Real Analysis Homework 3

1. Let F be an ordered field. (a) Prove that $0 < 1$ and deduce that $-1 < 0$.

Solution. (a) We start with the basic property of a field that states $0 \neq 1$. This property is given and does not need to be proven.

Now, let's assume for a contradiction that $0 \geq 1$. In ordered fields, we have the additive property which states if $a \leq b$, then $a + c \leq b + c$ for any a, b, c in the field.

Applying this to our contradiction assumption, we can add -1 to both sides to get $-1 \geq 0$.

But, -1 can also be expressed as $-1 = (-1)^2 - 1 = 1 - 1 = 0$.

This contradicts our result of $-1 \geq 0$. Therefore, the assumption that $0 \geq 1$ is false. We conclude that $0 < 1$.

Now, we want to prove that $-1 < 0$. This can be done using the multiplicative property of ordered fields, which states that if $a < b$ and $0 < c$, then $ac < bc$.

Applying this property with $a = -1$, $b = 0$, and $c = 1$, we get $-1 \times 1 < 0 \times 1$, or $-1 < 0$. So we have shown that $-1 < 0$.

The results together have established that in an ordered field, $0 < 1$ and $-1 < 0$.

(b) Prove that if $x, y, z \in F$ are such that $x < y$ and $0 < z$ then $xz < yz$.

Solution. From the order axioms, if $x < y$, then $x + z < y + z$ for all z. Let's take $z = -x$ to get $0 < y - x$. Also from order axioms, if $0 < a$ and $0 < b$, then $0 < ab$. Therefore, if $0 < y - x$ and $0 < z$, then $0 < z(y - x)$, which simplifies to $0 < yz - xz$. Adding xz to both sides gives us $xz < yz$. Expanding upon the solutiuon we're given that $x < y$ and $0 < z$. We know from the properties of ordered fields that if $a < b$ and $c > 0$, then $ac < bc$. This is known as the property of order preservation under multiplication by positive elements. Applying this property with $a = x$, $b = y$, and $c = z$, we obtain $xz < yz$. So we have proven the given statement.

(c) Prove that if $x, y, z \in F$ are such that $x < y$ and $z < 0$ then $xz > yz$.

Solution. (c) Similar to (b), we have $0 < y - x$. Now, if $z < 0$, then multiplying both sides by z and switching the inequality due to multiplication with a negative number, we get $0 > z(y - x)$, which simplifies to $0 > yz - xz$. Adding xz to both sides gives us $xz > yz$. Further providing clarity we're given that $x < y$ and $z < 0$. In ordered fields, there's a property that states if $a < b$ and $c < 0$, then $ac > bc$. This is essentially the same as the property used in part (b), but applied to negative multiplier. Applying this property with $a = x$, $b = y$, and $c = z$, we obtain $xz > yz$. So we have also proven this statement.

2. (a) Prove that $\sqrt{3} \notin \mathbb{Q}$ (mimic the proof for $\sqrt{2}$)

Solution. (a)Suppose, for the sake of contradiction, that $\sqrt{3}$ is a rational number. Then, $\sqrt{3} = p/q$ for some coprime integers p and q, with $q \neq 0$. Squaring both sides, we get $3 = p^2/q^2$, or equivalently, $p^2 = 3q^2$. This implies that p^2 is divisible by 3, so p must also be divisible by 3 (as the square of a number divisible by 3 is also divisible by 3). Let's write $p = 3k$ for some integer k. Substituting this into our equation, we have $(3k)^2 = 3q^2$, or $9k^2 = 3q^2$, simplifying to $3k^2 = q^2$. Now we see that q^2 is divisible by 3, so q must be divisible by 3 as well. But this contradicts the assumption that p and q are coprime (i.e., their only common divisor is 1). Therefore, our original assumption that $\sqrt{3}$ is rational must be false. Therefore, $\sqrt{3}$ is irrational.

(b)Prove that $\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$.

