DEVIN MAYA 7/23/23

Math 124: Introduction to Topology Asignment 12

1. EXERCISE 6.1.3

Solution. Given two paths and in the plane from x_p to x_q, we can define a straight-line homotopy from to similar to the homotopy from Example 6.1.2, but instead of mapping each point on to the origin, we map each point on to the corresponding point on . We can define this path homotopy as follows:

$f_s(t) = (1 - s)(t) + s(t)$ for s,t in [0, 1].

Here, $f_s(t)$ represents a point on the straight line between (t) and (t) at time t.

When s = 0, $f_s(t) = (t)$, which is the start of the homotopy, and when s = 1, $f_s(t) = (t)$, which is the end of the homotopy. For values of s between 0 and 1, $f_s(t)$ gives a path between (t) and (t).

The important thing to note here is that for each fixed t, $f_s(t)$ will be a straight line from (t) to (t). This means that every point on the path is continuously moved along a straight line to the corresponding point on the path . Thus, the path is continuously deformed into the path .

Hence, and are homotopic, or in other words,   .

2. EXERCISE 6.1.4

Solution. Given two paths and on the unit sphere S^2 with the same start and end points, and such that for every t, (t) and (t) are not antipodal (i.e., (t) does not equal -(t) for any t), we can define a homotopy from to in a manner similar to the previous exercise, but we need to take into account the fact that we are working on a sphere, not on a plane.

The homotopy will still involve the straight line joining (t) and (t), but since these lines will generally go off the sphere, we'll have to project these lines back onto the sphere.

To project a point p in $\mathbb{R}^3$ back onto the unit sphere S^2, we can normalize it by dividing it by its own norm, i.e., $p/||p||$.

Therefore, the explicit formula for the homotopy will be:

$f_s(t) = \frac{(1-s)(t)+s(t)}{||(1-s)(t)+s(t)||}$ for s,t in [0, 1].

For each fixed t, this gives us the point on the line from (t) to (t) that lies on the sphere.

When s = 0, $f_s(t) = (t)$, which is the start of the homotopy, and when s = 1, $f_s(t) = (t)$, which is the end of the homotopy. For values of s between 0 and 1, $f_s(t)$ gives a path from (t) to (t) on the sphere.

Hence, and are path homotopic on S^2, or in other words, .

3. EXERCISE 6.1.11

Solution. In this context, we are discussing paths and homotopies, which are continuous maps. The problem with the approach you mention, i.e., traveling along a path 'a', then staying put at a point 'p', and then traveling 'a' backwards, comes when we try to increase the time we stay put until it fills the entire interval [0, 1].

The trouble with this approach arises due to the requirement for continuity. When we "stay put" at a point, we are essentially trying to map a non-trivial interval [t1, t2] into a single point 'p'. This is not an issue until we reach the point where the interval [t1, t2] becomes the whole interval [0, 1].

When the interval becomes [0, 1], we have a discontinuity between this constant map and the previous maps, because those were non-constant over the interval [0, 1]. In terms of paths, we suddenly jump from a map that moves around to a map that doesn't move at all, which is a discontinuous transition.

The "staying put" map is fundamentally different from the "moving" maps, and there's no way to smoothly transition from one to the other while preserving continuity, which is required for a homotopy. Hence, this approach gets us into trouble and cannot be used to establish a homotopy.

4. EXERCISE 6.1.13

Solution. a. For a loop a in X based at p, the path $y^{-1}ay$ is indeed a loop based at q. Here's why:

The loop a starts and ends at the point p. The path y starts at p and ends at q, so its inverse y^{-1} starts at q and ends at p. Now, consider the path $y^{-1}ay$.

This starts at q (since it starts with y^{-1}), then travels to p (the end of y^{-1}), then loops around p (that's the loop a), then travels back to q (the path y). So, it's a path that starts and ends at q i.e., it's a loop based at q.

b. The map $[a] \to [y^{-1}ay]$ is indeed a group isomorphism from $\pi_1(X, p)$ to $\pi_1(X, q)$.

To prove that this is an isomorphism, we need to show two things: that the map is bijective (i.e., it has an inverse) and that it preserves the group operation.

