Détection de structures à l’aide de modèles probabilistes sur les graphes

Modèles

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21 juin 2024

My collaborators

On the R packages




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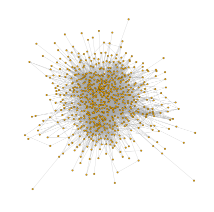
sbm	sbm, GREMLINS	blockmodels	colSBM

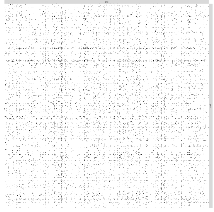
Other collaborators

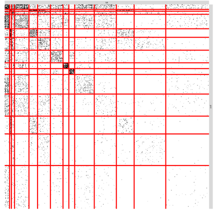
T. Tabouy (ex PhD student), E. Lazega (Sciences Po), L. Lacoste (new PhD student), E. Anakok (PhD student) + E. Thébault (iEES) + C. Fontaine (MNHN) + T. Vanrenterghem (INRAE), ANR Econet + ANR Pastodiv + GDR Resodiv

Recall

from the observation of a network determine structure







Outline

Stochastic and Latent Block Models

Other latent space models and other methods

Extensions of SBM

Outline

A first random graph model for network

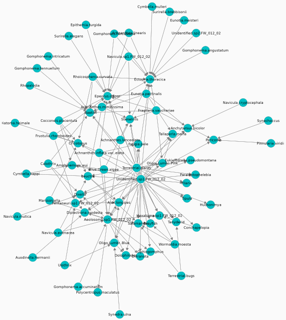
[Erdős and Rényi, 1959] Model for n nodes

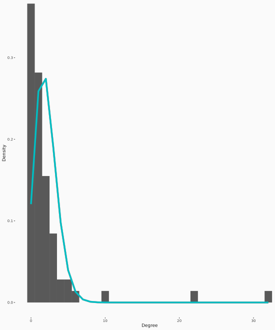
$$\forall 1 \leq i, j \leq n, \quad Y_{ij} \overset{i.i.d.}{\sim} \mathcal{B}\text{ern}(p),$$

where $p \in [0, 1]$ is the probability for a link to exist.

Consequence

$$\text{deg}(i) \sim_{i.i.d} \mathcal{B}\text{in}(n, p)$$





Not enough variability in the degree

Limitations of an ER graph to describe real networks

- Homogeneity of the connections
- Degree distribution too concentrated, no high degree nodes,
- All nodes are equivalent,
- No modularity, no hubs

Stochastic Block Model and Latent Block Model

Model on a simple network with n nodes:

SBM: [Nowicki and Snijders, 2001]

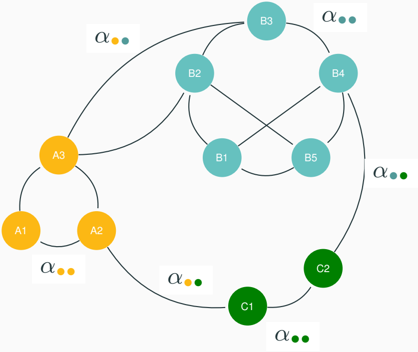
- Q blocks of nodes sharing similar connection structure,
- $\boldsymbol{Z} = (Z_1, \ldots, Z_n)$ independent latent variables s.t. $\mathbb{P}(Z_i = k) = \pi_k$ for $k \in \{1, \ldots, Q\}$ and $i \in \{1, \ldots, n\}$,
- $Y_{ij} | Z_i, Z_j \overset{ind}{\sim} \mathcal{F}(\alpha_{Z_i, Z_j})$ for all dyads (i, j)

Model on a bipartite network with n_1 and n_2 nodes:

LBM: [Govaert and Nadif, 2010]

- Q_1 and Q_2 blocks of nodes sharing similar connection structure,
- $\boldsymbol{Z}^1 = (Z_1^1, \ldots, Z_n^1)$ and $\boldsymbol{Z}^2 = (Z_1^2, \ldots, Z_m^2)$ independent latent variables s.t. $\mathbb{P}(Z_i^1 = k) = \pi_k^1$ for all $i \in \{1, \ldots, n\}$, $k \in \{1, \ldots, Q_1\}$ and $\mathbb{P}(Z_j^2 = l) = \pi_l^2$ for all $j \in \{1, \ldots, m\}$, $l \in \{1, \ldots, Q_2\}$
- $Y_{ij} | Z_i^1, Z_j^2 \overset{ind}{\sim} \mathcal{F}(\alpha_{Z_i^1, Z_j^2})$ for all dyads (i, j).

