DEVIN MAYA 7/14/23

Real Analysis Homework 2

1. (1) Which of the following functions are injective? Surjective? Bijective? Which ones are not? Explain why.

   (a) $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$ for all $n \in \mathbb{N}$.

   Solution. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n+1$ for all $n \in \mathbb{N}$. The function f is injective. We can verify this by assuming two distinct natural numbers $n_1, n_2 \in \mathbb{N}$, so $n_1 \neq n_2$. According to the function definition, we get $f(n_1) = n_1 + 1$ and $f(n_2) = n_2 + 1$. If $n_1 \neq n_2$, then $n_1 + 1 \neq n_2 + 1$, which implies that $f(n_1) \neq f(n_2)$. Hence, the function is injective.

   However, the function f is not surjective. This is because there is no natural number n such that $f(n) = 1$, as for any natural number n, $f(n) = n + 1$ and hence $f(n)$ will always be greater than 1. Therefore, f is not bijective, since it is not surjective.

   (b) $g : \mathbb{Z} \to \mathbb{N} \cup 0$ defined by $g(n) = n^2$ for all $n \in \mathbb{Z}$.

   Solution. The function g is not injective. We can see this by considering the negative and positive pair of any non-zero integer. For example, consider -1 and 1. We have $g(-1) = (-1)^2 = 1$ and $g(1) = 1^2 = 1$, hence $g(-1) = g(1)$, but $-1 \neq 1$ which contradicts the condition for injectivity.

   However, the function g is surjective. This can be seen by observing that for any non-negative integer $n \in \mathbb{N} \cup 0$, we can choose $x = \sqrt{n}$ or $x = -\sqrt{n}$ and $g(x) = x^2 = n$.

   Since the function is not injective, it is not bijective.

   (c) $i : (\mathbb{Q} \setminus 0) \to (\mathbb{Q} \setminus 0)$ defined by $i(x) = 1/x$ for all $x \in \mathbb{Q} \setminus 0$.

   Solution. The function i is injective. We can confirm this by assuming two distinct rational numbers $x_1, x_2 \in \mathbb{Q} \setminus 0$ such that $x_1 \neq x_2$. By the function definition, we have $i(x_1) = 1/x_1$ and $i(x_2) = 1/x_2$. If $x_1 \neq x_2$, then $1/x_1 \neq 1/x_2$, which implies $i(x_1) \neq i(x_2)$ and so the function is injective.

   The function i is also surjective. This is because for any $q \in \mathbb{Q} \setminus 0$, we can choose $x = 1/q$. We then have $i(x) = 1/(1/q) = q$. Hence, for every element in the codomain, there is a corresponding element in the domain.

   Therefore, the function i is bijective as it is both injective and surjective.

2. (2) Which of the following relations are reflexive? Symmetric? Transitive? Which ones are not? Explain why.

   (a) The relation $x < y$ on $\mathbb{N}$

Solution. (a)The relation $x < y$ on $\mathbb{N}$ is not reflexive because, for reflexivity, we require xRx for every x in the set. In this case, it would mean $n < n$ for all $n \in \mathbb{N}$, which is not true as no natural number is less than itself.

The relation is also not symmetric because, for symmetry, we require that if xRy, then yRx. This means if $n_1 < n_2$, then $n_2 < n_1$ must also hold. However, in the natural numbers, if n_1 is strictly less than n_2, then n_2 cannot be less than n_1.

The relation is transitive because, for transitivity, we require that if xRy and yRz, then xRz. If $n_1 < n_2$ and $n_2 < n_3$ for some $n_1, n_2, n_3 \in \mathbb{N}$, then it follows that $n_1 < n_3$, and hence the relation is transitive.

(b) The relation $\sim$ on $\mathbb{Z}$ defined by $n \sim m$ if $n - m \geq 0$ for all $n, m \in \mathbb{Z}$.

Solution. The relation $\sim$ on $\mathbb{Z}$ defined by $n \sim m$ if $n - m \geq 0$ for all $n, m \in \mathbb{Z}$.

The relation is reflexive because $n - n = 0 \geq 0$ for all $n \in \mathbb{Z}$. So, $n \sim n$ holds for all n, satisfying the reflexivity condition.

