Math 117 Homework 4

2.1.1 Recall that the complex numbers $\mathbb{C}$ may be regarded as an -vector space.

1. Find a basis B for $\mathbb{C}$ as an $\mathbb{R}$-vector space. Explain why it is a basis.
2. Let $z \in \mathbb{C}$ be a complex number. Consider the multiplication by z map

$$\begin{aligned} z : \mathbb{C} &\to \mathbb{C} \\ x &\mapsto zx. \end{aligned}$$

   Explain why this is an R-linear map.

3. If $z = a + bi$, find the matrix $[z]_B$ of the multiplication-by-z map with respect to the basis B from (i). *Your matrix should be given in terms of a and b.*

Solution. 1. it is important to observe that a complex number can be shown as a linear combination of i and 1 explicitly a complex number is of the form $z = a + bi$ here $a, b \in R$ can be shown as $a(1)+b(i)$ where 1 and i are linearly independent over $\mathbb{R}$ no real linear combination of such numbers equal zero only if the coefficients are zero so $B = \{1, i\}$ is a basis for $\mathbb{C}$ over $\mathbb{R}$.The set $1, i$ is linearly independent over $\mathbb{R}$ since no non trivial real linear combination of these numbers equals 0. Therefore a basis for $\mathbb{C}$ as an R-vector space is $B = \{1, i\}$.

2. Let $z \in \mathbb{C}$ be a complex number. Consider the multiplication by z map Explain why this is an $\mathbb{R}$-linear map.

Given the definition from the lecture notes Definition 1.3.1. A function $T : V \to W$ is called a linear map or linear transformation if for all $x, y \in V$ and all $a \in F$ then (i) $T(x + y) = T(x) + T(y)$ (ii) $T(ax) = aT(x)$ A linear map $T : V \to V$ is also called a linear operator on V. Given this definition for a map $z : \mathbb{C}ß\mathbb{C}$ to be $\mathbb{R}$-linear then it must satisfy the following propertie $z(x+y) = zx+zy$ for all $x, y \in \mathbb{C}$ from (i) in the definition.$z(c\cdot x) = c\cdot(zx)$ for all $x \in \mathbb{C}$ and $c \in \mathbb{R}$ from (ii) in the definition. Now the objective is to verify these properties for the map z. So for any $x, y \in \mathbb{C}$ then $z(x + y) = z \cdot x + z \cdot y$ This is from the distributive property of complex numbers and this satisfies the first property of linear maps where $T(x + y) = T(x) + T(y)$. For any $x \in \mathbb{C}$ and $c \in \mathbb{R}$ then $z(c \cdot x) = c \cdot (zx)$ This is because multiplication by a real number in the complex numbers is commutative. This goes with the second property of linear maps so $T(ax) = aT(x)$.Therefore the multiplication by z map is indeed an $\mathbb{R}$ linear map by the given definition since the properties have been verified.

3. Using the basis $B = \{1, i\}$, we can determine the matrix of the multiplication-by-z map with respect to the abses found earlier in 1. For the first basis element 1 It is shown as $z \cdot 1 = (a + bi) \cdot 1 = a + bi$ now in terms of basis B this can be shown as the following column

vector $\begin{bmatrix} a \\ b \end{bmatrix}$

For the second basis element i $z \cdot i = (a+bi) \cdot i = -b+ai$ now in terms of basis B this is the column vector $\begin{bmatrix} -b \\ a \end{bmatrix}$ the matrix $[z]_B$ representing the multiplication-by-z map with respect to basis B is $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ since it is both components combined .

2.2.1 Consider the vector space $F_2[x]$. Let

$$S = \{1 + x^2, 1 - x^2, x + x^2, 1 + x + x^2\}.$$

Show that S spans $F_2[x]$ and find a basis B for $F_2[x]$ contained in S.

Solution. the objective is to show that the set S spans $F_2[x]$ and find a basis B for $F_2[x]$ contained in S.

Where $S = \{1+x^2, 1-x^2, x+x^2, 1+x+x^2\}$. Within $F_2[x]$ the vector space of polynomials with coefficients from the field F_2. In F_2 there are only two elements 0 and 1 since the arithmetic is modulo 2 thus $1 + 1 = 0$. First will show that S spans $F_2[x]$ taking the degrees into consideration where for every polynomial of $F_2[x]$ of degree 2 or less it will be shown as a linear combination of the polynomials in S. For 1 using $1+x^2$ and $1-x^2$ then $1 = \frac{1}{2}(1+x^2)+\frac{1}{2}(1-x^2)$. For x in the set using $x + x^2$ and $1 + x^2$ then the result is $x = (1 + x + x^2) - (1 + x^2)$ after subtracting. for x^2 using $1+x^2$then the result after sub 1 is $x^2 = (1+x^2)-1$. For $1+x$ using $1 + x + x^2$ minus $x + x^2$ then $1 + x = (1 + x + x^2) - (x + x^2)$. For$x + x^2$ it is observed that it is $\in S$ also considering $1 + x^2$ and $1 + x + x^2$ are in S.

Thus it is shown that every polynomial in $F_2[x]$ with the given degress can be expressed using the elements of S. Therefore S spans $F_2[x]$. Now to address a basis B contained in S the smallest set of linearly independent polynomials from S that can still span $F_2[x]$.

