DEVIN MAYA 2/17/23

Math 100A Homework 5

1. CH 5 PROB. 6 Suppose $x \in \mathbb{R}$. If $x^3 - x > 0$, then $x > -1$.

Solution. Proving by contrapositive therefore the statement becomes if $x \leq -1$, then $x^3 - x \leq 0$.

Now assume $x \leq -1$ is true. For any value $x \in \mathbb{R}$ then the inequality becomes $x+1 \leq 0$ after adding 1 to both sides. Then factoring x from $x^3 - x$ it becomes $x \cdot (x^2 - 1)$. $x+1$ is negative and $x - 1$ is also negative due to $x + 1 \leq 0$. So $x^2 - 1 = (x + 1)(x - 1)$ is the product of two negative numbers and is therefore positive. Since $x \leq -1$, we know that x is also negative. Therefore, $x(x^2 - 1)$ is the product of a negative number and a positive number, and is therefore negative.

Therefore, if $x^3 - x > 0$, then $x > -1$ since $x^3 - x = x \cdot (x^2 - 1)$.

2. CH 5 PROB. 8 If $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$:

Solution. Proving by contrapositive enables the statement to become If $x < 0$, then $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$. Now assume $x < 0$ is true. Then factoring out a negative sign results in

$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 = -(x^5 + 4x^4 - 3x^3 + x^2 - 3x + 4)$

Now $x^5 + 4x^4 - 3x^3 + x^2 - 3x + 4 > 0$ must be true when $x < 0$. Each term except the constant 4 is a power of x multiplied by a coefficient of the same sign

$x^5 + 4x^4 - 3x^3 + x^2 - 3x + 4 = x^3(x^2 - 3x + 4) + x^2(4x - 3) + (4 - x) > 0$

Thus $x^3 < 0$, $4 - x > 0$ and $4x - 3 < 0$ since $x < 0$. Then each term in the expression is negative and their sum is negative. Thus $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$ when $x < 0$ Therefore, by contrapositive, if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

3. CH 5 PROB 16 Suppose x, y $\in Z$. If $x + y$ is even, then x and y have the same parity.

Solution. By contrapositive Suppose x and y have different parity. Then one of them is even and the other one is odd. Without loss of generality, assume that x is even and y is odd. Then there exist integers a and b such that $x = 2a$ and $y = 2b + 1$. Therefore, $x + y = 2a + 2b + 1 = 2(a + b) + 1$, which is odd since it is of the form $2k + 1$ where k is an integer.

Therefore if x and y have different parity, then $x + y$ is odd. Thus if $x + y$ is even, then x and y have the same parity.

4. CH 5 PROB 26 If $n = 2^k - 1$ for $k \in N$, then every entry in Row n of Pascal's Triangle is odd.

Solution. Suppose that $n = 2^k - 1$ for some positive integer k.The objective is to prove that every entry in Row n of Pascal's Triangle is odd. The i-th entry in Row n of Pascal's Triangle is given by the binomial coefficient $\binom{n}{i}$, which is equal to $\frac{n!}{i!(n-i)!}$. Since $n = 2^k - 1$, then $n!$ becomes $(2^k - 1)!$. Then the binomial coefficient $\binom{n}{i}$ is

$\binom{n}{i} = \frac{(2^k-1)!}{i!(2^k-1-i)!}$

By factoring out 2^k from the numerator $\binom{n}{i} = \frac{(2^k)!}{i!(2^k-i)!} \cdot \frac{1}{2^k-1}$

Now the denominator $2^k - 1$ is odd, since k is a positive integer and is an even number subtracted by 1 which is an odd parity result. The numerator $(2^k)!$ has 2^k as a factor k times. Then it has at least k factors of 2 where it becomes $2^k \cdot a$ for some odd integer a.

Substituting these expressions back into the equation for the binomial coefficient results in

$\binom{n}{i} = \frac{2^k \cdot a}{i!(2^k-i)! \cdot (2^k-1)}$

Thus the denominator is odd due to the product of odd integers. For $\binom{n}{i}$ to be an odd integer the numerator $2^k \cdot a$ must be an odd integer.Since 2^k is an even integer, so the only way for $2^k \cdot a$ to be odd is if a is odd. But since a is odd, it does not have any factors of 2.Thus neither does the numerator $2^k \cdot a$.

Therefore every entry in Row n of Pascal's Triangle is odd since the numerator is odd as shown.

5. CH 6 PROB 6 . If $a, b \in Z$, then $a^2 - 4b - 2 \neq 0$

Solution. Suppose for the sake of contradiction that there exist integers a and b such that $a^2 - 4b - 2 = 0$. Assume that a is even, which means that a^2 is also even. Thus $a^2 = 2k$ for some integer k. After substituting

$$2k = 4b + 2$$

Further simplifying to $k = 2b + 1$.

This means that k is odd since it is of form $k = 2b + 1$. But if k is odd then a^2 is also odd.Thus this is a contradiction since $a^2 = 4b + 2$. Now if a is odd $a^2 = 2k + 1$ for some integer k. After substituting $2k + 1 = 4b + 2$ Further simplifying, $k = 2b + \frac{1}{2}$ This means that k is not of the form $2b$ or $2b + 1$ thus a contradiction. In this case first a is even and then showed that this leads to a contradiction. Then that a is odd and showed that this also leads to a contradiction. Since both cases led to a contradiction the original statement $a^2 - 4b - 2 = 0$ is false, and that $a^2 - 4b - 2 \neq 0$ for any integers a and b.

