Homework 2

ESE 402/542

Due October 6, 2021 at 11:59pm

Type or scan your answers as a single PDF file and submit on Canvas.

Problem 1. Suppose that $X_1, X_2, ..., X_n$ are i.i.d. random variables in a sample with the density function

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left\{-\frac{|x|}{2\sigma}\right\}$$

(a) Use method of moments to estimate σ?

(b) Find the MLE estimate of σ?

(c) What is the asymptotic variance of the mle?

Problem 2. Given

$$I(\theta) = E\left[\frac{\partial}{\partial\theta}\log f(X|\theta)\right]^2$$

Under appropriate smoothness conditions, it can be proved that the probability distribution $\sqrt{nI(\theta_0)}\left(\hat{\theta} - \theta_0\right)$ tends to standard normal distribution.

(a) Show that $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(X|\theta)\right]$

(b) For the distribution in **Problem 1** find the confidence interval for the estimate $\hat{\sigma}$ (Hint: use standard normal property of $\sqrt{nI(\theta_0)}\left(\hat{\theta} - \theta_0\right)$ to find the confidence bounds)

(c) Suppose the distribution changes to a uniform distribution defined on on $[a,\ b]$ for $X_1, X_2, ..., X_n$ i.i.d random variables in a sample. Find the MLE estimate for parameters a and b.

Problem 3. Suppose $X_1, X_2, ..., X_n$ are i.i.d distributed in a sample with pdf

$$f(x|\theta) = \begin{cases} e^{-(x-\theta)}, & \text{if } x \geq \theta \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the method of moments estimate of θ.

(b) Find the mle of θ. (Hint: Be careful, and don't differentiate before thinking. For what values of is the likelihood positive?)

Problem 4. Suppose $X_1, X_2, ..., X_n \sim$ Poisson(λ). Given that the random variables are i.i.d, for $\theta = \exp(-\lambda)$,

1. Find an unbiased estimator of θ? (Note that it may not be the best estimator. Any unbiased estimator is fine).

2. Find the variance of the unbiased estimator you found and compare with the Cramer Rao lowerbound?

Problem 5. We have access to a file consisting of $n = 10^4$ numbers. The numbers are either 1, 2, or 3. Moreover, the value 1 appears $n_1 = 2600$ times in the file, the value 2 appears $n_2 = 5200$ times, and the value 3 appears $n_3 = 2200$ times. We know that these numbers are generated i.i.d. according to an unknown distribution.

(a) Let μ denote the mean of the distribution. Estimate the value of μ from sample data provided in the file and provide a 95% confidence interval.

(b) Assume now that the generating distribution of the data has the following form:

$$X = \begin{cases} 1, & \text{with probability } p_1, \\ 2, & \text{with probability } p_2, \\ 3, & \text{with probability } 1 - (p_1 + p_2). \end{cases}$$

We would like to estimate the value of the parameters p_1 and p_2. Consider the following estimator for the value of

$$p_1(X_1, \ldots, X_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{X_i = 1\},$$

where $\mathbb{1}\{A\}$ takes value 1 if A is true, and 0 otherwise (this is known as an indicator function for A).

Compute the estimate p_1 from the sample data provided in the file. Is this estimator an unbiased estimator for p_1? Justify your answer.

(c) Use the method of moments to estimate the value of p_1 and p_2 (you should compute the estimate from the sample data).

(d) Now, assume that the precise value of p_1 is given as $p_1 = \frac{1}{4}$. As a result, we now know that the distribution of the data has the form:

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{4}, \\ 2, & \text{with probability } p_2, \\ 3, & \text{with probability } \frac{3}{4} - p_2 \end{cases}$$

We would like to estimate the value of the parameter p_2 from data. Find the maximum likelihood estimator for p_2 and provide a 95% confidence interval.

Problem 6. Download data_HW2.csv and load it into Python. The numbers are observations drawn i.i.d. from an exponential distribution with unknown parameter λ. Include your code in your homework write up.

(a) Compute estimates for the sample mean and sample variance without using inbuilt functions. Compare your answers with inbuilt numpy functions

(b) Suppose now that the standard deviation is known to be 0.25. Compute a 90% confidence interval for the population mean.(python libraries can be used to calculate the confidence interval)