DEVIN MAYA 2/10/23

Math 100A Homework 4

1. CH 3.9 PROB. 2 You deal a pile of cards, face down, from a standard 52-card deck. What is the least number of cards the pile must have before you can be assured that it contains at least five cards of the same suit?

   Solution. Let n be the number of cards that you have, now sort the n number of cards into 4 different groups that represent the 4 different suits. The division principle says that one group contains $\lceil \frac{n}{4} \rceil$ at least 5 cards of the same suit. Therefore $\lceil \frac{n}{4} \rceil \geq 5$ will be true when there is at least 5 cards of the same suit. This inequality is true when $\frac{n}{4} > 4$. Therefor you need $n > 4 \cdot 4 = 16$, so when $n = 17$ you know you have at least $\lceil \frac{17}{4} \rceil = \lceil 4.25 \rceil = 5$ cards of the same color.

2. CH 3.9 PROB. 6 Given a sphere S, a great circle of S is the intersection of S with a plane through its center. Every great circle divides S into two parts. A hemisphere is the union of the great circle and one of these two parts. Show that if five points are placed arbitrarily on S, then there is a hemisphere that contains four of them.

   Solution. The problem calls for the division of two groups where n things must go into those two groups. Therefore by the division principle $\lceil \frac{n}{2} \rceil$ with at least 4 arbitrary points in one hemisphere. This can be shown by $\lceil \frac{n}{2} \rceil \geq 4$ will be true with such conditions when $\lceil \frac{n}{2} \rceil > 3$. Therefore you need $n > 3 \cdot 2 = 6$ points so that there is a hemisphere that contains at least 4 of them. When $n = 7$ the division principle shows $\lceil \frac{7}{2} \rceil = \lceil 3.5 \rceil = 4$ points on S.

3. CH 3.10 PROB 6 Five cards are dealt off of a standard 52-card deck and lined up in a row. How many such lineups are there in which all 5 cards are of the same suit?

   Solution. There are 13 cards in a suit with 4 different possible suits. Therefor there are 4 cases in which the first five cards dealt are of the same suit. This can be shown with the multiplication principle by case 1 where $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 154440$. Case 2 $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 154440$. Case 3 $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 154440$. Case 4 $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 154440$. Now by implementing the addition principle and adding the 4 cases together results in $154440 + 154440 + 154440 + 154440 = 617,760$ possible lineups.

4. CH 4 PROB 6 Suppose $a, b, c \in Z$. If $a|b$ and $a|c$, then a $|(b + c)$.

   Solution. Proof. Suppose $a, b, c \in Z$ where $a|b$ and $a|c$. By definition 4.4 $b = a \cdot d$ and $c = a \cdot e$ for some $d, e \in Z$. Then $(b + c) = (a \cdot d + a \cdot e) = a(d + e)$.

Therefore a $|(b+c)$ since $a|a(d+e)$.■

5. CH 4 PROB 8 . Suppose a is an integer. If $5|2a$, then $5|a$.

Solution. Proof. Suppose a is an integer where $5|2a$. By definition 4.4 $2a = 5 \cdot d$ for some d that is an integer. Then $a = \frac{5d}{2}$ and $5|2a = 5|(2 \cdot \frac{5d}{2})$ which is the same as saying $5|5d$. Likewise, $a = 5 \cdot (\frac{d}{2})$

Therefore $5|a$ since $5|(5 \cdot (\frac{d}{2}))$.■

6. CH 4 PROB 10 . Suppose a and b are integers. If $a|b$, then $a|(3b^3 - b^2 + 5b)$.

Solution. Proof. Suppose a, b are integers where $a|b$. By definition 4.4 $b = a \cdot d$ for some d that is an integer. Then $(3b^3 - b^2 + 5b) = 3(ad)^3 - (ad)^2 + 5(ad) = 3(a^3d^3) - a^2d^2 + 5ad$. Thus by factoring $3(a^3d^3) - a^2d^2 + 5ad$ becomes $a \cdot (3(a^2d^3) - ad^2 + 5d)$ where $(3(a^2d^3) - ad^2 + 5d)$ is an integer by definition.

Therefore $a|(3b^3 - b^2 + 5b)$ since $a|a \cdot (3(a^2d^3) - ad^2 + 5d)$.■

7. CH 4 PROB 14 If $n \in Z$, then $5n^2 + 3n + 7$ is odd. (Try cases.)

Solution. Proof. Suppose $n \in Z$ can be divided into two cases for when n is either even or odd in parity.

Case 1. Suppose n is even. Then $n = 2a$ for some a in Z by definition of an even integer. So $5n^2 + 3n + 7 = 5(2a)^2 + 3(2a) + 7 = 20a^2 + 6a + 7$ which is the same as $20a^2 + 6a + 7 = 2(10a^2 + 3a + 3) + 1$. Here $2(10a^2 + 3a + 3) + 1$ is of the form 2b+1 where b is some integer, then by definition $2(10a^2 + 3a + 3) + 1$ is odd when n is an even number.

Case 2. Suppose n is odd. Then $n = 2a + 1$ for some a in Z by definition of an odd integer. So $5n^2 + 3n + 7 = 5(2a + 1)^2 + 3(2a + 1) + 7 = 20a^2 + 26a + 15$ which is the same as $20a^2 + 6a + 7 = 2(10a^2 + 13a + 7) + 1$. Here $2(10a^2 + 3a + 3) + 1$ is of the form 2b+1 where b is some integer, then by definition $2(10a^2 + 3a + 3) + 1$ is odd when n is an odd number.

Therefore $5n^2 + 3n + 7$ is always odd for all $n \in Z$ by both cases. ■

8. CH 4 PROB 24 If n in N and $n \geq 2$, then the numbers $n!+2, n!+3, n!+4, n!+5, ..., n!+n$ are all composite. (Thus for any n ≥ 2, one can find $n-1$ consecutive composite numbers.This means there are arbitrarily large "gaps" between prime numbers.)

Solution. Proof. Suppose n in N and $n \geq 2$. Since $n! = n(n-1)!$ the expression $n! + n$ is equal to $n(n-1)! + n = n \cdot ((n-1)! + 1)$. Here after factoring out n from the expression results in $n \cdot ((n-1)!+1) = n \cdot b$. Then $b = ((n-1)!+1)$ where b is a natural number by definition. Let $x = n \cdot ((n-1)! + 1)$ by definition x is composite if and only if $x = n \cdot ((n-1)! + 1) = n \cdot b$ for $1 < n, b < x$. Thus $((n-1)! + 1) < n \cdot ((n-1)! + 1)$ is true since $n \geq 2$. So when $n = 2$, $((2-1)! + 1) < 2 \cdot ((2-1)! + 1)$ becomes $2 < 4$. Thus without loss of generality the same applies to values of n that are greater than 2.

Therefore the numbers $n! + 2, n! + 3, n! + 4, n! + 5, ..., n! + n$ are all composite since $x = n! + n = n \cdot ((n-1)! + 1)$ where x is a composite number. ■