Stochastic Block Model : illustration



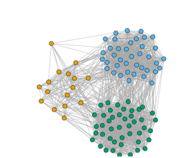
Parameters
Let n nodes divided into 3 clusters

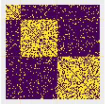
- $\{\bullet, \bullet, \bullet\}$ clusters
- $\pi_{\bullet} = \mathbb{P}(i \in \bullet),\, i = 1, \ldots, n$
- $\alpha_{\bullet\bullet} = \mathbb{P}(i \leftrightarrow j | i \in \bullet, j \in \bullet)$

$$\mathbf{Y} \sim \text{SBM}_n(Q, \boldsymbol{\pi}, \boldsymbol{\alpha})$$

Simulations under the SBM

$$\boldsymbol{\alpha} = \begin{pmatrix} 0.70 & 0.09 & 0.09 \\ 0.09 & 0.70 & 0.09 \\ 0.09 & 0.09 & 0.70 \end{pmatrix}$$





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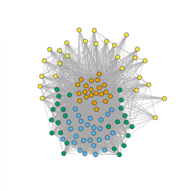

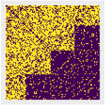

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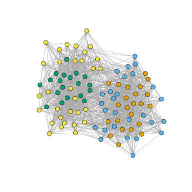

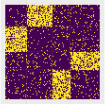

Likelihood for SBM

Complete likelihood $(\mathbf{Y})$ **et** (Z)

$$\begin{aligned}
\ell_c(\mathbf{Y}, \boldsymbol{Z}; \theta) &= p(\mathbf{Y}|\mathbf{Z}; \boldsymbol{\alpha}) p(\mathbf{Z}; \boldsymbol{\pi}) \\
&= \prod_{i,j} f_{\alpha_{Z_i,Z_j}}(Y_{ij}) \times \prod_i \pi_{Z_i} \\
&= \prod_{i,j} \alpha_{Z_i,Z_j}^{Y_{ij}} (1 - \alpha_{Z_i,Z_j})^{1-Y_{ij}} \prod_i \pi_{Z_i}
\end{aligned}$$

Marginal likelihood $(\mathbf{Y})$

$$\log \ell(\mathbf{Y}; \theta) = \log \sum_{\boldsymbol{Z} \in \boldsymbol{\mathcal{Z}}} \ell_c(\mathbf{Y}, \boldsymbol{Z}; \theta) \, .$$

Remark
$\mathcal{Z} = \{1, \ldots, Q\}^n \Rightarrow$ when Q and n increase, impossible to compute.

Standard tool to maximize the likelihood when latent variables involved
: EM algorithm.

Complete likelihood $(\mathbf{Y})$ et (Z)

$$\begin{aligned}\ell_c(\mathbf{Y},\boldsymbol{Z};\theta) &= p(\mathbf{Y}|\mathbf{Z};\boldsymbol{\alpha})p(\mathbf{Z};\boldsymbol{\pi}) \\ &= \prod_{i,j} f_{\alpha_{Z_i,Z_j}}(Y_{ij}) \times \prod_i \pi_{Z_i} \\ &= \prod_{i,j} \alpha_{Z_i,Z_j}^{Y_{ij}}(1-\alpha_{Z_i,Z_j})^{1-Y_{ij}} \prod_i \pi_{Z_i}\end{aligned}$$

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From EM to variational EM

Standard EM
At iteration (t) :

- **Step E**: compute

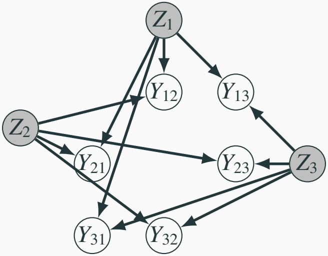
$$\mathcal{Q}(\theta|\theta^{(t-1)}) = \mathbb{E}_{\mathbf{Z}|\mathbf{Y},\theta^{(t-1)}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)\right]$$

- **Step M**:

$$\theta^{(t)} = \arg\max_{\theta} \mathcal{Q}(\theta|\theta^{(t-1)})$$

Limitation of EM

However, once conditioned by par $\mathbf{X}$, the Z are not independent anymore



$$p(\mathbf{Z}|\mathbf{X}, \theta^{(t-1)}) \neq \prod_{i=1}^{n} p(Z_i|\mathbf{X}, \theta^{(t-1)})$$

Variational EM : maximization of a lower bound

Idea : replace the complicated distribution $[\mathbf{Z}|\mathbf{Y},\theta]$ by a simpler one.

