ESE 542 homework 2 , due Oct. 6

Problem 1.

a) $E[X] = \int_{-\infty}^{\infty} x f(x|\sigma)\, dx$

$= \int_{-\infty}^{\infty} x \cdot \frac{1}{2\sigma} \exp\{-\frac{|x|}{\sigma}\} dx$

$= 0$ does not contain σ

$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x|\sigma) dx$

$= \int_{-\infty}^{\infty} x^2 \frac{1}{2\sigma} \exp\{-\frac{|x|}{\sigma}\} dx$

$= \int_0^{\infty} x^2 \frac{1}{\sigma} \exp\{-\frac{x}{\sigma}\} dx$ let $x = \sigma t$

$= \sigma^2 \int_0^{\infty} t^2 e^{-t} dt$

$= -\sigma^2 \int_0^{\infty} t^2 d e^{-t}$

$= -\sigma^2 t^2 e^{-t}\Big|_0^{+\infty} + \sigma^2 \int_0^{+\infty} 2t e^{-t} dt$

$= 0 - 2\sigma^2 (\int_0^{+\infty} t\, d e^{-t})$

$= -2\sigma^2 (t e^{-t}\Big|_0^{+\infty} - \int_0^{+\infty} e^{-t} dt)$

$= 2\sigma^2 \int_0^{+\infty} e^{-t} dt = -2\sigma\, e^{-t}\Big|_0^{+\infty}$

$= 2\sigma^2$

$2\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$ $\therefore \sigma = \sqrt{\frac{1}{2n}\sum_{i=1}^{n} X_i^2}$

b) $\text{lik}(\theta) = \prod_{i=1}^{n} f(x|\sigma)$

$\log(\prod_{i=1}^{n} f(x|\sigma)) = 0$

$\frac{\partial}{\partial\sigma} \log[(\frac{1}{2\sigma})^n \exp\{-\sum_{i=1}^{n} \frac{|x_i|}{\sigma}\}] = 0$

$\frac{\partial}{\partial\sigma}(-n\log(2\sigma) - \sum_{i=1}^{n} \frac{|x_i|}{\sigma}) = 0$

$-n \cdot \frac{2}{2\sigma} + \sum_{i=1}^{n} |x_i| \cdot \frac{1}{\sigma^2} = 0$

$\frac{n}{\sigma} = \sum_{i=1}^{n} \frac{1}{\sigma^2} |x_i|$

$n\sigma = \sum_{i=1}^{n} |x_i|$

$\sigma = \frac{1}{n}\sum_{i=1}^{n} |x_i|$

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right]$$

c) $$I(\sigma) = -E\left[\left(\frac{\partial^2}{\partial\sigma^2}\log\left(\left(\frac{1}{2\sigma}\right)\exp\left\{-\frac{|x|}{\sigma}\right\}\right)\right)\right] = -E\left[\frac{\partial^2}{\partial\sigma^2}\left(-\log(2\sigma) - \frac{|x|}{\sigma}\right)\right]$$

$$= -E\left[\frac{\partial}{\partial\sigma}\left(-\frac{1}{\sigma} + |x|\frac{1}{\sigma^2}\right)\right] = -E\left[\frac{1}{\sigma^2} - 2|x|\frac{1}{\sigma^3}\right]$$

$$= -\frac{1}{\sigma^2} + 2E[|x|]\frac{1}{\sigma^3}$$

asymtotic var :

$$\lim_{n\to\infty}\frac{1}{nI(\sigma_0)} = \frac{1}{nI(\sigma_{mle})} \qquad \sigma_{mle} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

$$= 1/\left(n\cdot\left(-\frac{1}{\sigma^2} + 2\cdot\frac{1}{\sigma^3}\cdot\frac{1}{n}\sum_{i=1}^{n}|x_i|\right)\right)$$

$$= \sigma^3/\left(-n\sigma + 2\sum_{i=1}^{n}|x_i|\right) = \frac{\left(\frac{1}{n}\right)^3\cdot\left(\sum_{i=1}^{n}|x_i|\right)^3}{-\sum_{i=1}^{n}|x_i| + 2\sum_{i=1}^{n}|x_i|} = \frac{\left(\frac{1}{n}\right)^3 t^3}{-t+2t} = \frac{\left(\frac{1}{n}\right)^3 t^{\cancel{3}\,2}}{\cancel{t}}$$

