Equivariant dendroidal Segal spaces and G-∞-operads

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Abstract

Things and stuff

Contents

1 Coloured Operads

1.1 Non-Equivariant Coloured Operads

Fix a closed symmetric monoidal category $\mathcal{V}$.

Definition 1.1. Fix a set $\mathfrak{C}$ of *colours.* A tuple $\xi = (c_1, \dots, c_n; c_0) \in \mathfrak{C}^{\times n} \times \mathfrak{C}$ is called a *signature* of $\mathfrak{C}$, and let $|\xi|$ denote the length n (so $\xi \in \mathfrak{C}^{\times |\xi|+1}$).

A $\mathfrak{C}$-*coloured operad* in $\mathcal{V}$ consists of the following data:

1. An object $\mathcal{O}(\xi) \in \mathcal{V}$ for each signature ξ.

2. For each $c \in \mathfrak{C}$, a *unit* $1_c \in \mathcal{O}(c; c)$.

3. For any signature $\xi \in \mathfrak{C}^{\times n+1}$ and $\sigma \in \Sigma_n$, a map $\mathcal{O}(\xi) \to \mathcal{O}(\sigma \cdot \xi)$, where Σ_n acts on the left of $\mathfrak{C}^{\times n+1}$ by acting on the first n coordinates. Explicitly, this is a map

$$\mathcal{O}(c_1, \ldots, c_n; c_0) \xrightarrow{\sigma} \mathcal{O}(c_{\sigma^{-1}1}, \ldots, c_{\sigma^{-1}n}; c_0).$$

4. For any compatible signatures $\xi = (c_1, \ldots, c_n; c_0)$, $\xi_i = (c_{i,1}, \ldots, c_{i,m_i}; c_i)$, a *composition* map

$$\mathcal{O}(\xi) \times \mathcal{O}(\xi_1) \times \ldots \times \mathcal{O}(\xi_n) \to \mathcal{O}(c_{1,1}, \ldots, c_{n,m_n}; c_0)$$

subject to all the compatibilities you'd expect.

A map of $\mathfrak{C}$-coloured operads is a compatible collection of maps $\{\mathcal{O}(\xi) \to \mathcal{O}'(\xi)\}_\xi$.

Let $\mathsf{Op}^{\mathfrak{C}}(\mathcal{V})$ denote the category of $\mathfrak{C}$-coloured operads in $\mathcal{V}$.

Definition 1.2. Given a map $f : \mathfrak{C}' \to \mathfrak{C}$ and a $\mathfrak{C}$-coloured operad $\mathcal{O}$, there is a natural $\mathfrak{C}'$-coloured operad $f^*(\mathcal{O})$, where

$$f^*(\mathcal{O})(c'_1, \ldots, c'_n; c'_0) = \mathcal{O}(f(c'_1), \ldots, f(c'_n); f(c'_0)).$$

A *map of coloured operads* $\mathcal{O}' \to \mathcal{O}$ is given by the data of a map of colours $f : \mathfrak{C}' \to \mathfrak{C}$, and a map of $\mathfrak{C}'$-coloured operads $\mathcal{O}' \to f^*(\mathcal{O})$.

Let $\mathsf{Op}(\mathcal{V})$ denote the category of coloured operads in $\mathcal{V}$.

Remark 1.3. The category $\mathsf{Op}(\mathcal{V})$ is isomorphic to the Grothendieck construction on the functor

$$\begin{array}{ccc} \mathsf{F} & \longrightarrow & \mathsf{Cat} \\ \mathfrak{C} & \longmapsto & \mathsf{Op}^{\mathfrak{C}}(\mathcal{V}). \end{array}$$

Notation 1.4. In previous work , $\mathsf{Op}(\mathcal{V})$ has been used to denote *single-coloured* operads specifically; that is, $\{*\}$-coloured operads. For this article, we will write these as $\mathsf{Op}^{\{*\}}(\mathcal{V})$. cite

1.2 Equivariant Coloured Operads

Definition 1.5. The category $\mathsf{Op}^G(\mathcal{V})$ of *G-coloured operads* in $\mathcal{V}$ is the category of G-objects in $\mathsf{Op}(\mathcal{V})$.

Remark 1.6. Unpacking this definition, we see $\mathcal{O} \in \mathsf{Op}^G(\mathcal{V})$ consists of the following data:

1. A G-set $\mathfrak{C}$ of colours.

2. For each signature ξ of $\mathfrak{C}$, an object $\mathcal{O}(\xi) \in \mathcal{V}$.

3. For each signature $\xi \in \mathfrak{C}^{\times n+1}$ and $(g, \sigma) \in G \times \Sigma_n$, a map $\mathcal{O}(\xi) \to \mathcal{O}((g, \sigma) \cdot \xi)$, where G acts on $\mathfrak{C}^{\times n+1}$ diagonally (across all $n+1$ coordinates), and Σ_n acts on the first n.

4. For each $c \in \mathfrak{C}$, a *unit* $1_c \in \mathcal{O}(c; c)^{G_c}$, where G_c is the stabilizer of c.

5. For compatible signatures ξ, ξ_1, ..., ξ_n, *composition maps*

$$\mathcal{O}(\xi) \otimes \mathcal{O}(\xi_1) \otimes \ldots \otimes \mathcal{O}(\xi_n) \to \mathcal{O}(\xi \circ (\xi_1, \ldots, \xi_n)),$$

such that composition is compatible with the G-action on each component as well as the appropriate actions of Σ, and is unital and associative.

Remark 1.7. Unlike in the single-coloured case, this is *not* the same as coloured operads in $\mathcal{V}^G$. Indeed, objects in $\mathsf{Op}(\mathcal{V}^G)$ have a G-fixed set of colours, and each level $\mathcal{O}(\xi)$ is a full G-set (though only a partial $\Sigma_{|\xi|}$-set).

