# CIS 520, Machine Learning, Fall 2019
# Homework 5 Solutions
# Due: Monday, November 2nd, 11:59pm
# Submit to Gradescope

**Instructions.** Please write up your responses to the following problems clearly and concisely. We require you to write up your responses using LaTeX; we have provided a LaTeX template, available on Canvas, to make this easier. **Submit your answers in PDF form to Gradescope. We will not accept paper copies of the homework.**

**Collaboration.** You are allowed and encouraged to work together. You may discuss the **written homework** to understand the problem and reach a solution in groups. However, **it is recommended that each student also write down the solution independently and without referring to written notes from the joint session.** You must understand the solution well enough to reconstruct it by yourself. (This is for your own benefit: you have to take the exams alone.)

## 1 Multiclass Adaboost and Support Vector Machines [30 points]

### 1.1 Adaboost: Theory

In this problem you will analyze the AdaBoost.M1 algorithm, a multiclass extension of AdaBoost. Given a training sample $S = ((x_1, y_1), \ldots, (x_m, y_m))$, where $x_i$ are instances in some instance space $\mathcal{X}$ and $y_i$ are multiclass labels that take values in $\{1, \ldots, K\}$, the algorithm maintains weights $D_t(i)$ over the examples $(x_i, y_i)$ as in AdaBoost, and on round $t$, gives the weighted sample $(S, D_t)$ to the weak learner. The weak learner returns a multiclass classifier $h_t : \mathcal{X} \to \{1, \ldots, K\}$ with weighted error less than $\frac{1}{2}$; here the weighted error of $h_t$ is measured as

$$\mathrm{er}_t = \sum_{i=1}^{m} D_t(i) \cdot \mathbf{1}(h_t(x_i) \neq y_i)\,.$$

Note that the assumption on the weak classifiers is stronger here than in the binary case, since we require the weak classifiers to do more than simply improve upon random guessing (there are other multiclass boosting algorithms that allow for weaker classifiers; you will analyze the simplest case here). For convenience, we will encode the weak classifier $h_t$ as $\widetilde{h}_t : \mathcal{X} \to \{\pm 1\}^K$, where

$$\widetilde{h}_{t,k}(x) = \begin{cases} +1 & \text{if } h_t(x) = k \\ -1 & \text{otherwise.} \end{cases}$$

