```
In[ ]:= eqncirc = x^2 + y^2 == a^2;
eqnx = x == r Sin[φ] + h2 Sin[α];
eqny = y == h1 - r Cos[φ] - h2 Cos[α];
elim = FullSimplify[Eliminate[{eqnx, eqny}, {α}]];
solve = Thread[FullSimplify[Solve[{eqncirc, elim}, {x, y}]]]
```

Out[ ]=

$$\Bigg\{\Big\{x \to \frac{1}{2 r \left(r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)} \Big(r^2 \operatorname{Sin}[\varphi] \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2 - h_2^2\right) - r \operatorname{Cot}[\varphi] \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)} + \operatorname{Csc}[\varphi] h_1 \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)}\Big),$$

$$x \to \frac{1}{2 r \left(r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)} \Big(r^2 \operatorname{Sin}[\varphi] \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2 - h_2^2\right) + r \operatorname{Cot}[\varphi] \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)} - \operatorname{Csc}[\varphi] h_1 \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)}\Big)\Big\},$$

$$\Big\{y \to \frac{1}{2 \left(r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)} \Big(-r^3 \operatorname{Cos}[\varphi] + r^2 (2 + \operatorname{Cos}[2 \varphi]) h_1 + h_1 \left(a^2 + h_1^2 - h_2^2\right) + r \operatorname{Cos}[\varphi] \left(-a^2 - 3 h_1^2 + h_2^2\right) - \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)}\Big),$$

$$y \to \frac{1}{2 \left(r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)} \Big(-r^3 \operatorname{Cos}[\varphi] + r^2 (2 + \operatorname{Cos}[2 \varphi]) h_1 + h_1 \left(a^2 + h_1^2 - h_2^2\right) + r \operatorname{Cos}[\varphi] \left(-a^2 - 3 h_1^2 + h_2^2\right) + \sqrt{\left(-r^2 \operatorname{Sin}[\varphi]^2 \left(\left(-a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right)^2 - 2 \left(a^2 + r^2 - 2 r \operatorname{Cos}[\varphi] h_1 + h_1^2\right) h_2^2 + h_2^4\right)\right)}\Big)\Big\}\Bigg\}$$

In[ ]:= `solve = solve /. Rule → Equal`

Out[ ]=

$$\Big\{\Big\{x == \frac{1}{2\,r\,(r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)}\Big(r^2\,\mathrm{Sin}[\varphi]\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2 - h_2^2) - r\,\mathrm{Cot}[\varphi]\,\sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))} + \mathrm{Csc}[\varphi]\,h_1\,\sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))}\Big),$$

$$x == \frac{1}{2\,r\,(r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)}\Big(r^2\,\mathrm{Sin}[\varphi]\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2 - h_2^2) + r\,\mathrm{Cot}[\varphi]\,\sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))} - \mathrm{Csc}[\varphi]\,h_1\,\sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))}\Big)\Big\},$$

$$\Big\{y == \frac{1}{2\,(r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)}\Big(-r^3\,\mathrm{Cos}[\varphi] + r^2\,(2 + \mathrm{Cos}[2\,\varphi])\,h_1 + h_1\,(a^2 + h_1^2 - h_2^2) + r\,\mathrm{Cos}[\varphi]\,(-a^2 - 3\,h_1^2 + h_2^2) - \sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))}\Big),$$

$$y == \frac{1}{2\,(r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)}\Big(-r^3\,\mathrm{Cos}[\varphi] + r^2\,(2 + \mathrm{Cos}[2\,\varphi])\,h_1 + h_1\,(a^2 + h_1^2 - h_2^2) + r\,\mathrm{Cos}[\varphi]\,(-a^2 - 3\,h_1^2 + h_2^2) + \sqrt{(-r^2\,\mathrm{Sin}[\varphi]^2\,((-a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)^2 - 2\,(a^2 + r^2 - 2\,r\,\mathrm{Cos}[\varphi]\,h_1 + h_1^2)\,h_2^2 + h_2^4))}\Big)\Big\}\Big\}$$

In[ ]:= $$\texttt{solve} \;/.\; \left\{\left(-a^2+r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)^2 \to A,\; a^2+r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2 \to B\right\}$$

