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Math 124: Introduction to Topology Asignment 8

1. EXERCISE 2.5.3

Solution. Here to draw the simple closed curve on your model of the surface. If the curve separates the surface into two disjoint, non-empty open sets, then the path disconnects the surface. If there exists a path between any two points in the surface that does not intersect the curve, then the curve does not disconnect the surface.

2. EXERCISE 2.5.4

Solution. a) a) It is possible to cut a torus along two simple closed curves simultaneously and not disconnect the surface. This is possible when the two curves are similar to latitude and longitude lines on a globe, where they intersect but do not divide the surface into separate pieces.

In terms of other surfaces, this depends on their topology. For instance, if we consider a sphere, any simple closed curve disconnects it. For a projective plane or Klein bottle, one simple closed curve does not disconnect it but two generally will, depending on their arrangement.

Solution. b) For a connected sum of m projective planes, it can be cut along $m - 1$ pairwise disjoint simple closed curves without disconnecting the surface. The reason for this is that each additional projective plane in the connected sum adds a 'handle' or 'hole' to the surface that can be cut along a new curve without disconnecting the surface. But cutting along one more curve such as the m numbered curve will disconnect the surface.

This is similar to the case of a torus which is a donut shape where n tori connected together forming nT can be cut along n disjoint simple closed curves without disconnecting the surface. When nT is cut along one more curve such as the $(n+1)$ numbered curve, it will disconnect the surface.

3. EXERCISE 2.6.3

Solution. a) T#P The TP represents a connected sum of a torus (T) and a projective plane (P). In terms of words, we can denote the torus as "aba$^{-}1b^{-}1$"

and the projective plane as "aba". Therefore, the connected sum can be represented by "aba$^{-}1b^{-}1\#aba$".

b) K#P

K#P represents a connected sum of a Klein bottle (K) and a projective plane (P). The word representation for a Klein bottle is "abab$^{-}1$", and for the projective plane is "aba". So the connected sum can be represented by "abab$^-1\#aba$".

c) 2T#2K This is a connected sum of two tori (2T) and two Klein bottles (2K). The word representation for the torus is "aba$^-1b^-1$"

and for the Klein bottle is "abab$^-1$".

Therefore, the connected sum can be represented as

"aba$^-1b^-1\#aba^-1b^-1\#abab^-1\#abab^-1$".

d) 3T2P This represents a connected sum of three tori (3T)

and two projective planes (2P). In terms of word

representation, the torus is "aba$^-1b^-1$"$andthe$

projective plane is "aba". So, the connected sum is

represented as "aba$^-1b^-1\#aba^-1b^-1\#aba^-1b^-1\#aba\#aba$".

Here the 2T#2K can be written as "a$_1b_1$

a$_1^-1b_1^-1a_2b_2a_2^-1b_2^-1a_3b_3a_3b_3$

a$_4b_4a_4b_4$".

Here we used different variables for each torus and Klein bottle.

4. EXERCISE 2.6.5 Show that, except for the sphere, all the vertices of the plane model in standard form correspond to a single point on the space model of the surface.

Solution. When constructing a surface using a polygonal with a plane model, we identify pairs of sides. The vertices of the polygon are the points where these sides meet, and due to the identification of the sides, all the vertices coincide to a single point in the space model of the surface.

To illustrate this in more detail, consider the standard polygonal schema for a torus a square with sides identified as "a" to "a" and "b" to "b". The vertices where "a" and "b" meet in the plane model are identified to a single point in the space model, i.e., the torus.

This property holds for every surface represented by a polygonal schema, except for the sphere. The sphere is unique in this regard because it is the only surface that can be represented by a polygon with no vertices.

In constructing a surface from a plane model in standard form, we identify certain pairs of edges. The vertices in the plane model are the points where these edges meet, and due to the identification of the edges, all the vertices coincide to a single point in the space model of the surface.

To illustrate this, let's consider the plane model for a torus, which is a square with opposite sides identified. In the plane model, the vertices are the points where the "a" and "b" edges meet. In the space model such as the torus these vertices all identify to a single point.

This property holds true for every surface that can be represented by a polygonal schema, with the exception of the sphere. The sphere is unique because it is the only surface that can be represented by a polygon with no vertices. Hence, except for the sphere, all vertices of the plane model in standard form correspond to a single point on the space model of the surface.