$$= \frac{1}{n^3}\left(\sum_{i=1}^{n}|x_i|\right)^2$$

Problem 2

(a) $$\frac{\partial^2}{\partial\theta^2}\log f(x|\theta) = \frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right] = \frac{\partial}{\partial\theta}\left[\frac{\frac{\partial}{\partial\theta}f(x|\theta)}{f(x|\theta)}\right] = \frac{\left(f(x|\theta)\frac{\partial^2}{\partial\theta^2}f(x|\theta)\right) - \left(\frac{\partial}{\partial\theta}f(x|\theta)\right)\left(\frac{\partial}{\partial\theta}f(x|\theta)\right)}{(f(x|\theta))^2}$$

$$= \frac{\frac{\partial^2}{\partial\theta^2}f(x|\theta)}{f(x|\theta)} - \left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)\left(\frac{\partial}{\partial\theta}\log f(x|\theta)\right)$$

$$\therefore -E\left[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)\right] = E\left[\frac{\partial}{\partial\theta}\log f(x|\theta)\right]^2 - E\left[\frac{\frac{\partial^2}{\partial\theta^2}f(x|\theta)}{f(x|\theta)}\right] = I(\theta) - \int\frac{\frac{\partial^2}{\partial\theta^2}f(x|\theta)}{f(x|\theta)}f(x|\theta)dx$$

$$= I(\theta) - \int\frac{\partial^2}{\partial\theta^2}f(x|\theta)dx = I(\theta) - \frac{\partial^2}{\partial\theta^2}\int f(x|\theta)dx = I(\theta) - 0 = I(\theta)$$

(b) A $100(1-\alpha)\%$ CI for the estimate $\hat{\sigma}$ is:

$$Pr\{\hat{\sigma} \in [\sigma_0 - \beta, \sigma_0 + \beta]\} = 1-\alpha$$

$$Pr\{\sigma_0 - \beta \le \hat{\sigma} \le \sigma_0 + \beta\} = 1-\alpha$$

$$Pr\{-\beta\cdot\sqrt{nI(\sigma_0)} \le (\hat{\sigma} - \sigma_0)\cdot\sqrt{nI(\sigma_0)} \le \beta\cdot\sqrt{nI(\sigma_0)}\} = 1-\alpha$$

$$\beta = \frac{Z_{\alpha/2}}{\sqrt{nI(\sigma_0)}}$$

Therefore the CI is $Pr\left\{\hat{\sigma} \in \left[\frac{1}{n}\sum_{i=1}^{n}|X_i| - \frac{Z_{\alpha/2}}{\sqrt{nI(\sigma_{mle})}}, \frac{1}{n}\sum_{i=1}^{n}|X_i| + \frac{Z_{\alpha/2}}{\sqrt{nI(\sigma_{mle})}}\right]\right\}$

(c) $$f(x|a,b) = \frac{1}{b-a}$$

$$lik(f) = \prod_{i=1}^{n}f(x|a,b) = \prod_{i=1}^{n}\frac{1}{b-a} = \left(\frac{1}{b-a}\right)^n$$

if $X_n \notin [a,b]$, $P(f) = 0$

To maximize $lik(f)$, need to minimize $(b-a)$

$\therefore \arg\max_{a,b} lik(f) \Rightarrow a = \min(X_i)$, $b = \max(X_i)$, $i = 1, 2, \ldots n$

Problem 3.

