a) Find basis for the four fundamental subspaces of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$.
b) Express the sol. set for $Ax = b$. where $b = (3,1)^T$. Is the sol. set a subspace?

1) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

a) $R(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\} = \mathbb{R}^2$

$N(A^T) = \left\{ y : \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \right\} \Rightarrow \underset{\sim}{y} = \underset{\sim}{0}$ no basis

$R(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

$N(A) = \left\{ y : \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} y = \underset{\sim}{0} \right\} \Rightarrow N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \Rightarrow x = \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{x_p} + k \underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{x_N} = \begin{pmatrix} 1+k \\ 1 \\ 1-k \end{pmatrix}$ $k \in \mathbb{R}$

No subspace, $\underset{\sim}{0}$ not included.

– Determine whether the quadratic form $f(x) = x_1^2 + 2x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$ is positive-definite. Is it positive-definite within the subspace $\text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$?

2) $f(x) = x_1^2 + 2x_2^2 + 2x_3^2 + 4x_1x_2 + 2x_1x_3 + 2x_2x_3$

$= \underset{\sim}{x}^T \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \underset{\sim}{x} = \underset{\sim}{x}^T Q \underset{\sim}{x}$ $\quad x \in \mathbb{R}^3$

leading principal minors: $\cancel{1 \cdot (2^2-1) - 2 \times (2^2-1) + 1 + 1 \times 0 = -3}$

$\det \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} < 0$ not p.d., $1 > 0$ ✓

For p. definiteness within the subspace spanned by $(1\ {-1}\ 1)^T$, $(0\ 1\ 1)^T$:

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 1 & 3 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 6 \end{bmatrix}$$

not p.d.

1- Suppose a wireless broadcast system has n transmitters. Transmitter j broadcasts at a power $p_j \geq 0$. There are m locations where the broadcast is to be received. The "path gain" from transmitter j to location i is $g_{i,j}$ (the power received from transmitter j at location i is $g_{i,j} p_j$). The total received power at location i is the sum of the received powers from all the transmitters. Formulate the problem of finding the minimum sum of the transmit powers subject to the requirement that the received power at each location is at least P.

2- Put the following problem into standard form:

$$\text{minimize} \quad z = 25x_1 + 30x_2, \quad \text{subject to} \begin{cases} 4x_1 + 7x_2 \geq 1 \\ 8x_1 + 5x_2 \leq 3. \end{cases}$$

3- Solve the following linear programming problem:

$$\min \; 2x_1 + 3x_2 - 4x_3 - 5x_4 + x_5 - 2x_6, \quad \text{sbj. to} \begin{cases} x_1 - 2x_2 + 2x_5 + 7x_6 = 3 \\ 3x_2 + x_4 - x_5 + 4x_6 = 5 \\ x_3 - 3x_5 + 2x_6 = 4 \\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{cases}$$

4- For the minimization of the cost function $f(\underset{\sim}{x}) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$, starting from the initial solution $(1,1)^T$, generate one iteration using i) steepest-descent ii) Newton method.

5- Put the following linear programming problem into standard form:

$$\max \quad x_1 + x_2, \qquad \text{subject to} \begin{cases} x_1 + 2x_2 \leq 10 \\ |x_1| - x_2 \leq 5 \\ x_2 \geq 1 \end{cases}$$

6- Solve the following linear programming problem using two-phase simplex method:

$$\min \; 2x_1 + 3x_2 + x_3, \qquad \text{subject to} \begin{cases} x_1 + x_2 + 2x_3 + x_4 = 5 \\ x_1 + x_2 + x_3 - x_4 = 5 \\ x_1 + 2x_2 + 2x_3 = 6 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

