Goniometrische formules: trucs en afleidingen

Begin met de t-formule voor $tg\,x = \dfrac{2t}{1-t^2}$ met $t = tg\left(\frac{x}{2}\right)$

$tg\,x = \dfrac{\sin x}{\cos x} = \dfrac{2t}{1-t^2}$ + de noemer voor sin en cos $\neq$ tg

$1 + t^2 \qquad 1 - t^2$

$$\Rightarrow tg\,x = \frac{\dfrac{2t}{1+t^2}}{\dfrac{1-t^2}{1+t^2}} \qquad \begin{aligned} \frac{2t}{1+t^2} &= \sin x \\ \frac{1-t^2}{1+t^2} &= \cos x \end{aligned}$$

Stel nu $\alpha = \frac{x}{2} \Rightarrow 2\alpha = x$ en $tg\left(\frac{x}{2}\right) = t = tg(\alpha)$

$$\Rightarrow \sin(2\alpha) = \frac{2\,tg\,\alpha}{1+tg^2\alpha} \qquad \Rightarrow tg(2\alpha) = \frac{2\,tg\,\alpha}{1-tg^2\alpha}$$

$$\Rightarrow \cos(2\alpha) = \frac{1-tg^2\alpha}{1+tg^2\alpha}$$

Met $tg\,\alpha = \dfrac{\sin\alpha}{\cos\alpha}$

$$\Rightarrow tg(2\alpha) = \frac{2\,\dfrac{\sin\alpha}{\cos\alpha}}{1-\dfrac{\sin^2\alpha}{\cos^2\alpha}} = 2\,\frac{\sin\alpha}{\cancel{\cos\alpha}} \cdot \frac{\cos^{\cancel{2}}\alpha}{\cos^2\alpha-\sin^2\alpha}$$

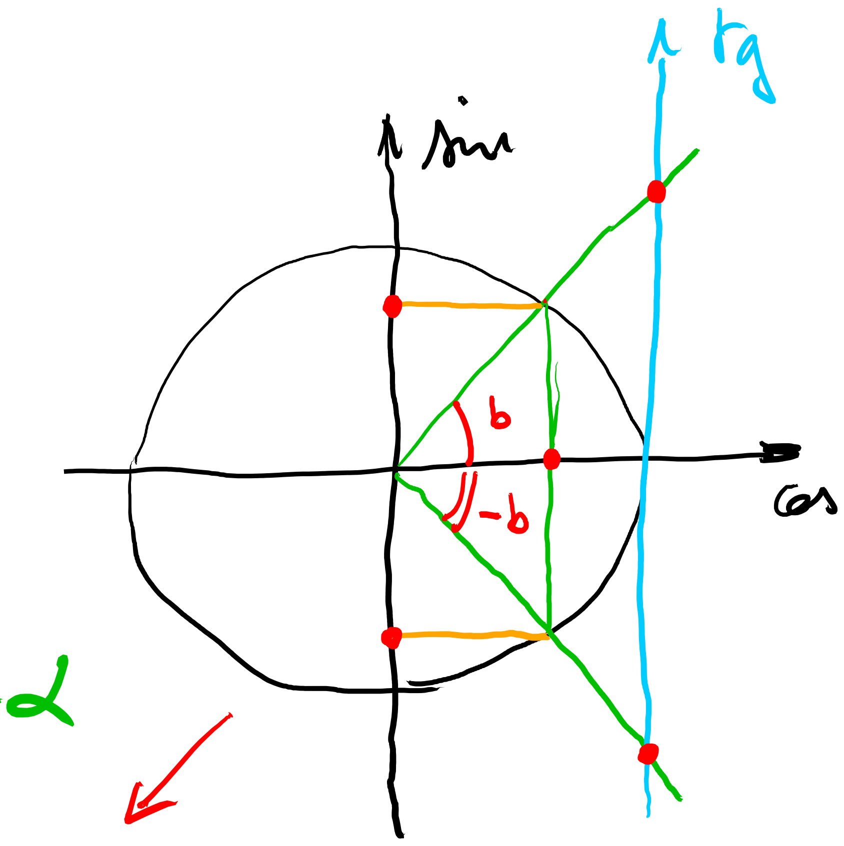
$$\Rightarrow tg(2\alpha) = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha-\sin^2\alpha} \qquad \begin{aligned} 2\sin\alpha\cos\alpha &= \sin(2\alpha) \\ \cos^2\alpha-\sin^2\alpha &= \cos(2\alpha) \end{aligned}$$

$$\begin{cases} \sin(2\alpha) = 2 \times \text{sin eerst, dan cos} = 2\sin\alpha\cos\alpha \\ \cos(2\alpha) = \text{cos eerst, lijkt op } \cos^2\alpha+\sin^2\alpha = 1 = \cos^2\alpha-\sin^2\alpha \end{cases}$$

$\sin(2\alpha) = \sin(\alpha+\alpha) \Rightarrow \sin(a+b)$

$= 2\sin\alpha\cos\alpha \Rightarrow \sin\alpha\cos\alpha + \sin\alpha\cos\alpha$

