# Homework #I

1) For the system of equations

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 10 \\ 1 \end{pmatrix}$$

a) Find a solution, if any, using elementary row operations.
b) Find a basis for the subspace spanned by the column vectors of the system matrix. What is its dimension?

2) For the system

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

find the conditions on a, b, c so that a solution exists.

3) Find the inverse of the matrix

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

a) using elementary row operations
b) using determinants and adjoint matrix.

4) Prove triangle inequality for the Euclidean norm in $\mathbb{R}^4$ space.

Introduction to Optimization HW # 2

1) Find the transformation matrix $T$ which switches from natural basis representation to the representation with respect to basis $\{b_1', b_2', b_3'\}$ in $\mathbb{R}^3$, when $b_1' = (1,0,0)^T$, $b_2' = (1,1,0)^T$, $b_3' = (1,1,1)^T$. Express the vector $\underset{\sim}{x} = (2,1,0)^T$ with respect to new basis.

2) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix}$. What is the derived diagonalizing matrix? What is the derived orthogonal basis for $\mathbb{R}^3$?

3) For the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$ find a basis for $\mathcal{N}(A)$ and $\mathcal{R}(A^T)$. Show that these subspaces are orthogonal complements.

4) For the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find the spectral norm and radius.

HW I

1a) Let us form the augmented matrix to apply elementary row operations.

$$\begin{bmatrix} 1 & 2 & 4 & 7 \\ 2 & 3 & 7 & 10 \\ -1 & 2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & -1 & -1 & -4 \\ 0 & 4 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 7 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

The last row is inconsistent: no solution

b) A basis can be found using elementary column operations:

$$\begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & -1 \\ -1 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -1 & 4 & 0 \end{bmatrix}$$

basis vectors: $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$

dim: 2

2) Again, applying elementary row operations

$$\begin{bmatrix} 1 & 2 & -3 & a \\ 2 & 6 & -11 & b \\ 1 & -2 & 7 & c \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & -4 & 10 & c-a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & a \\ 0 & 2 & -5 & b-2a \\ 0 & 0 & 0 & c+2b-5a \end{bmatrix}$$

The system will have solutions if $c+2b-5c = 0$.

3/a)
$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 8 & 2 & 2 & -1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/4 & 1/4 & -1/8 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/8 & 3/8 & 1/16 & 0 \\ 0 & 1 & 1/4 & 1/4 & -1/8 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3/4 & -5/16 & -3/8 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} 3/4 & -5/16 & -3/8 \\ 1/2 & -3/8 & -1/4 \\ -1 & 1 & 1 \end{bmatrix}$

b) $\mathrm{adj}(A) = \begin{bmatrix} -12 & 5 & 6 \\ -8 & 6 & 4 \\ 16 & -16 & -16 \end{bmatrix}$ $\det(A) = -16$, $A^{-1} = \frac{1}{\det(A)} \mathrm{adj}(A)$

4) The triangle inequality can be proven using Cauchy-Schwarz inequality. See page 18 of the textbook.

HW 2

1) $T = (\underset{\sim}{b}_1' \; \underset{\sim}{b}_2' \; \underset{\sim}{b}_3')^{-1} I = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$\underset{\sim}{x}' = T \underset{\sim}{x} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

new coordinate vector

Note: $\underset{\sim}{x} = 2\underset{\sim}{e}_1 + \underset{\sim}{e}_2 = \underset{\sim}{b}_1' + \underset{\sim}{b}_2'$   $\underset{\sim}{e}_i$: st. basis vector

2) $\det(\lambda I - A) = \det \begin{pmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-2 & 2 \\ 2 & 2 & \lambda-6 \end{pmatrix} = \lambda(\lambda-2)(\lambda-8)$

Eigenvalues are $8, 2, 0$. Rank$(A) = 2$ due to zero eigenvalue.

