ν Collective oscillations in 2D

October 29, 2024

1 Introduction

Collective oscillation of neutrinos is determined by the following equation.

$$i\left(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\rho(t,\boldsymbol{x},\boldsymbol{v}) = -\sqrt{2}G_{\mathrm{F}}\int\frac{dv'_z d\phi}{4\pi}(1-\boldsymbol{v}\cdot\boldsymbol{v}')(\rho(t,\boldsymbol{x},\boldsymbol{v}')-\bar{\rho}(t,\boldsymbol{x},\boldsymbol{v}')) \tag{1}$$

1.1 Two flavors

$$\rho = \begin{pmatrix}\rho_{ee} & \rho_{ex}\\ \rho_{ex}^* & \rho_{xx}\end{pmatrix} \tag{2}$$

Another way to represent ρ is,

$$\rho = \left(\frac{f_{\nu_e}+f_{\nu_x}}{2}\right)I + \left(\frac{f_{\nu_e}-f_{\nu_x}}{2}\right)\times\begin{pmatrix}a & S\\ S^* & -a\end{pmatrix} \tag{3}$$

2 Normal modes

Requiring $a^2+|S|^2=1$ and $|S|\ll|a|$, and taking the ansatz $S_{\boldsymbol{v}}(t,\boldsymbol{x}) = Q_{\boldsymbol{v}}^{\omega,\boldsymbol{k}}\exp[-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})]$ we have,

$$\left[(\omega-\epsilon_0)-(\boldsymbol{k}-\boldsymbol{\epsilon})\cdot\boldsymbol{v}\right]Q_{\boldsymbol{v}}^{\omega,\boldsymbol{k}} = -\mu\int\frac{dv'_z d\phi}{2\pi}(1-\boldsymbol{v}\cdot\boldsymbol{v}')G(\boldsymbol{v}')Q_{\boldsymbol{v}'}^{\omega,\boldsymbol{k}} \tag{4}$$

2.1 ID case

Assume ρ homogeneous in x and y and also posses azimuthal symmetry. In that case the eq.4 becomes,

$$\left[(\omega-\epsilon_0)-(k_z-\epsilon_z)v_z\right]Q_{v_z}^{\omega,k_z} = -\mu\int dv'_z(1-v_z v'_z)G(v'_z)Q_{v'_z}^{\omega,k_z} \tag{5}$$

The eq. 5 can be solved for ω as eigen value problem for given k_z.

2.1.1 Example

Consider $G(v_z) = g_{\nu_e}(v_z) - \alpha g_{\bar{\nu}_e}$, where $g_{\nu_e/\bar{,}nu_e}(v_z) \propto \exp[-\frac{(v_z-1)^2}{2\sigma_{\nu_e,\bar{\nu}_e}}]$. We choose values of ν_e and $\bar{\nu}_e$ to be 0.6 and 0.5 respectively. The ELN profile($G(v_z)$) for different values of α and corresponding $\mathrm{Im}(\omega)$ are given below.

2.2 3D Case

Describe the neutrino field evolution in $x-z$ plane. Phase-space distribution of the fields are describe using the velocity component along the z-direction and the azimuthal angle ϕ. In order to justify the treatment of the system only in the $x-z$ plane, we impose the constraint $\rho(v_z,\phi)=\rho(v_z,-\phi)$.

Now the eq. 4 assumes the form,

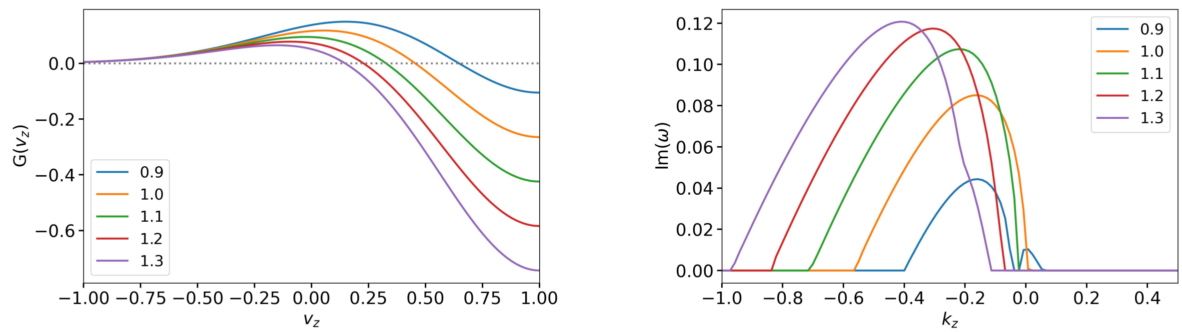

Figure 1: ELN (left) and corresponding $\mathrm{Im}(\omega)$ (right) for different values of α

$$
\begin{aligned}
&\left[(\omega-\epsilon_0)-(k_x-\epsilon_x)v_\perp c_\phi-(k_y-\epsilon_y)v_\perp s_\phi-(k_z-\epsilon_z)v_z\right]Q^{\omega,k_x,k_z}_{v_z}\\
&=-\mu\int\frac{dv'_z d\phi'}{2\pi}\left(1-v_\perp v'_\perp(c_\phi c_{\phi'}+s_\phi s_{\phi'})-v_z v'_z\right)G(v'_z,\phi')Q^{\omega,k_x,k_z}_{v'_z,\phi'}
\end{aligned}
\tag{6}
$$

