Devin Maya

1) considering the smallest case the 3 puzzle There are 24 ways in total to arrange the 3 square tiles in a 2x2 frame. where all are solvable

1	2
3	

There are 4! possible arrangements of tiles for a 2x2 case.

when examining

1	2
3	

the actions that can be done are sliding the blank tile to swap it with another number in the same row or column.

(1, 2, 3, _) (1, 2, _, 3)

Thus $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array} \xrightarrow{\text{row slide}} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline & 3 \\ \hline\end{array}$

$$\underset{(1,2,3,-)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array}} \xrightarrow{\text{column slide}} \underset{(1,-,3,2)}{\begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 2 \\ \hline\end{array}}$$

Now to swap the blank square with the only square which is not in its same row or column.

That is swap configuration 1,2,3,–

to –,2,3,1.

$$\overset{(1,2,3,-)}{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array}} \xrightarrow[\text{swap }(2,-)]{} \overset{(1,-,3,2)}{\begin{array}{|c|c|}\hline 1 & \\ \hline 3 & 2 \\ \hline\end{array}} \xrightarrow[\text{swap }(1,-)]{} \overset{(-,1,3,2)}{\begin{array}{|c|c|}\hline & 1 \\ \hline 3 & 2 \\ \hline\end{array}} \xrightarrow[\text{swap }(1,2)]{} \overset{(-,2,3,1)}{\begin{array}{|c|c|}\hline & 2 \\ \hline 3 & 1 \\ \hline\end{array}}$$

Thus it is shown that this can

also be applied to all possible configurations by applying the same swaps between values specifically for a 2x2 case all 24 configurations are solvable since any tile can be swapped to another tile regardless of its current position.

All 2x2 puzzles are solvable since any possible state out of 24 configurations is reachable from every other. Where as shown How moving just the blank square to all possible other squares the same can be done for the others where you can always make a sequence of swaps to solve any configuration. It is therefore shown that all the ways of assigning tiles can be placed into the solved (1,2,3,–) state after finitely many steps of sliding (swaps).

2) It is shown that the first row can be placed correctly regardless of the arrangement by the process of solving left to right.

If the   is not in the correct position then

from

B

3	X
Y	4
Z	

slide 4 →

B'

Y	3
	X
Z	4

→

B''

Y	3
X	4
	Z

B'''

3	4
Y	
X	Z

Where the sequence of moves are: $(3, x, y, 4, z, -) \to (y, 3, -, x, z, 4)$ $\to (y, 3, x, 4, -, z) \to (3, 4, y, -, x, z)$. Slide 4 down, slide x down, slide blank left, slide blank down will result in $B \to B'$. Then slide blank right, slide 4 up, slide blank left will result in $B' \to B''$. Finally, slide 3 left, slide 4 up, slide x down, slide y left will result in $B'' \to B'''$.

Now considering the remaining square has the first row solved in order 1, 2, 3, 4 then the same process can be applied to the second row where first the 5, 6, 7 are placed in order then with the similar sequence of moves 8 can be placed in the correct place. That is considering we are trying to solve for the second row

1	2	3	4
5	6	7	
⊗	⊗	y	x
⊗	⊗	z	8

we can just now consider the following

7	
y	x
z	8

C

where the same sequence can be applied to achieve the solved second row that B

from

7	
y	x
z	8

C

if you move x up, then 8 up the result is the same as the state solved for before in which

5 the first row was solved for via

$B \to B' \to B'' \to B'''$. Thus from $C \to C_2$

7	
y	x
z	8

C $\to$

7	x
y	8
z	

C_2 where C_2 is of the form that B was.

recall B

B

3	X
y	4
z	

$\to$ B'

y	3
	X
z	4

$\to$ B''

y	3
X	4
	z

B'''

3	4
y	
X	z

Thus applying the same sequence of steps to C_2

C_2

7	X
Y	8
Z	

$\rightarrow$ C_2'

Y	7
	X
Z	8

$\rightarrow$ C_2''

Y	7
X	8
	Z

C_2'''

7	8
Y	
X	Z

Thus it has been shown that the second row can also be arranged in the correct order.