To find eigenvectors we solve $\begin{pmatrix} \lambda-2 & -2 & 2 \\ -2 & \lambda-2 & 2 \\ 2 & 2 & \lambda-6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$

For $\lambda = 8$, we find $v_1 = v_2$, $v_3 = -2v_2$. The normalized eigen vector is $\underset{\sim}{v}_a = \frac{1}{\sqrt{6}}(1, 1, -2)^T$

For $\lambda = 2$, we find $v_1 = v_3$, $v_2 = v_3$. The normalized eigenvector is $\underset{\sim}{v}_b = \frac{1}{\sqrt{3}}(1,1,1)^T$

For $\lambda = 0$, we have $v_1 = -v_2$, $v_3 = 0$. The normalized eigenvector is $\underset{\sim}{v}_c = \frac{1}{\sqrt{2}}(1,-1,0)^T$

The diagonalizing matrix $V = [\underset{\sim}{v}_a, \underset{\sim}{v}_b, \underset{\sim}{v}_c]$

$$= \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Note: $V^T A V = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

An orthogonal basis for $\mathbb{R}^3$ is $\underset{\sim}{v}_a, \underset{\sim}{v}_b, \underset{\sim}{v}_c$

- HW 2 continued

3) Solution for $N(A)$ can be found solving

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \quad \begin{matrix} x_3 = -x_2 \\ x_2 = -x_1 \end{matrix}$$

$N(A)$ is the space of vectors $k\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ (single basis vectors)

A basis for $R(A^T)$ can be found using elementary row operations

$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{Basis for } R(A^T) \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Note that every basis vector of $R(A^T)$ is orthogonal to the vector $(1,-1,1)^T$ (i.e. the basis vector of $N(A)$).

4) $$A A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda-5 & -4 \\ -4 & \lambda-5 \end{pmatrix} = (\lambda-1)(\lambda-9) \Rightarrow \lambda_1 = 9$$

$$\|A\| = \sqrt{\lambda_1} = 3$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} = (\lambda-1)(\lambda-3)$$

$$\sigma(A) = 3 .$$

2.6 $\quad x_1 + x_2 + 2x_3 + x_4 = 1$
$\quad\quad x_1 - 2x_2 \quad\quad - x_4 = -2 \qquad b = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & -2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & -3 & -2 & -2 \end{bmatrix} \Rightarrow \text{rank}(A) = 2$$

$$[A\ b] = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -2 & 0 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & -3 & -2 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 & 3 \end{bmatrix}$$

$\Rightarrow \text{rank}([A\ b]) = 2 \Rightarrow$ A solution exists.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{aligned} x_1 &= -\tfrac{4}{3}x_3 - \tfrac{1}{3}x_4 \\ x_2 &= 1 - \tfrac{2}{3}x_3 - \tfrac{2}{3}x_4 \end{aligned}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -4/3 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} k_1 + \begin{pmatrix} -1/3 \\ -2/3 \\ 0 \\ 1 \end{pmatrix} k_2$$

$$= x_p + x_{N_A}$$

particular sol. + null space vector

3.7 One has to show only the closure properties.

Let $x, y \in V^\perp$ and $v \in V$

$$v^T(\alpha x + \beta y) = \alpha \cancel{v^T x}^{\,0} + \beta \cancel{v^T y}^{\,0} = 0$$

$\alpha, \beta \in \mathbb{R}$ $\qquad \Rightarrow$ subspace.

3.8
$$A = \begin{bmatrix} 4 & -2 & 0 \\ 2 & 1 & -1 \\ 2 & -3 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & -2 & 0 \\ 0 & 2 & -1 \\ 0 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = U$$

$$U\underset{\sim}{x} = \underset{\sim}{0} \Rightarrow \begin{aligned} x_1 &= \tfrac{1}{4}x_3 \\ x_2 &= \tfrac{1}{2}x_3 \end{aligned} \Rightarrow \underset{\sim}{x} = \begin{pmatrix} 1/4 \\ 1/2 \\ 1 \end{pmatrix} k$$

3.15 $\tilde{Q} = \frac{1}{2}(Q + Q^T)$. Leading principal minors of $\hat{Q}$ are $\Delta_1 = 1$, $\Delta_2 = -11.25 \Rightarrow$ indefinite.

3.17
a) $Q = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $\Delta_2 = -1$, $\Delta_3 = 2 \Rightarrow$ indefinite.