For simplicity, consider that neutrino beams are constrained to the angular bins such that,

$$
v_z\in\{v_z^-,v_z^+\},\quad \phi\in\{\phi^+,\phi^-\}
\tag{7}
$$

Also, while expanding we only consider v_x and v_z(we neglect quantities along $\hat{y}$). Generalization to full v_x, v_y and v_z is straight forward.

1. $v_z=v_z^+$, $\phi=\phi^+$:

$$
\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}-\left[(k_x-\epsilon_x)v_\perp^+c_{\phi+}+(k_z-\epsilon_z)v_z^+\right]Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}=\\
&-\tilde{\mu}\left\{g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}+g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right\}\\
&+\tilde{\mu}(v_\perp^+c_{\phi+})\left\{v_\perp^+\left(c_{\phi+}g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+c_{\phi-}g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_\perp^-\left(c_{\phi+}g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+c_{\phi-}g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}\\
&+\tilde{\mu}v_z^+\left\{v_z^+\left(g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_z^-\left(g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}
\end{aligned}
$$

2. $v_z=v_z^+$, $\phi=\phi^-$:

$$
\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}-\left[(k_x-\epsilon_x)v_\perp^+c_{\phi-}+(k_z-\epsilon_z)v_z^+\right]Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}=\\
&-\tilde{\mu}\left\{g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}+g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right\}\\
&+\tilde{\mu}(v_\perp^+c_{\phi-})\left\{v_\perp^+\left(c_{\phi+}g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+c_{\phi-}g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_\perp^-\left(c_{\phi+}g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+c_{\phi-}g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}\\
&+\tilde{\mu}v_z^+\left\{v_z^+\left(g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_z^-\left(g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}
\end{aligned}
$$

3. $v_z=v_z^-$, $\phi=\phi^+$:

$$
\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}-\left[(k_x-\epsilon_x)v_\perp^-c_{\phi+}+(k_z-\epsilon_z)v_z^-\right]Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}=\\
&-\tilde{\mu}\left\{g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}+g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right\}\\
&+\tilde{\mu}(v_\perp^-c_{\phi+})\left\{v_\perp^+\left(c_{\phi+}g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+c_{\phi-}g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_\perp^-\left(c_{\phi+}g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+c_{\phi-}g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}\\
&+\tilde{\mu}v_z^-\left\{v_z^+\left(g_{c_{\phi+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^+}+g_{c_{\phi-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^+}\right)+v_z^-\left(g_{c_{\phi+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi+},v_z^-}+g_{c_{\phi-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi-},v_z^-}\right)\right\}
\end{aligned}
$$

4. $v_z = v_z^-$, $\phi = \phi^-$:

$$(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} - \left[(k_x-\epsilon_x)v_\perp^- c_{\phi^-} + (k_z-\epsilon_z)v_z^-\right] Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} =$$
$$-\tilde{\mu}\left\{g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} + g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right\}$$
$$+\tilde{\mu}(v_\perp^- c_{\phi^-})\left\{v_\perp^+\left(c_{\phi^+}g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + c_{\phi^-}g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_\perp^-\left(c_{\phi^+}g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + c_{\phi^-}g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\}$$
$$+\tilde{\mu}v_z^-\left\{v_z^+\left(g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_z^-\left(g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\}$$

Now, if we define a vector $Q^{\omega,k_x,k_z} = \left(Q^{\omega,k_x,k_z}_{\phi^-,v_z^-}, Q^{\omega,k_x,k_z}_{\phi^+,v_z^-}, Q^{\omega,k_x,k_z}_{\phi^-,v_z^+}, Q^{\omega,k_x,k_z}_{\phi^+,v_z^+}\right)^T$ for each pair of (k_x, k_z) the above set equations can be written as following form.