1-

$$\min \; p_1 + p_2 + \dots + p_n$$

$$\text{sbj. to } \sum_{j=1}^{n} g_{ij}\, p_j \geq P \quad i = 1, 2, \dots, m.$$

$$p_1, p_2, \dots, p_n \geq 0$$

8

2-

)

$$\min \; 25x_1 + 30x_2 = z$$

$$\text{sbj. to } \begin{cases} 4x_1 + 7x_2 \geq 1 \\ 8x_1 + 5x_2 \leq 3 \end{cases}$$

x_1, x_2 free variables

$x_1 = u - v \qquad u, v, w, y \geq 0$

$x_2 = w - y$

$$\min \; 25u + 30w - 25v - 30y$$

$$4u + 7w - 4v - 7y - p = 1$$

$$8u + 5w - 8v - 5y + q = 3$$

$$u, v, w, y, p, q \geq 0$$

12

3-

$$\min\ 2x_1 + 3x_2 - 4x_3 - 5x_4 + x_5 - 2x_6 \quad \text{subj. to} \quad \begin{cases} x_1 - 2x_2 + 2x_5 + 7x_6 = 3 \\ 3x_2 + x_4 - x_5 + 4x_6 = 5 \\ x_3 - 3x_5 + 2x_6 = 4 \\ x_1, x_2, \dots, x_6 \geq 0 \end{cases}$$

Simplex tableau

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & b \end{array}$$
$$\begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 7 & 3 \\ 0 & 3 & 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 1 & 0 & -3 & 2 & 4 \\ 2 & 3 & -4 & -5 & 1 & -2 & 0 \end{bmatrix}$$

basic variables: x_1, x_3, x_4

Let us put the tableau into a standard (canonical) form

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 7 & 3 \\ 0 & 3 & 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 1 & 0 & -3 & 2 & 4 \\ 0 & 7 & -4 & -5 & -3 & -16 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 7 & 3 \\ 0 & 3 & 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 1 & 0 & -3 & 2 & 4 \\ 0 & 7 & 0 & -5 & -15 & -8 & 10 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 0 & 0 & 2 & 7 & 3 \\ 0 & 3 & 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 1 & 0 & -3 & 2 & 4 \\ 0 & 22 & 0 & 0 & \underline{-20} & 12 & 35 \end{bmatrix}$$

x_1 is the leaving var.
x_5 is the entering "

$$\sim \begin{bmatrix} 1/2 & -1 & 0 & 0 & 1 & 7/2 & 3/2 \\ 1/2 & 2 & 0 & 1 & 0 & 15/2 & 13/2 \\ 3/2 & -3 & 1 & 0 & 0 & 25/2 & 17/2 \\ 10 & 2 & 0 & 0 & 0 & 82 & 65 \end{bmatrix}$$

✓

$x_3 = 17/2$

$x_4 = 13/2$

$x_5 = 3/2$

cost $= -65$

25

5) max $x_1 + x_2$ sbj. to $\begin{cases} x_1 + 2x_2 \le 10 \\ |x_1| - x_2 \le 5 \\ x_2 \ge 1 \end{cases}$

$x_2' = x_2 - 1$

$x_2' + 1 = x_2 \quad \Rightarrow \quad x_2' \ge 0$

(or $x_2 - a_1 = 1$)

$|x_1| - x_2 \le 5 \Rightarrow |x_1| \le 5 + x_2$

$\Rightarrow |x_1| \le 6 + x_2'$ 15

$\Rightarrow x_1 \le 6 + x_2', \quad x_1 \ge -6 - x_2'$

$x_1 - x_2' \le 6 \qquad x_1 + x_2' \ge -6$

$x_1 - x_2' + x_3 = 6 \qquad x_1 + x_2' - x_4 = -6$

$x_1 = y_1 - y_2 \qquad y_1, y_2 \ge 0$

Prl max $y_1 - y_2 + x_2' + 1$

or min $y_2 - y_1 - x_2' - 1$

sbj to

$$y_1 - y_2 + 2x_2' + x_5 = 8$$

$$y_1 - y_2 - x_2' + x_3 = 6$$

$$-y_1 + y_2 - x_2' + x_4 = +6$$

$$x_2', x_3, x_4, x_5, y_1, y_2 \ge 0$$

6)