Or, $\mathrm{tr}\{Q\} = 0$. At least one eigenvalue is positive and
" " is negative.

b) $\mathcal{M} \equiv \mathcal{N}\{[1\,1\,1]\} = \mathrm{span}\left\{\begin{pmatrix}1\\-2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$

$u \in \mathcal{M} \Rightarrow u = \alpha\begin{pmatrix}1\\-2\\1\end{pmatrix} + \beta\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{bmatrix}1 & 1\\-2 & -1\\1 & 0\end{bmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix} = B\begin{pmatrix}\alpha\\\beta\end{pmatrix}$

$u^T Q u = (\alpha\ \beta)\, B^T Q B \begin{pmatrix}\alpha\\\beta\end{pmatrix} = (\alpha\ \beta)\, \hat{Q} \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ $\begin{pmatrix}\alpha\\\beta\end{pmatrix} \in \mathbb{R}^2$

$\hat{Q} = B^T Q B = \begin{bmatrix}-6 & -3\\-3 & -2\end{bmatrix}$ negative definite. Thus, $Q$ is negative definite in $\mathcal{M}$.

Other way: $(x_1, x_2, -x_1 - x_2)\begin{pmatrix}0 & 1 & 1\\1 & 0 & 1\\1 & 1 & 0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\-x_1-x_2\end{pmatrix}$

$= 2(x_1 x_2 - x_1(x_1 + x_2) - x_2(x_1 + x_2)) = -x_1^2 - x_2^2 - (x_1 + x_2)^2 < 0$
$\Rightarrow$ negative definite.

3.20 $Q = \begin{bmatrix}1 & \varepsilon & -1\\ \varepsilon & 1 & 2\\ -1 & 2 & 5\end{bmatrix}$ $\Delta_1 = 1$, $\Delta_2 = 1 - \varepsilon^2$
$\Delta_3 = -5\varepsilon^2 - 4\varepsilon$

$f = \underset{\sim}{x}^T Q \underset{\sim}{x}$

$f$ positive definite when $\varepsilon \in (-\frac{4}{5}, 0)$

4.2 Let $\|x_1\| \leq r$ and $\|x_2\| \leq r$. Then

$\|\alpha x_1 + (1-\alpha) x_2\| \leq \|\alpha x_1\| + \|(1-\alpha) x_2\|$ triangle inequality

$0 \leq \alpha \leq 1$

$= \alpha \|x_1\| + (1-\alpha)\|x_2\|$

$\leq \alpha r + (1-\alpha) r = r$

Thus, the set is convex.

# Fu work #1

Solutions

**1**

- For the matrix,

$$ = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 0 & 1 \\ 3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix},$$

find its rank by first transforming the matrix by means of the row elementary operations into an upper triangular form.

- Find the rank of the following matrix for different values of the parameter $\gamma$ by first transforming the matrix by means of the row elementary operations into an upper triangular form,

$$ = \begin{bmatrix} 1 & \gamma & 1 & 2 \\ 2 & 1 & \gamma & 5 \\ 1 & 10 & 6 & 1 \end{bmatrix}.$$

**Solution:**

- Performing a sequence of row elementary operations, we obtain

$$ = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 2 & 1 & 3 & 0 & 1 \\ 3 & 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 5 & 5 & 6 & 3 \\ 0 & 5 & 5 & 6 & 3 \\ 0 & 0 & 4 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 5 & 5 & 6 & 3 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \boldsymbol{B}$$

Because elementary operations do not change the rank of a matrix, hence rank( ) = rank($\boldsymbol{B}$). Therefore rank( ) = 3.

- Performing a sequence of row elementary operations, we obtain

$$= \begin{bmatrix} 1 & \gamma & 1 & 2 \\ 2 & 1 & \gamma & 5 \\ 1 & 10 & 6 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 6 & 1 \\ 1 & \gamma & 1 & 2 \\ 2 & 1 & \gamma & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 6 & 1 \\ 0 & \gamma \quad 10 & 5 & 1 \\ 0 & 21 & \gamma+12 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 10 & 6 \\ 0 & 1 & \gamma \quad 10 & 5 \\ 0 & 3 & 21 & \gamma+12 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 10 & 6 \\ 0 & 1 & \gamma \quad 10 & 5 \\ 0 & 0 & 3(\gamma \quad 3) & \gamma \quad 3 \end{bmatrix} = \boldsymbol{B}$$

Because elementary operations do not change the rank of a matrix, hence rank( ) = rank($\boldsymbol{B}$). Therefore rank( ) = 3 if $\gamma \neq 3$ and rank( ) = 2 if $\gamma = 3$.