In other words, $\widetilde{h}_t(x)$ is a $K$-dimensional vector that contains $+1$ in the position of the predicted class for $x$ and $-1$ in all other $(K-1)$ positions. On each round, AdaBoost.M1 re-weights examples such that examples misclassified by the current weak classifier receive higher weight in the next round. At the end, the algorithm combines the weak classifiers $h_t$ via a weighted majority vote to produce a final multiclass classifier $H$:

```
Algorithm **AdaBoost.M1**
**Inputs:** Training sample $S = ((x_1, y_1), \ldots, (x_m, y_m)) \in (\mathcal{X} \times \{1, \ldots, K\})^m$
            Number of iterations $T$
**Initialize:** $D_1(i) = \frac{1}{m} \;\; \forall i \in [m]$
For $t = 1, \ldots, T$:
  – Train weak learner on weighted sample $(S, D_t)$; get weak classifier $h_t : \mathcal{X} \to \{1, \ldots, K\}$
  – Set $\alpha_t \leftarrow \frac{1}{2} \ln \left( \frac{1 - \mathrm{er}_t}{\mathrm{er}_t} \right)$
  – Update:
        $D_{t+1}(i) \leftarrow \frac{D_t(i) \exp(-\alpha_t \, \widetilde{h}_{t,y_i}(x_i))}{Z_t}$
    where $Z_t = \sum_{j=1}^{m} D_t(j) \exp(-\alpha_t \, \widetilde{h}_{t,y_j}(x_j))$
**Output final hypothesis:**
    $H(x) \in \arg\max_{k \in \{1,\ldots,K\}} \underbrace{\sum_{t=1}^{T} \alpha_t \widetilde{h}_{t,k}(x)}_{F_{T,k}(x)}$
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You will show, in five parts below, that if all the weak classifiers have error $\mathrm{er}_t$ at most $\frac{1}{2} - \gamma$, then after $T$ rounds, the training error of the final classifier $H$, given by

$$\mathrm{er}_S[H] = \frac{1}{m} \sum_{i=1}^{m} \mathbf{1}(H(x_i) \neq y_i)\,,$$

is at most $e^{-2T\gamma^2}$ (which means that for large enough $T$, the final error $\mathrm{er}_S[H]$ can be made as small as desired).

**(a)** **[5 points]** Show that

$$D_{T+1}(i) \;=\; \frac{\frac{1}{m}\, e^{-F_{T,y_i}(x_i)}}{\prod_{t=1}^{T} Z_t}\,.$$

**(b)** **[5 points]** Show that

$$\mathbf{1}(H(x_i) \neq y_i) \;\leq\; \mathbf{1}\big(F_{T,y_i}(x_i) < 0\big)\,.$$

*Hint:* Consider separately the two cases $H(x_i) \neq y_i$ and $H(x_i) = y_i$. For the $H(x_i) \neq y_i$ case, you need to show $F_{T,y_i}(x_i) \leq 0$. Note that $\sum_{k=1}^{K} F_{T,k}(x_i) = -(K-2) \sum_{t=1}^{T} \alpha_t$, which will be a useful equality.

**(c)** **[5 points]** Show that

$$\mathrm{er}_S[H] \;\leq\; \frac{1}{m} \sum_{i=1}^{m} e^{-F_{T,y_i}(x_i)} \;=\; \prod_{t=1}^{T} Z_t\,.$$

*Hint:* For the inequality, use the result of part (b) above, and the fact that $\mathbf{1}(u < 0) \leq e^{-u}$; for the equality, use the result of part (a) above.

**(d)** **[5 points]** Show that for the given choice of $\alpha_t$, we have

$$Z_t \;=\; 2\sqrt{\mathrm{er}_t(1-\mathrm{er}_t)}\,.$$

**(e)** **[5 points]** Suppose $\mathrm{er}_t \le \frac{1}{2} - \gamma$ for all $t$ (where $0 < \gamma \le \frac{1}{2}$). Then show that

$$\mathrm{er}_S[H] \;\le\; e^{-2T\gamma^2}\,.$$

★ **SOLUTION:**

- part (a):

$$\begin{aligned}
D_{T+1}(i) &= \frac{D_T(i)\exp\{-\alpha_T \widetilde{h}_{T,y_i}(x_i)\}}{\sum_j^m D_T(j)\exp\{-\alpha_T \widetilde{h}_{T,y_i}(x_i)\}} \\
&= \frac{D_T(i)\exp\{-\alpha_T \widetilde{h}_{T,y_i}(x_i)\}}{Z_T}, \text{ by definition} \\
&= \frac{\exp\{-\alpha_T \widetilde{h}_{T,y_i}(x_i)\}}{Z_T} \times \frac{\exp\{-\alpha_{T-1} \widetilde{h}_{T-1,y_i}(x_i)\}}{Z_{T-1}} \times \\
&\cdots \times \frac{\exp\{-\alpha_2 \widetilde{h}_{2,y_i}(x_i)\}}{Z_2} \times \frac{\frac{1}{m}\exp\{-\alpha_1 \widetilde{h}_{1,y_i}(x_i)\}}{Z_1}, \text{ by expanding out the } D_t(i) \text{ 's} \\
&= \frac{\frac{1}{m}\exp\{-\sum_t^T \alpha_t \widetilde{h}_{t,y_i}(x_i)\}}{\prod_t^T Z_t} \\
&= \frac{\frac{1}{m}\exp\{-F_{T,y_i}(x_i)\}}{\prod_t^T Z_t}, \text{ by definition}
\end{aligned}$$

