Math 100A Homework 7

1. CH 9 PROB. 32 If $n, k \in \mathbb{N}$ and $\binom{n}{k}$ is a prime number, then $k = 1$ or $k = n - 1$.

Solution. Let $p = \binom{n}{k}$ be a prime number. We will prove that either $k = 1$ or $k = n - 1$ by contradiction. Therefore $k \neq 1 \neq n - 1$. So $1 < k < n - 1$

Assume that there exist $n, k \in \mathbb{N}$ such that $\binom{n}{k}$ is a prime number and $k \neq 1$ and $k \neq n - 1$. Then, we have $2 \leq k \leq n - 2$. Then $p = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)...(n-k-1)}{k(k-1)...1} = n(n-1) \cdot \frac{(n-2)...(n-k+1)}{k!}$ For some $k = n$, $\frac{n!}{k!(n-k)!}$ however this is a contradiction since p is prime and it cannot have be divided by another number other than 1 and itself. This demonstrates that it is not prime due to n(n-1) being a product of more than two numbers. Therefore the statement is proven by contradiction that $\binom{n}{k}$ is a prime number, then $k = 1$ or $k = n - 1$.

2. CH 9 PROB. 34 If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$ is true since

Solution. Suppose $X \subseteq A \cup B$. Then, for any $x \in X$ it must be that either $x \in A$ or $x \in B$. First possible case is that $x \in A$. Since X is a subset of $A \cup B$ then $x \in A \cup B$ and $x \in X \cap A$. Thus $X \subseteq A$. The second possibility is $x \in B$. Since X is a subset of $A \cup B$ then $x \in A \cup B$, and $x \in X \cap B$. Therefore $X \subseteq B$. Thus for both instances X is a subset of either A or B. Therefore If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$.

3. CH 10 PROB 4 If $n \in \mathbb{N}$, then $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$.

Solution. For $n = 1$, we have $1(1 + 1)/3 = 2/3$, and $1 \cdot 2 = 6/3 = 2$, so the base case holds.

Inductive implementation. Now assume the statement is true for some $k \in \mathbb{N}$, that is, $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k + 1) = \dfrac{k(k + 1)(k + 2)}{3}$. We want to show that the statement is true for $k + 1$ so $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (k + 1)((k + 1) + 1) = \dfrac{(k + 1)(k + 2)((k + 1) + 1)}{3}$.

Now applying to the starting equation given $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + (k + 1)((k + 1) + 1) = (1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + \cdots + k(k + 1)) + (k + 1)((k + 1) + 1) = \dfrac{k(k + 1)(k + 2)}{3} + (k + 1)(k + 2) = \dfrac{(k + 1)(k + 2)(k + 3)}{3}$.

Therefore If $X \subseteq A \cup B$, then $X \subseteq A$ or $X \subseteq B$ is true since the statement holds for $n = k + 1$. Then by induction the statement is true for all $n \in \mathbb{N}$.

4. CH 10 PROB 8 If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

Solution. The case being considered is For $n = 1$, we have $\frac{1}{2!} = \frac{1}{2}$ and $1 - \frac{1}{(1+1)!} = \frac{1}{2}$ then the case is true.

Now by implementing induction assuming the statement is true for some $k \in \mathbb{N}$ then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!} = 1 - \frac{1}{(k+1)!}$. The objective is to verify if for $k+1$ the case is true then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{(k+2)!} = 1 - \frac{1}{(k+2)!}$. By implementing the verification for the original equation given $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{(k+1)}{(k+2)!} = \left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{k}{(k+1)!}\right) + \frac{(k+1)}{(k+2)!} = 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{(k+2)!} = 1 - \frac{(k+2)-1}{(k+2)!} \ = 1 - \frac{1}{(k+2)!}$,

Thus verifying the induction.

Therefore the statement If $n \in \mathbb{N}$, then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ is true for all $n \in \mathbb{N}$.

5. CH 10 PROB 18 . Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$.

Solution. Let $n = 2$. Then $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ Thus true due to De Morgan's laws for sets so the base case is verified as true.

Implementing induction assuming that the statement is true for some integer $k \geq 2$ such as $\overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$.

It must be true when verifying $n = k+1$ so $\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{k+1}}$.

