+
```

```
                      (k-beetles$beetles[1])*exp(-r*beetles$days)))/2/sigma2)
  g[2] <- sum(2*(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                   log(beetles$beetles[1]+(k-beetles$beetles[1])*
                         exp(-r*beetles$days)))*
                beetles$days*(k-beetles$beetles[1])*exp(-r*beetles$days)/
                (beetles$beetles[1]+(k-beetles$beetles[1])*exp(-r*beetles$days))/
                2/sigma2)
  g[3] <- sum(-1/2/sigma2+(log(beetles$beetles)-log(k)-log(beetles$beetles[1])+
                             log(beetles$beetles[1]+(k-beetles$beetles[1])*
                                   exp(-r*beetles$days)))^2/2/sigma2^2)
  return(g)
}

fit <- constrOptim(theta = c(10, 0.1, 1), f = loglikeli, grad = grad.my,
                   ui = diag(1, 3), ci = rep(0, 3),
                   control = list(fnscale = -1), hessian = TRUE)
fit$par
```

```
## [1] 103.983292  12.124656   2.913935
```

```
fit$convergence
```

```
## [1] 0
```

```
Diag(-solve(fit$hessian))
```

```
## [1] 2.707176e+03 1.212466e+05 2.841633e+00
```

Using BFGS methods, set constrain to make k, r, $sigma^2$ nonnegative, we get the MLE estimates $\hat{k} = 103.98$, $\hat{r} = 12.12$, $\hat{\sigma}^2 = 2.91$. Using inverse of negative hessian, we have these estimates' variance to be 2.7×10^3, 1.2×10^5, 2.8 respectively.

But there's problem that results will change dramatically with initial values. I just tried several different initial values, get the estimates and compare their corresponding loglikelihood. I just choose the one with maximum loglikelihood.

Chapter 4

Exercise 4.8.1

4.1 E/M-Step Derivations

$$
\begin{aligned}
Q(\Psi|\Psi^{(k)}) &= \sum_{Z}\Big[p(Z|\mathbf{y},X,\Psi^{(k)})\log p(\mathbf{y},Z|X,\Psi)\Big] && (4.1)\\
&= \sum_{Z}\left[p(Z|\mathbf{y},X,\Psi^{(k)})\log\prod_{i=1}^{n}p(y_i,\mathbf{z}_i|\mathbf{x}_i,\Psi)\right] && (4.2)\\
&= \sum_{i=1}^{n}\sum_{Z}\Big[p(Z|\mathbf{y},X,\Psi^{(k)})\log p(y_i,\mathbf{z}_i|\mathbf{x}_i,\Psi)\Big] && (4.3)\\
&= \sum_{i=1}^{n}\sum_{\mathbf{z}_i}\Big[p(\mathbf{z}_i|\mathbf{y},X,\Psi^{(k)})\log p(y_i,\mathbf{z}_i|\mathbf{x}_i,\Psi)\Big] && (4.4)\\
&= \sum_{i=1}^{n}\sum_{\mathbf{z}_i}\Big[p(\mathbf{z}_i|y_i,\mathbf{x}_i,\Psi^{(k)})\log p(y_i,\mathbf{z}_i|\mathbf{x}_i,\Psi)\Big] && (4.5)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\Big[p(\mathbf{z}_i=(0,\cdots,1,\cdots,0)'|y_i,\mathbf{x}_i,\Psi^{(k)})\log p(y_i,\mathbf{z}_i=(0,\cdots,1,\cdots,0)'|\mathbf{x}_i,\Psi)\Big] && (4.6)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\Big[p(z_{ij}=1|y_i,\mathbf{x}_i,\Psi^{(k)})\log p(y_i,\mathbf{z}_i=(0,\cdots,1,\cdots,0)'|\mathbf{x}_i,\Psi)\Big] && (4.7)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\Big[E(z_{ij}|y_i,\mathbf{x}_i,\Psi^{(k)})\{\log\pi_j+\log\varphi(y_i-\mathbf{x}_i^T\boldsymbol{\beta}_j,0,\sigma^2)\}\Big] && (4.8)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}p_{ij}^{(k)}\{\log\pi_j+\log\varphi(y_i-\mathbf{x}_i^T\boldsymbol{\beta}_j,0,\sigma^2)\} && (4.9)
\end{aligned}
$$

where

$$
\mathbf{y}=(y_1,\cdots,y_n)'
$$

$$
Z=\begin{pmatrix}\mathbf{z}_1'\\ \vdots\\ \mathbf{z}_n'\end{pmatrix}=\begin{pmatrix}z_{11}&z_{12}&\cdots&z_{1m}\\ \vdots&\vdots&\vdots&\vdots\\ z_{n1}&z_{n2}&\cdots&z_{nm}\end{pmatrix}\quad X=\begin{pmatrix}\mathbf{x}_1'\\ \vdots\\ \mathbf{x}_n'\end{pmatrix}=\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1p}\\ \vdots&\vdots&\vdots&\vdots\\ x_{n1}&x_{n2}&\cdots&x_{np}\end{pmatrix}
$$

$$