3)

1	2	3	4
5	6	7	8
9	10	11	12

13	X	X	X

will be of
2 forms

10	11	12
14	15	

A

or

10	11	12
15	14	

B

By taking the section of the 4x4 puzzle that is unsolved for that is

1	2	3	4
5	6	7	8
9	X	X	X
13	X	X	X

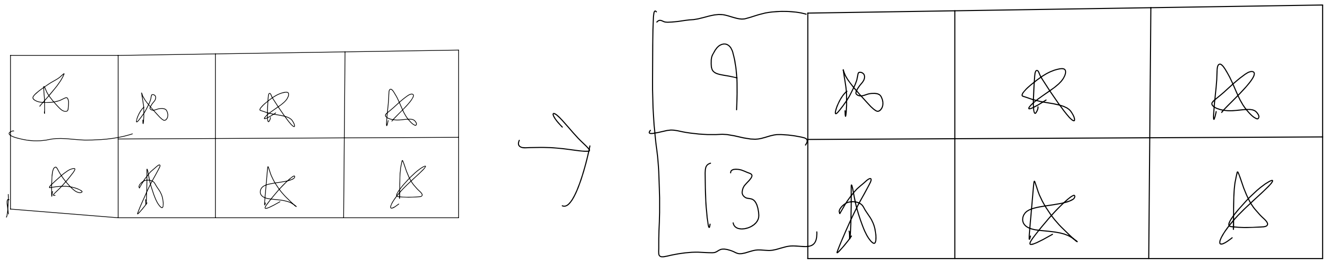
X	X	X
X	X	X

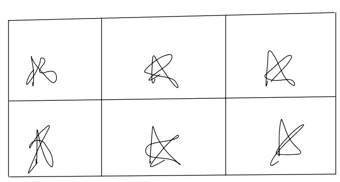
where the X's will take on the remaining values of 10, 11, 12, 14, 15, ...
It is taken that 9 & 13 can be solved for when considering

X	X	X	X
X	X	X	X

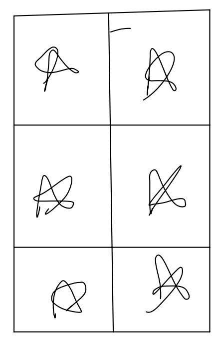
only the top two rows are solved for.

No matter the configuration of this puzzle it is always possible to place the 9 & 13 in the correct position here we have deviated from solving strictly each row at a time from left to right for the first two rows. Now we will be able to fix 9 & 13 in their correct squares via swaps which then results in solving for the 9 & 13 in their correct positions.



Where the X's are any remaining values that have not been configured correctly.  In order to solve the remaining component of the puzzle the same sequence of moves will be performed in order to solve for the correct configuration.

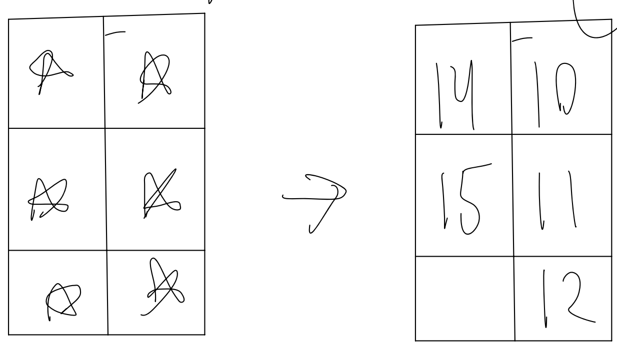
That is observe the remaining component with a 90° rotation in orientation becoming



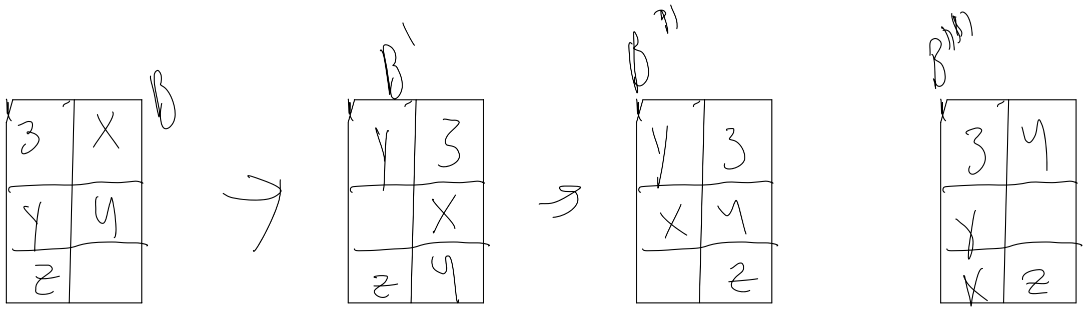
Now we have shown in problem 2 how any possible configuration for the second row solution is a result of performing the moves from $B \to B' \to B'' \to B'''$

