Asignment 3 Devin Maya

1.2.1 Objective is to show that a function $f: X \to Y$ is continuous if & only if for every closed subset A of Y, the pre-image $f^{-1}(A)$ is a closed subset of X. First assume that f is a continuous function. Let A be a closed subset of Y. Since A is closed the complement of A is in Y so $Y-A$ is open. We know f is continuous and also that the pre image of an open set is open thus $f^{-1}(Y-A)$ is open in X. $f^{-1}(A)$ has complement $f^{-1}(Y-A)$ in X. Therefore $f^{-1}(A)$ has an open complement and $f^{-1}(A)$ is closed.
Now the other direction. Assume that for every closed set A in Y its pre image $f^{-1}(A)$ is closed in X. Take an arbitrary open set B in Y then $Y-B$ is closed in

Y. We know for every closed set A in Y the pre image $f^{-1}(A)$ is closed in X thus $f^{-1}(Y-B)$ is closed in X. $f^{-1}(B)$ in X has complement $f^{-1}(Y-B)$ and we know the complement is closed thus $f^{-1}(B)$ is open. Therefore the pre image of every open set Y is open in X & f is continuous.

1.2.4 The objective is to show that for two continuous functions $f: X \to Y$ then $g: Y \to Z$ the composition function $g \circ f$ is $X \to Z$ is continuous.

Suppose both $f: X \to Y$ & $g: Y \to Z$ are continuous. The objective is to show that for any open set U in Z the pre image $g \circ f^{-1}(U)$ is open in X. We know g is continuous the pre image $g^{-1}(U)$ is an open set Y also since f is continuous the pre image of an open set in Y is open in X. Thus the pre image $f^{-1}(g^{-1}(U))$ is open in X. $f^{-1}(g^{-1}(U))$ is the same as $g \circ f^{-1}(U)$ since the pre image of U through composition $g \circ f$ is the set of all points in X which when U is applied there is a mapping of such points in U. This is the same as the set of points in X where there is a mapping from f to $g^{-1}(U)$ which is $f^{-1}g^{-1}(U)$. For every open set U in Z the pre image through composition $g \circ f$ is open in X. Therefore this aligns with the definition of continuity so the composition of continuous functions is continuous.

1.2.5 This question will be approached with a counter example in order to prove that A and B are closed in the gluing Lemma is neccessary. The approach will begin with making X = Real numbers and two intervals say P and Q that are not closed in the set of real numbers $A = (0,1]$ and $B = [1,2)$ also take $Z = \mathbb{R}$. Then both continuous functions are $f: A \to Z$ $G: B \to Z$ where $f(x) = 1$ for x in A and $G(x) = 0$ for x in B. We know both are continuous $f(x)$ and $G(x)$. Then $D: X \to Z$ the same in the gluing lemma thus $D(x) = f(x)$ for x in A and $D(x) = G(x)$ for x in B. Here it is observed that when $x = 1$ $D(x) = 1$ for x in A and $D(x)$ also is equal to 0 for x in B. So D is not continuous at $x = 1$ since the limit $D(x)$ when x approaches 1 is not equal when the limit approaches from either direction. Thus if f and G are continuous and $f(x) = G(x)$ for x in $A \cap B$ then we know that D is not continuous since A & B aren't closed. Therefore it must be that A & B are closed in the gluing lemma.

1.2.12 Here it is observed that the inverse of the wrapping function is not continuous since if you start at the 0 point and approach the point 1 from either side there is an evident discontinuity. Since the inverse doesn't map back to a single point it cannot be continuous.

1.2.23 First we will begin by letting $f: X \to Y$ be a homeomorphism and x a limit point of subset A of X.
It is known that x is a limit point of A in X thus $f(x)$ is a limit point of $f(A)$ in Y. This is because a homeomorphism preserves limit points.
The objective is to show that every open set U of Y with $f(x)$ U will intersect $f(A) - \{f(x)\}$ provides a non empty set as a result. The pre image of U under f is an open set in X containing x since f is a homeomorphism.
x is a limit point of A so $A - \{x\}$ now with the image of such intersection under f is $f(A) - \{f(x)\}$. We know

the result is non empty since there exists a bijection thus $f(x)$ is a limit point of $f(a)$.

1.2.24 we will begin by letting $f: X \to Y$ be a homeomorphism x is a limit point of subset A of X. If x is a limit point of A in X then $f(x)$ is a limit point of $f(A)$ in Y since a homeomorphism preserves the properties of limit points. Begin with any limit point y of $f(A)$ in Y. we know f is continuous and its inverse is as well. Then the pre image of y under f which is in X is also a limit point of A.

It is observed that the pre image is in A since it is closed it has all its limit points.. It is therefore closed in Y since $y = f(f^{-1}(y))$ is in $f(A)$ and $f(A)$ has all its limit points.

Extra prob. 1. Constructing a homeomorphism between $(0,1)$ and (a,b). First consider $f:(0,1) \to (a,b)$ where $f(x) = a + (b-a)x$.
It is observed that $f(x)$ is bijective and also continuous by linearity thus the inverse is continuous $f^{-1}:(a,b) \to (0,1)$ by $f^{-1}(x) = (x-a)\frac{1}{(b-a)}$
Therefore f is a homeomorphism between $(0,1)$ and (a,b)

Extra 2. The stereographic projection map $h: S - \{\text{north pole}\} \to R^2$ from the unit sphere S in R^3 to the plane R^2 is produced by $h(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ for all (x,y,z) in S with $z \neq 1$. Now the inverse stereographic projection map $h^{-1}: R^2 \to S - \{\text{north pole}\}$ is given by:

$h^{-1}(u,v) = \left(\frac{2u}{}, \frac{2v}{}, \frac{u^2+v^2+1}{}\right)$

(u^2+v^2+1) , u^2+v^2+1 , u^2+v^2+1

for all (u,v) in R^2. It is observed that the north pole of the sphere S is at the point $(0,0,1)$. The stereographic projection and its inverse are continuous thus there is a homeomorphism between the sphere $S - \{\text{north pole}\}$ and the plane R^2.

Assignment 3	**Section 1.2:** 1, 4, 5, 12, 23, 24. **Extra problems:** (1) Construct a homomorphism between open intervals (0,1) and (a,b). (2) Let S be the unit sphere in three-dimensional Euclidean space R^3 with coordinates (x,y,z). Identify the plane R^2 as those points in space with z=0. Write down the formula of the stereo projection map h: S-{north pole} à R^2. What is the formula for its inverse map?
Assignment 4	