Lecture 12 Wavelet Transform

Yuyao Zhang PhD
zhangyy8@shanghaitech.edu.cn
SIST Building 2 302-F

上海科技大学
ShanghaiTech University

Outline

– Discrete Wavelet Transform (DWT)（小波变换）

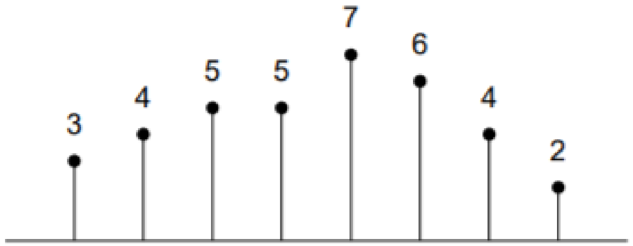
- An example for 1D-DWT
- generalization of 1D-DWT
- 2D-DWT

Discrete Wavelet Transform (DWT)

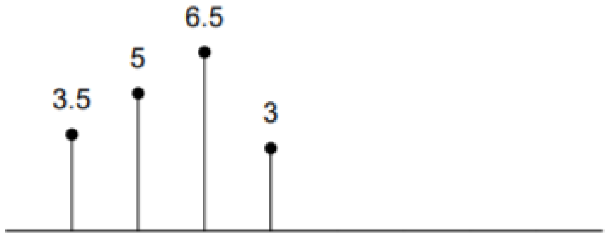
- Based on small waves called Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- Two types of DWT one for image processing （easy invertible) and one for signal processing (invertible but computational expensive).

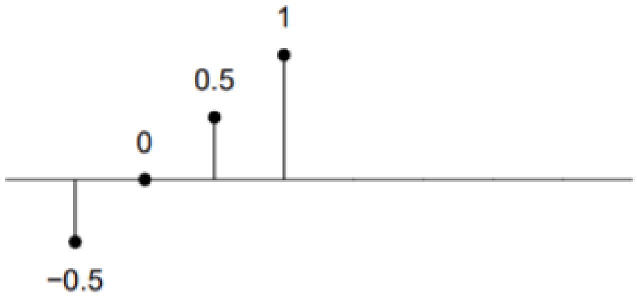
A simplest example

- We can decompose an eight-point signal x(n):



into two four-point signals:





$$c(n) = 0.5\,x(2n) + 0.5\,x(2n+1) \qquad d(n) = 0.5\,x(2n) - 0.5\,x(2n+1)$$

A simplest example

- The above process can be represented by a block diagram:

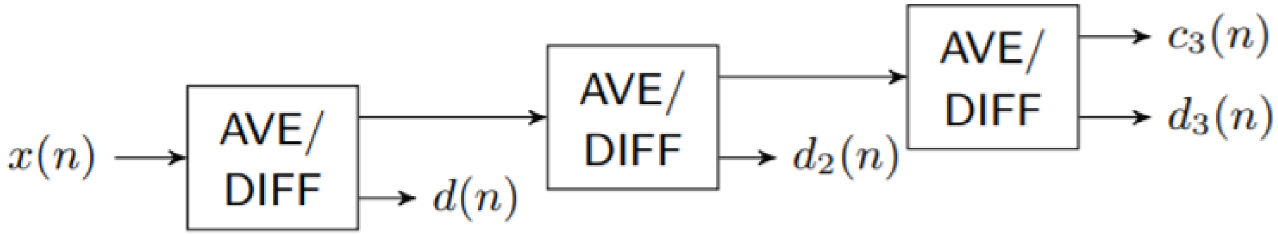
It is clear that this decomposition can be easily reversed:

$$y(2n) = c(n) + d(n)$$
$$y(2n+1) = c(n) - d(n)$$

Which is also represented by a block diagram:

A simplest example

- When we repeat the simple AVE/DIFF signal decomposition:



The Haar wavelet representation of the eight-point signal x[n] is simply the set of four output signals produced by this three-level operation :

$$
\begin{aligned}
c_3 &= [4.5] \\
d_3 &= [-0.25] \\
d_2 &= [-0.75, \quad 1.75] \\
d &= [-0.5, \quad 0, \quad 0.5, \quad 1]
\end{aligned}
$$

$$
\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2
\end{bmatrix}
$$

Haar Transform matrix

- When N=2 we have: $$\mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

- When N=4 we have: $$\mathbf{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}$$