Thus the remaining component

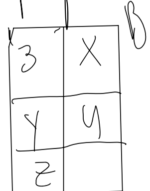
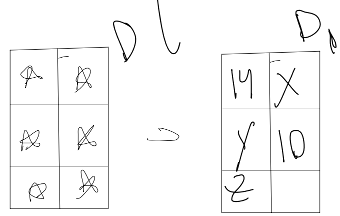
can also be solved for using the same approach because any remaing configuration



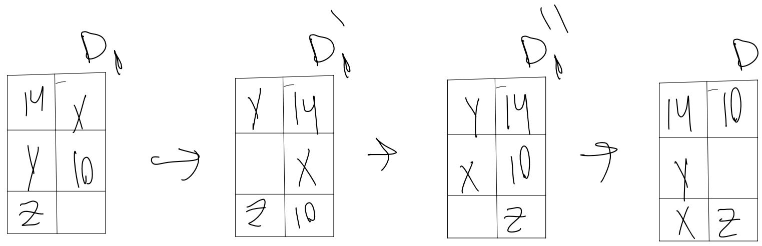
recall B



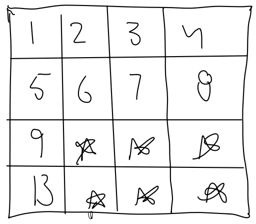
Thus first given any configuration the objective is to get in the form of B and apply the same sequence of moves

that is  from  is of form B.

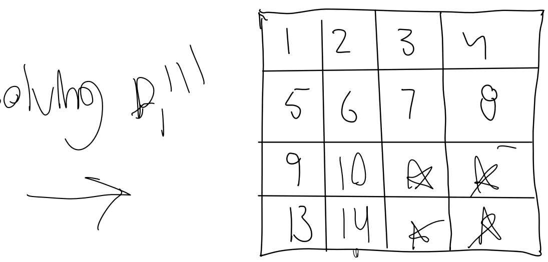
Now that we are in the following state D_1



Now observing the entire puzzle

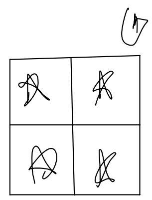


solving D_1'''

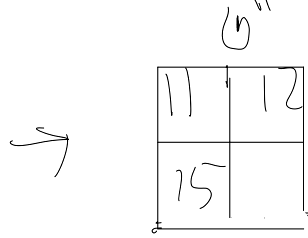


Since the remaining 2x2 contains the values (11, 12, 15, -) we can use the knowledge from problem 1 that there are 24 possible arrangements using those values and no matter the configuration it is always possible to solve for the correct final state.

That is from



will result in



as was shown in the first problem in solving for a 2x2 taking values (1, 2, 3, -) from any possible configuration you can achieve the solution with a finitely numbered amount of swaps.

Thus looking at the entire square considering we have solved for 6"

1	2	3	4
5	6	7	8
9	10	~~x~~	~~x~~
13	14	~~x~~	~~x~~

$\xrightarrow{\text{solved for 6"}}$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

where

this is of the form A (the solved square)

we have shown that after using some trick to fix 9 & 13 you will end up with configuration A.

4)

Prove when moving the blank spots vertically, the inversion number will change from an even number to an odd number or from an odd number to an even number.

Consider the case of moving the blank square horizontally we can assume the blank square is on the far left since it does not change the inversion number. That is consider the case for the top row $(a_1, a_2, a_3, \_\_)$ = 0 inversion

$(a_1, a_2, -, a_3) = 0$ inversion

$(a_1, -, a_2, a_3) = 0$ inversion

$(-, a_1, a_2, a_3) = 0$ inversion

Therefore moving the blank to the left or right does not change the inversion number. The same applies to other rows since we have shown when the blank spot moves horizontally the inversion number does not change. Now suppose that the blank square is in row i, where $1 \le i \le 4$

X	a_1	a_2	a_3
Y	a_4	a_5	a_6
	a_7	a_8	a_9
Z	a_{10}	a_{11}	a_{12}

Observe that if $1 \le i \le 3$ and we slide the blank square down then the only affected rows are i and row $i+1$ since $i+1$ is the row below i and we moved the blank square down which is true when $1 \le i \le 3$. Now observing that

If the blank square is in rows $2 \le i \le 4$ and the blank square is moved up then the rows that are affected are i and $i-1$. Thus the row in which the blank square is in and the row above it since its an upward slide. Now consider moving the blank down that is from the following cases.