6. CH 6 PROB 10 There exist no integers a and b for which $21a + 30b = 1$

Solution. Suppose, for the sake of contradiction, that there exist integers a and b such that $21a+30b=1$. We want to show that this assumption leads to a contradiction. Both 21 and 30 are divisible by 3. So the equation simplifies to $21a+30b=3(7a+10b)=1$

$3(7a+10b)=1$ means that 3 divides 1 by definition 4.4, which is a contradiction. Therefore, the statement there exist integers a and b such that $21a+30b=1$ must be false since there exist no integers a and b satisfying the equation.

7. CH 6 PROB 20 We say that a point $P=(x,y)$ in R^2 is rational if both x and y are rational. More precisely, P is rational if $P=(x,y)\in Q^2$. An equation $F(x,y)=0$ is said to have a rational point if there exists $x_0, y_0 \in Q$ such that $F(x_0,y_0)=0$. For example, the curve $x^2+y^2-1=0$ has rational point $(x_0,y_0)=(1,0)$. Show that the curve $x^2+y^2-3=0$ has no rational points.

Solution. Suppose, for the sake of contradiction, that there exists a rational point (x_0,y_0) on the curve. Then $x_0^2+y_0^2=3$. Since x_0 and y_0 are rational $x_0=\frac{a}{b}$ and $y_0=\frac{c}{d}$, where $a,b,c,d\in\mathbb{Z}$ and $b,d\neq 0$. Substituting the expressions into $x_0^2+y_0^2=3$ becomes $\frac{a^2}{b^2}+\frac{c^2}{d^2}=3$ further simplifying into $a^2d^2+b^2c^2=3b^2d^2$. Since 3 divides the right-hand side, it must also divide the left-hand side.Therefore, c^2 must be divisible by 3, which means that c must be divisible by 3. Thus $c=3e$ for some integer e. Substituting for c into the equation $(a^2-3b^2)d^2=3c^2b^2$ results in

$(a^2-3b^2)d^2=27b^2e^2$

Since 3 divides the left-hand side, it must also divide the right-hand side. This means that b must be divisible by 3. thus $b=3f$ for some integer f. Substituting for b into the equation $(a^2-3b^2)d^2=27b^2e^2$ results in

$(a^2-27f^2)d^2=81f^2e^2$

This means that a^2-27f^2 is divisible by 3. However a is not divisible by 3 and f is so a^2-27f^2 cannot be divisible by 3. This is a contradiction thus the statement there exists a rational point on the curve $x^2+y^2-3=0$ must be false. Therefore, the curve has no rational points. Therefore there are no rational solutions to the equation $x^2+y^2-3=0$.

8. CH 6 PROB 24 The number $log_2 3$ is irrational.

Solution. Suppose, for the sake of contradiction, that $\log_2 3$ is rational. Thus $\log_2 3$ is a fraction of two integers a and b in lowest terms, where $a\neq 0$. That is,

$\log_2 3=\frac{a}{b}$

Thus it can be rewritten to

$2^{\log_2 3}=2^{a/b}$

By raising the logarithm by the integer of its base simplifying the left-hand side results in

$3 = (2^{a/b})$

Now raising both sides of this equation to the power of q results in $3^b = 2^a$

So $3^b = 2^a$, where a and b are integers. 3 is an odd number since it is not divisible by 2. Therefore, if 3^b is odd, then 2^a must also be odd due to the equation. 2^a cannot be odd because any power of 2 is even. Thus $\log_2 3$ is rational leads to a contradiction and it must instead be irrational. This means that 2^a is an even number since it is equal to 3^b which is odd. It cannot be a fraction of two integers in which the denominator is a power of 2 since 3 is not a power of 2. Therefore $\log_2 3$ is rational leads to a contradiction, and so it must be irrational.

9. CH 7 PROB 12 There exists a positive real number x for which $x^2 < \sqrt{x}$

Solution. There exists a positive real number x for which $x^2 < \sqrt{x}$. Let $x = \frac{1}{4}$. Then this results in $x^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$ also, $\sqrt{x} = \sqrt{\frac{1}{4}} = \frac{1}{2}$. Since $x^2 < \sqrt{x}$ is equivalent to $\frac{1}{16} < \frac{1}{2}$, so $x = \frac{1}{4}$ satisfies the inequality.

Therefore there exists a positive real number x for which $x^2 < \sqrt{x}$, since $x = \frac{1}{4}$.

10. CH 7 PROB 20 There exists an $n \in N$ for which $11|(2^n - 1)$.

Solution. For 2^n the remainders of the powers of 2 when divided by 11 are

$2^1 = 2 \equiv 2 \pmod{11}$

$2^2 = 4 \equiv 4 \pmod{11}$

$2^3 = 8 \equiv 8 \pmod{11}$

$2^4 = 16 \equiv 5 \pmod{11}$

$2^5 = 32 \equiv 10 \pmod{11}$

$2^6 = 64 \equiv 9 \pmod{11}$

$2^7 = 128 \equiv 7 \pmod{11}$

$2^8 = 256 \equiv 3 \pmod{11}$

$2^9 = 512 \equiv 6 \pmod{11}$

$2^{10} = 1024 \equiv 1 \pmod{11}$

$2^{11} = 2048 \equiv 2 \pmod{11}$

Here the remainders repeat after 10 powers of 2. So $2^{10} \equiv 1 \pmod{11}$.

Therefore, $2^{10k} \equiv 1 \pmod{11}$ for any positive integer k. Thus $11 \mid (2^{10k} - 1)$ for any positive integer k. Thus when $k = 1$, $11 \mid (2^{10} - 1)$ will now be true since the remainder from the modulo 11 is subtracted due to the -1. Then there is an $n = 10$ for which $11 \mid (2^n - 1)$ since 11 is a divisor of $(2^n - 1)$ when $n = 10$.