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Some exact solutions to the nonlinear shallow-water wave equations

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These exact solutions correspond to time-dependent motions in parabolic basins. A characteristic feature is that the shoreline is not fixed. It is free to move and must be determined as part of the solution. In general, the motion is oscillatory and has the appropriate small-amplitude limit. For the case in which the parabolic basin reduces to a flat plane, there is a solution for a flood wave. These solutions provide a valuable test for numerical models of inundating storm tides.

1. Introduction

Exact solutions for nonlinear fluid motions with moving boundaries are quite rare. One classical example is the solution for the motion of a rotating, gravitating mass of fluid with an ellipsoidal bounding surface (Dirichlet 1860; Lamb 1945, §382). Another is the well-known solution for gravity wave motion such that the fluid particles follow closed circular orbits and the shape of the surface is trochoidal (Gerstner 1802; Rankine 1863; Lamb 1945, §251). More recent examples are the similarity solutions of Freeman (1972) and Sachdev (1980) for gravity wave motion in the hydraulic approximation and the generalizations of the Dirichlet ellipsoid by Longuet-Higgins (1972, 1976) to hyperbolic and parabolic boundaries. More closely akin to the solutions presented here are those of Carrier & Greenspan (1958) for water waves on a sloping beach, because, for both families of solutions, the motion is governed by the shallow-water equations and the shoreline is the moving boundary.

The solutions presented here complement the work of Ball (1964). His idea was first to make assumptions about the nature of the motion and then to solve for the basin in which that motion should be possible. That approach is also taken here, but, rather than using Lagrangian variables that are tied to the flow, the Eulerian equations are solved directly. The solutions that are obtained are similar to those sought by Ball, and, like his, require the shape of the basin to be parabolic.

These solutions can best be described as nonlinear normal mode oscillations of water in a parabolic basin. In one case the water's surface remains planar as it oscillates, and in another the surface is an oscillating paraboloid. Another solution corresponds to a flood wave caused by a parabolic mound spreading across a frictionless plane. The most general case can be thought of as a nonlinear superposition of simpler motions.

The only solutions considered here are those for which the water's surface is planar or parabolic. These have the appropriate small-amplitude limit, corresponding to linear normal mode oscillations within fixed boundaries (Thacker 1977; Lamb 1945, §§210, 212). Attempts to find other solutions for which the surface could be described by polynomials of higher than second degree led to overdetermined sets of equations. These attempts are not described here, but the conclusion to be made is that there is not a one-to-one correspondence between each of the infinite set of linear oscillations and a finite-amplitude counterpart. In fact, there may not be any polynomial solutions of degree higher than second.

An interesting feature of these solutions is that no bore forms as the water flows up the sloping sides of the basin. This is in agreement with the conclusion of Carrier & Greenspan (1958) that whether or not a wave breaks as it runs up a beach depends upon its initial shape and velocity distribution. Whereas their analysis was based upon one-dimensional motion in a basin with linearly varying depth, the analysis presented here is for two-dimensional motion, including the effects of a Coriolis force, in a parabolic basin.

2. The general case

The motion of water in shallow basins is governed by the shallow-water wave equations,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-fv+g\frac{\partial h}{\partial x}=0, \tag{1}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+fu+g\frac{\partial h}{\partial y}=0 \tag{2}$$

and

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}[u(D+h)]+\frac{\partial}{\partial y}[v(D+h)]=0. \tag{3}$$

The first two describe the evolution of the components u and v of velocity corresponding to the orthogonal directions x and y. The Coriolis parameter, f, accounts for the earth's rotation, and g is the acceleration of gravity. Equation (3) is the continuity equation. The surface elevation, h, is positive if it is above the equilibrium level, whereas the depth function, D, is positive *below* the equilibrium level. Thus, $D+h$ is the *total* depth of the fluid.

The instantaneous shoreline is determined by the condition, $D+h=0$. The moving shoreline separates a region in which the total depth is positive from another region in which it is negative. It follows from equation (3) that the volume of water within the region for which the total depth is positive remains constant in time as the shoreline moves about.

