Normal Mode Analysis νFFC-3D

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1 Introduction

Collective oscillation of neutrinos is determined by the following equation.

$$i\left(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right)\rho(t,\boldsymbol{x},\boldsymbol{v}) = -\sqrt{2}G_{\mathrm{F}}\int\frac{dv'_z d\phi}{4\pi}(1-\boldsymbol{v}\cdot\boldsymbol{v}')(\rho(t,\boldsymbol{x},\boldsymbol{v}')-\bar{\rho}(t,\boldsymbol{x},\boldsymbol{v}')) \tag{1}$$

1.1 Two flavors

$$\rho = \begin{pmatrix}\rho_{ee} & \rho_{ex}\\ \rho^*_{ex} & \rho_{xx}\end{pmatrix} \tag{2}$$

Another way to represent ρ is,

$$\rho = \left(\frac{f_{\nu_e}+f_{\nu_x}}{2}\right)I + \left(\frac{f_{\nu_e}-f_{\nu_x}}{2}\right)\times\begin{pmatrix}a & S\\ S^* & -a\end{pmatrix} \tag{3}$$

2 Normal modes

Requiring $a^2+|S|^2=1$ and $|S|\ll|a|$, and taking the ansatz $S_{\boldsymbol{v}}(t,\boldsymbol{x}) = Q^{\omega,\boldsymbol{k}}_{\boldsymbol{v}}\exp[-i(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})]$ we have,

$$\left[(\omega-\epsilon_0)-(\boldsymbol{k}-\boldsymbol{\epsilon})\cdot\boldsymbol{v}\right]Q^{\omega,\boldsymbol{k}}_{\boldsymbol{v}} = -\mu\int\frac{dv'_z d\phi}{2\pi}(1-\boldsymbol{v}\cdot\boldsymbol{v}')G(\boldsymbol{v}')Q^{\omega,\boldsymbol{k}}_{\boldsymbol{v}'} \tag{4}$$

2.1 ID case

Assume ρ homogeneous in x and y and also posses azimuthal symmetry. In that case the eq.4 becomes,

$$\left[(\omega-\epsilon_0)-(k_z-\epsilon_z)v_z\right]Q^{\omega,k_z}_{v_z} = -\mu\int dv'_z(1-v_zv'_z)G(v'_z)Q^{\omega,k_z}_{v'_z} \tag{5}$$

The Eq. (5) can be solved for ω as eigen value problem for given k_z.

2.1.1 Example

Consider $G(v_z) = g_{\nu_e}(v_z) - \alpha g_{\bar{\nu}_e}$, where $g_{\nu_e/\bar{,}nu_e}(v_z)\propto\exp[-\frac{(v_z-1)^2}{2\sigma_{\nu_e,\bar{\nu}_e}}]$. We choose values of ν_e and $\bar{\nu}_e$ to be 0.6 and 0.5 respectively. The ELN profile($G(v_z)$) for different values of α and corresponding Im(ω) are given below.

2.2 3D Case

Describe the neutrino field evolution in $x-z$ plane. Phase-space distribution of the fields are describe using the velocity component along the z-direction and the azimuthal angle ϕ. In order to justify the treatment of the system only in the $x-z$ plane, we impose the constraint $\rho(v_z,\phi)=\rho(v_z,-\phi)$.

Now the Eq. (4) assumes the form,

$$\begin{aligned}&\left[(\omega-\epsilon_0)-(k_x-\epsilon_x)v_\perp c_\phi-(k_y-\epsilon_y)v_\perp s_\phi-(k_z-\epsilon_z)v_z\right]Q^{\omega,k_x,k_z}_{v_z}\\ &= -\mu\int\frac{dv'_z d\phi'}{2\pi}\left(1-v_\perp v'_\perp(c_\phi c_{\phi'}+s_\phi s_{\phi'})-v_zv'_z\right)G(v'_z,\phi')Q^{\omega,k_x,k_z}_{v'_z,\phi'}\end{aligned} \tag{6}$$

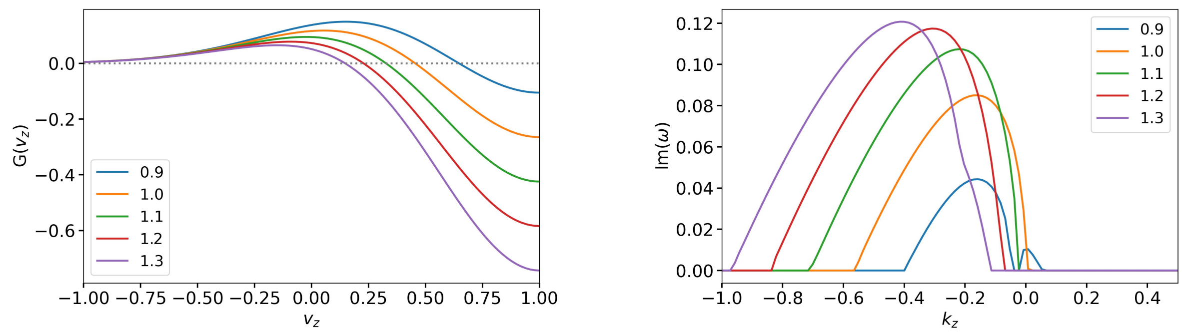

Figure 1: ELN (left) and corresponding $\mathrm{Im}(\omega)$ (right) for different values of α

