COLOURED GENUINE OPERADS

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ABSTRACT. Things and stuff

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1. COLOURED OPERADS

1.1. Non-Equivariant Coloured Operads

Fix a closed symmetric monoidal category $\mathcal{V}$.

Definition 1.1. Fix a set $\mathfrak{C}$ of *colours.* A tuple $\xi = (c_1, \dots, c_n; c_0) \in \mathfrak{C}^{\times n} \times \mathfrak{C}$ is called a *signature* of $\mathfrak{C}$, and let $|\xi|$ denote the length n (so $\xi \in \mathfrak{C}^{\times|\xi|+1}$).

A $\mathfrak{C}$-*coloured operad*[1] in $\mathcal{V}$ consists of the following data:

(1) An object $\mathcal{O}(\xi) \in \mathcal{V}$ for each signature ξ.
(2) For each $c \in \mathfrak{C}$, a *unit* $1_c \in \mathcal{O}(c; c)$.
(3) For any signature $\xi \in \mathfrak{C}^{\times n+1}$ and $\sigma \in \Sigma_n$, a map $\mathcal{O}(\xi) \to \mathcal{O}(\sigma \cdot \xi)$, where Σ_n acts on the left of $\mathfrak{C}^{\times n+1}$ by acting on the first n coordinates. Explicitly, this is a map
$$\mathcal{O}(c_1, \dots, c_n; c_0) \xrightarrow{\sigma} \mathcal{O}(c_{\sigma^{-1}1}, \dots, c_{\sigma^{-1}n}; c_0).$$
(4) For any compatible signatures $\xi = (c_1, \dots, c_n; c_0)$, $\xi_i = (c_{i,1}, \dots, c_{i,m_i}; c_i)$, a *composition* map
$$\mathcal{O}(\xi) \times \mathcal{O}(\xi_1) \times \dots \times \mathcal{O}(\xi_n) \to \mathcal{O}(c_{1,1}, \dots, c_{n,m_n}; c_0)$$

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[1] These are also known as *symmetric multicategories*

subject to all the compatibilities you'd expect.

A map of $\mathfrak{C}$-coloured operads is a compatible collection of maps $\{\mathcal{O}(\xi) \to \mathcal{O}'(\xi)\}_{\xi}$.

Let $\mathsf{Op}^{\mathfrak{C}}(\mathcal{V})$ denote the category of $\mathfrak{C}$-coloured operads in $\mathcal{V}$.

Definition 1.2. Given a map $f : \mathfrak{C}' \to \mathfrak{C}$ and a $\mathfrak{C}$-coloured operad $\mathcal{O}$, there is a natural $\mathfrak{C}'$-coloured operad $f^*(\mathcal{O})$, where

$$f^*(\mathcal{O})(c_1', \dots, c_n'; c_0') = \mathcal{O}(f(c_1'), \dots, f(c_n'); f(c_0')).$$

A *map of coloured operads* $\mathcal{O}' \to \mathcal{O}$ is given by the data of a map of colours $f : \mathfrak{C}' \to \mathfrak{C}$, and a map of $\mathfrak{C}'$-coloured operads $\mathcal{O}' \to f^*(\mathcal{O})$.

Let $\mathsf{Op}(\mathcal{V})$ denote the category of coloured operads in $\mathcal{V}$.

Remark 1.3. The category $\mathsf{Op}(\mathcal{V})$ is isomorphic to the Grothendieck construction on the functor

$$\begin{aligned} \mathsf{F} &\longrightarrow \mathsf{Cat} \\ \mathfrak{C} &\longmapsto \mathsf{Op}^{\mathfrak{C}}(\mathcal{V}). \end{aligned}$$

cite

Notation 1.4. In previous work , $\mathsf{Op}(\mathcal{V})$ has been used to denote *single-coloured* operads specifically; that is, $\{*\}$-coloured operads. For this article, we will write these as $\mathsf{Op}^{\{*\}}(\mathcal{V})$.

1.2. **Equivariant Coloured Operads**

Definition 1.5. The category $\mathsf{Op}^G(\mathcal{V})$ of *G-coloured operads* in $\mathcal{V}$ is the category of G-objects in $\mathsf{Op}(\mathcal{V})$.

