DATA STRUCTURES

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ALGORITHM

- What is an algorithm ?
  - An algorithm is a finite set of precise instructions for performing a computation or for solving a problem.
  - This is a rather vague definition.
- But this one is good enough for now…

CHARACTERISTICS

- **Input** from a specified set,
- **Output** from a specified set (solution),
- **Definiteness** of every step in the computation,
- **Correctness** of output for every possible input,
- **Finiteness** of the number of calculation steps,
- **Effectiveness** of each calculation step and
- **Generality** for a class of problems.

EXAMPLE

- We will use a pseudocode to specify algorithms, which slightly reminds us of Basic and Pascal.

- Example: an algorithm that finds the maximum element in a finite sequence

procedure max($a_1, a_2, \ldots, a_n$: integers)

max := a_1

for i := 2 **to** n

    if max < a_i **then** max := a_i

{max is the largest element}

EXAMPLE

- Example: a linear search algorithm, that is, an algorithm that linearly searches a sequence for a particular element.

procedure linear_search(x: integer; $a_1, a_2, \ldots, a_n$: integers)

i := 1

while ($i \leq n$ and $x \neq a_i$)

    i := i + 1

if $i \leq n$ **then** location := I

else location := 0

{location is the subscript of the term that equals x, or is zero if x is not found}

EXAMPLE

- If the terms in a sequence are ordered, a binary search algorithm is more efficient than linear search.
- The binary search algorithm iteratively restricts the relevant search interval until it closes in on the position of the element to be located.

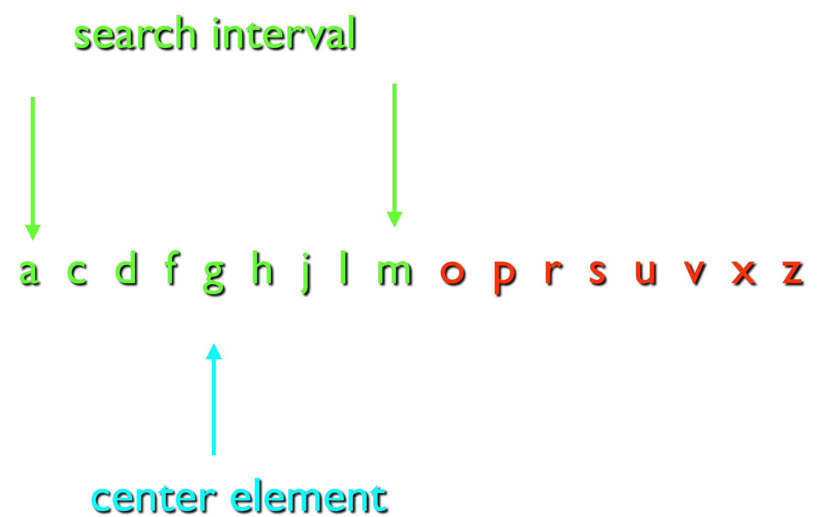
binary search for the letter ‘j’

search interval

a c d f g h j l m o p r s u v x z

center element

binary search for the letter ‘j’



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binary search for the letter ‘j’

search interval

a c d f g h j l m o p r s u v x z

center element

found !

EXAMPLE

procedure binary_search(x: integer; $a_1, a_2, \ldots, a_n$: integers)

$i := 1$ {i is left endpoint of search interval}

$j := n$ {j is right endpoint of search interval}

while $(i < j)$

Begin

  $m := \lfloor (i + j)/2 \rfloor$

  if $x > a_m$ **then** $i := m + 1$

  else $j := m$

end

if $x = a_i$ **then** location $:= i$

else location $:= 0$

{location is the subscript of the term that equals x, or is zero if x is not found}

COMPLEXITY

- In general, we are not so much interested in the time and space complexity for small inputs.
- For example, while the difference in time complexity between linear and binary search is meaningless for a sequence with $n = 10$, it is gigantic for $n = 2^{30}$.

COMPLEXITY

- Assume two algorithms A and B that solve the same class of problems.
- The time complexity of A is $5{,}000n$, the one for B is $\lceil 1.1^n \rceil$ for an input with n elements.
- For $n = 10$, A requires $50{,}000$ steps, but B only 3, so B seems to be superior to A.
- For $n = 1000$, however, A requires $5{,}000{,}000$ steps, while B requires $2.5 \cdot 10^{41}$ steps.