However, the relation is not symmetric because if $n \sim m$ for example $n - m \geq 0$ then it is not necessarily the case that $m \sim n$ such that $m - n \geq 0$. Now consider $2 \sim 1$ since $2 - 1 \geq 0$, but $1 \not\sim 2$ since $1 - 2 < 0$.

The relation is transitive because if $n_1 \sim n_2$ and $n_2 \sim n_3$, then we have $n_1 - n_2 \geq 0$ and $n_2 - n_3 \geq 0$. Adding these inequalities, we obtain $(n_1 - n_2) + (n_2 - n_3) = n_1 - n_3 \geq 0$, and hence $n_1 \sim n_3$.

(c) The relation $\equiv$ on $\mathbb{Z}$ defined by $n \equiv m$ if $m = n$ or $m = -n$ for all $n, m \in \mathbb{Z}$.

Solution. This relation is reflexive because $n = n$ for all $n \in \mathbb{Z}$. Thus, $n \equiv n$ holds for all n.

The relation is also symmetric because if $n \equiv m$, then either $m = n$ or $m = -n$. If $m = n$, then $n = m$ and so $n \equiv m$ implies $m \equiv n$. If $m = -n$, then $n = -m$, and so again $n \equiv m$ implies $m \equiv n$.

This relation is transitive because if $n_1 \equiv n_2$ and $n_2 \equiv n_3$, then either $n_2 = n_1$, $n_2 = -n_1$, $n_3 = n_2$, or $n_3 = -n_2$. If $n_2 = n_1$ and $n_3 = n_2$, then $n_3 = n_1$ and hence $n_1 \equiv n_3$. Similarly, other cases also lead to $n_1 \equiv n_3$. Therefore, the relation is transitive.

3. (3) Assume that $f : X \to Y$ is a function from X to Y. Define a relation $\sim$ on X by declaring $a \sim b$ if $f(a) = f(b)$, for any $a, b \in X$.

   (a) We have to prove that $\sim$ is an equivalence relation on X.

Solution. To prove that $\sim$ is an equivalence relation on X, we need to show that it's reflexive, symmetric, and transitive.

Reflexive: For any $x \in X$, since the same element is mapped to the same value under a function, we have $f(x) = f(x)$, so $x \sim x$. This shows that the relation is reflexive.

Symmetric: If $a \sim b$, then by definition $f(a) = f(b)$. But equality is symmetric, which means that $f(b) = f(a)$, so $b \sim a$. This demonstrates that the relation is symmetric.

Transitive: If $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$. By the transitivity of equality, this implies that $f(a) = f(c)$, so $a \sim c$. This verifies that the relation is transitive.

Hence, $\sim$ is an equivalence relation on X.

(b) If $x \in X$ and we set $y = f(x)$, then $f^{-1}(y) = [x]$, where $[x]$ denotes the equivalence class of x under the relation $\sim$.

Solution. If $x \in X$ and we set $y = f(x)$, prove that $f^{-1}(y) = [x]$, thus the preimage of the subset y under f is equal to the equivalence class of x under the relation defined above. To prove this, let $a \in X$. We say that $a \in f^{-1}(y)$ if and only if $f(a) = y$. But $f(a) = y$ if and only if $f(a) = f(x)$, because we defined $y = f(x)$. By the definition of our relation, $f(a) = f(x)$ holds if and only if $a \sim x$. That is, a belongs to the equivalence class of x, denoted by $[x]$. Therefore, every a in $f^{-1}(y)$ is in $[x]$, and every a in $[x]$ is in $f^{-1}(y)$, which means that $f^{-1}(y) = [x]$.

4. (4) Let A be a set. Prove that A is countable if and only if there is an injective function $i : A \to \mathbb{N}$.

Solution. Let A be a set. A set A is countable if and only if there is an injective function $i : A \to \mathbb{N}$. ($\Rightarrow$) first consider consider proving from left to right. If A is countable, by definition there exists a bijective function $i : A \to \mathbb{N}$ or a part of $\mathbb{N}$. This function is also injective because every bijective function is injective. Hence, if A is countable, there exists an injective function $i : A \to \mathbb{N}$.

($\Leftarrow$) Now the other direction. If there is an injective function $i : A \to \mathbb{N}$, each element of A corresponds to a unique element of $\mathbb{N}$, meaning A can be listed in the form of a sequence although possibly infinite which is the definition of a countable set. Thus, if there is an injective function $i : A \to \mathbb{N}$, then A is countable.