MASD 2021, Assignment 1

Hand-in in groups of 2 or 3 before 14.9.2021 at 10:00
One submission per group
Remember to include the names of all group memebers

Exercise 1 (Writing proofs). Apart from submitting your answers to all three exercises, submit your answer to this Exercise 1 separately as it is to be commented by another group. You will receive Exercise 1 for comments shortly after the hand-in deadline on September 8-th. Submit your comments as soon as possible but not later than 21.9.2021 at 10:00.

In this exercise, you will practice writing proofs. Remember that the proofs in this exercise (and any other proofs) must satisfy:

- Clearly stated assumptions (if there are any),
- Clearly stated claims (what do you want to prove),
- Clear logical arguments leading from assumptions to claims.

Your proofs will get peer-feedback from fellow students before they are corrected by your TA. *If you do not wish your fellow students to know your identity, do not put your names on the PDF that will be forwarded to another group.*

A function $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}$, has a limit L when x approaches $c \in D$ if for **every** number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

a) Suppose that the above implication holds for **some** fixed $\epsilon_0 > 0$. Prove that it holds for all $\epsilon \geq \epsilon_0$.

We need some definition before the remaining questions of this exercise can be formulated. A function: $f : D \to \mathbb{R}$ is **one-to-one** if for every pair of values $x_1, x_2 \in D$

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2)$$

Let R_f denote the range of a one-to-one function $f : D \to R$. f has a unique inverse function $f^{-1} : R_f \to D$ such that $f(f^{-1}(y)) = y$ for all $y \in R_f$. You can read more about inverse functions in section 1.5 of the textbook.

A function $f : D \to \mathbb{R}$ is **increasing** if

$$x_1 \leq x_2 \implies f(x_1) \leq f(x_2)$$

for every pair of values $x_1, x_2 \in I$. f is **strictly increasing** if

$$x_1 < x_2 \implies f(x_1) < f(x_2)$$

for every pair of values $x_1, x_2 \in I$. **Decreasing** and **strictly decreasing** functions are defined in analogous way.

Assume for the rest of this exercise that D is an open interval $]a, b[\subseteq \mathbb{R}$. Let $f : D \to \mathbb{R}$ be a strictly increasing and continuous function. It is intuitively clear that the range R_f of f is an open interval and the it is one-to-one (no need to prove that in this exercise). Hence f has the inverse function f^{-1}. Prove that

b) f^{-1} is strictly increasing.
c) f^{-1} is continuous.

Deliverables. The proofs.

Exercise 2 (Limits and continuity). Consider the function

$$f(x) = \begin{cases} x^2 & \text{when } x < 0, \\ x & \text{when } x \in [0, 2], \\ 5 & \text{when } x > 2, \end{cases}$$

a) In the supplied Jupyter notebook template `A1template.ipynb`, plot the function $f(x)$ on the interval $x \in [-5, 5]$, and based on your plot, decide if there are points $a \in [-5, 5]$ where f is not continuous? If yes, for which points it is not?
Include the plot and your claimed non-continuous points in your report.

b) Prove that your observations from a) are correct. That is:
   - Prove that f is continuous at all $a \in [-5, 5]$ where you claim that it is, *and*
   - Prove that f is not continuous at those points $a \in [-5, 5]$ where you claim that it is not.

   That is, you should have a proof of either continuity or non-continuity for every $a \in [-5, 5]$. Your proofs can be using any results mentioned Section 2.5 on the textbook.

Deliverables. Please submit a) the filled-out Jupyter template, and include the plot and the non-continuous points in your report; b) The proofs, following the same guidelines as in Exercise 1.

Exercise 3 (Limits and area of a disk). Let $n \in \mathbb{N}$, $n \geq 3$, denote the number of sides of a regular polygon P_n inscribed in a disk C with radius r and center O.

a) As $n \to \infty$, the area S_n of P_n approximates the area of C. We know that it is πr^2. Prove that $\lim_{n\to\infty} S_n = \pi r^2$. Hint: You may need that $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

Deliverables. The proof.