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Math 117 Linear Algebra in Game Theory

1 Introduction

Game theory is a branch of applied mathematics where it commonly analyzes strategic interactions in scenarios where the outcome for an individual depends on the choices of all participants. This discipline finds itself within other sectors such as economics, political science, biology, and computer science. A fundamental tool in this analysis is linear algebra by providing a structural and computational framework for game theory.

The appeal of game theory lies in its ability to model real life competitive situations, from market competition and political campaigns to simple rock paper scissors games. Its major contributors in the field are notable individuals like John Nash and John von Neumann where they have provided significant insights into understanding strategic human behavior. In the context of the course Math 117A, this topic ties into the study of matrices, vector spaces, and linear transformations, highlighting the practical applications of these concepts within the realm of game theory.

2 Mathematical Discussion

2.1 Payoff Matrices: Definition and Significance

Definition and Construction: In game theory, a payoff matrix is a tabular representation of the outcomes of a game for each player, given their chosen strategies and often times their respective payoffs for each outcome. For a two player game, the matrix is typically a square or rectangular grid where each cell of the matrix contains a pair of numbers representing the payoffs to the two players. These payoffs are with respect to the combination of strategies employed by the players.

For instance, consider a two player game where each player has Strategy A and Strategy B. The payoff matrix is represented by the following:

Player 1 Player 2	Strategy A	Strategy B
Strategy A	$(3, 2)$	$(1, 4)$
Strategy B	$(4, 1)$	$(2, 3)$

Here, if both players choose Strategy A, Player 1 receives a payoff of 3, and Player 2 receives a payoff of 2 so the format is read as (player 1 payoff, player 2 payoff) with respect to each distinct scenario.

Connection with Linear Algebra: The payoff matrix is really a linear algebraic structure observe that it is a matrix in the mathematical sense. Each strategy combination corresponds to a matrix element, and the entire game can be analyzed through matrix operations. This is particularly evident in games involving mixed strategies, where players choose a probability distribution over their strategies. In such cases, determining expected payoffs involves calculating dot products and matrix products which are fundamental operations in linear algebra.

2.2 Nash Equilibrium: Linear Algebraic Approach

Concept Introduction: The Nash equilibrium is named after John Nash where it is highly regarded as a central concept in game theory. It represents a stable state of a game where no player can gain by unilaterally deviating from their current strategy thus proving that they have a clear strategy in which there is no incentive to deviate from. At the Nash equilibrium each player's strategy is optimal given the strategies of the other players.

Significance in Game Theory: The Nash equilibrium provides a solution concept in non cooperative games. It's significant because it represents the decision points where players' strategies converge thus offering a prediction of how a game will be played based of the given players incentives. Nash's Existence Theorem states that every game with a finite number of strategies has at least one Nash equilibrium which is an important theorem to have in mind and will later extensively be discussed.

Linear Algebraic Techniques for Identification: Linear algebra plays a vital role in identifying Nash Equilibrium especially in games with mixed strategies where there is no clear pure strategy for the given player but with the assignment of probabilities to execute distinct strategies.For example imagine mixing your strategies in rock, paper, scissors where you obviously should never have the pure strategy of always picking rock instead you should mix in a way in which your oppenent will be unable to always counter you with paper. Here are two key techniques:

1. **Matrix Operations:** In games with mixed strategies, players' expected payoffs can be represented as matrix products. The Nash equilibrium occurs where the gradient of these payoff functions is zero, indicating that no unilateral change in strategy can increase a player's expected payoff. This can involve solving a system of linear equations which is fundamental to linear algebra.

2. **Eigenvector Analysis:** In certain strategic contexts, particularly in repeated games, Nash equilibria can be associated with eigenvectors of the payoff matrices. Here, the concept of eigenvalues and eigenvectors helps in understanding the long-term behavior of strategic interactions. For example, the principal eigenvector of a payoff matrix might indicate a dominant strategy or stable state in a dynamic game that is repeated.

3 Explicit Examples

3.1 Analyzing a Prisoner's Dilemma Game

The Prisoner's Dilemma is the most classic example in game theory that comes to mind that illustrates why two rational individuals might not strategically seek to cooperate even when it seems in their best interest to do so.