Let $\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}$ be any distribution on $\boldsymbol{Z}$

Central identity

$$\begin{aligned}
\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}) &= \log \ell(\mathbf{Y};\theta) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}, p(\cdot|\mathbf{Y};\theta)] \quad \leq \log \ell(\mathbf{Y};\theta) \\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] - \sum_{\mathbf{Z}} \mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}(\mathbf{Z}) \log \mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}(\mathbf{Z}) \\
&= \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}}[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] + \mathcal{H}\left(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}(\mathbf{Z})\right)
\end{aligned}$$

Note that:

$$\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}) = \log \ell(\mathbf{Y};\theta) \Leftrightarrow \mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}} = p(\cdot|\mathbf{Y};\theta)$$

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Variational EM

- Maximization of $\log \ell(\mathbf{Y}; \theta)$ w.r.t. θ replaced by maximization of the lower bound $\mathcal{I}_\theta(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}})$ w.r.t. τ and θ.
- **Benefit** : we choose $\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}$ such that the maximization calculus can be done explicitly
  - In our case: mean field approximation : neglect dependencies between the (Z_i)

$$P_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}}(Z_i = q) = \tau_{iq}$$

Variational EM

Algorithm
At iteration (t), given the current value $(\theta^{(t-1)}, \mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}^{(t-1)}})$,

- **Step 1** Maximization w.r.t. τ

$$
\begin{aligned}
\boldsymbol{\tau}^{(t)} &= \arg\max_{\boldsymbol{\tau}\in\mathcal{T}} \mathcal{I}_{\theta^{(t-1)}}(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}) \\
&= \arg\max_{\boldsymbol{\tau}\in\mathcal{T}} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}}\left[\log \ell_c(\mathbf{Y},\mathbf{Z};\theta^{(t-1)})\right] + \mathcal{H}\left(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}(\mathbf{Z})\right) \\
&= \arg\max_{\boldsymbol{\tau}\in\mathcal{T}} \log \ell(\mathbf{Y};\theta^{(t-1)}) - \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}, p(\cdot|\mathbf{Y};\theta^{(t-1)})] \\
&= \arg\min_{\boldsymbol{\tau}\in\mathcal{T}} \mathbf{KL}[\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}}, p(\cdot|\mathbf{Y};\theta^{(t-1)})]
\end{aligned}
$$

Variational EM

Algorithm

- **Step 2** Maximization w.r.t. θ

$$\begin{aligned}
\theta^{(t)} &= \arg\max_{\theta} \mathcal{I}_{\theta}(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}^{(t)}}) \\
&= \arg\max_{\theta} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}^{(t)}}} [\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)] + \mathcal{H}\left(\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}^{(t)}}(\mathbf{Z})\right) \\
&= \arg\max_{\theta} \mathbb{E}_{\mathcal{R}_{\mathbf{Y},\boldsymbol{\tau}^{(t)}}} [\log \ell_c(\mathbf{Y},\mathbf{Z};\theta)]
\end{aligned}$$

In practice

- Really fast
- Strongly depends on the initial values

A penalized likelihood criterion

- Selection of the number of clusters Q
- Integrated Classification Likelihood (ICL) [Biernacki et al., 2000]

$$ICL(\mathcal{M}_Q) = \log \ell_c(\mathbf{Y}, \hat{\mathbf{Z}}; \hat{\theta}_Q) - \text{Pen}(\mathcal{M}_Q)$$

  where

$$\hat{Z}_i = \underset{q \in \{1,\dots,Q\}}{\arg\max}\ \hat{\tau}_{iq}.$$

-

$$ICL(\mathcal{M}_Q) = \mathbb{E}_{p(\cdot|\mathbf{Y},\hat{\theta}_Q)}[\log \ell_c(\mathbf{Y}, \widehat{\mathbf{Z}}; \widehat{\theta}_Q)] - \text{Pen}(\mathcal{M}_Q)$$

  where

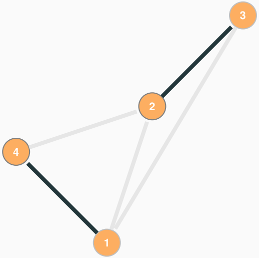
$$\text{Pen}(\mathcal{M}_Q) = \frac{1}{2}\left\{\underbrace{(Q-1)\log(n)}_{\text{Clust.}} + \underbrace{Q^2\log(n^2-n)}_{\text{Conn.}}\right\}$$

Penalty for bipartite networks

$$pen_{\mathcal{M}} = -\frac{1}{2}\left\{\underbrace{(K-1)\log(n) + (L-1)\log(p)}_{\text{Bi-Clust.}} + \underbrace{(KL)\log(np)}_{\text{Connection}}\right\}$$

Recall on missing value

Data: a graph G with missing data.



