Equivariant dendroidal sets and simplicial operads

Peter Bonventre, Luís A. Pereira

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Abstract

bla bla, generalizing CM13a [CM13]. In an appendix, we discuss Reedy categories in the equivariant context.

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1 Overview

The following are the categories currently with model structures and the right adjoints of already established Quillen equivalences.

$$\begin{tikzcd}
\mathsf{PreOp}^G & \\
\mathsf{sdSet}^G \arrow[u, "\gamma_*"] \arrow[r, "(-)_0"'] & \mathsf{dSet}^G
\end{tikzcd}$$

2 Equivariant dendroidal sets

2.1 Preliminaries

Definition 2.1. A map $f : S_0 \to T_0$ in Ω is called a *face map* if it is injective on underlying sets. A face map is *inner* if it is of the form $T_0 \smallsetminus E \hookrightarrow T_0$, where E is a subset of the set of inner edges of T. Fixing such a subset E, let $\mathscr{F}^E_{\text{Inn}}(T_0)$ denote the poset (under inclusion) of all inner face maps $S_0 \hookrightarrow T_0$ such that $E \subseteq T_0 \smallsetminus S_0$.

Definition 2.2. Given $S_0 \in \Omega$, $T \in \Omega_G$, and a map of forests $f : S_0 \to T$, let T_0 denote the component of T containing the image of S_0.

We say f is an *(inner, resp. outer) face map* if $f : S_0 \to T_0$ is an (inner, outer) face map on underlying trees.

Given a subset E of inner edges, let $\mathscr{F}^E_{\text{Inn}}(T)$ denote the subposet of faces such that miss all of E. Moveover, let $\mathscr{F}_{\text{Out}}(T)$ denote the subposet of outer faces.

Definition 2.3. Fix a set X. Let $\mathsf{im} : \mathsf{Set} \downarrow X \to \mathsf{Set}$ denote the *image* functor, sending a map $f : Y \to X$ to the subset $f(Y)$ of X.

Remark 2.4. This naturally extends to any category of presheaves on sets, in particular dSet and dSet_G. These arise naturally when considering horns and boundaries, as we will see that the natural map $\Omega[G \cdot_K U_0] \to \Omega[T]$ for a face U_0 of T is injective iff U_0 is *orbital*; see Lemma 2.12.

Definition 2.5. Fix $T \in \Omega_G$, and a component T_0 of T, with $H := \text{Stab}_G(T_0)$.

Let $\mathscr{F}(T)$ denote the poset (under inclusion) of face maps whose image is strictly containined in T_0. Given a G-closed subset E of inner edges, let $\mathscr{F}^E(T) = \mathscr{F}(T) \smallsetminus (T_0 \smallsetminus \bar{E})_{\bar{E} \subseteq E \cap T_0}$.[1] Define the *E-horn* of T to be the subdendroidal set

$$\Lambda^E[T] := \bigcup_{\mathscr{F}^E(T)/G} \mathsf{im}\left(\Omega[G \cdot_K S_0]\right) = \bigcup_{\mathscr{F}^E(T)/G} \bigcup_{g \in G} \Omega[gS_0]$$

where $K = \text{Stab}_H(S_0)$.

the colimit wasn't functorial as written: e.g. if U_0 is an equivariant subface of S_0, then $\Omega[G \cdot_K U_0]$ does not canonically map into $\Omega[G \cdot_L S_0]$. This can be fixed by making the poset larger (adding full orbits of certain subface subinclusions)

Remark 2.6. Equivalently, if $T \simeq G \cdot_H T_0$, then

$$\Lambda^E[T] \simeq G \cdot_H \Lambda^{E \cap T_0}[T_0] \simeq G \cdot_H \operatorname*{colim}_{\mathscr{F}^{E \cap T_0}(T_0)} \Omega[U_0].$$

Remark 2.7. If E is the entire set of inner edges, we denote $\Lambda^E[T]$ by $\partial^{out}\Omega[T]$, and call it the *outer boundary.*[2]

Definition 2.8. For $T \in \Omega_G$, let $\mathscr{F}_{\text{SC}}(T) \subseteq \text{Out}(T)$ denote the poset of outer faces with precisely one (non-equiviariant) vertex (that is, every element records precisely one generating broad relation $e^{\uparrow} \leq e$ from T). We define the *Segal core* of T to be the subdendroidal set

$$Sc[T] := \bigcup_{\mathscr{F}_{\text{SC}}(T)} \mathsf{im}\left(\Omega[G \cdot_K C_0]\right) = \bigcup_{\mathscr{F}_{\text{SC}}(T)} \bigcup_{g \in G} \Omega[gC_0],$$

where $K = \text{Stab}_H(C_0)$.

---

[1] In the case $E = Ge$ for a single inner edge $e \in T$, we have $\mathscr{F}^{Ge}(T) = \mathscr{F}(T) \smallsetminus (T_0 \smallsetminus \bar{e})_{\bar{e} \subseteq He}$.

[2] in [CM13], this was called the *external* boundary.

Remark 2.9. Explicitly, a map $Sc[T] \to X$ is given by an element in $X(G \cdot_K T_v)$ for each $v \in V(T_0)$, which are compatible on overlapping edges and under the action of H. Equivalently, a map $Sc[T] \to X$ is given by an element in $X(T_{Gv})$ for each $Gv \in V_G(T)$ which are compatible on overlapping edges.

Remark 2.10. We observe that, if $T \simeq G \cdot_H T_0$, then $Sc[T] \simeq G \cdot_H Sc[T_0]$.

Definition 2.11. A face map $f : S_0 \to T$ is called *orbital* if $f(S_0) \subseteq T$ is K-closed, where $K = \mathrm{Stab}_G(f(r_s))$, for r_S the root of S_0.

ORB_INJ_THM

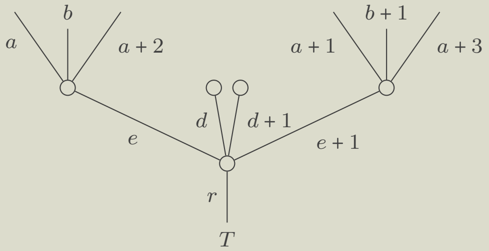
Lemma 2.12. *Let U_0 be a face of T with isotropy K. Then $\Omega[G \cdot_K U_0]$ is a subdendroidal set of $\Omega[T]$ iff U_0 is orbital.*

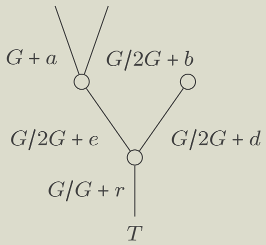
Proof. U_0 is orbital iff the (non-equivariant) subdendroidal sets $\Omega[gU_0]$ of $\Omega[T]$ are either disjoint or equal, and $\Omega[U_0] = \Omega[gU_0]$ iff $g \in K$. If U_0 is not orbital, then there exist $g \in G \setminus K$ such that $R_0 := U_0 \cap gU_0$ is a proper, non-empty subface of U_0, so $R_0 \in \Omega[U_0]$ and $\Omega[gU_0]$. Thus $R_0 \in \Omega[T]$ is hit at least twice. ☐

Definition 2.13. Fixing a component T_0 of T, let $\mathscr{F}_o(T)$ denote the poset (under inclusion) of orbital face maps whose image is strictly contained in T_0. Given an inner edge $e \in T$, let $\mathscr{F}_o^{Ge}(T) := \mathscr{F}_o(T) \setminus (T_0 \setminus \bar{e})_{\bar{e} \subseteq He}$ where $H = \mathrm{Stab}_G(T_0)$, and define the *Ge-orbital horn* to be the subdendroidal set

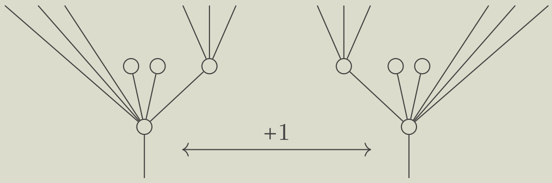
$$\Lambda_o^{Ge}[T] := \operatorname*{colim}_{\mathscr{F}_o^{Ge}(T)/G} \Omega[G \cdot_K U_0] = \bigcup_{\mathscr{F}_o^{Ge}(T)/G} \bigcup_{g \in G} \Omega[gU_0].$$

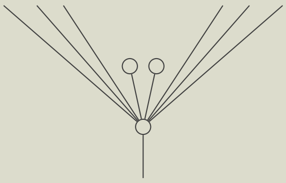
Example 2.14. To set up an example comparing the orbital horns with regular inner horns, let $G = C_4$ be the cyclic group with four elements, and consider the tree $T \in \Omega^G \subseteq \Omega_G$ below.



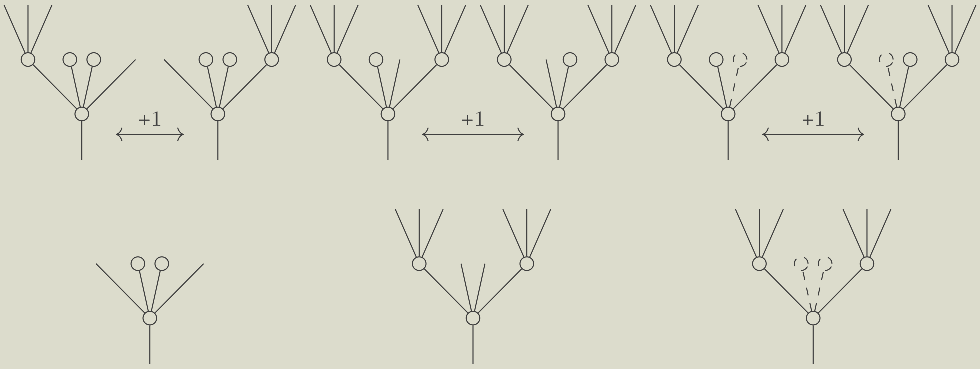


Consider the edge orbit Ge. The following are all proper faces of T which are *not* in the inner horn $\Lambda^{Ge}[T]$:

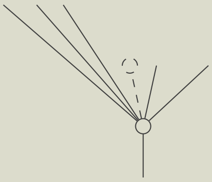




where the tree on the right is a face of both trees in the orbit on the left. Similarly, the maximal elements of $\Lambda^{Ge}[T]$ are displayed in the top row below, while the second row displays the maximal elements of $\Lambda_o^{Ge}[T]$; we observe that, in this particular case (where each orbit of vertices has at most two elements), these are all maximal subfaces of elements in the top row. For clarity, we have drawn using dashed lines the inner edges collapsed involving stumps.



However, these orbital faces are certainly *not* the only subfaces of the top row above. In particular, the following is a minimal face in $\Lambda^{Ge}[T]$ not in the orbital horn $\Lambda^{Ge}_o[T]$.



2.2 Proof of Proposition 2.35

need to think about hypersaturation for non-monomorphisms

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts. Moreover, following the discussion preceding [HHM16, Prop. 3.6.8], we will call a class of maps of dSet^G *hypersaturated* if it is further closed under the following cancellation property: if in

$$A \xrightarrow{f} B \xrightarrow{g} C$$

both f and gf are in the class, then so is g.

Notation 2.15. Given a class of morphisms $\mathcal{C}$ in $\mathcal{V}$, let $W(\mathcal{C})$ and $\hat{W}(\mathcal{C})$ denote the saturation and hypersaturation of $\mathcal{C}$.

Remark 2.16. If L is a left adjoint, then $L(\hat{W}(\mathcal{C})) \subseteq \hat{W}(L(\mathcal{C}))$. In particular, this applies to the "free G-object" functor $G \cdot (-)$.

The following lemma identifies some of the utility of hypersaturations.

Notation 2.17. Given a class of maps $\mathcal{C}$ in $\mathcal{V}$, let $\mathcal{C}^{\boxslash}$ (respectively $\mathcal{C}^{\boxslash !}$) denote the class of maps with the *(strict) right lifting property* (abreviated (S)RLP) against $\mathcal{C}$.