Scenario Description: Two criminals are arrested for a given crime and are sentenced to prison. Each prisoner is in solitary confinement with no means of communicating with the other. The prosecutors lack sufficient evidence to convict the pair on the evidence they have. Now their objective is to force the prisoners to betray the other by offering a reduced sentence. This scenario is represented as the following:

-If A and B both betray each other then each of them serves 2 years in prison.

-If A betrays B but B remains silent then A will be set free, and B will serve 3 years in prison and vice versa.

-If A and B both remain silent then both will only serve 1 year in prison which is the least time.

Payoff Matrix Construction: The payoff matrix for this game can be represented as follows with the years in prison represented as negative values since serving time is a negative outcome:

Prisoner (A,B)	Betray (B)	Remain Silent (S)
Betray (B)	$(-2, -2)$	$(0, -3)$
Silent (S)	$(-3, 0)$	$(-1, -1)$

Here, the first number in each pair represents Prisoner A's payoff and the second represents Prisoner B's payoff.

Linear Algebra in Nash Equilibrium Computation: To find the Nash equilibrium, where neither prisoner benefits from changing strategy, we analyze the matrix.

Consider each row (or column) as a vector representing the outcomes for Prisoner A (or B) for each of their strategies. The goal is to determine a stable pair of strategies (vectors) where neither prisoner's payoff can be increased by changing strategy alone.

- If Prisoner A betrays then Prisoner B has a higher payoff by betraying since it is the case of -2 vs. -3. - If Prisoner A remains silent then Prisoner B still has a higher payoff by betraying since it is the case of 0 vs. -1. - The same logic applies to Prisoner A when considering from Prisoner B's choices and perspective.

3.2 Matrix Operations in Strategy Analysis

In the Prisoner's Dilemma game, we can use linear algebra to represent the strategies of the prisoners and analyze their outcomes through matrix operations. Let's denote the strategies Betray and Remain Silent using vectors and examine the outcomes using matrix multiplication.

Strategy Representation:
- Prisoner A's strategy vector $\mathbf{a} = [1, 0]^T$, indicating choosing to "Betray".
- Prisoner B's strategy vector $\mathbf{b} = [0, 1]^T$, indicating choosing to "Remain Silent".
- The payoff matrix P for the Prisoner's Dilemma:

$$P = \begin{pmatrix} -2 & 0 \\ -3 & -1 \end{pmatrix}$$

The payoff is calculated using the formula: Payoff $= \mathbf{a}^T P \mathbf{b}$.
Performing the matrix multiplication:

$$\text{Payoff} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ -3 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

First, multiply $\mathbf{a}^T$ and P:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 & 0 \\ -3 & -1 \end{pmatrix} = \begin{pmatrix} 1 \times -2 + 0 \times -3 & 1 \times 0 + 0 \times -1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \end{pmatrix}$$

Next, multiply the resulting matrix with $\mathbf{b}$:

$$\begin{pmatrix} -2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2 \times 0 + 0 \times 1 = 0$$

Thus, the payoff for this specific strategy combination (Prisoner A betrays and Prisoner B remains silent) is 0. This result aligns with the interpretation of the Prisoner's Dilemma scenario: when one prisoner betrays and the other remains silent, the betraying prisoner (in this case, Prisoner A) receives no additional years in prison, which is represented by a payoff of 0.

Analysis: This linear algebraic approach allows for the ability to systematically compute the outcomes for any combination of strategies. By evaluating the resulting payoffs one can identify the Nash equilibrium which in the Prisoner's Dilemma occurs when both prisoners choose to betray.Through matrix operations, linear algebra not only provides a computational tool but also offers insight into the strategic structure of the game illustrating the relationship of the players' decisions and outcomes.