Out[ ]= $$\left\{\left\{x == \left(r^2\,\mathrm{Sin}[\varphi]\left(B-h_2^2\right)-r\,\mathrm{Cot}[\varphi]\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}+\mathrm{Csc}[\varphi]\,h_1\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}\right)\Big/\left(2\,r\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right),\right.\right.$$

$$x == \left(r^2\,\mathrm{Sin}[\varphi]\left(B-h_2^2\right)+r\,\mathrm{Cot}[\varphi]\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}-\mathrm{Csc}[\varphi]\,h_1\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}\right)\Big/\left(2\,r\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right)\Big\},$$

$$\Big\{y == \left(-r^3\,\mathrm{Cos}[\varphi]+r^2\,(2+\mathrm{Cos}[2\,\varphi])\,h_1+h_1\left(a^2+h_1^2-h_2^2\right)+r\,\mathrm{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)-\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}\right)\Big/\left(2\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right),$$

$$y == \left(-r^3\,\mathrm{Cos}[\varphi]+r^2\,(2+\mathrm{Cos}[2\,\varphi])\,h_1+h_1\left(a^2+h_1^2-h_2^2\right)+r\,\mathrm{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)+\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)}\right)\Big/\left(2\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right)\Big\}\Big\}$$

In[ ]:= $$\texttt{Simplify}\left[\%\;/.\;\left\{\sqrt{-r^2\,\mathrm{Sin}[\varphi]^2\left(A-2\,B\,h_2^2+h_2^4\right)} \to S\right\}\right]$$

Out[ ]= $$\left\{\left\{x == \frac{-r\,S\,\mathrm{Cot}[\varphi]+S\,\mathrm{Csc}[\varphi]\,h_1+r^2\,\mathrm{Sin}[\varphi]\left(B-h_2^2\right)}{2\,r\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)},\; x == \frac{r\,S\,\mathrm{Cot}[\varphi]-S\,\mathrm{Csc}[\varphi]\,h_1+r^2\,\mathrm{Sin}[\varphi]\left(B-h_2^2\right)}{2\,r\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)}\right\},\right.$$

$$\Big\{y == \left(-S-r^3\,\mathrm{Cos}[\varphi]+r^2\,(2+\mathrm{Cos}[2\,\varphi])\,h_1+h_1\left(a^2+h_1^2-h_2^2\right)+r\,\mathrm{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)\right)\Big/\left(2\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right),$$

$$y == \left(S-r^3\,\mathrm{Cos}[\varphi]+r^2\,(2+\mathrm{Cos}[2\,\varphi])\,h_1+h_1\left(a^2+h_1^2-h_2^2\right)+r\,\mathrm{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)\right)\Big/\left(2\left(r^2-2\,r\,\mathrm{Cos}[\varphi]\,h_1+h_1^2\right)\right)\Big\}\Big\}$$

```
In[ ]:= Simplify[% //. {r^2 Sin[φ] (B - h2^2) → Desc, r S Cot[φ] - S Csc[φ] h1 -> Besc,
          -r S Cot[φ] + S Csc[φ] h1 → -Besc, r^2 - 2 r Cos[φ] h1 + h1^2 → Z}]
```

Out[ ]=
$$\left\{\left\{x == \frac{-\text{Besc}+\text{Desc}}{2\,r\,Z},\ x == \frac{\text{Besc}+\text{Desc}}{2\,r\,Z}\right\},\right.$$
$$\left\{y == \frac{1}{2\,Z}\left(-S - r^3 \text{Cos}[\varphi] + r^2\,(2+\text{Cos}[2\,\varphi])\,h_1 + h_1\left(a^2+h_1^2-h_2^2\right) + r\,\text{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)\right),\right.$$
$$\left.\left.y == \frac{1}{2\,Z}\left(S - r^3 \text{Cos}[\varphi] + r^2\,(2+\text{Cos}[2\,\varphi])\,h_1 + h_1\left(a^2+h_1^2-h_2^2\right) + r\,\text{Cos}[\varphi]\left(-a^2-3\,h_1^2+h_2^2\right)\right)\right\}\right\}$$

```
In[ ]:= % /. {r^2 (2 + Cos[2 φ]) h1 + h1 (a^2 + h1^2 - h2^2) + r Cos[φ] (-a^2 - 3 h1^2 + h2^2) - r^3 Cos[φ] → L}
```

Out[ ]=
$$\left\{\left\{x == \frac{-\text{Besc}+\text{Desc}}{2\,r\,Z},\ x == \frac{\text{Besc}+\text{Desc}}{2\,r\,Z}\right\},\ \left\{y == \frac{L-S}{2\,Z},\ y == \frac{L+S}{2\,Z}\right\}\right\}$$