# Lecture 4 – Basic Statistics

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# Lecture X - today

Discrete random variables

Continues random variables

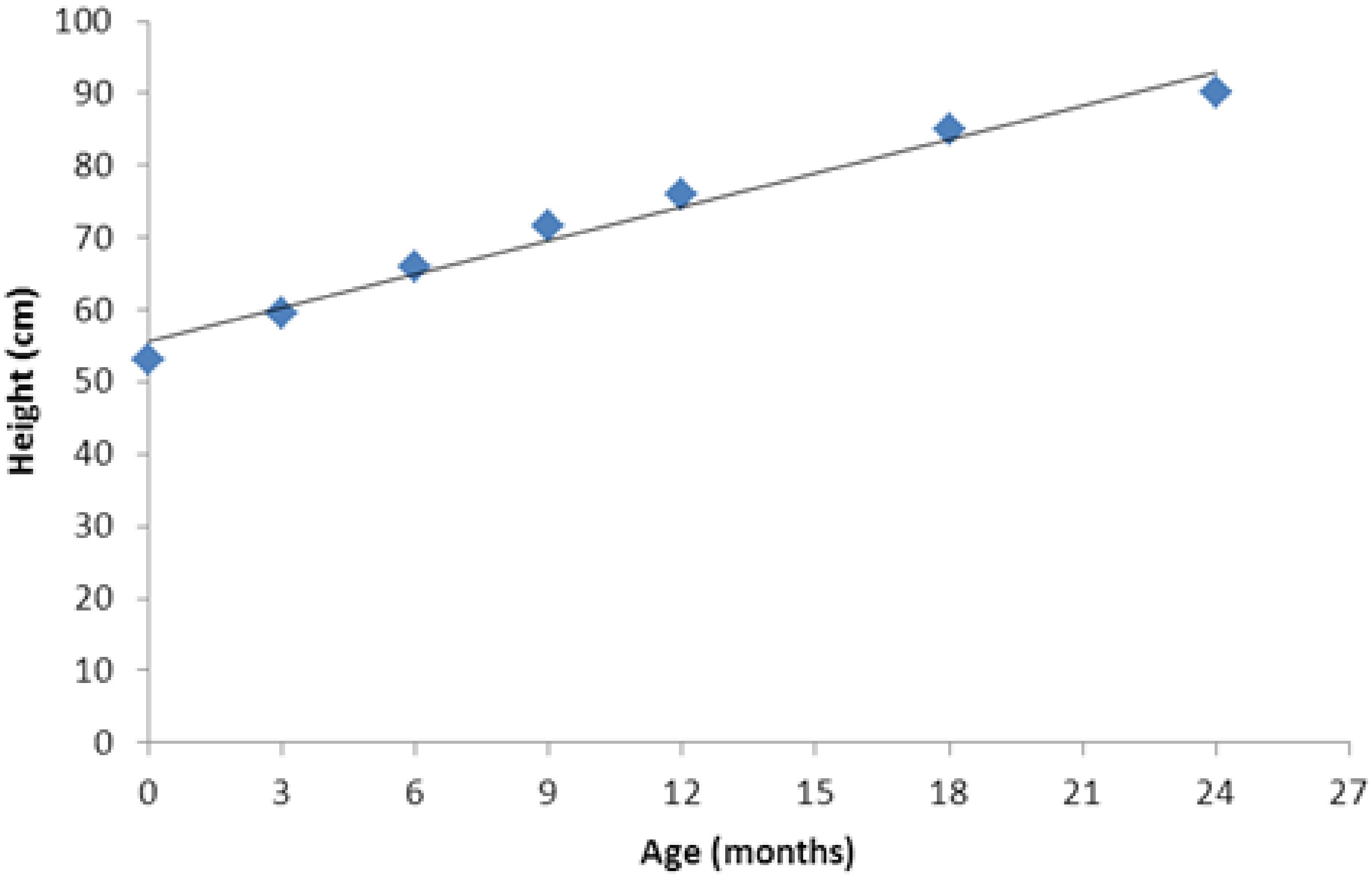
Mean and standard deviation

Bayes' rule

Simple distributions

# Random events

- Height vs age in children
- The linear model does not fit perfectly. Why?



# Random events

Children height depends on may factors:

- Genetics
- Diet
- Health
- etc.

Children height consists of deterministic and random parts

# Discrete random variables

**The total number of different outcomes is limited**

Example:

- Tossing a coin

Outcomes:

- $\Omega$ ={Head (H), Tail (T)}

If the coin is fair, the probability of outcomes:

- P(Y = H) = 0.5
- P(Y = T) = 0.5

The sum probability of all outcomes is always one

# Continues random variables

**The total number of different outcomes is unlimited**

Example:

- Throwing a coin on a round table to see how far from the center it will land

Outcomes:

- $\Omega$ =[0, R]

We cannot calculate probability for exact distance, but we can calculate probability for intervals

The probability for the complete interval [0, R] is again one

# Adding probabilities

**What is the probability of a die landing on x < 4?**

Outcomes:

$\Omega = \{$ ⚀ ⚁ ⚂ ⚃ ⚄ ⚅ $\}$

Probability:

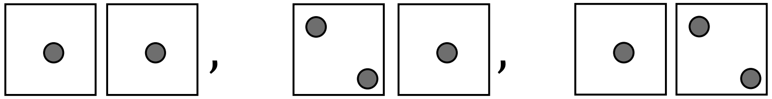
$$P(Y < 4) = P(Y = 1) + P(Y = 2) + P(Y = 3) = 1/6 + 1/6 + 1/6 = 0.5$$

Note that events should be **mutually exclusive**!

# Adding probabilities

**What is the probability of two dice landing on x < 4?**

Outcomes – 36 combinations:

$\Omega$ ={  }

Probability:

$P(Y < 4) = P(Y = 1) + P(Y = 2) + P(Y = 3) = 0/36 + 1/36 + 2/36 = 0.08333$

Note that outcome Y=3 can happen using different combinations of dice landing.

# Conditional probabilities

Example:

- Probability that a person gets university degree is 0.6 ($X = 1$)
- Probability that a person with university degree gets a well-paid job is 0.7 $Y = 1$
- Probability that a person without university degree gets a well-paid job is 0.4

Conditional probabilities:

- $P(Y = y \mid X = x)$ – probability that $Y = y$ happens considering that $X = x$ happened
- $P(Y = 0 \mid X = 0) = 1 – 0.4 = 0.6$
- $P(Y = 0 \mid X = 1) = 1 - 0.7 = 0.3$
- $P(Y = 1 \mid X = 0) = 0.4$
- $P(Y = 1 \mid X = 1) = 0.7$

# Joint probability

What is the probability that two random persons will get a university degree?

- These events are independent, so the joint probability is multiplication of individual probabilities:
  $$P(Y_1 = 1, Y_2 = 1) = P(Y_1 = 1) \cdot P(Y_2 = 1) = 0.6 \cdot 0.6 = 0.36$$

What is the probability that a person will get a university degree and a well-paid job?

- These events are dependent, we need to use conditional probabilities:
  $$P(Y = 1, X = 1) = P(Y = 1 | X = 1) \cdot P(X = 1) = 0.7 \cdot 0.6 = 0.42$$

What are the probabilities of other scenarios for an arbitrary person?