For simplicity, consider that neutrino beams are constrained to the angular bins such that,

$$v_z \in \left\{v_z^-, v_z^+\right\}, \quad \phi \in \left\{\phi^+, \phi^-\right\} \tag{7}$$

Also, while expanding we only consider v_x and v_z(we neglect quantities along $\hat{y}$). Generalization to full v_x, v_y and v_z is straight forward.

1. $v_z = v_z^+$, $\phi = \phi^+$:

$$\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} - \left[(k_x-\epsilon_x)v_\perp^+ c_{\phi^+} + (k_z-\epsilon_z)v_z^+\right] Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} = \\
&-\tilde{\mu}\left\{g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} + g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right\} \\
&+\tilde{\mu}(v_\perp^+ c_{\phi^+})\left\{v_\perp^+\left(c_{\phi^+}g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + c_{\phi^-}g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_\perp^-\left(c_{\phi^+}g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + c_{\phi^-}g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\} \\
&+\tilde{\mu}v_z^+\left\{v_z^+\left(g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_z^-\left(g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\}
\end{aligned}$$

2. $v_z = v_z^+$, $\phi = \phi^-$:

$$\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} - \left[(k_x-\epsilon_x)v_\perp^+ c_{\phi^-} + (k_z-\epsilon_z)v_z^+\right] Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} = \\
&-\tilde{\mu}\left\{g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} + g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right\} \\
&+\tilde{\mu}(v_\perp^+ c_{\phi^-})\left\{v_\perp^+\left(c_{\phi^+}g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + c_{\phi^-}g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_\perp^-\left(c_{\phi^+}g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + c_{\phi^-}g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\} \\
&+\tilde{\mu}v_z^+\left\{v_z^+\left(g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_z^-\left(g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\}
\end{aligned}$$

3. $v_z = v_z^-$, $\phi = \phi^+$:

$$\begin{aligned}
&(\omega-\epsilon_0)Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} - \left[(k_x-\epsilon_x)v_\perp^- c_{\phi^+} + (k_z-\epsilon_z)v_z^-\right] Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} = \\
&-\tilde{\mu}\left\{g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} + g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right\} \\
&+\tilde{\mu}(v_\perp^- c_{\phi^+})\left\{v_\perp^+\left(c_{\phi^+}g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + c_{\phi^-}g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_\perp^-\left(c_{\phi^+}g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + c_{\phi^-}g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\} \\
&+\tilde{\mu}v_z^-\left\{v_z^+\left(g_{c_{\phi^+},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+}\right) + v_z^-\left(g_{c_{\phi^+},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-}Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-}\right)\right\}
\end{aligned}$$

4. $v_z = v_z^-$, $\phi = \phi^-$:

$$
\begin{aligned}
&(\omega - \epsilon_0) Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} - \left[(k_x - \epsilon_x) v_\perp^- c_{\phi^-} + (k_z - \epsilon_z) v_z^-\right] Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} = \\
&- \tilde{\mu} \left\{ g_{c_{\phi^+},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} + g_{c_{\phi^+},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} \right\} \\
&+ \tilde{\mu}(v_\perp^- c_{\phi^-}) \left\{ v_\perp^+ \left(c_{\phi^+} g_{c_{\phi^+},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + c_{\phi^-} g_{c_{\phi^-},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} \right) + v_\perp^- \left(c_{\phi^+} g_{c_{\phi^+},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + c_{\phi^-} g_{c_{\phi^-},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} \right) \right\} \\
&+ \tilde{\mu} v_z^- \left\{ v_z^+ \left(g_{c_{\phi^+},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^+} + g_{c_{\phi^-},v_z^+} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^+} \right) + v_z^- \left(g_{c_{\phi^+},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^+},v_z^-} + g_{c_{\phi^-},v_z^-} Q^{\omega,k_x,k_z}_{c_{\phi^-},v_z^-} \right) \right\}
\end{aligned}
$$

Now, if we define a vector $Q^{\omega,k_x,k_z} = \left(Q^{\omega,k_x,k_z}_{\phi^-,v_z^-}, Q^{\omega,k_x,k_z}_{\phi^+,v_z^-}, Q^{\omega,k_x,k_z}_{\phi^-,v_z^+}, Q^{\omega,k_x,k_z}_{\phi^+,v_z^+} \right)^T$ for each pair of (k_x, k_z) the above set equations can be written as following form.

