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Math 117 Homework 8

4.3.1 Let $f, g \in F[\lambda]$ be two polynomial. We say
that a polynomial m is a *least common multiple* (LCM) of f and g if:

1. $f|m$ and $g|m$.
2. If $f|m'$ and $g|m'$, then $m|m'$.

Prove that any two polynomials have a least common multiple. If one of f or g is not zero, prove that the polynomial

$$\operatorname{lcm}(f, g) = \frac{fg}{\gcd(f, g)}$$

is an LCM of f and g. This is the unique monic LCM of f and g.

Solution. The objective is to prove that any two polynomials $f, g \in F[\lambda]$ have a least common multiple. Also if one of f or g is not zero then the polynomial $\operatorname{lcm}(f, g) = \frac{fg}{\gcd(f,g)}$ is a monic LCM of f and g.

First the objective is to prove that any two polynomials f and g have an LCM.

If either f or g is zero then the other polynomial is the LCM as zero divides any polynomial and any polynomial divides itself. If both f and g are non zero then consider the product fg. Observe that f divides fg and g divides fg thus fg is a common multiple of f and g. Any common multiple m' of f and g must have both f and g as factors and thus must also be a multiple of fg. Therefore fg is a least common multiple of f and g.

Now the objective is to prove that $\operatorname{lcm}(f, g) = \frac{fg}{\gcd(f,g)}$ is a unique monic LCM of f and g where at least one of f or g is nonzero. Both f and g divide their product fg so they also divide $\frac{fg}{\gcd(f,g)}$ since the greatest common divisor (gcd) of f and g is a factor of both f and g. Let m' be any common multiple of f and g. By the definition of gcd $\gcd(f, g)$ divides both f and g and hence divides m'. Therefore the product fg also divides $m' \cdot \gcd(f, g)$. It follows that $\frac{fg}{\gcd(f,g)}$ divides m'. To ensure that $\operatorname{lcm}(f, g)$ is monic we take the monic version of $\frac{fg}{\gcd(f,g)}$, which does not change the divisibility properties.

Therefore, $\operatorname{lcm}(f, g) = \frac{fg}{\gcd(f,g)}$ is the unique monic LCM of f and g when at least one of f or g is non-zero.

4.3.2 Let V be a finite-dimensional vector space.

Let $T : V \to V$ be a linear operator, and let $W, Z \subseteq V$ be two T- invariant subspaces. Prove that

$$\mu_{T,W+Z} = \operatorname{lcm}(\mu_{T,W}, \mu_{T,Z}).$$

Solution. First begin by letting V be a finite dimensional vector space and $T : V \to V$ be a linear operator. Let $W, Z \subseteq V$ be T invariant subspaces. The objective is to prove that $\mu_{T,W+Z} = \text{lcm}(\mu_{T,W}, \mu_{T,Z})$.

The minimal polynomial of T on a subspace S is denoted by $\mu_{T,S}$ it is the monic polynomial of smallest degree such that $\mu_{T,S}(T)$ restricted to S is the zero transformation. A subspace S is T-invariant if $T(s) \in S$ for all $s \in S$. The sum $W + Z$ is the set $\{w + z \mid w \in W, z \in Z\}$.

Since W and Z are T-invariant, $\mu_{T,W}(T)$ and $\mu_{T,Z}(T)$ act as zero on W and Z, respectively. For any $v \in W + Z$, $v = w + z$ for some $w \in W$, $z \in Z$. Thus $\mu_{T,W}(T)v = \mu_{T,W}(T)w + \mu_{T,W}(T)z = 0$ and similarly for $\mu_{T,Z}(T)v$. Thus both $\mu_{T,W}$ and $\mu_{T,Z}$ divide $\mu_{T,W+Z}$.

Let $\mu = \text{lcm}(\mu_{T,W}, \mu_{T,Z})$. By the properties of LCM μ is divisible by both $\mu_{T,W}$ and $\mu_{T,Z}$ and thus $\mu(T)$ acts as zero on both W and Z. Therefore for any $v \in W + Z$, $\mu(T)v = \mu(T)w + \mu(T)z = 0$. This shows that μ is a polynomial that annihilates T on $W + Z$.

Since μ is the LCM and is of smallest degree with the property that it annihilates T on both W and Z, it must also be the minimal polynomial that annihilates T on $W + Z$. Therefore, $\mu_{T,W+Z} = \mu$.