Adjacency matrix:

$$A = \begin{pmatrix} 0 & \mathtt{NA} & \mathtt{NA} & 1 \\ \mathtt{NA} & 0 & 1 & \mathtt{NA} \\ \mathtt{NA} & 1 & 0 & 0 \\ 1 & \mathtt{NA} & 0 & 0 \end{pmatrix}$$

Goal: Cluster nodes in spite of missing data and predict NA to $\{0, 1\}$ or predict most likely existing links.

Inferring the SBM from an observed network (Missing data)

[Timothée Tabouy and Chiquet, 2020].

Observation of a network: $n \times n$ binary matrix $\mathbf{R}$ such that $R_{ij} = 1$ if Y_{ij} is observed, $R_{ij} = 0$ otherwise ($Y_{ij} = \texttt{NA}$).

Observation process: [Rub76] MCAR, MAR or NMAR?

(MCAR)

(MAR or NMAR)

(NMAR)

(NMAR)

Inference under M(C)AR scheme: Likelihood on the observed data.

Inference under NMAR scheme

Need for accounting for the complete likelihood where we have missing data ($\mathbf{Y}^{\mathrm{m}}$) and latent variables $\boldsymbol{Z}$

Variational distribution on $(\mathbf{Y}^{\mathrm{m}}, \boldsymbol{Z})$ in the VEM algorithm:

$$\mathcal{R}_{(\mathbf{Y}^{\mathrm{m}},\mathbf{Z})} = \mathcal{R}_{(\mathbf{Y}^{\mathrm{m}})} \cdot \mathcal{R}_{(\mathbf{Z})} = \prod_{(i,j),Y_{ij}=NA} \nu_{ij}^{Y_{ij}} (1-\nu_{ij})^{1-Y_{ij}} \cdot \prod_{i=1}^{n} \prod_{k=1}^{Q} (\tau_{ik})^{\mathbb{I}_{Z_i=k}},$$

where

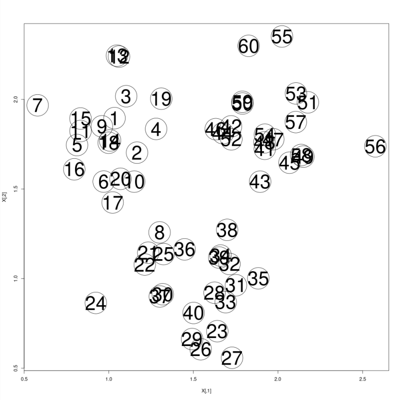
- ν_{ij}s and τ_{ik}s parameters to be optimized in the VE step,
- τ_{ik} is almost generic,
- ν_{ij} is specific to the sampling design.

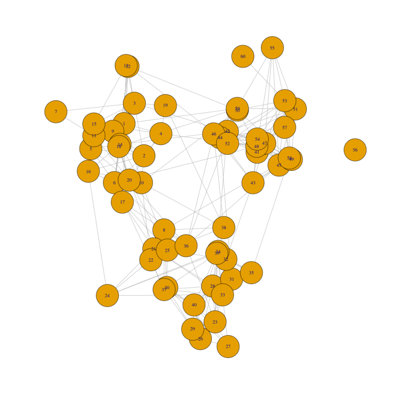
Contributions:

- Derived variational steps for some NMAR sampling schemes,
- Importance of accounting for sampling illustrated on synthetic and real data,
- Implementation in an `R` package `missSBM` [Tabouy et al., 2019].

Outline

- $\forall i \in \{1, \ldots, N\}$, $Z_i \overset{ind}{\sim}$ Mixture$\mathcal{N}((\mu_k)_k, (\Sigma_k)_k)$,
- $\forall (i,j)$,$Y_{ij}|Z_i, Z_j \overset{ind}{\sim} b(\exp(-\|Z_i - Z_j\|/\sigma^2))$.