Definition 1.8. Given a G-set $\mathfrak{C}$, let $\mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V})$ denote the category of $\mathfrak{C}$-*coloured operads* and maps which are the identity on colours.

Parallel to the non-equivariant case, $\mathsf{Op}^G(\mathcal{V})$ is isomorphic to the Grothendieck construction on the functor

$$\begin{aligned} \mathsf{F}^G &\longrightarrow \mathsf{Cat} \\ \mathfrak{C} &\longmapsto \mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V}). \end{aligned}$$

1.2.1 Categorical Description

Definition 1.9. Given a G-set X, let $B_X G$ denote the *translation category* of X, with object set X and morphisms $g : x \to g \cdot x$ for all pairs $(g, x) \in G \times X$.

We will denote $B_{\{*\}}G$ by G.

Remark 1.10. We observe that we have a natural diagonal map

$$F \times \mathsf{G} \hookrightarrow \mathsf{F} \wr \mathsf{G},$$

and so for any functor $F : \mathcal{C} \to \mathsf{F}$, we have an induced functor $F : \mathcal{C} \times \mathsf{G} \to \mathsf{F} \wr \mathsf{G}$.

Definition 1.11. Let $\mathfrak{C}\Sigma$ be the category

$$\mathfrak{C}\Sigma = \coprod_{n \geq 0} B_{\mathfrak{C}^{\times n} \times \mathfrak{C}}(G \times \Sigma_n).$$

We note that $\mathrm{Ob}(\mathfrak{C}\Sigma)$ is precisely the set of *signatures* in $\mathfrak{C}$. Further, we observe that this is equivalent to the pullback

$$\begin{CD} \mathfrak{C}\Sigma @>E>> \mathsf{F} \wr B_{\mathfrak{C}}G \\ @VVV @VVV \\ \Sigma \times \mathsf{G} @>E>> \mathsf{F} \wr \mathsf{G} \end{CD}$$

CSIGMA_EQ

where $E : \Omega \to \mathsf{F}$ sends a tree to its set of edges.

$B_{\mathfrak{C}}G = G \ltimes \mathfrak{C}$

More generally, let $\mathfrak{C}\Omega$ be the pullback

$$\begin{CD} \mathfrak{C}\Omega @>E>> \mathsf{F} \wr B_{\mathfrak{C}}G \\ @VVV @VVV \\ \Omega \times \mathsf{G} @>E>> \mathsf{F} \wr \mathsf{G} \end{CD} \tag{1.12}$$

COMEGA_EQ

We have a natural inclusion of categories $\mathfrak{C}\Sigma \hookrightarrow \mathfrak{C}\Omega$. Moreover, we will called elements of these categories *coloured trees* (or *coloured corollas*), and denote them by $(T, \mathfrak{c})$, where $\mathfrak{c} : E(T) \to \mathfrak{C}$ is a map of sets.

Remark 1.13. Unpacking definitions, we see that a map $(T, \mathfrak{c}) \to (S, \mathfrak{d})$ is given by a map $f : T \to S$ in Ω and an element $g \in G$, such that $g.\mathfrak{c}(e) = \mathfrak{d}(f(e))$ for all $e \in E(T)$.

$$\begin{CD} E(T) @>f>> E(S) \\ @V{\mathfrak{c}}VV @VV{\mathfrak{d}}V \\ \mathfrak{C} @>g>> \mathfrak{C} \end{CD}$$

In particular, we have maps of the form

$$g = (id, g) : (T, E(T) \to \mathfrak{C}) \to (T, E(T) \to \mathfrak{C} \xrightarrow{g\cdot} \mathfrak{C}).$$

Remark 1.14. $\mathfrak{C}\Omega$ is equivalent to the Grothendieck construction on the functor

$$\begin{aligned} \Omega^{op} \times G &\longrightarrow \mathsf{Cat} \\ T &\longmapsto \mathsf{Fun}(E(T), \mathfrak{C}). \end{aligned}$$

compare with genuine case: RHS equals $\mathsf{Fun}(\Phi(E(T)), \mathfrak{C}) = \mathsf{Fun}(\Phi(E(G \cdot T)), \mathfrak{C}))$

and a similar result holds for $\mathfrak{C}\Sigma$.

Remark 1.15. Note that we can replace the G-set $\mathfrak{C}$ with a *coefficient system* $\underline{\mathfrak{C}}$, substituting the rectangle of pullbacks below for COMEGA_EQ (1.12)

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega \arrow[r, "E"] \arrow[d] & \mathsf{F} \wr B_{\mathfrak{C}(G/e)}G \arrow[r] \arrow[d] & \mathsf{F} \wr \underline{\mathfrak{C}} \arrow[d] \\ \Omega \times G \arrow[r, "E"] & \mathsf{F} \wr G \arrow[r] & \mathsf{F} \wr O_G \end{tikzcd}$$

with $\mathfrak{C}(G/e) \hookrightarrow \underline{\mathfrak{C}}$ and $G \hookrightarrow O_G$ the natural inclusions.

compare $B_{\mathfrak{C}(G/e)}G = G \ltimes \mathfrak{C}(G/e)$ and $\underline{\mathfrak{C}} = O_G \ltimes \underline{\mathfrak{C}}$.

In this case, $\underline{\mathfrak{C}}\Omega = \mathfrak{C}(G/e)\Omega$.

Many of the natural functors around Ω and Σ have generalizations to the coloured setting, which can be built through a straightforward use of the universal property of pullbacks.

Definition 1.16. We have a natural *vertex* functor $V : \mathfrak{C}\Omega \to \Sigma \wr \mathfrak{C}\Sigma$, as colourings of a tree restrict to colourings of each vertex corolla.

Similarly, there is a *leaf-root* funct or $\mathsf{lr} : \mathfrak{C}\Omega \to \mathfrak{C}\Sigma$, where the colouring of $\mathsf{lr}(T)$ is a restrict of the colouring of T.