Remark 1.6. Unpacking this definition, we see $\mathcal{O} \in \mathsf{Op}^G(\mathcal{V})$ consists of the following data:

(1) A G-set $\mathfrak{C}$ of colours.
(2) For each signature ξ of $\mathfrak{C}$, an object $\mathcal{O}(\xi) \in \mathcal{V}$.
(3) For each signature $\xi \in \mathfrak{C}^{\times n+1}$ and $(g, \sigma) \in G \times \Sigma_n$, a map $\mathcal{O}(\xi) \to \mathcal{O}((g,\sigma)\cdot\xi)$, where G acts on $\mathfrak{C}^{\times n+1}$ diagonally (across all $n+1$ coordinates), and Σ_n acts on the first n.
(4) For each $c \in \mathfrak{C}$, a *unit* $1_c \in \mathcal{O}(c; c)^{G_c}$, where G_c is the stabilizer of c.
(5) For compatible signatures ξ, ξ_1, ..., ξ_n, *composition maps*

$$\mathcal{O}(\xi) \otimes \mathcal{O}(\xi_1) \otimes \dots \otimes \mathcal{O}(\xi_n) \to \mathcal{O}(\xi \circ (\xi_1, \dots, \xi_n)),$$

such that composition is compatible with the G-action on each component as well as the appropriate actions of Σ, and is unital and associative.

Remark 1.7. Unlike in the single-coloured case, this is *not* the same as coloured operads in $\mathcal{V}^G$. Indeed, objects in $\mathsf{Op}(\mathcal{V}^G)$ have a G-fixed set of colours, and each level $\mathcal{O}(\xi)$ is a full G-set (though only a partial $\Sigma_{|\xi|}$-set).

Definition 1.8. Given a G-set $\mathfrak{C}$, let $\mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V})$ denote the category of $\mathfrak{C}$-*coloured operads* and maps which are the identity on colours.

Parallel to the non-equivariant case, $\mathsf{Op}^G(\mathcal{V})$ is isomorphic to the Grothendieck construction on the functor

$$\begin{aligned} \mathsf{F}^G &\longrightarrow \mathsf{Cat} \\ \mathfrak{C} &\longmapsto \mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V}). \end{aligned}$$

1.2.1. *Categorical Description.*

Definition 1.9. Given a G-set X, let $B_X G$ denote the *translation category* of X, with object set X and morphisms $g : x \to g \cdot x$ for all pairs $(g, x) \in G \times X$.

We will denote $B_{\{*\}}G$ by G.

Remark 1.10. We observe that we have a natural diagonal map

$$F \times \mathsf{G} \hookrightarrow \mathsf{F} \wr \mathsf{G},$$

and so for any functor $F : \mathcal{C} \to \mathsf{F}$, we have an induced functor $F : \mathcal{C} \times \mathsf{G} \to \mathsf{F} \wr \mathsf{G}$.

Definition 1.11. Let $\mathfrak{C}\Sigma$ be the category

$$\mathfrak{C}\Sigma = \coprod_{n \geq 0} B_{\mathfrak{C}^{\times n} \times \mathfrak{C}}(G \times \Sigma_n).$$

We note that $\mathrm{Ob}(\mathfrak{C}\Sigma)$ is precisely the set of *signatures* in $\mathfrak{C}$. Further, we observe that this is equivalent to the pullback

$$\begin{tikzcd}
\mathfrak{C}\Sigma \arrow[r, "E"] \arrow[d] & \mathsf{F} \wr B_{\mathfrak{C}} G \arrow[d] \\
\Sigma \times \mathsf{G} \arrow[r, "E"] & \mathsf{F} \wr \mathsf{G}
\end{tikzcd}$$

where $E : \Omega \to \mathsf{F}$ sends a tree to its set of edges.

More generally, let $\mathfrak{C}\Omega$ be the pullback

$$\begin{tikzcd}
\mathfrak{C}\Sigma \arrow[r, "E"] \arrow[d] & \mathsf{F} \wr B_{\mathfrak{C}} G \arrow[d] \\
\Omega \times \mathsf{G} \arrow[r, "E"] & \mathsf{F} \wr \mathsf{G}
\end{tikzcd}$$

We have a natural inclusion of categories $\mathfrak{C}\Sigma \hookrightarrow \mathfrak{C}\Omega$. Moreover, we will called elements of these categories *coloured trees* (or *coloured corollas*), and denote them by $(T, \mathfrak{c})$, where $\mathfrak{c} : E(T) \to \mathfrak{C}$ is a map of sets.