- When N=8 we have $$\mathbf{H}_8 = \frac{1}{\sqrt{8}}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Haar Transform matrix

- The family of N Haar functions $\boldsymbol{h_k(t)}, (\boldsymbol{k = 0, \ldots, N-1})$ are defined on the interval $\mathbf{0 \leq t \leq 1}$. The shape of the specific function $\boldsymbol{h_k(t)}$ of a given index $\boldsymbol{k}$ depends on two parameters $\boldsymbol{p}$ and $\boldsymbol{q}$:

$$\boldsymbol{k = 2^p + q - 1}$$

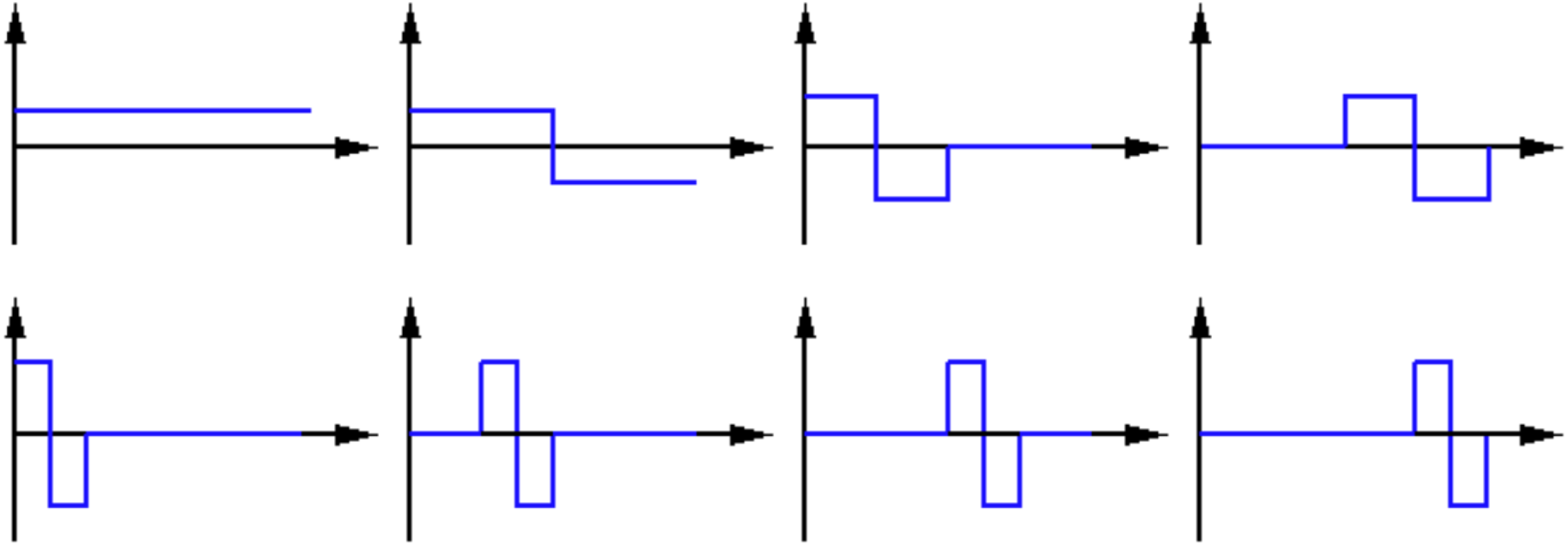
k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p	0	0	1	1	2	2	2	2	3	3	3	3	3	3	3	3
q	0	1	1	2	1	2	3	4	1	2	3	4	5	6	7	8

- When $\boldsymbol{k > 0}$, the Haar function is defined by:

$$h_k(t) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & (q-1)/2^p \leq t < (q-0.5)/2^p \\ -2^{p/2} & (q-0.5)/2^p \leq t < q/2^p \\ 0 & \text{otherwise} \end{cases}$$