The approach taken here is to assume that there are solutions for u and v of the form

$$u=u_0+u_x x+u_y y, \quad v=v_0+v_x x+v_y y, \tag{4), (5}$$

where u_0, u_x, u_y, v_0, v_x, v_y are functions only of time. Then equations (1) and (2) require that the solution for h have the form

$$h=h_0+h_x x+h_y y+\tfrac{1}{2}h_{xx}x^2+\tfrac{1}{2}h_{yy}y^2+\tfrac{1}{2}(h_{xy}+h_{yx})xy, \tag{6}$$

where

$$h_x=-\frac{1}{g}\left[\frac{du_0}{dt}+u_0u_x+v_0u_y-fv_0\right], \tag{7}$$

$$h_y=-\frac{1}{g}\left[\frac{dv_0}{dt}+u_0v_x+v_0v_y+fu_0\right], \tag{8}$$

$$h_{xx} = -\frac{1}{g}\left[\frac{du_x}{dt}+u_x^2+u_y v_x-fv_x\right], \tag{9}$$

$$h_{yy} = -\frac{1}{g}\left[\frac{dv_y}{dt}+u_y v_x+v_y^2+fu_y\right], \tag{10}$$

$$h_{xy} = -\frac{1}{g}\left[\frac{du_y}{dt}+u_x u_y+u_y v_y-fv_y\right], \tag{11}$$

$$h_{yx} = -\frac{1}{g}\left[\frac{dv_x}{dt}+u_x v_x+v_x v_y+fu_x\right], \tag{12}$$

where $h_{xy} = h_{yx}$, and where h_0 is a function only of t.

If equation (3) is to be satisfied, then D must be a polynomial similar to h. In particular, assume that

$$D = D_0\left(1-\frac{x^2}{L^2}-\frac{y^2}{l^2}\right), \tag{13}$$

so that the basin is an elliptical paraboloid. Inclusion of linear terms amounts to a shift at the co-ordinate origin, and a term proportional to xy corresponds to a rotation of the co-ordinate axes, so (13) is quite general. The equilibrium shoreline is determined by the condition $D = 0$; it is an ellipse,

$$\frac{x^2}{L^2}+\frac{y^2}{l^2} = 1. \tag{14}$$

Two special cases will be considered. One is for $l = L$, where the basin is a parabola of revolution, and the other is $l \gg L$, where the basin is a canal with a parabolic cross-section.

If the polynomials given by equations (4), (5), (6) and (13) are to satisfy equation (3), then the time-varying coefficients of the linearly independent terms must separately vanish. This requires that u_0, u_x, u_y, v_0, v_x, v_y and h_0 satisfy the following equations:

$$\frac{dh_0}{dt}+(u_x+v_y)(D_0+h_0)+u_0 h_x+v_0 h_y = 0, \tag{15}$$

$$\frac{dh_x}{dt}+(2u_x+v_y)h_x+v_x h_y+u_0\left(h_{xx}-\frac{2D_0}{L^2}\right)+v_0 h_{xy} = 0, \tag{16}$$

$$\frac{dh_y}{dt}+(2v_y+u_x)h_y+u_y h_x+v_0\left(h_{yy}-\frac{2D_0}{l^2}\right)+u_0 h_{xy} = 0, \tag{17}$$

$$\frac{dh_{xx}}{dt}+(3u_x+v_y)\left(h_{xx}-\frac{2D_0}{L^2}\right)+2v_x h_{xy} = 0, \tag{18}$$

$$\frac{dh_{yy}}{dt}+(3v_y+u_x)\left(h_{yy}-\frac{2D_0}{l^2}\right)+2u_y h_{xy} = 0, \tag{19}$$

$$\frac{dh_{xy}}{dt}+2(u_x+v_y)h_{xy}+u_y\left(h_{xx}-\frac{2D_0}{L^2}\right)+v_x\left(h_{yy}-\frac{2D_0}{l^2}\right) = 0. \tag{20}$$

These six equations, plus the requirement that $h_{xy} = h_{yx}$, i.e.

$$\frac{d}{dt}(v_x - u_y) + (u_x + v_y)(v_x - u_y + f) = 0, \tag{21}$$

determine the seven unknown functions of time.

