Game Sheet 2: Magic Squares

1) Prove that there is no magic square of order 2.

Proof. Let m be the magic constant of a magic square that is of order 2. Let a, b, c, d be positive distinct integers. That is none of them are equal to each other, so $a \neq b$ $b = c$ $c \neq a$ $b \neq d$. Thus a magic square of order 2 will be composed as follows:

a	b
c	d

The magical constant m is the sum of the numbers in each row, each column, and both diagonals. Then by taking the sum of the first row $m = a+b$ and the sum of the first column is $m = a+C$. Now by substituting m from the second row for the first equation

$a+b = a+c$ then $b = C$. Now by

taking the sum of the second row $m = c+d$
and the second column $m = b+d$

Now the first diagonal $m = a+d$ and
the second diagonal $m = c+b$.

Considering the second row $m = c+d$
since we know $b = c$ then $m = b+d$

Now considering the second digonal
and the first diagonal that is $c+b = a+d$

We have that $m = b+d$ and $m = a+d$
then $b+d = a+d$ thus $b = a$.

Now $c+b = a+d$ becomes
$c+a = a+d$ thus $c = d$.

This would then mean that the new
magic square would be composed of
positive integer values where $a = b$
and $c = d$. So a magic square
would be

a	a
c	c

or

b	b
d	d

This is absurd!!

Because a magic square must be composed of positive distinct integers. However, since we have shown that $a=b$ and $c=d$ it is not possible to construct a magic square of order 2. Therefore, there is no magic square of order 2 ∎

2) Show that for an order 3 magic square, the number 5 must be in the center of square.

Let m be the magic constant for an order 3 magic square. Let a,b,c,d,e,f,g,h,i be positive distinct integers, that is they do not equal each other. Thus a magic square is composed as follows:

a	b	c
d	e	f
g	h	i

A magic constant is $\frac{n(n^2+1)}{2}$ where n denotes the order of a magic square. Since we are dealing with an order 3 magic square $n=3$ thus the magic constant $m = \frac{n(n^2+1)}{2} = \frac{3(9+1)}{2} = 15$.

Considering that $m=15$ we know that $m = d+e+f$, $m = b+e+h$, $m = a+e+i$, and $m = g+e+f$. That is taking both diagonals and the sum of the row and column corresponding to the center value e. The objective is to prove $e=5$. We know that our magic constant is 15. Now each row in our magic square must equal 15 that is

$m = 15 = a+b+c$

$m = 15 = d+e+f$

$m = 15 = g+h+i$

So each row has a magic sum m. We know that all of these rows take values 1-9 since they are the 9 distinct positive integers for an order 3 magic square. Then we will compose an equation where the left hand side is each possible value for the magic square summed together and the right hand side is also the total sum of the magic square 3m since m = each row

and there are 3 rows

total. So

$$1+2+3+4+5+6+7+8+9 = 3m$$

$$45 = 3m$$

$$15 = m.$$

Thus if a magic square is to be composed it must have the magic constant $m = 15$ with distinct positive values 1-9.

Listing all the possible ways to sum 15 using 3 distinct integers that are 1-9 are as follows: $1+5+9=15$

$1+6+8=15$

$2+4+9=15$

$2+5+8=15$

$2+6+7=15$

$3+4+8=15$

$3+5+7=15$

$4+5+6=15$

We know that the center square e must appear in the column sum, row sum, and both diagonals. Thus some integer value must add to 15 from 1-9 in 4 distinct ways. By observing the previous sums of 15 the only integer value that appears 4 distinct ways is 5. That is from

$1+5+9=15$

$1+6+8=15$

$2+4+9=15$

$2+5+8=15$

$2+6+7=15$

$3+4+8=15$

$3+5+7=15$

$4+5+6=15$

So $e=5$ must be true as there is no other possible distinct way to sum 3 distinct integers 1-9 to get 15. Further observing the values for the first entry value given by a. a is involved in 3 sums which are the column $a+d+g=m$, the row sum $a+b+c=m$, and the diagonal $a+e+i=m$. Therefore the corner values a, c, g, i must apper distinctly 3 times as they contribute to the 3 previous sums. From the given list the corner values are 2,4,6,8.