Remark 1.12. Unpacking definitions, we see that a map $(T, \mathfrak{c}) \to (S, \mathfrak{d})$ is given by a map $f : T \to S$ in Ω and an element $g \in G$, such that $g.\mathfrak{c}(e) = \mathfrak{d}(f(e))$ for all $e \in E(T)$.

$$\begin{tikzcd}
E(T) \arrow[r, "f"] \arrow[d, "\mathfrak{c}"'] & E(S) \arrow[d, "\mathfrak{d}"] \\
\mathfrak{C} \arrow[r, "g"] & \mathfrak{C}
\end{tikzcd}$$

In particular, we have maps of the form

$$g = (id, g) : (T, E(T) \to \mathfrak{C}) \to (T, E(T) \to \mathfrak{C} \xrightarrow{g \cdot} \mathfrak{C}).$$

Remark 1.13. $\mathfrak{C}\Omega$ is equivalent to the Grothendieck construction on the functor

$$\Omega^{op} \times G \longrightarrow \mathsf{Cat}$$

$$T \longmapsto \mathsf{Fun}(E(T), \mathfrak{C}).$$

and a similar result holds for $\mathfrak{C}\Sigma$.

Many of the natural functors around Ω and Σ have generalizations to the coloured setting, which can be built through a straightforward use of the universal property of pullbacks.

Definition 1.14. We have a natural *vertex* functor $V : \mathfrak{C}\Omega \to \Sigma \wr \mathfrak{C}\Sigma$, as colourings of a tree restrict to colourings of each vertex corolla.

Similarly, there is a *leaf-root* funct or $\mathsf{lr} : \mathfrak{C}\Omega \to \mathfrak{C}\Sigma$, where the colouring of $\mathsf{lr}(T)$ is a restrict of the colouring of T.

Definition 1.15. The category $\mathsf{Sym}^{G,\mathfrak{C}}$ of *symmetric* $(G,\mathfrak{C})$*-sequences* is the category of functors $X : \mathfrak{C}\Sigma^{op} \to \mathcal{V}$.

Definition 1.16. Given $X \in \mathsf{Sym}^{G,\mathfrak{C}}$, let $\mathbb{F}^{\mathfrak{C}}X$ denote the left Kan extension below.

$$\begin{tikzcd}
\mathfrak{C}\Omega^{op} \arrow[r, "V"] \arrow[d, "\mathsf{lr}"'] & (\Sigma \wr \mathfrak{C}\Sigma)^{op} \arrow[r, "X"] \arrow[dl, Rightarrow] & (\Sigma \wr \mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\
\mathfrak{C}\Sigma^{op} \arrow[urrr, "\mathrm{Lan}=\mathbb{F}^{\mathfrak{C}} X"'] & & &
\end{tikzcd}$$

1.3. **Single-Coloured Operads**

We first show that this generalizes the free single-coloured operad monad. When $\mathfrak{C} = \{*\}$, we have $\mathfrak{C}\Omega = \Omega \times G$, and similarly $\mathfrak{C}\Sigma = \Sigma \times G$.

Notation 1.17. Given $X \in \mathsf{Cat}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{V}))$, let $\tilde{X}$ denote the adjoint functor in the isomorphic category $\mathsf{Cat}(\mathcal{C} \times \mathcal{D}, \mathcal{V})$.

SPAN_LAN_LEM **Lemma 1.18.** *Conisder the two spans below.*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "X"] \arrow[d, "p"] & \mathsf{Fun}(\mathcal{D}, \mathcal{V}) \\
\mathcal{E} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r, "\tilde{X}"] \arrow[d, "p \times \mathsf{id}"] & \mathcal{V} \\
\mathcal{E} \times \mathcal{D} &
\end{tikzcd}$$

Then $\mathrm{Lan}_p X$ *is adjoint to* $\mathrm{Lan}_{p\times\mathsf{id}} \tilde{X}$.

Proof. We have

$$\begin{aligned}
\widetilde{\mathrm{Lan}_p X}(e,d) &= (\mathrm{Lan}_p X(e))(d) = \left(\operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} X(c)\right)(d) = \operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} (X(c)(d)) = \operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} (\tilde{X}(c,d)) \\
&= \operatorname*{colim}_{\substack{\mathcal{C}\times\{d\}\downarrow(e,d) \\ p(c)\to e}} (\tilde{X}(c,d)) \cong \operatorname*{colim}_{\substack{\mathcal{C}\times\mathcal{D}\downarrow(e,d) \\ (p(c),d')\to(e,d)}} (\tilde{X}(c,d')) = \mathrm{Lan}_{p\times\mathsf{id}} \tilde{X}(c,d),
\end{aligned}$$

where the isomorphism holds by a straightforward finality argument. On maps, a similar argument holds. $\square$

