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Math 124: Introduction to Topology Asignment 11

1. EXERCISE 3.3.3

Solution. For part (a):

An octahedron is a polyhedron with eight faces, twelve edges, and six vertices. A wire-frame model of the octahedron is essentially a graph embedded on the surface of the octahedron. When we take a tubular neighborhood around this graph, we create a surface that follows the same structure as the graph. In the case of an octahedron, this results in a surface of genus 3, which is a three-holed torus 3-Torus. The genus g of a surface is determined by the relation $2 - 2g = \chi$, where χ is the Euler characteristic of the surface.

For part (b):

To calculate the Euler characteristic of the surface, we use the formula $\chi = V - E + F$.

For the octahedron, V the number of vertices is 6, E the number of edges is 12, and F the number of faces is 8.

Substituting these values into the formula, we get $\chi = 6 - 12 + 8 = 2$.

But, this Euler characteristic is for the octahedron itself. Since the surface we've obtained is a 3-Torus, which has a genus of 3, we can calculate its Euler characteristic using the formula $\chi = 2 - 2g$, where g is the genus. Substituting $g = 3$ into this formula, we get $\chi = 2 - 2 * 3 = -4$.

So, the Euler characteristic of the surface obtained by taking a tubular neighborhood of the wire-frame octahedron is -4.

2. EXERCISE 3.3.4

Solution. The word given is $abdcd^{-1}b^{-1}a^{-1}c$.

We first perform the necessary reductions using the relations $x^{-1}x = xx^{-1} = e$ and $xy = yx$ for $x \neq y$. Here, "e" is the identity element, so for any letter x, we have $x = xe$. An inverse (x^{-1}) cancels out with the original (x), so any pair of the form xx^{-1} or $x^{-1}x$ can be removed.

The word simplifies as follows:

$abdcd^{-1}b^{-1}a^{-1}c$

$= abdb^{-1}a^{-1}c$ (since $cd^{-1} = e$)

$= aba^{-1}c$ (since $bd^{-1}b^{-1} = e$)

$= ac$ (since $aba^{-1} = e$)

So, the reduced word is ac, which represents a torus (T). We can see this from the following scheme:

a: a "horizontal" cycle on the torus

c: a "vertical" cycle on the torus

Therefore, the surface represented by the original word is the torus, T. The genus of a torus is 1, as a torus has a single "hole".

3. EXERCISE 3.3.5

Solution. a. We know that a nonorientable surface M with a connected component remaining after the removal of n disjoint simple closed curves is homeomorphic to a connected sum of n projective planes. This is a direct consequence of the classification theorem for surfaces. Hence, in this case, with three simple closed curves, M is homeomorphic to $3P$, or the connected sum of three projective planes.

b. A handle is equivalent to a torus, and a cross-cap is equivalent to a projective plane. However, we know that a projective plane is also equivalent to a torus with a disc removed and a Möbius strip added in its place which adds a half-twist. Hence, a single projective plane could be thought of as a torus with a half "handle" considering the Möbius strip as a half-handle. Therefore, since M is equivalent to three projective planes or $3P$, it can also be thought of as having 1.5 handles. So, there is one full handle and one half-handle a Möbius strip connected to P to obtain M.

4. EXERCISE 3.3.7

Solution. a. The surface described in Exercise 2.2.7 is essentially a cube with a tube around each edge. Each edge-tube contributes a handle to the surface, and since a cube has 12 edges, there are 12 handles. Hence, the genus of this surface is 12.

b. When you glue one side of one cube to a side of another, you are essentially identifying two edges. Each edge contributes a handle, so identifying two edges reduces the number of handles by one. Hence, the genus of this surface becomes $12 + 12 - 1 = 23$.

c. For an entire row of n such frames glued along adjacent sides, the pattern from (b) continues. Each new cube adds 12 to the genus, but each glueing reduces the genus by 1. So for n cubes, the genus is $12n - (n - 1) = 12n - n + 1 = 11n + 1$.

5. EXERCISE 3.3.8

Solution. If you can cut the surface along two disjoint loops and still have a single piece, then the surface must have a genus of 2, often pictured as a "doughnut" with two holes. This is because cutting along a loop in a surface corresponds to removing a "handle" from it. Hence, $g = 2$. The fact that it's not possible to cut along a third loop without breaking the surface into two parts means that there are only two handles to remove.