**2** Consider the following system of equations,

$$\left.\begin{array}{rcl} x_1 + x_2 + 2x_3 + x_4 & = & 1 \\ x_1 \quad 2x_2 \quad x_4 & = & 2 \end{array}\right\}$$

Use Theorem 2.1 to check if the system has a solution. Then, use the method of the proof of Theorem 2.2 to find a general solution to the system.

**Solution:** We represent the given system of equations in the form $\quad \boldsymbol{x} = \boldsymbol{b}$, where

$$= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix}, \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \text{ and } \qquad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Using row elementary operations yields

$$= \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & 2 & 2 \end{bmatrix}, \text{and}$$

$$[\quad, \boldsymbol{b}] = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 & 3 \end{bmatrix},$$

from which rank$(A) = 2$ and rank$[\quad, \boldsymbol{b}] = 2$.

Therefore, by Theorem 2.1 the system has a solution.

We next represent the system of equations as

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \quad 2x_3 \quad x_4 \\ 2 + x_4 \end{bmatrix}$$

Assigning arbitrary values to $x_3$ and $x_4$ ($x_3 = d_3$, $x_4 = d_4$), we get

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{1} \begin{bmatrix} 1 \quad 2x_3 \quad x_4 \\ 2 + x_4 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \quad 2x_3 \quad x_4 \\ 2 + x_4 \end{bmatrix} \\ &= \begin{bmatrix} \frac{4}{3}d_3 \quad \frac{1}{3}d_4 \\ 1 \quad \frac{2}{3}d_3 \quad \frac{2}{3}d_4 \end{bmatrix}. \end{aligned}$$

Therefore, a general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}d_3 \quad \frac{1}{3}d_4 \\ 1 \quad \frac{2}{3}d_3 \quad \frac{2}{3}d_4 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} d_3 + \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} d_4 + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

where $d_3$ and $d_4$ are arbitrary values.

**3** Find the nullspace of

$$= \begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

**Solution:** The null space of is $\mathcal{N}(\ ) = \{\boldsymbol{x} \in {}^3 : \ \boldsymbol{x} = 0\}$. Using elementary row operations and back-substitution, we can solve the system of equations:

$$\begin{bmatrix} 4 & 2 & 0 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} 4x_1 \quad 2x_2 &= 0 \\ 2x_2 \quad x_3 &= 0 \end{aligned}$$

$$\Rightarrow \qquad x_2 = \frac{1}{2}x_3\ , \qquad x_1 = \frac{1}{2}x_2 = \frac{1}{4}x_3 \qquad \Rightarrow \qquad \boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \end{bmatrix} x_3\ .$$

Therefore, $\mathcal{N}(\quad) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} c : c \in \quad \right\}$.

**4** Find the transformation matrix $\boldsymbol{T}$ from $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ to $\{\boldsymbol{e}_1', \boldsymbol{e}_2', \boldsymbol{e}_3'\}$, where

(a) $\boldsymbol{e}_1' = \boldsymbol{e}_1 + 3\boldsymbol{e}_2 \quad 4\boldsymbol{e}_3$, $\boldsymbol{e}_2' = 2\boldsymbol{e}_1 \quad \boldsymbol{e}_2 + 5\boldsymbol{e}_3$, $\boldsymbol{e}_3' = 4\boldsymbol{e}_1 + 5\boldsymbol{e}_2 + 3\boldsymbol{e}_3$.

(b) $\boldsymbol{e}_1 = \boldsymbol{e}_1' + \boldsymbol{e}_2' + 3\boldsymbol{e}_3'$, $\boldsymbol{e}_2 = 2\boldsymbol{e}_1' \quad \boldsymbol{e}_2' + 4\boldsymbol{e}_3'$, $\boldsymbol{e}_3 = 3\boldsymbol{e}_1' + 5\boldsymbol{e}_3'$.

