Asignment 1 Devin Maya

1a) For any sets A, B, C $\quad A \cup (B \cup C) = (A \cup B) \cup C$

Lets consider an element x where x is a member of $A \cup (B \cup C)$ then x is either in A or in $(B \cup C)$. Now if it is in $(B \cup C)$ then it is either in B or in C. Thus x is in A or B or C now this is the same as x being a member of $(A \cup B) \cup C$. Therefore $A \cup (B \cup C) = (A \cup B) \cup C$. Since $(A \cup B) \cup C$ shows that x is in A or B or C.

b) For any sets A, B, C $\quad A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$

Let x be an element in $A \cup (B \setminus C)$ then x is in A and in $(B \setminus C)$. Here x is in A and in B but not in C. Observe that $(A \cap B) \setminus (A \cap C)$ is an element x is in A and B minus A and C here it is shown that x is in A and B but not in C. Therefore it is shown that

$A \cap (B \setminus C) = (A \cap B) \setminus (A \cap C)$.

c) For any set A, B, C $(A \setminus C) \cap (B \setminus C) = (A \cap B) \setminus C$

Here let an element x be in $(A \setminus C) \cap (B \setminus C)$ it is then true that x is in A but not C and x is in B but not C. Now observing $(A \cap B) \setminus C$ it is shown that such an element is in A and B but not C. Therefore if x is in $(A \setminus C) \cap (B \setminus C)$ it is also in $(A \cap B) \setminus C$ since x is in A and B but not C in both cases.

d) We know that $B \subseteq A$. Since B is a subset of A then we will analyze $(A \setminus B) \cup B$. Here $(A \setminus B)$ means there is an element x in A but not in B now after considering it with the union B $(\cup B)$ it is the entire set A again so $(A \setminus B) \cup B = A$. Now observing the set A here any element in B is in A since $B \subseteq A$ thus $(A \setminus B) \cup B = A$ has been shown to be true if and only if $B \subseteq A$.

2) Let $f: X \to Y$ be a function. If C is a subset of X then $f(C) := \{f(x) \mid x \in C\}$

a) we will prove that for any function $f: X \to Y$ and for all subsets $A, B \subseteq X$ that $f(A \cap B) \subseteq f(A) \cap f(B)$

Here we will take an element p that is arbitrary that is a member of $f(A \cap B)$ thus there is an element x that is in A and in B such that $f(x) = p$. p is in $f(A)$ and in $f(B)$ so p is in $f(A) \cap f(B)$. Thus every element of $f(A \cap B)$ is also in $f(A) \cap f(B)$ this is the same as saying that $f(A \cap B) \subseteq f(A) \cap f(B)$. Therefore the statement has been proven.

b) $f: X \to Y$ and subsets $A, B \subseteq X$ for which the containment in part A is proper.

we will adress this with providing an example for a function that is $f(x) = x^2$ where $x \in \mathbb{R}$. Here the given range of Y is $[0, \infty)$ and providing two subsets of X where $A = (-2,0)$ and $B = (0,2)$ observing part a specifically $f(A \cap B)$. Here the intersect of A and B is the set of elements that are both in A and B. However considering the case given there are no real numbers that are both greater than -2 and less than 0 while also considering greater than 0 and less than 2 at the same time. Thus $A \cap B$ is the empty set since there are no such elements that satisfy the given conditions. Here if you apply an empty set to the function f then the result is an empty set so $f(A \cap B) = \emptyset$. Now analyzing $f(A) \cap f(B)$ here $f(A)$ is the image of A under the function f and $f(B)$ is the image of B under the function f. Observing that $(-2)^2 = 4$ and $0^2 = 0$ by plugging in $A = (-2,0)$ into the given function thus $f(A) = [0,4)$. Now performing the same operation for $B = (0,2)$ here $0^2 = 0$ and $2^2 = 4$ so $f(B) = [0,4)$. Now by taking the intersection of $f(A) \cap f(B)$ it will be the set of elements that they both have in common respectively which is $[0,4)$. Therefore $f(A) \cap f(B) = [0,4)$ and it is observed that $f(A) \cap f(B) = [0, 4)$ has proper subset $f(A \cap B) = \emptyset$ since both sets are not equal the containment is proper thus the statement has been proven.