Definition 1.17. The category $\mathsf{Sym}^{G,\mathfrak{C}}$ of *symmetric* $(G, \mathfrak{C})$*-sequences* is the category of functors $X : \mathfrak{C}\Sigma^{op} \to \mathcal{V}$.

Definition 1.18. Given $X \in \mathsf{Sym}^{G,\mathfrak{C}}$, let $\mathbb{F}^{\mathfrak{C}}X$ denote the left Kan extension below.

$$\begin{tikzcd} \mathfrak{C}\Omega^{op} \arrow[r, "V"] \arrow[d, "\mathsf{lr}"'] & (\Sigma \wr \mathfrak{C}\Sigma)^{op} \arrow[r, "X"] \arrow[dl, Rightarrow] & (\Sigma \wr \mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\ \mathfrak{C}\Sigma^{op} \arrow[urrr, "\mathrm{Lan} = \mathbb{F}^{\mathfrak{C}}X"'] & & & \end{tikzcd}$$

1.3 Single-Coloured Operads

We first show that this generalizes the free single-coloured operad monad. When $\mathfrak{C} = \{*\}$, we have $\mathfrak{C}\Omega = \Omega \times G$, and similarly $\mathfrak{C}\Sigma = \Sigma \times G$.

Notation 1.19. Given $X \in \mathsf{Cat}(\mathcal{C}, \mathsf{Fun}(\mathcal{D}, \mathcal{V}))$, let $\tilde{X}$ denote the adjoint functor in the isomorphic category $\mathsf{Cat}(\mathcal{C} \times \mathcal{D}, \mathcal{V})$.

SPAN_LAN_LEM **Lemma 1.20.** *Conisder the two spans below.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "X"] \arrow[d, "p"] & \mathsf{Fun}(\mathcal{D},\mathcal{V}) \\ \mathcal{E} & \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{C}\times\mathcal{D} \arrow[r, "\tilde{X}"] \arrow[d, "p\times\mathsf{id}"] & \mathcal{V} \\ \mathcal{E}\times\mathcal{D} & \end{tikzcd}$$

Then $\mathrm{Lan}_p X$ *is adjoint to* $\mathrm{Lan}_{p\times\mathsf{id}}\tilde{X}$.

Proof. We have

$$\widetilde{\mathrm{Lan}_p X}(e,d) = (\mathrm{Lan}_p X(e))(d) = \left(\operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} X(c)\right)(d) = \operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} (X(c)(d)) = \operatorname*{colim}_{\substack{\mathcal{C}\downarrow e \\ p(c)\to e}} (\tilde{X}(c,d))$$

$$= \operatorname*{colim}_{\substack{\mathcal{C}\times\{d\}\downarrow(e,d) \\ p(c)\to e}} (\tilde{X}(c,d)) \cong \operatorname*{colim}_{\substack{\mathcal{C}\times\mathcal{D}\downarrow(e,d) \\ (p(c),d')\to(e,d)}} (\tilde{X}(c,d')) = \mathrm{Lan}_{p\times\mathsf{id}}\tilde{X}(c,d),$$

where the isomorphism holds by a straightforward finality argument. On maps, a similar argument holds. □

Notation 1.21 (BP17 [BP17])**.** Let $\mathbb{F}'$ denote the *free single-coloured operad monad* on $\mathcal{V}$, given by the left Kan extension of the following diagram.

$$\begin{tikzcd} \Omega^{op} \arrow[r, "V"] \arrow[d, "\mathsf{lr}"'] & (\Sigma\wr\Sigma)^{op} \arrow[r, "X"] \arrow[dl, Rightarrow] & (\Sigma\wr\mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\ \Sigma^{op} \arrow[urrr, "\mathrm{Lan}=\mathbb{F}'X"'] & & & \end{tikzcd}$$

Proposition 1.22. $\mathbb{F}^{\{*\}}$ *is a monad, and moreover the category of* $\mathbb{F}^{\{*\}}$*-algebras in* $\mathsf{Fun}(\Sigma\times G,\mathcal{V})$ *is equivalent to the category of* $\mathbb{F}'$*-algebras in* $\mathsf{Fun}(\Sigma,\mathcal{V}^G)$.

Proof. Let $\tau:\tilde{X}\mapsto X$ denote the isomorphism of categories $\mathsf{Fun}(\Sigma\times G,\mathcal{V})\xrightarrow{\tau}\mathsf{Fun}(\Sigma,\mathcal{V}^G)$. Then $\mathbb{F}^{\{*\}} = \tau^{-1}\mathbb{F}'\tau$ by SPAN_LAN_LEM 1.20, and so $\mathbb{F}^{\{*\}}$ is in fact a monad, and the the isomorphism lifts to an isomorphism on the category of algebras. □

1.4 General Case

Theorem 1.23. *For every* G*-set* $\mathfrak{C}$, $\mathbb{F}^{\mathfrak{C}}$ *is a monad, with category of algebras given by* $\mathsf{Op}^{G,\mathfrak{C}}(\mathcal{V})$.

Proof. This will be a corollary of Genuine Coloured stuff

□

2 Coloured Genuine Equivariant Operads

Throughout this section, we will abuse notation, and refer to a coefficient system and its associated (Grothendieck) category over O_G by the same name.

Idea: we have a *coefficient system* $\underline{\mathfrak{C}}$ of colours, and a *signature* will consist of a tuple $\xi = (x_1, \dots, x_n; x_0)$ with $x_i \in \mathfrak{C}(G/H_i)$ for subgroups $H_i \leq H_0 \leq G$.

2.1 Coloured G-Trees

Definition 2.1. The *edge orbit* functor $E_G : \Omega_G \to \mathsf{F} \wr O_G$ sends a G-tree T to the tuple $(E_G(T), (G/G_e)_{Ge \in E_G(T)})$ with G_e denoting $\mathrm{Stab}_G(e)$, and where we have canonical representatives for elements in $E_G(T)$ by choosing $e \in Ge$ minimal with respect to the planar structure on T.