Haar Transform matrix

$$\mathbf{H}_8 = \frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$



上海科技大学
ShanghaiTech University

Generalization of 1D-DWT

- Discrete Wavelet Transform (DWT):

$$W_\varphi(j_0,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\,\varphi_{j_0,k}(n)$$

$$W_\psi(j,k) = \frac{1}{\sqrt{M}}\sum_n f(n)\,\psi_{j,k}(n) \qquad j \geq j_0$$

- Inverse Discrete Wavelet Transform (IDWT):

$$f(n) = \frac{1}{\sqrt{M}}\sum_k W_\varphi(j_0,k)\,\varphi_{j_0,k}(n) + \frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\,\psi_{j,k}(n)$$

Where

$\varphi_{j_0,k}(n)$: scaling function (尺度函数) $\psi_{j,k}(n)$: Wavelet (小波)

$W_\varphi(j_0,k)$: Approximation coefficients (近似系数) $W_\psi(j,k)$: detail coefficients (细节系数)

上海科技大学 ShanghaiTech University

2D-DWT

- Define 2D wavelet function: Directionally sensitive wavelet

$$\psi^H(x,y) = \psi(x)\varphi(y) \qquad \psi^V(x,y) = \varphi(x)\psi(y) \qquad \psi^D(x,y) = \psi(x)\psi(y)$$

- 2D-DWT

$$W_\varphi(j_0,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\varphi_{j_0,m,n}(x,y)$$

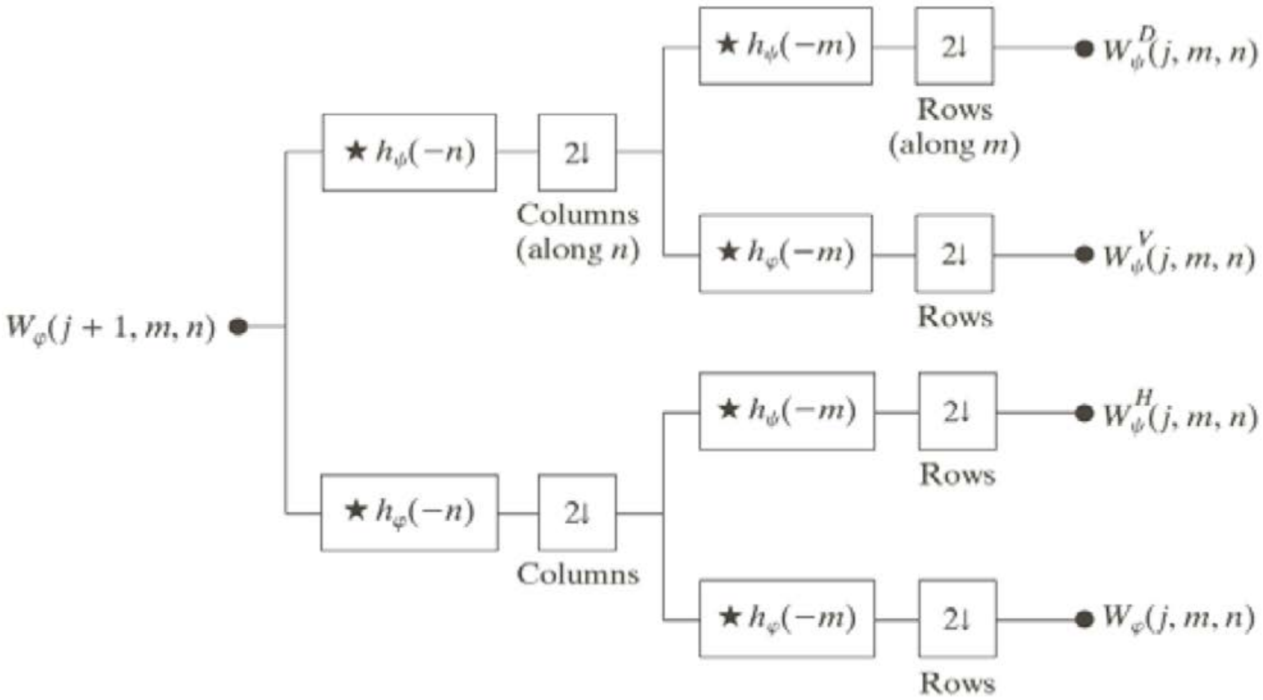
$$W_\psi(j,m,n) = \frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\psi^i_{j,m,n}(x,y) \qquad i = \{H,V,D\}$$

