Markov Chains

SI252 Reinforcement Learning

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Outline

1. Discrete-Time Markov Chain
2. Continuous-Time Markov Chain

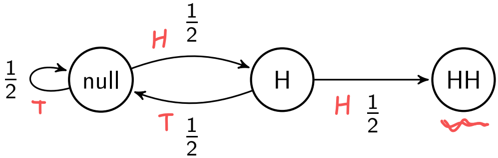
First-Step Analysis

- A systematic technique in Markov chain analysis
- Main applications are the computation of:
  - Hitting probabilities
  - Mean hitting and absorption times
  - Mean first return times
  - Average number of returns to a given state

Toss A Coin till HH Appears

fair coin

$e_s = E[\text{waiting time for HH} \mid \text{intial state} = s]$

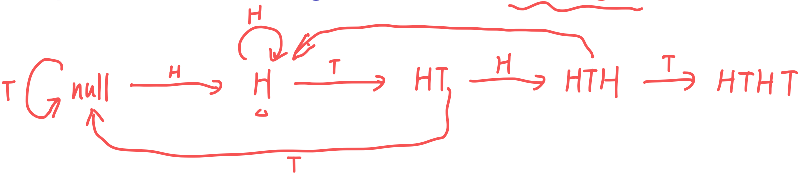


$$\begin{cases} e_{null} = \frac{1}{2}(1 + e_H) + \frac{1}{2}(1 + e_{null}) \\ e_H = \frac{1}{2}(1 + e_{HH}) + \frac{1}{2}(1 + e_{null}) \\ e_{HH} = 0 \end{cases}$$

$\Rightarrow e_H = 4$

$e_{null} = 6$

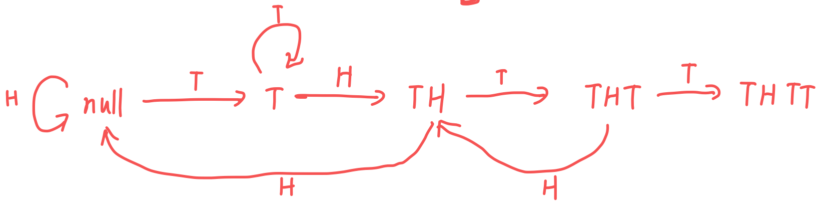
Expected Waiting Time for HTHT



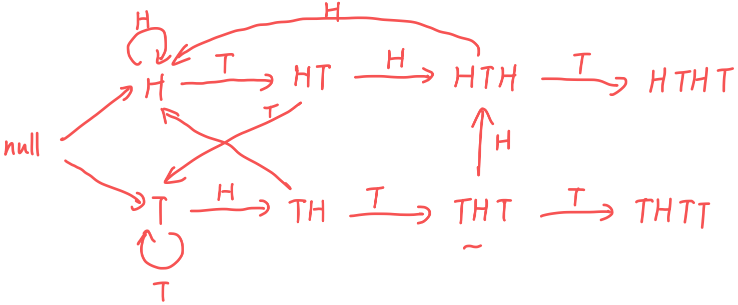
$$\begin{cases} e_{null} = \frac{1}{2}(e_H + 1) + \frac{1}{2}(e_{null} + 1) \\ e_H = \frac{1}{2}(e_{HT} + 1) + \frac{1}{2}(e_H + 1) \\ e_{HT} = \frac{1}{2}(e_{HTH} + 1) + \frac{1}{2}(e_{null} + 1) \\ e_{HTH} = \frac{1}{2}(e_{HTHT} + 1) + \frac{1}{2}(e_H + 1) \\ e_{HTHT} = 0 \end{cases} \Rightarrow e_{null} = 20$$