Definition 2.2. Let $\underline{\mathfrak{C}}$ be a G-coefficient system of sets. Then the category $\underline{\mathfrak{C}}\Omega_G$ of $\underline{\mathfrak{C}}$-*coloured* G-*trees* is defined to be the pullback below.

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega_G \arrow[r] \arrow[d] & \mathsf{F} \wr \underline{\mathfrak{C}} \arrow[d] \\ \Omega_G \arrow[r, "E_G"] & \mathsf{F} \wr O_G \end{tikzcd}$$

COMEGA_G_EQ

The category $\underline{\mathfrak{C}}\Sigma_G$ of $\underline{\mathfrak{C}}$-coloured corollas is the subcategory defined similarly, with Ω_G replaced with Σ_G.

Explicitly, objects of $\underline{\mathfrak{C}}\Omega_G$ are pairs $(T, \mathfrak{c})$ of a G-tree T and a map $\mathfrak{c} : E_G(T) \to \underline{\mathfrak{C}}$ over O_G. That is, each orbit of edges Ge (with e minimal) is assigned a "colour" $\mathfrak{c}(Ge) \in \underline{\mathfrak{C}}(G/G_e)$. Morphisms $(T, \mathfrak{c}) \to (S, \mathfrak{d})$ are given by maps of trees $\varphi : T \to S$ such that, for every edge orbit Ge of T, we have

$$\mathfrak{c}(Ge) = \varphi_e^* g_e^* \mathfrak{d}(Gf),$$

where $\varphi_e : G/G_e \to G/G_{\varphi(e)}$ is the map in O_G induced by φ, and $\varphi(e) = g_e f$ for $f \in Gf \in E_G(S)$ minimal; as g_e is unique modulo G_f, g_e^* is well-defined.

Remark 2.3. Alternatively, consider the Grothendieck construction on the functor

$$\begin{aligned} \mathsf{F}^{G,op} &\longrightarrow \mathsf{Set} \\ A &\longmapsto \mathsf{Set}^{O_G^{op}}(\Phi(A), \underline{\mathfrak{C}}), \end{aligned}$$

where $\Phi : \mathsf{Set}^G \to \mathsf{Set}^{O_G^{op}}$ sends a G-set X to its fixed-point system $G/H \mapsto X^H$. We will denote this by $\mathsf{F}^G \wr \underline{\mathfrak{C}}$. Then $\underline{\mathfrak{C}}\Omega_G$ is also isomorphic to the pullback

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega_G \arrow[r] \arrow[d] & \mathsf{F}^G \wr \underline{\mathfrak{C}} \arrow[d] \\ \Omega_G \arrow[r, "E"] & \mathsf{F}^G. \end{tikzcd}$$

We note that the class of morphisms in F^G in the image of E (restricted to Ω_G^0) are those isomorphic to an adjunction counit $G \cdot_H A|_H \to A$.

In this case, a colouring is a map $\mathfrak{c} : \Phi E(T) \to \underline{\mathfrak{C}}$ of coefficient systems, and morphisms are maps $\varphi : T \to S$ such that $\mathfrak{c}(G/H, e) = \mathfrak{d}(G/H, e)$ for all $e \in E(T)^H$.

$$\begin{tikzcd} \Phi A \arrow[rr, "f"] \arrow[dr, "\mathfrak{c}"'] & & \Phi B \arrow[dl, "\mathfrak{d}"] \\ & \underline{\mathfrak{C}} & \end{tikzcd}$$

It is easy to show this is equivalent to requiring that $\mathfrak{c}(G/G_e, e) = \varphi_e^* \mathfrak{d}(G/G_{\varphi(e)}, \varphi(e))$.

figure out whether first or "alternatively" is more useful as the chosen construction

Similarly, $\underline{\mathfrak{C}}\Omega_G$ is isomorphic to the Grothendieck construction on the functor

$$\begin{aligned} \Omega_G^{op} &\longrightarrow \mathsf{Cat} \\ T &\longmapsto \mathsf{Set}^{O_G^{op}}(\Phi(E(T)), \underline{\mathfrak{C}}), \end{aligned}$$

$\underline{\mathfrak{C}}\Sigma_G$ can be defined similarly, with the relevant sources restricted to $\Sigma_G \subseteq \Omega_G$.

Remark 2.4. $\underline{\mathfrak{C}}\Omega_G$ is also a root fibration — that is, a split Grothendieck fibration over the orbit category.

cite reading material

Formally, as $\mathsf{F} \wr (-)$ and pullbacks preserve such fibrations, and these are compatible under composition, this follows from the natural maps $\underline{\mathfrak{C}}\Omega_G \to \Omega_G \to O_G$. Explicitly, $\underline{\mathfrak{C}}\Omega_G(G/H)$ has as objects those pairs $(T, \mathfrak{c})$ such that $T \simeq G \cdot_H T_*$ for $T_* \in \Omega^H$. Maps $\varphi : (T, \mathfrak{c}) \to (S, \mathfrak{d})$ in each fiber are called *root-fixed*: as maps in Ω_G, they are *rooted* ($Gr_T \to Gr_S$ is a planar isomorphism), and moreover $\mathfrak{c}(Gr_T) = \mathfrak{d}(Gr_S)$.

Given $q : G/H \to G/K$ in the orbit category, the chosen Cartesian maps are the induced root pullback maps $q : q^*T \to T$ on G-trees, with the colouring of q^*T defined as follows: for $b \in E(q^*T)$, minimal in it's G-orbit, we have $q(b) = ga$ for some $g \in G$ and $a \in E(T)$ minimal in its orbit. Moreover, as this g is unique modulo G_a, we have that there is a well-defined map $g_* : G/G_{q(b)} \to G/G_a$, and as q induces a unique map $q_b^* : G/G_b \to G/G_{q(b)}$, we have

$$(q^*\mathfrak{c})([b]) q^* g^* \mathfrak{c}([a]).$$

Alternatively, on $\Phi E(q^*T)$, we have $(q^*\mathfrak{c})(G/H, b) = q_b^*(\mathfrak{c}(G/H, q(b)))$.