3) a) Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = n^2 - 2n + 1$. The objective is to find $f^{-1}(\mathbb{Z})$ if $\mathbb{Z} = \{m \in \mathbb{Z} \mid m \geq 0\}$. Here it is observed that the inverse image $f^{-1}\mathbb{Z}$ will produce the set of all integers since you can take any integer n and plug it into the function will provide a non negative result. The inverse image $f^{-1}(\mathbb{Z})$ is the set of all elements n in the domain such that $f(n) \in \mathbb{Z}$

for every integer n, $f(n)$ will be in the set $\mathbb{Z}$ thus the inverse image of $\mathbb{Z}$ under f is all integers. so $f^{-1}\mathbb{Z} = \mathbb{Z}$

b) Let $f : \mathbb{Z} \to \mathbb{Z}$ be the function defined by $f(n) = (-1)^n n^2$ and let Z be the set $\{n \in \mathbb{Z} \mid m < 0\}$.
The objective is to find the inverse image $f^{-1}Z$ which is the set of all elements n in the domain such that $f(n) \in Z$. Here the function $f(n) = (-1)^n n^2$ is negative when n is odd parity since the square of any integer is positive and -1 raised to an odd parity power is always negative so the overall result is a negative value. Therefore the inverse image of Z by function f has all odd integers that are negative
$f^{-1}(Z) = \{\ldots, -5, -3, -1\}$

4a) $W \subseteq f^{-1}(f(W))$ for all $W \subseteq X$. This means that for any subset W of X, W is a subset of the inverse image of its image under the given function f. Begin by letting an arbitrary element x in W. Then $f(x)$ is in $f(W)$ since x is in W. Now the inverse image of a set Y by a function f is $f^{-1}(Y)$ which is the set of all elements in the domain that map to elements in Y.

we know $f(x)$ is in $f(W)$ then x is in the inverse image of $f(W)$ so x is in $f^{-1}(f(W))$. Therefore every element of W is in $f^{-1}(f(W))$ thus W is a subset of $f^{-1}(f(W))$.

b) Here $f(f^{-1}(Z)) \subseteq Z$ for all $Z \subseteq Y$ means that for any subset Z of Y the image under f of the inverse image of Z is a subset of Z. Begin by taking an arbitrary element y in $f(f^{-1}(Z))$ thus there exists some element x in $f^{-1}(Z)$ such that $f(x)=y$. Then if x is in $f^{-1}(Z)$ this implies that $f(x)$ is in Z. Observe that $f(x)$ is in Y thus y is in Z, Therefore every element of $f(f^{-1}(Z))$ is in Z thus $f(f^{-1}(Z))$ is a subset of Z.

c) $f^{-1}(Z_1 \cap Z_2) = f^{-1}(Z_1) \cap f^{-1}(Z_2)$ for all $Z_1, Z_2 \subseteq Y$ means that for any subsets Z_1 and Z_2 of Y the inverse image of the intersection of Z_1 and Z_2 is equal to the intersection of the inverse images of Z_1 and Z_2. Begin by taking an arbitrary element x in $f^{-1}(Z_1 \cap Z_2)$ thus $f(x)$ is in $Z_1 \cap Z_2$ which means that $f(x)$ is in Z_1 and $f(x)$ is in Z_2. Then x is in $f^{-1}(Z_1)$ and also in $f^{-1}(Z_2)$. So x is in $f^{-1}(Z_1) \cap f^{-1}(Z_2)$. Therefore every element of $f^{-1}(Z_1 \cap Z_2)$ is in $f^{-1}(Z_1) \cap f^{-1}(Z_2)$ so $f^{-1}(Z_1 \cap Z_2)$ is a subset of $f^{-1}(Z_1) \cap f^{-1}(Z_2)$. Now by approaching the opposite direction if x is in $f^{-1}(Z_1) \cap f^{-1}(Z_2)$ then x is in $f^{-1}(Z_1)$ and x is in $f^{-1}(Z_2)$ so $f(x)$ is in Z_1 and $f(x)$ is in Z_2 thus $f(x)$ is in $Z_1 \cap Z_2$. Then x is in $f^{-1}(Z_1 \cap Z_2)$, therefore $f^{-1}(Z_1) \cap f^{-1}(Z_2)$ is a subset of $f^{-1}(Z_1 \cap Z_2)$. We have now shown that both sets are subsets of each other therefore the two sets are equal so $f^{-1}(Z_1 \cap Z_2) = f^{-1}(Z_1) \cap f^{-1}(Z_2)$

for all $Z_1, Z_2 \subseteq Y$.