Equations (16)–(20) are second order and equations (15) and (21) are first order. Therefore, 12 initial conditions are needed. These correspond to the initial values of u_0, u_x, u_y, v_0, v_x, v_y, h_0, h_x, h_y, h_{xx}, h_{yy}, and h_{xy}, which fully define the initial fields u, v and h.

In the following sections various cases will be considered, corresponding to particular choices for initial values and for the ellipticity of the basin.

3. Oscillations for which the surface remains planar

Assume that $u_x = u_y = v_x = v_y = 0$, so that $h_{xx} = h_{yy} = h_{xy} = h_{yx} = 0$. Then only three functions, u_0, v_0 and h_0, must be determined. Equations (18)–(21) are identically satisfied, and equations (15)–(17) become

$$\frac{dh_0}{dt} - \frac{1}{g}\left[u_0 \frac{du_0}{dt} + v_0 \frac{dv_0}{dt}\right] = 0, \tag{22}$$

$$\frac{d^2u_0}{dt^2} - f\frac{dv_0}{dt} + \frac{2gD_0}{L^2}u_0 = 0 \tag{23}$$

and

$$\frac{d^2v_0}{dt^2} + f\frac{du_0}{dt} + \frac{2gD_0}{l^2}v_0 = 0. \tag{24}$$

Note that equations (23) and (24) are linear and have constant coefficients, so u_0 and v_0 vary sinusoidally with frequency, ω, satisfying the dispersion equation,

$$\left(\omega^2 - \frac{2gD_0}{L^2}\right)\left(\omega^2 - \frac{2gD_0}{l^2}\right) - f^2\omega^2 = 0. \tag{25}$$

If the basin is a parabola of revolution, $l = L$, then there are two basic solutions:

$$\left.\begin{aligned} u &= -\eta\omega_1 \sin\omega_1 t, \quad v = -\eta\omega_1 \cos\omega_1 t, \\ h &= 2\eta\frac{D_0}{L}\left(\frac{x}{L}\cos\omega_1 t - \frac{y}{L}\sin\omega_1 t - \frac{\eta}{2L}\right), \\ \omega_1 &= \frac{f}{2} + \left[\frac{f^2}{4} + \frac{2gD_0}{L^2}\right]^{\frac{1}{2}}; \end{aligned}\right\} \tag{26}$$

and

$$\left.\begin{aligned} u &= -\eta\omega_2 \sin\omega_2 t, \quad v = \eta\omega_2 \cos\omega_2 t, \\ h &= 2\eta\frac{D_0}{L}\left(\frac{x}{L}\cos\omega_2 t + \frac{y}{L}\sin\omega_2 t - \frac{\eta}{2L}\right), \\ \omega_2 &= -\frac{f}{2} + \left[\frac{f^2}{4} + \frac{2gD_0}{L^2}\right]^{\frac{1}{2}}; \end{aligned}\right\} \tag{27}$$

the constant, η, determines the amplitude of the motion.

For equation (26), the shoreline consists of the points (x, y) satisfying

$$(x-\eta\cos\omega_1 t)^2+(y+\eta\sin\omega_1 t)^2 = L^2, \tag{28}$$

and, for equations (27),

$$(x-\eta\cos\omega_2 t)^2+(y-\eta\sin\omega_2 t)^2 = L^2. \tag{29}$$

In either case the moving shoreline is a circle in the x, y plane, and the motion is such that the centre of the circle orbits the centre of the basin. For the lower-frequency mode, the orbit is counterclockwise and, for the higher-frequency mode, clockwise. In both cases, the surface remains planar as the water oscillates. Note that if $\eta > L$, then the motion is such that the bottom of the basin remains dry and centrifugal force holds the water to the sloping sides of the basin.