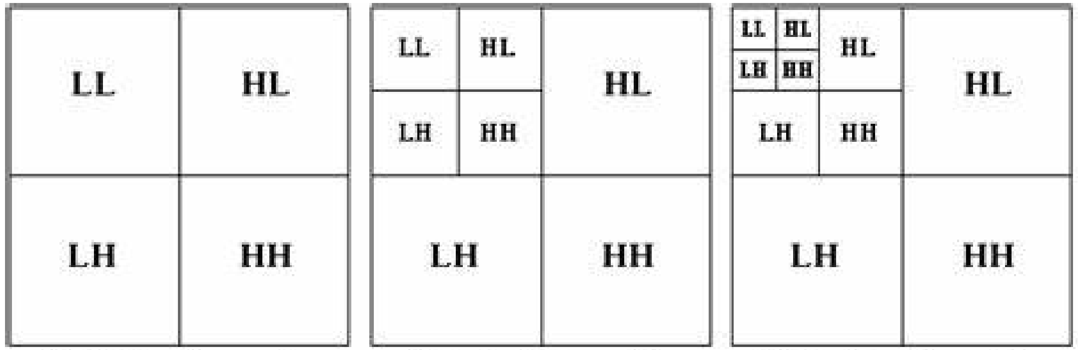
- 2D-IDWT

$$f(x,y) = \frac{1}{\sqrt{MN}}\sum_{m}\sum_{n} W_\varphi(j_0,m,n)\,\varphi_{j_0,m,n}(x,y)$$

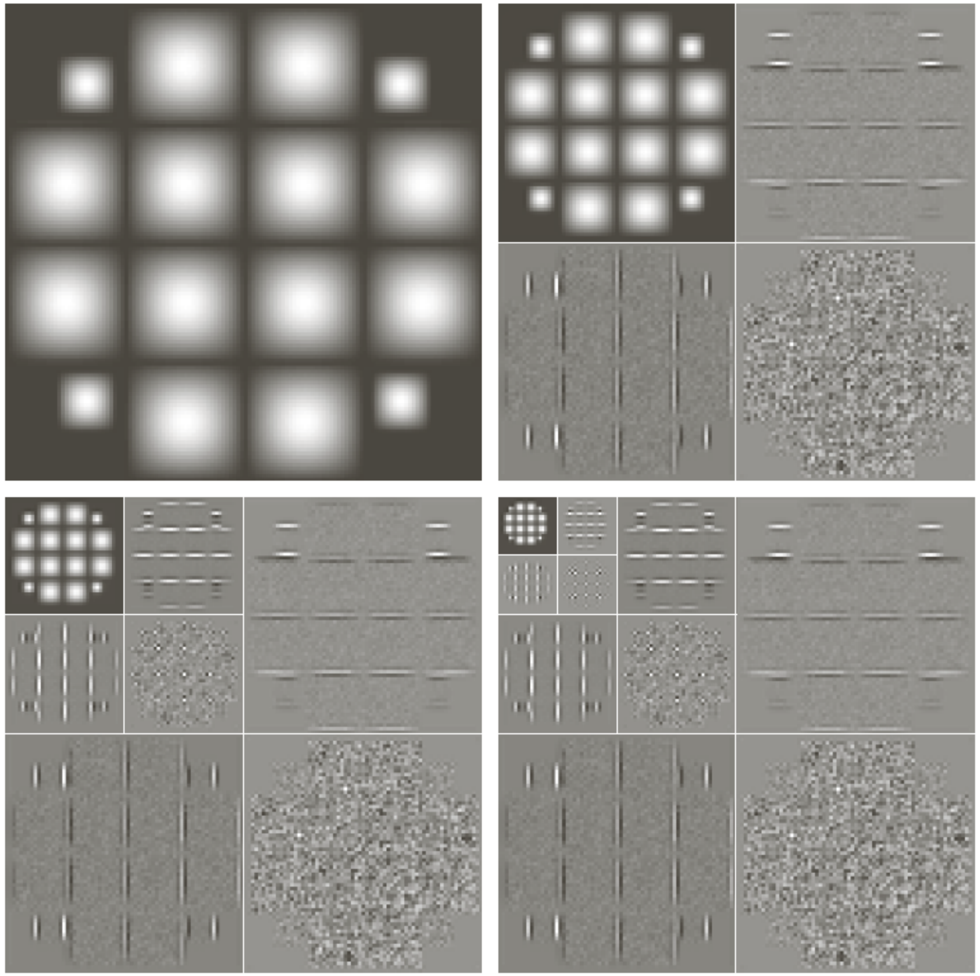
$$+\frac{1}{\sqrt{MN}}\sum_{i=\{H,V,D\}}\sum_{j=j_0}^{\infty}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} W_\psi(j,m,n)\,\psi^i_{j,m,n}(x,y)$$