Expected Waiting Time for THTT

THTH



Expected Waiting Time for Either HTHT or THTT



$$e_{null} = \frac{1}{2}(e_H + 1) + \frac{1}{2}(e_T + 1)$$

$$e_H = \frac{1}{2}(e_H + 1) + \frac{1}{2}(e_{HT} + 1)$$

$$e_T = \frac{1}{2}(e_T + 1) + \frac{1}{2}(e_{TH} + 1)$$

$$e_{HT} = \frac{1}{2}(e_{HTH} + 1) + \frac{1}{2}(e_T + 1)$$

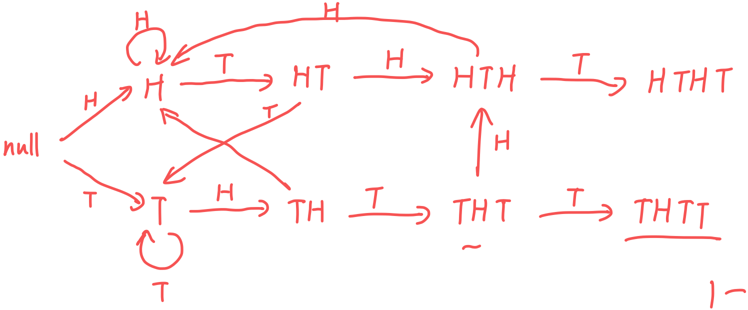
$$e_{TH} = \frac{1}{2}(e_{THT} + 1) + \frac{1}{2}(e_{HTH} + 1)$$

$\cdots$

$$e_{HTHT} = e_{THTT} = 0$$

$$\Rightarrow e_{null} = ?$$

Which Pattern (THTT or HTHT) is More Likely to Occur First?



$1 - \frac{9}{14} = \frac{5}{14}$

$P_s = Pr(\text{HTHT eventually occurs} \mid \text{initial state} = s)$

$$\begin{cases} P_{null} = P_H \cdot \frac{1}{2} + P_T \cdot \frac{1}{2} \\ P_H = P_{HT} \cdot \frac{1}{2} + P_H \cdot \frac{1}{2} \\ P_T = P_{TH} \cdot \frac{1}{2} + P_T \cdot \frac{1}{2} \\ \cdots \end{cases}$$

$P_{HTHT} = 1$

$P_{THTT} = 0$

$\Rightarrow P_{null} = \frac{9}{14} > \frac{5}{14}$

HTHT ✓

Recurrent and Transient States

Let $f_{j,j}$ denote the probability that the Markov chain starting at state j eventually enters state j.

Definition (Recurrent and Transient States)

State j is said to be recurrent if the Markov chain started in j eventually revisits j. That is, $f_{j,j} = 1$.
State j is said to be transient if there is positive probability that the Markov chain started in j never returns to j. That is, $f_{j,j} < 1$.

Example

$\{a, b\}$: recurrent

$\{c\}$: transiant

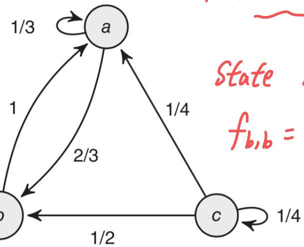
$f_{j,j}$

$$f_{c,c} = P(T_c < \infty \mid X_0 = c)$$
$$= P(X_1 = c \mid X_0 = c)$$
$$= \frac{1}{4} < 1$$

State a: $\{X_0, X_1, \dots\}$

$$T_j \triangleq \min\{n > 0 : X_n = j\}$$

$$f_{a,a} = P(\exists\, n \in (0, \infty) \text{ s.t. } X_n = a \mid X_0 = a)$$
$$= P(\underline{T_a < \infty \mid X_0 = a}) = 1$$



State b:

$$f_{b,b} = P(T_b < \infty \mid X_0 = b)$$
$$= 1 - P(\underline{T_b = \infty \mid X_0 = b})$$
$$= 1 - 0 = 1$$

$$P(T_b = n \mid X_0 = b)$$
$$= P(X_n = b, \underline{X_{n-1} = a, X_{n-2} = a \cdots X_1 = a} \mid X_0 = b)$$
$$= (P_{aa})^{n-2} P_{ab} = \left(\frac{1}{3}\right)^{n-2} \frac{2}{3}$$
$$\lim_{n \to \infty} P(T_b = n \mid X_0 = b) = 0$$

Expected Number of Visits

$I_n = I\{X_n = j \mid X_0 = j\} \ \forall n \geq 0$

$\sum_{n=0}^{\infty} I_n$: number of visits to j

1. State j is recurrent if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n = \infty.$$

$E\left(\sum_{n=0}^{\infty} I_n\right) = \sum_{n=0}^{\infty} E(I_n)$

$= \sum_{n=0}^{\infty} P(X_n = j \mid X_0 = j)$

$= \boxed{\sum_{n=0}^{\infty} P_{jj}^n} = \infty$

2. State j is transient if and only if

$$\sum_{n=0}^{\infty} P_{jj}^n < \infty.$$

$\sum_{n=0}^{\infty} P_{jj}^n \sim \text{Geom}(1 - f_{jj})$

$E\left(\sum_{n=0}^{\infty} P_{jj}^n\right) = \frac{1}{1 - \underbrace{f_{jj}}_{<1}} < \infty$