Other solutions can be obtained by superposing these two solutions. For example,

$$\left.\begin{aligned} u &= -\eta(\omega_1\sin\omega_1 t+\omega_2\sin\omega_2 t),\\ v &= -\eta(\omega_1\cos\omega_1 t-\omega_2\cos\omega_2 t),\\ h &= 2\eta\frac{D_0}{L}\left\{\frac{x}{L}(\cos\omega_1 t+\cos\omega_2 t)+\frac{y}{L}(\sin\omega_1 t-\sin\omega_2 t)-\frac{\eta}{L}[1+\cos(\omega_1+\omega_2)t]\right\}.\end{aligned}\right\} \tag{30}$$

In this case the nonlinearity results in a term with frequency given by the sum of the two basic frequencies. The boundary is again a circle given by

$$[x-\eta(\cos\omega_1 t+\cos\omega_2 t)]^2+[y+\eta(\sin\omega_1-\sin\omega_2 t)]^2 = L^2. \tag{31}$$

In the limit, $(\frac{1}{2}f)^2 \ll 2gD_0/L^2$, the co-ordinates of the centre of the circle are

$$x = 2\eta\cos\overline{\omega}t\cos\tfrac{1}{2}ft, \quad y = -2\eta\cos\overline{\omega}t\sin\tfrac{1}{2}ft, \tag{32}$$

where $\overline{\omega} = \frac{1}{2}(\omega_1+\omega_2) \doteq (2gD_0)^{\frac{1}{2}}/L$. Thus the centre oscillates along a line passing through the centre of the basin which precesses at the angular velocity of the rotating basin. The motion is like that of a Foucault pendulum. Again, the surface remains planar as it oscillates, and for large η the water runs far up the sides of the basin.

Now suppose that $l \gg L$, so that the basin is a canal. In this case there is a solution,

$$\left.\begin{aligned} &u = -\eta\omega\sin\omega t, \quad v = -\eta f\cos\omega t,\\ &h = 2\eta\frac{D_0}{L}\cos\omega t\left(x-\frac{\eta}{2L}\cos\omega t\right), \quad \omega = \left(f^2+\frac{2gD_0}{L^2}\right)^{\frac{1}{2}}.\end{aligned}\right\} \tag{33}$$

The shorelines are

$$x = \eta\cos\omega t \pm L. \tag{34}$$

The water sloshes back and forth across the canal and the flow oscillates along the canal. Again, the surface is a tilting plane.

Solutions can also be found for basins with elliptical cross-sections. For example, if rotation is neglected so that $f = 0$, then there are two modes that correspond to motions along the axes of the ellipse:

$$\left.\begin{aligned} &u = -\eta\omega_1\sin\omega_1 t, \quad v = 0,\\ &h = 2\eta\frac{D_0}{L}\cos\omega_1 t\left(\frac{x}{L}-\frac{\eta}{2L}\cos\omega_1 t\right), \quad \omega_1^2 = \frac{2gD_0}{L^2};\end{aligned}\right\} \tag{35}$$

and

$$\left.\begin{aligned} &u = 0, \quad v = -\eta\omega_2 \sin\omega_2 t, \\ &h = 2\eta\frac{D_0}{l}\cos\omega_2 t\left(\frac{y}{l}-\frac{\eta}{2l}\cos\omega_2 t\right), \quad \omega_2^2 = \frac{2gD_0}{l^2}. \end{aligned}\right\} \tag{36}$$

In each case the surface remains planar and the boundary remains elliptical as the water sloshes up and down the sides of the basin. Solutions for $f \neq 0$ can also be found, but they will not be presented here.

The shallow water equations, (1)–(3), are based on the hydrostatic assumption that vertical accelerations are negligible compared to g. This restricts the validity of these solutions by demanding that $\eta \ll L^2/2D_0$.

4. Oscillations for which the surface is curved

Now assume that $u_0 = v_0 = 0$. This restricts the motion to convergence toward, divergence from, and rotation about the centre of the basin. In this case $h_x = h_y = 0$, and the five unknown functions, u_x, u_y, v_x, v_y and h_0, are determined by equations (15) and (18)–(21).

