AOC 2021 Day 6

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Let $f_{k,n}$ denote the amount of fish with timer value k on day n. The problem gives us recurrence relations

$$\begin{aligned}
f_{0,n} &= f_{1,n-1} \\
f_{1,n} &= f_{2,n-1} \\
f_{2,n} &= f_{3,n-1} \\
f_{3,n} &= f_{4,n-1} \\
f_{4,n} &= f_{5,n-1} \\
f_{5,n} &= f_{6,n-1} \\
f_{6,n} &= f_{7,n-1} + f_{0,n-1} \\
f_{7,n} &= f_{8,n-1} \\
f_{8,n} &= f_{0,n-1}
\end{aligned}$$

We can express everything in terms of $f_{0,n}$ as

$$\begin{aligned}
f_{0,n} &= f_{0,n-9} + f_{0,n-7} \\
f_{1,n} &= f_{0,n-8} + f_{0,n-6} \\
f_{2,n} &= f_{0,n-7} + f_{0,n-5} \\
f_{3,n} &= f_{0,n-6} + f_{0,n-4} \\
f_{4,n} &= f_{0,n-5} + f_{0,n-3} \\
f_{5,n} &= f_{0,n-4} + f_{0,n-2} \\
f_{6,n} &= f_{0,n-3} + f_{0,n-1} \\
f_{7,n} &= f_{0,n-2} \\
f_{8,n} &= f_{0,n-1}
\end{aligned}$$

conversely we find

$$\begin{aligned}
f_{0,n} &= f_{0,n} \\
f_{0,n-1} &= f_{8,n} \\
f_{0,n-2} &= f_{7,n} \\
f_{0,n-3} &= f_{6,n} - f_{8,n} \\
f_{0,n-4} &= f_{5,n} - f_{7,n} \\
f_{0,n-5} &= f_{4,n} - f_{6,n} + f_{8,n} \\
f_{0,n-6} &= f_{3,n} - f_{5,n} + f_{7,n} \\
f_{0,n-7} &= f_{2,n} - f_{4,n} + f_{6,n} - f_{8,n} \\
f_{0,n-8} &= f_{,n} - f_{3,n} + f_{5,n} - f_{7,n}
\end{aligned}$$

We can then create a recurrence relation as

$$F_n = \begin{pmatrix} f_{0,n} \\ f_{0,n-1} \\ f_{0,n-2} \\ f_{0,n-3} \\ f_{0,n-4} \\ f_{0,n-5} \\ f_{0,n-6} \\ f_{0,n-7} \\ f_{0,n-8} \end{pmatrix} = \begin{pmatrix} f_{0,n-9} + f_{0,n-7} \\ f_{0,n-1} \\ f_{0,n-2} \\ f_{0,n-3} \\ f_{0,n-4} \\ f_{0,n-5} \\ f_{0,n-6} \\ f_{0,n-7} \\ f_{0,n-8} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} F_{n-1}$$

We can solve this relation by diagonalizing the matrix, after all

$$F_n = AF_{n-1} \implies F_n = CDC^{-1}F_{n-1} = CDC^{-1}CDC^{-1}F_{n-2} = CD^2C^{-1}F_{n-2} = \ldots = CD^nC^{-1}F_0$$

Using a script to calculate C and D gives

$$C = \begin{bmatrix} \Lambda_0^8 & \Lambda_1^8 & \Lambda_2^8 & \Lambda_3^8 & \Lambda_4^8 & \Lambda_5^8 & \Lambda_6^8 & \Lambda_7^8 & \Lambda_8^8 \\ \Lambda_0^7 & \Lambda_1^7 & \Lambda_2^7 & \Lambda_3^7 & \Lambda_4^7 & \Lambda_5^7 & \Lambda_6^7 & \Lambda_7^7 & \Lambda_8^7 \\ \Lambda_0^6 & \Lambda_1^6 & \Lambda_2^6 & \Lambda_3^6 & \Lambda_4^6 & \Lambda_5^6 & \Lambda_6^6 & \Lambda_7^6 & \Lambda_8^6 \\ \Lambda_0^5 & \Lambda_1^5 & \Lambda_2^5 & \Lambda_3^5 & \Lambda_4^5 & \Lambda_5^5 & \Lambda_6^5 & \Lambda_7^5 & \Lambda_8^5 \\ \Lambda_0^4 & \Lambda_1^4 & \Lambda_2^4 & \Lambda_3^4 & \Lambda_4^4 & \Lambda_5^4 & \Lambda_6^4 & \Lambda_7^4 & \Lambda_8^4 \\ \Lambda_0^3 & \Lambda_1^3 & \Lambda_2^3 & \Lambda_3^3 & \Lambda_4^3 & \Lambda_5^3 & \Lambda_6^3 & \Lambda_7^3 & \Lambda_8^3 \\ \Lambda_0^2 & \Lambda_1^2 & \Lambda_2^2 & \Lambda_3^2 & \Lambda_4^2 & \Lambda_5^2 & \Lambda_6^2 & \Lambda_7^2 & \Lambda_8^2 \\ \Lambda_0 & \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 & \Lambda_5 & \Lambda_6 & \Lambda_7 & \Lambda_8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} \Lambda_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \Lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Lambda_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Lambda_8 \end{bmatrix}$$

where Λ_i is a zero of the polynomial $x^9 - x^2 - 1$. This is hard to compute though, so we can instead use a script to do the matrix multiplication for us.