Example: Simple Random Walk

$\sum_{n=0}^{\infty} P_{00}^{n}$ $\quad P_{00}^{n} = 0$ if n is odd

$P_{00}^{2n} = P(n \text{ steps to left}, n \text{ steps to right}) = \binom{2n}{n} p^n (1-p)^n$

Example (Simple Random Walk)

A random walk on the integer line starts at 0 and moves left, with probability p, or right, with probability $1 - p$. For $0 < p < 1$, the process is an irreducible Markov chain, as every state is accessible from every other state. Is the chain recurrent or transient?

Example: Simple Random Walk (Solution)

Method 1:

Stirling's approximation. $n! \approx n^n e^{-n}\sqrt{2\pi n}$ for large n

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1$$

$$\binom{2n}{n} = \frac{(2n)!}{n!\cdot n!} \approx \frac{(2n)^{2n} e^{-2n}\sqrt{2\pi\cdot 2n}}{(n^n e^{-n}\sqrt{2\pi n})^2} = \frac{4^n}{\sqrt{\pi n}}$$

$$\sum_{n=0}^{\infty} P_{00}^{2n} = \sum_{n=0}^{\infty} \binom{2n}{n} p^n (1-p)^n \approx \sum_{n=1}^{\infty} \frac{4p(1-p)^n}{\sqrt{\pi n}}$$

$$p=\frac{1}{2}: \quad \sum_{n=0}^{\infty} P_{00}^{2n} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$$

$$p\neq\frac{1}{2}: \quad \sum_{n=0}^{\infty} P_{00}^{2n} \approx \sum_{n=1}^{\infty} \frac{\epsilon^n}{\sqrt{\pi n}} < \infty$$

$$\epsilon = 4p(1-p)$$

Example: Simple Random Walk (Solution)

Method 2:

Lemma: $\forall\, n \geq 1$, $\dfrac{4^n}{2n+1} < \binom{2n}{n} < 4^n$

Proof: 1° $\displaystyle\sum_{k=0}^{2n} \binom{2n}{k} = (1+1)^{2n} = 4^n \Rightarrow \binom{2n}{n} < 4^n$

2° $\displaystyle\max_{0 \leq k \leq 2n} \binom{2n}{k} = \binom{2n}{n}$

$$4^n = \sum_{k=0}^{2n} \binom{2n}{k} \leq \sum_{k=0}^{2n} \binom{2n}{n} = (2n+1) \cdot \binom{2n}{n}$$

$$\Rightarrow \quad \binom{2n}{n} < 4^n < (2n+1)\binom{2n}{n}$$

$$\Rightarrow \quad \frac{4^n}{2n+1} < \binom{2n}{n} < 4^n$$

Example: Simple Random Walk (Solution)

By the lemma and $\sum_{n=0}^{\infty} P_{00}^{2n} = \sum_{n=0}^{\infty} \underbrace{\binom{2n}{n}} p^n (1-p)^n$

$$\underbrace{\sum_{n=1}^{\infty} \frac{4^n}{2n+1} p^n (1-p)^n}_{\Delta} < \sum_{n=0}^{\infty} P_{00}^{2n} < \underbrace{\sum_{n=0}^{\infty} \boxed{4^n p^n (1-p)^n}} \quad (\underbrace{4p(1-p)}_{\Delta \mu})^n$$

If $p = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{4^n}{2n+1} p^n (1-p)^n = \sum_{n=1}^{\infty} \frac{1}{2n+1} = \infty$

$$\sum_{n=0}^{\infty} 4^n p^n (1-p)^n = \sum_{n=1}^{\infty} 1 = \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} P_{00}^{2n} = \infty$$

If $p \neq \frac{1}{2}$: $4p(1-p) < (p+1-p)^2 = 1$ $\quad \mu < 1$

$$\sum_{n=0}^{\infty} \mu^n < \infty$$

Example: Simple Random Walk (Solution)

Limiting Distribution

Definition (Limiting Distribution)