Solution. Suppose, for contradiction, that $\sqrt{2} + \sqrt{3}$ is a rational number. Then $\sqrt{2} + \sqrt{3} = r$ for some $r \in \mathbb{Q}$. Squaring both sides gives us $2 + 2\sqrt{6} + 3 = r^2$, or $2\sqrt{6} = r^2 - 5$. This simplifies to $\sqrt{6} = (r^2 - 5)/2$. The right-hand side is a quotient of rational numbers, so it is rational. But this would imply that $\sqrt{6}$ is rational, which contradicts the known fact that $\sqrt{6}$ is irrational. Therefore, our original assumption that $\sqrt{2} + \sqrt{3}$ is rational must be false. Hence, $\sqrt{2} + \sqrt{3}$ is irrational.

3. Prove that the infimum (greatest lower bound) of a set, if it exists, is unique.

Solution. The infimum of a set is the greatest lower bound of that set. By definition, it's a lower bound of the set, meaning it's less than or equal to every element in the set, and it's greater than or equal to every other lower bound of the set. Let's see why this forces the infimum, if it exists, to be unique.

Let's suppose to the contrary that a set A has two different infima, say l_1 and l_2. That is, both l_1 and l_2 are lower bounds of the set A and no other lower bound of A is greater than either of them.

By the definition of a lower bound, all elements of A are greater than or equal to both l_1 and l_2. This means that l_1 and l_2 are also lower bounds of each other (since they can be considered as elements of A).

Moreover, since each of l_1 and l_2 is an infimum of A, each is greater than or equal to any lower bound of A. Therefore, $l_1 \geq l_2$ and $l_2 \geq l_1$. The only way both these inequalities can hold simultaneously is if $l_1 = l_2$.

So we're led to a contradiction. We initially assumed that l_1 and l_2 are different infima of the set A, but our reasoning forced us to conclude that they must be equal.

Therefore, our initial assumption must be wrong and a set cannot have two different infima. This shows that if a set A has an infimum, it must be unique.It is further shown by letting A be a nonempty set in an ordered field F, and let l_1 and l_2 be lower bounds of A such that $l_1 > l_2$. Then, l_1 cannot be an infimum of A because the infimum is the greatest lower bound, and there exists a greater lower bound, l_2, than l_1. Hence, the infimum of a set, if it exists, is unique.

4. If $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$, prove that

$$x^n - y^n = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}.$$

Solution. This is a variation of the formula for the difference of powers, which can be proven using mathematical induction.

The base case, where $n = 1$, is trivially true. Assume the formula holds for some n. We need to prove that it also holds for $n + 1$.

The left-hand side of our expression is $x^{n+1} - y^{n+1}$.

We can write the left-hand side as: $x(x^n - y^n) + y^n(x - y)$.

By the inductive hypothesis, we have $x^n - y^n = (x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k}$.

Substitute this back into our expression to obtain:

$x(x - y) \sum_{k=0}^{n-1} x^k y^{n-1-k} + y^n(x - y)$.

We can rewrite this as:

$x \sum_{k=0}^{n-1} x^k y^{n-1-k} - y \sum_{k=0}^{n-1} x^{k+1} y^{n-1-k} + xy^n - y^{n+1}$.

The next step requires re indexing the second sum by replacing k by $k + 1$ gives us:

$x \sum_{k=0}^{n-1} x^k y^{n-1-k} - \sum_{k=1}^{n} x^k y^{n-k} + xy^n - y^{n+1}$.

And now, notice that we can rewrite this as:

$x^n - y^n + x^n y - y^{n+1}$.

We can then factor out $(x - y)$ to obtain the formula we wanted to prove:

$(x - y) \left(\sum_{k=0}^{n-1} x^k y^{n-k} + x^n \right)$,

which is exactly the formula for $n + 1$, as required, so the proof is complete.

Therefore, the formula is valid for all $n \in \mathbb{N}$. This is a proof by mathematical induction. Each step was justified either by basic arithmetic, the definition of $\sum$, or the inductive hypothesis which is the assumption that the formula holds for n.

Also further demonstarating the proof and providing the solution

$x^n - y^n = (x-y)\sum_{k=0}^{n-1} x^k y^{n-1-k}$.

