Units

EMOOPIC

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Maxwell Equations

In Yee algorithm, using Gaussian unit style with $c = 1$, i.e.

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} = -\frac{\partial B}{\partial \tilde{t}} \tag{0.1}$$

$$\nabla \times B = \frac{4\pi}{c} J + \frac{1}{c}\frac{\partial E}{\partial t} = 4\pi \tilde{J} + \frac{\partial E}{\partial \tilde{t}}. \tag{0.2}$$

We choose to keep cm as the length unit, so that the subsequent two Maxwell equations are unaffected:

$$\nabla \cdot E = 4\pi\rho \tag{0.3}$$

$$\nabla \cdot B = 0. \tag{0.4}$$

This means that the time has the unit of (1 cm) / (speed of light), i.e.

$$1\tilde{s} = \frac{1\text{cm}}{c_0\text{cm/s}} \approx 3.34 \times 10^{-11} \text{ s}, \tag{0.5}$$

where $c_0 \approx 2.9979 \times 10^{10}$ is the speed of light in unit of cm/s.

Velocity and acceleration

The velocity $\tilde{v}$ is now normalized automatically to the speed of light, since it is measured in cm / (distance light travels in 1 cm). Namely,

$$\tilde{v} = \frac{v}{c_0\text{cm/s}}. \tag{0.6}$$

The non-relativistic thermal velocity v_{th}, defined as the mean of the magnitude of the velocity, is then

$$
\begin{aligned}
\tilde{v}_{th} &= \frac{v_{th}}{c} = \sqrt{\frac{8T}{\pi m c^2}} && (0.7)\\
&\approx 2.232 \times 10^{-3} \sqrt{\frac{T/1\mathrm{eV}}{m/m_e}}. && (0.8)
\end{aligned}
$$

We then have the Lorentz acceleration as

$$
\begin{aligned}
a &= \frac{1}{c_0^2} \frac{q}{m} \left(E + \tilde{v} \times B \right) && (0.9)\\
&\approx 1.1 \times 10^{-21} \frac{q}{m} \left(E + \tilde{v} \times B \right). && (0.10)
\end{aligned}
$$

The small acceleration reflects the timescale separation between light waves and plasma motion.

Charge, Mass, and Field

It is convenient to normalize charge q to electron charge e, and normalize mass m to electron mass m_e. Namely,

$$\tilde{q} = \frac{q}{e},\ \tilde{m} = \frac{m}{m_e}. \tag{0.11}$$

Recall the value

$$\frac{e}{m_e} \approx 5.2728 \times 10^{17} \mathrm{statC/g}. \tag{0.12}$$

Having normalized charge and mass, it is convenient to normalize the electric field E to kilo statV/cm (KstatV/cm), and normalize the magnetic field B to kilo Gauss (KG). Namely,

$$\tilde{E} = \frac{E}{\mathrm{KstatV/cm}},\ \ \tilde{B} = \frac{B}{\mathrm{KG}}. \tag{0.13}$$

It is useful to note

$$\frac{1 \cdot \mathrm{KstatV/cm}}{c_0 \mathrm{cm/s} \cdot \mathrm{KG}} = 1. \tag{0.14}$$

With such normalization, there is no large or small coefficient in the Newton's equation with Lorentz force. Explicitly, we have

$$
\begin{aligned}
\tilde{a} &= \frac{a}{\mathrm{cm} \cdot \tilde{s}^{-2}} && (0.15)\\
&= \frac{\tilde{q}}{\tilde{m}} (\tilde{E} + \tilde{v} \times \tilde{B}) \times \frac{e \cdot \mathrm{KstatV/cm}}{m_e \cdot \mathrm{cm} \cdot \tilde{s}^{-2}} && (0.16)\\
&\approx 0.58668774 \frac{\tilde{q}}{\tilde{m}} (\tilde{E} + \tilde{v} \times \tilde{B}). && (0.17)
\end{aligned}
$$

To get a sense of how large the electric and magnetic fields are, it is useful to note

$$\tilde{E} = 1 \quad \leftrightarrow \quad E \approx 299.79\text{KV/cm}, \tag{0.18}$$
$$\tilde{B} = 1 \quad \leftrightarrow \quad B = 0.1\text{T}, \tag{0.19}$$

where KV/cm stands for kilo volte/cm, and T stands for tesla. Notice that a constant field $\tilde{E} = 1$ is a very large electric field, which can accelerate an electron to relativistic velocity within 1 $\tilde{s}$. In comparison, a constant field $\tilde{B} = 1$ is moderate magnetic field that can be easily produced in laboratory.

Charge and Current Density

Using the above units, the current density due to a single fluid species can be expressed as

$$\begin{aligned} \tilde{J} &= \frac{J}{c} = \frac{qvn}{c} && (0.20)\\ &= \tilde{q}\tilde{v}\tilde{n} \times e\text{cm}^{-3} && (0.21)\\ &= 4.8032 \times 10^{-10}\frac{\text{StatC}}{\text{cm}^3}\tilde{q}\tilde{v}\tilde{n}. && (0.22) \end{aligned}$$

That is to say, current densities used in the program is in units statC/cm^3.
Similarly, the charge density can be expressed as

$$\begin{aligned} \rho &= qn = \tilde{q}\tilde{n} \times e\text{cm}^{-3} && (0.23)\\ &= 4.8032 \times 10^{-10}\frac{\text{StatC}}{\text{cm}^3}\tilde{q}\tilde{n}. && (0.24) \end{aligned}$$

Notice that the charge density has the same unit as the current density.

Super Particles

It is rare that one has the numerical resources to run a PIC code with physical number of particles. Therefore, super particles are typically used, each representing N_s number of real particles. The mass and charge of a super particle is thereof N_s time those of a real particle:

$$m_s = N_s m, \; e_s = N_s e. \tag{0.25}$$

Since the charge-to-mass ratio remains the same, acceleration of a super particle is no different than that of a real particle. Hence, the velocity of super particle should not be scaled:

$$v_s = v. \tag{0.26}$$

Consequently, the temeprature of super particles should be N_s times larger

$$T_s \propto m_s v_{Ts}^2 = N_s m v_T^2 \propto N_s T. \tag{0.27}$$

Nevertheless, since super particle density n_s, necessary for representing physical density n_0, is reduced to

$$n_s = n_0/N_s, \tag{0.28}$$

the plasma frequency and Debye length remains unchanged:

$$\omega_{ps}^2 = \frac{4\pi n_s e_s^2}{m_s} = \frac{4\pi(n_0/N_s)(N_s e)^2}{N_s m} = \omega_p^2, \tag{0.29}$$

$$\lambda_{Ds}^2 = \frac{T_s}{4\pi n_s e_s^2} = \frac{N_s T}{4\pi n_0/N_s(N_s e)^2} = \lambda_D^2. \tag{0.30}$$

As the super fraction $N_s = n_0/n_s$ becomes closer to 1, the simulation becomes closer to the reality.