Notation 1.19 ([BP17] [2])**.** Let $\mathbb{F}'$ denote the *free single-coloured operad monad* on $\mathcal{V}$, given by the left Kan extension of the following diagram.

$$\begin{tikzcd}
\Omega^{op} \arrow[r, "V"] \arrow[d, "\mathsf{lr}"'] & (\Sigma \wr \Sigma)^{op} \arrow[r, "X"] \arrow[dl, Rightarrow] & (\Sigma \wr \mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\
\Sigma^{op} \arrow[urrr, "\mathrm{Lan}=\mathbb{F}' X"'] & & &
\end{tikzcd}$$

Proposition 1.20. *$\mathbb{F}^{\{*\}}$ is a monad, and moreover the category of $\mathbb{F}^{\{*\}}$-algebras in $\mathsf{Fun}(\Sigma \times G, \mathcal{V})$ is equivalent to the category of $\mathbb{F}'$-algebras in $\mathsf{Fun}(\Sigma, \mathcal{V}^G)$.*

Proof. Let $\tau : \tilde{X} \mapsto X$ denote the isomorphism of categories $\mathsf{Fun}(\Sigma \times G, \mathcal{V}) \xrightarrow{\tau} \mathsf{Fun}(\Sigma, \mathcal{V}^G)$. Then $\mathbb{F}^{\{*\}} = \tau^{-1}\mathbb{F}'\tau$ by SPAN_LAN_LEM 1.18, and so $\mathbb{F}^{\{*\}}$ is in fact a monad, and the the isomorphism lifts to an isomorphism on the category of algebras. □

1.4. **General Case**

Theorem 1.21. *For every G-set $\mathfrak{C}$, $\mathbb{F}^{\mathfrak{C}}$ is a monad, with category of algebras given by $\mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V})$.*

2. Coloured Genuine Equivariant Operads

Throughout this section, we will abuse notation, and refer to a coefficient system and its associated (Grothendieck) category over O_G by the same name.

Definition 2.1. Let $\underline{\mathfrak{C}}$ be a coefficient system of sets. Then the category $\underline{\mathfrak{C}}\Omega_G$ of $\underline{\mathfrak{C}}$-*coloured trees* is defined to be the pullback below.

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega_G \arrow[r] \arrow[d] & \mathsf{F} \wr \underline{\mathfrak{C}} \arrow[d] \\ \Omega_G \arrow[r, "E_G"] & \mathsf{F} \wr O_G \end{tikzcd}$$

The category $\underline{\mathfrak{C}}\Sigma_G$ of $\underline{\mathfrak{C}}$-coloured corollas is the subcategory defined similarly, with Ω_G replaced with Σ_G.

Remark 2.2. Consider the Grothendieck construction on the functor

$$\begin{aligned} \mathsf{F}^{G,op} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathsf{Set}^{O_G^{op}}(\Phi(A), \underline{\mathfrak{C}}), \end{aligned}$$

where $\Phi : \mathsf{Set}^G \to \mathsf{Set}^{O_G^{op}}$ sends a G-set X to its fixed-point system $G/H \mapsto X^H$. We will denote this by $\mathsf{F}^G \wr \underline{\mathfrak{C}}$. Then $\underline{\mathfrak{C}}\Omega_G$ is also isomorphic to the pullback

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega_G \arrow[r] \arrow[d] & \mathsf{F}^G \wr \underline{\mathfrak{C}} \arrow[d] \\ \Omega_G \arrow[r, "E"] & \mathsf{F}^G. \end{tikzcd}$$

We note that the class of morphisms in F^G in the image of E are those isomorphic to an adjunction counit $G \cdot_H A|_H \to A$.

Equivalently, $\underline{\mathfrak{C}}\Omega_G$ is isomorphic to the Grothendieck construction on the functor

$$\begin{aligned} \Omega_G^{op} &\longrightarrow \mathsf{Cat} \\ T &\longmapsto \mathsf{Set}^{O_G^{op}}(\Phi(E(T)), \underline{\mathfrak{C}}), \end{aligned}$$

$\underline{\mathfrak{C}}\Sigma_G$ can be defined similarly, with the relevant sources restricted to $\Sigma_G \subseteq \Omega_G$.