(Generalised) random dot product graph

Alternative to the distance between latent positions, the dot product can be used:

$$\forall (i,j),\ Y_{ij}|Z_i, Z_j \overset{ind}{\sim} b(Z_i \cdot Z_j = Z_i^\top Z_j).$$

[Rubin-Delanchy et al., 2022] proposed a generalisation:

$$\forall (i,j),\ Y_{ij}|Z_i, Z_j \overset{ind}{\sim} b(Z_i I_{p,q} Z_j)$$

with

$$I_{p,q} \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}.$$

Modularity

In [Newman, 2006], definition of modularity, for a given clustering:

$$Mod = \frac{1}{C} \sum_{i,j} \left[A_{ij} - \frac{d_i d_j}{C} \right] \delta_{ij}$$

where

- $C = \sum_i \sum_j A_{ij}$,
- d_i is the degree of species i (i.e. $d_i = \sum_j A_{ij}$),
- δ_{ij} are dummy variables indicating whether species/nodes i and j are assumed to belong to the same module/community/cluster.

Goal: look for the partitioning of nodes into communities/modules that maximizes the associated modularity score.

Algos: edge-betweenness algorithm (EB), the leading-eigenvector algorithm (LE) and the Louvain algorithm (ML).

Laplacian of a graph

For $G = (V, E)$ an undirected graph s.t. $V = \{1, \ldots, N\}$ and A the corresponding adjacency matrix.

- Degree of a vertex/node: $d_i = \sum_j A_{ij}$,
- Unnormalized Laplacian: $L = D - A$ with $D = \mathrm{diag}(d_1, \ldots, d_N)$,

Properties:

- for $x \in \mathbb{R}^n,\ x^\top L x = \frac{1}{2} \sum_j^N A_{ij}(x_i - x_j)^2$,
- L is symmetric and positive definite,
- the smallest eigenvalue is 0 and associated with the vector $\mathbb{1}$,
- the order of multiplicity of 0 is the number of connected components.

[Von Luxburg, 2007]

Normalized Laplacians:

$$
\begin{aligned}
L_{sym} &= D^{-1/2} L D^{-1/2} = I_N - D^{-1/2} A D^{-1/2} \\
L_{rw} &= D^{-1} L = I_N - D^{-1} A
\end{aligned}
$$

Properties:

- for $x \in \mathbb{R}^n,\ x^\top L_{sym} x = \frac{1}{2} \sum_j^N A_{ij} (x_i/\sqrt{d_i} - x_j/\sqrt{d_j})^2$,
- L_{sym} and L_{rw} are symmetric and positive definite,
- the smallest eigenvalue is 0,
- the order of multiplicity of 0 is the number of connected components.

Spectral Clustering:

Input: Adjacency Matrix $A \in \mathbb{R}^{N \times N}$, number k of clusters to construct.

- Compute the unnormalized Laplacian L.
- Compute the first k eigenvectors $u_1, \dots, u_k$ of L.
- Let $U \in \mathbb{R}^{N \times k}$ be the matrix containing the vectors $u_1, \dots, u_k$ as columns.
- For $i = 1, \dots, N$, let $z_i \in \mathbb{R}^k$ be the vector corresponding to the i-th row of U.
- Cluster the points $(z_i)_{i=1,\dots,N}$ in $\mathbb{R}^k$ with the k-means algorithm into clusters $C_1, \dots, C_k$.

Outline

Stochastic and Latent Block Models

Other latent space models and other methods

Extensions of SBM

Degree corrected SBM

[Karrer and Newman, 2011]

$$\mathbb{P}(Y_{i,j} = 1 | Z_i = k, Z_j = l) = 1/\big(1 + \exp(-\alpha_{kl} - \nu_i - \nu_j)\big)$$

with

- $(\nu_i)_{1 \leq i \leq n}$ parameters to be estimated,
- find clusters besides the degree/popularity.

Mixed membership SBM

[Airoldi et al., 2008]

For each node $p \in N$:

- Draw a K-dimensional mixed membership vector $\vec{\pi}_p \sim \text{Dirichlet}(\vec{\alpha})$.

For each pair of nodes $(p, q) \in N \times N$:

- Draw membership indicator for the initiator, $\vec{z}_{p \to q} \sim \text{Multinomial}(\vec{\pi}_p)$.
- Draw membership indicator for the receiver, $\vec{z}_{q \to p} \sim \text{Multinomial}(\vec{\pi}_q)$.
- Sample the value of their interaction, $Y(p, q) \sim \text{Bernoulli}(\vec{z}^T_{p \to q} A \vec{z}_{q \to p})$.

