Hw 4 Devin Maya

1.3.2 We will prove proposition 1.3.1. The objective is to show that if X is a topological space with the fixed point property and $h: X \to Y$ is a homeomorphism then Y has the fixed point property. Assume X has the fixed point property thus every continuous function $f: X \to X$ has at least one fixed point. It must be shown that Y also has the fixed point property so for every continuous function $g: Y \to Y$ there is a point y in Y such that $g(y) = y$. Assume $g: Y \to Y$ is a continuous function then consider $f: X \to X$ as $f = h^{-1} \circ g \circ h$. This is since h is a homeomorphism so the composition of a continuous function is also continuous thus f is a continuous function from X to X. It is known that homeomorphism is bijective and continuous. Observing that X has the fixed point property there is a point $x \in X$ where $f(x) = x$ by implementing the definition of f then $h^{-1} \circ g \circ h(x) = x$. After using h on the result will produce $g \circ h(x) = h(x)$ next $h(x)$ will be substituded with y to obtain $g(y) = y$. observe that this is the definition of a fixed point in Y. Therefore we know $g: Y \to Y$ has at least one fixed point for every continuous function. Thus Y has the fixed point property which is a topological property so Proposition 1.3.1 has been proved.

1.3.4 The objective is to show that $[0,1]$ and S^1 are not homeomorphic by using proposition 1.3.1 and 1.3.3. Specifically 1.3.3 mentions that the unit interval $[0,1]$ has the fixed property thus for any continuous function $f: [0,1] \to [0,1]$ there is a point x in $[0,1]$ such that $f(x) = x$. It is observed that the circle S^1 does not have such a property. $g: S^1 \to S^1$ is a continuous function in which the given circle has different rotations based on the given angle. Thus the function has no fixed points since the points on S^1 are changed through such rotations. By implementing proposition 1.3.1 the property X "has the fixed point property" is a topological property. Thus where there is a homeomorphism between two given spaces both will either have such fixed point property or they will not. Oberve $[0,1]$ does have the fixed point property and S^1 doesn't thus they are not homeomorphic.

Therefore $[0,1]$ and S^1 are not homeomorphic by using the given propositions.

1.3.6 a) The annulus $\{(x,y) \mid 1 \le x^2+y^2 \le 2\}$

If you consider a rotation on the origin in the plane where such an angle is not a multiple of 2π then there is a continuous map from the annulus and there are no fixed points present since every point is moved by the rotations. Therefore the given space does not have the fixed point property.

b) Now the open square $\{(x,y) \mid |x| < 1$ and $|y| < 1\}$

$f:(x,y) \to x + \frac{1}{(1-|x|, y)}$ when $|x| < 1$ and $y = y$ when $|x| = 1$. It is observed that such a function has no fixed points since every point in the interior of the given square is moved by the function the square is mapped to itself. When $|x| = 1$ the function is not a defined function so the limit boundaries of the square are not impacted thus the open square does not have the fixed point property.

1.3.7 It is known that the fixed point property is dependent on whether there exists a continuous function mapping from the space to itself that doesn't map any point to itself. If you remove the interiors of some small disks from the closed unit disk in R^2 the resulting space will or won't have the fixed point property since it is dependent on the properties. If a single disk from the interior is removed then the resulting space has the fixed point property. Observe the result after pushing the points towards the removed disk results in a continuous function with fixed points. Now if an infinite number of disks are removed it is possible for the fixed point property to not hold. Implementing the fixed point theorem for the closed unit disk every continuous function from a closed disk to itself has at least one fixed point.

1.3.8 The torus will be adressed by taking a point along the surface and moving it around the surface where points would slide into each other while having one fixed. Thus the fixed point theorem holds for the torus.

For the sphere the fixed point theorem holds since for any continuous function mapping the sphere to itself there exists a point that maps to itself.

