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Math 117 Homework 7

3.5.1 Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear operator on V. Let B be a basis for V, and let $A = [T]_B$. Prove that $\det(T) = \det(A)$

Solution. Let V be an n dimensional vector space over a field F and let $T : V \to V$ be a linear operator on V. Suppose $B = \{v_1, v_2, \dots, v_n\}$ is a basis for V. The matrix representation of T with respect to basis B is $A = [T]_B$. This matrix A is constructed in a way where the i-th column is the coordinate vector of $T(v_i)$ with respect to the basis B. The determinant of a linear operator T is defined as the determinant of any of its matrix representations. This definition is well defined because the determinant of a linear operator is independent of the choice of basis used for the matrix representation. Thusdet(T) is the same regardless of the basis chosen to represent T as a matrix. Now the objective is to prove that $\det(T) = \det(A)$ Since $A = [T]_B$ represents the linear operator T in the basis B the determinant of T can be computed as the determinant of this matrix A. The determinant of A denoted as $\det(A)$ is a scalar value that characterizes the linear transformation T for volume scaling and orientation change in the vector space V. By the basis independence property of the determinant of a linear operator observe that $\det(T)$ which is the determinant of T as a linear operator, must equal ($\det(A)$ which is the determinant of the matrix representation of T in basis B.

Given that $A = [T]_B$ is the matrix representation of T in basis B and the determinant of a linear operator is independent of the basis choice then the determinant of T as a linear operator is equal to the determinant of its matrix representation A. Thus $\det(T) = \det(A)$ is proven.

4.1.1 Let F be a field.

1. Prove that a non-zero polynomial $f \in F[x]$ of degree ≤ 3 is reducible if and only if it has a root in F.
2. Give an example of a field F and a reducible polynomial $f \in F[x]$ with no roots in F.

Solution. The objective is to prove that a non zero polynomial $f \in F[x]$ of degree $\deg(f) \leq 3$ is reducible in $F[x]$ if and only if it has a root in F. If f has a root, then it is reducible Suppose $f \in F[x]$ is a non-zero polynomial of degree 3 and has a root $a \in F$.By the Factor Theorem, $(x - a)$ is a factor of $f(x)$. Therefore, $f(x) = (x - a)g(x)$ for some polynomial $g(x) \in F[x]$. Since $\deg(f) \leq 3$ and $\deg(x - a) = 1$, $\deg(g) \leq 2$. Thus $f(x)$ is expressed as the product of two non constant polynomials in $F[x]$ proving it is reducible.

If f is reducible then it has a root Now suppose $f(x)$ is reducible in $F[x]$. Then $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are non constant polynomials in $F[x]$ with $\deg(g), \deg(h) < \deg(f)$. Since $\deg(f) \leq 3$ then at least one of $g(x)$ or $h(x)$ must be of degree 1 or the result would be a product with degree greater than 3. Assume $\deg(g) = 1$ then $g(x) = a(x - b)$ for some $a, b \in F$ $a \neq 0$. Thus b is a root of $f(x)$ since $f(b) = g(b)h(b) = 0$. Therefore f has a root in F. A non zero polynomial $f \in F[x]$ of degree $\deg(f) \leq 3$ is reducible in $F[x]$ if and only if it has a root in F.

Example of a Reducible Polynomial Without Roots in F To provide an example of a field F and a reducible polynomial $f \in F[x]$ which has no roots in F.Field $F = \mathbb{R}$ the field of real numbers. Polynomial Consider $f(x) = x^2 + 1$. Observe in $\mathbb{R}[x]$ the polynomial $f(x) = x^2 + 1$ has no roots since $x^2 + 1 > 0$ for all $x \in \mathbb{R}$. However $f(x)$ is reducible in $\mathbb{C}[x]$ which is the polynomials over the complex numbers since it can be factored as $(x + i)(x - i)$ in $\mathbb{C}[x]$. The polynomial $f(x) = x^2 + 1$ over $\mathbb{R}$ is an example of a reducible polynomial in an extension field complex numbers but has no roots in the original field $\mathbb{R}$.

