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Real Analysis Homework 5

1. (1) If $S \subseteq \mathbb{R}^n$, recall that S' denotes the derived set of S, that is, S' is the set of all accumulation points of S. Prove: (a) If $S \subseteq T \subseteq \mathbb{R}^n$, then $S' \subseteq T'$.

Solution.

Proof. Let x be an arbitrary point in S'. By definition of an accumulation point, for every open neighborhood U of x, U contains a point in S different from x itself. Now, since $S \subseteq T$, every point of S is also a point in T. Hence, for every open neighborhood of x, U contains a point of T different from x itself. Thus, x is an accumulation point of T. Therefore, $x \in T'$.

Since x was arbitrary, this implies that every element of S' is also an element of T'. Thus, $S' \subseteq T'$. □

(b) If S and T are both subsets of $\mathbb{R}^n$, then $(S \cup T)' = S' \cup T'$.

Solution. Prove that if S and T are both subsets of $\mathbb{R}^n$, then $(S \cup T)' = S' \cup T'$.

Solution:

Proof. We'll prove this in two parts: $(S \cup T)' \subseteq S' \cup T'$ and $S' \cup T' \subseteq (S \cup T)'$.

1. Proof of $(S \cup T)' \subseteq S' \cup T'$:

Let x be any point in $(S \cup T)'$. This means for every open neighborhood U of x, U contains points from $S \cup T$ other than x. These points, by definition of union, must belong to either S or T.

If for some neighborhoods, the points belong to S and for others to T, then x is an accumulation point for both S and T, and hence $x \in S' \cup T'$.

If all such points belong only to S or only to T, then x is an accumulation point for either S or T, which again implies $x \in S' \cup T'$.

Therefore, $(S \cup T)' \subseteq S' \cup T'$.

2. Proof of $S' \cup T' \subseteq (S \cup T)'$:

Let x be any point in $S' \cup T'$. Without loss of generality, assume $x \in S'$. This means for every open neighborhood U of x, U contains points from S different from x. These points clearly belong to $S \cup T$ as well. Thus, x is also an accumulation point of $S \cup T$, implying $x \in (S \cup T)'$.

A similar argument holds if $x \in T'$.

Thus, $S' \cup T' \subseteq (S \cup T)'$.

Combining the two parts, we have $(S \cup T)' = S' \cup T'$. □

2. (2) If $S \subseteq \mathbb{R}^n$, prove that the collection of isolated points of S is countable (hint: use a covering theorem).

Solution.

Proof. An isolated point of a set S is a point $x \in S$ such that there exists an open ball B centered at x such that $B \cap S = \{x\}$.

Given an isolated point x of S, there exists an open ball $B(x, r)$ for some $r > 0$ such that $B(x, r) \cap S = \{x\}$. Since the set of rational numbers is dense in $\mathbb{R}$, we can find a rational number r_x with $0 < r_x < r$ such that $B(x, r_x) \cap S = \{x\}$.

For each isolated point x, consider the ball $B(x, r_x)$. Since two different isolated points cannot have overlapping balls (by the definition of isolated points), each isolated point can be uniquely associated with a ball of a rational radius.

Consider the set Q^n of points in $\mathbb{R}^n$ with rational coordinates. This set is countable. For every ball $B(x, r_x)$, choose a point $p \in Q^n$ that lies within this ball. Since the number of rational radii is countable and Q^n is countable, the set of such points p is also countable.

Thus, we have associated every isolated point of S with a unique point in a countable set, which implies that the collection of isolated points of S must also be countable.

Therefore, the set of isolated points of S is countable. □

3. (3) The collection $\mathcal{O}$ of open intervals of the form $(1/n, 2/n)$, where $n = 2, 3, \ldots$, is an open cover of the interval $(0, 1)$. Prove directly (i.e., without the Heine-Borel Theorem) that no finite subcollection of $\mathcal{O}$ covers $(0, 1)$.

Solution.