p_{ij}^{(k)}=E(z_{ij}|y_i,\mathbf{x}_i,\Psi^{(k)})=\frac{\pi_j^{(k)}\varphi(y_i-\mathbf{x}_i^T\boldsymbol{\beta}_j^{(k)},0,\sigma^{2^{(k)}})}{\sum_{j=1}^{m}\pi_j^{(k)}\varphi(y_i-\mathbf{x}_i^T\boldsymbol{\beta}_j^{(k)},0,\sigma^{2^{(k)}})}
$$

The elaboration of the above steps are

- Step1→Step2: Use independence among $(y_i, \mathbf{z}_i)$
- Step3→Step4: Marginal density of $\mathbf{z}_i$
- Step4→Step5: Use the fact $\mathbf{z}_i \perp (y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)|y_i$, we can get rid of $(y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_n)$
- Step6→Step7: Easy to see conditional joint density is equal to condition marginal density

For M-step, since we have

$$\begin{aligned}
Q(\Psi|\Psi^{(k)}) &= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)}\{\log \pi_j + \log \varphi(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j, 0, \sigma^2)\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \pi_j - \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \sqrt{2\pi}\sigma - \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \frac{(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j)^2}{2\sigma^2} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \frac{\pi_j}{\sqrt{2\pi}} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \sigma^2 - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \frac{(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j)^2}{\sigma^2} \\
&= I_1 - \frac{1}{2}I_2 - \frac{1}{2}I_3
\end{aligned}$$

From the above, we can see only I_3 contains $\boldsymbol{\beta}_j$ and

$$I_3 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \frac{(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j)^2}{\sigma^2} = \sum_{j=1}^{m}\sum_{i=1}^{n} p_{ij}^{(k)} \frac{(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j)^2}{\sigma^2}$$

To minimize I_3, we only need to fix j and optimize with regard to $\boldsymbol{\beta}_j$. We can directly use the formula from generazied least square method and obtain

$$\boldsymbol{\beta}_j^{(k+1)} = (X'V^{-1}X)^{-1}X'V^{-1}\mathbf{y} = \left(\sum_{i=1}^{n}\mathbf{x}_i\mathbf{x}_i^T p_{ij}^{(k)}\right)^{-1}\left(\sum_{i=1}^{n}\mathbf{x}_i p_{ij}^{(k)} y_i\right) \quad j = 1, \cdots, m$$

where

$$V^{-1} = diag(p_{1j}^{(k)}, \cdots, p_{nj}^{(k)})$$

Only I_2 and I_3 contains σ^2, since

$$I_2 + I_3 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \sigma^2 + \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \frac{(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j)^2}{\sigma^2}$$

We minimize it with regard to σ^2 given $\boldsymbol{\beta}_j = \boldsymbol{\beta}_j^{(k+1)}$ and obtain

$$\sigma^{2^{(k+1)}} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)}(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j^{(k+1)})^2}{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)}} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)}(y_i - \mathbf{x}_i^T\boldsymbol{\beta}_j^{(k+1)})^2}{n}$$

Only I_1 contains π_j and

$$I_1 = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} \log \frac{\pi_j}{\sqrt{2\pi}} = -\frac{1}{2}\log(2\pi)\sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}^{(k)} + \sum_{j=1}^{m}\left(\sum_{i=1}^{n} p_{ij}^{(k)}\right)\log \pi_j$$

In order to maximize I_1 under constraint $\pi_1 + \cdots \pi_m = 1$. We use Lagrange multiplier method

$$L(\pi_1, \cdots, \pi_m) = \sum_{j=1}^{m}\left(\sum_{i=1}^{n} p_{ij}^{(k)}\right)\log \pi_j - \lambda\left(\sum_{j=1}^{m}\pi_j - 1\right)$$

with λ a Lagrange multiplier. We can obtain

$$\pi_j^{(k+1)} = \frac{\sum_{i=1}^n p_{ij}^{(k)}}{\sum_{j=1}^m \sum_{i=1}^n p_{ij}^{(k)}} = \frac{\sum_{i=1}^n p_{ij}^{(k)}}{n} \quad j = 1, \cdots, m$$

4.2 A Function to Implement EM Algorithm