Remark 2.5. We note that any *planar* map of coloured G-trees is always *colour-fixed*, in that $\mathfrak{c}(Ge) = \mathfrak{d}(G\varphi(e))$ for all $Ge \in E_G(T)$.

Remark 2.6. A *quotient* map in $\underline{\mathfrak{C}}\Omega_G$ is any morphism such that the underlying map in Ω_G is a quotient.

We have natural inclusions on the left

$$\begin{tikzcd} \underline{\mathfrak{C}}\Sigma \arrow[r] \arrow[d, "\iota"] & \underline{\mathfrak{C}}\Omega \arrow[d] \\ \underline{\mathfrak{C}}\Sigma_G \arrow[r] & \underline{\mathfrak{C}}\Omega_G \end{tikzcd} \qquad \begin{tikzcd} \Sigma \times G \arrow[r] \arrow[d] & \Omega \times G \arrow[d] \\ \Sigma_G \arrow[r] & \Omega_G \end{tikzcd}$$

which forget to the uncoloured inclusions on the right. Specifically, $U \mapsto G \cdot U$ and, as $E_G(G \cdot U) = E(U)$, the associated colouring map is simply $\mathfrak{c}$ again. On morphisms, (φ, g) maps to $(\varphi)_G \circ g$.

2.2 Planar Strings and Stuff

Need to strike a balance between what to show explicitly, and what to just state. §3.4 and §4 from [BP17] extend almost formally, though phrasing it as such...
We still have natural span $\underline{\mathfrak{C}}\Sigma_G \leftarrow \underline{\mathfrak{C}}\Omega_G^0 \rightarrow \mathsf{F}_s \wr \underline{\mathfrak{C}}\Sigma_G$, such that the left arrow is a map of rooted fibrations. This, plus whats already in §3.4 and §4, may be enough to just formally push through.

Generalizing [BP17, Remark 3.78]

otherwise, have to force on the non-equivariant trees the correct isotropy of their colours. If not, we just see $\Phi\mathfrak{C}(G/e)$, and not the whole coefficient system.

Definition 2.7. Given $(T, \mathfrak{c}) \in \underline{\mathfrak{C}}\Omega_G$, a *planar (resp. rooted) T-substitution datum* is a tuple $((U_{v_{Ge}}, \mathfrak{c}_{v_{Ge}}))_{v_{Ge} \in V_G(T)}$ of $\underline{\mathfrak{C}}$-coloured G-trees along with planar (resp. rooted colour-fixed) tall maps $T_{v_{Ge}} \rightarrow U_{v_{Ge}}$.

A map of planar (resp. rooted) T-substitution data $(U_{v_{Ge}}) \rightarrow (V_{v_{Ge}})$ is a compatible tuple of planar (resp. rooted colour-fixed) tall maps $(U_{v_{Ge}} \rightarrow V_{v_{Ge}})$. Let $\mathsf{Sub}_p(T)$ and $\mathsf{Sub}(T)$ denote the categories of planar (resp. rooted) T-substituion datum.

Lemma 2.8 (cf. [BP17, Prop. 3.41])**.** *Let $(T, \mathfrak{c}) \in \underline{\mathfrak{C}}\Omega_G$ be a $\underline{\mathfrak{C}}$-coloured G-tree. There are isomorphisms of categories*

$$\begin{array}{ccc} \mathsf{Sub}_p(T) & \rightleftarrows & (T, \mathfrak{c}) \downarrow \underline{\mathfrak{C}}\Omega_G^{pt} \\ (U_{v_{Ge}}) & \longmapsto & ((T, \mathfrak{c}) \rightarrow \operatorname{colim}_{Sc_G(T)} U_{(-)}). \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}(T) & \rightleftarrows & (T, \mathfrak{c}) \downarrow \underline{\mathfrak{C}}\Omega_G^{r} \\ (U_{v_{Ge}}) & \longmapsto & ((T, \mathfrak{c}) \rightarrow \operatorname{colim}_{Sc_G(T)} U_{(-)}). \end{array} \tag{2.9}$$

SUB_EQUIV_EQ

where $\underline{\mathfrak{C}}\Omega_G^{pt}, \underline{\mathfrak{C}}\Omega_G^{r}$ are the categories of planar tall (resp. rooted) maps under $(T, \mathfrak{c})$.

Proof. This follows as in [BP17, Prop. 3.41], going by induction on $n = |V_G(T)|$. Let $(U_T, \mathfrak{c}_{U_T})$ denote the colimit, if it exists. If n is 0 or 1, T is terminal in $Sc_G(T)$, and the colouring $\mathfrak{c}_{U_T}$ is just $\mathfrak{c}$. Otherwise, we have a decomposition $T = R \amalg_{Ge} S$ with the planar ordering on Ge in R, S, and T the same, $E_G(T) = E_G(R) \amalg_{Ge} E_G(S)$ $\mathfrak{c}_R = \mathfrak{c}|_{E_G(R)}$, $\mathfrak{c}_S = \mathfrak{c}|_{E_G(S)}$, such that the existance of U_T and $\mathfrak{c}_{U_T}$ follow from the existance of the pushout below in $\underline{\mathfrak{C}}\Omega_G^{pt,cf}$.

$$\begin{tikzcd} (\eta_{Ge}, \mathfrak{c}) \arrow[r] \arrow[d] & (U_S, \mathfrak{c}_{U_S}) \arrow[d, dashed] \\ (U_R, \mathfrak{c}_{U_R}) \arrow[r, dashed] & (U_T, \mathfrak{c}_{U_T}) \end{tikzcd}$$

