Math 100A Homework 8

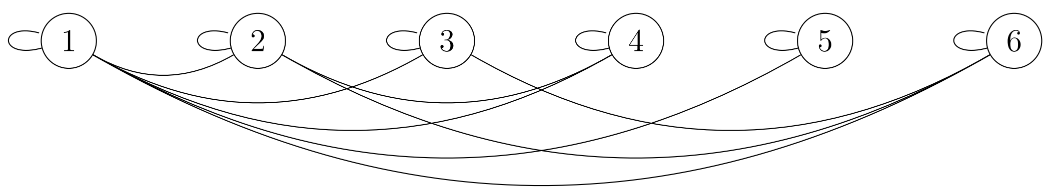
1. CH 11.1 PROB. 2 . Let $A = \{1, 2, 3, 4, 5, 6\}$. Write out the relation R that expresses | (divides) on A. Then illustrate it with a diagram.

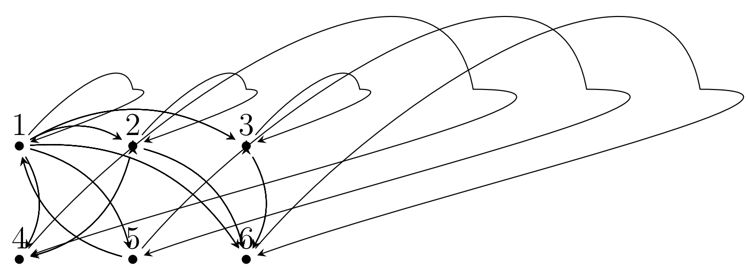
Solution. Let $A = \{1, 2, 3, 4, 5, 6\}$. The relation R that expresses | (divides) on A is:

$$R = \{(a, b) : a, b \in A \text{ and } a \mid b\}$$

$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 2),$

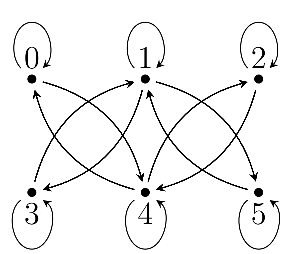
(2,4), (2,6), (3,3), (3,6), (4,4), (4,6), (5,5), (6,6)}





(this is with arrows I had trouble with arrows displaying on the previous diagram)

2. CH 11.1 PROB. 4 Here is a diagram for a relation R on a set A. Write the sets A and R.



Solution. Here $A = \{0, 1, 2, 3, 4, 5\}$

Where $R = \{(0, 0), (0, 4), (1, 1), (1, 3), (1, 5), (2, 2), (2, 4),$

(3,3), (3,1), (4,2), (4,0),(4,4),(5,5), (5,1)$\}$

3. CH 11.2 PROB 8 Define a relation on $\mathbb{Z}$ as xRy if $|x - y| < 1$. Is R reflexive? Symmetric? Transitive? If a property does not hold, say why. What familiar relation is this?

Solution. The relation R is reflexive because for any when inputted the inequality holds since $|x - x| = 0 < 1$ thus xRx.

The relation R is symmetric because the objective is to show that yRx which means we need to show that $|y - x| < 1$. Thus xRy so $|x - y| < 1$ results in $|-1| \cdot |x - y| < 1$ therefore $|y - x| < 1$ for yRx.

The relation R is transitive because for any if xRy and yRz then $|x - y| < 1$ and $|y - z| < 1$ then $|x - z| \leq |x - y| + |y - z| < 2$ therefore xRz. Also accounting for xRZ then $|x - z| < 1$ thus the relation is transitive.

This relation is the set of integers that when you take the difference of the two it is at most equal to 1. Therefore the two integers differ by at most 1.

4. CH 11.2 PROB 14 Suppose R is a symmetric and transitive relation on a set A, and there is an element $a \in A$ for which aRx for every $x \in A$. Prove that R is reflexive.

Solution. The objective is to show for every $x \in A$, xRx holds. Since we are given that aRx for every $x \in A$, we can use the symmetry and transitivity of R to show that xRx holds for any $x \in A$.

For any $x \in A$. Since $a \in A$ and aRx then xRa due to symmetry. Then xRx since aRx and xRa due to transitivity. aRa when $x = a$. Given R is symmetric aRx then xRa for every $x \in A$. Then aRa and aRx results in xRa for any $x \in A$.

Since R is transitive if xRa and aRy for any $x, y \in A$ results in xRy. Since aRa for any $a \in A$ then aRa and aRa for any $a \in A$ thus resulting in aRa by transitivity.

Therefore we have shown that aRa for any $a \in A$ thus R is reflexive.

5. CH 11.3 PROB 6 . There are five different equivalence relations on the set $A = \{a, b, c\}$. Describe them all.

Solution. aRa, bRb, and cRc is one such relation where $R = \{(a, a), (b, b), (c, c)\}$

Then the case for aRb, bRa, aRc, cRa, bRc, and cRb is

$R = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b)\}$.

The relation where only a and b are related: $R = \{(a, b), (b, a), (a, a), (b, b), (c, c)\}$.