Bijectivity: We need to show that every loop based at q can be written in the form $y^{-1}ay$ for some loop a based at p, and vice versa. Given a loop b based at q, consider the loop yby^{-1} based at p. Then the map sends $[yby^{-1}]$ to $[(y^{-1})(yby^{-1})y] = [b]$. So, the map is surjective. Similarly, we can show that the map is injective.

Preserves the group operation: We need to show that the map sends the product of two loops based at p to the product of their images under the map. Let a and b be two loops based at p. Then the map sends $[ab]$ to $[y^{-1}(ab)y] = [y^{-1}ay][y^{-1}by]$. So, the map preserves the group operation.

Therefore, the map $[a] \to [y^{-1}ay]$ is indeed a group isomorphism from $\pi_1(X,p)$ to $\pi_1(X,q)$.

5. EXERCISE 6.1.14

Solution. Let's use the hint provided and consider the loop aB^{-1} based at x_0. Since X is simply connected, we know that the fundamental group $\pi_1(X,x_0)$ is trivial, which means that every loop is null homotopic. So, there exists a path homotopy $H_1 : I \times I \to X$ from aB^{-1} to the constant loop c_{x_0} at x_0, where $I = [0,1]$.

We can write the constant loop c_{x_0} as BB^{-1}, so we also have a path homotopy $H_2 : I \times I \to X$ from BB^{-1} to aB^{-1}.

By concatenating H_1 and H_2, we get a path homotopy $H : I \times I \to X$ from a to B. Here is how you define the concatenation:

For $(s,t) \in I \times I$ with $t \in [0, 0.5]$, define $H(s,t) = H_1(s, 2t)$. This is essentially just running H_1 at double speed.

For $(s,t) \in I \times I$ with $t \in [0.5, 1]$, define $H(s,t) = H_2(s, 2t-1)$. This is running H_2 at double speed but started at the midpoint.

You can verify that H is continuous, and it satisfies $H(s,0) = a(s)$ and $H(s,1) = B(s)$ for all $s \in I$. So, it is indeed a path homotopy from a to B.

6. EXERCISE 6.1.15

Solution. To show that a space is simply connected, we need to prove two things: the space is path connected and its fundamental group is trivial.

Path connectedness: By definition, a convex set in $\mathbb{R}^n$ is path connected because for any two points x, y in A, the straight line segment from x to y lies in A. This line segment is a continuous map from the interval $[0,1]$ into A.

Trivial fundamental group: Consider any loop α based at x_0 in A. We can define a homotopy $H : [0,1] \times [0,1] \to A$ by $H(s,t) = (1-t)\alpha(s) + tx_0$. This is well-defined and continuous since it is based on the straight line homotopy between the points $\alpha(s)$ and x_0, both of which lie in the convex set A for all s,t. At $t = 0$, this gives the loop α, and at $t = 1$, it gives the constant loop at x_0. Therefore, α is null-homotopic, and every loop in A is null-homotopic. So, the fundamental group $\pi_1(A, x_0)$ is trivial.

These two facts together imply that A is simply connected.

As $\mathbb{R}^n$ itself is convex, it is also simply connected.

7. EXERCISE 6.1.16

Solution. a. To see that the converse is not true, consider a subset of $\mathbb{R}^2$ that is formed by removing an open line segment from a disk, say the segment $((0,0),(1,0))$ from the unit disk centered at $(0,0)$. We can choose $a_0 = (0,0)$ to show that this space is star-convex, because any line segment from a_0 to another point in the set is contained within the set. However, this set is not convex because the line segment between points $(0.5, 0.1)$ and $(0.5, -0.1)$ is not entirely contained in the set.

b. To show a star-convex set is simply connected, we can use a similar reasoning as for convex sets.

Path connectedness: Given any two points x, y in the set, we can form a path by first moving along the line segment from x to a_0 and then along the line segment from a_0 to y.

Trivial fundamental group: Consider any loop α based at x_0 in the set. We can define a homotopy $H : [0,1] \times [0,1] \to A$ by $H(s,t) = (1-t)\alpha(s) + ta_0$. As in the previous case, this gives the loop α at $t = 0$ and the constant loop at x_0 at $t = 1$, showing that any loop is null-homotopic and the fundamental group is trivial.

So, any star-convex subset of $\mathbb{R}^n$ is simply connected.

