DEVIN MAYA 8/25/23

Real Analysis Homework 7

1. (1) Prove that the sequence $(x_n)_{n\in\mathbb{N}}$ defined by $x_n = \frac{n^2-1}{n^2+1}$ is Cauchy.

Solution.

Proof. A sequence x_n is Cauchy if for every $\varepsilon > 0$, there exists a natural number N such that for all natural numbers $m, n \geq N$ we have $|x_n - x_m| < \varepsilon$.The sequence x_n is defined by $x_n = \frac{n^2-1}{n^2+1} = 1 - \frac{2}{n^2+1}$.

Now to compute $|x_n - x_m|$ for $n, m \geq 1$

$$|x_n - x_m| = \left|1 - \frac{2}{n^2+1} - \left(1 - \frac{2}{m^2+1}\right)\right| = \left|\frac{2}{m^2+1} - \frac{2}{n^2+1}\right| = 2\left|\frac{1}{m^2+1} - \frac{1}{n^2+1}\right|.$$

Herethe sequence $1/(n^2+1)$ is decreasing and approaches 0 as n approaches infinity. Thus for $m > n$ the result is

$$0 < \frac{1}{m^2+1} - \frac{1}{n^2+1} \leq \frac{1}{n^2+1}.$$

Then for $m, n \geq N$ the result is

$$|x_n - x_m| < 2\left(\frac{1}{N^2+1}\right).$$

given $\varepsilon > 0$ choosing N such that $2/(N^2+1) < \varepsilon$. This can be done by choosing N as the smallest integer greater than or equal to $\sqrt{2/\varepsilon - 1}$.

Then for $m, n \geq N$ the result is $|x_n - x_m| < \varepsilon$ where by definition the sequence x_n is Cauchy.

Therefore, the sequence $x_n = 1 - \frac{2}{n^2+1}$ is Cauchy. □

the choice of N ensures that the difference $|x_n - x_m|$ is less than ε for all $m, n \geq N$ which is what was needed to prove.

2. (2) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $\begin{cases} 1 & \text{if } x \leq 0, \\ -1 & \text{if } x > 0. \end{cases}$

Prove that $\lim_{x\to 0} f(x)$ does not exist.

Solution. The question asks to prove that the limit of $f(x)$ as x approaches 0 does not exist where $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x \leq 0, \\ -1 & \text{if } x > 0. \end{cases}$$

Proof. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by what was given above in the problem. The objective is to prove that $\lim_{x\to 0} f(x)$ does not exist.

To prove this it is enough to show that the limit from the left is not equal to the limit from the right.

Considering the limit of $f(x)$ as x approaches 0 from the left ($x \to 0^-$) here the result is

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 1 = 1.$$

Now considering the limit of $f(x)$ as x approaches 0 from the right ($x \to 0^+$) the result is

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (-1) = -1.$$

Since the limit from the left (1) is not equal to the limit from the right (-1) the limit of $f(x)$ as x approaches 0 does not exist.

Therefore concluding that $\lim_{x\to 0} f(x)$ does not exist. $\square$

(3) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q} \cap [0, 1]$. Prove that $f(x) = 0$ for all $x \in [0, 1]$.

Solution.

Proof. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(x) = 0$ for all $x \in \mathbb{Q} \cap [0, 1]$.

The objective is to prove that $f(x) = 0$ for all $x \in [0, 1]$.

Let $x \in [0, 1]$ be an arbitrary point. Since the rational numbers $\mathbb{Q}$ are dense in the real numbers $\mathbb{R}$ there exists a sequence (q_n) of rational numbers in $[0, 1]$ such that $q_n \to x$. Since $f(q_n) = 0$ for all n by the definition of the sequence limit

$$f(x) = f\left(\lim_{n\to\infty} q_n\right).$$

Now since f is continuous by the definition of function continuity the result is

$$f(x) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} 0 = 0.$$

Therefore $f(x) = 0$ for all $x \in [0, 1]$.

This proof demonstrates that for any point x in $[0, 1]$ there is a sequence of rational numbers that converges to x and since f is continuous and $f(q_n) = 0$ for all n it follows that $f(x) = 0$ for all $x \in [0, 1]$. $\square$

(4) Let $g : [0,1] \to \mathbb{R}$ be the function defined by

$$g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ x & \text{if } x \in \mathbb{Q}. \end{cases}$$

Prove that g is continuous at $x = 0$.

Solution. The objective is to prove that the function $g : [0,1] \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} 0 & \text{if } x \notin \mathbb{Q}, \\ x & \text{if } x \in \mathbb{Q}, \end{cases}$$

is continuous at $x = 0$.

A function g is continuous at $x = 0$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ whenever $|x - 0| < \delta$.

Proof. It is observable that $g(0) = 0$ because $0 \in \mathbb{Q}$. For $x \in [0,1]$ it is known that $g(x)$ is either 0 or x, so $|g(x)| = |x|$ or 0. Thus $|g(x) - g(0)| = |g(x)|$ and the objective is to show that this can be made smaller than any $\varepsilon > 0$.

For any $\varepsilon > 0$ choose $\delta = \varepsilon$. Then for any $x \in [0,1]$ with $|x - 0| = |x| < \delta = \varepsilon$ the result is

$$|g(x) - g(0)| = |g(x)| \leq |x| < \varepsilon.$$

Therefore for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|g(x) - g(0)| < \varepsilon$ whenever $|x - 0| < \delta$. Hence g is continuous at $x = 0$.

It is also important to notice that this does not imply that g is continuous at any other point in [0, 1]. It is observable that g is discontinuous at every other point in [0, 1]. $\square$

(5) Let $f : U \to \mathbb{R}$ be continuous on an open set $U \subseteq \mathbb{R}^n$. Suppose that for some $p \in U$ we have $f(p) > 0$. Prove that there is an n-ball $B(p; r) \subseteq U$ such that $f(x) > 0$ for all $x \in B(p; r)$.

Solution. To prove this implementation of the definition of continuity will be of great use. A function $f : U \to \mathbb{R}$ is continuous at a point $p \in U$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(p)| < \varepsilon$ whenever $|x - p| < \delta$ where $|x - p|$ denotes the Euclidean distance between x and p.

Proof. Given $f(p) > 0$ let $\varepsilon = \frac{f(p)}{2} > 0$. Since f is continuous at p there exists a $\delta > 0$ such that for all x in U with $|x - p| < \delta$ then resulting in $|f(x) - f(p)| < \frac{f(p)}{2}$.

Since U is open and f is continuous on U there exists some $r > 0$ such that the n-ball $B(p; r)$ is entirely contained in U. Let $r' = \min(r, \delta)$ and consider the n-ball $B(p; r')$ centered at p with radius r'. For any x in $B(p; r')$ then the result is $|x - p| < r' \leq \delta$ so $|f(x) - f(p)| < \frac{f(p)}{2}$. Then $f(x) > f(p) - \frac{f(p)}{2} = \frac{f(p)}{2} > 0$ so $f(x) > 0$ for all x in $B(p; r')$.

Therefore there exists an n-ball $B(p; r') \subseteq U$ such that $f(x) > 0$ for all x in $B(p; r')$. This proof shows that if a continuous function is positive at a point in an open set then it must also be positive at all points in some neighborhood of that point. □