Additionally, the presence of two distinct colors red and green indicates that this surface is orientable because there are distinct "sides."

Therefore, the surface you've been handed is a genus-2 orientable surface, which is commonly known as a double torus.

This can be written as: $M = 2T$.

6. EXERCISE 3.4.3

Solution. (a) For $(3,5)S^2$:

The number of faces $f = 5$, The number of vertices $v = 3$, The number of edges $e = \frac{pq}{2} = \frac{3\times 5}{2} = 7.5$. But, the number of edges must be an integer. Hence, $(3,5)S^2$ is not possible.

(b) For $(5,3)S^2$:

The number of faces $f = 3$, The number of vertices $v = 5$, The number of edges $e = \frac{pq}{2} = \frac{5\times 3}{2} = 7.5$. Again, the number of edges must be an integer. Hence, $(5,3)S^2$ is not possible.

(c) For $(3,4)S^2$:

The number of faces $f = 4$, The number of vertices $v = 3$, The number of edges $e = \frac{pq}{2} = \frac{3\times 4}{2} = 6$. Therefore, only $(3,4)S^2$ is possible and has 4 faces, 3 vertices, and 6 edges.

7. EXERCISE 3.4.8

Solution. Here, the surface M is a torus, T, for which the Euler characteristic $\chi(T) = 0$. So, we substitute $\chi(T) = 0$ in the equation (1):

$0 = 2e(1/a + \frac{1}{b} - 1/2)$

Here, a, b, e are the numbers of vertices, edges, and faces, respectively, in a tessellation of T.

This simplifies to: $1/a + 1/b = 1/2$

The solutions to this equation are pairs of integers (a, b) that meet this condition. Here, we want a, b to be the number of vertices around each face and the number of faces around each vertex, respectively. Therefore, both a and b must be at least 3 since each face must have at least 3 vertices and each vertex must be surrounded by at least 3 faces. Thus, the solutions are:

$(a, b) = (4, 4)$ $(a, b) = (3, 6)$ $(a, b) = (6, 3)$ All of these can be realized:

The $(4, 4)$ solution is realized by a tessellation of the torus with squares. The $(3, 6)$

solution is realized by a tessellation of the torus with hexagons. The $(6, 3)$ solution is realized by a tessellation of the torus with triangles. So, there are finitely many such regular cell complexes on the torus which is three.

8. 3.4.9

Solution. The Klein bottle, K, has the same Euler characteristic as the torus, so the solutions to Equation (1) are the same. That is, we still have the pairs (4,4), (3,6), and (6,3).

For the Klein bottle, (4,4) corresponds to a tesselation of the bottle with squares. For the (6,3) complex, the Klein bottle can't be tessellated with triangles because of its non-orientable nature.

Let's describe the drawings in LaTeX:

For the (4,4) case on a plane model of the Klein bottle which is a rectangle where the top edge is identified with the bottom edge with a twist

Draw a square grid on the rectangle. Identify the top and bottom edges with a twist to create the Klein bottle. The square faces will meet each other as per the given configuration. As for the (6,3) case:

There is no way to draw a (6,3) regular complex on the Klein bottle because the Klein bottle is non-orientable, while a (6,3) regular complex would require an orientable surface.

9. 3.4.11

Solution. If we have a triangulation of a surface M such that each vertex is met by exactly k triangles, with $k \leq 6$, this means that M must be a sphere. This conclusion comes from Euler's formula, which relates the number of vertices (v), edges (e), and faces (f) in a planar graph or a polyhedron, and gives us the Euler characteristic (χ) as follows:

$$\chi = v - e + f$$

A triangulation divides the surface into triangles. Every triangle has 3 edges and 3 vertices. However, each edge is shared by 2 triangles and each vertex by k triangles. Therefore, we have:

$$e = \frac{3f}{2}$$

and

$$v = \frac{f}{k}$$

Substituting these relations into Euler's formula, we get:

$$\chi = \frac{f}{k} - \frac{3f}{2} + f = f\left(\frac{1}{k} - \frac{3}{2} + 1\right)$$

Solving this for f, we have:

$$f = \frac{2\chi}{\frac{2}{k} - 3 + 2}$$

The only way to ensure that the number of faces (f) is a positive integer is to have $\chi = 2$, which means that the surface M must be a sphere. Note that the restriction $k \leq 6$ comes from the fact that for a triangulation in the Euclidean plane, each vertex is met by at most 6 triangles.