HYPER_LP_THM

Lemma 2.18. *Let $\mathcal{C}$ be a class of maps in $\mathcal{V}$. Then $W(\mathcal{C}) = {}^{\boxslash}(\mathcal{C}^{\boxslash})$, $\mathcal{C}^{\boxslash} = W(\mathcal{C})^{\boxslash}$ and $\mathcal{C}^{\boxslash !} = \hat{W}(\mathcal{C})^{\boxslash !}$.*

Proof. Is a straightforward exercise to show that if $X \to Y$ has the (S)RLP with respect to a map $A \to B$ (resp. maps $A_\alpha \to A_{\alpha+1}$), then $X \to Y$ has the (S)RLP with respect to any pushout or retract of $A \to B$ (resp. the transfinite composite).

- For the pushout, the map $C \to X$ always provides one component of the new lift, while $B \to D \to X$ is always a lift over $A \to B$; hence strictness implies strictness.

- For the retract, again we have that $B \to D \to X$ is a lift over $A \to B$, and hence strictness implies this composite is uniquely determined; composing with the section map yields that $D \to X$ is then also uniquely determined.

- Both the definition of the lift for the composite and its uniqueness are immediate.

Now, suppose we have a diagram as on the left

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d, equal] \\
B \arrow[r] \arrow[d] & X \arrow[d] \\
C \arrow[r] \arrow[uur, dashed, "\exists ! H"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[d] \arrow[urr, dashed, "\exists ! G"] & X \arrow[r] \arrow[d] & Y \arrow[d, equal] \\
C \arrow[r] \arrow[uurr, dashed, bend right, "H"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

such that $X \to Y$ has the SRLP against $A \to B$ and $A \to C$. By assumption, there exist unique lifts H and G as denoted by the dashed arrows. It is immediate that both $B \to X$ and $B \to C \xrightarrow{H} X$ are candidates for lift G, and hence by strictness all three are equal. Thus H is also a lift of the bottom-left square. Since any lift of the bottom square is also a lift of the rectangle, strictness for $A \to B$ implies strictness for $B \to C$. $\square$

Definition 2.19. Given $T \simeq G \cdot_H T_0$, let $U_0 \subseteq T_0$ be a (non-equivariant) subtree, and let $K = \mathrm{Stab}_H(U_0)$. Suppose we have another subdendroidal set $X \subseteq \Omega[T]$ which contains all proper outer faces of U_0, and an inner edge edge $e \in U_0$. We say that $Ke \subseteq U_0$ is a *characteristic edge orbit* of $X \subseteq \Omega[T] \leftarrow \Omega[G \cdot_K U_0]$ if we have, for any $R_0 \in \mathscr{F}_{\mathrm{Inn}}(U_0)$ containing e as an inner edge,

$$R_0 \in \mathscr{F}_{\mathrm{Inn}}(U_0) \cap X \text{ if and only if } R_0/\bar{K}e \in \mathscr{F}_{\mathrm{Inn}}(U_0) \cap X,$$

where $\bar{K} = \mathrm{Stab}_K(R_0)$.

`CHAR_HORN_THM`

Proposition 2.20. *Given T, T_0, H, U_0, K, and X as above. If Ke is a characteristic edge orbit of $X \subseteq \Omega[T] \leftarrow \Omega[G \cdot_K U_0]$ then $X \to X \cup \mathsf{im}\,(\Omega[G \cdot_K U_0])$ is inner G-anodyne.*

Proof. If $\mathsf{im}(\Omega[G \cdot_K U_0]) = \bigcup_G \Omega[gU_0]$ is already contained in X, we are done. Assuming otherwise, it suffices to show that for all $C \subseteq C'$ K-closed convex subsets of $\mathscr{F}^{Ke}_{\mathrm{Inn}}(U_0) \smallsetminus X$, the map $X_C \to X_{C'}$ is inner G-anodyne, where we define

$$X_C \coloneqq X \cup \bigcup_{C/K} \mathsf{im}\,(\Omega[G \cdot_L U_0 \smallsetminus E]) = X \cup \left(G \cdot_K \bigcup_{C/K} \Omega[U_0 \smallsetminus E] \right),$$

with the unions taking place in $\Omega[T]$. Indeed, once $C = \mathscr{F}^{Ke}_{\mathrm{Inn}}(U_0) \smallsetminus X$, we have the pushout

$$\begin{tikzcd}
G \cdot_K \left(\Lambda^{Ke}[U_0] \right) \arrow[r] \arrow[d, hook] & X_C \arrow[d] \\
G \cdot_K \Omega[U_0] \simeq \Omega[G \cdot_K U_0] \arrow[r] & X \cup \mathsf{im}\,(\Omega[G \cdot_K U_0]) .
\end{tikzcd}$$

Moreover, it suffices to consider $C' = C \cup G \cdot_K (U_0 \smallsetminus D)$ for a set of inner edges D, without loss of generality $e \notin D$ and $U_0 \smallsetminus D$ is not in the domain. Let $\bar{K} = \mathrm{Stab}_H(D)$.

We first claim that (the image of) $\Lambda^{Ke}[U_0 \smallsetminus D]$ is in the domain. If S_0 is an outer face of $U_0 \smallsetminus D$, then S_0 factors through an outer face of U_0, and so S_0 is in X. Further, if $S_0 = U_0 \smallsetminus (D \cup E)$ with $E \cap Ke = \varnothing$, then concavity implies that S_0 is in the domain, as required.

Second, we claim that no face $S_0 = U_0 \smallsetminus D \cup \bar{e}$, with $\bar{e} \subseteq Ke$, is in the domain. Suppose $U_0 \smallsetminus D \cup \bar{e}$ is contained in some $U_0 \smallsetminus E$ already attached. Then, since $E \cap Ke = \varnothing$ we have $U_0 \smallsetminus D \subseteq U_0 \smallsetminus E$, so $U_0 \smallsetminus D$ is also in the domain, a contradiction. Further, if $U_0 \smallsetminus D \cup \bar{e}$ is in X, then so is $U_0 \smallsetminus D \cup Ke$, and hence so is $U_0 \smallsetminus D$ (by definition of characteristic edge orbit), also a contradiction.

Now, all faces $U_0 \smallsetminus D \cup \bar{e}$ with $\varnothing \subseteq \bar{e} \subseteq Ke$ have stabilizer $\bar{K}$ (else $D \cap Ke \neq \varnothing$). Thus $X_C \to X_{C'}$ is the pushout of

$$G \cdot_{\bar{K}} \left(\Lambda^{\bar{K}e}[U_0 \smallsetminus D] \to \Omega[U_0 \smallsetminus D] \right),$$

{CHAR_HORN_EQ}

and hence is anodyne, as required. ☐

ORB_HORN_THM

Proposition 2.21. *$\Lambda^{Ge}_o[T] \to \Omega[T]$ is inner G-anodyne, and hence the (hyper)saturated class of the orbital horn inclusions is contained in the (hyper)saturated class of the horn inclusions.*

Proof. Let $\mathrm{Out}^{-o}(T)$ be the poset of outer faces U_0 of T which are not in $\Lambda^{Ge}_0[T]$. It suffices to show that for any G-convex subsets $B \subseteq B'$ of $\mathrm{Out}^{-o}(T)$, the map

$$\left(\Lambda^{Ge}_o[T]\right)_B := \Lambda^{Ge}_o[T] \cup \bigcup_{B/G} \mathsf{im}\,(\Omega[G \cdot_K U_0]) \to \Lambda^{Ge}_o[T] \cup \bigcup_{B'/G} \mathsf{im}\,(\Omega[G \cdot_K U_0]) =: (\Lambda^{Ge}_o[T])_{B'}$$

is inner G-anodyne. Again, suffices to show when $B' = B \cup G \cdot_K \{U_0\}$, where $K = \mathrm{Stab}_G(U_0)$, and without loss of generality $e \in U_0$. For the final reduction, Proposition 2.20 implies it suffices to show Ke is a characteristic edge orbit for $(\Lambda^{Ge}_o[T])_B \subseteq \Omega[T] \leftarrow \Omega[G \cdot_K U_0]$. To that end, let $U_0 \smallsetminus D$ be an inner face with stabilizer $\bar{K}$.

For $B = \varnothing$, we suppose $U_0 \smallsetminus \bar{K}e \in \Lambda^{Ge}_o[T]$. This implies that

$$K_r := \mathrm{Stab}_H(r_U) \leq \mathrm{Stab}_H(U_0 \smallsetminus D \cup \bar{K}e) \leq \mathrm{Stab}_H(U_0) \leq Kr,$$

so the outer face U_0 is K_r-closed and not in $\Lambda^{Ge}_0[T]$, and hence we conclude that U_0 must be T_0. However, if $\Lambda^{Ge}_0[T] \neq \Lambda^{Ge}[T]$, T_0 is not a minimal outer face not in $\Lambda^{Ge}_0[T]$. Thus, in these cases, Ke is a characteristic edge orbit vacuously. If in fact $\Lambda^{Ge}_o[T] = \Lambda^{Ge}[T]$, then $T \simeq G \cdot T_0$, so $H = \bar{K} = K_r = \{e\}$. Now, since $U_0 \smallsetminus D \cup e \in \Lambda^{Ge}_0[T]$, $D \smallsetminus e \neq \varnothing$, so $U_0 \smallsetminus D \in \Lambda^{Ge}_0[T]$.

For general B, we know the domain contains all outer faces of U_0 by convexity. Now, let $U_0 \smallsetminus D$ be an inner face, with stabilizer $\bar{K}$. Then $U_0 \smallsetminus D \cup \bar{K}e$ is in the domain if either $U_0 \smallsetminus D \cup \bar{K}e$ is contained in some $\Omega[G \cdot_L R_0]$, or $U_0 \smallsetminus D \cup \bar{K}e \in \Lambda^{Ge}_o[T]$. But then U_0 is contained in R_0 since both are outer faces, or again we apply the previous lemma. Both lead to contradictions, and thus the result follows. ☐

The reverse inclusion is only true on the level of hypersaturations.

Definition 2.22. Let $\hat{W}_o$ be the hypersaturation of the orbital horn inclusions.

HORN_ORB_THM

Proposition 2.23. *For any $T \in \Omega_G$, the map $\Lambda^{Ge}[T] \to \Omega[T]$ is in $\hat{W}_o$.*

Proof. We go by induction on $|G| \times |\mathscr{F}_{\mathrm{Inn}}(T)|$ When $G = \{e\}$, the orbital horns are regular inner horns, and so the result holds.

replace
$Inn(T)$
with
$Inn_G(T)$?
$E(T)$?
$V(T)$?
$V_G(T)$?

Now, the proof of Proposition 2.20 says that, for any face $U_0 \subseteq T_0$, if Ke is a characteristic edge orbit for $X \subseteq \Omega[T] \leftarrow \Omega[G \cdot_K U_0]$, then

$$X \to X \cup \mathsf{im}\,(\Omega[G \cdot_K U_0])$$

{CHAR_ATTACH_EQ}

is cellular on maps of the form in CHAR_HORN_EQ (2.2), where D is a set of inner edges. Thus, by induction, any such maps CHAR_ATTACH_EQ (2.2) are in W_o. Thus, the proof of Proposition 2.21 implies, in particular, that

$$\Lambda_o^{Ge}[T] \to \Lambda_o^{Ge}[T] \cup \partial^{out}(T) =: A$$

is in $\hat{W}_o$. It then suffices to show that, for any G-convex subposets $B \subseteq B' \subseteq \mathscr{F}^{Ge,-o}_{\mathrm{Inn}}(T)$,

$$A_B := A \cup \bigcup_{B/G} \mathsf{im}\,(\Omega[G \cdot_K U_0]) \to A \cup \bigcup_{B'/G} \mathsf{im}\,(\Omega[G \cdot_K U_0]) =: A'_B$$

{ORB_HORN_ATTACH

is in $\hat{W}_o$, where $\mathscr{F}^{Ge,-o}_{\mathrm{Inn}}(T)$ is the poset of inner faces of T which contain He and are not in $\Lambda_o^{Ge}[T]$. Indeed, for $B = \mathscr{F}^{Ge,-o}_{\mathrm{Inn}}(T)$, $A_B = \Lambda^{Ge}[T]$, so the result follows from the cancelation property in the definition of hypersaturation.