Other extensions exist...

Multilayer networks

- Large variety of multilayer networks,
- SBM as a probabilistic generative model easy to extend to numerous cases.
- e.g. dynamic or spatial SBM ([Matias and Miele, 2017, Longepierre and Matias, 2019]) or Topic SBM [Bouveyron et al., 2018].

Our contributions

- Multiplex network [Barbillon et al., 2017, Lazega et al., 2016],
- Multilevel network [Chabert-Liddell et al., 2019],
- Multipartite network [Bar-Hen et al., 2018].

Adaptation of the VEM algorithm and ICL criterion to select the numbers of blocks.

Multiplex network

In collaboration with A. Bar-Hen, S. Donnet and E. Lazega [Barbillon et al., 2017, Lazega et al., 2016].

Multiple relations between individuals:

$$\mathbf{Y}_{ij}|Z_i, Z_j \overset{ind}{\sim} \text{Bern}^M((\alpha^w_{Z_i,Z_j})_w) \quad \text{with} \sum_w \alpha^w_{Z_i,Z_j} = 1.$$

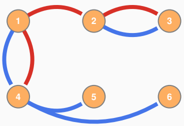


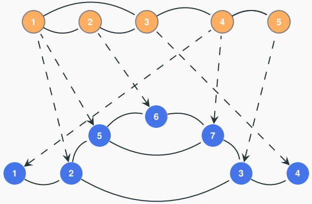
Figure 1: Illustration of a multiplex network. For each dyad, two kinds of link may exist. They are respectively displayed by red and blue edges.

Model inference implemented in the `R` package: `sbm`.

Application to a network of French researchers in cancerology (advice relation and indirect relation through the labs of the researchers).

Multilevel network

In collaboration with S. Donnet and S.-C. Chabert-Liddell (Ph.D. Thesis) and E. Lazega [Chabert-Liddell et al., 2019].



- Organizational level: SBM for $(\mathbf{Y}^O, \mathbf{Z}^O)$,
- Individual level: SBM for $(\mathbf{Y}^I, \mathbf{Z}^I)$,
- Interlevel dependence $i \in \{1, \ldots, n_I\}$, $k \in \{1, \ldots, K_I\}$,
  $\mathbb{P}(Z_i^I = k | Z_j^O, A_{ij} = 1) \overset{ind}{=} \gamma_{kZ_j^O}$, where A is the affiliation matrix.

Implemented in S.-C. Chabert-Liddell's `R` package: `MLVSBM`.

Application to a dataset of a program trade fair (Organizations = audiovisual firms, individuals = sales representatives).

Generalized Multipartite Networks

In collaboration with A. Bar-Hen and S. Donnet [Bar-Hen et al., 2018].

- Pre-specified functional groups (colors of nodes),
- Looking for blocks within functional groups,
- Each network between 2 functional groups is either an SBM or an LBM.

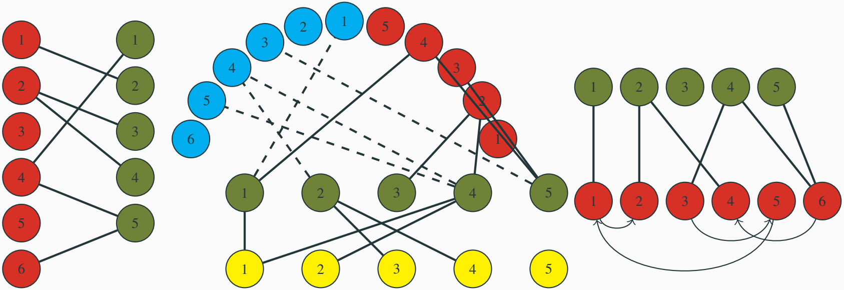
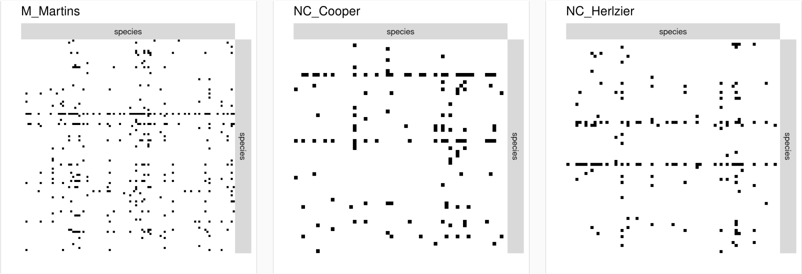


Figure 2: Illustrations of bipartite (left), multipartite (center) and generalized multipartite networks (right). The colors stand for the different functional groups.