# Joint probability

$$P(Y = y, X = x) =$$
$$P(Y = y|X = x) \cdot P(X = x) = P(X = x|Y = y) \cdot P(Y = y)$$

Let's check what are the values of $P(Y = y|X = x)$ and $P(X = x|Y = y)$ for our example with university degrees and incomes?

| | Degree | No degree |
|---|---|---|
| Well-paid | 0.6 · 0.7 | 0.4 · 0.4 |
| Low-paid | 0.6 · 0.3 | 0.4 · 0.6 |

| | Degree | No degree |
|---|---|---|
| Well-paid | 0.42 | 0.16 |
| Low-paid | 0.18 | 0.24 |

# Added, conditional and joint probability: example 1

Input:

- A woman was killed, and her husband is a suspect
- The husband was abusing the wife
- Defense attorney statement:
    - Only 0.01% of the men who abuse their wives end up murdering them
    - Therefore, the fact that Simpson abused his wife is irrelevant to the case (By irrelevant, he means that the probability of abusiveness importance is very low)
- Why is this a wrong use of conditional probability?

- First statement:
    - P(husband murders wife | husband abuses wife) = 0.0001
- Second statement:
    - P(husband abused wife | woman is murdered) = ?

# Added, conditional and joint probability: example 2

Input:

- You have a database of all life events for each Dane for this day
- You found a person who had a flat tire and lost 100DKK in the same day
- The probability of having a flat tire is $10^{-5}$
- The probability of losing 100DKK is $10^{-7}$
- The events are independent (joint probability = $10^{-12}$), so you deduce that there is a conspiracy against this person
- Is this a correct deduction?

# Added, conditional and joint probability: example 2

Issues:

- You did not check a specific person, but checked all Danes until you find a suitable one
- You first found a specific person, instead of first defining "bad"-events of interest (**Multiple comparisons problem**)
- Basically, you were looking for any person who experienced two arbitrary "bad" events in the database

- What is the probability of finding such a person by a coincidence?

# Added, conditional and joint probability: example 2

Probability of >one “bad” events happening for a selected person:

- First, we need a table with statistics of all events we consider “bad”:
    - Flat tire = $10^{-5}$
    - Losing >=50DKK = $10^{-6}$
    - Injury = $2 \cdot 10^{-5}$
    - Etc.
- Second, we compute the statistics that none of the events happen. The events are independent so their joint probability is the multiplication (we use the rule: $P(z) = 1 - P(\bar{z})$):

$$\text{P(nothing “bad”)} = \underbrace{(1 - 10^{-5})}_{\text{Tire is alright}}\underbrace{(1 - 10^{-6})}_{\text{Did not lose any banknote}}\underbrace{(1 - 2 \cdot 10^{-5})}_{\text{No injury}} \cdots = 0.99$$

# Added, conditional and joint probability: example 2

Probability of >one “bad” events happening for a selected person:

- Third, we compute that one “bad” event happen. Can we add probabilities of individual events?

$$\text{P(one “bad” event)} = 10^{-5} + 10^{-6} + 2 \cdot 10^{-5} + \cdots$$

- We cannot add probabilities, because the events are not **mutually exclusive**!

$$\begin{aligned}\text{P(one “bad” event)} = {} & 10^{-5}(1 - 10^{-6})(1 - 2 \cdot 10^{-5}) \ldots + \\ & (1 - 10^{-5})10^{-6}(1 - 2 \cdot 10^{-5}) \ldots + \\ & (1 - 10^{-5})(1 - 10^{-6})2 \cdot 10^{-5} \ldots + \cdots = 0.00998\end{aligned}$$

# Added, conditional and joint probability: example 2

Probability of >one “bad” events happening for a selected person:

- Finally:

P(>one “bad” event) = 1 – (P(nothing “bad”) + P(one “bad” event))=

1 – (0.99 + 0.00998) = **0.00002**

What is the probability that we will find at least one such a person in the database?

- We need to compute the probability that there is no such a person, and subtract it from 1. There are 5,814,461 people in the database, bad events occurring to them are independent:

P(at least one person found) =

$$1 - (1 - 0.00002)^{5{,}814{,}461} = 1 - 3 \cdot 10^{-51} \approx 1$$

- We will certainly find such a person in the database!

# Bayes' rule

From joint probability formulation:

$$P(Y = y, X = 1) = P(Y = y|X = x) \cdot P(X = x) = P(X = x|Y = y) \cdot P(Y = y)$$

We can get Bayes' rule:

$$P(X = x|Y = y) = \frac{P(Y = y|X = x) \cdot P(X = x)}{P(Y = y)}$$

# Bayes' rule

Returning to our example of university degrees and success (probability of university degree, on condition of well-paid job):

$$P(X = 1|Y = 1) = \frac{P(Y = 1|X = 1) \cdot P(X = 1)}{P(Y = 1)} = \frac{0.7 \cdot 0.6}{0.42 + 0.16} = 0.72$$

| | Degree | No degree |
|---|---|---|
| Well-paid | 0.6 · 0.7 | 0.4 · 0.4 |
| Low-paid | 0.6 · 0.3 | 0.4 · 0.6 |

| | Degree | No degree |
|---|---|---|
| Well-paid | 0.42 | 0.16 |
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# Bayes' rule: example

Input:

The probability of a certain medical test being positive is 90% if a patient has disease D. The 1% of the population have the disease and the test records a false positive 5% of the time. If a random person receives a positive test, what is the probability of D for him?

**5 minutes to think**

# Bayes' rule: example

Input:

The probability of a certain medical test being positive is 90%, if a patient has disease D. 1% of the population have the disease, and the test records a false positive 5% of the time. If a random person receives a positive test, what is the probability of D for him?

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+)}$$

$$P(D) = 0.01; \qquad P(+|D) = 0.9$$

$$P(+) = \underset{0.9}{P(+|D)} \cdot \underset{0.01}{P(D)} + \underset{0.05}{P(+|no\ D)} \cdot \underset{0.99}{P(no\ D)}{=}0.0585$$

$$P(D|+) = \frac{P(+|D) \cdot P(D)}{P(+)} = \frac{0.9 \cdot 0.01}{0.0585} \approx 0.15$$

# Expectation: mean

A calculating student wants to find a most prosperous job and compares two education specialties A and B. He lives in a small city and there is only one company in A and one in B, so he will have to go to a specific company after finishing.

He has got an access to the database of salaries for people working in company A and B:

Salaries in A = [6800, 3150, 2700, 4700, 7100, 5800, 2000]

Salaries in B = [5500, 4500, 3900, 3800, 4800, 4500, 5900]

How to estimate the optimal strategy?