Thus we have shown that $\mu_{T,W+Z} = \text{lcm}(\mu_{T,W}, \mu_{T,Z})$, proving the statement.

4.3.3 Let V be a finite-dimensional vector space.

Let $T : V \to V$ be a linear operator. Prove that if T is not an isomorphism, then there exists a non-zero operator $U : V \to V$ such that $TU = 0$. Hint: Show that the constant term of χ_T is zero, and use Cayley-Hamilton.

Solution. Begin by letting V be a finite dimensional vector space and $T : V \to V$ be a linear operator. Our goal is to prove that if T is not an isomorphism then there exists a non zero operator $U : V \to V$ such that $TU = 0$.

The objective is to determine the Constant Term of the Characteristic Polynomial of T Define the characteristic polynomial of T as $\chi_T(\lambda) = \det(T - \lambda I)$. If T is not an isomorphism since it is not invertible which implies that $\det(T) = 0$. Hence the constant term of $\chi_T(\lambda)$ which is $\chi_T(0) = \det(T)$, is zero.

Next the objective is to apply the Cayley-Hamilton Theorem According to the Cayley-Hamilton Theorem, T satisfies its characteristic polynomial so $\chi_T(T) = 0$ expanding this the result is $\chi_T(T) = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_1T + a_0I = 0$, where $a_0, a_1, \ldots, a_n$ are the coefficients of χ_T and notably $a_0 = 0$ as demonstrated earlier.

Now addressing the Operator U Since $a_0 = 0$ we can construct U as follows $U = a_nT^{n-1} + a_{n-1}T^{n-2} + \cdots + a_1I$. Then multiplying T by U the result is $TU = T(a_nT^{n-1} + a_{n-1}T^{n-2} + \cdots + a_1I) = a_nT^n + a_{n-1}T^{n-1} + \cdots + a_1T = 0$.

Now to show that U is Non Zero We need to show that U is not the zero operator. Since T is not an isomorphism it has a non trivial kernel which implies that for some $k < n$ and T^k is non zero. Therefore U as shown is also non zero.

We have successfully constructed a non zero operator $U : V \to V$ such that $TU = 0$ therefor proving the statement.

4.4.1 Let V be an n-dimensional vector space. A linear operator $T : V \to V$ is called *unipotent* if $T - 1$ is nilpotent. Prove that if T is unipotent, then

$$\chi_T(\lambda) = (\lambda - 1)^n.$$

We are not necessarily working over the complex numbers, so be sure to verify that χ_T splits.

Solution. Let V be an n dimensional vector space and let $T : V \to V$ be a unipotent linear operator. By definition this means that $T - 1$ is nilpotent. We aim to prove that the characteristic polynomial of T is $\chi_T(\lambda) = (\lambda - 1)^n$.

Now addressing the nilpotency of $T - 1$ Since $T - 1$ is nilpotent there exists some smallest positive integer $k \leq n$ such that $(T - 1)^k = 0$. This implies that all eigenvalues of $T - 1$ are 0 as nilpotent operators only have 0 as their eigenvalue. Characteristic Polynomial of $T - 1$ The characteristic polynomial of $T - 1$ is given by $\chi_{T-1}(\lambda) = \det((T - 1) - \lambda I)$. Since $T - 1$ is nilpotent $\chi_{T-1}(\lambda) = \lambda^n$ as all its eigenvalues are 0. Relationship Between χ_T and χ_{T-1} we need to establish the relationship between $\chi_T(\lambda)$ and $\chi_{T-1}(\lambda)$. We have $\chi_T(\lambda) = \det(T - \lambda I) = \det((T - 1) - (\lambda - 1)I)$. By substituting $\mu = \lambda - 1$ into $\chi_{T-1}(\mu)$ we get $\chi_{T-1}(\mu) = \mu^n \Rightarrow \chi_{T-1}(\lambda - 1) = (\lambda - 1)^n$. Hence we have shown that $\chi_T(\lambda) = (\lambda - 1)^n$. This result indicates that all eigenvalues of T are 1 consistent with the definition of a unipotent operator.

Verification that χ_T to verify that χ_T splits we observe that $(\lambda - 1)^n$ is a polynomial with all its roots being 1, which are elements of the field over which V is defined. Therefore χ_T indeed splits into linear factors.

Therefore if T is a unipotent operator on an n dimensional vector space V, then $\chi_T(\lambda) = (\lambda - 1)^n$.