In particular, suppose that the basin is circular, $l = L$, and that $u_x = v_y$ and $u_y = -v_x$. This reduces the number of unknown functions to three, and they are determined by the following equations:

$$\frac{d^2u_x}{dt^2}+\left(\frac{8gD_0}{L^2}+f^2\right)u_x+6u_x\frac{du_x}{dt}+4u_x^3 = 0, \tag{37}$$

$$\frac{dv_x}{dt}+2u_x\left(v_x+\frac{f}{2}\right) = 0, \tag{38}$$

$$\frac{dh_0}{dt}+2u_x(h_0+D_0) = 0. \tag{39}$$

Equations (37)–(39) have simple, exact solutions,

$$u_x = \frac{\omega}{2}\frac{A\sin\omega t}{1-A\cos\omega t}, \tag{40}$$

$$v_x+\frac{f}{2} = \left(v_{x0}+\frac{f}{2}\right)\frac{1-A}{1-A\cos\omega t} \tag{41}$$

and

$$h_0+D_0 = (\eta+D_0)\frac{1-A}{1-A\cos\omega t}, \tag{42}$$

where

$$\omega^2 = \frac{8gD_0}{L^2}+f^2. \tag{43}$$

The initial value of v_x is v_{x0} and the initial value of h_0 is η; u_x is initially zero. The constant A depends on the values of v_{x0} and η.

Since $h_x = h_y = 0$, $h_{xx} = h_{yy}$ and $h_{xy} = -h_{yx}$, the surface is a parabola of revolution,

$$h = h_0+\tfrac{1}{2}h_{xx}(x^2+y^2), \tag{44}$$

where

$$h_{xx} = -\frac{1}{g}\left(\frac{du_x}{dt} + u_x^2 - v_x^2 - fv_x\right). \tag{45}$$

The shoreline is a circle with radius

$$R = L\left[\frac{D_0 + h_0}{D_0 - \frac{1}{2}L^2 h_{xx}}\right]^{\frac{1}{2}}. \tag{46}$$

Since the volume remains constant in time,

$$V = \tfrac{1}{2}\pi R^2 (D_0 + h_0) = \tfrac{1}{2}\pi L^2 \eta, \tag{47}$$

so another expression for the radius of the circular shoreline is

$$R = L\left[\frac{D_0}{D_0 + h_0}\right]^{\frac{1}{2}}. \tag{48}$$

The radius varies inversely with the square root of the total depth at the centre of the basin. When the surface is convex and the central elevation is above the equilibrium level, the shoreline contracts to a lower level, and, when the surface is concave with the central elevation below the equilibrium level, the shoreline expands to a higher level. The radius goes to zero as the central elevation goes to infinity, and it becomes arbitrarily large as the shape of the surface approaches the shape of the basin.

Equations (45), (46), and (48) can be combined to provide an equation relating A to v_{x0} and η,

$$\left[\left(1 + \frac{\eta}{D_0}\right)^2 + \frac{f^2L^2}{8gD_0}\left(1 + \frac{2v_{x0}}{f}\right)^2\right] \bigg/ \left(1 + \frac{f^2L^2}{8gD_0}\right) = \frac{1+A}{1-A}. \tag{49}$$

Because A is always less than unity, the solutions given by (40)–(43) do not diverge. With the initial condition, $2v_{x0}/f = \eta/D_0$, equation (49) yields

$$A = \frac{(D_0 + \eta)^2 - D_0^2}{(D_0 + \eta)^2 + D_0^2}. \tag{50}$$

With A given by (50) and ω given by (43), the complete solution for the motion is:

$$\left.\begin{aligned} u &= \frac{1}{1 - A\cos\omega t}\left[\tfrac{1}{2}\omega x A \sin\omega t - \tfrac{1}{2} f y\left((1-A^2)^{\frac{1}{2}} + A\cos\omega t - 1\right)\right],\\ v &= \frac{1}{1 - A\cos\omega t}\left[\tfrac{1}{2} f x\left((1-A^2)^{\frac{1}{2}} + A\cos\omega t - 1\right) + \tfrac{1}{2}\omega y A \sin\omega t\right],\\ h &= D_0\left\{\frac{(1-A^2)^{\frac{1}{2}}}{1 - A\cos\omega t} - 1 - \frac{x^2+y^2}{L^2}\left[\frac{1-A^2}{(1-A\cos\omega t)^2} - 1\right]\right\}. \end{aligned}\right\} \tag{51}$$