**Solution:**

(a)

$$\begin{bmatrix} \boldsymbol{e}_1' & \boldsymbol{e}_2' & \boldsymbol{e}_3' \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & \ 1 & 5 \\ \ 4 & 5 & 3 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{T} = \begin{bmatrix} \boldsymbol{e}_1' & \boldsymbol{e}_2' & \boldsymbol{e}_3' \end{bmatrix}^{1} \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 3 & \ 1 & 5 \\ \ 4 & 5 & 3 \end{bmatrix}^{1} = \frac{1}{42}\begin{bmatrix} 28 & 14 & 14 \\ 29 & 19 & 7 \\ \ 11 & 13 & 7 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_1' & \boldsymbol{e}_2' & \boldsymbol{e}_3' \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & \ 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$

Therefore,

$$T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}.$$

**5** Given two bases, $\{e_1, e_2, e_3, e_4\}$ and $\{e'_1, e'_2, e'_3, e'_4\}$ of $^4$, where $e'_1 = e_1$, $e'_2 = e_1 + e_2$, $e'_3 = e_1 + e_2 + e_3$, $e'_4 = e_1 + e_2 + e_3 + e_4$, and the matrix representation of a linear transformation in $\{e_1, e_2, e_3, e_4\}$ of the form

$$= \begin{bmatrix} 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 0 & 0 & 3 \end{bmatrix}.$$

Find the matrix representation of the linear transformation in the basis $\{e'_1, e'_2, e'_3, e'_4\}$.

**Solution:**

$$\begin{bmatrix} e'_1 & e'_2 & e'_3 & e'_4 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Therefore, the transformation matrix from $\{e_1\ ,\ e_2\ ,\ e_3\ ,\ e_4\}$ to $\{e'_1\ ,\ e'_2\ ,\ e'_3\ ,\ e'_4\}$ is

$$T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{\ 1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now consider a linear transformation $L : \ ^4 \to \ ^4$, and let be its representation with respect to $\{e_1\ ,\ e_2\ ,\ e_3\ ,\ e_4\}$, and $B$ its representation with respect to $\{e'_1\ ,\ e'_2\ ,\ e'_3\ ,\ e'_4\}$. Let $y = \ x$ and $y' = Bx'$. Then,

$$y' = Ty = T(\ x) = T \ (T^{\ 1}x') = (T \ T^{\ 1})x'\ .$$

Therefore,

$$
\boldsymbol{B} = \boldsymbol{T} \quad \boldsymbol{T}^{\ 1} = \begin{bmatrix} 5 & 3 & 4 & 3 \\ 3 & 2 & 1 & 2 \\ 1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 4 \end{bmatrix} .
$$

**6** Given two bases, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ of $\ ^3$, where $\boldsymbol{e}_1 = 2\boldsymbol{e}'_1 + \boldsymbol{e}'_2 \quad \boldsymbol{e}'_3$, $\boldsymbol{e}_2 = 2\boldsymbol{e}'_1 \quad \boldsymbol{e}'_2 + 2\boldsymbol{e}'_3$, $\boldsymbol{e}_3 = 3\boldsymbol{e}'_1 + \boldsymbol{e}'_3$, and the matrix representation of a linear transformation in $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of the form

$$
= \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} .
$$

Find the matrix representation of the linear transformation in the basis $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$.

**Solution:**

$$
\begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}'_1 & \boldsymbol{e}'_2 & \boldsymbol{e}'_3 \end{bmatrix} \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} .
$$

Therefore, the transformation matrix from $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ to $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is

$$
\boldsymbol{T} = \begin{bmatrix} 2 & 2 & 3 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} ,
$$

and the representation of the linear transformation with respect to $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3\}$ is

$$
\boldsymbol{B} = \boldsymbol{T} \quad \boldsymbol{T}^{\ 1} = \begin{bmatrix} 3 & 10 & 8 \\ 1 & 8 & 4 \\ 2 & 13 & 7 \end{bmatrix} .
$$