By induction, U_S, U_R, $\mathfrak{c}_{U_S}$, $\mathfrak{c}_{U_R}$ exist (with unique choices such that $(U_{v_{Ge}}, \mathfrak{c}_{U_{v_{Ge}}}) \hookrightarrow (U_R, \mathfrak{c}_{U_R})$ is planar [and colour-fixed]). Forgetting colours, this is an equivariant grafting diagram, and hence the G-tree U_T exists. Moreover, we have $E_G(U_T) = E_G(U_S) \amalg_{Ge} E_G(U_R)$, and so we have a well-defined colouring

$$\mathfrak{c}_{U_T}(Gf) = \begin{cases} \mathfrak{c}_{U_R}(Gf) & Gf \in E_G(R) \\ \mathfrak{c}_{U_S}(Gf) & Gf \in E_G(S) \end{cases}$$

since the overlap Ge is in T, and hence it is dictated that $\mathfrak{c}_{U_T}(Ge) = \mathfrak{c}(Ge)$. □

Lemma 2.10 (cf. [BP17, Lemma 3.63])**.** $\underline{\mathfrak{C}}\Omega_G^0 \rightarrow \mathsf{F}_s \wr \underline{\mathfrak{C}}\Sigma_G$ *sends root pullbacks to pullbacks over* $\mathsf{F}_s \wr O_G$.

Proof. Exactly as in *loc cite*, with the additional note that the colouring of $\psi^* T$ is precisely such that each $(\psi^* T)_{v_{Ge}} \to T_{v_{G\varphi(e)}}$ is a pullback in $\underline{\mathfrak{C}}\Sigma_G$. □

Definition 2.11. The category $\underline{\mathfrak{C}}\Omega^n_G$ of *coloured planar n-strings* is the category whoses objects are strings

$$(T_0, \mathfrak{c}_0) \xrightarrow{\varphi_1} (T_1, \mathfrak{c}_1) \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} (T_n, \mathfrak{c}_n)$$

where $(T_i, \mathfrak{c}_i) \in \underline{\mathfrak{C}}\Omega_G$ and the φ_i are all coloured planar tall maps, while arrows are commutative diagrams of quotient maps.

Remark 2.12. We observe

1. $\underline{\mathfrak{C}}\Omega^\bullet_G \to \underline{\mathfrak{C}}\Sigma_G$ is an augmented simplicial object in categories.
2. $\underline{\mathfrak{C}}\Omega^n_G \to O_G$ is a root fibration.
3. We have a vertex functor $V_G : \underline{\mathfrak{C}}\Omega^{n+1}_G \to \mathsf{F}_s \wr \underline{\mathfrak{C}}\Omega^n_G$ by

$$\big((T_0, \mathfrak{c}_0) \to (T_1, \mathfrak{c}_1) \to \dots \to (T_n, \mathfrak{c}_n)\big) \mapsto \big((T_{1,v_{Ge}}, \mathfrak{c}_1) \to \dots \to (T_{n,v_{Ge}}, \mathfrak{c}_n)\big)_{v_{Ge} \in V_G(T_0)}$$

where we write abusively denote by $T_{i,v_{Ge}}$ the G-tree $(T_{i,\bar{\varphi}_i(f)^{\uparrow} \leq \bar{\varphi}_i(f)})_{f \in Ge}$ and by $\mathfrak{c}_i$ the restriction to any of its sub-G-trees.

Alternatively, regarding the source above as a string of $n-1$ arrows in $(T_0, \mathfrak{c}_0) \downarrow \underline{\mathfrak{C}}\Omega^{pt}_G$, the image under V_G can be recognized as the inverse image under (2.9).

Proposition 2.13 (cf. [BP17, Prop 3.82])**.** *For any $n \geq 0$, the commutative diagram*

$$\begin{tikzcd}
\underline{\mathfrak{C}}\Omega^n_G \arrow[r, "V_G"] \arrow[d, "d_{1,\dots,n}"'] & \mathsf{F}_s \wr \underline{\mathfrak{C}}\Omega^{n-1}_G \arrow[d, "\mathsf{F} \wr d_{0,\dots,n-1}"] \\
\underline{\mathfrak{C}}\Omega^0_G \arrow[r, "V_G"'] & \mathsf{F}_s \wr \underline{\mathfrak{C}}\Sigma_G
\end{tikzcd}$$

is a pullback diagram in Cat.

Proof. □

Proposition 2.14 (cf. [BP17, Lemma 4.28])**.** *$N_{\mathfrak{C}}$ on spans preserves right Kan extensions over* $\mathsf{F} \wr \mathcal{A} \downarrow \mathsf{F} \wr \underline{\mathfrak{C}}\Sigma_G$.

Proof. Stuff □

Similarly, [BP17, Prop 3.47, 3.90, 4.12, 4.15, 4.26, 4.30] naturally generalized to the coloured-setting, replacing all instances of Ω^n_G or Σ_G with $\underline{\mathfrak{C}}\Omega^n_G$ and $\underline{\mathfrak{C}}\Sigma_G$. In particular, this yields the following definitions and proposition.

Definition 2.15 (cf. [BP17, Defn 4.3])**.** Let $\mathsf{WSpan}^l(\mathcal{C}, \mathcal{D})$ (resp. $\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})$) denote the category of *left (resp. right) weak spans*, with objects

$$\mathcal{C} \xleftarrow{k} \mathcal{A} \xrightarrow{X} \mathcal{D}$$

and arrows those diagrams as on the left (resp. right) below

$$\begin{tikzcd} & \mathcal{A}_1 \arrow[dl, "k_1"'] \arrow[dr, "X_1"] \arrow[dd, "i"'] & \\ \mathcal{C} & & \mathcal{D} \arrow[dl, Rightarrow, shorten=4mm] \\ & \mathcal{A}_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] & \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{A}_1 \arrow[dl, "k_1"'] \arrow[dr, "X_1"] \arrow[dd, "i"'] & \\ \mathcal{C} & & \mathcal{D} \arrow[dl, Rightarrow, shorten=4mm] \\ & \mathcal{A}_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] & \end{tikzcd}$$

denoted by $(i, \varphi) : (k_1, X_1) \to (k_2, X_2)$, with composition defined in the natural way.