$1+5+9=15$

$1+6+8=15$

$2+4+9=15$

$2+5+8=15$

$2+6+7=15$

$3+4+8=15$

$3+5+7=15$

$4+5+6=15$

With the same approach to the edges column sum $b+e+h=15$ and its raw sum $a+b+c=15$

where the edge values b, d, f, h appear in two distinct sums. That is the only possible values that distinctly apper twice are 3, 9, 7, 1. From

$1 + 5 + 9 = 15$

$1 + 6 + 8 = 15$

$2 + 4 + 9 = 15$

$2 + 5 + 8 = 15$

$2 + 6 + 7 = 15$

$3 + 4 + 8 = 15$

$3+5+7=15$

$4+5+6=15$

Thus we have shown that a magic square of order 3 has a magic constant $m=15$ and with a given center value we shall call e. where the center value $e=5$ since we have shown e must odd up to 15 in 4 distinct ways using values 1-9.

The only possible ways are $2+5+8=15$,

$3+5+7=15$

$4+5+6=15$

$1+5+9=15$

Therefore the center must be 5.

3) ①

4	9	2
3	5	7
8	1	6

given a magic square we will rotate it counter clockwise by 90°

② 90

2	7	6
9	5	1
4	3	8

performing another rotation results in

③

6	1	8
7	5	3
2	9	4

Another rotation of ④

8	3	4
1	5	9
6	7	2

. Cannot rotate again or else we will return to ① original square.

Now taking original square again and performing a flip from ① → ⑤

① flip ⑤

4	9	2
3	5	7
8	1	6

→

2	9	4
7	5	3
6	1	8

Taking 5 and performing 90° counter clockwise rotations results in 6, 7, 8

5

2	9	4
7	5	3
6	1	8

→

6

4	3	8
9	5	1
2	7	6

90°

→

7

8	1	6
3	5	7
4	9	2

90°

→

8

6	7	2
1	5	9
8	3	4

performing any other rotation or flip will result in some variation we have already seen. There are only 8 ways to obtain a magic square from rotation or reflection. That is because we know that the center must be 5 from problem 2 we also showed that the corners are 2,4,6,8 when paired by their distinct sum (2,8),(4,6) also the edges were shown to take values 1,3,9,7 when paired by their distinct sums (9,1),(7,3) There are only 8 possible ways to achieve these combinations

that is $(2,8)$ $(8,2)$ $(4,6)$ $(6,4)$
$(9,1)$ $(1,9)$ $(7,3)$ $(3,7)$.

Since the combinations show all possible ways to achieve the sum of 15.

Example using $(2,8)$ & $(8,2)$

$$2+5+8=15$$
$$8+5+2=15$$

This is an account of a flip. We have shown all possible flips and rotations of order 3 magic square.

4) given matrix A ~~&~~ & B

A

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

B

7	14	2	11
9	4	16	5
12	1	13	8
6	15	3	10

observing A with corner values of 1,4,13,16 and B corner values 7,6,11,10 there is no possible way to obtain B from A given rotations or reflections. It is not possible to make the corner values of matrix A & B equal to each other by performing all possible rotations or by reflections since the two matrices have values that are not equal to each other.

List more order 4 magic squares using other strategies.

By subtracting 17 which is 1/2 the magic constant value of order 4 square since $\frac{4(4^2+1)}{2} = 34$

$m = \frac{\quad}{2} =$

$\frac{m}{2} = 17$ we will subtract every value of the magic square by 17 by creating an equivalent new magic square therefore from $A \to A'$

A

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

$\xrightarrow{-17}$

A'

-16	-2	-3	-13
-5	-11	-10	-6
-9	-7	-6	-12
-4	-14	-15	-1

If you observe the new magic square was created by subtracting half the magic constant.

without loss of generality the same can be done to B that is $B \to B'$

B

7	14	2	11
9	4	16	5
12	1	13	8
6	15	3	10

$\xrightarrow{-17}$

B'

-10	-3	-15	-6
-8	-13	-1	-12
-5	-16	-4	-9
-11	-2	-14	-7

By the given strategy we have constructed more magic squares.

Implementing another strategy to generate more order 4 magic squares we will swap rows or columns to obtain new magic squares thus from A → A''

A

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

$\xrightarrow{\text{swap row } 1 \,\&\, 2}$

A''

12	6	7	9
1	15	14	4
8	10	11	5
13	3	2	16

we have constructed a new magic square A'' this method can be done with other rows being swapped or other columns being swapped.

column example just to be explicit using A thus $A \to A'''$

A

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

swap 1st & 2nd column $\longrightarrow$

A'''

15	1	14	4
6	12	7	9
10	8	11	5
3	13	2	16

we have shown more order 4 magic squares that are essentially different from A and B other than using rotation or reflection