P:

x	a_1	a_2	a_3
	a_4	a_5	a_6

$(x, a_1, a_2, a_3, -, a_4, a_5, a_6)$ $\mathrm{Inv}(P) = 0$

↓ ↑

-3 ↑ ↓ $+3$

Q:

	a_1	a_2	a_3
x	a_4	a_5	a_6

$(-, a_1, a_2, a_3, x, a_4, a_5, a_6)$

$\mathrm{Inv}(Q) = 3$

Observing that in case P if $x \le a_1$ then when the blank moves up $x \le a_1$ is not true because it is now out of order with inversion of 3.

Now observing case Q if $x \leq a_i$ then when the blank square moves down $x \leq a_i$ is now true again. Since the order is now correct again the inversion number is 0 again.

Thus the difference between the two cases when moving the blank up or down is a negative or positive inversion number change. As observed from $P \to Q$ resulted in inversion +3 and $Q \to P$ resulted in taking away 3 inversion. Thus when moving up or down inversions will either add or subtract some amount from the inversion number between two distinct cases.

Now Let $A = (a_1, a_2, a_3)$

The inversion number is 0 since all elements are in correct order

Let $xA = (x, a_1, a_2, a_3)$ & $Ax = (a_1, a_2, a_3, x)$

The inversion number of xA & Ax are

$$\operatorname{Inv}(xA) = \operatorname{Inv}(A) + |\{a_i : a_i \leq x\}|$$

Where the set xS is $|\{a_i : a_i \leq x\}|$

$$\operatorname{Inv}(Ax) = \operatorname{Inv}(A) + |\{a_i : x \leq a_i\}|$$

where the set Sx is $|\{a_i : x \leq a_i\}|$

case 1: If $|xS| = 0$ it is the correct order

then $|Sx| = 3$ where 3 will be added to the inversion number

Case 2: If $|xS| = 1$

then $|Sx| = 2$ where $|Sx| - |xS| = 2-1 = 1$ will be added to the inversion

Case 3: If $|xS| = 2$

then $|Sx| = 1$ where $|Sx| - |xS| = 1-2 = -1$

Here 1 will be subtracted from the inversion number

Case 4: If $|xS| = 3$

then $|Sx| = 0$ where $|Sx| - |xS| = 0-3 = -3$

Here 3 will be subtracted from the inversion number.

Note that all the cases are adding

or subtracting by an odd number thus the parity of an odd number where in this case we have shown 1, -1, 3, 3 are the odd numbers. An odd number plus or minus another odd number equals an even number. Thus the parity when the blank spot moves vertically will change the inversion number from an even number to an odd number, or from an odd number to an even number since each vertical movement will result in a change of inversion by adding either 1, -1, 3, 3 depending on the case.

5) If the inversion number of the puzzle is even and the blank square is at the second or the 4th row, or if the inversion number of the puzzle is odd and the blank square is at the first or the third row, the puzzle is solvable.

Suppose the blank square is in the second or 4th row and the inversion number is 0 since it is an even number. This means the square will be in order with the blank square in either the second or fourth rows. Thus the second row will be of form (5, 6, 7, 8) and the fourth row will be (13, 14, 15, –) which is the solved state of the puzzle with correct order an inversion number of 0.

Therefore the puzzle is solvable if the blank square is in the second or fourth row. Since we previously showed in problem 4 when moving the blank horizontally the inversion number doesn't change. Now considering that the blank square can either move horizontally or vertically. We have shown that horizontal moves of the blank square don't change the inversion number or the given row.

Now when you move the blank square vertically as shown in problem 4 both the inversion number and the row change.

Therefore, as shown by presenting the solved case where the inversion number 0 is even, it will stay even in all subsequent moves. If the sum is odd in the initial configuration of the inversion number then it is unsolvable because there would not be any legal moves to solve for such a case.

That is case

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

cannot reach a solved state.

For a 5x5 case: The same principles and approach to solvability in a 5x5 case should be the same. That is the puzzle is solvable if and only if the inversion number is even and the blank square is in an even row. Noting that there are a lot more cases to consider since there is an extra row and column however the principles discussed of inversion and the solvability of the puzzle will stay consistent in the solvability of a 5x5 puzzle.