We may assume that $B' = B \cup (G \cdot_K \{U_0\})$, for $U_0 = T_0 \smallsetminus D$. Then, it suffices to show that Ke is a characteristic edge orbit for $A_B \subseteq \Omega[T] \leftarrow \Omega[G \cdot_K U_0]$. Since A_B contains all outer faces of T (and hence all outer faces of U_0), let $R_0 = T_0 \smallsetminus D \cup E$ be an inner face of U_0 with isotropy $\bar{K}$, and suppose $R_0 \smallsetminus \bar{K}e$ is in A_B. We see $R_0 \smallsetminus \bar{K}e \in \Lambda_o^{Ge}[T]$ iff $D \cup E$ contains a full H-orbit Ht of edges, in which case

$$R_0 \smallsetminus \bar{K}e = T_0 \smallsetminus D \cup E \cup Ke \hookrightarrow T_0 \smallsetminus Ht \in \Lambda_o^{Ge}[T].$$

But then we also have $R_0 = T_0 \smallsetminus D \cup E \hookrightarrow T_0 \smallsetminus Ht$, as required. Moreover, no inner face factors through an outer face of T. Finally, if $R_0 \smallsetminus \bar{K}e$ is in some $S_0 = T_0 \smallsetminus D'$ already attached, then $D \cup E \cup \bar{K}e \supseteq D'$; but since $D' \cap Ke = \varnothing$, this implies $D \cup E \supseteq D'$, and hence R_0 also factors through S_0, as required. Thus Ke is a characteristic edge orbit, and the result is proven. ☐

We will now compare Segal core inclusions with the inner horn inclusions.

SC_IN_GHORN_THM

Proposition 2.24. *For all $T \in \Omega_G$, $Sc[T] \to \Omega[T]$ is inner G-anodyne.*

Proof. Let $\mathsf{Out}^{>1}(T)$ denote the subposet $\mathsf{Out}(T) \smallsetminus \mathscr{F}_{\mathrm{SC}}(T)$. It suffices to show that for all G-convex $B \subseteq B' \subseteq \mathsf{Out}^{>1}(T)$, the map

$$Sc[T] \cup \bigcup_{B/G} \mathsf{im}(\Omega[G \cdot_K U_0]) \to Sc[T] \cup \bigcup_{B'/G} \mathsf{im}(\Omega[G \cdot_K U_0])$$

is inner G-anodyne. In particular, we may assume $B' = B \cup G \cdot_K U_0$. Then the above map is the pushout over

$$G \cdot_K \left(\partial^{out}\Omega[U_0] \to \Omega[U_0]\right),$$

which by Per17 [Per17, Proposition 6.17] is inner G-anodyne, and thus the result follows. ☐

SC_IN_OHORN_THM

Proposition 2.25. *For all $T \in \Omega_G$, $Sc[T] \to \Omega[T]$ is in W_o.*

Proof. Define $\mathscr{F}'_o(T)$ denote the poset of orbital non-vertex-inclusion outer faces $\mathscr{F}^{Ge}_o(T) \smallsetminus \mathscr{F}_{SC}(T)$. It suffices to show that for all G-convex $B \subseteq B' \subseteq \mathscr{F}'_o(T)$, the map

$$Sc[T] \cup \bigcup_{B/G} \mathsf{im}\,(\Omega[G \cdot_K U_0]) \to \bigcup_{B'/G} \mathsf{im}\,(\Omega[G \cdot_K U_0])$$

is in W_o; in particular, we may assume $B' = B \cup G \cdot_K U_0$.

Now, since B is convex, every orbital face of U_0 is in the domain. Continuing, any face in $\Omega[G \cdot_K U_0] \cap Sc[T]$ is in $Sc[G \cdot_K U_0] \subseteq \partial_o^{out}[G \cdot_K U_0]$. Moreover, any face R_0 in $\Omega[G \cdot_K U_0] \cap \Omega[G \cdot_L V_0]$ for V_0, another orbital face, is a face of their intersection (where we may assume R_0 injects into both U_0 and V_0), which is also orbital. Thus, the above map is the pushout over

$$\partial_o^{out}[G \cdot_K U_0] \to \Omega[G \cdot_K U_0].$$

☐

The above combined with Proposition 2.21 immediately implies the following.

Corollary 2.26. *Segal core inclusions are inner G-anodyne.*

Definition 2.27. Let $\hat{W}_{SC} = \hat{W}(SCI_G)$ denote the hypersaturation of the Segal core inclusions.

GHORN_IN_SC_THM

Lemma 2.28. *The inner G-horn inclusions are contained in $\hat{W}_{SC}$.*

Proof. It suffices to show that $\Lambda^{Ge}[T] \to \Omega[T]$ is in $\hat{W}_{SC}$ for any $T \in \Omega_G$ and inner edge e of T.

Suppose we have a decomposition $T \simeq G \cdot_H T_0$, where $H = \mathrm{Stab}(T_0)$. We go by induction on $|H| \times |V_H(T_0)|$.

If $H = e$, then, since $G \cdot SCI \subseteq SCI_G$, and by CM13a [CM13, 2.5] we know $IHI \subseteq \hat{W}(SCI)$, we conclude that

$$G \cdot IHI \subseteq G \cdot (\hat{W}(SCI)) \subseteq \hat{W}(G \cdot SCI) \subseteq \hat{W}(SCI_G) = \hat{W}_{SC}.$$

If $H \neq e$, and $V_H(T_0) = 2$, then there is only one inner edge orbit, so the Segal core is an inner horn. Thus, we may assume that $V_H(T_0) > 2$.

By Lemma 2.28, we see that

$$Sc[T] \to \partial^{out}\Omega[T]$$

{SC_POUT_EQ}

is cellular on maps of the form $\partial^{out}[G \cdot_K U_0] \to \Omega[G \cdot_K U_0]$ for outer faces U_0 of T, which are themsevles inner G-anodyne on inner horn inclusions in trees either with strictly smaller isotropy (e.g. $K \leq H$) or with a strictly smaller number of G-vertices. Thus, by induction, SC_POUT_EQ (2.2) is in $\hat{W}_{SC}$.

Now consider

$$\partial^{out}\Omega[T] \to \Lambda^{Ge}[T].$$

{POUT_GHORN_EQ}

By Per17 [Per17, Proposition 6.17], this is built by pushouts over maps of the form $\partial^{out}\Omega[G \cdot_K U_0] \to \Omega[G \cdot_K U_0]$ for $U_0 \in \mathscr{F}^{He}_{\mathsf{Inn}}[T_0]$, and so again by induction on $|H| \times |V_H(T_0)|$, these maps are all in $\hat{W}_{SC}$.

Combining the above, we have that $Sc[T] \to \Lambda^{Ge}[T]$ is in $\hat{W}_{SC}$, and hence by the closure property of hypersaturations, the inner horn inclusion $\Lambda^{Ge}[T] \to \Omega[T]$ is in $\hat{W}_{SC}$, as desired. □

The following is an equivariant generalization of CM13a [CM13, Props. 2.4 and 2.5], summarizing the results (in particular, Proposition 2.21, Proposition 2.23 Proposition 2.24, Proposition 2.25, and Lemma 2.28) of this subsection.

Proposition 2.29. *The following sets of maps generate the same hypersaturated class:*

- *the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the orbital G-inner horn inclusions $\Lambda^{Ge}_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the G-segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.*

Moreover, one also has the following:

(a) *orbital G-inner horn inclusions are in the saturation of G-inner horn inclusions;*

(b) *G-segal core inclusions are in the saturation of both orbital G-inner horn and G-inner horn inclusions.*

As a corollary of the above, we can fully characterize the image of the nerve functor $N : \mathsf{Op}^G \to \mathsf{dSet}^G$.

Definition 2.30. *$X \in \mathsf{dSet}^G$ is called a (strict) inner G-Kan complex* if X has the strict right lifting property against inner G-horn inclusions (that is, $(X \to *) \in (IHI_G)^{\boxslash(!)}$). Let $\mathsf{(S)Kan}_G \subseteq \mathsf{dSet}^G$ denote the full subcategory.

Similarly, let $\mathsf{(S)Kan}$ denote the category of (strict) inner Kan complexes in dSet (that is, $(X \to *) \in (IHI)^{\boxslash(!)}$). We will denote their image under $\iota_!$ by $(S)Kan^G = G \cdot (S)Kan$.

make sure we say $(\iota_!)\Lambda^E[T_0] = \Lambda^{GE}[G \cdot T_0]$ somewhere

Remark 2.31. If IHI denotes the set of inner horn inclusions in dSet, with image $\iota_! IHI = G{\cdot}IHI$ in dSet^G, then it is immediate that $(G \cdot IHI)^{\boxslash} = Kan^G$ and $(G \cdot IHI)^{\boxslash !} = SKan^G$.

We will in fact show that $(G \cdot IHI)^{\boxslash !} = (IHI_G)^{\boxslash !}$. First, a lemma.

Lemma 2.32. *Let $GIHI$ denote the class of generalized inner horn inclusions $\Lambda^E[T_0] \to \Omega[T]$. If $X \in (G\ cdotIHI)^{\boxslash !}$, then $X \in (G \cdot GIHI)^{\boxslash !}$.*

Proof. We go by induction on $|V(T_0)| \times |E|$, where the base case is immediate. Now, given $e \in E$, the square below is a pushout.

$$\begin{tikzcd}
\iota_!\Lambda^{E\setminus e}[T_0/e] \arrow[r] \arrow[d] & \iota_!\Lambda^{E}[T_0] \arrow[r, "f"] \arrow[d, Rightarrow] & X \\
\iota_!\Omega[T_0/e] \arrow[r] \arrow[urr, dotted, "\exists! F"] & \iota_!\Lambda^{E\setminus e}[T_0] \arrow[ur, dotted, "\exists! F'"'] &
\end{tikzcd}$$

Moreover, by induction, any map f induces a unique lift F. The universal property of the pushout then implies that we have a unique lift F'.

Now, we have the following diagram

$$\begin{tikzcd}
\iota_!\Lambda^{E}[T_0] \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d, equal] \\
(i_G)_!\Lambda^{E\setminus e}[T_0] \arrow[r, dotted, "\exists! F'"] \arrow[d, "j"'] & X \\
\iota_!\Omega[T_0] \arrow[uur, dotted, "\exists \psi"] \arrow[ur, dotted, "\exists! \phi"'] &
\end{tikzcd}$$

where the dotted arrows are lifts: F' is the unique lift of f against i, ϕ is the unique lift of F' against j, and ψ is some lift of f over ji. This implies that ψj is a lift of f over i, so by uniquenss $\psi j = F'$, but then ψ is a lift of F' over j, and hence $\psi = \phi$, Thus the lift of f over ji is uniquely determined. ☐

`SRLP_IHI_THM` **Proposition 2.33.** $(IHI_G)^{\boxslash !} = (G \cdot IHI)^{\boxslash !}$. *Thus* $\mathsf{Kan}_G = \mathsf{Kan}^G$

Proof. Any inner G-horn inclusion $i : \Lambda^{Ge}[T] \to \Omega[T]$ has a decomposition description as

$$j/N : (\iota_!\Lambda^{He}[T_0])/N \to (\iota_!\Omega[T_0])/N,$$

where $e \in T_0$, $T \simeq G \cdot_H T_0 \simeq (G \cdot T_0)/N$, N a graph subgroup of $G \times \mathsf{Aut}(T_0)$, and $j \in GIHI$. Now, for any span $\Omega[T] \xleftarrow{j/N} \Lambda^{Ge}[T] \xrightarrow{f}$, consider the following diagram, where $n \in N$, and the dashed arrows are lifts.

$$\begin{tikzcd}
\iota_!\Lambda^{He}[T_0] \arrow[rr] \arrow[dr, "n"] \arrow[dd, "j"'] & & \iota\Lambda^{H.e}[T_0]/N \simeq \Lambda^{Ge}[T] \arrow[r, "f"] \arrow[dd, "/N=i"] & X \\
& \iota_!\Lambda^{H.e}[T_0] \arrow[ur, two heads] \arrow[dd] & & \\
\iota_!\Omega[T_0] \arrow[rr] \arrow[dr, "n"] \arrow[urrr, dotted, bend left=15, "\exists!\phi"] & & \iota_!\Omega[T_0]/N \simeq \Omega[T] & \\
& \iota_!\Omega[T_0] \arrow[ur, two heads] \arrow[uuurr, dotted, bend right=40, "\exists!\phi"'] & &
\end{tikzcd}$$

Uniqueness implies that $\phi = n \cdot \phi$ for all $n \in N$, and hence factors through $\Omega[T]$, finishing the proof. $\square$

Corollary 2.34. *Fix $X \in \mathsf{dSet}^G$. Then X is a strict inner G-Kan complex iff $X \simeq N^G(\mathcal{O})$ for some $\mathcal{O} \in \mathsf{Op}^G$.*

Proof. $N^G(\mathcal{O})(T)$ is given by maps $\mathsf{Op}^G(\Omega(T), \mathcal{O})$, and since $\Omega(T)$ is a free coloured operad on it's vertices, we may conclude that

$$N^G(\mathcal{O})(T) \simeq \mathsf{Op}^G(\Omega(T), \mathcal{O}) \simeq \mathsf{dSet}^G(Sc[T], N^G\mathcal{O}).$$

Thus $N^G(\mathcal{O})$ is in $(SCI_G)^{\boxslash !}$, and hence is in $(IHI_G)^{\boxslash !}$ by Lemmas 2.18 and 2.28.