Implemented in an `R` package: `GREMLINS`.
Application for ecological interaction (see practical session).

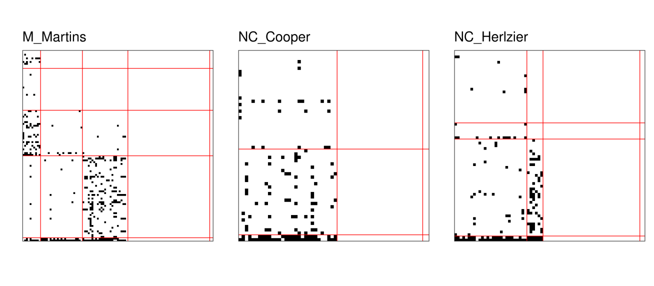
Towards collection of networks: Three foodwebs

- Pine-firest stream food webs issued from Maine, North-Caroline and New-Zealand [Thompson and Townsend, 2003]
- Involve respectively 105, 58 and 71 species.
- $Y_{ij} = 1$ if i is eaten by j. Directed relation



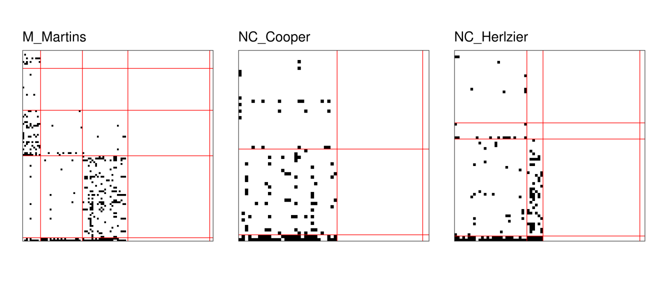
- Look for similarities and differences between network structures.

Separate SBMs



- Fitted SBM on each separately
- Reordered the matrices following the blocks
- Label the blocks following the average out-degrees order

Towards collection of networks: Separate SBMs



- Two bottom groups in each matrix are basal species : eaten by many species and not eating anybody.
-
    - **Martins**: has a separation into 5 blocks, the third one is a medium trophic level, which preys on basal species and is highly preyed by species of the 1st block.
    - **Cooper**. Higher trophic levels grouped together in the same block (lack of statistical power).
    - **Herlzier**: higher trophic level is separated into 2 blocks determined on how much they prey on the less preyed basal block.

Collection of networks: Joint modeling of the networks

- Need to model jointly the networks
- Identify the groups playing the same role through out the networks, with an unsupervised strategy.
- Let $(\mathbf{Y}^m)_{m=1,\dots,M}$ denote the collection of networks each involving n_m nodes.
- $(\mathbf{Y}^m)$ independent.
- $$\mathbf{Y}^m \sim \text{SBM}_{n_m}(Q^m, \boldsymbol{\pi}^m, \boldsymbol{\alpha}^m)$$
- Conditions on the parameters $(\boldsymbol{\pi}^m)_{m=1,\dots,M}$ and $(\boldsymbol{\alpha}^m)_{m=1,\dots,M}$

First naive model

iid-colSBM

$$\mathbf{Y}^m \sim \mathsf{SBM}_{n_m}(Q, \boldsymbol{\pi}, \boldsymbol{\alpha})$$

with $\pi_q > 0 \; \forall q \in \{1, \dots, Q\}$ and $\sum_{q=1}^{Q} \pi_q = 1$.

- $(Q-1) + Q^2$ unknown parameters, M clustering
- Maybe too strict

Same structure of connection α, specific proportions of blocks in each network

π-colSBM

$$\mathbf{Y}^m \sim \text{SBM}_{n_m}(Q, \boldsymbol{\pi}^m, \boldsymbol{\alpha})$$

On the block proportions

- $\pi_q^m \geq 0$
- If $\pi_q^m = 0$ then block q is not represented in network m

π-colSBM: different proportions

$M = 2$ networks

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} \qquad \begin{matrix} \pi^1 = [.25, .25, .50] \\ \pi^2 = [.20, .50, .30] \end{matrix}.$$

- Same connection structure between blocks
- Different block proportions
- $2 \times (3-1) + 3^2 = 15$ parameters.