$$\omega Q^{\omega,k_x,k_z} = \left[\epsilon_0 A_0 + (k_x - \epsilon_x) A_x + (k_z - \epsilon_z) A_z\right] Q^{\omega,k_x,k_z} - \tilde{\mu} \left[I_0 - I_x - I_z\right] Q^{\omega,k_x,k_z} \tag{8}$$

where, $\tilde{\mu} = \mu \left(\frac{\Delta v_z \Delta \phi}{2\pi} \right)$ and

$$A_0 = I_{N\times N}, \quad \text{where}, N = n_{v_z} \times n_\phi, \tag{9a}$$

$$A_x = \text{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix} \cos(\phi^-) \\ \cos(\phi^+) \end{bmatrix} \right) \tag{9b}$$

$$A_z = \text{diag}\left(\begin{bmatrix} v_z^- \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \tag{9c}$$

$$I_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} G(v_z^-, \phi^-) \\ G(v_z^-, \phi^+) \\ G(v_z^+, \phi^-) \\ G(v_z^+, \phi^+) \end{bmatrix} \tag{9d}$$

$$I_x = \begin{bmatrix} v_\perp^- \cos(\phi^-) \\ v_\perp^- \cos(\phi^+) \\ v_\perp^+ \cos(\phi^-) \\ v_\perp^+ \cos(\phi^+) \end{bmatrix} \otimes \begin{bmatrix} v_\perp^- \cos(\phi^-) G(v_z^-, \phi^-) \\ v_\perp^- \cos(\phi^+) G(v_z^-, \phi^+) \\ v_\perp^+ \cos(\phi^-) G(v_z^+, \phi^-) \\ v_\perp^+ \cos(\phi^+) G(v_z^+, \phi^+) \end{bmatrix} \tag{9e}$$

$$I_z = \begin{bmatrix} v_z^- \\ v_z^- \\ v_z^+ \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix} v_z^- G(v_z^-, \phi^-) \\ v_z^- G(v_z^-, \phi^+) \\ v_z^+ G(v_z^+, \phi^-) \\ v_z^+ G(v_z^+, \phi^+) \end{bmatrix} \tag{9f}$$

The full 3D case is then,

$$\omega Q^{\omega,k_x,k_z} = \left[\epsilon_0 A_0 + (k_x - \epsilon_x) A_x + (k_y - \epsilon_y) A_y + (k_z - \epsilon_z) A_z\right] Q^{\omega,k_x,k_z} - \tilde{\mu} \left[I_0 - I_x - I_y - I_z\right] Q^{\omega,k_x,k_z} \tag{10}$$

$$A_0 = I_{N\times N}, \quad \text{where}, N = n_{v_z} \times n_\phi, \tag{11a}$$

$$A_x = \text{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix} \cos(\phi^-) \\ \cos(\phi^+) \end{bmatrix} \right) \tag{11b}$$

$$A_y = \text{diag}\left(\begin{bmatrix} v_\perp^- \\ v_\perp^+ \end{bmatrix} \otimes \begin{bmatrix} \sin(\phi^-) \\ \sin(\phi^+) \end{bmatrix} \right) \tag{11c}$$

$$A_z = \text{diag}\left(\begin{bmatrix} v_z^- \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \tag{11d}$$

$$I_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} G(v_z^-, \phi^-) \\ G(v_z^-, \phi^+) \\ G(v_z^+, \phi^-) \\ G(v_z^+, \phi^+) \end{bmatrix} \tag{11e}$$

$$I_x = \begin{bmatrix} v_\perp^- \cos(\phi^-) \\ v_\perp^- \cos(\phi^+) \\ v_\perp^+ \cos(\phi^-) \\ v_\perp^+ \cos(\phi^+) \end{bmatrix} \otimes \begin{bmatrix} v_\perp^- \cos(\phi^-) G(v_z^-, \phi^-) \\ v_\perp^- \cos(\phi^+) G(v_z^-, \phi^+) \\ v_\perp^+ \cos(\phi^-) G(v_z^+, \phi^-) \\ v_\perp^+ \cos(\phi^+) G(v_z^+, \phi^+) \end{bmatrix} \tag{11f}$$

$$I_y = \begin{bmatrix} v_\perp^- \sin(\phi^-) \\ v_\perp^- \sin(\phi^+) \\ v_\perp^+ \sin(\phi^-) \\ v_\perp^+ \sin(\phi^+) \end{bmatrix} \otimes \begin{bmatrix} v_\perp^- \sin(\phi^-) G(v_z^-, \phi^-) \\ v_\perp^- \sin(\phi^+) G(v_z^-, \phi^+) \\ v_\perp^+ \sin(\phi^-) G(v_z^+, \phi^-) \\ v_\perp^+ \sin(\phi^+) G(v_z^+, \phi^+) \end{bmatrix} \tag{11g}$$

$$I_z = \begin{bmatrix} v_z^- \\ v_z^- \\ v_z^+ \\ v_z^+ \end{bmatrix} \otimes \begin{bmatrix} v_z^- G(v_z^-, \phi^-) \\ v_z^- G(v_z^-, \phi^+) \\ v_z^+ G(v_z^+, \phi^-) \\ v_z^+ G(v_z^+, \phi^+) \end{bmatrix} \tag{11h}$$