DEVIN MAYA 2/24/23

Math 100A Homework 6

1. CH 7 PROB. 26 The product of any n consecutive positive integers is divisible by $n!$.

Solution. **Proof:** Suppose we have n consecutive positive integers starting with k. Then the integers can be written as $k, k+1, k+2, \ldots, k+n-1$. The product of the n integers is $k(k+1)(k+2)\cdots(k+n-1)$ Then in order to check that the product is divisible by $n!$. $n!$ can be written as $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$ Then by grouping the terms in the product of n consecutive integers the result is $k(k+1)(k+2)\cdots(k+n-1)$ $= k \cdot (k+1) \cdot (k+2) \cdots [(k+n-2)+1] \cdot (k+n-1) = \frac{(k+n-1)!}{(k-1)!}$ Since $(k+n-2)+1 = k+n-1$ then the product is $k(k+1)(k+2)\cdots(k+n-1) = \frac{(k+n-1)!}{(k-1)!} = \frac{(k+n-1)(k+n-2)\cdots(k+1)k}{(n-1)!}$ Therefore since $(k+n-1)(k+n-2)\cdots(k+1)k$ is a multiple of $n!$ then the product of any n consecutive positive integers is divisible by $n!$.

2. CH 7 PROB. 30 Suppose $a, b, p \in Z$ and p is prime. Prove that if $p|ab$ then $p|a$ or $p|b$. (Suggestion: Use the proposition on page 152.)

Solution. Proposition: If $a, b \in \mathbb{N}$ then there exist integers k and l for which $\gcd(a, b) = ak+bl$. Suppose a, b, and p are integers and p is prime. Assume that p does not divide a. thus the objective is that p must divide b.Since p does not divide a then $\gcd(a, p) = 1$. By the proposition given, there exist integers k and l such that $ak + pl = 1$. After multiplying the used proposition both sides by b the equation becomes $akb + plb = b$. Since p divides plb and p divides ab then $p|akb+plb = b(ak+pl) = b$ Therefore if p does not divide a then p must divide b. Without loss of generality after applying the same logic to the other case if p does not divide b then p must divide a. Therefore we have shown that if p divides ab then p must divide either a or b.

3. CH 7 PROB 34 If gcd(a, c) = gcd(b, c) = 1, then gcd(ab, c) = 1. (Suggestion: Use the proposition on page 152.)

Solution. Proposition: If $a, b \in \mathbb{N}$, then there exist integers k and l for which $\gcd(a, b) = ak + bl$. Suppose $\gcd(a, c) = \gcd(b, c) = 1$. The objective is that $\gcd(ab, c) = 1$. Assume due to contradiction that $\gcd(ab, c) = d > 1$. Then there exist integers x and y such that $d = ax + by$ and d divides both ab and c. Since $\gcd(a, c) = 1$ after using the given proposition to find integers k and l such that $ak + cl = 1$. Multiplying both sides by b, we get: $akb + clb = b$ Note that $\gcd(b, c) = 1$ since $\gcd(a, c) = \gcd(b, c) = 1$. Therefore, d cannot divide both b and c. If d divides b, then we have $d|axb$ and $d|b$, so d divides ab. If d divides c, then we have $d|clb$ and $d|c$, so d divides ab. In either case, we have a contradiction, since d cannot divide both ab and c. Therefore, we must have $\gcd(ab, c) = 1$.

4. CH 8 PROB 12 If A, B, and C are sets, then $A - (B \cap C) = (A - B) \cup (A - C)$.

Solution. To prove that $A - (B \cap C) = (A - B) \cup (A - C)$. Then any element that is in $A - (B \cap C)$ is also in $(A - B) \cup (A - C)$ and without loss of generality the other way around as well. Then for any $x \in A - (B \cap C)$ also $x \in (A - B) \cup (A - C)$. For any $y \in (A - B) \cup (A - C)$ also $y \in A - (B \cap C)$. Let $x \in A - (B \cap C)$. This means that $x \in A$ and $x \notin B \cap C$. Since $x \notin B \cap C$ also $x \notin B$ or $x \notin C$. Thus either $x \in A - B$ or $x \in A - C$ then $x \in (A - B) \cup (A - C)$. Therefore for any $x \in A - (B \cap C)$ also $x \in (A-B) \cup (A-C)$. now let $y \in (A-B) \cup (A-C)$. This means that either $y \in A-B$ or $y \in A - C$. Suppose without loss of generality that $y \in A - B$. Then $y \in A$ and $y \notin B$. Since $y \notin B$ also $y \notin B \cap C$. Thus $y \in A - (B \cap C)$ then $y \in (A - B) \cup (A - C)$ also $y \in A - (B \cap C)$.