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$. A limiting distribution for the Markov chain is a probability distribution $\boldsymbol{\lambda}$ with the property that for all i and j,

$$\lim_{n\to\infty} P_{ij}^n = \lambda_j. \quad \forall i$$

Equivalent Definitions

The definition of limiting distribution is equivalent to each of the following:

1. For any initial distribution, and for all j,

$$\lim_{n\to\infty} P\left(X_n = j\right) = \lambda_j.$$

2. For any initial distribution $\boldsymbol{\alpha}$,

$$\lim_{n\to\infty} \boldsymbol{\alpha}\boldsymbol{P}^n = \boldsymbol{\lambda}.$$

3. $$\lim_{n\to\infty} \boldsymbol{P}^n = \boldsymbol{\Lambda},$$

$$\Lambda = \begin{bmatrix} \lambda \\ \lambda \\ \vdots \\ \lambda \end{bmatrix}$$

where $\boldsymbol{\Lambda}$ is a stochastic matrix all of whose rows are equal to $\boldsymbol{\lambda}$.

Example: Two-State Markov Chain

$\lim_{n \to \infty} P^n$

Example (Two-State Markov Chain)

The transition matrix for a general two-state chain is

$$\boldsymbol{P} = \begin{array}{c c} & \begin{array}{cc} 1 & \quad 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

for $0 < p, q < 1$. Find its limiting distribution.

Example: Two-State Markov Chain (Solution)

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$$

① If $p+q=1$:

$$P = \begin{pmatrix} 1-p & p \\ 1-p & p \end{pmatrix} \qquad P^n = P \qquad \lambda = (1-p,\ p)$$

② If $p+q \neq 1$:

$$P = A \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} A^{-1} \qquad \lambda_1 = 1 \qquad \lambda_2 = 1-p-q$$

$$\Rightarrow P^n = A \begin{pmatrix} \lambda_1^n & \\ & \lambda_2^n \end{pmatrix} A^{-1}$$

$$= \frac{1}{p+q} \begin{pmatrix} q + p\,\underline{(1-p-q)^n} & p - p\,\underline{(1-p-q)^n} \\ q - q\,\underline{(1-p-q)^n} & p + q\,\underline{(1-p-q)^n} \end{pmatrix}$$

$$0 < p, q < 1 \Rightarrow |1-p-q| < 1 \Rightarrow \lim_{n\to\infty} P^n = \frac{1}{p+q} \begin{pmatrix} q & p \\ q & p \end{pmatrix}$$

$$\Rightarrow \lambda = \left(\frac{q}{p+q},\ \frac{p}{p+q} \right)$$

Proportion of Time in Each State

$\{X_0, X_1, \cdots\}$ transition matrix P limiting distribution λ

For state j, define $I_k \triangleq I\{X_k = j\}$ $\forall k \in \{0, 1, 2, \cdots\}$

$\sum_{k=0}^{n-1} I_k$

Two perspectives of Limiting Distribution:

- Long-term probability that a Markov chain hits each state
- Long-term proportion of time that the chain visits each state

$$\lim_{n\to\infty} E\left(\frac{1}{n}\sum_{k=0}^{n-1} I_k \,\Big|\, X_0 = i\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} E(I_k \mid X_0 = i)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P(X_k = j \mid X_0 = i)$$

$$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_{ij}^{k}$$

Cesaro's lemma

$$= \lim_{n\to\infty} P_{ij}^{n} = \lambda_j.$$

Cesaro's lemma:

$\{x_n\}$ if $x_n \to x$ as $n \to \infty$

then $(x_1 + \cdots + x_n)/n \to x$ as $n \to \infty$

Proportion of Time in Each State (Proof)

Definition (Stationary Distribution)

Let $X_0, X_1, \ldots$ be a Markov chain with transition matrix $\boldsymbol{P}$. A stationary distribution is a probability distribution $\boldsymbol{\pi}$, which satisfies

$$\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}.$$

That is

$$\pi_j = \sum_i \pi_i P_{ij}, \text{ for all } j.$$

Limiting Distributions are Stationary Distributions

Lemma (Limiting Distributions are Stationary Distributions)

Assume that $\boldsymbol{\pi}$ is the limiting distribution of a Markov chain with transition matrix $\boldsymbol{P}$. Then, $\boldsymbol{\pi}$ is a stationary distribution.