To prove this, we have:

$\overline{A_1 \cup A_2 \cup \cdots \cup A_{k+1}} \ = \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}} \ = \overline{A_1 \cup A_2 \cup \cdots \cup A_k} \cap \overline{A_{k+1}} \ = (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}) \cap \overline{A_{k+1}} \ = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k} \cap \overline{A_{k+1}}$.

Thus the statement has been verified as true after implementing Induction Therefore for all $n \geq 2$, $\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}$ as verified by induction.

6. CH 10 PROB 22 If $n \in \mathbb{N}$, then $(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8})(1 - \frac{1}{16}) \cdots (1 - \frac{1}{2^n}) \geq \frac{1}{4} + \frac{1}{2^{n+1}}$.

Solution. Let $n = 1$. Then $(1 - \frac{1}{2}) = \frac{1}{2} \geq \frac{1}{4} + \frac{1}{4} = \frac{1}{2^{1+1}}$. So the statement is true for $n = 1$.

Implementing Induction assuming that the statement is true for some positive integer $k \geq 1$ then $(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^k}) \geq \frac{1}{4} + \frac{1}{2^{k+1}}$. The statement must be verified as true for $n = k+1$ then $(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^k})(1 - \frac{1}{2^{k+1}}) \geq \frac{1}{4} + \frac{1}{2^{k+2}}$.

Thus in order to implement this $(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^k})(1 - \frac{1}{2^{k+1}}) \ = (1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^k} \cdot \frac{2}{2})(1 - \frac{1}{2^{k+1}}) \ = (1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^{k+1}}) \quad + \frac{1}{2^{k+1}}(1 - \frac{1}{2})(1 - \frac{1}{4})(1 - \frac{1}{8}) \cdots (1 - \frac{1}{2^k}) \ \geq \frac{1}{4} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}}\left(\frac{1}{4} + \frac{1}{2^{k+1}}\right)$ In the induction step, you first multiply both sides of the inequality for k by $(1 - \frac{1}{2^{k+1}})$ to get the inequality for $k+1$. Since $(1-\frac{1}{2^{k+1}}) > 0$ for all k. then using the distributive property to expand the product $(1-\frac{1}{2})(1-\frac{1}{4})\cdots(1-\frac{1}{2^{k+1}})$ and rearranging the terms to obtain the inequality for $k+1$. Then simplify the resulting expression for the inequality for $k+1$ to show that it is greater than $\frac{1}{4}+\frac{1}{2^{k+2}}$, which is true since by the inductive hypothesis $= \frac{1}{4}+\frac{1}{2^{k+1}}+\frac{1}{4}\cdot\frac{1}{2^{k+1}}+\frac{1}{2^{2(k+1)}} \;\; = \frac{1}{4}+\frac{1}{2^{k+2}}+\frac{1}{2^{2(k+1)}} \;\; > \frac{1}{4}+\frac{1}{2^{k+2}}$.

Therefore by the implementation of induction, the statement If $n \in \mathbb{N}$, then $(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{8})(1-\frac{1}{16})\cdots(1-\frac{1}{2^n}) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$ is true.

7. CH 10 PROB 24 $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for each natural number n.

Solution. The initial main case verified is When $n=1$ then $\sum_{k=1}^{1} k\binom{1}{k} = 1\cdot\binom{1}{1} = 1$ and $1\cdot 2^{1-1} = 1\cdot 1 = 1$ So the statement is verified as true for $n=1$. Now implementing induction assuming that the statement holds for some $n \geq 1$ then $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$.

The statement must be verified for $n+1$ then $\sum_{k=1}^{n+1} k\binom{n+1}{k} = (n+1)2^n$. The sum rewritten as follows $\sum_{k=1}^{n+1} k\binom{n+1}{k} = \sum_{k=1}^{n} k\binom{n+1}{k} + (n+1)\binom{n+1}{n+1}$.