(a). $E[X] = \int_{-\infty}^{+\infty} x f(x|\theta)\, dx$

$= \int_{\theta}^{+\infty} x e^{-(x-\theta)}\, dx = \int_{\theta}^{+\infty} x e^{-x+\theta}\, dx$

$= -\int_{\theta}^{+\infty} x\, d e^{-x+\theta}$

$= -[x e^{-x+\theta}\Big|_{\theta}^{+\infty} - \int_{0}^{+\infty} e^{-x+\theta}\, dx]$

$= \theta + \int_{\theta}^{+\infty} e^{-x+\theta}\, dx = \theta - e^{-x+\theta}\Big|_{\theta}^{+\infty}$

$= \theta + 1$

$\theta + 1 = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\theta = \frac{1}{n}\sum_{i=1}^{n} X_i - 1$

(b). $lik(\theta) = \prod_{i=1}^{n} f(x|\theta)$

if $x < \theta$, $lik(\theta) = 0$

if $x \geq \theta$, $\theta_{ml} = \underset{\theta}{\operatorname{argmax}}\left(\prod_{i=1}^{n} f(x|\theta)\right)$

$= \underset{\theta}{\operatorname{argmax}}\left(\prod_{i=1}^{n} e^{-x+\theta}\right)$, $x \geq \theta$

because $-x+\theta \leq 0$ and $g(x) = e^{x}$ increase.

$\theta_{ml} = \min(X_i)$, $i = 1, 2 \cdots n$

Problem 4. Suppose $X_1, X_2, ..., X_n \sim$ Poisson(λ). Given that the random variables are i.i.d, for $\theta = \exp(-\lambda)$,

1. Find an unbiased estimator of θ? (Note that it may not be the best estimator. Any unbiased estimator is fine).

Since $X_1, X_2 \cdots X_n \sim Poisson(\lambda)$, $E[X] = \lambda$, $Var[X] = \lambda$

$\theta = e^{-\lambda}$, Poisson's pdf is $f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ $\therefore f(x=0) = \theta$

unbiased estimator $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}(X_i = 0)$

Poisson's pdf is $f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$

2. Find the variance of the unbiased estimator you found and compare with the Cramer Rao lowerbound?

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i = 0\}$ $\quad \therefore Var(\hat{\theta}) = \frac{1}{n}\hat{\theta}(1-\hat{\theta})$

The Cramer-Rao lower bound is $Var(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

$I(\theta) = E[(\frac{\partial}{\partial\theta}\log f(x|\theta))^2]$ (*) or $E[\frac{\partial^2}{\partial\theta^2}\log f(x|\theta)]$

$\theta = e^{-\lambda} \Rightarrow \lambda = -\log\theta$ $\quad \therefore f(x|\theta) = \frac{(-\log\theta)^x}{x!}\theta$

$\frac{\partial}{\partial\theta}\log f(x|\theta) = \frac{\partial}{\partial\theta}\log(\frac{-x\log\theta}{x!}\theta)$

$= \frac{\partial}{\partial\theta}\log(-\log\theta)^x - \frac{\partial}{\partial\theta}\log x! + \frac{\partial}{\partial\theta}\log\theta$

$= x \cdot \frac{1}{\log\theta} \cdot \frac{1}{\theta} - 0 + \frac{1}{\theta}$

$= \frac{x(1+\log\theta)}{\theta\log\theta}$

$\frac{\partial}{\partial\theta}\log(-\log\theta)^x$

$= \frac{\partial}{\partial\theta}\log t^x, \; t = -\log\theta$

$= \frac{\partial}{\partial\theta} x\log t, \; t = -\log\theta$

$= x \cdot \frac{1}{t} \cdot t'$

$= x \cdot \frac{1}{-\log\theta} \cdot (-\frac{1}{\theta})$

$E[x] = Var[X] = \lambda \quad \therefore E[X] = -\log\theta,$

$E[X^2] = Var[X] + (E[X])^2 = -\log\theta + (\log\theta)^2$

(*) $= E[(\frac{x(1+\log\theta)}{\theta\log\theta})^2] = (\frac{1}{\theta\log\theta})^2 E[X^2 + 2X\log\theta + (\log\theta)^2]$

$= (\frac{1}{\theta\log\theta})^2[-\log\theta + (\log\theta)^2 - 2(\log\theta)^2 + (\log\theta)^2]$

$= (\frac{1}{\theta\log\theta})^2[-\log\theta] = \frac{-1}{\theta^2\log\theta}$

$\frac{1}{nI(\theta)} = \frac{-\hat{\theta}^2\log\hat{\theta}}{n}$

Since $\max \hat{\theta} - \hat{\theta}\log\hat{\theta} = \hat{\theta}(1-\log\hat{\theta})|_{\hat{\theta}=1} = 1$