phase one

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 5 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 5 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 1 & 0 & 0 & 5 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 5 \\ 1 & 2 & 2 & 0 & 0 & 0 & 1 & 6 \\ -3 & -4 & -5 & 0 & 0 & 0 & 0 & -16 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & \frac{5}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & -\frac{3}{2} & -\frac{1}{2} & 1 & 0 & \frac{5}{2} \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ -\frac{1}{2} & -\frac{3}{2} & 0 & \frac{5}{2} & \frac{5}{2} & 0 & 0 & -\frac{7}{2} \end{pmatrix}$$ 25

$$\begin{pmatrix} \frac{1}{2} & 0 & 1 & 1 & 1 & 0 & -\frac{1}{2} & 2 \\ +\frac{1}{2} & 0 & 0 & -1 & 0 & 1 & -\frac{1}{2} & 2 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ -\frac{1}{2} & 0 & 0 & 1 & 1 & 0 & \frac{3}{2} & -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 2 & -1 & 4 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$x_1 = 4$, $x_2 = 1$, x_3 , $x_4 = 0$

phase two

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 2 & 3 & 1 & 0 & 0 \end{pmatrix} \qquad 2x_1 + 3x_2 + x_3$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 3 & 1 & 4 & -8 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 7 & -11 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & -2 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 5 & -11 \end{pmatrix}$$

$$x_1 = 4, \quad x_2 = 1, \quad x_3 = 0 \quad x_4 = 0$$

direct way

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 1 & 1 & 1 & -1 & 5 \\ 1 & 2 & 2 & 0 & 6 \\ 2 & 3 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 5 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & -2 & -10 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -3 & -1 & -11 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & -2 & 4 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 5 & -11 \end{pmatrix}$$

$x_1 = 4$, $x_2 = 1$,
$x_3 = 0$, $x_4 = 0$

Solve the following linear programming problem in two-phase simplex method.

3) $\max\ 3x_1 - x_2$ s.t. $\begin{cases} 2x_1 + x_2 \geq 2 \\ x_1 + 3x_2 \leq 2 \\ x_2 \leq 4 \end{cases}$ $x_1, x_2 \geq 0$

$\min\ -3x_1 + x_2$ s.t. $\begin{cases} 2x_1 + x_2 - x_3 = 2 \\ x_1 + 3x_2 + x_4 = 2 \\ x_2 + x_5 = 4 \end{cases}$

Two initial basic variables are already generated. We need one more.

Art. problem $\min\ x_6$ s.t. $\begin{cases} 2x_1 + x_2 - x_3 + x_6 = 2 \\ x_1 + 3x_2 + x_4 = 2 \\ x_2 + x_5 = 4 \end{cases}$ $x_i \geq 0$

x_1	x_2	x_3	x_4	x_5	x_6	
2	1	-1	0	0	1	2
1	3	0	1	0	0	2
0	1	0	0	1	0	4
0	0	0	0	0	1	0

2	1	-1	0	0	1	2
1	3	0	1	0	0	2
0	1	0	0	1	0	4
-2	-1	1	0	0	0	-2

↑ x_1 entering x_6 leaving

1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	$\frac{1}{2}$	1
0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	1
0	1	0	0	1	0	4
0	0	0	0	0	1	0

x_1	x_2	x_3	x_4	x_5	
1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1
0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	1
0	1	0	0	1	4
-3	1	0	0	0	0

↑

1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1
0	$\frac{5}{2}$	$\frac{1}{2}$	1	0	1
0	1	0	0	1	4
0	$\frac{5}{2}$	$-\frac{3}{2}$	0	0	3

x_4 leaving ↑ x_3 entering

1	3	0	1	0	2
0	5	1	2	0	2
0	1	0	0	1	4
0	10	0	3	0	6

$x_1 = 2$, $x_3 = 2$, $x_5 = 4$

$x_2 = 0$, $x_4 = 0$ cost $= -6$

The KKT conditions define a complementary problem. The linear complementary problem LCP) for quadratic programming QP) has a very similar structure, as discussed in a later section.