Therefore any element that is in $A - (B \cap C)$ is also $(A - B) \cup (A - C)$ and the other way around. Thus $A - (B \cap C) = (A - B) \cup (A - C)$.

5. CH 8 PROB 20 .

$$\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$$

Solution. To prove that $\{9^n : n \in \mathbb{Q}\} = \{3^n : n \in \mathbb{Q}\}$ any element that is in $\{9^n : n \in \mathbb{Q}\}$ is also in $\{3^n : n \in \mathbb{Q}\}$ and the other way around. Thus for any $x \in 9^n : n \in \mathbb{Q}$ then $x \in 3^n : n \in \mathbb{Q}$. For any $y \in 3^n : n \in \mathbb{Q}$ then $y \in 9^n : n \in \mathbb{Q}$. Let $x \in 9^n : n \in \mathbb{Q}$. This means that there exists a rational number q such that $x = 9^q$. q then can be written as $q = \frac{m}{n}$ where m and n are prime integers. Then $x = 9^q = (3^2)^q = 3^{2q} = 3^{2(m/n)} = (3^m)^{(2/n)}$. Since m and n are prime $2/n$ is also a rational number. Thus x can be written in the form 3^n for some rational number n then $x \in 3^n : n \in \mathbb{Q}$. Now let $y \in 3^n : n \in \mathbb{Q}$. This means that there exists a rational number q such that $y = 3^q$. q can be written as $q = \frac{m}{n}$ where m and n are prime integers. Then$y = 3^q = (3^2)^{(m/n)} = 9^{(m/n)}$. Since m and n are prime, m/n is also a rational number. Thus y can be written in the form 9^n for some rational number n then $y \in 9^n : n \in \mathbb{Q}$. Therefore any element that is in $\{9^n : n \in \mathbb{Q}\}$ is also in $\{3^n : n \in \mathbb{Q}\}$ and the other way around. Thus $9^n : n \in \mathbb{Q} = 3^n : n \in \mathbb{Q}$.

6. CH 8 PROB 22 Let A and B be sets. Then $A \subseteq B$ if and only if $A \cap B = A$.

Solution. Suppose $A \subseteq B$. Then must prove $A \cap B = A$. For the first case $A \cap B \subseteq A$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. Since $A \subseteq B$ then $x \in B$. Therefore $x \in A$ and $x \in B$ implies $x \in A \cap B$ thus $A \cap B \subseteq A$. Now the case for $A \subseteq A \cap B$. Let $y \in A$. Since $A \subseteq B$ then $y \in B$. Therefore $y \in A$ and $y \in B$ implies $y \in A \cap B$ thus $A \subseteq A \cap B$. Since $A \cap B \subseteq A$ and $A \subseteq A \cap B$ then $A \cap B = A$. Suppose $A \cap B = A$. Then $A \subseteq B$

must be examined. Let $x \in A$. Since $A \cap B = A$ then $x \in B$. Therefore $x \in A$ and $x \in B$ implies $x \in A \cap B$ thus $A \subseteq B$. Therefore $A \subseteq B$ if and only if $A \cap B = A$ since both cases of the statement have been proved.

7. CH 8 PROB 24 $\bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2] = [3, 5]$

Solution. To prove that $\bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2] = [3, 5]$ it must be that $[3, 5]$ is within intersection and that the intersection is within in $[3, 5]$. Then first $[3, 5]$ is within the intersection. So $3 \leq 3 - x^2 \leq 5 + x^2 \leq 5$ for all $x \in \mathbb{R}$. This can be rewritten as $3 \leq 5 + x^2$ and $3 \leq 3 - x^2$ which is true for all $x \in \mathbb{R}$. Therefore $[3, 5]$ is within in the intersection. Now to show that the intersection is within in $[3, 5]$. Suppose $a \in \bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2]$. This means that $3 - x^2 \leq a \leq 5 + x^2$ for all $x \in \mathbb{R}$. This is true for $x = 0$ resulting in $3 \leq a \leq 5$. Therefore $a \in [3, 5]$, and the intersection is within $[3, 5]$. Therefore $\bigcap_{x \in \mathbb{R}} [3 - x^2, 5 + x^2] = [3, 5]$ since $[3, 5]$ is within the intersection and that the intersection is within $[3, 5]$