Therefore, the Nash equilibrium is for both prisoners to betray each other then leading to the payoff of (-2, -2). This outcome is not the optimal scenario of (-1, -1) if both had remained silent but it is the stable state of the game where no player has an incentive to deviate unilaterally. This example demonstrates how linear algebra, particularly the use of matrices, plays a crucial role in analyzing and solving game theory problems. The Nash equilibrium provides a predictable outcome based on the assumption that each player knows the payoff matrix and tries to maximize their own payoff regardless of the other player's decision.

4 Main Theorem and Proof

4.1 Theorem: Nash's Existence Theorem for Finite Games

Every finite game a game with a finite number of players and strategies has at least one Nash equilibrium.

4.2 Proof:

Nash's Existence Theorem is a vital result in game theory where its full proof is complex and involves advanced concepts such as fixed point theorems. The objective is to present a simplified version of the proof that captures the essence of the theorem.

Key Concepts:

- A **Nash Equilibrium** is a set of strategies, one for each player, such that no player has anything to gain by deviating from their strategy.
- A **finite game** is one where each player has a finite set of strategies.

Proof Overview: Nash's Existence Theorem is fundamental in game theory which makes use of a generalization of Brouwer's Fixed Point Theorem, and is steeped in the study of non-linear and high-dimensional spaces. The proof utilizes a fixed-point theorem, specifically Brouwer's Fixed Point Theorem, which states that for any continuous function mapping a compact convex set to itself, there is a point such that $f(x) = x$. Thus the proof will also involve concepts within the field of topology in order to prove the given theorem.

1. **Strategy Space**: Consider the set of all possible strategy combinations in the game as a multidimensional space where each dimension corresponds to the strategies available to a player. Since the game is finite this space is a compact and convex subset of Euclidean space.

2. **Best Response Function**: Define a function, called the best response function, for each player. This function takes a set of strategies one for each of the other players and returns the best strategy for the player in response thus the strategy that maximizes that player's payoff given the others' strategies.

3. **Continuous Function Construction**: Construct a continuous function F from the strategy space to itself. This function maps a point a set of strategies, one for each player to another point, where each coordinate is a strategy for a player and it is the best response to the strategies at the original point for the other players.

4. **Using Brouwer's Fixed Point Theorem**: By Brouwer's Fixed Point Theorem, the function F has at least one fixed point. A fixed point of F is a set of strategies where each player's strategy is the best response to the strategies of the others.

5. **Existence of Nash Equilibrium**: The fixed point of F corresponds to a Nash equilibrium. At this point, no player can benefit by unilaterally deviating from their strategy, as their current strategy is already the best response to the others'.

Therefore, every finite game must have at least one Nash equilibrium, as guaranteed by the fixed point of the function F.

This theorem is significant as it assures the existence of at least one point of stability given Nash equilibrium in any finite game, which is a fundamental concept in strategic decision making scenarios.

5 Linear Programming in Zero-Sum Games

5.1 Concept

In the realm of game theory zero sum games are when one player's gain is precisely another players loss. These games represent conflicting scenarios where the interests of players are diametrically opposed. In such scenarios linear programming emerges as a crucial tool to determine optimal strategies for the players involved.

The core idea behind using linear programming in zero sum games is to convert the strategic decisions and payoffs into a set of linear equations and inequalities. This mathematical formulation encapsulates the game displaying possibilities into a framework with the ability to implement optimization techniques. The strategies which are represented by variables in these linear expressions are subject to constraints that reflect the rules and structure of the game. The payoffs are often formulated as a function to be maximized or minimized Linear programming serves as a bridge between the abstract strategic aspects of the game and a concrete solution approach. It provides a systematic method to navigate through the strategic choices by identifying it maximizing one's gain or minimizing one's loss.

5.2 Significance

The significance of applying linear programming in zero sum games extends far beyond academics. In economics, it equips decision makers with a framework to analyze market competitions, auctions, and bidding strategies, where understanding and outmaneuvering the competition can make a substantial difference.

In military strategy where scenarios often approximate the zero sum structure linear programming helps in resource allocation, attack planning, and defensive tactics, where the optimal deployment of limited resources can be the difference between victory and defeat. Linear programming's role in zero-sum games is not just of theoretical importance but is a vital tool in the arsenal of strategists across various fields.

6 References

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