$\boxed{P \to P\bar{F}}$

$$PX(\vec{c}) = \left(\begin{array}{l} X(\vec{c}) \xrightarrow{\eta^{\bar{F}}_X(\vec{c})} X(\vec{c}) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq \vec{c}}} \bigotimes_v X(\vec{T}_v) \\ 1_v \sqcup X(c_j;c) \xrightarrow{\mathrm{id} \,\sqcup\, \eta_{\bar{F}}(c_j;c)} 1_v \sqcup \left(X(c_j;c) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq (c_j;c)}} \bigotimes_v X(\vec{T}_v) \right) \end{array} \right) = P\bar{F}X(\vec{c})$$

$$\psi = P\eta^{\bar{F}}_X$$

$\boxed{\vec{c} \neq (c_j;c)}$

$$\begin{tikzcd}
PPX(\vec{c}) \arrow[r, "P\eta_{\bar{F}}(\vec{c})"] \arrow[d, equal] & PP\bar{F}X(\vec{c}) \arrow[rr, "\eta^{\bar{F}}_{P\bar{F}X}"] & & P\bar{F}P\bar{F}X(\vec{c}) \\
X(\vec{c}) \arrow[r] \arrow[dd] \arrow[rrd, phantom, "\circlearrowleft"] & X(\vec{c}) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq \vec{c}}} \bigotimes X(\vec{T}_v) \arrow[r] & P\bar{F}X(\vec{c}) \arrow[d, equal] & \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq \vec{c}}} \bigotimes P\bar{F}X(\vec{T}_v) \arrow[ddl] \\
& & X(\vec{c}) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq \vec{c}}} \bigotimes X(\vec{T}_v) \arrow[d, "\mathrm{id}"] & \\
X(\vec{c}) \arrow[rr] \arrow[d, equal] & & X(\vec{c}) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq \vec{c}}} \bigotimes X(\vec{T}_v) \arrow[d, equal] & \\
PX(\vec{c}) \arrow[rr] & & P\bar{F}X(\vec{c}) &
\end{tikzcd}$$

$\boxed{\vec{c} = (c_j;c)}$

$$\begin{tikzcd}
PPX(c_j;c) \arrow[r, "P(\eta)"] \arrow[d, equal] & PP\bar{F}X(c_j;c) \arrow[rr, "P(\eta)"] & & P\bar{F}P\bar{F}X(c_j;c) \\
1_v \sqcup (1_v \sqcup X(c_j;c)) \arrow[r, hook] \arrow[dd, "\nabla"] \arrow[rrd, phantom, "\circlearrowleft"] & 1_v \sqcup \left(1_v \sqcup \left(X(c_j;c) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq (c_j;c)}} \bigotimes_v X(\vec{T}_v)\right)\right) \arrow[r] & 1_v \sqcup \Big(P\bar{F}X(c_j;c) \arrow[d, equal] & \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq (c_j;c)}} \bigotimes P\bar{F}X(\vec{T}_v) \Big) \arrow[ddl] \\
& & 1_v \sqcup X(c_j;c) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq (c_j;c)}} \bigotimes_v X(\vec{T}_v) \arrow[d, hook] & \\
1_v \sqcup X(c_j;c) \arrow[rr, hook] \arrow[d, equal] & & 1_v \sqcup X(c_j;c) \sqcup \coprod\limits_{\substack{[T] \\ \vec{T} \neq (c_j;c)}} \bigotimes X(\vec{T}_{v}) \arrow[d, equal] & \\
PX(c_j;c) & & P\bar{F}X(c_j;c) &
\end{tikzcd}$$

$$\boxed{\bar F \Rightarrow P\bar F}$$

$$\bar F X(\vec c) = X(\vec c) \amalg \coprod_{\substack{[T],\\ T\neq \vec c}} \bigotimes_v X(\vec T_v) \xrightarrow{\ \eta^P_{\bar F X}\ } \begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix} \amalg X(\vec c) \amalg \coprod_{\substack{[T],\\ T\neq \vec c}} \bigotimes_v X(\vec T_v) = P\bar F X(\vec c)$$

$$\begin{tikzcd}
\bar F \bar F X(\vec c) = \coprod_{\substack{[T]\downarrow \vec c,\\ T\neq\eta}} \bigotimes_v \bar F X(\vec T_v) \arrow[r] \arrow[d] \arrow[dr] & \begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix} \amalg \coprod_{\substack{[T]\downarrow \vec c,\\ T\neq\eta}} \bigotimes_v P\bar F X(\vec T_v) \arrow[d] \arrow[dl, phantom, "\checkmark"] \\
\bar F X(\vec c) = \coprod_{\substack{[S]\downarrow \vec c\\ S\neq\eta}} \bigotimes_v X(\vec S_v) \arrow[r] & \begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix} \amalg \coprod_{\substack{[S]\downarrow \vec c\\ S\neq\eta}} \bigotimes_v X(\vec S_v)
\end{tikzcd}$$

$$\boxed{P\bar F \xrightarrow{P\eta\bar F} P\bar F P\bar F = \overline{F F} \xRightarrow{\mu} P\bar F \text{ is the identity}}$$

$$P\bar F X(\vec c) \xrightarrow{P\eta\bar F} P\bar F P\bar F X(\vec c) \xrightarrow{\ \mu\ } P\bar F X(\vec c)$$

$$P\bar F X(\vec c) = \underbrace{\begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix}}_{*} \amalg \underbrace{X(\vec c) \amalg \coprod_{\substack{[T]\downarrow \vec c\\ T\neq\eta}} \bigotimes_v X(\vec T_v)}_{**}$$

$$P\bar F P\bar F X(\vec c) = \underbrace{\begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix}}_{*} \amalg P\bar F X(\vec c) \amalg \coprod_{\substack{[S]\downarrow \vec c\\ S\neq\eta}} P\bar F X(\vec S_v) = \begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix} \amalg \underbrace{X(\vec c) \amalg \coprod_{\substack{[T]\downarrow \vec c\\ T\neq\eta}} \bigotimes_v X(\vec T_v)}_{**}$$

$$\longrightarrow \underbrace{\begin{pmatrix}1_v\\ \text{or}\\ \phi\end{pmatrix}}_{*} \amalg \underbrace{X(\vec c) \amalg \coprod_{\substack{[T]\downarrow \vec c\\ T\neq\eta}} \bigotimes_v X(\vec T_v)}_{**} = P\bar F X(\vec c)$$

$\checkmark\checkmark$

$X \rightrightarrows Y \leftrightarrows Z$ in $\mathrm{Sym}_{\mathcal{C}}$

iff $X(\vec c) \rightrightarrows Y(\vec c) \leftrightarrows Z(\vec c)$ in $\mathcal{D}$

$\bar F P \xrightarrow{\bar F \eta_{\bar F}} \bar F P \bar F \Rightarrow \bar F$