5. EXERCISE 2.6.6 Which of the following words represent legitimate gluing instructions for a plane model of a surface? And if a surface, is it orientable? a. abac

   b. $a_1a_2a_1^{-1}a_3a_4a_2^{-1}a_3$

   c. $b_1a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$

   d. $b_1^{-1}a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$

   e. $ab^{-1}a^{-1}bcb^{-1}c$

Solution. a) abac

This does not satisfy condition (1), because 'a' appears twice, but 'a^{-1}' does not appear at all. So, it's not a legitimate gluing instruction.

b) $a_1a_2a_1^{-1}a_3a_4a_2^{-1}a_3$

This is not a legitimate gluing instruction. 'a_1' and 'a_1^{-1}' are properly paired, and 'a_2' and 'a_2^{-1}' are properly paired, but 'a_3' appears twice, and 'a_4' does not have a pair. Thus, the condition (1) is not satisfied.

c) $b_1a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$

This is a legitimate gluing instruction. Each symbol and its inverse appear exactly once, satisfying condition (1), and each symbol appears before its inverse, satisfying condition (2). The surface represented by this word is orientable because each symbol appears directly before its inverse.

d)$b_1^{-1}a_2b_3a_3^{-1}b_3^{-1}a_3a_2^{-1}a_1^{-1}b_1^{-1}a_1$

This is not a legitimate gluing instruction because the symbol 'b_1^{-1}' appears before its corresponding 'b_1'. The condition (2) is violated.

e) $ab^-1a^-1bcb^-1c$

This is a legitimate gluing instruction. Each symbol and its inverse appear exactly once, satisfying condition (1), and each symbol appears before its inverse, satisfying condition

(2). The surface represented by this word is non-orientable because 'b' appears directly before b^-1, violating the rule that a symbol should appear before its inverse.

6. EXERCISE 2.6.7 Determine all the different ways a Square can serve as a plane model for a surface (that is, the sides are identified in pairs). Which surfaces are represented this way?

Solution. The task of this exercises is to examine how a square can serve as a plane model for a surface when its sides are identified in pairs. A square has four sides, and there are several ways to pair and identify them. In order to generate a surface without boundary, each vertex of the square must be incident to an even number of identified edges.

Torus $S^1 \times S^1$: Identify opposite sides with the same orientation. In symbols, this is often written as $abab^{-1}$, where a and b are the two pairs of opposite sides.

Klein Bottle: Identify one pair of opposite sides with the same orientation and one pair with opposite orientation. This is usually written as $abab$.

Real Projective Plane $\mathbb{R}P^2$: Identify both pairs of opposite sides with opposite orientation. This can be denoted by $aabb$.

Sphere S^2: The sphere cannot be represented by a square with identified sides.

7. EXERCISE 2.7.3 Use the fact that $P\#T$ is homeomorphic to 3P to show $kP\#nT$ is homeomorphic to mP for some m, as long as $k > 0$. Express m as a function of k and n. The proof of the classification theorem for surfaces is now complete.

Solution. Given $P\#T$ is homeomorphic to $3P$, we can use this result to substitute T with $3P - P = 2P$ in the expression $kP\#nT$.

Now, the expression $kP\#nT$ can be rewritten as $kP\#2nP$, which simplifies to $(k+2n)P$. So, we can say that $kP\#nT$ is homeomorphic to $(k + 2n)P$.

Therefore, if $kP\#nT$ is homeomorphic to mP for some m, then $m = k + 2n$. Given the fact that PT is homeomorphic to 3P, this means that adding a torus to a projective plane is equivalent to adding three projective planes.

We can use this fact to express kP#nT as mP for some m.

If we consider the operation of adding a torus to the projective plane kP, for each torus we add, we are effectively adding three projective planes.

So if we start with k projective planes and add n toruses, the equivalent number of projective planes m that we would have is given by the formula:

m = k + 3n

This gives us an equivalent number of projective planes as a function of k and n, showing that kP nT is homeomorphic to mP for some m, as long as k > 0.

This result is an important part of the classification theorem for surfaces, which states that every closed, connected surface is homeomorphic to one of the following: the sphere, a connected sum of toruses, or a connected sum of projective planes. This exercise demonstrates that by using the operation of connected sum, we can express a complex surface as a 'sum' of simpler ones.

8. 2.7.4

Solution. The exercise involves the process of simplifying words using the five steps from the proof of the classification theorem for surfaces. which begins with Deformation of words such as exchanging a letter for its inverse Permutation of words Insertion of cancellation pairs Removal of cancellation pairs Use of the word relations for standard forms of surfaces The surfaces to consider for standard forms are the sphere S^*, the torus T represented by the word $aba^{-1}b^{-1}$, and the projective plane P represented by the word $abab^{-1}$.

a. $dac^{-1}bca^{-1}b^{-1}d^{-1}$ First, notice that we have $c^{-1}c$ and $a^{-1}a$, we can cancel these out to get $dab^{-1}d^{-1}$. Now, since we are allowed to permute letters, we can exchange d and a to get $ad^{-1}b^{-1}$, which is $aba^{-1}b^{-1}$ when $d = a$ and $b = b$. So the word represents a Torus T.