Conversely, by Proposition 2.33 and [MW09, Theorem 6.1, Proposition 6.10], we have that $X \simeq N_d \circ \tau_d \circ X$ (where we consider X as a functor $G \to \mathsf{Kan}$). $\square$

2.3 Weak Equivalences in dSet^G

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts. Moreover, following the discussion preceding [HHM16, Prop. 3.6.8], we will call a class of maps of dSet^G *hypersaturated* if it is further closed under the following cancellation property: if in

$$A \xrightarrow{f} B \xrightarrow{g} C$$

both f and gf are in the class, then so is g.

The following is an equivariant generalization of [CM13, Props. 2.4 and 2.5].

HYPER PROP **Proposition 2.35.** *The following sets of maps generate the same hypersaturated class:*

- *the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the orbital G-inner horn inclusions $\Lambda^{Ge}_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the G-segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.*

Moreover, one also has the following:

(a) orbital G-inner horn inclusions are in the saturation of G-inner horn inclusions;

(b) G-segal core inclusions are in the saturation of both orbital G-inner horn and G-inner horn inclusions.

Remark 2.36. Setting $G = e$ and slicing over the stick tree η in the previous result one recovers the well known claim that the hypersaturation of the simplicial inner horns $\{\Lambda^i[n] \to \Delta[n] : 0 < i < n\}$ coincides with the hypersaturation of the simplicial Segal core inclusions $\{Sc[n] \to \Delta[n] : n \geq 2\}$.

HYPERSATKAN REM

Remark 2.37. We will also make use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns $\{\Lambda^i[n] \to \Delta[n] : 0 \leq i \leq n\}$ coincides with the hypersaturation of all vertex inclusion maps $\{\Delta[0] \to \Delta[n]\}$. Indeed, call the latter hypersaturation S. An easy argument shows that the Segal core inclusions $\{Sc[n] \to \Delta[n]\}$ are in S and thus so are all inner horn inclusions. On the other hand, the skeletal filtration of the left horns $\Lambda^0[n]$ is built exclusively out of left horn inclusions, and thus since $\Delta[0] = \Lambda^0[1] \to \Delta[1]$ is in S so are all left horn inclusions $\Lambda^0[n] \to \Delta[n]$. The case of right horn inclusions $\Lambda^n[n] \to \Delta[n]$ is dual.

Notation 2.38. Given subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$, $1 \leq i \leq k$ we write $C_{\amalg_i H_0/H_i}$ for the G-corolla encoding the H_0-set $H_0/H_1 \amalg \cdots \amalg H_0/H_k$.

The following is the equivariant generalization of CM13a [CM13, Thm. 3.5].

TFAE PROP

Proposition 2.39. *Let $X \to Y$ be a map between G-∞-operads. The following are equivalent:*

(a) for all G-corollas C_A and $H \leq G$ the maps

$$k(\Omega[C_A], X) \to k(\Omega[C_A], Y), \qquad k(\Omega[G/H \cdot \eta], X) \to k(\Omega[G/H \cdot \eta], Y)$$

are weak equivalences in sSet*;*

(b) for all G-trees T the maps

$$k(\Omega[T], X) \to k(\Omega[T], Y)$$

are weak equivalences in sSet*;*

(c) for all normal G-dendroidal sets A, the maps

$$k(A, X) \to k(A, Y)$$

are weak equivalences in sSet*;*

(d) $f : X \to Y$ is a weak equivalence in dSet^G.

PROF DEF

Definition 2.40. Let X be a G-∞-operad. A *G-profile* on X is a map

$$\partial\Omega[C] \to X$$

for some G-corolla $C \in \Sigma_G$.

More explicitly, a G-profile consists of:

- subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$ for $1 \leq i \leq k$;
- objects $x_i \in X(\eta)^{H_i}$ for $0 \leq i \leq k$.

To simplify notation, we will prefer to denote a G-profile as $(x_1, \cdots, x_k; x_0)$, and refer to it as a *C-profile*.

MAPSPACE DEF

Definition 2.41. Given a G-∞-operad and a C-profile $(x_1, \cdots, x_k; x_0)$ we define the space of maps $X(x_1, \cdots, x_k; x_0)$ to be given by the pullback

$$\begin{tikzcd}
X(x_1, \cdots, x_k; x_0) \arrow[r] \arrow[d] & Hom(\Omega[C], X) \arrow[d] \\
G \cdot \eta \arrow[r, "(x_1, \cdots, x_k; x_0)"'] & \prod_{0 \le i \le k} X^{H_i}
\end{tikzcd}$$

Noting that there are equivalences of categories (the first of which is an isomorphism)

$$(\mathsf{dSet}_G)/G \cdot \eta \simeq \mathsf{sSet}^{B_G} \simeq \mathsf{sSet},$$

one sees that $X(x_1, \cdots, x_k; x_0)$ can indeed be regarded as a simplicial set (in fact, this is a Kan complex).

Definition 2.42. Let $f : X \to Y$ be a map of G-∞-operads.

The map f is called *fully faithful* if, for each C-profile $(x_1, \cdots, x_k; x_0)$ one has that

$$X(x_1, \cdots, x_k; x_0) \to Y(f(x_1), \cdots, f(x_k); f(x_0))$$

is weak equivalence in sSet.

The map f is called *essentially surjective* if for each subgroup $H \le G$ the map of categories $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ are essentially surjective.

The following is the equivariant generalization of [CM13, Thm. 3.11 and Remark 3.12].

COMSQ THM

Theorem 2.43. *A map $f : X \to Y$ of G-∞-operads is fully faithful iff for all G-corollas $C \in \Sigma_G$ the commutative squares of Kan complexes*

$$\begin{tikzcd}
k(\Omega[C], X) \arrow[r] \arrow[d, "p"'] & k(\Omega[C], Y) \arrow[d, "q"] \\
k(\partial\Omega[C], X) \arrow[r, "f_*"'] & k(\partial\Omega[C], Y)
\end{tikzcd}$$

{COMSQ EQ}

are homotopy pullback squares.

Hence, f is a weak equivalence in dSet^G *iff f is both fully faithful and essentially surjective.*

Proof. Noting that the 0-simplices of $k(\partial\Omega[C], X)$ are precisely the C-profiles $(x_1, \cdots, x_k, x_0)$, fully faithfulness can be reinterpreted as saying that all fiber maps $p^{-1}(x_1, \cdots, x_k, x_0) \to q^{-1}(f(x_1), \cdots, f(x_k), f(x_0))$ are weak equivalences. But since p, q are Kan fibrations, this is equivalent to the condition that (2.43) is a homotopy pullback (see [CM13, Lemma 3.9]), and the first half follows.

For the second half, note first that the bottom map in (2.43) can be rewritten as

$$\prod_{0 \le i \le k} k(G/H_i \cdot \eta, X) \to \prod_{0 \le i \le k} k(G/H_i \cdot \eta, Y).$$

Assume first that f is a weak equivalence. Proposition 2.39 then implies that the horizontal maps in (2.43) are weak equivalences, so that the square is a pull back square, and thus f is fully faithful. That f is essentially surjective follows from the identity $k(G/H \cdot \eta, X) = k(\iota^*(Z^H))$, so that $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ is essentially surjective at the level of maximal groupoids, and this suffices for essential surjectivity.

Assume now that f is fully faithful and essentially surjective. Since (2.43), Proposition 2.39 now implies that one needs only check that the maps of Kan complexes

$$k\left(G/H\cdot\eta,X\right)\to k\left(G/H\cdot\eta,Y\right)\qquad\text{or}\qquad k\left(\iota^*\left(X^H\right)\right)\to k\left(\iota^*\left(Y^H\right)\right)$$

{KANMAP EQ}

are weak equivalences. As before, essential surjectivity is equivalent to the fact that the maps (2.3) induce surjections on connected components. Hence, it now suffices to show that for each 0-simplex $x\in X^H$ the top map of loop spaces in

$$\begin{tikzcd}
\Omega(k(\iota^*X^H),x) \arrow[r] \arrow[d] & \Omega(k(\iota^*Y^H),f(x)) \arrow[d] \\
X(x;x) \arrow[r] & Y(f(x);f(x))
\end{tikzcd}$$

{OMEGASQ EQ}

is a weak equivalence. Noting that the bottom map in (2.3) is a weak equivalence since F is fully faithful and that the vertical maps are the inclusion of the connected components corresponding to automorphisms of x in $\tau(\iota^*X^H)$. It thus suffices to check that the top map in (2.3) is an isomorphism on π_0, and this follows since the map of categories $\tau(\iota^*(X^H))\to\tau(\iota^*(Y^H))$ is fully faithful. ☐

3 Equivariant simplicial dendroidal sets

The results in [CM13, §4] concerning the simplicial Reedy model structure all generalized mutatis mutandis.

COMBMODSTR PROP

Proposition 3.1. *Suppose $\mathcal{C}$ admits two model structures (C,W_1,F_1) and (C,W_2,F_2) with the same class of cofibrations, and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

Then there is a smallest common left Bousfield localization (C,W,F) and for any (C,W,F)-local c,d objects one has that $c\to d$ is in W iff it is in W_1 iff it is in W_2.

Moreover, an object X is local in the common left Bousfield localization iff it is simultaneously fibrant in both of the two initial model structures.

Proof. The model structure (C,W,F) can be obtained by either localizing (C,W_1,F_1) with regards to the generating trivial cofibrations of (C,W_2,F_2) or vice-versa. That the two processes yield the same model structure follows from from the universal property of left Bousfield localizations [Hir03, Prop. 3.4.18]. The claim concerning local c,d follows from the local Whitehead theorem [Hir03, Thm. 3.3.8], stating that the local equivalences between local objects match the initial weak equivalences.

That local objects are fibrant in both model structures follows since $C\cap W$ contains both $C\cap W_1$ and $C\cap W_2$ (in fact, this shows that local fibrations are fibrations in both model structures). The converse claim follows from the observation that fibrant objects in any model structure are local with respect to the weak equivalences in that same model structure. ☐

The prototypical example of Proposition 3.1 is given by the category ssSet of bisimplicial sets together with the two possible Reedy structures (over the Kan model structure on sSet).

Explicitly, writing the levels of $X\in\mathsf{ssSet}$ as $X_{n,m}$ one can either form a Reedy model structure with respect to the *horizontal index* n or with respect to the *vertical index* m. In either case, the generating cofibrations are then given by the maps

$$\left(\partial\Delta[n]\to\Delta[n]\right)\,\square\,\left(\partial\Delta[m]\to\Delta[m]\right),\qquad n,m\geq 0.$$

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, \left(\Lambda^j[m] \to \Delta[m]\right), \qquad n \geq 0, m \geq j \geq 0.$$

{GTRCOHOR EQ}

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square\, (\partial\Delta[m] \to \Delta[m]), \qquad n \geq i \geq 0, m \geq 0.$$

{GTRCOVER EQ}

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, i.e. maps inducing Kan equivalences of simplicial sets $X_{n,\bullet} \to Y_{n,\bullet}$ for each $n \geq 0$, and dually for the vertical Reedy structure.