$\Rightarrow \sin(a+b) = \underline{\sin a} \cdot \underline{\cos b} + \underline{\sin b} \cdot \underline{\cos a}$



$\Rightarrow \sin(a+(-b)) = \sin a \cdot \cos b - \underline{\sin b} \cdot \cos a$

$\cos(2\alpha) = \cos(\alpha+\alpha) \Rightarrow \cos(a+b)$

$= \cos^2\alpha - \sin^2\alpha \Rightarrow \cos\alpha\cos\alpha + \sin\alpha\sin\alpha$

$\Rightarrow \cos(a+b) = \cos a \cdot \cos b - \sin a \cdot \sin b$

$\Rightarrow \cos(a+(-b)) = \cos a \cdot \cos b - \sin a\ \underline{\sin b}$

$$\begin{cases} \sin(-b) = -\sin b \\ \cos(-b) = \cos b \\ tg(-b) = -tg\,b \end{cases}$$

$tg(2\alpha) = tg(\alpha+\alpha) \Rightarrow tg(a+b)$

$$= \frac{tg\,\alpha + tg\,\alpha}{1 - tg\,\alpha \cdot tg\,\alpha}$$

$$\Rightarrow tg(a+b) = \frac{tg\,a + tg\,b}{1 - tg\,a \cdot tg\,b}$$

$$\Rightarrow tg(a+(-b)) = \frac{tg\,a - tg\,b}{1 + tg\,a \cdot tg\,b}$$

Simpson en omgekeerde Simpson

A: $\sin(a+b) = \sin a \cos b + \sin b \cos a$

B: $\sin(a-b) = \sin a \cos b - \sin b \cos a$

C: $\cos(a+b) = \cos a \cos b - \sin a \sin b$

D: $\cos(a-b) = \cos a \cos b + \sin a \sin b$

$$\begin{cases} a+b = x \\ a-b = y \end{cases}$$

$$\begin{cases} a = \dfrac{x+y}{2} \\ b = \dfrac{x-y}{2} \end{cases}$$

$\Rightarrow$ A+B $\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$

$\sin(x) + \sin(y) = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$

$\Rightarrow$ A-B $\sin(a+b) - \sin(a-b) = 2 \sin b \cos a$

$\sin(x) - \sin(y) = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$

$\Rightarrow$ C+D $\cos(a+b) + \cos(a-b) = 2 \cos a \cos b$

$\cos(x) + \cos(y) = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$

$\Rightarrow$ C-D $\cos(a+b) - \cos(a-b) = -2 \sin a \sin b$

$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$

$$\cos a . \cos b = \frac{1}{2}\left(\cos(a+b) + \cos(a-b)\right)$$

idem voor de anderen!

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha \Rightarrow \cos(2\alpha) + 1 = \cos^2\alpha - \cancel{\sin^2\alpha} + \cos^2\alpha + \cancel{\sin^2\alpha}$$

$$\Rightarrow 2\cos^2\alpha = 1 + \cos(2\alpha) \Rightarrow \boxed{\cos^2\alpha = \frac{1}{2}\left(1+\cos(2\alpha)\right)}$$

$$\cos(2\alpha) - 1 = \cancel{\cos^2\alpha} - \sin^2\alpha - \left(\cancel{\cos^2\alpha} + \sin^2\alpha\right)$$

$$\Rightarrow 2\sin^2\alpha = 1 - \cos(2\alpha) \Rightarrow \boxed{\sin^2\alpha = \frac{1}{2}\left(1-\cos(2\alpha)\right)}$$

Stel $\alpha = \frac{x}{2}$

teken → goniometrische cirkel tekenen!

$$\Rightarrow \cos^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1+\cos x\right) \Rightarrow \boxed{\cos\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1+\cos x}{2}}}$$

$$\Rightarrow \sin^2\left(\frac{x}{2}\right) = \frac{1}{2}\left(1-\cos x\right) \Rightarrow \boxed{\sin\left(\frac{x}{2}\right) = \pm\sqrt{\frac{1-\cos x}{2}}}$$

$$\Rightarrow \boxed{\operatorname{tg}\left(\frac{x}{2}\right) = \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \pm\sqrt{\frac{1-\cos x}{1+\cos x}}}$$

$$= \sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1+\cos x}{1+\cos x}} = \sqrt{\frac{1-\cos^2\alpha}{(1+\cos x)^2}}$$

$$= \sqrt{\frac{\sin^2 x}{(1+\cos x)^2}} = \boxed{\frac{\sin x}{1+\cos x} = \operatorname{tg}\left(\frac{x}{2}\right)}$$

$$= \sqrt{\frac{1-\cos x}{1+\cos x}\cdot\frac{1-\cos x}{1-\cos x}} = \sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}}$$

$$= \sqrt{\frac{(1-\cos x)^2}{\sin^2 x}} = \boxed{\frac{1-\cos x}{\sin x} = \operatorname{tg}\left(\frac{x}{2}\right)}$$