$\forall$ initial distribution α,

$$\pi = \lim_{n\to\infty} \alpha P^n = \lim_{n\to\infty} \alpha (P^{n-1} \cdot P) = \left(\lim_{n\to\infty} \alpha P^{n-1}\right) \cdot P$$

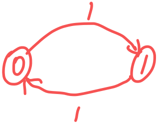
$$= \left(\lim_{n\to\infty} \alpha P^n\right) \cdot P$$

$$= \pi \cdot P$$

$$\Rightarrow \pi = \pi P$$

The Converse is NOT True.

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$



$$(\pi_1 \ \ \pi_2)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\pi_1, \pi_2) \Rightarrow \pi_1 = \pi_2 = \frac{1}{2}$$

$$\pi = \left(\frac{1}{2} \ \ \frac{1}{2}\right)$$

Check the Detailed Balance Equation

Theorem (Check the Detailed Balance Equation)

If for an irreducible Markov chain with transition matrix $\boldsymbol{P} = (P_{ij})$*, there exists a probability solution* $\boldsymbol{\pi}$ *to the detailed balance equations*

$$\pi_i P_{ij} = \pi_j P_{ji}$$

for all pairs of states i, j*, then this Markov chain is positive recurrent, time-reversible and the solution* $\boldsymbol{\pi}$ *is the unique stationary distribution.*

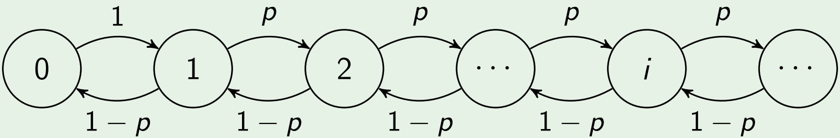
Example: Simple Random Walk

$1 \cdot \pi_0 = (1-p)\pi_1$

$p\,\pi_1 = (1-p)\pi_2$ $\quad \cdots\cdots \quad \pi_i\, p = \pi_{i+1}(1-p) \;\; \forall\, i \geq 1$

Example (Simple Random Walk)

Consider a negative drift simple random walk, restricted to be non-negative, in which $q_{01} = 1$, $q_{i,i+1} = p < 0.5$, $q_{i,i-1} = 1 - p$ for $i \geq 1$. Corresponding state diagram is:



$\Rightarrow \quad \pi_n = \frac{\pi_0}{1-p}\left(\frac{p}{1-p}\right)^{n-1} \qquad \sum_{n=0}^{\infty} \pi_n = 1 \qquad \pi_0 = \frac{1-2p}{2(1-p)}$

$\pi_n = \left(\frac{1}{2} - p\right)\left(\frac{p}{1-p}\right)^{n-1} \;\; \forall\, n \geq 1$

Outline

Ergodicity of CTMC

- An irreducible CTMC with a finite state space is positive recurrent.
- For an irreducible CTMC with transition rate matrix $Q = \{q_{ij}\}$, if there exists a probability solution π to the detailed balance equation

$$\pi_i q_{i,j} = \pi_j q_{j,i}, \ \forall i, j \in S$$

  then this Markov chain is positive recurrent, time-reversible, and the solution π is the unique stationary distribution.

Example: M/M/1 Queue

$\pi_0 q_{01} = \pi_1 q_{10} \Rightarrow \pi_0 \lambda = \pi_1 \mu \Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0$

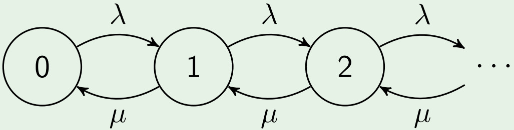
$\pi_1 q_{12} = \pi_2 q_{21} \Rightarrow \pi_1 \lambda = \pi_2 \mu \Rightarrow \pi_2 = \frac{\lambda}{\mu}\pi_1 = \left(\frac{\lambda}{\mu}\right)^2 \pi_0$

$\vdots$

Example

$\pi_{i+1} = \left(\frac{\lambda}{\mu}\right)^{i+1} \pi_0$

M/M/1 Queue Consider the following CTMC, find the stationary distribution $\pi = (\pi_0, \pi_1, \cdots)$ when $\lambda < \mu$.



$\frac{\lambda}{\mu} < 1 \quad \sum_{n=0}^{\infty} \pi_n = 1 \Rightarrow \pi_0 = 1 - \frac{\lambda}{\mu} \quad \pi_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

$\forall n > 0$