recall adjunctions with Lan and Ran, canonical op-isos, etc

Definition 2.16 (cf. [BP17, Defn 4.16]). Suppose $\mathcal{V}$ is a symmetric monoidal category with diagonals . We define an endofunctor $N_{\underline{\mathfrak{C}}}$ on $\mathsf{WSpan}^r(\underline{\mathfrak{C}}\Sigma_G, \mathcal{V}^{op})$ by letting $N_{\underline{\mathfrak{C}}}(\underline{\mathfrak{C}}\Sigma_G \leftarrow \mathcal{A} \to \mathcal{V}^{op})$ be given by the span

recall

$$\begin{tikzcd} \underline{\mathfrak{C}}\Omega^0_G \wr \mathcal{A} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \mathcal{A} \arrow[r] \arrow[d] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] & \mathcal{V}^{op} \\ \underline{\mathfrak{C}}\Omega^0_G \arrow[r, "V_G"'] \arrow[d] & \mathsf{F} \wr \underline{\mathfrak{C}}\Sigma_G & & \\ \underline{\mathfrak{C}}\Sigma_G & & & \end{tikzcd}$$

where the given square is a pullback, and on arrows in the natural way.

Moreover, we have a multiplication $\mu : N_{\underline{\mathfrak{C}}} \circ N_{\underline{\mathfrak{C}}} \Rightarrow N_{\underline{\mathfrak{C}}}$ given by the natural isomorphism

$$\begin{tikzcd} \underline{\mathfrak{C}}\Sigma_G \arrow[d, equal] & \underline{\mathfrak{C}}\Omega^1_G \wr A \arrow[l] \arrow[r, "V_G"] \arrow[d, "d_0"'] & \mathsf{F} \wr \underline{\mathfrak{C}}\Omega^0_G \wr A \arrow[r, "\mathsf{F}\wr V_G"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] & \mathcal{V}^{op} \arrow[d, equal] \\ \underline{\mathfrak{C}}\Sigma_G & \underline{\mathfrak{C}}\Omega^0_G \wr A \arrow[l] \arrow[rr, "V_G"'] \arrow[urr, Leftrightarrow, "\pi_0"'] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\otimes^{op}"'] \arrow[urr, Leftrightarrow, "\alpha"'] & & \mathcal{V}^{op} \end{tikzcd}$$

MULTDEFSPAN EQ

and a unit $\eta : id \Rightarrow N_{\underline{\mathfrak{C}}}$ give by the strictly commuting diagram

$$\begin{tikzcd} \underline{\mathfrak{C}}\Sigma_G \arrow[d, equal] & A \arrow[l] \arrow[r, equal] \arrow[d, "s_{-1}"'] & A \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[d, equal] \\ \underline{\mathfrak{C}}\Sigma_G & \underline{\mathfrak{C}}\Omega^0_G \wr A \arrow[l] \arrow[r, "V_G"'] & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"'] & \mathcal{V}^{op}. \end{tikzcd}$$

UNITSPAN EQ

Proposition 2.17 (cf. [BP17, Prop 4.19]). *$(N_{\underline{\mathfrak{C}}}, \mu, \eta)$ is a monad on* $\mathsf{WSpan}^r(\underline{\mathfrak{C}\mathfrak{C}}\Sigma_G, \mathcal{V}^{op})$.

Definition 2.18. The *genuine $\underline{\mathfrak{C}}$-coloured operad monad* is the monad $\mathbb{F}_{G,\underline{\mathfrak{C}}}$ on $\mathsf{Sym}_{G,\underline{\mathfrak{C}}}(\mathcal{V}) = \mathsf{Fun}(\underline{\mathfrak{C}}\Sigma^{op}_G, \mathcal{V})$ given by

$$\mathbb{F}_{G,\underline{\mathfrak{C}}} = \mathrm{Lan} \circ N_{\underline{\mathfrak{C}}} \circ \iota$$

with multiplication and unit given by

$$\mathsf{Lan} \circ N_{\underline{\mathfrak{C}}} \circ \iota \circ \mathsf{Lan} \circ N_{\underline{\mathfrak{C}}} \circ \iota \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ N_{\underline{\mathfrak{C}}} \circ N_{\underline{\mathfrak{C}}} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\underline{\mathfrak{C}}} \circ \iota$$

$$id \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\underline{\mathfrak{C}}} \circ \iota.$$

We will write $\mathsf{Op}_{G,\underline{\mathfrak{C}}}(\mathcal{V})$ for the category $\mathsf{Alg}_{\mathbb{F}_{G,\underline{\mathfrak{C}}}}(\mathsf{Sym}_{G,\underline{\mathfrak{C}}}(\mathcal{V}))$ of *genuine $\underline{\mathfrak{C}}$-coloured operads.*

2.3 Genuine $\mathfrak{C}$-coloured operads

Come back : Something about profiles.

come back: Combine with above

Remark 2.19. Given $X \in \mathsf{dSet}_G$ with $X(\eta_{G/H}) = \mathfrak{C}(G/H)$, we have that $\underline{\mathfrak{C}}\Sigma_G$ is equal to the category of *profiles* $\partial\Omega[C] \to X$, where C ranges over all of Σ_G.

come back

2.4 Comparison with $\mathfrak{C}$-coloured operads

Given $(T = (T_i)_I, \mathfrak{c}) \in \underline{\mathfrak{C}}\Omega_G$, we define $\mathfrak{c}_i : E(T_i) \to \mathfrak{C}(G/e)$ by

$$\mathfrak{c}_i(e) = g^* q_e^*(\mathfrak{c}[f]),$$

where $e \in Gf$ (with f minimal in the planar structure on T), $g \in G$ minimal such that $ge = f$, $q : G \to r(T)$ the unique quotient map preserving minimal elements, and $q_e : G/G_e \to G/G_{q(e)}$ the induced map.