Mean value –

$$\mathbf{E}_{P(x)}\{X\} = \sum_{x} xP(x)$$

# Expectation: mean

Salaries in A = [6800, 3150, 2700, 4700, 7100, 5800, 2000]
Salaries in B = [5500, 4500, 3900, 3800, 4800, 4500, 5900]

Mean value for company A:

$$\mathbf{E}_{P(x)}\{X\} = \sum_x xP(x) =$$

$$6800\frac{1}{7} + 3150\frac{1}{7} + 2700\frac{1}{7} + 4700\frac{1}{7} + 7100\frac{1}{7} + 5800\frac{1}{7} + 2000\frac{1}{7} = \mathbf{4607}$$

Mean value for company B:

$$\mathbf{E}_{P(y)}\{Y\} = \sum_y yP(y) =$$

$$5500\frac{1}{7} + 4500\frac{1}{7} + 3900\frac{1}{7} + 3800\frac{1}{7} + 4800\frac{1}{7} + 4500\frac{1}{7} + 5900\frac{1}{7} = \mathbf{4700}$$

# Expectation: variance

Although the means for A and B are relatively similar, the actual salaries in A and B behave differently, in A salaries are in [2000:7100] while in B salaries are in [3800:5900] intervals

Salaries in A = [6800, 3150, 2700, 4700, 7100, 5800, 2000]
Salaries in B = [5500, 4500, 3900, 3800, 4800, 4500, 5900]

Variance value:

$$\mathrm{var}\{X\} = \mathbf{E}_{P(x)}\left\{\left(X - \mathbf{E}_{P(x)}(X)\right)^2\right\} =$$

$$\sum_{y}\left(x - \mathbf{E}_{P(x)}(X)\right)^2 P(x)$$

# Expectation: variance

Salaries in A = [6800, 3150, 2700, 4700, 7100, 5800, 2000]
Salaries in B = [5500, 4500, 3900, 3800, 4800, 4500, 5900]

Variance value for company A:

$$\text{var}\{X\} = \mathbf{E}_{P(x)}\left\{\left(X - \mathbf{E}_{P(x)}(X)\right)^2\right\} = 3573163$$

Variance value for company B:

$$\text{var}\{Y\} = \mathbf{E}_{P(y)}\left\{\left(Y - \mathbf{E}_{P(y)}(Y)\right)^2\right\} = 517143$$

# Standard deviation

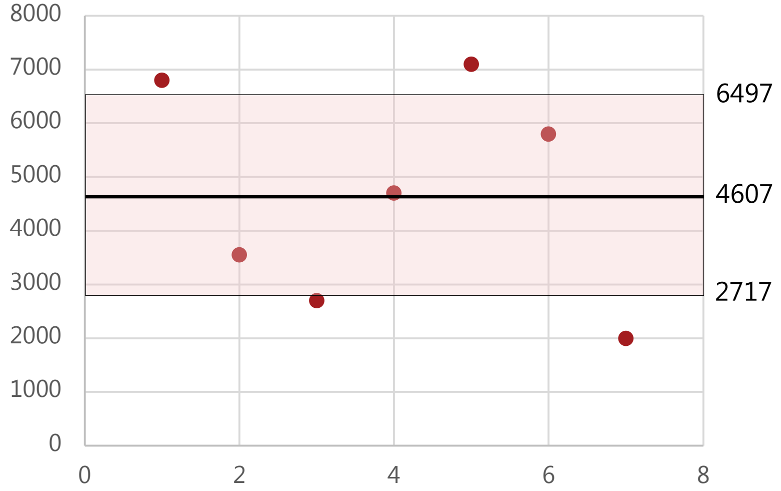
Standard deviation (SD) is the square root of variance.

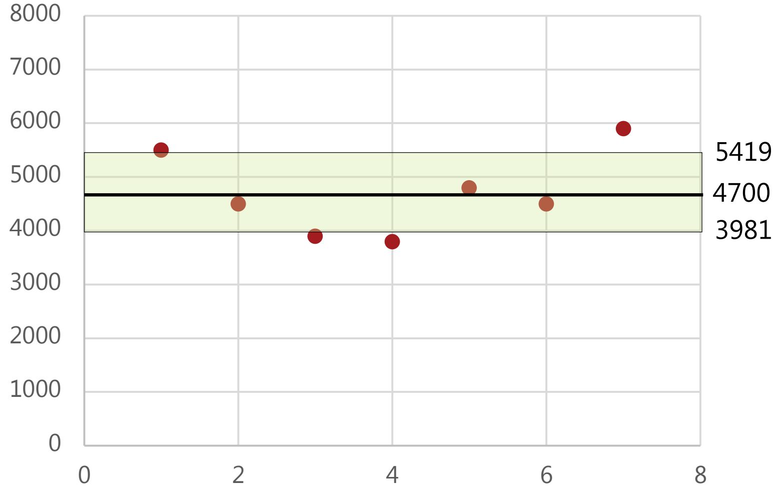
Standard deviation for company A:

$$\sigma_X = \mathbf{E}_{P(x)}\left\{\left(X - \mathbf{E}_{P(x)}(X)\right)^2\right\}^{0.5} = 1890$$

Standard deviation for company B:

$$\sigma_Y = \mathbf{E}_{P(y)}\left\{\left(Y - \mathbf{E}_{P(y)}(Y)\right)^2\right\}^{0.5} = 719$$





# Expectation: vector form variables

Mean and variance can be computed for random variables in the vector form.

Let’s say we have a set of students that passed math and English exams. How does a random student of such population look like?

| | Math | English |
|---|---|---|
| **Student 1** | **80** | **40** |
| **Student 2** | **60** | **80** |
| **Student 3** | **50** | **70** |
| **Student 4** | **40** | **70** |
| **Student 5** | **20** | **90** |
| **Student 6** | **50** | **70** |

Mean grades =
**[50, 70]**

Variance for individual grades =
**[333.3, 233.3]**

SD for individual grades =
**[18.3, 15.3]**

# Expectation: vector form variables

For vector form random variables we can also compute covariance matrix (pairwise variances between induvial components):

$$\text{cov}\{x\} = \mathbf{E}_{P(x)}\left\{\left(x - \mathbf{E}_{P(x)}\{x\}\right)\left(x - \mathbf{E}_{P(x)}\{x\}\right)^T\right\}$$

Matrix of [6x2] size

Mean grades = **[50, 70]**

| | Math | English |
|---|---|---|
| **Student 1** | **80** | **40** |
| **Student 2** | **60** | **80** |
| **Student 3** | **50** | **70** |
| **Student 4** | **40** | **70** |
| **Student 5** | **20** | **90** |
| **Student 6** | **50** | **70** |

| Math | English |
|---|---|
| **80-50** | **40-70** |
| **60-50** | **80-70** |
| **50-50** | **70-70** |
| **40-50** | **70-70** |
| **20-50** | **90-70** |
| **50-50** | **70-70** |

→

| Math | English |
|---|---|
| **30** | **-30** |
| **10** | **10** |
| **0** | **0** |
| **-10** | **0** |
| **-30** | **20** |
| **0** | **0** |