In the next result we refer to the localized model structure given by Proposition COMBMODSTR PROP 3.1 as the *joint Reedy model structure.*

SSSETJREE PROP

Proposition 3.2. *Suppose that* $X, Y \in \mathsf{ssSet}$ *are horizontal Reedy fibrant. Then:*

(i) for each fixed m *all vertex maps* $X_{\bullet,m} \to X_{\bullet,0}$ *are trivial Kan fibrations;*

(ii) any vertical Reedy fibrant replacement $\tilde{X}$ *of* X *is in fact fibrant in the joint Reedy model structure;*

(iii) a map $X \to Y$ *is a horizontal weak equivalence iff it is a joint weak equivalence;*

(iv) the canonical map $X_{n,0} \to X_{n,n}$ *is a Kan equivalence.*

Proof. (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form $(\partial\Delta[n] \to \Delta[n]) \square (\Delta[0] \to \Delta[m])$.

(ii) follows since by (i) $\tilde{X}$ is then local (with the vertical Reedy model structure as the initial model structure) with respect to all maps of the form $\Delta[0] \times (\Delta[0] \to \Delta[m])$, and thus by Remark HYPERSATKAN REM 2.37 it is fibrant in the joint Reedy model structure (add a remark about this).

(iii) follows from (ii) since the localizing maps $X \to \tilde{X}$, $Y \to \tilde{Y}$ are horizontal equivalences.

For (iv), note first that the diagonal functor $\delta^* \colon \mathsf{ssSet} \to \mathsf{sSet}$ is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure). But noting that all objects are cofibrant, and regarding $X_{n,0}$ as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map $X_{n_0} \to X$ is a horizontal weak equivalence in ssSet. ☐

WEAKDIAG COR

Corollary 3.3. *A map* $f \colon X \to Y$ *in* ssSet *is a joint equivalence iff it induces a Kan equivalence on diagonals* $\delta^*(X) \to \delta^*(Y)$ *in* sSet.

Proof. Since horizontal Reedy fibrant replacement maps $X \to \tilde{X}$ are diagonal equivalences, one reduces to the case of X, Y horizontal Reedy fibrant.

But Proposition SSSETJREE PROP 3.2 (i) and (iii) then combine to say that $X \to Y$ is a joint equivalence iff $X_{\bullet,0} \to Y_{\bullet,0}$ is a Kan equivalence, so that the result follows from Proposition SSSETJREE PROP 3.2 (iv). ☐

SSETSSETADJ COR

Corollary 3.4. *The adjunction*

$$\delta_! \colon \mathsf{sSet} \rightleftarrows \mathsf{ssSet} \colon \delta^*$$

is a Quillen equivalence.

Moreover if a map $f \colon X \to Y$ *in* ssSet *has the right lifting property against both sets of maps in* GTRCOHOR EQ (3) *and* GTRCOVER EQ (3), *then* $\delta^*(f)$ *is a Kan fibration in* sSet.

Note that the "moreover" claim in this result is not quite formal, since the maps in GTRCOHOR EQ (3), GTRCOVER EQ (3) are not known to be generating trivial cofibrations for the joint model structure in ssSet.

Proof. Recall that $\delta_!$ can be characterized as the unique colimit preserving functor such that $\delta_!(\Delta[n]) = \Delta[n] \times \Delta[n]$.

The claim that $\delta_!$ preserves cofibrations thus follows provided that $\delta_!(\partial\Delta[n] \to \Delta[n])$ is a monomorphism for all $n \geq 0$. This holds since: (i) any two face inclusions $F_1 \to \Delta[n]$, $F_2 \to \Delta[n]$ factor through a minimal face inclusion $F \to \Delta[n]$ (indeed, faces are indexed by subsets of $\{0,1\cdots,n\}$); (ii) for any face inclusion one has $\delta_!(F \to \Delta[n]) = \left(F^{\times 2} \to \Delta[n]^{\times 2}\right)$, which is a monormorphism.

The claim that $\delta_!$ preserves trivial cofibrations easily follows from Remark 2.37 together with Corollary 3.3, but here we give a harder argument needed to establish the stronger "moreover" claim. Namely, we will argue that the maps $\delta_!\left(\Lambda^i[n] \to \Delta[n]\right)$ are built cellularly out of the maps in (3), (3). One has a factorization

$$\delta_!\Lambda^i[n] \to \Lambda^i[n] \times \Delta[n] \to \Delta[n]^{\times 2}$$

where the second map is clearly built cellularly out of maps in (3), and we claim that the first map is likewise built cellularly out of maps in (3). Indeed, this first map be built by iteratively attaching the maps

$$\left(\Lambda^F[n] \to \Lambda^i[n]\right) \square \left(\Lambda^i F \to F\right)$$

where F ranges over the poset $\mathsf{Face}_{\supsetneq\{i\}}$ of faces of $\Delta[n]$ strictly containing $\{i\}$ (note that when $F = \Delta[n]$ then $\Lambda^F[n] = \varnothing$, so that these maps can not in general be built out of the maps in (3)).

Lastly, the Quillen equivalence condition is that for all $X \in \mathsf{sSet}$ and joint fibrant $Y \in \mathsf{ssSet}$ a map $X \to \delta^* Y$ is a weak equivalence iff $\delta_! X \to Y$ is. But by Corollary 3.3 this reduces to showing that the unit maps $X \to \delta^*\delta_! X$ are weak equivalences, and this latter claim easily follows by cellular induction on X (recall that since sSet is left proper the pushouts attaching cells are homotopy pushouts). □

HYPERSIMPL REM

Remark 3.5. Hypersaturations can be used to simplify the lifting condition required in the previous result. Indeed, note first that $X \to Y$ will have the lifting condition against the maps in (3) iff all maps $X^L \to X^K \times_{Y^K} Y^L$ are Kan fibrations in sSet for $K \to L$ a monomorphism in sSet. But hence $X \to Y$ will also have the lifting condition against the maps in (3) provided $X^L \to X^K \times_{Y^K} Y^L$ is also a *trivial* Kan fibration when $K \to L$ is an horn inclusion. But one readily checks that horn inclusions can now be replaced with any set with the same hypersaturation. In particular, by Remark 2.37 it suffices to check that the maps $X_n \to X_0 \times_{Y_0} Y_n$ induced by the vertex maps $[0] \to [n]$ are trivial Kan fibrations.

Remark 3.6. The adjunction $\delta^*\colon \mathsf{ssSet} \rightleftarrows \mathsf{sSet}\colon \delta_*$ is also a Quillen equivalence, but we will not need this fact.

We now turn to our main application of Proposition 3.1, the category $\mathsf{sdSet}^G = \mathsf{Set}^{\Delta^{op} \times \Omega^{op} \times G}$ of G-equivariant simplicial dendroidal sets.

Using the fact that Δ is a (usual) Reedy category and the model structure on dSet^G given by [Per17, Thm. 2.1] yields a model structure on sdSet^G that we will refer to as the *simplicial Reedy model structure*.

On the other hand, in the context of Definition A.1, $\Omega^{op} \times G$ is a generalized Reedy category such that the families $\{\mathcal{F}^\Gamma_U\}_{U \in \Omega}$ of G-graph subgroups are Reedy-admissible (see Example A.5), and hence using the underlying Kan model structure on sSet, Theorem A.7 yields a model structure on sdSet^G that we will refer to as the *equivariant dendroidal Reedy model structure*, or simply as *dendroidal Reedy model structure* for the sake of brevity.

Proposition 3.7. *Both the simplicial and dendroidal Reedy model structures on* sdSet^G *have generating cofibrations given by the maps*

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T])\,, \qquad n \geq 0, T \in \Omega_G.$$

Further, the dendroidal Reedy structure has generating trivial cofibrations the maps

$$\big(\Lambda^i[n] \to \Delta[n]\big) \,\square\, (\partial\Omega[T] \to \Omega[T])\,, \qquad n \geq i \geq 0, T \in \Omega_G.$$

`{DENDTRIVCOF EQ}`

while the simplicial Reedy structure has generating trivial cofibrations the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (A \to B)\,, \qquad n \geq 0$$

`{SIMPTRIVCOF EQ}`

for $\{A \to B\}$ *a set of generating trivial cofibrations of* dSet^G*.*

Proof. For the claims concerning the dendroidal Reedy structure, note that the presheaves $\Omega[T] \in \mathsf{dSet}^G$ are precisely the quotients $(G \cdot \Omega[U])/K$ for $U \in \Omega$ and $K \leq G \times \Sigma_U$ a G-graph subgroup, so that $\partial\Omega[T] \to \Omega[T]$ represents the maps $X_U^K \to (M_U X)^K$ for $X \in \mathsf{dSet}^G$.

The claims concerning the simplicial Reedy structure are immediate. ☐

`JOINTFIBCHAR COR`

Corollary 3.8. *The joint fibrant objects* $X \in \mathsf{sdSet}^G$ *have the following equivalent characterizations:*

(i) X *is both simplicial Reedy fibrant and dendroidal Reedy fibrant;*

(ii) X *is simplicial Reedy fibrant and all maps* $X_0 \to X_n$ *are equivalences in* dSet^G*;*

(iii) X *is dendroidal Reedy fibrant and all maps*

$$X^{\Omega[T]} \to X^{Sc[T]} \qquad \text{and} \qquad X^{\Omega[T]} \to X^{\Omega[T]\otimes J_d}$$

for $T \in \Omega_G$ *are Kan equivalences in* sSet*.*

Proof. (i) simply repeats the last part of Proposition `COMBMODSTR PROP` 3.1. In the remainder we write $K \to L$ for a generic monomorphism in sSet and $A \to B$ a generic normal monomorphism in dSet^G.

For (ii), note first that X is simplicial fibrant iff $X^L \to X^K$ is always a fibration in dSet^G. Hence, such X will have the right lifting property againt all maps in `DENDTRIVCOF EQ` (3.7) iff $X^L \to X^K$ is a trivial fibration whenever $K \to L$ is anodyne. But Remark `HYPERSATKAN REM` 2.37 implies that it suffices to verify this for the vertex inclusions $\Delta[0] \to \Delta[n]$.

For (iii), note first that X is dendroidal fibrant iff $X^B \to X^A$ is always a Kan fibration in sSet. Therefore, X will have the right lifting property against all maps `SIMPTRIVCOF EQ` (3.7) iff $X^B \to X^A$ is a trivial Kan fibration whenever $A \to B$ is a generating trivial of dSet^G. By adjunction, this is equivalent to showing that $X^L \to X^K$ is a fibration in dSet^G for any monomorphism $K \to L$ in sSet. Moreover, by the fibration between fibrant objects part of `Per17` [Per17, Prop. 8.8] (see also the beginning of `Per17` [Per17, §8.1]) it suffices to verify that the maps $X^L \to X^K$ have the right lifting property against the maps

$$\Lambda^{Ge}\Omega[T] \to \Omega[T], \quad T \in \Omega_G, e \in Inn(T) \qquad \text{and} \qquad \Omega[T] \otimes (\{i\} \to J_d)\,, \quad T \in \Omega_G, i = \{0,1\}$$

and it thus suffices to check that $X^B \to X^A$ is a trivial Kan fibration whenever $A \to B$ is one of these maps. Proposition `HYPER PROP` 2.35 now finishes the proof. ☐

We now obtain the following partial analogue of Proposition `SSSETJREE PROP` 3.2. Note that the equivalences in the simplicial Reedy model structure are the dendroidal equivalences and vice versa.

SDSETG COR

Corollary 3.9. *Suppose that $X, Y \in \mathsf{sdSet}^G$ are dendroidal Reedy fibrant. Then:*

(i) for each fixed m all vertex maps $X_{\bullet,m} \to X_{\bullet,0}$ are trivial fibrations in dSet^G;

(ii) any simplicial Reedy fibrant replacement $\tilde{X}$ of X is in fact fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a dendroidal weak equivalence iff it is a joint weak equivalence;

(iv) regarding X_0 as a simplicially constant object in sdSet^G, the map $X_0 \to X$ is a dendroidal equivalence, and thus a joint equivalence. (iv) follows from (i).

Proof. The proof adapts that of Proposition 3.2. (i) follows since X then has the right lifting property with respect to all maps $(\Delta[0] \to \Delta[m]) \square (\partial\Omega[T] \to \Omega[T])$. (ii) follows from (i) and the characterization in Corollary 3.8 (ii). (iii) follows from (ii) since the simplicial fibrant replacement maps $X \to \tilde{X}$ are dendroidal equivalences. □

INCOAGJ THM

Theorem 3.10. *The inclusion/0-th level adjunction*

$$\iota\colon \mathsf{dSet}^G \rightleftarrows \mathsf{sdSet}^G \colon (-)_0,$$

where sdSet^G is given the joint Reedy model structure, is a Quillen equivalence.