**7** Find the basis in which the matrix

$$
= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 5 & 2 & 1 \\ 1 & 1 & 0 & 3 \end{bmatrix}
$$

is diagonal

**Solution:** Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ be a set of linearly independent eigenvectors of cor- responding to the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\lambda_4$. Let $T = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3\ \boldsymbol{v}_4]$. Then,

$$\begin{aligned}
\boldsymbol{T} &= [\,\boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\,] = [\quad \boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \boldsymbol{v}_3 \quad \boldsymbol{v}_4\,] \\
&= [\,\lambda_1\boldsymbol{v}_1\ \ \lambda_2\boldsymbol{v}_2\ \ \lambda_3\boldsymbol{v}_3\ \ \lambda_4\boldsymbol{v}_4\,] = [\,\boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \ \boldsymbol{v}_3\ \ \boldsymbol{v}_4\,] \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}.
\end{aligned}$$

Hence, $\quad \boldsymbol{T} = \boldsymbol{T} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$, or $\boldsymbol{T}^{\ 1}\ \ \boldsymbol{T} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$.

Therefore, the matrix has a diagonal form with respect to the basis formed by a linearly independent set of eigenvectors.

Because

$$\det(\quad) = (\lambda \quad 2)(\lambda \quad 3)(\lambda \quad 1)(\lambda + 1),$$

the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$, $\lambda_3 = 1$, and $\lambda_4 = \quad 1$.

From $\quad \boldsymbol{v}_i = \lambda_i \boldsymbol{v}_i$, where $\boldsymbol{v}_i \neq 0$ $(i = 1, 2, 3, 4)$, the corresponding eigenvectors are

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 9 \\ 1 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{v}_4 = \begin{bmatrix} 24 \\ 12 \\ 1 \\ 9 \end{bmatrix}.$$

Therefore, the basis is $\{\boldsymbol{v}_1\ ,\ \boldsymbol{v}_2\ ,\ \boldsymbol{v}_3\ ,\ \boldsymbol{v}_4\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 24 \\ 12 \\ 1 \\ 9 \end{bmatrix} \right\}$.

**8** Determine if the following quadratic forms are positive definite, negative definite, positive semidefinite, negative semidefinite, or indefinite:

(a) $f(x_1, x_2, x_3) = x_2^2$;

(b) $f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 \quad x_1x_3$;

(c) $f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

**Solution:**

(a)

$$f(x_1, x_2, x_3) = x_2^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\boldsymbol{Q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = 0$. Therefore, the quadratic form is positive semidefinite.

(b)

$$f(x_1, x_2, x_3) = x_1^2 + 2x_2^2 \quad x_1x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 2 & 0 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$ and the eigenvalues of $\boldsymbol{Q}$ are $\lambda_1 = 2$, $\lambda_2 = (1 \quad \sqrt{2})/2$, and $\lambda_3 = (1 + \sqrt{2})/2$. Therefore, the quadratic form is indefinite.

(c)

$$f(x_1, x_2, x_3) = x_1^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $Q = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and the eigenvalues of $Q$ are $\lambda_1 = 0$, $\lambda_2 = 1 - \sqrt{3}$, and $\lambda_3 = 1 + \sqrt{3}$. Therefore, the quadratic form is indefinite.

3. Compute the linear, $l(x_1, x_2)$, and quadratic, $q(x_1, x_2)$, approximations of the function

$$f = f(x_1, x_2) = x_1^3 + x_1 x_2 - x_1^2 x_2^2,$$

at the point $\boldsymbol{x}^{(0)} = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$.

**Answer:**

(i) We use the first-order Taylor series expansion to obtain a linear approximation of $f$,

$$l(\boldsymbol{x}) = f\left(\boldsymbol{x}^{(0)}\right) + \nabla f\left(\boldsymbol{x}^{(0)}\right)^T \left(\boldsymbol{x} - \boldsymbol{x}^{(0)}\right),$$

where

$$\nabla f\left(\boldsymbol{x}^{(0)}\right) = \left. \begin{bmatrix} 3x_1^2 + x_2 - 2x_1 x_2^2 \\ x_1 - 2x_1^2 x_2 \end{bmatrix} \right|_{\boldsymbol{x}=\boldsymbol{x}^{(0)}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$