# Expectation: vector form variables

For vector form random variables we can also compute covariance matrix (pairwise variances between induvial components):

$$\mathrm{cov}\{x\} = \mathbf{E}_{P(x)}\left\{\left(x - \mathbf{E}_{P(x)}\{x\}\right)\left(x - \mathbf{E}_{P(x)}\{x\}\right)^T\right\}$$

$$\begin{bmatrix} 30 & -30 \\ 10 & 10 \\ 0 & 0 \\ -10 & 0 \\ -30 & 20 \\ 0 & 0 \end{bmatrix} \mathrm{X} \begin{bmatrix} 30 & 10 & 0 & -10 & -30 & 0 \\ -30 & 10 & 0 & 0 & 20 & 0 \end{bmatrix} = \begin{bmatrix} 2000 & -1400 \\ -1400 & 1400 \end{bmatrix}$$

$$\begin{bmatrix} 333 & -233 \\ -233 & 233 \end{bmatrix}$$

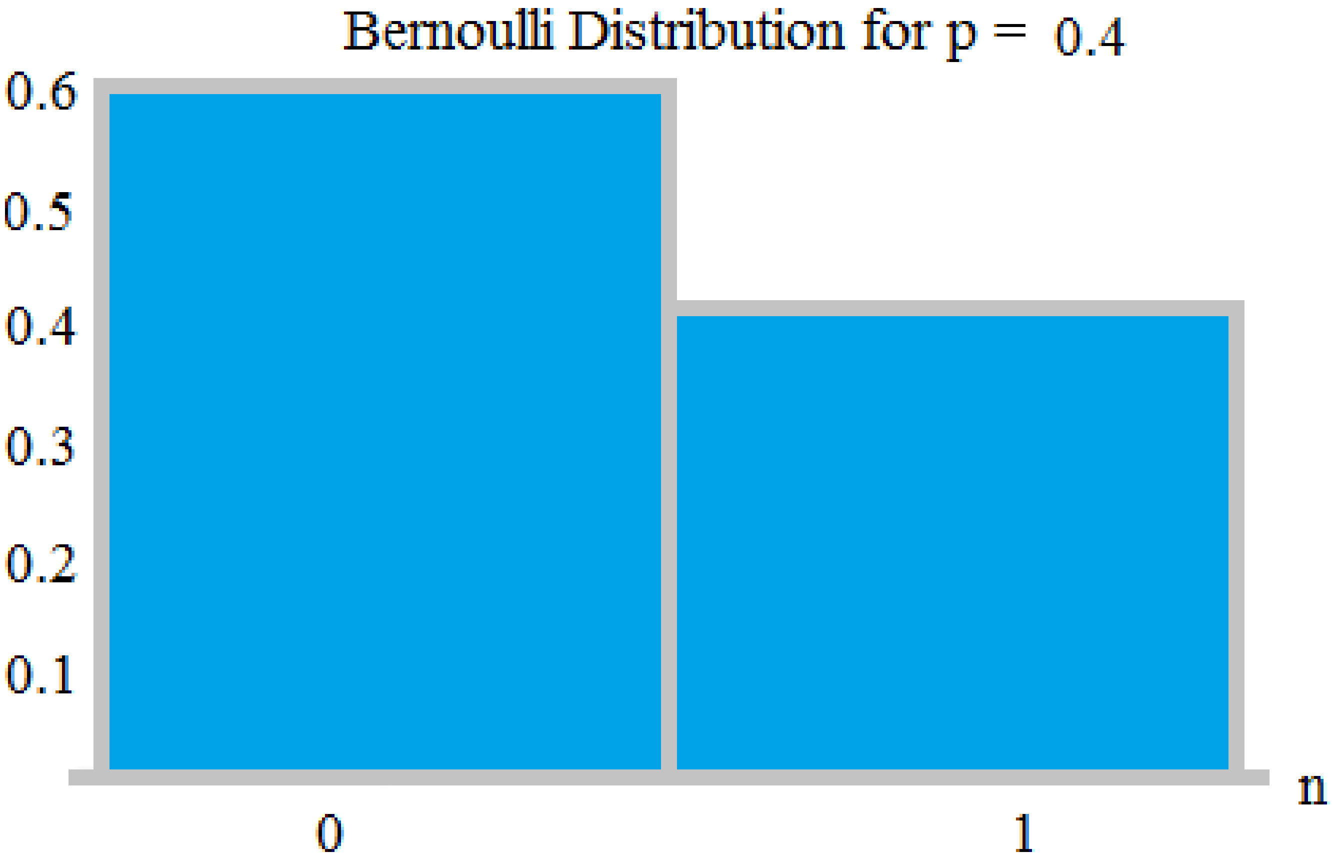
Normalized to # students

**What is the meaning of the covariance matrix?**

# Simple distributions: Bernoulli distribution

Coin tossing is a good example

$P(X = x) = p^x * (1-p)^{1-x}$



# Simple distributions: Binominal distribution

We toss a coin N times, what is the probability of getting x tails?