The relation where only a and c are related: $R = \{(a,c),(c,a),(a,a),(c,c),(b,b)\}$.

Lastly the relation where only b and c are related: $R = \{(b,c),(c,b),(b,b),(c,c),(a,a)\}$.

6. CH 11.3 PROB 10 Suppose R and S are two equivalence relations on a set A. Prove that $R \cap S$ is also an equivalence relation. (For an example of this, look at Figure 11.2. Observe that for the equivalence relations R_2, R_3 and R_4, we have $R_2 \cap R_3 = R_4$.)

Solution. Suppose R and S are two equivalence relations on a set A. Prove that $R \cap S$ is also an equivalence relation. (For an example of this, look at Figure 11.2. Observe that for the equivalence relations R_2, R_3 and R_4, we have $R_2 \cap R_3 = R_4$.)

To prove that $R \cap S$ is an equivalence relation the objective is the check for the three properties that are symmetry,transitivity, and reflexivity.

Checking for reflexivity. For any $a \in A$ then $a \in R \cap S$ since R and S are both equivalence relations and therefore contain a in their equivalence classes. Thus aRa and aSa hold true which means $a \in R \cap S$ results in aRa and aSa so $R \cap S$ is reflexive. Reflexivity holds true for the case.

Checking for symmetry. Suppose $a, b \in A$ with $R \cap S$. Then aRb and aSb hold true since R and S are equivalence relations. Also bRa and bSa by the symmetry of R and S. Therefore b,a on $R \cap S$ holds true and $R \cap S$ is symmetric. Symmetry holds true for the case.

Checking for transitivity.Suppose $a, b, c \in A$ with a,b on $R \cap S$ and b,c $R \cap S$. Then aRb, bRa, bSc, and cSb hold true since R and S are equivalence relations. R Results in aRc due to transitivity and by transitivity of S results in aSc. Therefore a,c on $R \cap S$ holds true and $R \cap S$ is transitive.The case for transitivity is therefore True.

Therefore $R \cap S$ is an equivalence relation after checking the three properties each hold true.

7. CH 11.3 PROB 14 Suppose R is a reflexive and symmetric relation on a finite set A. Define a relation S on A by declaring xSy if and only if for some $n \in \mathbb{N}$ there are elements $x_1, x_2, \ldots, x_n \in A$ satisfying xRx_1, x_1Rx_2, x_2Rx_3, $x_3Rx_4, \ldots, x_{n-1}Rx_n$, and x_nRy. Show that S is an equivalence relation and $R \subseteq S$. Prove that S is the unique smallest equivalence relation on A containing R.

Solution. The objective is to show S is an equivalence relation and $R \subseteq S$ then that S is the unique smallest equivalence relation on A containing R. For the objective is to show S is an equivalence relation the properties of S is reflexive, symmetric, and transitive must be verified. When verifying reflexivity let $x \in A$. Since R is reflexive then xRx. Then there exists a (x) such that xRx. Therefore xSx and S is reflexive. When verifying symmetry let $x, y \in A$ such that xSy. Then there exists a $(x_1, x_2, \ldots, x_n)$ such that

xRx_1, x_1Rx_2, …, $x_{n-1}Rx_n$, and x_nRy. Since R is symmetric then yRx_n, x_nRx_{n-1}, …, x_2Rx_1, and x_1Rx. Thus $(y, x_n, x_{n-1}, \ldots, x_1, x)$ is satisfying yRx, xRx_1, …, $x_{n-1}Rx_n$, and x_nRy. Therefore ySx and S is symmetric. When checking transitivity let $x, y, z \in A$ such that xSy and ySz. Then there exist $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_m)$ such that xRx_1, x_1Rx_2, …, $x_{n-1}Rx_n$, and x_nRy, and yRy_1, y_1Ry_2, …, $y_{m-1}Ry_m$, and y_mRz. Combining the two sequences results in $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$ thus xRx_1, x_1Rx_2, …, $x_{n-1}Rx_n$, x_nRy_1, y_1Ry_2, …, $y_{m-1}Ry_m$, and y_mRz. Therefore xSz and S is transitive. Therefore S is an equivalence relation.

The next objective is to demonstrate $R \subseteq S$. Let $x, y \in A$ such that xRy then xRx due to reflexivity, and xRy. Then there exists a (x, y) satisfying xRx and xRy. Therefore xSy and $R \subseteq S$. the final objective is to show that S is the unique smallest equivalence relation on A containing R. Let P be an equivalence relation on A containing R then it must be shown that $S \subseteq P$. xSy then xPy. Suppose xSy, then there exist $n \in \mathbb{N}$ and elements $x_1, x_2, \ldots, x_n$ in A such that xRx_1, x_1Rx_2, …, $x_{n-1}Rx_n$, and x_nRy. Since $R \subseteq P$ and P is transitive then $xRx_1Rx_2 \cdots x_{n-1}Rx_nRy$, so xTy. Therefore $S \subseteq P$.Since S is an equivalence relation on A containing R, and any equivalence relation on A containing R must contain S. Thus S is the unique smallest equivalence relation on A containing R. Thus it has been shown that S is an equivalence relation and $R \subseteq S$ and that S is the unique smallest equivalence relation on A containing R.