4 8 Treatment of GE and EQ Constraints

GE and EQ constraints need some special consideration in the simplex initiation. We present the steps involved in the solution by considering a problem involving LE, GE, and EQ constraints.

$$
\begin{array}{lrrrl}
\text{maximize} & x_1 & x_2 & 2x_3 & \\
\text{subject to} & 2x_1 & x_2 & 2x_3 \le 8 & [1] \\
 & x_1 & x_2 & x_3 \ge 2 & [2] \\
 & -x_1 & x_2 & 2x_3 = 1 & [3] \\
 & x_1 \ge 0 & x_2 \ge 0 & x_3 \ge 0 &
\end{array}
\quad 4.13)
$$

We first bring the problem into standard form by introducing a slack variable x_4 and a surplus variable x_5 as

$$
\begin{array}{lrrrrrl}
\text{maximize} & x_1 & x_2 & 2x_3 & 0x_4 & 0x_5 & \\
\text{subject to} & 2x_1 & x_2 & 2x_3 & x_4 = 8 & & 1) \\
 & x_1 & x_2 & x_3 - x_5 = 2 & & & 2) \\
 & -x_1 & x_2 & 2x_3 & = 1 & & 3) \\
 & x_1 \ge 0 & x_2 \ge 0 & x_3 \ge 0 & x_4 \ge 0 & x_5 \ge 0 &
\end{array}
\quad 4.14)
$$

In the preceding standardized form, a basic feasible starting point is not readily available. That is, a basic feasible solution whereby two of the variables are set to zero and the constraints then furnish the three nonnegative basic variables is not apparent, as was the case with only LE constraints. Our first step in the initiation of the simplex method is to establish a basic feasible solution. Two different approaches to tackle such problems will now be discussed. The Two-Phase approach and the Big M method presented in the following need the introduction of artificial variables.

The Two-Phase Approach

In the two-phase approach, we introduce *artificial variables* x_6 and x_7 in Eqs. 2) and 3) in 4.14), and set up an auxiliary problem that will bring these artificial variables to zero. This auxiliary LP problem is the Phase I of the simplex method.

Phase I for Problems with Artificial Variables

One *artificial variable* for each constraint of the GE and EQ type are added to the standard form in 4.14), and the objective function is set as the minimization of the sum of the artificial variables.

$$
\begin{array}{lllllll}
\text{minimize} & x_6 & x_7 & & & & \\
\text{subject to} & 2x_1 & x_2 & 2x_3 & x_4 = 8 & & 1) \\
 & x_1 & x_2 & x_3 - x_5 & & x_6 = 2 & 2) \\
 & -x_1 & x_2 & 2x_3 & x_7 = 1 & & 3) \\
 & x_1 \geq 0 & x_2 \geq 0 & x_3 \geq 0 & x_4 \geq 0 & & \\
 & x_5 \geq 0 & x_6 \geq 0 & x_7 \geq 0 & & &
\end{array} \qquad 4.15)
$$

This set is in canonical form with the slack and artificial variables in the basis, excepting that coefficients of x_6 and x_7 are not zero in the objective function. Recall that the objective function must be written only in terms of the nonbasic variables. Since this is a minimization problem, we write the first row coefficients same as mentioned previously.

Initial Tableau for Phase I

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
f	0	0	0	0	0	1	1	0
x_4	2	1	2	1	0	0	0	8
x_6	1	1	1	0	−1	1	0	2
x_7	−1	1	2	0	0	0	1	1

To make the coefficients of x_6 and x_7 zero in the first row, the last two equations are to be subtracted from the first row. Equivalently, we can use the constraints 2) and 3) in 4.15) to write $f = x_6 \quad x_7 = \; 2 - x_1 - x_2 - x_3 \quad x_5) \quad 1 \quad x_1 - x_2 - 2x_3) = -2x_2 - 3x_3 \quad x_5 \quad 3$.