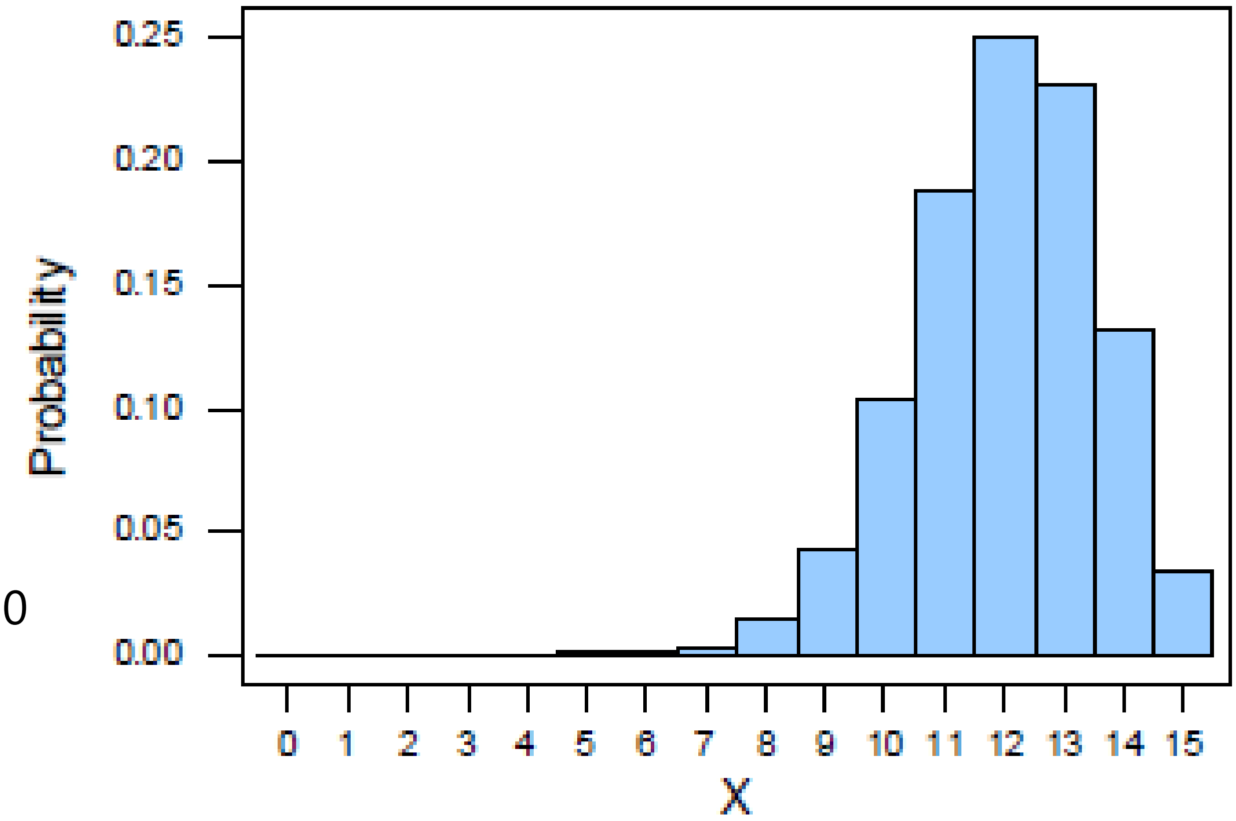
P(X = x) =

$$\binom{n}{x} p^{x} q^{n-x}$$

*q = 1 - p

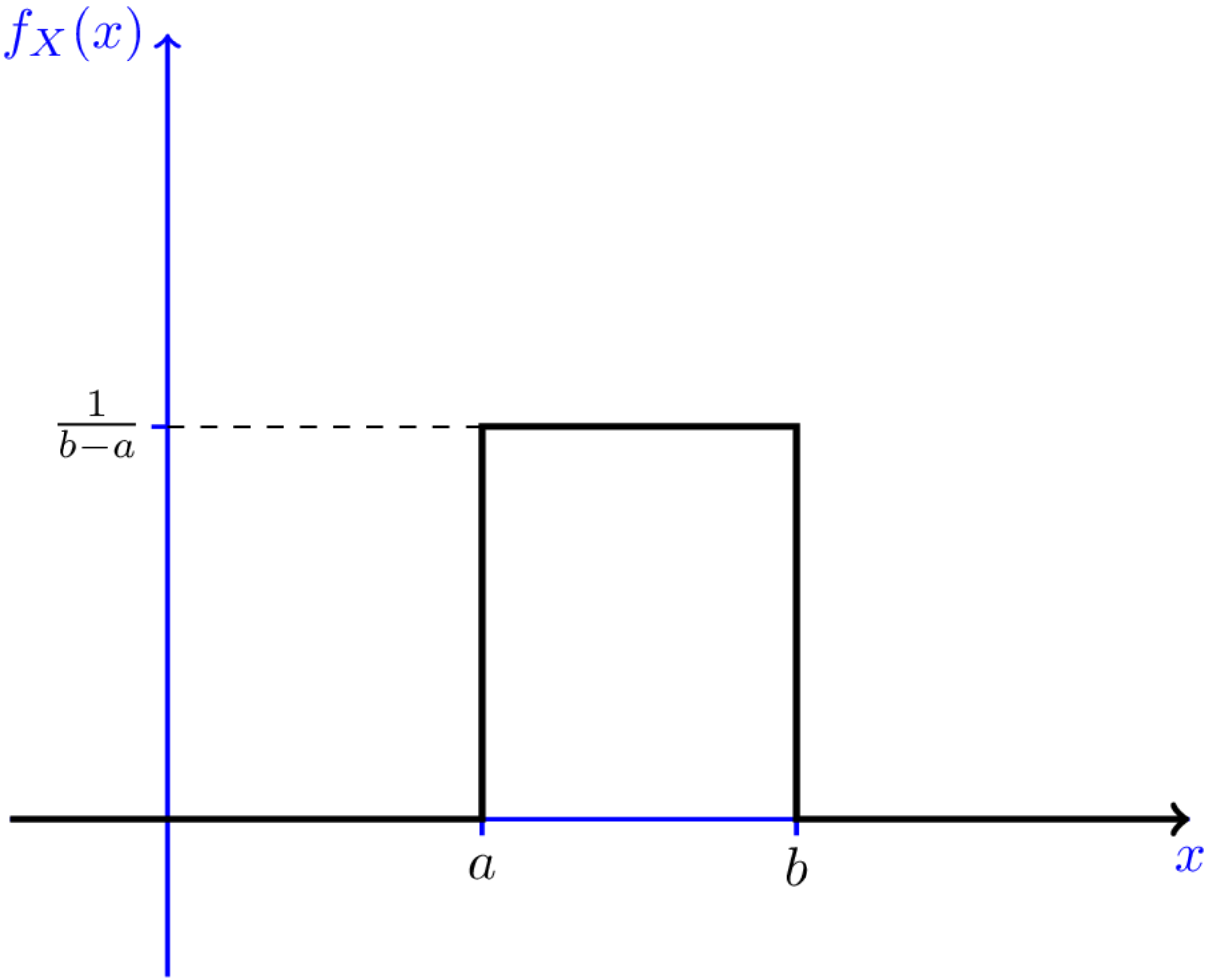
Let’s say we toss a coin 10 times, what is the probability of 5 tails?

Binomial distribution with n = 15 and p = 0.8



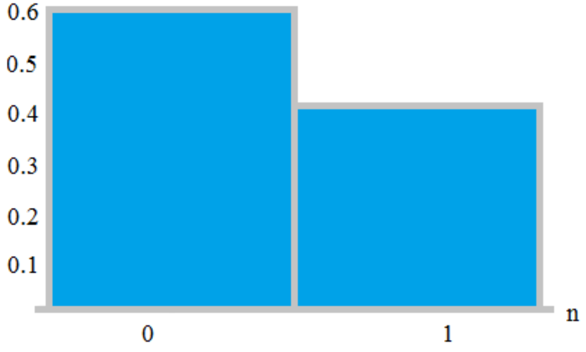
# Simple distributions: Uniform distribution

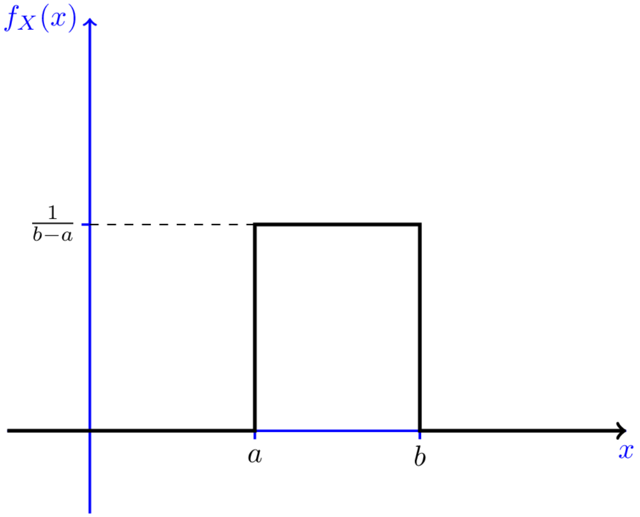
Let’s say we choose a random real number between a and b

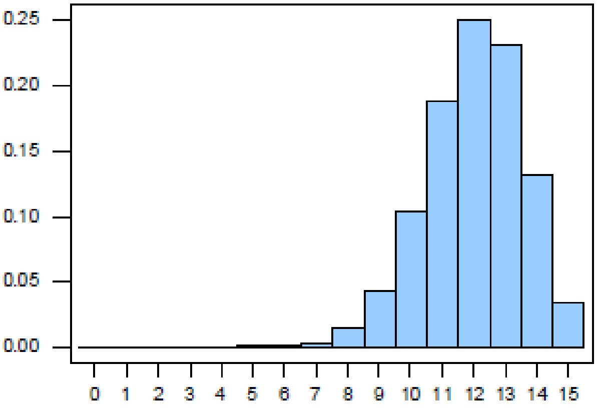


# Probability density function

PDF estimates the likelihood that the value of the random variable would be equal to a specified number





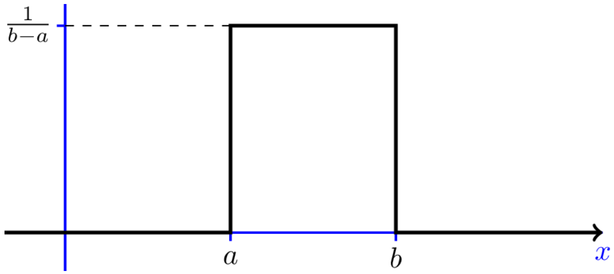


# Simple distributions: Uniform distribution

Let’s say we choose a random real number between 0 and 1:

- What is the probability of getting a specific number like 0.23423432432?

