DEVIN MAYA 3/17/23

Math 100A Homework 9

1. CH 12.1 PROB. 8 Consider the set $f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4\}$. Is this a function from $\mathbb{Z}$ to $\mathbb{Z}$?n Explain?

Solution. The set $f = (x, y) \in \mathbb{Z} \times \mathbb{Z} : x + 3y = 4$ can also be considered as a function from $\mathbb{Z}$ to $\mathbb{Z}$. This is the function $f : \mathbb{Z} \to \mathbb{Z}$ defined as $y = \frac{4-x}{3}$

2. CH 12.2 Prob 4 A function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as $f(n) = (2n, n + 3)$. Verify whether this function is injective and whether it is surjective.

Solution. First the objective is to verify if it is injective or One-to-One Suppose $m, n \in \mathbb{Z}$ and $f(m) = f(n)$. Then $(2m, m + 3) = (2n, n + 3)$ So $2m = 2n \quad \text{and} \quad m + 3 = n + 3$ Thus $m = n$ for both by dividing by two on the first equation and subtracting three for the second equation. Thus f is injective since $f(m) = f(n)$, then $m = n$.

Now the objective is to verify surjectivity For any element $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, there exists an $n \in \mathbb{Z}$ such that $f(n) = (a, b)$.

Since $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ find n such that $(2n, n + 3) = (a, b)$ So $2n = a \quad \text{and} \quad n + 3 = b$ Solving the first equation for n results in $n = \frac{a}{2}$ Since a is an integer then n will be an integer if a is even. Substituting n into the second equation results in $\frac{a}{2} + 3 = b$ This equation is satisfied for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ where a is even. If a is odd then the equation does not result in an integer. Thus a must be even for it to satisfy the equation. Thus f is not surjective.

Therefore function $f(n) = (2n, n + 3)$ is injective but not surjective.

3. CH 12.2 PROB 16 This question concerns functions $f : \{A, B, C, D, E\} \to \{1, 2, 3, 4, 5, 6, 7\}$. How many such functions are there? How many of these functions are injective? How many are surjective? How many are bijective?

Solution. Function f can be described as a list $(f(A), f(B), f(C), f(D), f(E))$ where there are seven choices for each entry. By the multiplication principle the total number of functions f is $7^5 = 16807$. If f is injective, then this list can't have any repetition so there are $7!/(7 - 5)! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ injective functions. Since the domain has fewer elements than the codomain $5 < 7$ it is not surjective. Thus there are 0 surjective functions. Therefore it is not a bijective function since the domain and codomain do not have the same number of elements $5 \neq 7$ so there are 0 bijective functions. Therefore there are 16807 total functions, 2520 injective functions, 0 surjective functions, and 0 bijective functions.

4. CH 12.4 PROB 8 Consider the functions $f, g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined as $f(m, n) = (3m - 4n, 2m + n)$ and $g(m, n) = (5m + n, m)$. Find the formulas for $g \circ f$ and $f \circ g$.

Solution. $g \circ f$: $g \circ f(m, n) = g(f(m, n)) = g(3m - 4n, 2m + n)$. Then

$g(3m-4n, 2m+n) = (5(3m-4n)+(2m+n), 3m-4n) = (15m-20n+2m+n, 3m-4n)$

Simplifying results in $g \circ f(m, n) = (17m - 19n, 3m - 4n)$

$f \circ g$: $f \circ g(m, n) = f(g(m, n)) = f(5m + n, m)$. Then

$f(5m + n, m) = (3(5m + n) - 4m, 2(5m + n) + m) = (15m + 3n - 4m, 10m + 2n + m)$

Simplifying results in $f \circ g(m, n) = (11m + 3n, 11m + 2n)$

Therefore $g \circ f(m, n) = (17m - 19n, 3m - 4n)$ and $f \circ g(m, n) = (11m + 3n, 11m + 2n)$

5. CH 12.5 PROB 2 . In Exercise 9 of Section 12.2, you proved that $f : \mathbb{R} - 2 \to \mathbb{R} - 5$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective. Now find its inverse.

Solution. The objective is to solve for x thus Multiply both sides by $(x - 2)$ results in $y(x - 2) = 5x + 1$

$xy - 2y = 5x + 1$

$xy - 5x = 1 + 2y$

$x(y - 5) = 1 + 2y$

$x = \frac{1+2y}{y-5}$

Therefore the inverse function $f^{-1} : \mathbb{R} - 5 \to \mathbb{R} - 2$ is

$f^{-1}(y) = \frac{1+2y}{y-5}$

6. CH 12.5 PROB 8 Is the function $\theta : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ defined as $\theta(X) = \overline{X}$ bijective? If so, find θ^{-1}.

Solution. Verifying for injectivity first let $\theta(X) = \theta(Y)$. Then $\overline{X} = \overline{Y}$. Then $X = Y$. Therefore θ is injective.

Verifying for surjectivity let $Y \in \mathcal{P}(\mathbb{Z})$. $X \in \mathcal{P}(\mathbb{Z})$ such that $\theta(X) = Y$. If $X = \overline{Y}$ then $\theta(X) = \theta(\overline{Y}) = \overline{\overline{Y}} = Y$.Since the complement of a complement results in the regular set. Therefore θ is surjective.

therefore it is bijective since θ is both injective and surjective.

The inverse $\theta^{-1} : \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})$ is

$\theta^{-1}(Y) = \overline{Y}$

7. CH 12.6 PROB 6 Given a function $f : A \to B$ and a subset $Y \subseteq B$, is $f(f^{-1}(Y)) = Y$ always true? Prove or give a counterexample.

Solution. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^4$ or any even powered function such as $f(x) = x^2$ and let $Y = 1$. Then $f^{-1}(Y) = -1, 1$ and $f(f^{-1}(Y)) = f(-1, 1) = 1$. $Y = 1$ does not equal$f(f^{-1}(Y))$ so $f(f^{-1}(Y)) = Y$ is not true for all cases. Therefore disproving statement.

8. CH 12.6 PROB 10 Given $f : A \to B$ and subsets $Y, Z \subseteq B$, prove $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$.

Solution. First it will be shown that $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$ Let $a \in f^{-1}(Y \cap Z)$. By Definition 12.9 this means $f(a) \in Y \cap Z$. Thus $f(a) \in Y$ and $f(a) \in Z$. Since $f(a) \in Y$ then $a \in f^{-1}(Y)$by Definition 12.9. Since $f(a) \in Z$ then $a \in f^{-1}(Z)$. Then $a \in f^{-1}(Y) \cap f^{-1}(Z)$. Results in $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$.

Now it will be shown that $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$ let $a \in f^{-1}(Y) \cap f^{-1}(Z)$. Then $a \in f^{-1}(Y)$ and $a \in f^{-1}(Z)$. By Definition 12.9 $f(a) \in Y$ and $f(a) \in Z$ thus $f(a) \in Y \cap Z$. By Definition 12.9, $f(a) \in Y \cap Z$ means $a \in f^{-1}(Y \cap Z)$. Results in $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$.

Therefore $f^{-1}(Y \cap Z) = f^{-1}(Y) \cap f^{-1}(Z)$ since $f^{-1}(Y \cap Z) \subseteq f^{-1}(Y) \cap f^{-1}(Z)$ and $f^{-1}(Y) \cap f^{-1}(Z) \subseteq f^{-1}(Y \cap Z)$.