8. 6.2.1

Solution. The loop $\alpha(t)$ corresponds to the path in $\mathbb{R}$, denoted as $\tilde{\alpha}(t)$, that begins at 0 and is wrapped n times around S^1 by the covering map $p : \mathbb{R} \to S^1$.

a. $\alpha(t) = (\cos 2\pi nt, \sin 2\pi nt)$ for $0 \le t \le 1$, n a positive integer.

In this case, the path wraps around the circle n times in the counterclockwise direction, starting and ending at $(1,0)$. The corresponding path $\tilde{\alpha}(t)$ in $\mathbb{R}$ that starts at 0 is given by $\tilde{\alpha}(t) = nt$. The endpoint of this path is n, which corresponds to the image of the loop under the covering map.

b. $\alpha(t) = (\cos 2\pi nt, \sin 2\pi nt)$ for $0 \le t \le 1$, n a negative integer.

Here, the path wraps around the circle $|n|$ times in the clockwise direction. The corresponding path $\tilde{\alpha}(t)$ in $\mathbb{R}$ that starts at 0 is again given by $\tilde{\alpha}(t) = nt$. The endpoint of this path is n, which corresponds to the image of the loop under the covering map.

c. $\alpha(t) = \begin{cases} (\cos 4\pi t, \sin 4\pi t) & \text{for } 0 \le t \le 1/2 \ (\cos 4\pi t, -\sin 4\pi t) \\ \text{for } 1/2 \le t \le 1 \end{cases}$

In this case, the path first wraps twice around the circle in the counterclockwise direction, then wraps twice in the opposite direction. This corresponds to a path $\tilde{\alpha}(t)$ in $\mathbb{R}$ that starts at 0, goes up to 2 for $t = 1/2$ and then comes back down to 0 for $t = 1$. The endpoint of this path is 0, which corresponds to the image of the loop under the covering map

9. 6.2.2

Solution. From Exercise 6.2.1(a), we know that for $a(t) = (\cos 2\pi t, \sin 2\pi t)$, the corresponding path $\tilde{a}(t)$ in $\mathbb{R}$ that starts at 0 is $\tilde{a}(t) = t$. This path ends at 1.

Now, we look at $B(t) = (\cos 4\pi t, \sin 4\pi t)$, which wraps around the circle twice in the counterclockwise direction. We want to find a path $\tilde{B}(t)$ in $\mathbb{R}$ that starts at the point where $\tilde{a}$ ends (which is 1) and gets mapped to B by p. Such a path would be $\tilde{B}(t) = 1+2t$. This path ends at 3.

If we were to lift the product path aB to a path in $\mathbb{R}$ beginning at 0 and getting mapped by p to aB, we can use the concatenation of the paths $\tilde{a}$ and $\tilde{B}$, which is a path that starts at 0 and ends at 3.

This is the same endpoint as $\tilde{B}$.

The key point here is that the endpoint of the lifted product path in $\mathbb{R}$ depends on the specific lifts chosen for a and B, which depend on their starting points. So the lifted product path ends at the same place as $\tilde{B}$, not because a and B are path-homotopic, but because of the particular lifts chosen.

10. 6.2.8

Solution. The fundamental group of a topological space gives us information about its "shape" or structure, and it is a useful tool for distinguishing between different topological spaces.

The fundamental group of the unit disc D^2 (or indeed any convex subset of $\mathbb{R}^n$) is the trivial group, since any loop in the disc can be continuously shrunk to a point without leaving the disc.

On the other hand, the fundamental group of the circle S^1 is $\mathbb{Z}$, the group of integers. This is because a loop can go around the circle an integer number of times, either in the positive direction (which we can represent with a positive integer) or the negative direction (which we can represent with a negative integer). The operation in this group is addition: if one loop goes around the circle m times and another goes around n times, then their concatenation goes around $m + n$ times.

Since the fundamental groups of D^2 and S^1 are not isomorphic ($0 \ncong \mathbb{Z}$), these spaces cannot be homeomorphic. By the theorem that homeomorphic spaces have isomorphic fundamental groups, if two spaces are homeomorphic, their fundamental groups must be isomorphic. Thus, this provides a proof that D^2 is not homeomorphic to S^1 using fundamental groups.