Math 117 Homework 5

3.1.1 Prove Lemma 3.1.16. Suppose that V is finite-dimensional. Let $x \in V$ be a vector. If $\omega(x) = 0$ for all $\omega \in V^*$, then $x = 0$.

Solution. Suppose V is an n dimensional vector space over a field F, and let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. The dual space V^* consists of all linear functionals mapping V to F. observe the function δ_{ij} is defined as follows: $\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$ This function is used to describe the action of the dual basis on the original basis so for each basis vector v_i there is a unique dual vector $\omega_i \in V^*$ such that $\omega_i(v_j) = \delta_{ij}$. This means $\omega_i(v_i) = 1$ and $\omega_i(v_j) = 0$ for all $j \neq i$. consider a vector $x \in V$ such that $x \neq 0$ observe since $\{v_1, v_2, \ldots, v_n\}$ spans V then it can be expressed x uniquely as a linear combination of the basis vectors $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, where the coefficients a_i are elements of F and at least one coefficient a_k is non zero.Now the definition of the dual basis the dual vector ω_k corresponding to v_k acts on x as

$\omega_k(x) = \omega_k(a_1v_1 + a_2v_2 + \cdots + a_nv_n) = a_1\omega_k(v_1) + a_2\omega_k(v_2) + \cdots + a_k\omega_k(v_k) + \cdots + a_n\omega_k(v_n) = a_1 \cdot 0 + a_2 \cdot 0 + \cdots + a_k \cdot 1 + \cdots + a_n \cdot 0 = a_k$.

from the assumption $a_k \neq 0$ then $\omega_k(x) \neq 0$ which is a contradiction because we were given that $\omega(x) = 0$ for all $\omega \in V^*$.

The contradiction is from the shown false assumption that $x \neq 0$. Therefore it must be that $x = 0$ therefore if $\omega(x) = 0$ for all $\omega \in V^*$, then x must be the zero vector in V.

3.1.2 Recall that the *real part* of a complex number $z = a + bi$ is defined to be $\Re(z) = a$. Given $w \in \mathbb{C}$, consider the -linear map

$$\omega_w : \mathbb{C} \to \mathbb{R}$$
$$\omega_w(z) = \Re(wz).$$

Show that every $\mathbb{R}$-linear map $\omega : \mathbb{C} \to \mathbb{R}$ is of the form ω_w for some $w \in \mathbb{C}$.(This gives an isomorphism $C^* \cong C$ as real vector spaces.))

Solution. Consider an arbitrary $\mathbb{R}$ linear map $\omega : \mathbb{C} \to \mathbb{R}$. By the definition of linearity for any complex numbers $z, z_1, z_2 \in \mathbb{C}$ and any real number $r \in \mathbb{R}$ the map ω satisfies the following requirements $\omega(z_1 + z_2) = \omega(z_1) + \omega(z_2)$ and $\omega(rz) = r\omega(z)$. The objective is to find a complex number $w \in \mathbb{C}$ such that ω can be expressed as $\omega(z) = \Re(wz)$ for all $z \in \mathbb{C}$. Now the standard basis for $\mathbb{C}$ is as a two dimensional real vector space by $\{1, i\}$. the complex number w as $w = \omega(1) - i\omega(i)$. Here this w satisfies the property that $\omega(z) = \Re(wz)$ for all $z \in \mathbb{C}$.To verify let $z = a + bi$ be any $mathbbC$. The action of ω on z using its $\mathbb{R}$-linearity is $\omega(z) = \omega(a + bi) = a\omega(1) + b\omega(i)$. now the product wz $wz = (\omega(1) - i\omega(i))(a + bi) = (a\omega(1) + b\omega(i)) + i(b\omega(1) - a\omega(i))$.

The real part of this product is precisely the real part of z scaled by $\omega(1)$ and the imaginary part of z scaled by $-\omega(i)$ so $\Re(wz) = a\omega(1) + b\omega(i)$.

Thus$\omega(z) = \Re(wz)$, here the map ω can be represented by the real part of the multiplication by w, and hence every $\mathbb{R}$ linear map from $\mathbb{C}$ to $\mathbb{R}$ can be expressed as ω_w for some $w \in \mathbb{C}$.

