DEVIN MAYA 10/13/23

Math 117 Homework 2

1.3.1. Proposition 1.3.3 (Properties of Linear Maps) Let $T : V \to W$ be a linear map. Then:

(i) $T(0) = 0$.

(ii) T preserves linear combinations: For any $a_1, \ldots, a_n \in F$ and $x_1, \ldots, x_n \in V$,

$$T(a_1x_1 + \ldots + a_nx_n) = a_1T(x_1) + \ldots + a_nT(x_n).$$

Proof. Exercise. Hint: For (ii), use induction on n.

Solution.

(i) $T(0) = 0$: Here the problem provides a linear map that the objective is to prove that the given linear map when receiving o as an input for V will output 0 for W.

let x to be any vector in V and observing that when multiplying a vector by zero scalar results in the zero vector. Thus $0 \cdot x = 0$ so implementing the knowledge of the linearity of T then combining with multiplication of 0 $T(0 \cdot x) = 0 \cdot T(x)$ Then $T(0) = 0$

Therefore the zero vector in V under T is the zero vector in W.

(ii) The objective is to show that $T(a_1x_1 + \ldots + a_nx_n) = a_1T(x_1) + \ldots + a_nT(x_n)$ We will use the tip provided and implement the use of induction n.

Then the Base Case $n = 1$ Observe that $T(a_1x_1) = a_1T(x_1)$ Now for the Inductive Hypothesis assume the statement holds for some $n = z$ thus $T(a_1x_1 + \ldots + a_zx_z) = a_1T(x_1) + \ldots + a_kT(x_z)$ Now after performing the inductive step $n = z+1$ By implementing the linearity of T and the given inductive hypothesis then $T(a_1x_1 + \ldots + a_{z+1}x_{z+1}) = T(a_1x_1 + \ldots + a_zx_z + a_{z+1}x_{z+1}) = T(a_1x_1 + \ldots + a_zx_z) + T(a_{z+1}x_{z+1}) = a_1T(x_1) + \ldots + a_zT(x_z) + a_{z+1}T(x_{z+1})$ Thus the inductive step holds for $n = z + 1$ Therefore the statement is true for all n and thus the statement has been proven.

1.3.2 Let $T : V \to W$ be a linear map.

(i) Prove that for any subspace $X \subseteq V$, $T(X)$ is a subspace of W.

(ii) Prove that for any subspace $Y \subseteq W$, $T^{-1}(Y)$ is a subspace of V.

Solution. Observing that the linear map for a given vector $x \in \bar{x} \in V$. We know $T(x) \in T(V) \in W$ also that $x, y \in T(X)$ and say $u, v \in \bar{x}$.

(i) Let X be a subspace of V. To show that $T(X)$ is a subspace of W there must be requirements that should be met. Since X is a subspace of V and 0 is always an element of any subspace then $0 \in X$. Observe that $T(0) = 0$ and the zero vector of $W \in T(X)$ then the result is that $T(X)$ non empty. Now the objective is to show that $T(X)$ is closed under vector addition and multiplication. Let u and v be any two vectors in X. We know that X is a subspace thus you can see that $u+v \in X$. Now implementing $T(u+v) = T(u)+T(v)$. Now since u and v are in X then $T(u)$ and $T(v)$ are in $T(X)$. Thus $T(u)+T(v) \in T(X)$. Therefore it has been shown that $T(X)$ is closed under addition now it must be shown for multiplication. Here the next objective is to show that $T(X)$ is closed under scalar multiplication. Let u be a vector $\in X$ and a be a scalar $\in F$. Since X is a subspace then $au \in X$. $T(au) = aT(u)$. We know that $T(u) \in T(X)$ then $aT(u)$ is also in $T(X)$. Therefore $T(X)$ is closed under scalar multiplicationand thus $T(X)$ is a subspace of W.

(ii) The objective is to prove that $T^{-1}(Y)$ is a subspace of V this will be done with the same approach from (i) where the requirements must be shown. Observing that Y contains the zero vector of W then there is u in V such that $T(u) = 0$. This means $u \in T^{-1}(Y)$ which results in $T^{-1}(Y)$ being non-empty. Let u and v be any vectors $\in T^{-1}(Y)$. Then in order to show the addition requirement $T(u)$ and $T(v)$ are both in Y. We know that Y is a subspace thus $T(u)+T(v) \in Y$. Observing that T is linear then $T(u+v) = T(u)+T(v)$ This shows that $u+v$ is in $T^{-1}(Y)$ where $T^{-1}(Y)$ is closed under addition. Now the objective is to show that $T^{-1}(Y)$ is closed under scalar multiplication. Let u be a vector $\in T^{-1}(Y)$ and a be a scalar $\in F$. Thus $T(u) \in Y$. We know that Y is a subspace then $aT(u)$ is also in Y. $T(au) = aT(u)$ which is the desired requirement. Therefore au is in $T^{-1}(Y)$ so $T^{-1}(Y)$ is closed under scalar multiplication and $T^{-1}(Y)$ is a subspace of V.