π-colSBM: nested structures

$\pi_q^m \geq 0$

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{12} & \alpha_{22} & \alpha_{23} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix} \qquad \begin{array}{c} \pi^1 = [.25, .25, .50] \\ \pi^2 = [.40,\ 0\ , .60] \end{array}.$$

- Blocks 1 and 3 are represented in the two networks while block 2 only exists in network 1.
- $3 - 1 + 3 - 2 + 3^2 = 14$ parameters

π-colSBM: partially nested structures

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \cdot \\ \alpha_{31} & \cdot & \alpha_{33} \end{pmatrix} \qquad \begin{matrix} \pi^1 = [.25, .75, \ 0 \ ] \\ \pi^2 = [.40, \ 0 \ , .60] \end{matrix}.$$

- The two networks share block 1 (for instance super predators or basal species)
- The remaining nodes of each network not equivalent in terms of connectivity.
- Blocks 2 and 3 never interact because their elements do not belong to the same network and so α_{23} and α_{32} are not required to define the model.
- $(2-1)+(2-1)+7 = 11$ parameters.

Summary

M independent networks.

$$\mathbf{Y}^m \sim \text{SBM}(Q^m, \boldsymbol{\pi}^m, \boldsymbol{\alpha}^m)$$

Model name	Block prop.	Connexion param.	Nb of param.
iid-colSBM	$\pi_q^m = \pi_q, \pi_q > 0$	$\alpha_{qr}^m = \alpha_{qr}$	$(Q-1) + Q^2$
π-*colSBM*	$\pi_q^m, \pi_q^m \geq 0$	$\alpha_{qr}^m = \alpha_{qr}$	$\leq M(Q-1) + Q^2$
δ-*colSBM*	$\pi_q^m = \pi_q, \pi_q > 0$	$\alpha_{qr}^m = \delta^m \alpha_{qr}$	$(Q-1) + Q^2 + (M-1)$
$\delta\pi$-*colSBM*	$\pi_q^m, \pi_q^m \geq 0$	$\alpha_{qr}^m = \delta^m \alpha_{qr}$	$\leq M(Q-1) + Q^2 + M - 1$
sep-SBM	$\pi_q^m, \pi_q^m > 0$	α_{qr}^m	$\sum_{m=1}^M (Q_m - 1) + Q_m^2$


colSBM
πcolSBM
δcolSBM
δπcolSBM

Inference and model selection

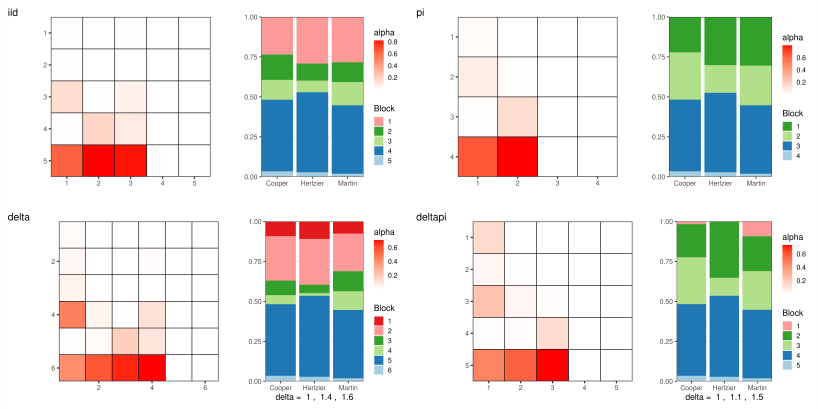
VEM algorithm

- Direct extension of VEM previously described for iid-colSBM and π-colSBM
- Less obvious with $\delta_m\alpha$: M step not explicit.
- Sensitive to initializations: need to match blocks among networks.

Model Selection

- ICL can be directly extended for iid-colSBM and the δ-colSBM
- for $\pi(\delta)$-colSBM, taking into account empty blocks...

Our 4 consensus models



Top left : iid (-1966). Top right: π-colSBM (-1982) Bottom-left: δ-colSBM (-1969). Bottom-right: $\delta\pi$-colSBM (-1989)

- separated SBMs gives an ICL of -2080.
- iid-colSBM : prefered model. Make 5 blocks
- π-colSBM: block proportion quite similar. Make no use of its flexibility

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