First Step Is To Bring the Equations to Canonical Form

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
f	0	−2	−3	0	1	0	0	−3
x_4	2	1	2	1	0	0	0	8
x_6	1	1	1	0	−1	1	0	2
x_7	−1	1	[2]	0	0	0	1	1

Phase I is solved using the simplex method. The tableaus leading to the minimum, are given as follows. If the minimum objective of zero cannot be achieved, the original problem does not have a feasible solution.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
f	$-\frac{3}{2}$	$-\frac{1}{2}$	0	0	1	0	$\frac{3}{2}$	$-\frac{3}{2}$
x_4	3	0	0	1	0	0	-1	7
x_6	$[\frac{3}{2}]$	$\frac{1}{2}$	0	0	-1	1	$-\frac{1}{2}$	$\frac{3}{2}$
x_3	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs
f	0	0	0	0	0	1	1	0
x_4	0	-1	0	1	2	-2	0	4
x_1	1	$\frac{1}{3}$	0	0	$-\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	1
x_3	0	$\frac{2}{3}$	1	0	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1

This point with $x_1 = 1$ and $x_3 = 1$ and all other variables at zero correspond to the point A shown in Fig. 4.5. The minimum function value is zero. Also note that the artificial variables are not in the basis. Phase II can be initiated now.

Phase II

Since all artificial variables are nonbasic, the columns corresponding to those variables can be dropped. First, the original objective function negative for maximization problem) is put in the first row, and elementary row operations are performed to get zero values in the entries corresponding to basic variables.

Initial Tableau with Objective Function Added and Artificial Variables Dropped

	x_1	x_2	x_3	x_4	x_5	rhs
f	-1	-1	-2	0	0	0
x_4	0	-1	0	1	2	4
x_1	1	$\frac{1}{3}$	0	0	$-\frac{2}{3}$	1
x_3	0	$\frac{2}{3}$	[1]	0	$-\frac{1}{3}$	1

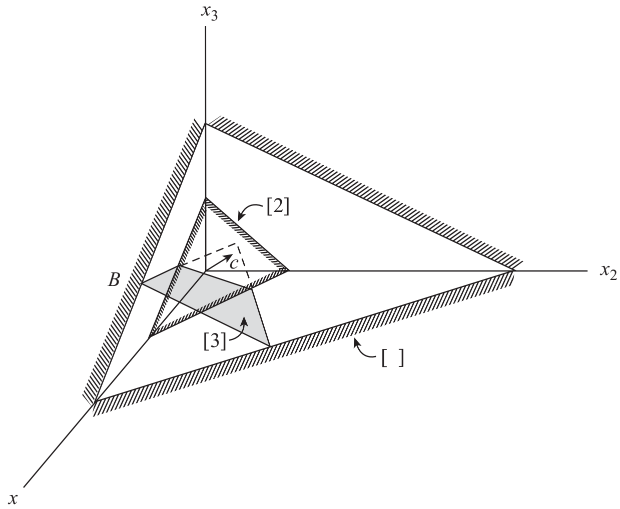

Figure 4.5. Feasible region in the original space.

Coefficients of Basic Variables in the First Row Made Zero

	x_1	x_2	x_3	x_4	x_5	rhs
f	0	$\frac{2}{3}$	0	0	$-\frac{4}{3}$	3
x_4	0	-1	0	1	[2]	4
x_1	1	$\frac{1}{3}$	0	0	$-\frac{2}{3}$	1
x_3	0	$\frac{2}{3}$	1	0	$-\frac{1}{3}$	1

The coefficients in the first row are nonnegative and thus, the solution is optimal.

Second Tableau

	x_1	x_2	x_3	x_4	x_5	rhs
f	0	0	0	$\frac{2}{3}$	0	$\frac{17}{3}$
x_5	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2
x_1	1	0	0	$\frac{1}{3}$	0	$\frac{7}{3}$
x_3	0	$\frac{1}{2}$	1	$\frac{1}{6}$	0	$\frac{5}{3}$

In general, there could exist artificial variables in the basis at zero value. First, redundant rows in the tableau must be detected and deleted. Second, the artificial variables must be pivoted out. See programs implementations.