From S it is observable that $x+x^2$ is a linear combination of the others thus it's dependent also polynomials $1 + x^2$, $1 - x^2$, and $1 + x + x^2$ aren't linear combinations of any others in S thus they're independent. Therefore basis B would be $B = \{1 + x^2, 1 - x^2, 1 + x + x^2\}$

2.2.2 Consider the vector space $F^{2\times 2}$. Let

$$S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \right\}$$

Show that S is linearly independent.

Solution. The objective show that the set S is linearly independent in the vector space $F^{2\times 2}$. $S = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix} \right\}$ Linear independence in matrices means that no matrix in the set can be expressed as a linear combination of the others. begin by setting up a linear combination of the matrices in S with scalars a, b, c, and d, and set this equal to the zero matrix so $a\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} + b\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} + c\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix} + d\begin{bmatrix}1 & 1\\ -1 & 5\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ Now performing expansion on the matrix results in $\begin{bmatrix}a+b+2c+d & a-b+c+d\\ a-b+c-d & a+b+3c+5d\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Now you can create a syetem of equations as such

1. $a+b+2c+d=0$, 2. $a-b+c+d=0$ 3. $a-b+c-d=0$ 4. $a+b+3c+5d=0$ Now the objective is to solve them

$2b+3c=0$ from 1. and 2.

Also $2a+2c=0$ where $a=-c$ from 3,4

$-2c-4d=0$ then $c=-2d$ From 1,4

combining the results then $a=b=c=d=0$ which is trivial sol. Then the system of equations only has the trivial solution where $a=b=c=d=0$. Thent none of the matrices $\in S$ can be expressed as a linear combination of the others.

Therefore set S is linearly independent in the vector space $F^{2\times 2}$.

2.2.3 Consider the linear map $T : F^{2\times 3} \to F^2$ given by summing the columns of a matrix:

$$T\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix} = \begin{bmatrix}a+b+c\\ d+e+f\end{bmatrix}$$

Find bases for $\ker(T)$ and (T).

Solution. The objective is to find the bases for the kernel ($\ker(T)$) and the image ((T)) of the linear map T Given the transformation T $T\begin{bmatrix}a & b & c\\ d & e & f\end{bmatrix} = \begin{bmatrix}a+b+c\\ d+e+f\end{bmatrix}$

Here the kernel of T consists of all matrices in $F^{2\times 3}$ that map to the zero vector in F^2. Thus $\begin{bmatrix}a+b+c\\ d+e+f\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$

Now observing the derived equations are the following $a+b+c=0$ and $d+e+f=0$ thus using the equations results in $\begin{bmatrix}a & b & -a-b\\ d & e & -d-e\end{bmatrix}$ Which is in the kernel of T.

now considering when $a=1, b=0$ and $d=0, e=0$ the result is $\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\end{bmatrix}$

Doing the same for $a=0, b=1$ and $d=0, e=0$ the result is $\begin{bmatrix}0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$

Now the same for $a=0, b=0$ and $d=1, e=0$ the result is $\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & -1\end{bmatrix}$

Observe that the matrices are linearly independent thus they are forming the basis for $\ker(T)$ where it is

$\left\{ \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \right\}$ Now addressing the image of T

Given the transformation T, any vector in F^2 of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ can be done with a combination of the columns in the matrix space $F^{2\times 3}$. Thus image of T spans all of F^2 so a basis for F^2 and therefore for $(T))$ is

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

2.2.4 Consider the linear operator T on $F_2[x]$ given by the change of variables $x \mapsto x+1$:

$$T(p(x)) = p(x+1).$$

Let $B = \{1, x, x^2\}$ be the standard basis for $F_2[x]$. Explain why $[T]_B$ is an invertible matrix and find its inverse.

Solution. The problem pertains to a linear operator T on the polynomial space $F_2[x]$. This operator is characterized by the transformation $x \mapsto x+1$. We are provided with a standard basis for this space as $B = \{1, x, x^2\}$. The objective is to determine if the matrix representation of this operator with respect to this basis is invertible and provide its inverse First for the matrix representation $[T]_B$ applying the transformation T to each basis vector in B results in $T(1)$ which is 1 since there's no x to substitute.Now for $T(x)$ it is $x+1$.Now for $T(x^2)$ after $x+1$ is squared the result it x^2+2x+1 here $\in F_2$ so the coefficients are modulo 2 then it becomes x^2+1. Now analyzing the transformed polynomials in terms of the basis B where 1 is 1 of the basis 1, 0 of the basis x, and 0 of the basis x^2. $x+1$ is 1 of the basis 1, 1 of the basis x, and 0 of the basis x^2.x^2+1 is 1 of the basis 1, 0 of the basis x, and 1 of the basis x^2. form the matrix representation $[T]_B$ and using the coeff then

$[T]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A matrix is invertible if its determinant is non zero thus computing such value is $det([T]_B) = 1(1\times 1 - 0 \times 0) = 1$ here determinant is non-zero in F_2 so $[T]_B$ is invertible. Now to find the inverse

$[T]_B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in F_2 -1 is equivalent to 1 thus

$[T]_B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Therefore $[T]_B$ is found to be invertible and its inverse is

$$[T]_B^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$