$$\omega Q^{\omega,k_x,k_z} = \left[\epsilon_0 A_0 + (k_x-\epsilon_x)A_x + (k_z-\epsilon_z)A_z\right]Q^{\omega,k_x,k_z} - \tilde{\mu}\left[I_0 - I_x - I_x\right]Q^{\omega,k_x,k_z} \tag{8}$$

where, $\tilde{\mu} = \mu\left(\frac{\Delta v_z \Delta\phi}{2\pi}\right)$ and

$$A_0 = I_{N\times N}, \quad \text{where}, N = n_{v_z}\times n_\phi, \tag{9a}$$

$$A_x = \mathrm{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix}\cos(\phi^-) \\ \cos(\phi^+)\end{bmatrix}\right) \tag{9b}$$

$$A_z = \mathrm{diag}\left(\begin{bmatrix} v_z^- \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix}1 \\ 1\end{bmatrix}\right) \tag{9c}$$

$$I_0 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix} \otimes \left[G(v_z^-,\phi^-),\ G(v_z^-,\phi^+),\ G(v_z^+,\phi^-),\ G(v_z^+,\phi^+)\right] \tag{9d}$$

$$I_x = \begin{bmatrix} v_\perp^-\cos(\phi^-) \\ v_\perp^-\cos(\phi^+) \\ v_\perp^+\cos(\phi^-) \\ v_\perp^+\cos(\phi^+)\end{bmatrix} \otimes \left[v_z^-\cos(\phi^-)G(v_z^-,\phi^-),\ v_z^-\cos(\phi^+)G(v_z^-,\phi^+),\ v_z^+\cos(\phi^-)G(v_z^+,\phi^-),\ v_z^+\cos(\phi^+)G(v_z^+,\phi^+)\right] \tag{9e}$$

$$I_z = \begin{bmatrix} v_z^- \\ v_z^- \\ v_z^+ \\ v_z^+\end{bmatrix} \otimes \left[v_z^- G(v_z^-,\phi^-),\ v_z^- G(v_z^-,\phi^+),\ v_z^+ G(v_z^+,\phi^-),\ v_z^+ G(v_z^+,\phi^+)\right] \tag{9f}$$

The full 3D case is then,

$$\omega Q^{\omega,k_x,k_z} = \left[\epsilon_0 A_0 + (k_x-\epsilon_x)A_x + (k_y-\epsilon_y)A_y + (k_z-\epsilon_z)A_z\right]Q^{\omega,k_x,k_z} - \tilde{\mu}\left[I_0 - I_x - I_x\right]Q^{\omega,k_x,k_z} \tag{10}$$

$$A_0 = I_{N\times N}, \quad \text{where}, N = n_{v_z}\times n_\phi, \tag{11a}$$

$$A_x = \mathrm{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix}\cos(\phi^-) \\ \cos(\phi^+)\end{bmatrix}\right) \tag{11b}$$

$$A_y = \mathrm{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix}\sin(\phi^-) \\ \sin(\phi^+)\end{bmatrix}\right) \tag{11c}$$

$$A_z = \mathrm{diag}\left(\begin{bmatrix} v_z^- \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix}1 \\ 1\end{bmatrix}\right) \tag{11d}$$

$$I_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \left[G(v_z^-, \phi^-),\ G(v_z^-, \phi^+),\ G(v_z^+, \phi^-),\ G(v_z^+, \phi^+) \right] \tag{11e}$$

$$I_x = \begin{bmatrix} v_\perp^- \cos(\phi^-) \\ v_\perp^- \cos(\phi^+) \\ v_\perp^+ \cos(\phi^-) \\ v_\perp^+ \cos(\phi^+) \end{bmatrix} \otimes \left[v_z^- \cos(\phi^-) G(v_z^-, \phi^-),\ v_z^- \cos(\phi^+) G(v_z^-, \phi^+),\ v_z^+ \cos(\phi^-) G(v_z^+, \phi^-),\ v_z^+ \cos(\phi^+) G(v_z^+, \phi^+) \right] \tag{11f}$$

$$I_y = \begin{bmatrix} v_\perp^- \sin(\phi^-) \\ v_\perp^- \sin(\phi^+) \\ v_\perp^+ \sin(\phi^-) \\ v_\perp^+ \sin(\phi^+) \end{bmatrix} \otimes \left[v_z^- \sin(\phi^-) G(v_z^-, \phi^-),\ v_z^- \sin(\phi^+) G(v_z^-, \phi^+),\ v_z^+ \sin(\phi^-) G(v_z^+, \phi^-),\ v_z^+ \sin(\phi^+) G(v_z^+, \phi^+) \right] \tag{11g}$$

$$I_z = \begin{bmatrix} v_z^- \\ v_z^- \\ v_z^+ \\ v_z^+ \end{bmatrix} \otimes \left[v_z^- G(v_z^-, \phi^-),\ v_z^- G(v_z^-, \phi^+),\ v_z^+ G(v_z^+, \phi^-),\ v_z^+ G(v_z^+, \phi^+) \right] \tag{11h}$$