Now ω and w have been shown to correspond thus showing that there is an isomorphism between the dual space $\mathbb{C}^*$ and $\mathbb{C}$ itself when these are regarded as real vector spaces.

3.2.1 Let $x \in F^n$ and $y \in F^m$. The emphouter product of x and y is the matrix

$$x \otimes y = yx^\top \in F^{m\times n}.$$

1. Prove that $x \otimes y$ has rank at most 1.
2. Prove that if $A \in F^{m\times n}$ has rank at most 1, then there exist $x \in F^n$ and $y \in F^n$ such that $A = x \otimes y$.

(Hint: A has rank at most 1 if and only if Col(A) is spanned by a single vector.)

Solution. To prove that $x \otimes y$ has rank at most 1 the objective is to show that every column of $x \otimes y$ is a scalar multiple of y.

Given $x \in F^n$ and $y \in F^m$ the outer product $x \otimes y$ is the matrix $yx^\top$ here the j column of $x \otimes y$ is given by $x_j y$ where x_j is the j entry of x. Then every column of $x \otimes y$ is a scalar multiple of y so all the columns are in the span of y. the columns of $x \otimes y$ are linearly dependent then the rank of $x \otimes y$ is at most 1. Now $A \in F^{m\times n}$ has rank at most 1. From the use of the hint tthen the column space of A so $\mathrm{Col}(A)$ is spanned by a single vector $y \in F^m$.

Let a_1 be the first non zero column of A since $\mathrm{Col}(A)$ is spanned by y then a_1 as $a_1 = c_1 y$ for some non zero scalar $c_1 \in F$. Now $x \in F^n$ such that its first entry is c_1 and all other entries are determined by the corresponding columns of A then $a_j = c_j y$ for $j = 1, \ldots, n$ where c_j is the scalar multiple of y that gives the $j$column a_j of A.

Thus A can be written as $x \otimes y$ because A and $x \otimes y$ have the same columns scalar multiples of y. Therefore any matrix A with rank at most 1 there exist vectors $x \in F^n$ and $y \in F^m$ such that $A = x \otimes y$.

3.2.2 Consider the bilinear form on $_2[x]$ defined by

$$f(p, q) = \int_{-1}^{2} p(x)q(x)\mathrm{d}x.$$

Find the matrix $[f]$ of this form with respect to the standard basis of $R_2[x]$.

Solution. observe that the standard basis for $\mathbb{R}_2[x]$ is $\{1, x, x^2\}$. The matrix $[f]$ is determined by evaluating f on all pairs of these basis vectors. the entry $[f]_{ij}$ is given by $f(b_i, b_j)$ where $b_1 = 1$, $b_2 = x$, and $b_3 = x^2$. $[f]_{11} = f(1, 1) = \int_{-1}^{2} 1 \cdot 1\,\mathrm{d}x = 3, [f]_{12} = f(1, x) = \int_{-1}^{2} x\,\mathrm{d}x = 0, [f]_{13} = f(1, x^2) = \int_{-1}^{2} x^2\,\mathrm{d}x = \frac{9}{5}, [f]_{21} = f(x, 1) = \int_{-1}^{2} x\,\mathrm{d}x = 0, [f]_{22} = f(x, x) = \int_{-1}^{2} x^2\,\mathrm{d}x = \frac{4}{3}, [f]_{23} = f(x, x^2) = \int_{-1}^{2} x^3\,\mathrm{d}x = 0, [f]_{31} = f(x^2, 1) = \int_{-1}^{2} x^2\,\mathrm{d}x = \frac{9}{5}, [f]_{32} = f(x^2, x) =$

$\int_{-1}^{2} x^3 \, \mathrm{d}x = 0, [f]_{33} = f(x^2, x^2) = \int_{-1}^{2} x^4 \, \mathrm{d}x = \frac{33}{5}$.

therefore the matrix $[f]$ with respect to the standard basis $\{1, x, x^2\}$ is

$$[f] = \begin{pmatrix} 3 & 0 & \frac{9}{5} \\ 0 & \frac{4}{3} & 0 \\ \frac{9}{5} & 0 & \frac{33}{5} \end{pmatrix}.$$