- part (b): Consider first the case when $H(x_i) = y_i$. Then $\mathbf{1}(H(x_i) \neq y_i) = 0$, and the inequality trivially holds. Now consider the case $H(x_i) \neq y_i$. Then we claim that $F_{T,y_i}(x_i) \le 0$ (in which case both sides of the inequality will be 1, and the inequality will again hold). To see why this must be true, note that if $F_{T,y_i}(x_i) > 0$, then we must have have $F_{T,y_i}(x_i) > F_{T,k}(x_i)$ for all $k \neq y_i$ (otherwise we would get $\sum_{k=1}^K F_{T,k}(x_i) > -(K-2)\sum_{t=1}^T \alpha_t$); but this would give $H(x_i) = y_i$ and would contradict our assumption. Therefore we must have $F_{T,y_i}(x_i) \le 0$, and the inequality holds.

- part (c):

  Showing $er_s[H] \le \frac{1}{m}\sum_i^m \exp\{-F_{T,y_i}(x_i)\}$:

$$\begin{aligned}
er_s[H] &= \frac{1}{m}\sum_i^m \mathbf{1}(H(x_i) \neq y_i) \\
&\le \frac{1}{m}\sum_i^m \mathbf{1}(F_{T,y_i}(x_i) < 0), \text{ from part b} \\
&\le \frac{1}{m}\sum_i^m \exp\{-(F_{T,y_i}(x_i)\}, \text{ using the given fact that } \mathbf{1}(u < 0) \le e^{-u}
\end{aligned}$$

  Showing $\frac{1}{m}\sum_i^m \exp\{-(F_{T,y_i}(x_i)\} = \prod_t^T Z_t$:

$$\begin{aligned}
\frac{1}{m}\sum_{i}^{m}\exp\{-(F_{T,y_i}(x_i)\} &= \sum_{i}^{m}\prod_{t}^{T} Z_t D_{T+1}(i), \text{ by the definition of } D_{T+1}(i) \text{ shown in part a} \\
&= \prod_{t}^{T} Z_t \sum_{i}^{m} D_{T+1}(i), \text{ since the product does not depend on } i \\
&= \prod_{t}^{T} Z_t, \text{ since } D(i)\text{'s are softmax weights constructed to sum to 1 over the examples}
\end{aligned}$$

- Part (d) Showing $Z_t = 2\sqrt{er_t(1-er_t)}$:

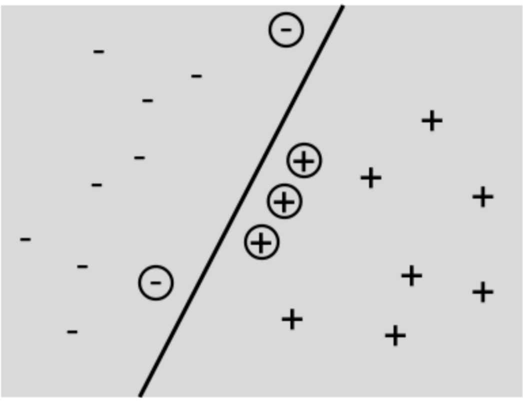
$$\begin{aligned}
Z_t &= \sum_{i}^{m} D_t(i)\exp\{-\alpha_t \widetilde{h}_{t,y_i}(x_i)\} \\
&= \sum_{i}^{m} D_t(i)\exp\{-\alpha_t(\mathbf{1}(h_t(x_i)=y_i)-\mathbf{1}(h_t(x_i)\neq y_i))\}, \text{ converting } \widetilde{h} \text{ to the two indicator function cases} \\
&= \sum_{i}^{m} D_t(i)\exp\{-\alpha_t\mathbf{1}(h_t(x_i)=y_i)\}\exp\{\alpha_t(\mathbf{1}(h_t(x_i)\neq y_i)\}, \text{ note that the indicators bring in either } e^{-\alpha_t} \text{ or } e^{\alpha_t} \\
&= \sum_{i}^{m} D_t(i)\exp\{-\alpha_t\}\mathbf{1}(h_t(x_i)=y_i) + \sum_{i}^{m} D_t(i)\exp\{\alpha_t\}\mathbf{1}(h_t(x_i)\neq y_i) \\
&\quad \text{since the indicators are disjoint and are } \{0,1\}, \text{ making this term equivalent to the single summation} \\
&= \sum_{i}^{m} D_t(i)(1-\mathbf{1}(h_t(x_i)\neq y_i))\exp\{-\alpha_t\} + \sum_{i}^{m} D_t(i)\mathbf{1}(h_t(x_i)\neq y_i)\exp\{\alpha_t\} \\
&= e^{-\alpha_t}\sum_{i}^{m} D_t(i)(1-\mathbf{1}(h_t(x_i)\neq y_i)) + e^{\alpha_t}\sum_{i}^{m} D_t(i)\mathbf{1}(h_t(x_i)\neq y_i) \\
&= e^{-\alpha_t}[\sum_{i}^{m} D_t(i) - \sum_{i}^{m} D_t(i)\mathbf{1}(h_t(x_i)\neq y_i))] + e^{\alpha_t}er_t, \text{ by definition of } er_t \\
&= e^{-\alpha_t}(1-er_t) + e^{\alpha_t}er_t, \text{ by definition of } er_t \\
&= (1-er_t)\sqrt{er_t/(1-er_t)} + er_t\sqrt{(1-er_t)/er_t}, \text{ by definition of } \alpha_t \\
&= \sqrt{er_t(1-er_t)} + \sqrt{(1-er_t)er_t} \\
&= 2\sqrt{er_t(1-er_t)}
\end{aligned}$$

- Part (e) Showing $er_s[H] \le e^{-2T\gamma^2}$:

$$
\begin{aligned}
er_s[H] &= \frac{1}{m}\sum_i^m \mathbf{1}(H(x_i) \neq y_i) \\
&\le \prod_t^T Z_t, \text{ from part c} \\
&\le \prod_t^T 2\sqrt{er_t(1-er_t)}, \text{ from part d} \\
&\le \prod_t^T 2\sqrt{(\frac{1}{2}-\gamma)(\frac{1}{2}+\gamma)}, \text{ from the given upper bound of } er_t \\
&\le \prod_t^T 2\sqrt{\frac{1}{4}-\gamma^2} \\
&= \prod_t^T \sqrt{1-4\gamma^2} \\
&\le \prod_t^T \sqrt{e^{-4\gamma^2}}, \text{ since } 1-x \le e^{-x} \\
&= \prod_t^T e^{-2\gamma^2} = e^{-2\gamma^2 T}
\end{aligned}
$$

## 1.2 Support Vector Machine

**[5 points]** Consider a binary classification problem in a 2-dimensional instance space $\mathcal{X} = \mathbb{R}^2$. You are given a linearly separable training set containing 10 positive and 10 negative training examples. You run the hard-margin SVM algorithm and obtain the separating hyperplane below (support vectors are circled):



What is the largest number of data points that can be removed from the training set without changing the hard margin SVM solution? Explain your solution.