$\therefore \hat{\theta} - \hat{\theta}\log\hat{\theta} \leq 1$

$1 - \hat{\theta} \geq -\hat{\theta}\log\hat{\theta}$

$\frac{\hat{\theta}}{n}(1-\hat{\theta}) \geq -\frac{\hat{\theta}^2\log\hat{\theta}}{n}$

i.e. $Var(\hat{\theta}) \geq \frac{1}{nI(\theta)}$

$\cancel{\frac{1}{n}}\theta(1-\theta) \geq \frac{-\theta^2\log\theta}{\cancel{n}}$

$1-\theta \geq -\theta\log\theta$

$\theta - 1 \leq \theta\log\theta$

Problem 5. We have access to a file consisting of $n = 10^4$ numbers. The numbers are either 1, 2, or 3. Moreover, the value 1 appears $n_1 = 2600$ times in the file, the value 2 appears $n_2 = 5200$ times, and the value 3 appears $n_3 = 2200$ times. We know that these numbers are generated i.i.d. according to an unknown distribution.

(a) Let μ denote the mean of the distribution. Estimate the value of μ from sample data provided in the file and provide a 95% confidence interval.

$$\hat{\mu} = (1\times n_1 + 2\times n_2 + 3\times n_3) \div (n_1+n_2+n_3)$$
$$= (1\times 2600 + 2\times 5200 + 3\times 2200) \div (2600+5200+2200)$$
$$= 1.96$$

$$E[\mu^2] = (1^2\times n_1 + 2^2\times n_2 + 3^2\times n_3) \div (n_1+n_2+n_3) = 4.32$$
$$Var[\mu^2] = E[\mu^2] - (E[\mu])^2 = 4.32 - (1.96)^2 = 0.4784$$

for 95% CI, $Z_{\alpha/2} = Z_{0.025} = 1.96$, $\sigma = \sqrt{0.4784}$, $n = 10000$

$$Pr\{\hat{\mu} - \beta \le \mu \le \hat{\mu} + \beta\} = 1-\alpha \Rightarrow Pr\left\{\frac{-\beta}{\sigma/\sqrt{n}} \le \frac{\mu-\hat{\mu}}{\sigma/\sqrt{n}} \le \frac{\beta}{\sigma/\sqrt{n}}\right\} = 1-\alpha$$
$$\frac{\beta}{\sigma/\sqrt{n}} = Z_{\alpha/2} \Rightarrow \beta \approx 0.01356$$

95% Confidence Interval is $[1.946, 1.974]$

(b) Assume now that the generating distribution of the data has the following form:

$$X = \begin{cases} 1, & \text{with probability } p_1, \\ 2, & \text{with probability } p_2, \\ 3, & \text{with probability } 1-(p_1+p_2). \end{cases}$$

We would like to estimate the value of the parameters p_1 and p_2. Consider the following estimator for the value of

$$p_1(X_1, \ldots, X_n) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{X_i = 1\},$$

where $\mathbb{1}\{A\}$ takes value 1 if A is true, and 0 otherwise (this is known as an indicator function for A).

Compute the estimate p_1 from the sample data provided in the file. Is this estimator an unbiased estimator for p_1? Justify your answer.

$$p_1(X_1,\cdots,X_n) = \frac{1}{10000}(2600\times 1 + 5200\times 0 + 2200\times 0) = 0.26$$

This estimator is unbiased

because: $E[p_1(X_1,\cdots,X_n)] = E\left[\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}\{X_i=1\}\right]$, $X_1 \ldots X_n$ iid

$$= \frac{1}{n}E[n_1\times 1] = p_1$$

(c) Use the method of moments to estimate the value of p_1 and p_2 (you should compute the estimate from the sample data).