8. CH 11.4 PROB 4 Suppose P is a partition of a set A. Define a relation R on A by declaring xRy if and only if $x, y \in X$ for some $X \in P$. Prove R is an equivalence relation on A. Then prove that P is the set of equivalence classes of R.

Solution. R is an equivalence relation on A if the three properties reflexivity, symmetry, and transitivity are verified as true.

verifying reflexivity for any $x \in A$, xRx since x is in the partition X that contains x. therefore xRx, and R is reflexive since for any $x \in A$ results in $x \in X$ for some $X \in P$ since P is a partition of A.

verifying for Symmetry. Now suppose xRy for some $x, y \in A$. Then there exists some $X \in P$ such that $x, y \in X$. Since X is a set, $y, x \in X$. Therefore yRx and R is symmetric since for any $x, y \in A$, if xRy, then yRx since x and y are in the same partition.

verifying for transitivity. Now suppose xRy and yRz for some $x, y, z \in A$. Then there exist $X, Y \in P$ such that $x, y \in X$ and $y, z \in Y$. Since P is a partition, $X = Y$ or $X \cap Y = \emptyset$. If $X = Y$, then $x, z \in X$, and xRz. If $X \cap Y = \emptyset$ then y belongs to two different sets in the partition. Therefore, R is transitive since for any $x, y, z \in A$, if xRy and yRz, then xRz since x, y, and z are all in the same partition.

Therefore R is an equivalence relation on A.

Now the objective is to show that P is the set of equivalence classes of R. Thus each element in A belongs to exactly one equivalence class and that each equivalence class corresponds to exactly one element in P must be shown.

Let $[x]$ denote the equivalence class of x under R, $[x] = y \in A \mid xRy$. For every element in A belongs to exactly one equivalence class. Since P is a partition of A there exists a unique $X \in P$ such that $x \in X$. Then by definition of R xRy for all $y \in X$. Thus x belongs to the equivalence class $[x] = X$. If y belongs to $[x]$ then xRy, which means y also belongs to X. Thus every element in A belongs to exactly one equivalence class.

Now the objective is to show each equivalence class corresponds to exactly one element in P. Let $X \in P$ and $x \in X$ be an element in X. then $[x] = X$. So $[x] \subseteq X$ let $y \in [x]$ be some value. Then by definition of $[x]$, xRy, thus y belongs to the same partition as x. Since x belongs to X and y belongs to the same partition as x then $y \in X$. Therefore $[x] \subseteq X$. Now for $X \subseteq [x]$, $y \in X$ will take on a value. Since x and y belong to the same partition we have xRy then y belongs to the same equivalence class as x. Therefore $X \subseteq [x]$. Therefore each equivalence class corresponds to exactly one element in P Since $[x] = X$ for any $X \in P$ and $x \in X$.

Therefore P is the set of equivalence classes of R.

9. CH 11.5 PROB 4 Write the addition and multiplication tables for $\mathbb{Z}_6$.

Solution. The set $\mathbb{Z}_6$ consists of the integers $0, 1, 2, 3, 4, 5$ under addition and multiplication modulo 6. The addition and multiplication tables for $\mathbb{Z}_6$ are shown below:

$\times$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

10. Ch 11.5 Prob 6. Suppose $[a], [b] \in \mathbb{Z}_6$ and $[a] \cdot [b] = [0]$. Is it necessarily true that either $[a] = [0]$ or $[b] = [0]$? What if $[a], [b] \in \mathbb{Z}_7$?

Solution. For $\mathbb{Z}_6$ if $[a] \cdot [b] = [0]$ it is not necessarily true that either $[a] = [0]$ or $[b] = [0]$. As shown in the previous problem $[2] \cdot [3] = [0]$ in $\mathbb{Z}_6$ where $[2]$, $[3] \neq [0]$.

We know that $[a] \cdot [b] = [ab]$. Since $[ab] = [0]$, it follows that ab is a multiple of 6. Thus a and b must both be multiples of 2 or 3 Since they are divisors of 6.

If a and b are both multiples of 2 or both multiples of 3 then $[a] = [0]$ or $[b] = [0]$ respectively so the statement is true in $\mathbb{Z}_6$. Thus verifying the case provided. Now if a and b are not both multiples of 2 or not both multiples of 3, then $[a] \cdot [b] \neq [0]$. If $a = 2$ and $b = 4$, then $[a] = [2]$ and $[b] = [4]$, but $[a] \cdot [b] = [2] \cdot [4] = [8] = [2] \neq [0]$.

In $\mathbb{Z}_7$, the statement is true. Since 7 is prime the only divisors of 7 are 1 and 7 thus the only multiples of a, b are 0 and 7. Thus ab can only be 0 if at least one of a or b is 0 which means that either $[a] = [0]$ or $[b] = [0]$.