★ **SOLUTION:** 17: You need to keep three support vectors to specify the hyperplane., E.g. both the negative support vectors and any one of the positive ones. (Try this and see if you can rotate the hyperplane at all.)

| | |
|---|---|
| Stated Points: | 30 |
| Section Points: | 30 |
| Total Points So Far: | 30 |
| CORRECT TOTAL | |

# 2 Programming [20 points]

For this question, refer the Jupyter Notebook. You will be using functions from sklearn and call them in the notebook. We will be using the MNIST digits dataset for a classification task. Familiarize yourself with the dataset on the sklearn website, here.

## 2.1 Random Forests

**[3 points]**

Tabulate the prediction results on the test set in Table 1.

| **n_estimators** | **Accuracy (%)** |
|---|---|
| 1 | — |
| 5 | — |
| 10 | — |
| 50 | — |
| 100 | — |
| 500 | — |

Table 1: Accuracy for the Random Forests classification problem on the test set

## 2.2 Kernel SVM

**[3 points]**

Tabulate the prediction results on the test set in Table 2.

| **kernel** | **Accuracy (%)** |
|---|---|
| Linear | — |
| Poly | — |
| RBF | — |

Table 2: Accuracy for the kernel SVM classification problem on the test set

## 2.3 Multi Layer Perceptron

**[3 points]**

Tabulate the prediction results on the test set in Table 3.

| Network Architecture | Accuracy (%) |
|---|---|
| (3) | — |
| (10) | — |
| (10,10,10) | — |
| (20,50,20) | — |

Table 3: Accuracy for the MLP classification problem on the test set

## 2.4 AdaBoost

**[3 points]**

Tabulate the prediction results on the test set in Table 4.

| n_estimators | Accuracy (%) |
|---|---|
| 1 | — |
| 5 | — |
| 10 | — |
| 50 | — |
| 100 | — |
| 150 | — |

Table 4: Accuracy for the AdaBoost classification problem on the test set

## 2.5 Short Answer

**[8 points]** Write a short paragraph describing your results for each of the four models with regards to performance trends and explain the influence of hyperparameters on these results.

★ **SOLUTION:**

1. **Random Forest**

| n_estimators | Accuracy (%) |
|---|---|
| 1 | 73.74 |
| 5 | 92.59 |
| 10 | 95.29 |
| 50 | 96.63 |
| 100 | 97.64 |
| 500 | 97.64 |

Table 5: Accuracy for the Random Forests classification problem on the test set

2. **kernel**

| kernel | Accuracy (%) |
|---|---|
| Linear | 98.32 |
| Poly | 99.33 |
| RBF | 98.65 |

Table 6: Accuracy for the kernel SVM classification problem on the test set

3. **MLP**

| Network Architecture | Accuracy (%) |
|---|---|
| (3) | 85.86 |
| (10) | 94.61 |
| (10,10,10) | 92.93 |
| (20,50,20) | 96.30 |

Table 7: Accuracy for the MLP classification problem on the test set

4. **AdaBoost**

| n_estimators | Accuracy (%) |
|---|---|
| 1 | 43.77 |
| 5 | 82.49 |
| 10 | 83.16 |
| 50 | 89.90 |
| 100 | 92.93 |
| 150 | 92.59 |

Table 8: Accuracy for the AdaBoost classification problem on the test set

5. It is an open question. As long as the answer is reasonable.

   Possible answers should help address some of the following points:

   - How does the number of tree influence the performance? (Possible Answer: As the number of trees $n$ increases, the accuracy increases)
   - What are the differences between different kernels?
   - How did you choose the architecture for your network? (Possible Answer: Sometimes more layers in the network may not be needed)
   - Are there other hyperparameters that can influence model performance? (Possible Answer: Max depth, learning rate for AdaBoost model)

| | |
|---|---|
| Stated Points: | 20 |
| Section Points: | 20 |
| Total Points So Far: | 50 |
| CORRECT TOTAL | |