$E[X] = \int_{-\infty}^{\infty} x f(x|\theta)\, dx$

$$\begin{cases} E[X] = P_1 + 2P_2 + 3P_3 = 1.96 \\ E[X^2] = P_1 + 4P_2 + 9P_3 = 4.32 \\ P_3 = 1 - P_1 - P_2 \end{cases} \Rightarrow \begin{cases} P_1 = 0.26 \\ P_2 = 0.52 \end{cases}$$

(d) Now, assume that the precise value of p_1 is given as $p_1 = \frac{1}{4}$. As a result, we now know that the distribution of the data has the form:

$\propto f(x|P_2)$

$$X = \begin{cases} 1, & \text{with probability } \frac{1}{4}, \\ 2, & \text{with probability } p_2, \\ 3, & \text{with probability } \frac{3}{4} - p_2 \end{cases}$$

$P_1 = 0.25$
$P_2 = 0.527$
$P_3 = 0.223$

We would like to estimate the value of the parameter p_2 from data. Find the maximum likelihood estimator for p_2 and provide a 95% confidence interval.

The likelihood of observing the data from file is

$$P(D|P_2) = P_2^{n_2} \left(\tfrac{3}{4} - P_2\right)^{n_3}$$

$$\underset{P_2}{\operatorname{argmax}}\, P(D|P_2) = \underset{P_2}{\operatorname{argmax}} \log P(D|P_2) = \underset{P_2}{\operatorname{argmax}} \log\left(P_2^{n_2} \left(\tfrac{3}{4} - P_2\right)^{n_3}\right) \quad (*)$$

$$\frac{d}{dP_2}(*) = 0 \Rightarrow \frac{\partial}{\partial P_2}\left(n_2 \log(P_2) + n_3 \log\left(\tfrac{3}{4} - P_2\right)\right) = 0$$

$Z_{\alpha/2} = 1.96$

$$\frac{n_2}{P_2} - \frac{n_3}{\frac{3}{4} - P_2} = 0 \Rightarrow \hat{P}_{2\,MLE} = 0.527$$

$$Pr\{\hat{P}_2 - \beta \le P_2 \le \hat{P}_2 + \beta\} = \underbrace{1-\alpha}_{0.95} \Rightarrow Pr\{-\beta \cdot \sqrt{nI(P_2)} \le (P_2 - \hat{P}_2)\sqrt{nI(P_2)} \le \beta\sqrt{nI(P_2)}\} = 1-\alpha$$

$$\frac{\partial}{\partial P_2} \log f(x=2|P_2) = \frac{1}{P_2} \qquad \frac{\partial}{\partial P_1} \log f(x=1|P_2) = 0 \qquad \frac{\partial}{\partial P_3} \log f(x=3|P_2) = -\frac{1}{0.75 - P_2}$$

$$I(P_2) = E\left[\left(\frac{\partial}{\partial \theta} f(x|P_2)\right)^2\right] = \frac{1}{n_1+n_2+n_3}\left(\frac{n_2}{P_2^2} - \frac{1}{(0.75-P_2)^2}\right) = \frac{1}{n} \times 62963$$

$$\beta = \frac{Z_{\alpha/2}}{\sqrt{nI(\theta_0)}} = 0.0078$$

95% CI is $[0.5192, 0.5348]$

Problem 6.

Q6-(a)

```
# Compute estimates for the sample mean and sample variance
# mymean = sum(data) / len(data) # piazza said sum() is built-in
total, cnt = 0, 0
for d in data:
    total += d
    cnt += 1
mymean = total / cnt
totalerr = 0
for d in data: totalerr+= (d-mymean)**2
myvar = totalerr / cnt
print('Without inbuilt functions:','Sample mean is %f and sample variance is %f'%(mymean,myvar))
npmean, npvar = np.mean(data), np.var(data)
print('With inbuilt numpy functions:','Sample mean is %f and sample variance is %f'%(npmean,npvar))
```

✓ 0.2s

```
Without inbuilt functions: Sample mean is 4.234069 and sample variance is 18.185304
With inbuilt numpy functions: Sample mean is 4.234069 and sample variance is 18.185304
```

Q6-(b)

```
import scipy.stats as st
st.norm.interval(alpha=0.90, loc=mymean, scale=0.25)
```

✓ 0.7s

```
(3.8228558437678597, 4.645282657243596)
```