Proof. It is clear that the inclusion preserves both normal monomorphisms and all weak equivalences, hence the adjunction is Quillen. Consider any map $\iota(A) \to X$ with X joint fibrant and perform a trivial cofibration followed by fibration factorization on the left

$$\iota(A) \xrightarrow{\sim} \widetilde{\iota(A)} \to X \qquad A \xrightarrow{\sim} \widetilde{\iota(A)}_0 \to X_0$$

for the simplicial Reedy model structure. Corollary 3.8 (ii) now implies that $\widetilde{\iota(A)}$ is in fact joint fibrant and thus that the leftmost composite above is a joint equivalence iff $\widetilde{\iota(A)} \to X$ is a dendroidal equivalence in sdSet^G iff $\widetilde{\iota(A)}_0 \to X_0$ is an equivalence in dSet^G iff the rightmost composite is an equivalence in dSet^G. □

CONCRECOM REM

Remark 3.11. Given a G-∞-operad X, a joint fibrant model in sdSet^G is given by $X^{J_d(m)}$. Moreover, note that it follows from [Per17, Cor. 8.21] that $u_*(X^{J_d(m)}) = X^{(J_d(m))} \to X^{(\Delta_m)}$ is a trivial fibration in $\mathsf{sdSet_G}$, so that $X^{J_d(m)}(T) \sim k(\Omega[T], X)$.

4 Pre-operads

Recall that the category PreOp of *pre-operads* is the full subcategory $\mathsf{PreOp} \subset \mathsf{sdSet}$ of those X such that $X(\eta)$ is a discrete simplicial set. Writing γ^* for the inclusion one has left and right adjoints $\gamma_!$ and γ_*

$$\begin{tikzcd}
\mathsf{PreOp}^G \arrow[r, "\gamma^*"'] & \mathsf{sdSet}^G \arrow[l, bend right, "\gamma_!"'] \arrow[l, bend left, "\gamma_*"]
\end{tikzcd}$$

described as follows [CM13, §7]: $\gamma_! X(T) = X(T)$ if $T \notin \Delta$ while $\gamma_! X([n])$ for $[n] \in \Delta$ is given by the pushout on the left below; $\gamma_* X(T)$ is given by the pullback on the right below.

$$\begin{tikzcd}
X(\eta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \pi_0 X(\eta) \arrow[d] \\
X([n]) \arrow[r] & \gamma_! X([n])
\end{tikzcd}
\qquad
\begin{tikzcd}
\gamma_* X(T) \arrow[r] \arrow[d] & X(T) \arrow[d] \\
\coprod_{Edges(T)} X(\eta)_0 \arrow[r] \arrow[ur, phantom, "\lrcorner", very near end] & \coprod_{Edges(T)} X(\eta)
\end{tikzcd}$$

`BOUNDRED REM` **Remark 4.1.** Let $A \to B$ denote a monomorphism in sdSet^G of the form

$$(\partial\Delta[n] \to \Delta[n]) \square (\partial\Omega[T] \to \Omega[T]), \qquad n \geq 0, T \in \Omega_G, T \ncong G/H \cdot \eta.$$

`{BOUNDRED EQ}`

Then one has an isomorphism $A(\eta) \simeq B(\eta)$ so that the square below is a pushout square.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \gamma_! A \arrow[d] \\ B \arrow[r] & \gamma_! B \end{tikzcd}$$

`{GAMMASH EQ}`

In the next results we write I' for the set of maps in (4.1).

`GENSET LEM` **Lemma 4.2.** *The normal monomorphisms of* PreOp^G *are the saturation of the set of maps* $\{\varnothing \to G/H \cdot \eta | H \leq G\} \cup \gamma_!(I')$.

Proof. Using the cellular filtration in sdSet^G, any monomorphism $A \to B$ in PreOp^G can be written as a transfinite composition of pushouts of maps in $\{\varnothing \to G/H \cdot \eta\} \cup I'$. But, since the squares (4.1) are pushouts, the same also holds for the maps $\{\varnothing \to G/H \cdot \eta\} \cup \gamma_!(I')$. ☐

`TRIVFIB LEM` **Lemma 4.3.** *Any map in* PreOp^G *which has the right lifting property against all normal monomorphisms in* PreOp^G *is a joint equivalence in* sdSet^G.

Proof. We need simply adapt the proof of [CM13, Lemma 8.12] mutatis mutandis.

Choose a normalization E_∞ of $*$ in sdSet^G, i.e. a normal object such that $E_\infty \to *$ is a trivial fibration. Regarding E_∞ as a simplicially constant object in sdSet^G, a map $X \to Y$ in PreOp^G will have the right lifting property against all monomorphisms iff so does $E_\infty \times (X \to Y)$, so that one is free to assume X, Y are normal.

One is thus free to pick a section $s: Y \to X$ of $p: X \to Y$ and regarding $J_d \in \mathsf{dSet}^G$ as a simplicially constant object of sdSet^G our assumption yields the lift below, so that p is a homotopy equivalence.

$$\begin{tikzcd} X \coprod X \arrow[r, "(id_X, sp)"] \arrow[d] & X \arrow[d, "p"] \\ X \otimes J_d \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

☐

Theorem 4.4. *The category* Preop^G *of* G*-preoperads has a model structure such that*

- *the cofibrations are the normal monomorphisms;*
- *the weak equivalences are the maps that become joint equivalences when regarded as maps on* sdSet^G.

Proof. One repeats the proof of the non-equivariant analogue [CM13, Thm. 8.13], applying J. Smith's theorem [Bek00, Thm. 1.7] with the required set of generating cofibrations the set $\{\varnothing \to G/H \cdot \eta | H \leq G\} \cup \gamma_!(I')$ given by Lemma 4.2. Indeed, conditions c0 and c2 in [Bek00] are inherited from sdSet^G and c1 follows from Lemma 4.3. The technical "solution set" condition c3 follows from [Bek00, Prop. 1.15] since weak equivalences are accessible, being the preimage by γ^* of the weak equivalences in sdSet^G (see [Lur09, Cor. A.2.6.5] and [Lur09, Cor. A.2.6.6]). ☐

ANOQUEQUIV THM

Theorem 4.5. *The adjunction*

$$\gamma^*\colon \mathsf{PreOp}^G \rightleftarrows \mathsf{sdSet}^G\colon \gamma_*$$

is a Quillen equivalence.

Proof. It it tautological that the left adjoint γ^* preserves and detects cofibrations and weak equivalences, so it suffices to show that for all fibrant $X \in \mathsf{sdSet}^G$ there exists $Y \in \mathsf{PreOp}^G$ and a weak equivalence $\gamma^* Y \to X$. But by Corollary 3.9 it suffices to take $Y = X_0$, regarded as a simplicially trivial object. ☐

We will find it useful to also have a characterization of the fibrant objects in PreOp^G. In doing so, it becomes useful to consider a fourth model structure on the category sdSet^G whose fibrant objects "interpolate" between the fibrant objects in the two model structures in Theorem 4.5. This is the model structure of equivariant dendroidal Segal spaces, that we discuss in the next section.

5 Equivariant dendroidal Segal spaces

Definition 5.1. The *equivariant Segal space model structure* on the category sdSet^G, which we denote sdSet^G_S, is the left Bousfield localization of the dendroidal Reedy model structure with respect to the equivariant Segal core inclusions

$$Sc[T] \to \Omega[T], \qquad T \in \Omega_G.$$

Remark 5.2. By Proposition 2.35 this model structure can equivalently be obtained by localizing with respect to the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$.

Notation 5.3. We will refer to the fibrant objects in sdSet^G_S as *equivariant dendroidal Segal spaces*, or just *dendroidal Segal spaces.* Further, a pre-operad $X \in \mathsf{PreOp}^G$ is called *fibrant* if $\gamma^* X$ is a dendroidal Segal space.

Remark 5.4. If $X \in \mathsf{sdSet}^G$ is a dendroidal Segal space, then $\gamma_* X \in \mathsf{PreOp}^G$ is fibrant. Indeed, the equality $Sc[T](\eta) = \Omega[T](\eta)$ implies that the square below is a pullback square.

$$\begin{tikzcd}
\gamma_* X(T) \arrow[r] \arrow[d] & X(T) \arrow[d] \\
(\gamma_* X)^{Sc[T]} \arrow[r, phantom, "\lrcorner" near end, shift left=3] \arrow[r] & X^{Sc[T]}
\end{tikzcd}$$

The following is a variation on Definition 2.41

MAPSPACESEG DEF

Definition 5.5. Given a dendroidal Segal space $X \in \mathsf{sdSet}^G_S$ and a C-profile $(x_1, \cdots, x_n; x_0)$ on X (defined exactly as in Definition 2.40) we define the space of maps $X(x_1, \cdots, x_n; x_0)$ via the pullback

$$\begin{tikzcd}
X(x_1, \cdots, x_k; x_0) \arrow[r] \arrow[d] & X(C) \arrow[d] \\
\Delta[0] \arrow[r, "{(x_1, \cdots, x_k; x_0)}"'] & \prod_{0 \le i \le k} X(\eta)^{H_i}
\end{tikzcd}$$

Definition 5.6. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. The *homotopy genuine operad* $ho(X) \in \mathsf{dSet_G}$ is defined by

$$ho(X) = \pi_0 \left(u_* \left(\gamma_* X \right) \right).$$

Remark 5.7. It is immediate that $X(x_1, \cdots, x_k; x_0) = \gamma_* X(x_1, \cdots, x_k; x_0)$, so that both of the previous definitions depend only on the fibrant pre-operad $\gamma_* X$.

Remark 5.8. It is important not to confuse Definitions MAPSPACE DEF 2.41 and MAPSPACESEG DEF 5.5. Indeed, when X is a dendroidal Segal, its 0-th level X_0 is a G-∞-operad, and one can thus form two "spaces of maps" $X_0(x_1, \cdots, x_k; x_0)$ (cf. Definition MAPSPACE DEF 2.41) and $X(x_1, \cdots, x_k; x_0)$ (cf. Definition MAPSPACESEG DEF 5.5). The constructions leading to these spaces are quite different. When X is complete Segal, the fact that these two spaces are homotopic follows from Remark CONCRECOM REM 3.11, since X must then be both dendroidally and simplicially equivalent to $X_0^{J_d(m)}$. The claim that this holds without completeness is harder, with the rest of the section devoted to establishing this.

Remark 5.9. Writing ι for the inclusion $\Delta \to \Omega$ and ι_G for the composite inclusion $\Delta \times \mathsf{O}_G \to \Omega \times \mathsf{O}_G \to \Omega_G$, one has that $\iota_G^* ho(X)$ is the G-coefficient system of categories formed by the homotopy categories $ho\left(\iota^* \left(X^H\right)\right) = \pi_0\left(\iota^* \gamma_* X^H\right)$.

Definition 5.10. A map $f\colon X \to Y$ of equivariant dendroidal Segal spaces is called

- *fully faithful* is for all $C \in \Sigma_G$ and C-profile $(x_1, \cdots, x_n; x_0)$ on X the map
$$X(x_1, \cdots, x_k; x_0) \to Y\left(f(x_1), \cdots, f(x_k); f(x_0)\right)$$
is a Kan equivalence in sSet;
- *essentially surjective* if the map $\iota_G^* ho(X) \to \iota_G^* ho(Y)$ is essentially surjective on all category levels of the G-coefficient system;
- a *DK-equivalence*[3] if it is both fully faithful and essentially surjective.

Remark 5.11. This definition depends only on the map $\gamma_* X \to \gamma_* Y$ of fibrant pre-operads.

Definition 5.12. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. We call a point $f \in X(C_{H/H})_0$ a H-equivalence if the corresponding class

$$[f] \in ho(X)(C_{H/H}) = ho\left(\iota^* \left(X^H\right)\right) = \pi_0\left(\iota^* \gamma_* X^H\right)$$

is an isomorphism.

DK-equivalences will provide an explicit description of complete/joint equivalences between dendroidal Segal objects (and thus also between fibrant pre-operads), as we will prove in whatever below. We now introduce some auxiliary notions.

