DEVIN MAYA 8/04/23

Real Analysis Homework 4

1. (1) Determine all of the accumulation points of the following subsets of $\mathbb{R}$: (a) $\mathbb{Z}$.

Solution. The set of integers, $\mathbb{Z}$, has no accumulation points since for every integer $n \in \mathbb{Z}$, there is a neighborhood around n where no other integers are contained. The set of accumulation points is the empty set: $\emptyset$.

(b) $\frac{1}{n} \mid n \in \mathbb{N}$.

Solution. $\frac{1}{n} \mid n \in \mathbb{N}$ The accumulation points of this set are the points that are approached by a subsequence of the sequence $\frac{1}{n}$. Since the sequence converges to 0 as $n \to \infty$, 0 is the only accumulation point. The set of accumulation points is 0.

(c) $(-1)^n + \frac{1}{m} \mid n, m \in \mathbb{N}$.

Solution. The set consists of two subsequences. For every $m \in \mathbb{N}$, one subsequence converges to $1 + \frac{1}{m}$, and the other converges to $-1 + \frac{1}{m}$. Since both m and n are unrestricted over the natural numbers, the accumulation points will be the union of these two sequences as $m \to \infty$.

The set of accumulation points is $1, -1 \cup \frac{1}{m} \mid m \in \mathbb{N} \cup -\frac{1}{m} \mid m \in \mathbb{N}$.

2. (2) Determine all of the accumulation points of the following subsets of $\mathbb{R}^2$: (a) $v \in \mathbb{R}^2 \mid |v| > 1$.

Solution. The set described here is the exterior of the unit circle in $\mathbb{R}^2$. Since the complement of the set is closed and bounded, the set itself will have accumulation points in all points that are not within the unit circle.

The set of accumulation points is $v \in \mathbb{R}^2 \mid |v| \geq 1$

(b) $(x, y) \in \mathbb{R}^2 \mid x < -1$

Solution. This set describes the entire half-plane of points where the x-coordinate is less than -1. Every point in this half-plane is an accumulation point since any neighborhood of a point in this set will contain infinitely many other points from this set.

The set of accumulation points is $(x, y) \in \mathbb{R}^2 \mid x < -1$.

3. (3) Let $S \subseteq \mathbb{R}^n$. Prove that $\text{int}(S)$ is an open subset of $\mathbb{R}^n$, and that $\text{int}(S)$ is the largest open subset of $\mathbb{R}^n$ contained in S in the sense that if U is an open subset of $\mathbb{R}^{\ltimes}$ and $U \subseteq S$, then $U \subseteq \text{int}(S)$.

Solution.

Proof. 1. **int(S) is open:** The interior of a set S, denoted $\text{int}(S)$, is defined as the union of all open sets contained in S. Since the union of open sets is open, $\text{int}(S)$ must be an open set.

2. **int(S) is the largest open subset of $\mathbb{R}^n$ contained in S:** Let U be an open subset of $\mathbb{R}^{\ltimes}$ such that $U \subseteq S$. By definition of the interior, U is one of the open sets whose union forms $\text{int}(S)$, so $U \subseteq \text{int}(S)$.

$\square$

The solution is a direct implementation of the definitions and properties of open sets and the interior of a set. The interior of S is open, and it is the largest open subset of $\mathbb{R}^n$ contained in S.

4. (4) Prove that the only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset$ and $\mathbb{R}$.

Solution.

Proof. Let $S \subseteq \mathbb{R}$ be both open and closed.

- If $S = \emptyset$, then S is both open and closed by definition, so the statement holds for the empty set.
- If $S = \mathbb{R}$, then S is also both open and closed by definition, so the statement holds for the entire real line.
- Now, assume that S is non-empty and not equal to $\mathbb{R}$. Then, its complement $S^c = \mathbb{R} \setminus S$ is also non-empty.
  - Since S is open, S^c is closed.
  - Since S is closed, S^c is open.

  Thus, both S and S^c are non-empty open sets. But this contradicts the connectedness of $\mathbb{R}$, which means that there is no separation of $\mathbb{R}$ into two non-empty disjoint open sets.

Therefore, the only possibilities for S are $\emptyset$ and $\mathbb{R}$. $\square$

The only subsets of $\mathbb{R}$ that are both open and closed are $\emptyset$ and $\mathbb{R}$, as shown by contradiction and the connectedness property of $\mathbb{R}$. Hence, the only possibilities for S are $\emptyset$ and $\mathbb{R}$.

5. (5) Let F be a collection of subsets of $\mathbb{R}^n$, and set $S = \bigcup_{A \in F} A$. Prove or disprove: if x is an accumulation point of S, then x is an accumulation point of at least one set A in F.

Solution.

We will prove the statement.

Assume x is an accumulation point of S. By definition, every open neighborhood of x intersects S at some point other than x. Since S is the union of all sets $A \in F$, any open neighborhood of x must intersect at least one of the sets in F at some point other than x.

Now, let's take a sequence of open neighborhoods $\{U_k\}$ that converges to x, meaning each U_k contains points from S other than x. Since S is the union of all $A \in F$, at least one of these sets, say $A' \in F$, must intersect infinitely many of the open neighborhoods $\{U_k\}$ at points other than x.

Then, x is an accumulation point of A', as for any open neighborhood of x, there is a corresponding U_k that intersects A' at a point other than x.

Hence, x is an accumulation point of at least one set A in F, and the statement is proven.

The statement is true. If x is an accumulation point of S, then x is an accumulation point of at least one set A in F. To further add upon the proof we will go through the structure and steps implemented.

Assume x is an accumulation point of S, meaning that every open neighborhood of x contains at least one point of S different from x.

Now, we will show that there exists a set $A' \in F$ such that x is an accumulation point of A':

- Let U be any open neighborhood of x.
- Since x is an accumulation point of S, there exists a point $y \in U$ such that $y \neq x$ and $y \in S$.
- Since $S = \bigcup_{A \in F} A$, there exists $A' \in F$ such that $y \in A'$.
- Since U was an arbitrary open neighborhood of x, it follows that every open neighborhood of x intersects A' at some point other than x.

Hence, x is an accumulation point of A', and thus, x is an accumulation point of at least one set A in F. The statement is proven.