In the small-amplitude limit the solution has the correct limit (Thacker 1977),

$$\left.\begin{aligned} u &= \frac{\eta}{2D_0}(\omega x \sin\omega t - f y \cos\omega t),\\ v &= \frac{\eta}{2D_0}(f x \cos\omega t + \omega y \sin\omega t),\\ h &= \eta\left(1 - \frac{2(x^2+y^2)}{L^2}\right)\cos\omega t. \end{aligned}\right\} \tag{52}$$

It is also easy to imagine a solution for which the water converges along the x axis while it diverges along the y axis and vice versa. This requires that $f = 0$ and $u_y = -v_x$, so that $h_{xy} + h_{yx} = 0$. In this case there are four unknown functions, u_x, v_x, v_y, and h_0, which are governed by the four equations

$$\frac{d^2u_x}{dt^2} + \frac{6gD_0}{L^2}u_x + 5u_x\frac{du_x}{dt} + 3u_x^3 - 2v_x\frac{dv_x}{dt} + v_y\left[\frac{du_x}{dt} + u_x^2 - v_x^2 - \frac{2gD_0}{L^2}\right] = 0, \tag{53}$$

$$\frac{d^2v_y}{dt^2} + \frac{6gD_0}{L^2}v_y + 5v_y\frac{dv_y}{dt} + 3v_y^3 - 2v_x\frac{dv_x}{dt} + u_x\left[\frac{dv_y}{dt} + v_y^2 - v_x^2 - \frac{2gD_0}{L^2}\right] = 0, \tag{54}$$

$$\frac{dv_x}{dt} + (u_x + v_y)\,v_x = 0, \tag{55}$$

and

$$\frac{dh_0}{dt} + (u_x + v_y)(h_0 + D_0) = 0. \tag{56}$$

If $u_x = v_y$, then (53) and (54) are redundant and these four equations reduce to the three for the previous case with $f = 0$. However, there should also be a solution for which u_x and v_y have opposite signs.

Now suppose that the basin is a canal, $l \gg L$. There should be a solution with the flow alternately converging toward and diverging from the centre of the canal. In this case, $u_y = v_y = 0$, and the three unknown functions, u_x, v_x and h_0, are determined by the three equations

$$\frac{d^2u_x}{dt^2} + \left(\frac{6gD_0}{L^2} + f^2\right)u_x + 5u_x\frac{du_x}{dt} + 3u_x^3 - 2fu_xv_x = 0, \tag{57}$$

$$\frac{dv_x}{dt} + u_x(v_x + f) = 0 \tag{58}$$

and

$$\frac{dh_0}{dt} + u_x(h_0 + D_0) = 0. \tag{59}$$

Equations (57)–(59) are similar to (37)–(39) when $f = 0$. However, there seem to be no closed-form solutions for these equations. Numerical solutions indicate that the frequency of the oscillations depends on the amplitude of the motion.

5. A parabolic flood wave

If $D_0 = 0$, then equations (37)–(39) govern the motion of a parabolic mound of water spreading over a frictionless horizontal surface. So long as $f \neq 0$, equations (51), with $\omega = f$, are the solutions for u, v, and h. The flow is oscillatory, spreading outward until the Coriolis force turns it back and it returns to the initial state.

When the Coriolis force is absent, equations (51) no longer apply. In this case the solution is

$$\left.\begin{aligned} u &= \frac{xt}{t^2+T^2}, \quad v = \frac{yt}{t^2+T^2}, \\ h &= \eta\left[\frac{T^2}{t^2+T^2} - \frac{x^2+y^2}{R_0^2}\left(\frac{T^2}{t^2+T^2}\right)^2\right], \end{aligned}\right\} \tag{60}$$

which corresponds to the initial conditions $u = v = 0$ and $h = \eta(1 - (x^2+y^2)/R_0^2)$.