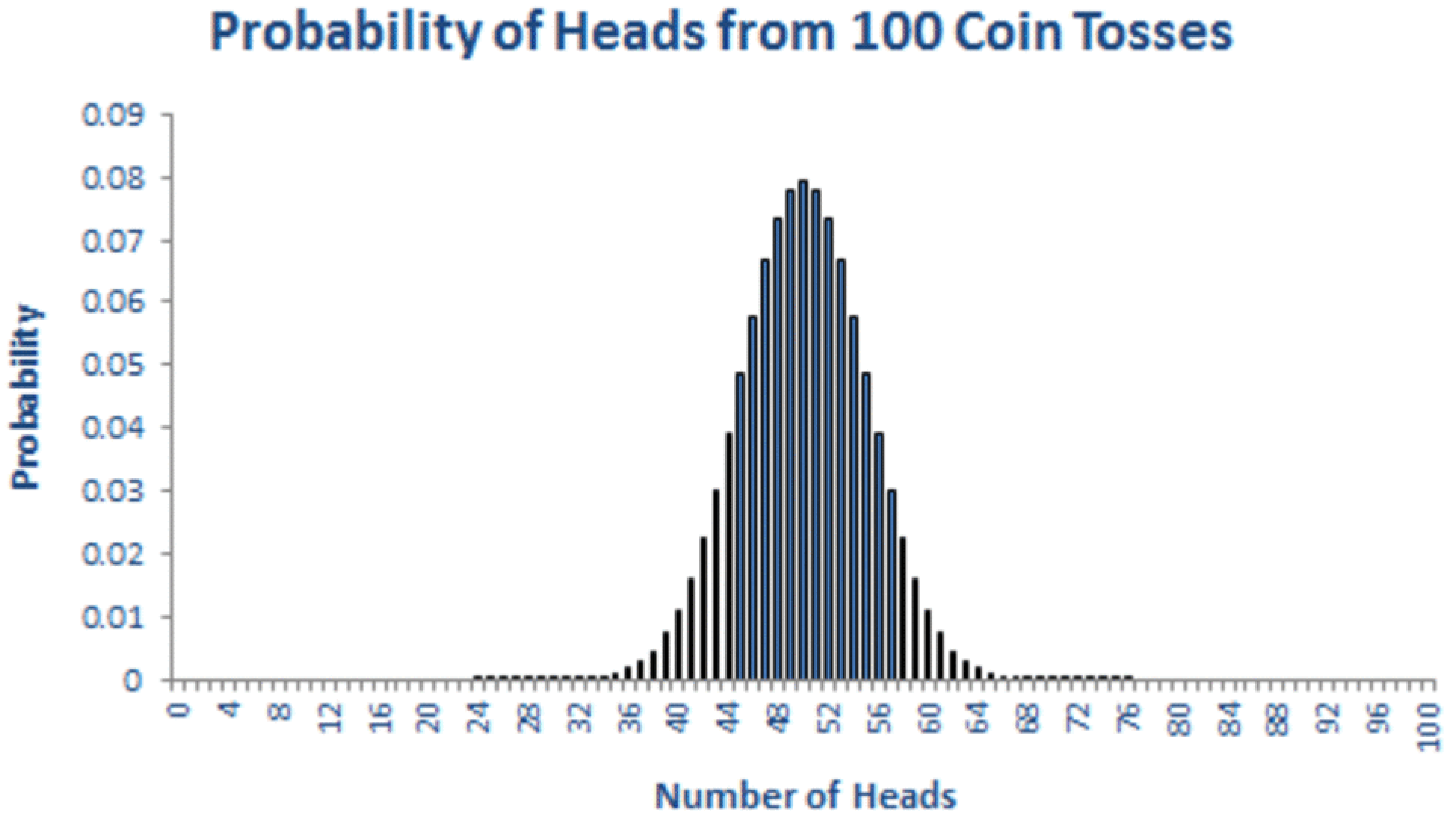
- The probability of getting a specific real number is zero.
- But we can compute the probability of getting a number in an interval from [0.2,  0.3]



$$P(0.2 \le x \le 0.3) = \int_{0.2}^{0.3} \frac{1}{1-0}\,dx = \frac{1}{1}(0.3-0.2) = 0.1$$

# Simple distributions: Normal distribution

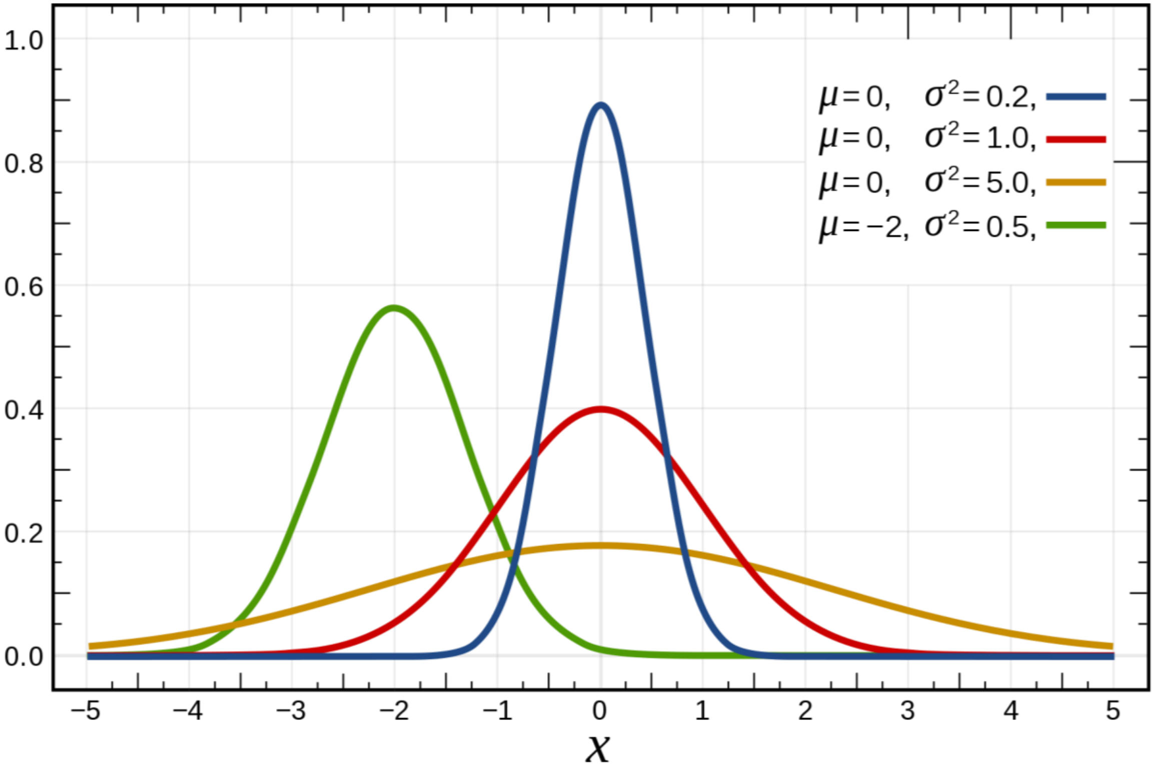
If we toss coin 100 times and count heads:



# Simple distributions: Normal distribution

Normal distribution:

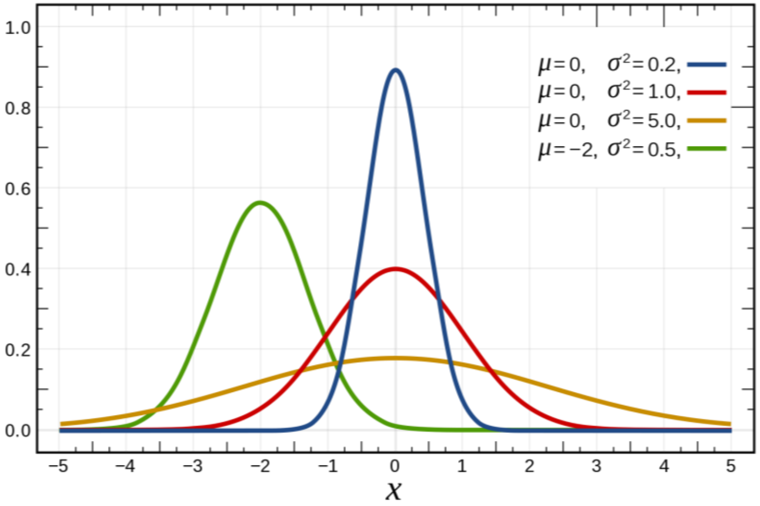
$$f(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



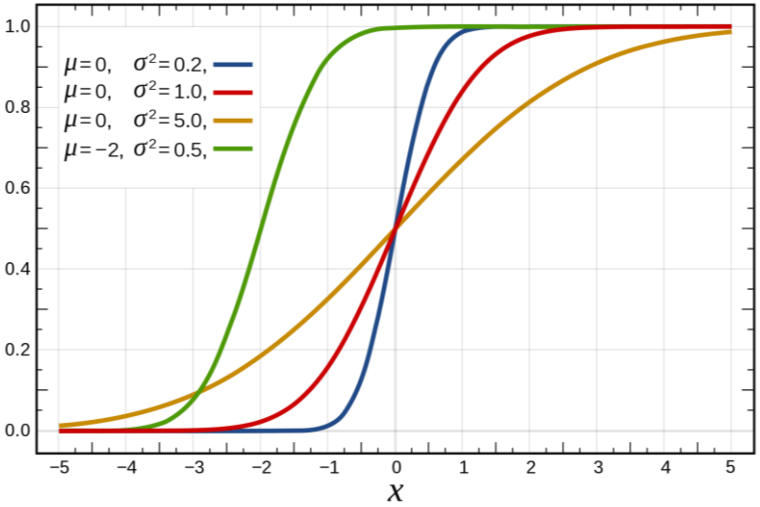
# Cumulative distribution function

Plot probability of getting x ≤ t:

$$P(x \leq t) = \int_{-\infty}^{t} f(x)\,dx$$



Probability density function

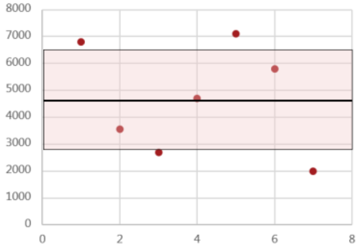


Cumulative distribution function

# Calculating student example 2

The same formulation, but now the student lives in a big city and there are N companies in field A and N in B. This time he does not have lists of salaries, but knows the means and standard deviations:

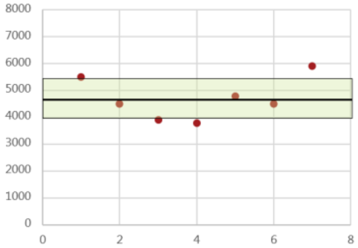
- Company A – $\{\mu, s\} = [4607,\ 1890]$
- Company B – $\{\mu, s\} = [4700,\quad 719]$



Which field is better to choose from if:

- N = 1. The only one position is available from A or B.
- N = 10. Ten positions are available from A (or from B). If he rejects a position, he cannot return to it.

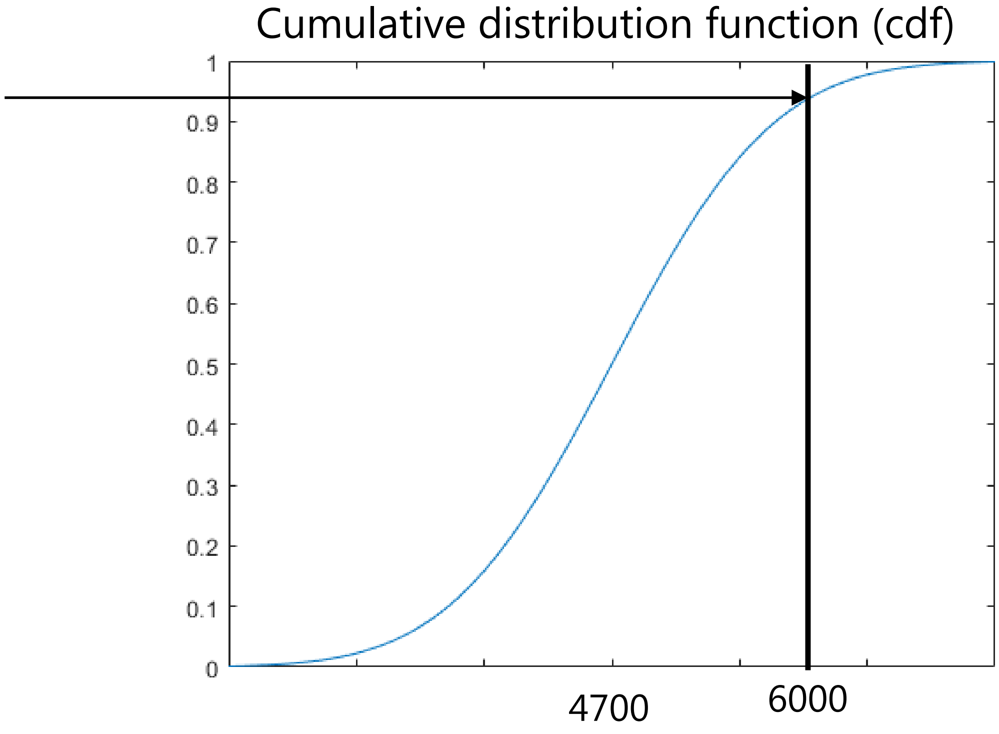
**What could be the student’s strategy?**



# Calculating student example 2

- One possible strategy is to define a salary threshold and accept the first job that surpasses the threshold. If none found among first N-1 offers, accept the last one

Probability of getting salary below a certain threshold



# Calculating student example 2

- Let’s select 6000 as the threshold. Probability of getting a job offer earning ≥ 6000:
  - P(S ≥ 6000 | A) = 1 - scipy.stats.norm.cdf(6000, 4607, 1890) = 0.231
  - P(S ≥ 6000 | B) = 1 - scipy.stats.norm.cdf(6000, 4700, 719) = 0.035
- How to calculate the probability of getting one job offer with S ≥ 6000 in N=10 attempts?
  - First calculate the probably of not getting such a job in 10 attempts:
    - P(S ≥ 6000, N = 10 | A) = $(1 – 0.231)^{10}$ = $(0.769)^{10}$ = 0.073
    - P(S ≥ 6000, N = 10 | B) = $(1 – 0.035)^{10}$ = $(0.965)^{10}$ = 0.698
  - The probability of getting such a job is:
    - For Company A = 1 - 0.073 = 0.927
    - For Company B = 1 - 0.698 = 0.302
- It is clearly better to select from Company in field A if he wants to increase the chances of getting a highly-paid job

# Calculating student example 2

- What about the risks? Can we estimate the probability of ending up with a bad last option?