1.3.12 Here the path connections are topological properties that ensure there is a continuous path between two points in a set. First we will begin with two points such as p and q in X = A ∪ B there are some cases to consider such as where both p and q belong to A or they belong to B. Here since A and B are path connected there is a continuous path from p to q within the set A or B this can be drawn thus X is path connected. Now considering p is in A and q is in B. Here since A is path connected it is possible to draw a continuous path from p to X_0 within A. Also since B is path connected a continuous path can be drawn from X_0 to q within B. By considering both paths together they will give a continuous path from p to q within X. Now considering p is in B and q is in A this specific case is also a reverse version of the previous case examined. Therefore in each examined case it is possible to draw a continuous path from any point p to any point q within X. Thus X is path connected.

1.3.19 ex 1.1.15) The rationals $\mathbb{Q}$ as a subset of the real line $\mathbb{R}^1$. A path in the real line is continuous and the image of a continuous function from a connected space to $\mathbb{R}$ is connected. Here there is no connected subset of $\mathbb{R}$ whose points are all rational besides sets with a single point. Therefore it is not possible to find a path inside $\mathbb{Q}$ that connects two different points.

ex 1.1.16) Topologists comb. Observe that there isn't a path from a point in I to a point in any J_k, $k>0$ since such a path must go through the ridges or teeth of the comb and they are disjoint from I and each other. Therefore it is not path connected.

ex 1.1.17) The Hawaiian earring. Observing that any two points on a single circle C_k are path connected it is not possible to draw a path that connects a point on circle C_k to a point C_I which is on a different circle. This is because such a drawn path would exit point C_k where the space is not part of the set. Therefore the path is not connected.

1.3.28 The objective is to show that boundedness is not a topological property. Begin by considering $\mathbb{R}$ and the interval $(0,1)$ by defining a homeomorphism $f: \mathbb{R} \to (0,1)$ by $f(x) = \frac{1}{1+e^{-x}}$ it is observed that $\mathbb{R}$ is unbounded and $(0,1)$ is bounded. Therefore boundedness is not a topological property since it is not preserved under homeomorphism.

1.3.29 Observe that $[0,1]$ is compact since it is closed and a bounded subset of $\mathbb{R}$. It is also observed that $[0,1)$ is not compact since the intervals $\frac{1}{n,1}$ where $n = 2,3,4$ there is no finite subcover. Therefore by implementing compactness $[0,1]$ and $[0,1)$ cannot be homeomorphic.

1.3.30 Through the implementation of compactness property it will be shown that D^2 and $\mathbb{R}^2$ are not homeomorphic. Here the closed disk $D^2 = \{(x,y) \mid x^2+y^2 \leq 1\}$ is compact since it is a closed and bounded subset of $\mathbb{R}^2$ but $\mathbb{R}^2$

is unbounded so it is not compact. Since compactness is a topological property D^2 and $\mathbb{R}^2$ are not homeomorphic since such a property is preserved.

1.3.31 The sets that are compact:

In $[0,1]$ it is closed and bounded so it is compact

For $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ is not compact since there is no finite subcover.

The set $\mathbb{Q}$ is not compact since it is not bounded

The topologists comb is not bounded so it is not compact.

The Hawaiian earring is compact since it is a closed subset of the compact set D^2.

1.3.33 By assuming that the sequence $\{X_n\}$ has infinitely many distinct values if this is not true then it is constant then it converges to the given constant value. The sequence is in the compact subset X of $\mathbb{R}^n$ then the set of all values $\{X_n\}$ are also a subset of X and are bounded. The implementation of the Bolzano-Weierstrass property any bounded

Sequence in R^n has a convergent subsequence thus such a limit point must be in X since X is closed and the limit of a sequence of points in a closed set is also in that set by compactness.

1.3.34 a) The figure eight ∞ and the circle S^1 are not homeomorphic since the figure eight has an intersection point where if removed the figure eight becomes disconnected while for the circle there is no such case after removing a point. Therefore they do not share the same property of being connected thus they are not homeomorphic.

b) Observe that by implementing the property of compactness S^2 is closed and bounded so it is compact and R^2 is not bounded. Therefore the property of compactness is preserved under homeomorphism so R^2 & S^2 are not homeomorphic.