# Homework 7

ESE 402/542

Ungraded

**Problem 1.** Assume we have 1000 data points that are 50-dimensional, i.e. $\{x_i \in \mathbb{R}^{50}\}_{i=1}^{1000}$. Suppose the singular values of the data matrix are given by $\sigma_1 = 10, \sigma_2 = 9, \sigma_3 = 8, \ldots, \sigma_{10} = 1, \sigma_{11} = 0, \sigma_{12} = 0, \ldots$. We would like to find a low-dimensional representation of the data points in a way that at least 50% of the energy of the data is kept. What is the minimum value of the dimension that we can use for the low-dimensional representation of the data?

**Problem 2.** Let $\mathcal{H}$ be the class of functions of the form $h_{a,b} : \mathbb{R} \mapsto \{0, 1\}$, where

$$h_{a,b}(x) = \mathbb{1}_{\{x \in [a,b]\}}$$

for all $a, b \in \mathbb{R}$. Our goal will be to show that $\text{VCdim}(\mathcal{H}) = 2$.

(a) Let $C = \{c_1, \ldots, c_k\} \subset \mathbb{R}$. Recall that the restriction of $\mathcal{H}$ to $C$, denoted by $\mathcal{H}_C$, is the set of all binary $k$-tuples that can be derived from evaluating the functions in $\mathcal{H}$ on the set $C$. That is,

$$\mathcal{H}_C = \{(h(c_1), \ldots, h(c_k)) : h \in \mathcal{H}\}$$

Compute $\mathcal{H}_C$ for $C = \{1, 2\}$, and $C = \{1, 2, 3\}$.

(b) For any set $C$ with $|C| = 2$, show that $|\mathcal{H}_C| = 4$. Also, can we say that for any set $C$ with $|C| = 3$ we have $|\mathcal{H}_C| < 8$?

(c) Recall that a function class $\mathcal{H}$ shatters a set $C$ if $|\mathcal{H}_C| = 2^{|C|}$. Given the specific choice of $\mathcal{H}$ as above, does $\mathcal{H}$ shatter any set $C$ of size 2? Is there a set $C$ of size 3 that is shattered by $\mathcal{H}$?

(d) Recall that the VC dimension of $\mathcal{H}$ is the maximal size of a set $C$ that can be shattered by $\mathcal{H}$. What is the VC dimension of the class $\mathcal{H}$ considered in the question?

**Problem 1.** *Short answer:*

(a) In your own words, define what it means for a hypothesis class to be *PAC learnable.*

(b) Let $\mathcal{H}$ be the class of *all* functions from $\mathbb{R}$ to $\{0, 1\}$. What is the VC dimension of $\mathcal{H}$?

(c) For a finite function class $\mathcal{H}$ with $m$ functions $h_1, \ldots, h_m$, such that $h_i : \mathcal{X} \to \{0, 1\}$, explain why $\text{VCdim}(\mathcal{H}) \leq \log_2 m$. Are there cases where the VC dimension is exactly $\log_2 m$?