2D-DWT





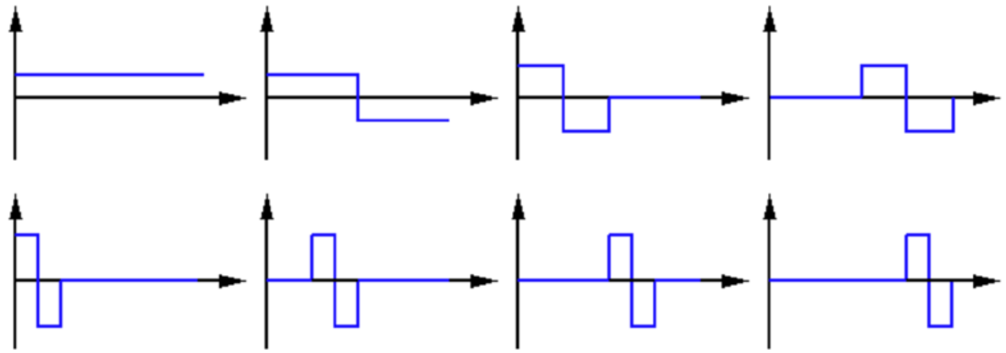
(a) Single Level Decomposition (b) Two Level Decomposition (c) Three Level Decomposition



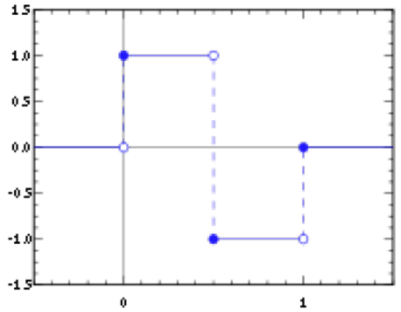
上海科技大学 ShanghaiTech University

Mother Wavelet (母小波)

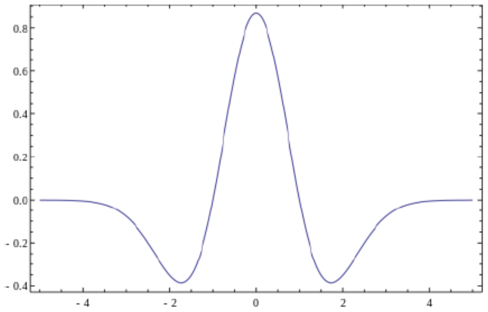
- Mother Wavelet should satisfy:
  - $\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$
  - $\int_{-\infty}^{\infty} |\psi(t)| dt < \infty$
  - $\int_{-\infty}^{\infty} \psi(t) dt = 0$



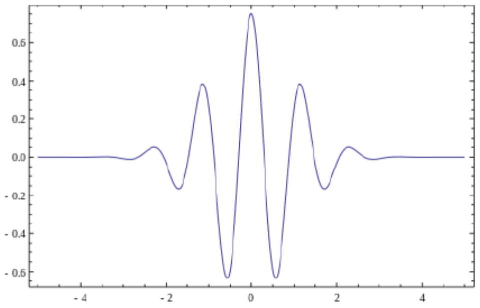
Haar



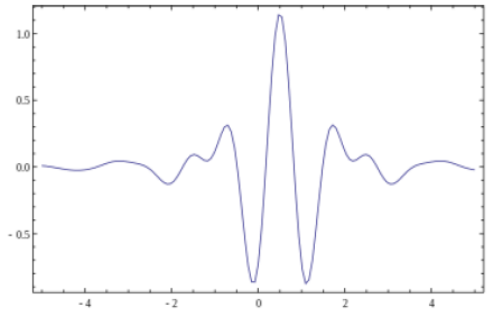
Mexican Hat



Morlet



Meyer



Take home message

- Based on small waves called Wavelets-1) limited; 2) oscillation.
- Key idea: Translation & Scaling.
- Localized both time/space and frequency.
- Efficient for noise reduction and image compression.
- JPEG2000, FBI finger printing databased.