- Using cdf, we can estimate the probability of getting an offer with S < 3500?
  - P(S < 3500 | A) = scipy.stats.norm.cdf(3500, 4607, 1890) = 0.279
  - P(S < 3500 | B) = scipy.stats.norm.cdf(3500, 4700, 719) = 0.048
  - Does this mean that it is better to go to filed B if he wants to minimize risks of getting a low-paid job?

- We also need to take into account the probability of rejecting first N-1 offers:
  - (1 - P(S ≥ 6000, N = 9 | A)) * P(S < 3500 | A) = 0.026
  - (1 - P(S ≥ 6000, N = 9 | B)) * P(S < 3500 | B) = 0.034
- It is still better to go to the field A

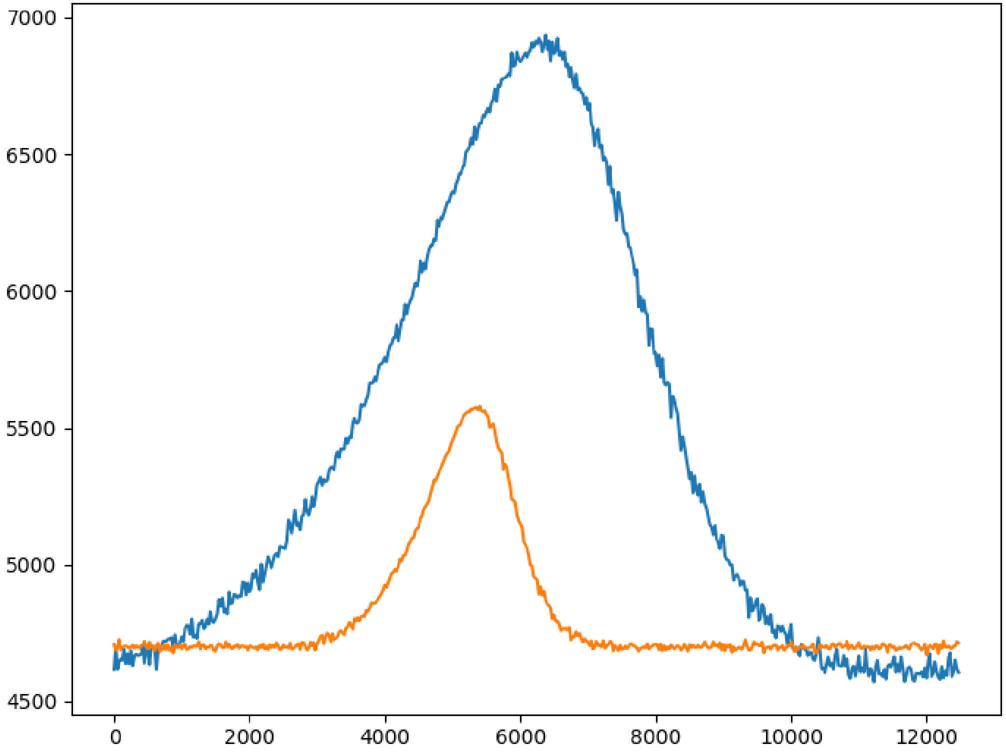
# Calculating student example 2

| | Field A | | Field B | |
|---|---|---|---|---|
| | **Success** | **Fail** | **Success** | **Fail** |
| Success: S > 6000; Fail: S < 3500; N = 10 | **0.927** | **0.026** | 0.302 | 0.034 |
| Success: S > 7000; Fail: S < 3500; N = 10 | **0.662** | 0.105 | 0.007 | **0.047** |
| Success: S > 7000; Fail: S < 3500; N = 25 | **0.933** | **0.021** | 0.017 | 0.047 |
| Success: S > 6000; Fail: S < 3000; N = 10 | **0.927** | 0.019 | 0.302 | **0.006** |
| Success: S > 8000; Fail: S < 3500; N = 10 | **0.309** | 0.200 | <0.001 | **0.048** |
| Success: S > 15000; Fail: S < 3500; N = 10 | **<0.001** | 0.249 | 0 | **0.048** |

- The students needs to select the salary threshold according to the number of jobs available.
- Is there a winning strategy for the field A over field B?

# Calculating student example 2

- Empirically, it is best to set up threshold around 6300, and, in average, get 6900 working in the field A.
- For the field B, you can maximum get around 5580 while requesting 5350



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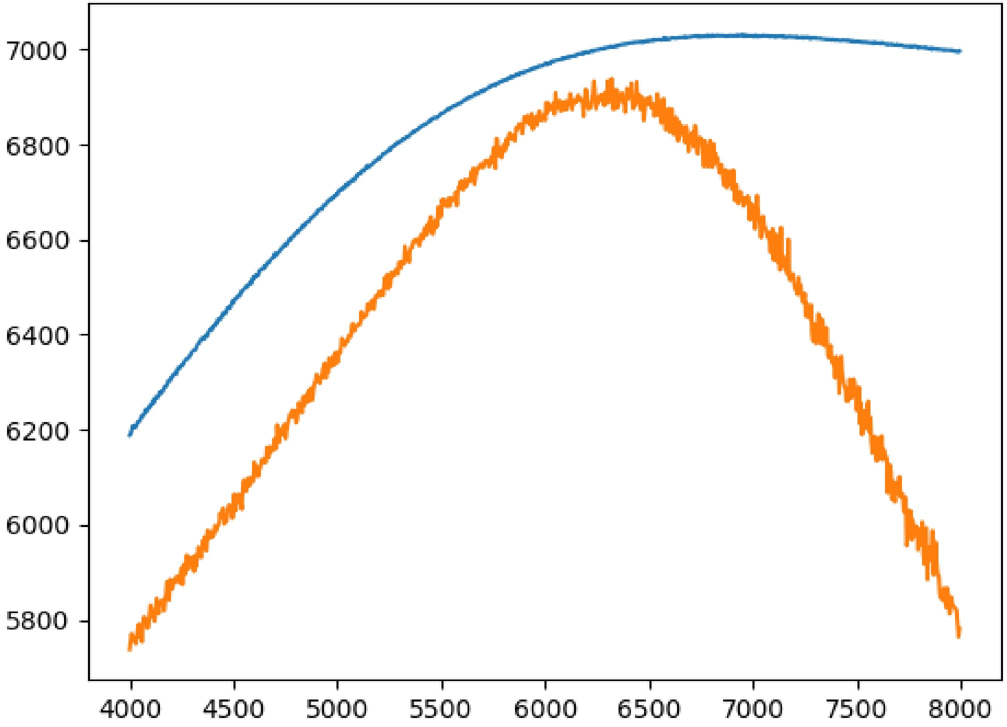
- Can this be improved?
- Is it wise to request 6300 if all N = 10 attempts left and if you already got 7 rejects?
- The solution is to slowly reduce the requested salary with every reject. The mean obtained salary grows from 6900 to 7030.



# Questions?