Suppose $C, D \in \Sigma_G$ are G-corollas that can be grafted, i.e. that C has a leaf orbit and D a root orbit both isomorphic to G/H. Denote this orbit as Ge and write $T = C \amalg_{Ge} D$ for the grafted G-tree. For any dendroidal Segal space X one then has $X^{Sc[T]} \simeq X(C) \times_{X(\eta)^H} X(D)$ and one can hence form the section in the middle row below

$$\begin{tikzcd}
\{\varphi\} \times X(z_1, \cdots, z_l; e) \arrow[r, dashed, "\varphi \circ_{Ge} (-)"] \arrow[d, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[d, hook] \\
X(C) \times_{X(\eta)^H} X(D) \arrow[r, dashed, bend left] & X(T) \arrow[l, two heads, "\sim"] \arrow[r, two heads] & X(T - Ge) \\
X(e, y_2, \cdots, y_k; x) \times \{\psi\} \arrow[r, dashed, "(-) \circ_{Ge} \psi"'] \arrow[u, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[u, hook]
\end{tikzcd}$$

{HOMOTCIRC EQ}

---

[3] Here DK stands for Dwyer and Kan.

thus defining maps $\varphi \circ_{Ge} (-)$ (resp. $(-) \circ_{Ge} \psi$) for any choice of $\varphi \in X(e, y_2, \cdots, y_k; x)$ (resp. $\psi \in X(z_1, \cdots, z_l; e)$).

GENOPHO PROP

Proposition 5.13. *The maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are well defined up to homotopy. Further, if $\varphi, \bar{\varphi} \in X(e, y_2, \cdots, y_k; x)$ are homotopic (i.e. in the same connected component), then the maps $\varphi \circ_{Ge} (-)$, $\bar{\varphi} \circ_{Ge} (-)$ are homotopic, and likewise for ψ.*

In particular, $\varphi \circ_{Ge} \psi \in X(z_1, \cdots, z_l, y_2, \cdots, y_k; x)$ is well defined up to homotopy.

Lastly, the maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are functorial in maps $f \colon X \to Y$ between dendroidal Segal spaces.

Proof. This is immediate once one notes that, writing $E = T - Ge$ for the "composite G-corolla", all solid maps in (5) are compatible with the projections to $X^{\partial\Omega[E]}$.

For functoriality, one simply notes that either type of section as on the leftmost diagram

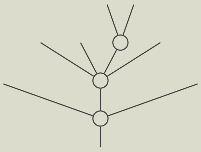
$$\begin{tikzcd}
X(T) \arrow[rr] \arrow[d, two heads, "\sim"] & & Y(T) \arrow[d, two heads, "\sim"] & & & Y(T) \arrow[d, two heads, "\sim"] \\
X(C) \underset{X(\eta)^H}{\times} X(D) \arrow[u, dashed, bend left] \arrow[rr] & & Y(C) \underset{Y(\eta)^H}{\times} Y(D) \arrow[u, dashed, bend left] & X(C) \underset{X(\eta)^H}{\times} X(D) \arrow[rr] \arrow[urr, dashed] & & Y(C) \underset{Y(\eta)^H}{\times} Y(D)
\end{tikzcd}$$

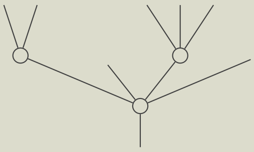
induces lifts as on the rightmost diagram. However, a standard argument shows that all such lifts are homotopic over $X^{\partial\Omega[E]}$. □

We will now show that the operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy the obvious compatibilities one expects, but we will find it convenient to first package these compatibilities into a common format. In the categorical case (corresponding to linear trees), there are three types of compatibilities, corresponding to homotopies

$$\varphi \circ (\psi \circ (-)) \sim (\varphi \circ \psi) \circ (-) \qquad \varphi \circ ((-) \circ \psi) \sim (\varphi \circ (-)) \circ \psi \qquad ((-) \circ \varphi) \circ \psi \sim (-) \circ (\varphi \circ \psi)$$

but in the operadic case there are instead five cases, corresponding to the different possible roles of the nodes in G-trees with exactly three G-nodes, whose *orbital* representation falls into one of the two cases illustrated below.





Since all these compatibilities can be simultaneously encoded in terms of such trees, we will simply refer to all types of compatibility as "associativity".

In the next result, note that a G-tree T with three G-nodes contains precisely two inner edge orbits Ge and Gf. We will write $T[Ge]$ (resp. $T[Gf]$) for the unique orbital outer face of T with Ge (resp. Gf) has its single inner edge orbit.

Proposition 5.14. *The operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy all associativity conditions with respect to 3-nodal G-trees.*

Proof. For any 3-nodal G-tree T with inner edge orbits Ge, Gf, consider the following diagram, where all solid maps are fibrations, and the maps labelled ~ are trivial fibrations.

$$\begin{tikzcd}
X^{\mathsf{Sc}[T]} \arrow[r, dashed, bend left] \arrow[d, dashed, bend right] & X^{\mathsf{Sc}_{T[Ge]}[T]} \arrow[l, two heads, "\sim"] \arrow[r, two heads] \arrow[d, dashed, bend right] & X^{\mathsf{Sc}[T-Ge]} \arrow[d, dashed, bend right] \\
X^{\mathsf{Sc}_{T[Gf]}[T]} \arrow[u, two heads, "\sim"'] \arrow[r, dashed, bend left] \arrow[d, two heads] & X^{\Omega[T]} \arrow[l, two heads, "\sim"] \arrow[u, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, two heads] & X^{\Omega[T-Ge]} \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \\
X^{\mathsf{Sc}[T-Gf]} \arrow[r, dashed, bend left] & X^{\Omega[T-Gf]} \arrow[l, two heads, "\sim"] \arrow[r, two heads] & X^{\Omega[T-Ge-Gf]}
\end{tikzcd}$$

{FOURSQ EQ}

Noting that the following three diagrams, where $\Lambda_o^{Ge,Gf}[T]$ is a generalized orbital G-horn and $\Lambda_{o,c}^{Ge}[T]$, $\Lambda_{o,c}^{Gf}[T]$ are characteristic orbital G-horns (cf. ref), are pullbacks

$$\begin{tikzcd}
X^{\mathsf{Sc}[T]} \arrow[dr, phantom, very near start, "\ulcorner"] & X^{\mathsf{Sc}_{T[Ge]}[T]} \arrow[l, two heads, "\sim"] & X^{\mathsf{Sc}_{T[Ge]}[T]} \arrow[r, two heads] & X^{\mathsf{Sc}[T-Ge]} \arrow[dl, phantom, very near start, "\urcorner"] & X^{\mathsf{Sc}_{T[Gf]}[T]} \arrow[d, two heads] & X^{\Lambda_{o,c}^{Ge}[T]} \arrow[l, two heads, "\sim"] \arrow[d, two heads] \\
X^{\mathsf{Sc}_{T[Gf]}[T]} \arrow[u, two heads, "\sim"'] & X^{\Lambda_o^{Ge,Gf}[T]} \arrow[l, two heads, "\sim"] \arrow[u, two heads, "\sim"'] & X^{\Lambda_{o,c}^{Gf}[T]} \arrow[u, two heads, "\sim"'] \arrow[r, two heads] & X^{\Omega[T-Ge]} \arrow[u, two heads, "\sim"'] & X^{\mathsf{Sc}[T-Gf]} \arrow[ur, phantom, very near start, "\llcorner"] & X^{\Omega[T-Gf]} \arrow[l, two heads, "\sim"]
\end{tikzcd}$$

one sees that: (i) sections in the top left square of (5) can be chosen to be compatible in the sense that the two composites $X^{Sc[T]} \to X^{\Omega[T]}$ coincide; (ii) sections in the top right and bottom left squares of (5) can be chosen to be compatible in the sense that the two composites $X^{Sc[T-Ge]} \to X^{\Omega[T]}$ and $X^{Sc[T-Gf]} \to X^{\Omega[T]}$ coincide. Note that we do not claim (or need) that (i) and (ii) hold simultaneously. We thus conclude that the possible choices of maps $X^{\mathsf{Sc}[T]} \to X^{\Omega[T-Ge-Gf]}$ given by outer paths in (5) are homotopic. All desired forms of associativity follow from taking fibers of these maps over the objects $X^{\partial\Omega[T-Ge-Gf]}$.

Actually, I may just be able to do without these arguments. ☐

Remark 5.15. While in the non-equivariant case the associativity conditions in the previous result capture all the key compatibilities of the $\varphi \circ_e (-)$, $(-) \circ_e \psi$ operations, in the equivariant case there are further "compatibilities with pullback of G-trees", which are closely related to the genuine equivariant operads introduced in [BP17]. However, describing these extra compatibilities would require using G-trees with more than 3 G-nodes, and since such compatibilites are not needed for our current goals, we omit their discussion.

Corollary 5.16. *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-3.*

$$\begin{tikzcd}
X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Z \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. The non trivial claim is that when f and gf are DK-equivalences then so is g, or more precisely, that the maps

$$Y(y_1, \cdots, y_n; y_0) \to Z(g(y_1), \cdots, g(y_n); g(y_0))$$

are weak equivalences even if the y_i are not in the image of f. But this follows from the functoriality in Proposition 5.13, essential surjectivity (note that when $y_i \in Y(\eta)^{H_i}$ one needs to use H_i-equivalences), and the fact that by the previous corollary the maps $f \circ_{Ge} (-)$, $(-) \circ_{Ge} f$ are weak equivalences whenever f is a H-equivalence. ☐

`26COR` **Corollary 5.17.** *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-6, i.e. whenever gf and hg are DK-equivalences then so are f, g, h, hgf.*

Proof. The hypothesis implies together with the 2-out-of-6 condition for the Kan model structure in sSet imply that g is fully faithful for objects in the image of f. But this easily implies that f is a DK equivalence, and thus 2-out-of-3 concludes the proof. □

We now recover the following from Rez01 [Rez01] (add commentary)

Proposition 5.18. *Let $X \in \mathsf{ssSet}$ be a Segal space. Then:*

(i) equivalences define a subset of connected components $X_1^h \subset X_1$;

(ii) the pullbacks

$$\begin{tikzcd} X_n^h \arrow[r] \arrow[d] & X_n \arrow[d] \\ X_1^h \times_{X_0} \cdots \times_{X_0} X_1^h \arrow[r] & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{tikzcd} \qquad \text{\{XHDEF EQ\}}$$

define a Segal space $X^h \subset X$, consisting of a union of connected components at each level;

(iii) the maps $X_2^h \xrightarrow{(d_2,d_1)} X_1^h \times_{X_0} X_1^h$, $X_2^h \xrightarrow{(d_0,d_1)} X_1^h \times_{X_0} X_1^h$ are trivial fibrations;

(iv) the map $X^J \to X^{\Delta^1} = X_1$ factors through a weak equivalence $X^J \to X_1^h$.

Proof. For (i), note first that given $f\colon x \to y$ in $X_{1,0}$, then $[f]$ has a left inverse iff one can find a lift p as on the leftmost diagram below. But for any path H between f and f' in X_1

$$\begin{tikzcd} & X_2 \arrow[d, "(d_2,d_1)"] \\ \Delta[0] \arrow[ru, "p"] \arrow[r, "(f,s_0(x))"'] & X_1 \times_{X_0} X_1 \end{tikzcd} \qquad \begin{tikzcd} \Delta[0] \arrow[r, "p"] \arrow[d, "0"'] & X_2 \arrow[d, "(d_2,d_1)"] \\ \Delta[1] \arrow[ru, dashed] \arrow[r, "(H,s_0 d_1(H))"'] & X_1 \times_{X_0} X_1 \end{tikzcd}$$

one can form the lift in the rightmost diagram, showing that f' is also left-invertible. The situation for right inverses is identical, thus (i) follows.