b. $ca^{-1}b^{-1}cdab^{-1}d$ We see that c and c^{-1} cancel out, so we have $a^{-1}b^{-1}dab^{-1}d$. Now we permute the letters and let $a = d$, and $b = a$ to get $aba^{-1}b^{-1}$. So this word represents the Torus T.

c. $cda^{-1}dc^{-1}a$ We see that d and d^{-1} cancel out, so we have $caa^{-1} = caa^{-1}c^{-1}c = aba^{-1}b^{-1}$ where $a = a$, $b = c$. So this word represents the Torus T.

d. $abcdf^{-1}fd^{-1}g^{-1}gcee^{-1}b^{-1}a^{-1}$ Firstly, eliminate the inverses that are together: $abcdeedcba^{-1}$. Now, we permute and get: $aba^{-1}b^{-1}cdc^{-1}d^{-1}e^{-1}e$. We have four instances of the word $aba^{-1}b^{-1}$, thus this word represents a connected sum of 4 tori: $4T$.

9. 2.7.5

Solution. This exercise involves using the rules of connected sums and topology properties of surfaces.

a. Express $K\#S^2\#T\#2P$ as a connected sum of projective planes or as a connected sum of tori.

First, the connected sum of any surface with a sphere is homeomorphic to the original surface itself. Thus, $K\#S^2$ is homeomorphic to K.

Secondly, a torus T connected with two projective planes $2P$ is homeomorphic to three projective planes, i.e., $T\#2P = 3P$.

Thus, $K\#S^2\#T\#2P$ is equivalent to $K\#3P$.

Finally, a Klein bottle K is equivalent to a connected sum of two projective planes, i.e., $K = PP$.

So, $K3P$ is equivalent to $(P\#P)\#3P = 5P$, which is a connected sum of five projective planes.

b. Write the surface in part (a) as a connected sum of five (not necessarily distinct) surfaces. Can you do so in more than one way (ignoring order)?

The surface in part (a), $5P$, can be represented as a connected sum of five (not necessarily distinct) surfaces in multiple ways:

i. $5P = P\#P\#P\#P\#P$: As a connected sum of five projective planes.

ii. $5P = K\#P\#P\#P$: As a connected sum of a Klein bottle and three projective planes.

iii. $5P = K\#K\#P$: As a connected sum of two Klein bottles and a projective plane.

iv. $5P = T\#T\#P$: As a connected sum of two tori and a projective plane.

This are just a few examples and there could be more combinations depending on how the connected sums are distributed and grouped.

10. 2.7.6

Solution. The surface $2K$ represents the connected sum of two Klein bottles. We know that each Klein bottle can be represented as a connected sum of two projective planes, that is, $K = P\#P$. Therefore, $2K$ can be written as $P\#P\#P\#P$, which is the connected sum of four projective planes.

In terms of distinct ways to represent this, assuming that Klein bottles (K), projective planes (P), and tori (T) are distinct, we have the following possibilities:

$P\#P\#P\#P$: This is expressing it as a connected sum of four projective planes. $K\#P\#P$: This is expressing it as a connected sum of a Klein bottle and two projective planes. $K\#K$: This is expressing it as a connected sum of two Klein bottles, which is the original form. $T\#P$: This is expressing it as a connected sum of a torus and a projective plane, since $T\#P$ equals K. So, there are 4 distinct ways to write $2K$ as the connected sum of four not necessarily distinct surfaces, ignoring the order.

11. 2.7.7

Solution. The surface mP represents the connected sum of m projective planes.

First, let's observe the relationship between the projective planes (P), tori (T), and Klein bottles (K)

$T\#P = K$: A torus connected with a projective plane is homeomorphic to a Klein bottle.

$K\#P = 2P$: A Klein bottle connected with a projective plane is homeomorphic to two projective planes.

$K\#K = T\#T\#T\#P\#P = 4T$: Two Klein bottles connected together is homeomorphic to four tori.

From these observations, we can write mP as $kT\#P$ or $KT\#K$ for some integer k. The specific integer k and the configuration will depend on the value of m. For instance, if $m = 4$, we can use $K\#K = 4T$ to express mP as $4T$. And if $m = 5$, we can use $K\#K\#P = 4T\#P$ to express mP as $4T\#P$.

For a general integer m, we need to find the quotient and remainder when m is divided by 2. If $m = 2k$, then $mP = kT\#kT$. If $m = 2k + 1$, then $mP = kT\#kT\#P = (kT\#P)(kT\#P)\#P = KT\#KT\#P = (KT\#P)\#KT = KT\#K$.

So, mP can indeed be written as $kT\#P$ or $KT\#K$ for some integer k.