8. CH 8 PROB 28 $\{12a + 25b : a, b \in \mathbb{Z}\} = \mathbb{Z}$

Solution. Let $x \in 12a + 25b : a, b \in \mathbb{Z}$. Then $x = 12a + 25b$ for some $a, b \in \mathbb{Z}$. Then $\gcd(12, 25) = 1$. Since 12 and 25 have no common factors except 1. There exist integers k and l such that $12k + 25l = 1$. Then multiplying both sides of the equation by x results in $x(12k + 25l) = x$. Distributing results in $12xk + 25xl = x$. Since $x = 12a + 25b$ after substitution $12(ak) + 25(bl) = 12a + 25b = x$. Therefore $x \in \mathbb{Z}$ due to x equaling two integers being added together. Therefore $12a + 25b : a, b \in \mathbb{Z} \subseteq \mathbb{Z}$. Now let $x \in \mathbb{Z}$. Then $\gcd(12, 25) = 1$. Without loss of generality there exist integers k and l such that $12k + 25l = 1$. Multiplying both sides of the equation by x results in $x(12k + 25l) = x$. Distributing results in $12xk + 25xl = x$. Since $12xk$ is divisible by 25 then $12xk = 25m$ for some integer m. After substituting $25m + 25xl = x$. Factoring out 25 results in $25(m + xl) = x$. Therefore x again equals two integers being added and thus $x \in 12a + 25b : a, b \in \mathbb{Z}$. Therefore $\mathbb{Z} \subseteq 12a + 25b : a, b \in \mathbb{Z}$.

9. CH 9 PROB 12 . If $a, b, c \in N$ and ab, bc and ac all have the same parity, then a, b and c all have the same parity.

Solution. Suppose for contradiction that a, b, and c do not have the same parity. Then a and b are even while c is odd. Thus $a = 2m$ and $b = 2n$ for some $m, n \in \mathbb{N}$ and $c = 2k + 1$ for some $k \in \mathbb{N}$. Since ab and bc have the same parity both must be even. Thus resulting in $ab = (2m)(2n) = 4mn$ $bc = (2n)(2k + 1) = 4nk + 2n$. Since ab and bc have the same parity $2n$ must be even. Since n is even and $n = 2d$ for some $d \in \mathbb{N}$. Substituting bc results in $bc = (2n)(2k + 1) = 4dk + 2n$. After simplifying then becomes

$bc = 4dk + 4d$ Where the equation is even.. Since a is even and c is odd ac is even. However then ac and bc do not have the same parity thus resulting in a contradiction. Therefore a, b, and c must all have the same parity since a, b, and c do not have the same parity is false due to contradiction.

10. CH 9 PROB 14 If A and B are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

Solution. Suppose $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \subseteq A$ and $X \subseteq B$. Therefore $X \subseteq A \cap B$. So $X \in \mathcal{P}(A \cap B)$ then $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$. Now suppose $Y \in \mathcal{P}(A \cap B)$. This means that $Y \subseteq A \cap B$. Therefore, $Y \subseteq A$ and $Y \subseteq B$. So $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$ then $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Therefore $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$. Therefore $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ since $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$

11. CH 9 PROB 24 The inequality $2^x \geq x + 1$ is true for all positive real numbers x.

Solution. The inequality $2^x \geq x + 1$ is true for all positive real numbers x.

To disprove it a value of x for which the inequality is false must be found. For the minimum case where the inequality is satisfied $x = 1$. Then $2^x = 2$ and $x + 1 = 2$ so the inequality becomes $2 \geq 2$. In this case the inequality is true for the value $x = 1$. Now considering a positive real number $0 \leq x \leq 1$ such as $x = 1/2 = .5$. Then $2^x = \sqrt{2}$ and $x + 1 = 1.5$ so the inequality becomes $\sqrt{2} < 1.5$. in this case the statement is shown to be false. Therefore the statement is false and the inequality $2^x \geq x + 1$ is not true for all positive real numbers x since in the case where $0 \leq x \leq 1$ is applied.