Then $(T_i, \mathfrak{c}_i) \in \underline{\mathfrak{C}}\Omega$, and moreover $i \mapsto (T_i, \mathfrak{c}_i)$ yields a well-defined functor $B_I G \to \underline{\mathfrak{C}}\Omega$.

Remark 2.20. The colouring $\mathfrak{c}_i$ is *almost* the composite

$$E(T_i) \to E_{G_i}(T_i) \xrightarrow{\simeq} E_G(T) \to \mathfrak{C} \to G \ltimes \mathfrak{C}(G/e)$$

where G_i is the stabilizer in G of T_i, and $E_{G_i}(T_i) \to E_G(T)$ is the canonical isomorphism sending $eG_i \to Gf$ with $f \in Ge$ minimal. However, this composite does not record the "twisting" action by the element g_e.

With that, we have the formula

$$\iota_* Y(T, \mathfrak{c}) = \left(\textstyle\prod_I Y(T_i, \mathfrak{c}_i)\right)^G.$$

Remark 2.21 (cf. [BP17, Rem 4.35])**.** Equivalently, the essential image of ι_* are those sheaves $X \in \mathsf{Sym}_{G,\mathfrak{C}}(\mathcal{V})$ such that the canonical map

$$X(C, \mathfrak{c}) \xrightarrow{\simeq} X(q^*(C, \mathfrak{c}))^\Gamma$$

is an isomorphism, where $q : G \to r(C)$ is the unique map preserving the minimal element, and $\Gamma \leq \mathsf{Aut}(q^*(C, \mathfrak{c}))$ the subgroup preserving the quotient map $q^* C \to C$ under precomposition.

Remark 2.22. Alternatively, $\mathfrak{c}_i$ is the composite

$$E(T_i) \to E(T) \to \mathfrak{C}(G/e).$$

Come BACK

DO STUFF.

3 In dSet_G

Definition 3.1. Define the *genuine operadic nerve* $N : \mathsf{Op}_G \to \mathsf{dSet}_G$ by

$$N\mathcal{P}(T) = \mathrm{Hom}_{\mathsf{Op}_G}(T, \mathcal{P})$$

where we think of T as the operad $T \in \mathsf{Op}^G \hookrightarrow \mathsf{Op}_G$.

Remark 3.2. We note that $N\mathcal{P} \in (SCI)^{\boxslash !}$, as $T \in \mathsf{Op}_G$ is a free $\mathbb{F}_G$-algebra on its vertices.

Remark 3.3. We can rephrase the definition of being an $\mathbb{F}_G$-algebra in terms of $N\mathcal{P}$. For $\mathcal{P} \in \mathsf{Sym}_G$ a G-symmetric sequence, a genuine G-operad structure on $\mathcal{P}$ is given by:

- Composition Maps:
  maps $N\mathcal{P}(T) \to \mathcal{P}(\mathsf{lr}(T))$ for all $T \in \Omega_G$.

- Naturality under restriction and conjugation:
  maps $N\mathcal{P}(T_1) \to N\mathcal{P}(T_0)$ for all quotient maps $T_0 \to T_1$ in $\Omega_{G,0}$, such that the following commutes:

$$\begin{tikzcd} N\mathcal{P}(T_1) \arrow[r] \arrow[d] & \mathcal{P}(\mathsf{lr}(T_1)) \arrow[d] \\ N\mathcal{P}(T_0) \arrow[r] & \mathcal{P}(\mathsf{lr}(T_0)). \end{tikzcd}$$

- Associativity under $\mathbb{F}_G$:
  maps $N\mathcal{P}(T_1) \to N\mathcal{P}(T_0)$ for all planar tall maps $T_0 \to T_1$ in Ω^t_G, such that the analogus diagram (with the right vertical map the identity) commutes.[1]

The above reflects the following result.

Proposition 3.4. Op_G *is equivalent to the subcategory of* $\mathsf{dSet_G}$ *spanned by those* X *such that*

1. $X(H/H) = \{*\}$ *for all* $H \leq G$.
2. $X(T) \cong \otimes_{T_v \in V(T)} X(T_v)$.

Proof. The fact that $N\mathcal{P} \in (SCI_G)^{\boxslash !}$ is immediate, as remarked above.

For the reverse direction, we will follow the construction of the homotopy operad as in [MW09, §6], replacing their use of inner horn inclusions with *orbital* inner G-horn inclusions, to show that any $X \in (OHI)^{\boxslash !}$ is in the image of N; the result will then follow from [BP18, HYPER PROP].

In fact, interpreting all of their pictures are as *orbital* representations of G-trees yields that, for all $C \in \Sigma_G$

- $\sim_{Ge}$ is an equivalence relation on $X(C)$ for all $Ge \in E_G(C)$.
- The relations $\sim_{Ge}$ and $\sim_{Ge'}$ are equal for all $e, e' \in E(C)$.
- $[h] \circ [f] = [h \circ f]$ yields a well-defined composition map.

  come back

need to show naturality, check associativity of composition

□

[1] As in [BP17], we note that "associativity" under $\mathbb{F}_G$ includes both the usual notion of associativity of our composition maps, but also unitality; this is recorded here by the fact that degeneracies are always planar tall.

4 Scratchwork

4.1 Colored simplicial tensors and cotensors

References

BP17 [BP17] Peter Bonventre and Luís Alexandre Pereira. Genuine equivariant operads. arXiv preprint: 1707.02226, 2017.

BP18 [BP18] Peter Bonventre and Luís Alexandre Pereira. Equivariant dendroidal segal spaces and G-∞-operads. arXiv preprint: 1801.02110, 2018.

MW09 [MW09] I. Moerdijk and I. Weiss. On inner Kan complexes in the category of dendroidal sets. *Adv. Math.*, 221(2):343–389, 2009.