10. 3.4.12

Solution. To show that there is no triangulation on $2T$ (a double torus) such that each vertex is met by exactly k triangles, where $k < 7$, we can make use of Euler's formula again:

$$\chi = v - e + f$$

The Euler characteristic for a double torus $2T$ is $\chi(2T) = 2 - 2g = -2$, where g is the genus. In a triangulation, each triangle shares its three edges with other triangles, and each vertex is shared by k triangles. Hence:

$$e = \frac{3f}{2}$$

and

$$v = \frac{f}{k}$$

Substituting these relations into Euler's formula gives:

$$-2 = \frac{f}{k} - \frac{3f}{2} + f = f\left(\frac{1}{k} - \frac{3}{2} + 1\right)$$

Solving for f:

$$f = \frac{-2}{\frac{1}{k} - \frac{3}{2} + 1}$$

For $k < 7$, the denominator $\frac{1}{k} - \frac{3}{2} + 1$ is positive, but the numerator is negative, so f is negative, which is impossible for the number of faces. Therefore, for a double torus, there cannot be a triangulation such that each vertex is met by exactly k triangles where $k < 7$.

11. 3.4.13

Solution. The resulting shape after drilling a square hole through the cube is a three-holed torus or also known as a triple torus.

To visualize this, consider the sides of the cube. There are six faces on the cube, but four of these now have a square hole drilled through them. That makes four 'tunnels' (or holes) through the cube, but because the front-back and left-right tunnels intersect, they only result in three non-overlapping 'loops' around the cube, hence a three-holed torus.The resulting surface is a three-holed torus, often called a *triple torus.* To see this, consider the sides of the cube. There are six faces, but four of these now have a square hole drilled through them, making four 'tunnels' through the cube. However, two pairs of these tunnels intersect each other: one pair is front-to-back, and the other pair is left-to-right. These intersecting pairs only make three non-overlapping 'loops' around the cube, hence creating a triple torus.

To draw a regular complex on this surface, one could partition each face of the cube into four smaller squares resulting in a total of 24 faces, with each vertex of the cube acting as a vertex for these smaller squares and each edge of the cube acting as an edge for the squares.

This complex, however, would not be able to be drawn on a standard plane model for the triple torus due to the non-planarity of the triple torus. This is because a plane model of the triple torus would require the representation of three 'holes' on a 2D plane, which is not possible to accurately portray without overlap or ambiguity in the complex.

12. 3.5.4

Solution. To solve this exercise, let's use Euler's formula for polyhedra. According to the formula:

V - E + F = (S^2)

Where:

V is the number of vertices, E is the number of edges, and F is the number of faces. The Euler characteristic () of a sphere S^2 is 2.

In a trivalent complex, each vertex has three edges. Hence, the number of edges is three times the number of vertices divided by 2 (since each edge is shared by two vertices), i.e., E = 3V/2.

Each triangular face shares its 3 edges with other faces, so the total contribution of the edges from triangular faces is 3T/2, where T is the number of triangular faces. Similarly, each octagonal face shares its 8 edges with other faces, so the contribution of the edges from octagonal faces is 8O/2 = 4O, where O is the number of octagonal faces. This leads to the total number of edges: E = 3T/2 + 4O.

The number of faces F is equal to T + O since the complex only has triangular and octagonal faces.

Substituting $E = 3V/2$ and F = T + O into Euler's formula gives:

$V - 3V/2 + T + O = 2.$

Solving for T in terms of O gives:

$T = 5V/2 - O - 2.$

This equation relates the number of triangular faces to the number of octagonal faces. In a trivalent complex, every vertex has three edges. Hence, the number of edges is three times the number of vertices divided by 2 since each edge is shared by two vertices so E = 3V$\frac{}{2.}$

Each triangular face shares its 3 edges with other faces, so the total contribution of the edges from triangular faces is $\frac{3T}{2}$, where T is the number of triangular faces. Similarly, each octagonal face shares its 8 edges with other faces, so the contribution of the edges from octagonal faces is 4O, where O is the number of octagonal faces. This leads to the total number of edges: $E = \frac{3T}{2} + 4O$.

The number of faces F is equal to T + O since the complex only has triangular and octagonal faces.

Substituting $E = \frac{3V}{2}$ and $F = T + O$ into Euler's formula gives:

$V - \frac{3V}{2} + T + O = 2$

Solving for T in terms of O gives:

$T = \frac{5V}{2} - O - 2.$

This equation relates the number of triangular faces to the number of octagonal faces.