1.3.3. Let $T : V \to W$ and $U : W \to X$ be two linear maps.

(i) Prove that the composite function $U \circ T : V \to X$ is a linear map.

(ii) Prove that if T is bijective, then the inverse function $T^{-1} : W \to V$ is a linear map.

Solution. $\forall x, y \in T^{(-1)}(y)$ then $x+y \in T^{-1}(Y)$ then $ax \in T^{-1}(y)$

i) To prove $U \circ T$ is a linear map the objective is to demonstrate that it preserves both vector addition and scalar multiplication. begin by doing addition so by letting $v_1, v_2 \in V$. $T(v_1+v_2) = T(v_1)+T(v_2)$ by observing the linear map T Now applying U thus $U(T(v_1+v_2)) = U(T(v_1)+T(v_2)) \implies U(T(v_1)+T(v_2)) = U(T(v_1))+U(T(v_2))$ Then $(U \circ T)(v_1+v_2) = (U \circ T)(v_1)+(U \circ T)(v_2)$. Therefore the addition requirement has been shown. Now for multiplication begin with $v \in V$ and let a be a scalar. $T(a \cdot v) = a \cdot T(v)$ now applying U $U(T(a \cdot v)) = a \cdot U(T(v))$ Then $(U \circ T)(a \cdot v) = a \cdot (U \circ T)(v)$. Therefore $U \circ T$ is a linear map. ii) The same approach from i) will be implemented for T^{-1} So by letting $w_1, w_2 \in W$ and $v_1, v_2 \in V$from the bijection mapping in T^{-1}. Now observe that $T(v_1) = w_1$ and $T(v_2) = w_2$.

We know that T is a bijective and linear map thus $T(v_1+v_2) = T(v_1)+T(v_2)$ thus $T(v_1+v_2) = w_1+w_2$ v_1+v_2 is the pre-image of w_1+w_2 Therefore $T^{-1}(w_1+w_2) = v_1+v_2$.

For a scalar a and a vector $w \in W$ such that $w = T(v)$ for some $v \in V$

$T(a \cdot v) = a \cdot T(v) \implies T(a \cdot v) = a \cdot w$ Observing that T is bijective then $a \cdot v$ is the pre-image of $a \cdot w$ under T. Thus $T^{-1}(a \cdot w) = a \cdot v$. Therefore T^{-1} is a linear map.

(iii) Proposition 1.4.1. The map Φ defined above is linear. Moreover, $\text{im}(\Phi) = Z + Z'$, $\ker(\Phi) = \{(z, -z) : z \in Z \cap Z'\}$.

Solution. The statement provided has given a map $\Phi : Z \oplus Z' \to V$ defined as $\Phi(z, z') = z + z'$.The objective is to prove The map Φ is linear, the image of Φ is $Z + Z'$, and the kernel of Φ is $\{(z, -z) : z \in Z \cap Z'\}$. Begin by showing that Φ is linear So for all $z_1, z_2 \in Z$ and $z'_1, z'_2 \in Z'$ the objective is to show $\Phi(z_1 + z_2, z'_1 + z'_2) = \Phi(z_1, z'_1) + \Phi(z_2, z'_2)$ Now for any scalar α and vectors $z \in Z$ and $z' \in Z'$ the objective is to show $\Phi(\alpha z, \alpha z') = \alpha\Phi(z, z')$ Now for the addition requirement $\Phi(z_1 + z_2, z'_1 + z'_2) = (z_1 + z_2) + (z'_1 + z'_2)$ by definition of $\Phi = z_1 + z'_1 + z_2 + z'_2$ from the associative property $= \Phi(z_1, z'_1) + \Phi(z_2, z'_2)$ Here is the next observation $\Phi(\alpha z, \alpha z') = \alpha z + \alpha z'$ from $\Phi = \alpha(z + z') = \alpha\Phi(z, z')$ Therefore Φ satisfies both requirements so Φ is linear. Now for image of Φ observing $z + z'$ where $z \in Z$ and $z' \in Z'$.

For each (z, z') then $\Phi(z, z') = z + z'$. This means that every element of the form $z + z' \in \text{im}(\Phi)$. Also observing every vector in $\text{im}(\Phi)$ is of the form $z + z'$. Thus $\text{im}(\Phi) = Z + Z'$. The kernel of Φ is the set of all vectors from $Z \oplus Z'$ that map to the zero vector in V. For any (z, z') such that $\Phi(z, z') = 0$. From Φ $z + z' = 0$ so $z' = -z$ For $z' = -z$ to be verified for all requirements and the mapping for the zero vector in V so both z and z' must be in their given subspaces. Thus $z \in Z \cap Z'$. Therefore any such z paired with $-z$ is in the kernel of Φ and $\ker(\Phi) = \{(z, -z) : z \in Z \cap Z'\}$.