Proof. Assume for the sake of contradiction that a finite subcollection $\mathcal{F}$ of $\mathcal{O}$ covers the interval $(0, 1)$. Let the intervals in $\mathcal{F}$ be denoted by $(1/n_1, 2/n_1), (1/n_2, 2/n_2), \ldots, (1/n_k, 2/n_k)$ where $n_1, n_2, \ldots, n_k$ are distinct positive integers.

Among the indices $n_1, n_2, \ldots, n_k$, let N be the largest. This implies that for any i, $n_i \leq N$.

Now, consider the number $\frac{1}{N+1}$, which is an element of the interval $(0, 1)$. For every interval $(1/n_i, 2/n_i)$ in $\mathcal{F}$, we deduce that $\frac{1}{N+1} \leq \frac{1}{n_i}$ due to $n_i \leq N$. Consequently, $\frac{1}{N+1}$ is not included in any interval from $\mathcal{F}$.

This observation is in contradiction with our initial assumption that $\mathcal{F}$ covers the entire interval $(0,1)$, as $\frac{1}{N+1}$ lies within $(0,1)$ but is excluded from all intervals in $\mathcal{F}$.

Hence, our conclusion is that no finite subcollection of $\mathcal{O}$ covers the interval $(0,1)$. $\square$

4. (4) Prove that every finite subset of $\mathbb{R}^n$ is compact.

Solution.

Proof. Let S be a finite subset of $\mathbb{R}^n$ consisting of the elements $x_1, x_2, \ldots, x_k$.

To establish the compactness of S, we must demonstrate that for any open cover $\mathcal{O}$ of S, there exists a finite subcover.

Given any open cover $\mathcal{O}$ of S, for each point x_i in S, by the nature of the cover, there exists some open set O_i in $\mathcal{O}$ such that x_i is an element of O_i.

Now, take the collection $\{O_1, O_2, \ldots, O_k\}$. This is a finite subcollection of $\mathcal{O}$ which, by definition, covers all of S. This follows since each point x_i in S is contained in its corresponding open set O_i.

Consequently, every open cover $\mathcal{O}$ of S possesses a finite subcover. Therefore, by definition, S is compact. $\square$

5. (5) Let $S, T \subseteq \mathbb{R}^n$ such that S is closed and T is compact. Prove that $S \cap T$ is compact.

Solution.

Proof. To establish the compactness of $S \cap T$, we need to demonstrate that for any open cover $\mathcal{O}$ of $S \cap T$, there exists a finite subcover.

Given any open cover $\mathcal{O}$ of $S \cap T$, since T is compact, there exists a finite subcollection $\mathcal{O}_T = \{O_1, O_2, \ldots, O_k\}$ of $\mathcal{O}$ that covers T, i.e.,

$$T \subseteq \bigcup_{i=1}^{k} O_i.$$

Now, consider the intersection $S \cap T$. Since every point of $S \cap T$ lies in T, the finite subcollection $\mathcal{O}_T$ also covers $S \cap T$.

Therefore, the collection $\mathcal{O}_T$ is a finite subcover of $S \cap T$ for every open cover $\mathcal{O}$ of $S \cap T$. This proves that $S \cap T$ is compact. $\square$

6. 6) Prove that the intersection of an arbitrary collection of compact subsets of $\mathbb{R}^n$ is compact.

Solution.

Proof. Let $\mathcal{C}$ be an arbitrary collection of compact subsets of $\mathbb{R}^n$ and let

$$K = \bigcap_{C \in \mathcal{C}} C.$$

We want to show that K is compact.

Let $\mathcal{O}$ be an arbitrary open cover of K.

Now, take an arbitrary set C from $\mathcal{C}$. Since $K \subseteq C$, every open set that covers K also covers the part of C that intersects with K. As C is compact, there exists a finite subcollection $\mathcal{O}_C \subseteq \mathcal{O}$ that covers C.

Since $K \subseteq C$, this same finite subcollection $\mathcal{O}_C$ covers K. Note that this argument is true for any C chosen from $\mathcal{C}$.

Thus, for any open cover $\mathcal{O}$ of K, we can always find a finite subcover that covers K. Hence, K is compact. $\square$