Objects of $\underline{\mathfrak{C}}\Omega_G$ are pairs $(T, \mathfrak{c})$ of a G-tree T with a map $\mathfrak{c} : E_G(T) \to \underline{\mathfrak{C}}$ over O_G. That is, each orbit of edges $[e]$ is assigned a "colour" $\mathfrak{c}([e]) \in \underline{\mathfrak{C}}(G/G_{[e]})$, where $G_{[e]}$ is the stabilizer in G of e. Morphisms $(T, \mathfrak{c}) \to (S, \mathfrak{d})$ are given by quotients of trees $q : S \to T$ such that, for every edge orbit $[e]$ of S, we have

$$\mathfrak{c}([e]) = q_{[e]}^* \mathfrak{d}([q(e)]),$$

where $q_{[e]} : G/G_{[e]} \to G/G_{[q(e)]}$ is the map in O_G induced by q.

3. In $\mathsf{dSet_G}$

Definition 3.1. Define the *genuine operadic nerve* $N : \mathsf{Op}_G \to \mathsf{dSet}_G$ by

$$N\mathcal{P}(T) = \mathrm{Hom}_{\mathsf{Op}_G}(T, \mathcal{P})$$

where we think of T as the operad $T \in \mathsf{Op}^G \hookrightarrow \mathsf{Op}_G$.

Remark 3.2. We note that $N\mathcal{P} \in (SCI)^{\boxslash !}$, as $T \in \mathsf{Op}_G$ is a free $\mathbb{F}_G$-algebra on its vertices.

Remark 3.3. We can rephrase the definition of being an $\mathbb{F}_G$-algebra in terms of $N\mathcal{P}$. For $\mathcal{P} \in \mathsf{Sym}_G$ a G-symmetric sequence, a genuine G-operad structure on $\mathcal{P}$ is given by:

- Composition Maps:
  maps $N\mathcal{P}(T) \to \mathcal{P}(\mathsf{lr}(T))$ for all $T \in \Omega_G$.
- Naturality under restriction and conjugation:
  maps $N\mathcal{P}(T_1) \to N\mathcal{P}(T_0)$ for all quotient maps $T_0 \to T_1$ in $\Omega_{G,0}$, such that the following commutes:

$$\begin{array}{ccc} N\mathcal{P}(T_1) & \longrightarrow & \mathcal{P}(\mathsf{lr}(T_1)) \\ \downarrow & & \downarrow \\ N\mathcal{P}(T_0) & \longrightarrow & \mathcal{P}(\mathsf{lr}(T_0)). \end{array}$$

- Associativity under $\mathbb{F}_G$:
  maps $N\mathcal{P}(T_1) \to N\mathcal{P}(T_0)$ for all planar tall maps $T_0 \to T_1$ in Ω^t_G, such that the analogus diagram (with the right vertical map the identity) commutes.[2]

The above reflects the following result.

Proposition 3.4. Op_G *is equivalent to the subcategory of* $\mathsf{dSet_G}$ *spanned by those* X *such that*

(1) $X(H/H) = \{*\}$ *for all* $H \leq G$.
(2) $X(T) \cong \otimes_{T_v \in V(T)} X(T_v)$.

Proof. The fact that $N\mathcal{P} \in (SCI_G)^{\boxslash !}$ is immediate, as remarked above.

For the reverse direction, we will follow the construction of the homotopy operad as in MW08 [3, §6], replacing their use of inner horn inclusions with *orbital* inner G-horn inclusions, to show that any $X \in (OHI)^{\boxslash !}$ is in the image of N; the result will then follow from BP_Segal [1, HYPER PROP].

In fact, interpreting all of their pictures are as *orbital* representations of G-trees yields that, for all $C \in \Sigma_G$

- $\sim_{Ge}$ is an equivalence relation on $X(C)$ for all $Ge \in E_G(C)$.
- The relations $\sim_{Ge}$ and $\sim_{Ge'}$ are equal for all $e, e' \in E(C)$.
- $[h] \circ [f] = [h \circ f]$ yields a well-defined composition map.
  come back

need to show naturality, check associativity of composition

□

[2] As in BP17 [2], we note that "associativity" under $\mathbb{F}_G$ includes both the usual notion of associativity of our composition maps, but also unitality; this is recorded here by the fact that degeneracies are always planar tall.

References

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MW08 [3] I. Moerdijk and I. Weiss. On inner Kan complexes in the category of dendroidal sets. *Adv. Math.*, 221(2):343–389, 2009.