
Units

EMOOPIC

January 15, 2017

MAXWELL EQUATIONS

In Yee algorithm, using Gaussian unit style with $c = 1$, i.e.

$$\nabla \times E = -\frac{1}{c} \frac{\partial B}{\partial t} = -\frac{\partial B}{\partial \tilde{t}} \quad (0.1)$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c} \frac{\partial E}{\partial t} = 4\pi \tilde{J} + \frac{\partial E}{\partial \tilde{t}}. \quad (0.2)$$

We choose to keep cm as the length unit, so that the subsequent two Maxwell equations are unaffected:

$$\nabla \cdot E = 4\pi \rho \quad (0.3)$$

$$\nabla \cdot B = 0. \quad (0.4)$$

This means that the time has the unit of (1 cm) / (speed of light), i.e.

$$1\tilde{s} = \frac{1\text{cm}}{c_0\text{cm/s}} \approx 3.34 \times 10^{-11} \text{ s}, \quad (0.5)$$

where $c_0 \approx 2.9979 \times 10^{10}$ is the speed of light in unit of cm/s.

VELOCITY AND ACCELERATION

The velocity \tilde{v} is now normalized automatically to the speed of light, since it is measured in cm / (distance light travels in 1 cm). Namely,

$$\tilde{v} = \frac{v}{c_0\text{cm/s}}. \quad (0.6)$$

The non-relativistic thermal velocity v_{th} , defined as the mean of the magnitude of the velocity, is then

$$\tilde{v}_{th} = \frac{v_{th}}{c} = \sqrt{\frac{8T}{\pi mc^2}} \quad (0.7)$$

$$\approx 2.232 \times 10^{-3} \sqrt{\frac{T/1\text{eV}}{m/m_e}}. \quad (0.8)$$

We then have the Lorentz acceleration as

$$a = \frac{1}{c_0^2} \frac{q}{m} (E + \tilde{v} \times B) \quad (0.9)$$

$$\approx 1.1 \times 10^{-21} \frac{q}{m} (E + \tilde{v} \times B). \quad (0.10)$$

The small acceleration reflects the timescale separation between light waves and plasma motion.

CHARGE, MASS, AND FIELD

It is convenient to normalize charge q to electron charge e , and normalize mass m to electron mass m_e . Namely,

$$\tilde{q} = \frac{q}{e}, \quad \tilde{m} = \frac{m}{m_e}. \quad (0.11)$$

Recall the value

$$\frac{e}{m_e} \approx 5.2728 \times 10^{17} \text{statC/g}. \quad (0.12)$$

Having normalized charge and mass, it is convenient to normalize the electric field E to kilo statV/cm (KstatV/cm), and normalize the magnetic field B to kilo Gauss (KG). Namely,

$$\tilde{E} = \frac{E}{\text{KstatV/cm}}, \quad \tilde{B} = \frac{B}{\text{KG}}. \quad (0.13)$$

It is useful to note

$$\frac{1 \cdot \text{KstatV/cm}}{c_0 \text{cm/s} \cdot \text{KG}} = 1. \quad (0.14)$$

With such normalization, there is no large or small coefficient in the Newton's equation with Lorentz force. Explicitly, we have

$$\tilde{a} = \frac{a}{\text{cm} \cdot \tilde{s}^{-2}} \quad (0.15)$$

$$= \frac{\tilde{q}}{\tilde{m}} (\tilde{E} + \tilde{v} \times \tilde{B}) \times \frac{e \cdot \text{KstatV/cm}}{m_e \cdot \text{cm} \cdot \tilde{s}^{-2}} \quad (0.16)$$

$$\approx 0.58668774 \frac{\tilde{q}}{\tilde{m}} (\tilde{E} + \tilde{v} \times \tilde{B}). \quad (0.17)$$

To get a sense of how large the electric and magnetic fields are, it is useful to note

$$\tilde{E} = 1 \leftrightarrow E \approx 299.79 \text{KV/cm}, \quad (0.18)$$

$$\tilde{B} = 1 \leftrightarrow B = 0.1 \text{T}, \quad (0.19)$$

where KV/cm stands for kilo volte/cm, and T stands for tesla. Notice that a constant field $\tilde{E} = 1$ is a very large electric field, which can accelerate an electron to relativistic velocity within 1 \tilde{s} . In comparison, a constant field $\tilde{B} = 1$ is moderate magnetic field that can be easily produced in laboratory.

CHARGE AND CURRENT DENSITY

Using the above units, the current density due to a single fluid species can be expressed as

$$\tilde{J} = \frac{J}{c} = \frac{qvn}{c} \quad (0.20)$$

$$= \tilde{q}\tilde{v}\tilde{n} \times e \text{cm}^{-3} \quad (0.21)$$

$$= 4.8032 \times 10^{-13} \frac{\text{KStatC}}{\text{cm}^3} \tilde{q}\tilde{v}\tilde{n}. \quad (0.22)$$

That is to say, current densities used in the program is in units KstatC/cm³. Similarly, the charge density can be expressed as

$$\rho = qn = \tilde{q}\tilde{n} \times e \text{cm}^{-3} \quad (0.23)$$

$$= 4.8032 \times 10^{-13} \frac{\text{KStatC}}{\text{cm}^3} \tilde{q}\tilde{n}. \quad (0.24)$$

Notice that the charge density has the same unit as the current density.

SUPER PARTICLES

It is rare that one has the numerical resources to run a PIC code with physical number of particles. Therefore, super particles are typically used, each representing N_s number of real particles. The mass and charge of a super particle is thereof N_s time those of a real particle:

$$m_s = N_s m, \quad e_s = N_s e. \quad (0.25)$$

Since the charge-to-mass ratio remains the same, acceleration of a super particle is no different than that of a real particle. Hence, the velocity of super particle should not be scaled:

$$v_s = v. \quad (0.26)$$

Consequently, the temepature of super particles should be N_s times larger

$$T_s \propto m_s v_{Ts}^2 = N_s m v_T^2 \propto N_s T. \quad (0.27)$$

Nevertheless, since super particle density n_s , necessary for representing physical density n_0 , is reduced to

$$n_s = n_0/N_s, \quad (0.28)$$

the plasma frequency and Debye length remains unchanged:

$$\omega_{ps}^2 = \frac{4\pi n_s e_s^2}{m_s} = \frac{4\pi(n_0/N_s)(N_s e)^2}{N_s m} = \omega_p^2, \quad (0.29)$$

$$\lambda_{Ds}^2 = \frac{T_s}{4\pi n_s e_s^2} = \frac{N_s T}{4\pi n_0/N_s (N_s e)^2} = \lambda_D^2. \quad (0.30)$$

As the super fraction $N_s = n_0/n_s$ becomes closer to 1, the simulation becomes closer to the reality.