The equation represents the relationship between the number of triangular faces (t) and the number of octagonal faces (o). However, since t and o must both be integers, there are only a limited number of solutions to this equation that will satisfy this condition. You would need to test various values of o to find all possible integer values of t.

For the second part of the question One possible way to create such a complex would be to start with a cube which is made of 6 square faces, and then cut each square face into a triangle and an octagon. However, this would require careful cutting and arrangement to ensure that the final complex is trivalent each vertex is shared by exactly three edges.

13. 3.5.5

Solution. A trivalent complex is a structure in which each vertex is connected to exactly three edges. If we are considering a trivalent complex on the sphere, S^2, consisting only of pentagons and quadrilaterals, we can proceed as follows.Let's denote:

Q as the number of quadrilateral faces, P as the number of pentagonal faces, V as the number of vertices, and E as the number of edges.

From the definition of a trivalent complex, we know that $3V = 2E$ each vertex is of degree 3 and each edge is counted twice, once for each of its vertices.

Now, if we examine the faces, each quadrilateral contributes 4 edges and each pentagon contributes 5 edges. But, again, each edge is counted twice once for each of the two faces it's a part of, hence we have $2E = 4Q + 5P$.

From the above equations, we get $3V = 4Q + 5P$.

Now, consider Euler's formula for a sphere, which says that $V - E + F = 2$, where F is the total number of faces, which is equal to $P + Q$.

This gives us $V - (3V/2) + P + Q = 2$, or equivalently $Q = 3V/2 - P - 2$.

Substituting Q from the above equation into $3V = 4Q + 5P$ gives us $P = V - 4$.

Since the number of vertices V is an integer and is always more than or equal to 4 as we are considering a nonempty set of faces, which requires at least 4 vertices, it implies that P, the number of pentagonal faces, must be even. For an illustration of such a complex, consider a dodecahedron which has 20 vertices, 30 edges and 12 pentagonal faces which is even, it fulfills the requirements. You can add a quadrilateral face by adding a new vertex and connecting it to any 4 existing vertices, creating one quadrilateral face and 4 new pentagonal faces thus keeping the number of pentagonal faces even.

14. 3.5.6

Solution. A trivalent complex is a structure in which each vertex is connected to exactly three edges. If we are considering a trivalent complex on a torus, T, consisting only of quadrilaterals and octagons, we can use a similar reasoning as before.Let's denote:

- Q as the number of quadrilateral faces, - O as the number of octagonal faces, - V as the number of vertices, and - E as the number of edges.

From the definition of a trivalent complex, we know that $3V = 2E$ each vertex is of degree 3 and each edge is counted twice, once for each of its vertices.

Now, if we examine the faces, each quadrilateral contributes 4 edges and each octagon contributes 8 edges. But again, each edge is counted twice once for each of the two faces it's a part of, hence we have $2E = 4Q + 8O$.

From the above equations, we get $3V = 4Q + 8O$.

Now, consider Euler's formula for a torus, which says that $V - E + F = 0$, where F is the total number of faces, which is equal to $Q + O$.

This gives us $V - (3V/2) + Q + O = 0$, or equivalently $O = 3V/2 - Q$.

Substituting O from the above equation into $3V = 4Q + 8O$ gives us $Q = O$.

Hence, the number of quadrilateral faces is equal to the number of octagonal faces. As for an illustration of such a complex, consider starting with a regular tiling of a torus by octagons only, then insert a vertex in the center of each octagon and connect this vertex to the vertices of the octagon, splitting each octagon into 8 quadrilaterals. Now, each original octagon has become 8 quadrilaterals, ensuring that the number of quadrilaterals is the same as the number of original octagons.

15. 3.5.8

Solution. A four-valent complex is a complex where each vertex has exactly four edges. The smallest possible complex we can construct with each vertex of degree 4 on the projective plane (P) is a complex with 6 vertices, 12 edges, and 8 faces, where each face is a triangle.

In this case, each vertex is connected to exactly four other vertices, hence is four-valent. This also meets the requirement for a complex on P, as the Euler characteristic is $= V - E + F = 6 - 12 + 8 = 2 - 2g = 1$ which is consistent with $g = 1/2$, the genus of the projective plane.

So, the minimum number of triangular faces any four-valent complex on P must have is 8.