We can expand the right-hand side:

$(x-y)\sum_{k=0}^{n-1} x^k y^{n-1-k}$

$= \sum_{k=0}^{n-1}(x^{k+1}y^{n-1-k} - x^k y^{n-k})$

$= x^n + \sum_{k=1}^{n-1}(x^k y^{n-k} - x^k y^{n-k}) - y^n$

$= x^n - y^n$.

Therefore, the original equation is correct.

5. Find the supremum and infimum (if they exist) of the set

$$A = \{x \in \mathbb{R} \mid 3x^2 - 10x + 3 < 0\}.$$

Solution. First, the quadratic inequality can be factored:

$3x^2 - 10x + 3 = (3x-1)(x-3) < 0$.

This quadratic equation has roots at $x = 1/3$ and $x = 3$. Since it is an upwards opening parabola as the coefficient of x^2 is positive, the inequality is satisfied in between the roots of the equation. This gives us the interval $(1/3, 3)$.

Next, we consider the elements of the set A:

$A = x \in \mathbb{R} \mid 3x^2 - 10x + 3 < 0 = x \in \mathbb{R} \mid 1/3 < x < 3$.

As for the supremum and infimum of this set, we can note that the supremum and infimum do not belong to the set because x is strictly greater than $1/3$ and strictly less than 3. But, they are the least upper bound and the greatest lower bound of the set.

Therefore, the supremum of the set A is 3 and the infimum is $1/3$. These are the least upper and greatest lower bounds of the interval $(1/3, 3)$.

This result is consistent with the general fact that the supremum and infimum of an open interval (a, b) are b and a, respectively. The inequality $3x^2 - 10x + 3 < 0$ gives us the roots $x = 1/3$ and $x = 3$. So, the solutions to the inequality are $1/3 < x < 3$. Thus, the infimum is $1/3$ and the supremum is 3. So the supremum of this set, denoted as $\sup(A)$, is 3 and the infimum of this set, denoted as $\inf(A)$, is $1/3$.

6. Let A and B be nonempty subsets of $\{x \in \mathbb{R} \mid x > 0\}$. Assume that both A and B are bounded above. Let $a = \sup(A)$ and $b = \sup(B)$. Define $A \cdot B = \{xy \mid x \in A \text{ and } y \in B\}$. Prove that $A \cdot B$ is bounded above and $\sup(A \cdot B) = ab$.

Solution. Let's denote $C = A \cdot B = xy \mid x \in A$ and $y \in B$. We aim to prove that the set C is bounded above and that its supremum the least upper bound is ab.

Firstly, we know that A and B are nonempty subsets of $x \in \mathbb{R} \mid x > 0$. That means all elements of A and B are positive real numbers.

We also know that both A and B are bounded above, by a and b respectively. By definition of upper bounds, we have that for all $x \in A$ and $y \in B$, $x \leq a$ and $y \leq b$. Since x and y are both positive, it follows that $xy \leq ab$. This shows that ab is an upper bound of C.

Now we need to show that ab is the least upper bound of C, or in other words, it is the supremum of C.

To prove this, we need to show that for any $c < ab$, c is not an upper bound of C. Thus let's consider $c = ab - \epsilon$, where $\epsilon > 0$. Since a and b are supremums of A and B respectively, there exist $x \in A$ and $y \in B$ such that $x > a - \frac{\epsilon}{2b}$ and $y > b - \frac{\epsilon}{2a}$.

Note that both $a - \frac{\epsilon}{2b}$ and $b - \frac{\epsilon}{2a}$ are positive, so x and y are positive as well. It follows that $xy > \left(a - \frac{\epsilon}{2b}\right)\left(b - \frac{\epsilon}{2a}\right) = ab - \epsilon$, which is c.

Therefore, c is not an upper bound of C, so ab is the least upper bound of C.

Hence, we have shown that C is bounded above and $\sup(C) = ab$.

By letting A and B be bounded above by a and b respectively. Then, for every $x \in A$ and $y \in B$, we have $x \leq a$ and $y \leq b$. Therefore, $xy \leq ab$ which implies $A \cdot B$ is bounded above and $\sup(A \cdot B) = ab$.