From $\binom{n+1}{k} = \binom{n}{k-1}+\binom{n}{k}$ after immplementing for the term $\sum_{k=1}^{n} k\binom{n+1}{k} = \sum_{k=1}^{n} k(\binom{n}{k-1}+\binom{n}{k}) = \sum_{k=0}^{n-1}(k+1)\binom{n}{k} + \sum_{k=1}^{n} k\binom{n}{k}$. After simplifying by substituting $k+1$ with k produces $\sum_{k=0}^{n-1}(k+1)\binom{n}{k} = \sum_{k=1}^{n} k\binom{n}{k-1}$. Now from the initial hypothesis and the simplified result. $\sum_{k=1}^{n+1} k\binom{n+1}{k} = \sum_{k=0}^{n} k\binom{n+1}{k} = \sum_{k=1}^{n} k\binom{n+1}{k} + (n+1)\binom{n+1}{n+1} = \sum_{k=1}^{n} k\binom{n}{k-1} + \sum_{k=1}^{n} k\binom{n}{k} + (n+1) = n2^{n-1}+n+1 = (n+1)2^n$. So $n2^{n-1}$ for $\sum_{k=1}^{n} k\binom{n}{k}$ in the simplified expression when n= n+1, and then simplified further to obtain $(n+1)2^n$.

Therefore, by induction, the statement $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for each natural number n. is true for all natural numbers n.Since the base case was verified and proved the inductive step.

8. CH 10 PROB 26 Concerning the Fibonacci sequence,prove that $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$

Solution. The main case to test for is $n=1$ so $\sum_{k=1}^{1} F_k^2 = F_1^2 = 1$ then $F_1F_{1+1} = F_1F_2 = 1$ thus the case has been verified as true. The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, Where $F_1 = 1, F_2 = 1$

Implementing Induction assuming that $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$ for some $n \geq 1$. Then the objective is to verify for $\sum_{k=1}^{n+1} F_k^2 = F_{n+1}F_{n+2}$ to be true.

Thus by plugging in n+1 for n $\sum_{k=1}^{n+1} F_k^2 = \sum_{k=1}^{n} F_k^2 + F_{n+1}^2$

Through the implementation of inductive hypothesis F_nF_{n+1} for $\sum_{k=1}^{n} F_k^2$ thus producing $\sum_{k=1}^{n+1} F_k^2 = F_nF_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1}F_{n+2}$ since $F_n + F_{n+1} = F_{n+2}$.

Therefore by the implementation of induction $\sum_{k=1}^{n} F_k^2 = F_nF_{n+1}$ for all $n \in \mathbb{N}$.

9. CH 10 PROB 32 The number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}. For example, for $n = 2$ there are three such numbers (00, 01, and 10), and $3 = F_{2+2} = F_4$. Also, for $n = 3$ there are five such numbers (000, 001, 010, 100, 101), and $5 = F_{3+2} = F_5$.

Solution. For $n = 1$ there are two possible binary numbers 0 and 1, and both have no consecutive 1s. Thus the statement holds for $n = 1$. For $n = 2$, we have three possible binary numbers (00), (01), and (10) and all of them have no consecutive 1s in such numbers. Thus the statement holds for $n = 2$.

Implementing inductive step assuming that the statement holds for all k such that $1 \leq k \leq n$ for some $n \geq 2$. the objective is to verify that $n + 1$ is true. Consider an $(n + 1)$ digit binary number that has no consecutive 1s in such numbers. This can be shown in two different parts the first n digits and the last digit. Here it must be that the first n digits must themselves be a binary number with no consecutive 1's. After implementing the induction hypothesis there are F_{n+2} such numbers.

Now considering the other case where the last digit. If it is a 0, then the entire $(n + 1)$-digit number is a binary number with no consecutive 1s in such numbers. If the last digit is a 1, then the second to last digit must be a 0 so the last two digits are (10). We can now consider the first $n-1$ digits as a binary number with no consecutive 1s in such numbers and through the implementation of induction there are F_{n+1} such numbers. Therefore the total number of $(n + 1)$ digit binary numbers with no consecutive 1s is $F_{n+2} + F_{n+1} = F_{n+3}$. Thus adding a 0 to the end of an n-digit number is equivalent to counting the number of $(n + 1)$-digit numbers that have no consecutive 1s and end in 0, while adding 10 to the end of an $(n - 1)$-digit number is equivalent to counting the number of $(n + 1)$ digit numbers that have no consecutive 1s and end in (10).

Therefore by the implementation of strong induction the statement for all $n \in \mathbb{N}$ thus the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}.