Devin Maya HW1 Combinatorics.

1) If there is only one pile and the pile contains n objects then when $n>1$ you should go first and remove $n-1$ objects from the pile to win. Now if $n=1$ then you should not go first since you would be forced to take the last object. Thus if $n=1$ you should go second.

Easy case 2: If there are K piles such that K = odd number of piles then you should go second to win. Now if K = even number of piles then you should go first. This is considering each K pile only has one object.

2) Now there are $k=2$ piles where each pile has n and m objects respectivily.

When $K=2$ and $n=m=1$ this is the answer to problem 2 where you should go first and remove a pile.

When $k=2$ and $n=m$ where $n,m>1$ you should move second. If there are two piles and $n=m$ where both n and m are greater than 1 the player should play second in order to ensure victory.

By going second in this case there is always a winning move since the player that goes first when $n=m$ where $n,m>1$ loses. This is because when $n=m$ any move player 1 makes will result in a player 2 win when playing optimally.

That is if player 1 removes p objects from a pile such that $n \neq m$ and $n,m>1$ then player 2 will just remove p objects from the other pile such that $n=m$ and $n,m>1$ again. Moves that result in different cases such as player 1 removes all objects from a pile or any other number of objects has been explored already.

To be more explicit for any case where $n \neq m$ and either n or m have 1 object then you should immediately remove the pile with more than 1 object resulting in the singleton pile. Thus for such a case there is 1st move advantage.

when $k=2$ and $n \neq m$ where $n, m > 1$ you should go first. The winning strategy for this case as your first move is to make $n=m$, that is if $n>m$ then take p objects such that $n-p=m$ resulting in $n=m$. Also if $m>n$ then take p objects such that $m-p=n$ resulting in $m=n$. Considering the result of this strategy is to take advantage of the scenario when $n=m$ where $n, m > 1$, the player whose turn it is when such scenario is true loses. Thus when the scenario is not true on your turn the optimal strategy is to remove some number of objects p such that either $m-p=n$ or $n-p=m$ resulting in the scenario where $m=n$ and $n, m > 1$.

Thus the state in which $m \neq n$ where $m, n > 1$ the winning or optimal move is to remove p objects such that $m=n$. Player

2 in this state is now losing as examined before.

When $m=1$ or $n=1$ with $k=2$ piles this is a winning state because Player 1 will remove all from the pile for which $n\neq1$ or $m\neq1$ resulting in the win with a singleton pile left.

3) yes you can definitly relate the digit wise sum to the conditions in problem 2 for which you should go first.

when there are k number of piles and $n=m$ where $n,m>1$ was said to be the winning state for the player who can on his turn set $n=m$. Thus for any case where $n=m$ and $n,m>1$ their digit-wise sum is equal to 0.

That is take $n=m$ where $n=2$ with $k=2$ piles.

2 2

digit wise sum.

$$\begin{array}{cccc} 2^3 & 2^2 & 2^1 & 2^0 \\ 0 & 0 & 1 & 0 \\ +\,0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

This example as examined before if the game starts with 2 objects in 2 piles respectively the player that goes first is in a losing state that is if player 2 is playing optimally you cannot get out of a losing state. When a player sets the digit-wise sum to 0 by removing p objects such that $m = n$ where $m, n > 1$ then the player is in a winning state. That is if the game begins in a 0 sum when you take the digit-wise sum of the numbers and the two numbers are not equal to 1 then you are already in a losing state because player 2 will play optimally by replying with the move that restores the digit wise sum of 0.

Now when examining the specific case where $n=m=1$ and $k=2$ piles

that is 1 1

$$\begin{array}{rccc} & 2^2 & 2^1 & 2^0 \\ & 0 & 0 & 1 \\ + & 0 & 0 & 1 \\ \hline & 0 & 0 & 0 \end{array}$$

digit wise sum

As mentioned before in this game state player 1 will win by taking all from one pile resulting in a win with a singleton pile left.

Thus after taking the binary expansion of both n and m then when you do the digit-wise sum there are two cases.

That is case 1 take the digit-wise sum when $n=m$ will result in 0. When the digit wise sum of the numbers is 0 you should not go first, since the second player can maintain the digit-wise sum to be 0 and force the win.

Now for case 2 take the digit-wise sum when $n \neq m$ will result in a sum that is not 0. Thus when this is true you should go first and play optimally by maintaining the imbalanced sum which forces the second player to lose.

Explicitly stating for 2 piles with n, m objects

n (digit-wise sum) $m = Q$

If $Q \neq 0$ you should go first

If $Q = 0$ you should go second

Ex: take $n = 3$ $m = 7$

n: in binary 0011

m: in binary 0111

n (digit-wise sum) m = 0100 = Q

(sum)

Where a is not equal to zero thus you should go first.

Examining the single pile case in question 1 performing the (digit-wise sum) operation does not apply but logically if there is one pile with a singleton you should not go first. As it is a direct loss by construction of the game.

4) Now generalizing above to the cases where $k>2$ piles the same will apply.

take three piles (A, B, C) to find the game

State Q perform the (digit-wise sum)

of all the piles.

$$\begin{cases} \text{If } Q \neq 0 & \text{you should go first} \\ \text{If } Q = 0 & \text{you should go second.} \end{cases}$$

EX: take $A = 3$ $B = 7$

$C = 5$

A: Binary 0011

B: Binary 0111

C: Binary 0101

A (digit wise sum) B digit wise sum C = 0

$$
\begin{array}{r}
0011 \\
0111 \\
+\ 0101 \\
\hline
0001
\end{array}
= 0
$$

since $0 \neq 0$ you should go first.

player 1 in piles of 3

When $Q \neq 0$ should select the pile for which you can restore a zero digit wise sum

That is taking the example above when $A=3$ $B=7$ $C=5$

take away 1 from C in order to obtain a zero sum.

That is now $A=3$ $B=7$ $C=4$

$$\begin{array}{r} 011 \\ 0111 \\ 0100 \\ \hline 0000 \end{array} = 0$$ setting winning state.

You want to be setting $Q=0$.

The strategy to win in cases with any number of piles is to perform the digit wise sum of all piles resulting in some number Q. If $Q \neq 0$ then you should go

first and optimally
play each round by
keeping the imbalance
since you set the digit-wise sum
to 0 this will force the
oponent to Lose.

If $\theta = 0$ then
you should go second
since your oponent can
maintain the digit wise

sum to be 0 on their turn forcing a win for them.

5) Now if the problem changes to "whoever takes the last object wins" you would adjust the strategy by reversing the strategy. That is if the digit-wise sum 0 of

all the piles = 0 then you should now go first since the opponent will imbalance Q on their move and you will play optmally by restoring the balance. Now if $Q \neq 0$ then you should not go first since your opponent can maintain the imbalance and force you to take the last object.

Also including the reverse for what was examined before.

when $m=1$ or $n=1$ with $k=2$ piles this now losing state because player 1 will remove all from the pile for which $n \neq 1$ or $m \neq 1$ resulting in the loss with a singleton pile left for player 2 to take and win.

That is when k is odd with singleton objects you now want to go first and when k is even you now want to go second.