```
regmix_em <- function(y, xmat, pi.init, beta.init, sigma.init,
                      control = list(maxiter = 100, tol = .Machine$double.eps^0.2)){

  xmat <- as.matrix(xmat)

  n <- nrow(xmat)
  p <- ncol(xmat)
  m <- length(pi.init)

  pi <- pi.init
  beta <- beta.init
  sigma <- sigma.init

  maxiter <- control$maxiter
  tol <- control$tol
  conv <- 1

  P <- matrix(NA, nrow = n, ncol = m)
  beta.new <- matrix(NA, nrow = p, ncol = m)

  for (i in 1:maxiter) {
    for (j in 1:n) {
      P[j,] <- pi * dnorm(y[j] - xmat[j,] %*% beta, 0, sigma)/
        sum(pi * dnorm(y[j] - xmat[j,] %*% beta, 0, sigma))
    }

    pi.new <- apply(P, MARGIN = 2, mean)

    for (j in 1:m) {
      beta.new[,j] <- solve(t(xmat) %*% diag(P[,j]) %*% xmat) %*% t(xmat) %*% diag(P[,j]) %*% y
    }

    sigma.new <- sqrt(sum(P * (y %*% t(rep(1, m)) - xmat %*% beta.new)^2)/n)

    conv <- sum(abs(pi.new - pi)) + sum(abs(beta.new - beta)) + abs(sigma.new - sigma)
    if(conv < tol) break

    pi <- pi.new
    beta <- beta.new
    sigma <- sigma.new

  }

  if(i == maxiter)
  message("Reached the maximum iteration!")
```

```
  list(pi = pi.new, beta = beta.new, sigma = sigma.new, conv = conv, iter = i)

}
```

4.3 Data Generation and Parameters Estimation

After I carried out the following code, I found parameters won't be updated after the second iteration. Tracing back to E-Step Derivation, we can see if $\beta_1 = \cdots = \beta_m$, then $\mathbf{p}_{\cdot j}^{(k)}$ and $\pi_j^{(k)}$ will remain the same at all times.

```
regmix_sim <- function(n, pi, beta, sigma) {
    K <- ncol(beta)
    p <- NROW(beta)
    xmat <- matrix(rnorm(n * p), n, p) # normal covaraites
    error <- matrix(rnorm(n * K, sd = sigma), n, K)
    ymat <- xmat %*% beta + error # n by K matrix
    ind <- t(rmultinom(n, size = 1, prob = pi))
    y <- rowSums(ymat * ind)
    data.frame(y, xmat)
}

n <- 400
pi <- c(.3, .4, .3)
beta <- matrix(c( 1,  1,  1,
                 -1, -1, -1), 2, 3)
sigma <- 1
set.seed(1205)
dat <- regmix_sim(n, pi, beta, sigma)

fit <- regmix_em(y = dat[,1], xmat = dat[,-1],
          pi.init = pi / pi / length(pi),
          beta.init = beta * 0,
          sigma.init = sigma / sigma,
          control = list(maxiter = 500, tol = 1e-5))
fit
```

```
## $pi
## [1] 0.3333333 0.3333333 0.3333333
##
## $beta
##            [,1]       [,2]       [,3]
## [1,]  0.3335660  0.3335660  0.3335660
## [2,] -0.4754645 -0.4754645 -0.4754645
##
## $sigma
## [1] 1.732492
##
## $conv
## [1] 0
##
## $iter
## [1] 2
```

Thus we change the initial values of $\beta_1, \cdots, \beta_m$. And we can see this time after 83 iterations, the algorithm converged and I got the following consequences.

```
fit1 <- regmix_em(y = dat[,1], xmat = dat[,-1],
          pi.init = pi / pi / length(pi),
          beta.init = matrix(1:6, 2, 3),
          sigma.init = sigma / sigma,
          control = list(maxiter = 500, tol = 1e-5))
fit1
```

```
## $pi
## [1] 0.3454017 0.3858262 0.2687721
##
## $beta
##            [,1]      [,2]       [,3]
## [1,] -0.9136801 0.8796636  0.9912061
## [2,] -1.1990374 0.9341887 -1.2424685
##
## $sigma
## [1] 1.023598
##
## $conv
## [1] 9.786183e-06
##
## $iter
## [1] 83
```