4.1.2 Let V be an n-dimensional vector space, and let $T : V \to V$ be a linear operator on V. Use the Leibniz determinant formula to prove that the function

$$\chi_T(\lambda) = \det(T - \lambda)$$

is a polynomial of degree *exactly* n. This is the *characteristic polynomial* of T.

Solution. The objective is to prove that the characteristic polynomial $\chi_T(\lambda) = \det(T - \lambda I)$ is a polynomial of degree exactly n where $T : V \to V$ is a linear operator on an n-dimensional vector space V. By implementing the Leibniz formula for the determinant which is the determinant of a square matrix as a sum over all permutations of the matrix indices. Let $A = [T]_B$ be the matrix representation of T with respect to some basis B of V.The matrix $T - \lambda I$ is represented by $A - \lambda I$, where I is the $n \times n$ identity matrix.The characteristic polynomial is defined as $\chi_T(\lambda) = \det(A - \lambda I)$. Applying the Leibniz determinant formula $\det(A - \lambda I) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (a_{i,\sigma(i)} - \lambda \delta_{i,\sigma(i)})$, where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$, $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ, and δ_{ij} is the Kronecker delta.Each term in the sum is a product of entries from $A - \lambda I$. The polynomial in λ resulting from each term will have a degree at most equal to the number of diagonal entries $(a_{ii} - \lambda)$ in the product.The highest degree term in λ when all entries in the product are diagonal entries of $A - \lambda I$. This will result in a λ^n term. Since there is at least one term in the sum that results in a λ^n term specifically, the term corresponding to the identity permutation the degree of the polynomial $\chi_T(\lambda)$ is at least n. Also no term in the sum can result in a polynomial of degree greater than n, because each product contains exactly n factors each of which contributes at most one λ to the product. Therefore the degree of $\chi_T(\lambda)$ is exactly n.

4.2.1 Let A be an F-algebra which is a finite- dimensional vector space over F. Let $T \in A$. Recall that $\mu_T \in F[x]$ denotes the minimal polynomial of T.

1. Prove that if $p \in F[x]$ is irreducible with $p(T) = 0$, then μ_T is a scalar multiple of p.

2. Give an example of an algebra A and an element $T \in A$ whose minimal polynomial is reducible.

Solution. To solve this problem we will first prove that if a polynomial $p \in F[x]$ is irreducible and annihilates a linear operator T, then the minimal polynomial of T, μ_T is a scalar multiple of p. Then to provide an example of an algebra and an element whose minimal polynomial is reducible. If $p \in F[x]$ is irreducible with $p(T) = 0$ for some $T \in A$ then the minimal polynomial μ_T of T is a scalar multiple of p.

Given $p \in F[x]$ is an irreducible polynomial and $p(T) = 0$ for some $T \in A$. To Prove The minimal polynomial μ_T is a scalar multiple of p. Definition of Minimal Polynomial The minimal polynomial μ_T is the monic polynomial of smallest degree that satisfies $\mu_T(T) = 0$. Irreducibility of pSince p is irreducible and $p(T) = 0$, p must divide any polynomial q for which $q(T) = 0$. This is because the only divisors of an irreducible polynomial are itself and constants scalar multiples.Division of μ_T by p Therefore, p divides μ_T. Observe μ_T is the polynomial of smallest degree that annihilates T thus μ_T must be a scalar multiple of p as any polynomial of lesser degree would contradict the minimality of μ_T then μ_T is a scalar multiple of p.

2. Example of a Reducible Minimal Polynomial

Algebra A Consider $A = M_2(F)$ the algebra of all 2×2 matrices over the field F. Element T Let T be the matrix $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ Minimal Polynomial of T Computation Observe that $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ thus $T^2 = 0$. Minimal Polynomial The polynomial $\mu_T(x) = x^2$ satisfies $\mu_T(T) = T^2 = 0$. Reducible The polynomial x^2 is reducible over F as it can be factored as $x \cdot x$.Therefore in this algebra $\mu_T(x) = x^2$ is the minimal polynomial of T, and it is reducible over F.