- Algorithm B cannot be used for large inputs, while algorithm A is feasible.
- What is important is the **growth** of the complexity functions.
- The growth of time and space complexity with increasing input size n is a suitable measure for the comparison of algorithms.

COMPARISON OF COMPLEXITY

Input Size	Algorithm A	Algorithm B
n	5,000n	$\lceil 1.1^n \rceil$
10	50,000	3
100	500,000	13,781
1,000	5,000,000	$2.5 \cdot 10^{41}$
1,000,000	$5 \cdot 10^9$	$4.8 \cdot 10^{41392}$

GROWTH OF FUNCTION

- The growth of functions is usually described using the big-O notation.

- **Definition:** Let f and g be functions from the integers or the real numbers to the real numbers. We say that $f(x)$ is $O(g(x))$ if there are constants C and k such that

$$|f(x)| \leq C|g(x)|, \text{ whenever } x > k.$$

- When we analyze the growth of complexity functions, f(x) and g(x) are always positive.
- Therefore, we can simplify the big-O requirement to

$$f(x) \leq C \cdot g(x) \quad \text{whenever } x > k.$$

- If we want to show that $f(x)$ is $O(g(x))$, we only need to find one pair (C, k).

GROWTH OF FUNCTION

- The idea behind the big-O notation is to establish an **upper boundary** for the growth of a function *f(x)* for large *x*. This boundary is specified by a function *g(x)* that is usually much **simpler** than *f(x)*.

- We accept the constant *C* in the requirement

  $$f(x) \leq C \cdot g(x) \text{ whenever } x > k,$$

  because ***C* does not grow with *x*.**

- We are only interested in large x, so it is OK if $f(x) > C \cdot g(x)$ for $x \leq k$.

GROWTH OF FUNCTION EXAMPLE

Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$.

For $x > 1$ we have:

$$x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2$$

$$\Rightarrow \quad x^2 + 2x + 1 \leq 4x^2$$

Therefore, for $C = 4$ and $k = 1$:

$f(x) \leq Cx^2$ whenever $x > k$.

$\Rightarrow f(x)$ is $O(x^2)$.

GROWTH OF FUNCTION EXAMPLE

Question: If $f(x)$ is $O(x^2)$, is it also $O(x^3)$?

Yes. x^3 grows faster than x^2, so x^3 grows also faster than $f(x)$.

We are always interested in the **smallest** simple function $g(x)$ for which $f(x)$ is $O(g(x))$.

- "Popular" functions g(n) are $n \log n$, 1, 2^n, n^2, $n!$, n, n^3, $\log n$
- Listed from slowest to fastest growth:
  - $1 < \log n < n < n \log n < n^2 < n^3 < 2^n < n!$

TRACTABLE, INTRACTABLE, UNSOLVABLE

- A problem that can be solved with polynomial worst-case complexity is called **tractable**.
- Problems of higher complexity are called **intractable.**
- Problems that no algorithm can solve are called **unsolvable**.

- More details can be found in the study of Computability and Complexity.

USEFUL RULES OF BIG-O

- For any polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, where $a_0, a_1, \ldots, a_n$ are real numbers, $f(x)$ is $O(x^n)$.

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 + f_2)(x)$ is $O(max(g_1(x), g_2(x)))$.

If $f_1(x)$ is $O(g(x))$ and $f_2(x)$ is $O(g(x))$, then $(f_1 + f_2)(x)$ *is* $O(g(x))$.

If $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$, then $(f_1 f_2)(x)$ *is* $O(g_1(x)\, g_2(x))$.

EXAMPLE

- What does the following algorithm compute?

procedure who_knows($a_1, a_2, \ldots, a_n$: integers)

m := 0

for i := 1 to n-1

  for j := i + 1 to n

    if $|a_i - a_j| > m$ **then** m := $|a_i - a_j|$

{m is the maximum difference between any two numbers in the input sequence}

Comparisons: $n-1 + n-2 + n-3 + \ldots + 1 = (n-1)n/2 = 0.5n^2 - 0.5n$

Time complexity is $O(n^2)$.

EXAMPLE

- Another algorithm solving the same problem:

  procedure max_diff($a_1, a_2, \ldots, a_n$: integers)

  min := a_1

  max := a_1

  for i := 2 to n

    if a_i < min **then** min := a_i

    else if a_i > max **then** max := a_i

  m := max - min

  Comparisons: $2n - 2$

  Time complexity is $O(n)$.

THANKS