Thus η is the initial height of the centre of the parabolic mound, and R_0 is the initial radius of the mound. At time $t = T$, the central height is $\frac{1}{2}\eta$. An analysis similar to that leading to equation (49) provides an equation

$$\frac{R_0}{T} = (2g\eta)^{\frac{1}{2}}. \tag{61}$$

The radius of the spreading mound is

$$R = R_0\left(\frac{t^2+T^2}{T^2}\right)^{\frac{1}{2}}, \tag{62}$$

and the velocity of the boundary is

$$\frac{dR}{dt} = \frac{R_0}{T}\left(\frac{t^2}{t^2+T^2}\right)^{\frac{1}{2}}. \tag{63}$$

For large times, $t \gg T$, the velocity of the edge is $(2g\eta)^{\frac{1}{2}}$.

When $D_0 < 0$, so that the basin is inverted and the flow is a flood wave spreading down a hill, equations (51) still apply so long as $\omega^2 = f^2 + 8gD_0/L^2 > 0$. If the Coriolis force is large enough, a parabolic mound of water spreading down the hill should turn and climb to its initial position. If $f^2 = -8gD_0/L^2$, equations (60) apply and the edge should reach a terminal velocity as the parabolic mound spreads down the hill. If $f^2 < -8gD_0/L^2$, then it is reasonable to expect that the downhill flow continues indefinitely to accelerate.

6. Concluding remarks

In addition to the several cases for which complete solutions for exact nonlinear motion have been given, others exist which correspond to different initial conditions. In the general case, the motion is governed by seven coupled, nonlinear, ordinary differential equations, (15)–(21). These equations can be integrated numerically without difficulty to provide accurate solutions for arbitrary initial conditions.

On the other hand, numerical integration of the partial differential equations (1)–(7) is much more difficult. For that reason, these exact solutions are particularly valuable. They provide a standard against which it is possible to compare the computations of numerical models. For example, storm tides are shallow water flows having a moving shoreline, and numerical models are used to predict the extent of the inundation. If such models are unable to simulate these solutions, they are unlikely to provide an accurate storm surge forecast.

REFERENCES

BALL, F. K. 1964 An exact theory of simple finite shallow water oscillations on a rotating earth. *Hydraulics and Fluid Mechanics* (ed. R. Silvester), pp. 293–305. MacMillan.

CARRIER, G. F. & GREENSPAN, H. P. 1958 Water waves of finite amplitude on a sloping beach. *J. Fluid Mech.* **4**, 97–109.

DIRICHLET, G. L. 1860 Untersuchungen über ein Problem der Hydrodynamik. *Abh. Kän. Gest. Wiss. Göttingen* **8**, 3–42.

FREEMAN, N. C. 1972 Simple waves on shear flows: similarity solutions. *J. Fluid Mech.* **56**, 257–263.

GERSTNER, F. J. VON 1802 Theorie der Wellen. *Adh. d. k. bohm. Ges. d. Wiss.* Reprinted in *Ann. der Physik*, 1809, **32**, 412–440.

LAMB, H. 1945 *Hydrodynamics*, 6th edn. Dover.

LONGUET-HIGGINS, M. S. 1972 A class of exact, time-dependent, free-surface flows. *J. Fluid Mech.* **55**, 529–543.

LONGUET-HIGGINS, M. S. 1976 Self-similar, time-dependent flows with a free surface. *J. Fluid Mech.* **73**, 603–620.

RANKINE, W. M. J. 1863 On the exact form of waves near the surface of deep water. *Phil. Trans. Roy. Soc.* A **153**, 127–138.

SACHDEV, P. L. 1980 Exact, self-similar, time-dependent free surface flows under gravity. *J. Fluid Mech.* **96**, 797–802.

THACKER, W. C. 1977 Irregular grid finite-difference techniques: simulations of oscillations in shallow circular basins. *J. Phys. Oceanogr.* **7**, 284–292.