For (ii), the fact that $X_\bullet^h$ is closed under the simplicial operators follows since the composite of equivalences is an equivalence. In fact, this further implies that the bottom row in the pullback XHDEF EQ (5.18) could have been replaced with $(X^h)^{sk_1\Delta[n]} \to X^{sk_1\Delta[n]}$, from which it follows that the squares

$$\begin{tikzcd} (X^h)^K \arrow[r] \arrow[d] & X^K \arrow[d] \\ (X^h)^{sk_1 K} \arrow[r] & X^{sk_1 K} \end{tikzcd}$$

are pullbacks. Since $sk_1(\partial\Delta[n]) = sk_1\Delta[n]$ if $n \geq 2$ it follows that the maps $X_n^h \to (X^h)^{\partial\Delta[n]}, n \geq 2$ are Kan fibrations, and since the map $X_1^h \to X_0 \times X_0$ is certainly a Kan fibration, X^h is indeed Reedy fibrant. The Segal condition is obvious from the pullback XHDEF EQ (5.18).

For (iii), it suffices by symmetry to show the first claim. Moreover, one reduces to showing show that for any choice of section in the following diagram the top composite is a weak equivalence.

$$\begin{tikzcd}
X_1^h \times_{X_0} X_1^h \arrow[r, dashed, bend left, "\sim"] \arrow[dr, two heads, "{(id,d_0)}"'] & X_2^h \arrow[l, two heads, "{(d_2,d_0)}"] \arrow[r, two heads, "{(d_2,d_1)}"'] & X_1^h \times_{X_0} X_1^h \arrow[dl, two heads, "{(id,d_0)}"] \\
& X_1^h \times X_0 &
\end{tikzcd}$$

But this composite is a map of fibrations over $X_1^h \times X_0$ with the map between the fibers over $(f\colon x \to y, z)$ computing the map $(-)\circ f\colon X^h(y;z) \to X^h(x;z)$, which is a Kan equivalence since $f \in X_1^h$ is an equivalence. Thus the composite is an equivalence, establishing (iii).

Lastly, for (iv) we first note that (iii) can be restated as saying that X^h is local with respect to the outer horn inclusions $\Lambda^0[2] \to \Delta[2]$ and $\Lambda^2[2] \to \Delta[2]$, and that hence by Remarks ANHYPER REM 6.1 and CONTGR REM 6.2 the map $(X^h)^J \to X_1^h$ is a Kan equivalence. Hence, the only remaining claim is that $(X^h)^J = X^J$, which is clear. ☐

Remark 5.19. The inclusion $X^h \to X$ is a Reedy fibration. justify

JDDK PROP **Proposition 5.20.** *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then the map $X \to X^{J_d}$ is a DK-equivalence.*

Proof. Note first that for any $T \in \Omega_G$ the map $X^{J_d}(T) \to X^{\Omega[1]}(T)$ can be rewritten as $\left(X^{\Omega[T]}\right)^{J_d} \to \left(X^{\Omega[T]}\right)^{\Omega[1]} = \left(X^{\Omega[T]}\right)_1$, and since $X^{\Omega[C]}$ is a (simplicial) Segal space the previous result implies that this map is a weak equivalence onto a subset of components (such maps are also called homotopy monomorphisms). It thus follows that for any G-corolla $C_{\amalg_i H_0/H_i}$ the horizontal maps in the rightmost square below are homotopy monomorphisms.

$$\begin{tikzcd}
X(C) \arrow[r] \arrow[d, two heads] & X^{J_d}(C) \arrow[r] \arrow[d, two heads] & X^{\Omega[1]}(C) \arrow[d, two heads] \\
\prod_{0\le i\le k} X(\eta)^{H_i} \arrow[r] & \prod_{0\le i\le k} X^{J_d}(\eta)^{H_i} \arrow[r] & \prod_{0\le i\le k} X^{\Omega[1]}(\eta)^{H_i}
\end{tikzcd}$$ {BIGSQ EQ}

Since the claim that $X \to X^{J_d}$ is a DK-equivalence is the statement that the leftmost square induces weak equivalences on fibers, it thus suffices to show that so does the composite square.

Now note that we can rewrite $X^{\Omega[1]}(C) = X^{\Omega[1]\otimes\Omega[C]}$ and that there is a pullback diagram

$$\begin{tikzcd}
X^{\Omega[1]\otimes\Omega[C]} \arrow[r, two heads] \arrow[d, two heads] & X(C \star \eta) \arrow[d, two heads] \\
X(\eta \star C) \arrow[r, two heads] & X(C)
\end{tikzcd}$$

Noting that the required cube is projective fibrant, one reduces to checking that the following squares induce weak equivalences on fibers, and this claim is clear from the top right vertical trivial fibrations (which are instances of the Segal condition).

$$\begin{tikzcd}
X(C) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(C \star \eta) \arrow[d, two heads, "\sim"] \\
& X(C) \underset{X(\eta)^{H_0}}{\times} X([1])^{H_0} \arrow[d, two heads] \\
\prod\limits_{0 \leq i \leq n} X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod\limits_{1 \leq i \leq n} X(\eta)^{H_i} \right) \times X([1])^{H_0}
\end{tikzcd}
\qquad
\begin{tikzcd}
X(C) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(C \star \eta) \arrow[d, two heads, "\sim"] \\
& \prod\limits_{1 \leq i \leq n} X([1])^{H_i} \underset{\prod\limits_{1 \leq i \leq n} X(\eta)^{H_i}}{\times} X(C) \arrow[d, two heads] \\
\prod\limits_{0 \leq i \leq n} X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod\limits_{1 \leq i \leq n} X([1])^{H_i} \right) \times X(\eta)^{H_0}
\end{tikzcd}$$

□

Definition 5.21. Two maps $f, f' \colon A \rightrightarrows B$ between dendroidal Segal spaces are called J-homotopic, written $f \sim_J f'$, if there is a H such that the two composites $A \xrightarrow{H} B^J \rightrightarrows B$ are f, f'.

Further, a map $f \colon X \to Y$ of dendroidal Segal spaces is called a J-homotopy equivalence if there is $g \colon Y \to X$ such that $gf \sim_J id$, $fg \sim_J id$.

Remark 5.22. It follows from Proposition JDDK PROP 5.20 and 2-out-of-3 that if $f \sim_J f'$ then f is a DK-equivalence iff $f'$is. Thus by 2-out-of-6 all J-homotopy equivalences are DK-equivalences.

ALLXJK REM **Remark 5.23.** Let X be a Segal space. All simplicial operators $X^{J_n} \to X^{J_m}$ are induced from equivalences of groupoids $\widetilde{[m]} \to \widetilde{[n]}$, and one easily checks that these operators are thus J-homotopy equivalences and thus also DK-equivalences.

COMPLE PROP **Proposition 5.24.** *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then there is a complete dendroidal Segal space $\tilde{X}$ and complete equivalence $X \to \tilde{X}$ such that*

(i) $X \to \tilde{X}$ is a monomorphism and a DK-equivalence;

(ii) $X_0(\eta) \to \tilde{X}_0(\eta)$ is an isomorphism.

Proof. Our argument will mostly adapt the construction of the completion functor in Rez01 [Rez01, §10.4].

Firstly, we let $X^{J_\bullet} \in (\mathsf{sdSet}^G)^{\Delta^{op}} = \mathsf{ssdSet}^G$ be the object whose k-th level (in the new simplicial direction) is X^{J_k}. Since $J_\bullet$ is a Reedy cofibrant cosimplicial object, $X^{J_\bullet}$ is Reedy fibrant with respect to the dendroidal Space model structure on sdSet^G. In particular, it follows that $X^{J_\bullet} \to \mathsf{csk}_\eta X^{J_\bullet}$ is a fibration in ssdSet^G.

In particular, this implies that for each $T \in \Omega_G$ and vertex map $[0] \to [k]$ the induced square

$$\begin{tikzcd}
X^{J_k}(T) \arrow[r, two heads] \arrow[d, two heads] & X(T) \arrow[d, two heads] \\
\prod\limits_{e_i \in E_G(T)} \left(X^{J_k}(\eta)\right)^{H_i} \arrow[r, two heads] & \prod\limits_{e_i \in E_G(T)} \left(X(\eta)\right)^{H_i}
\end{tikzcd}$$

is an (injective) fibrant square, which by Remark ALLXJK REM 5.23 induces weak equivalences on fibers, so that the map from $X^{J_k}(T)$ to the pullback of the remaining diagram is a trivial Kan fibration. By Remark HYPERSIMPL REM 3.5 we have just shown that for each fixed $T \in \Omega_G$ the map

$$X^{J_\bullet}(T) \to \left(\mathsf{csk}_\eta X^{J_\bullet}\right)(T)$$

{MOREOVER EQ}

satisfies the "moreover" condition in Corollary 3.4. Therefore, applying δ^* to (5) yields a Kan fibration, so that all fibers of this map are in fact homotopy fibers.

We now write $\tilde{X}$ for any dendroidal Reedy fibrant replacement of the diagonal $\delta^*\left(X^{J_\bullet}\right)$, which we note can always be chosen so that $\delta^*\left(X^{J_\bullet}\right) \to \tilde{X}$ is a monomorphism and $\tilde{X}_0(\eta) = \left(\delta^*\left(X^{J_\bullet}\right)\right)_0(\eta) = X_0(\eta)$ (this follows since fibrant replacements in the Kan model structure in sSet can be chosen to preserve 0-simplices, since existence of lifts against the horn inclusions $\Delta[0] = \Lambda^0[1] \to \Delta[1]$, $\Delta[0] = \Lambda^1[1] \to \Delta[1]$ is automatic).

To see that $\tilde{X}$ is a complete Segal space, note that there is a composite $X_0^{J_\bullet} \to \delta^*\left(X^{J_\bullet}\right) \to \tilde{X}$ where the first map is a dendroidal Reedy equivalence by Proposition 3.2(iv) and the second by definition of $\tilde{X}$. But since $X_0^{J_\bullet}$ is a complete Segal space, so is $\tilde{X}$.

For the remaining claim that the composite $X = X^{J_0} \to \delta^*\left(X^{J_\bullet}\right) \to \tilde{X}$ is a DK equivalence, though the first map is no longer a dendroidal Reedy equivalence, it is nonetheless an equivalence on fibers over $\prod_{e_i \in E_G(T)} \left(X(\eta)\right)^{H_i}$ for each $T \in \Omega_G$. And since we established above that the fibers of $\delta^*\left(X^{J_\bullet}\right)(T)$ are homotopy fibers, these are equivalent to the fibers of $\tilde{X}(T)$ (since Reedy replacement does not change the homotopy fibers), and thus $X \to \tilde{X}$ is indeed fully faithful. Essential surjectivity is trivial since the objects coincide. The monomorphism condition is clear. ☐

Corollary 5.25. *A map of $X \to Y$ of dendroidal Segal spaces is a joint equivalence iff it is a DK equivalence.*

Proof. By the (proof of) previous result one is free to, via a zigzag, replace X, Y with $X_0^{J_\bullet}$, $Y_0^{J_\bullet}$. But by Theorem 3.10 this map is a joint equivalence iff $X_0 \to Y_0$ is an equivalence in dSet^G, which by Theorem 2.43 holds iff this is a fully faithful and essentially surjective map of G-∞-operads. But it is easy to check that $X_0 \to Y_0$ is a fully faithful and essentially surjective map of G-∞-operads iff $X_0^{J_\bullet} \to Y_0^{J_\bullet}$ is a fully faithful and essentially surjective map of dendroidal Segal spaces. ☐

Corollary 5.26. *A pre-operad $X \in \mathsf{PreOp}^G$ is fibrant iff $\gamma^*(X)$ is fibrant in the Segal space model structure on sdSet^G.*

Proof. We start with the "only if" direction. Recall that $\gamma^* X$ is a dendroidal Segal space if it has the right lifting property against the maps of the form

$$(\Lambda^i[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]) \qquad (\partial\Delta[n] \to \Delta[n]) \,\square\, (Sc[T] \to \Omega[T]). \tag{SOMEMAPS EQ}$$

With the exception of the first type of maps when $T = \eta$, in which case the lifting condition is automatic since $\gamma^* X(\eta)$ is discrete, all other maps induce isomorphisms at the η-level, so that by (4.1) applying $\gamma_!$ to these maps yields trivial cofibrations in PreOp^G. Thus, if $X \in \mathsf{PreOp}^G$ is fibrant, an adjunction argument shows that $\gamma^*(X)$ indeed has the lifting property against all maps (5), i.e. $\gamma^*(X)$ is a dendroidal Segal space.

For the "if" direction, form the completion $\gamma^* X \to \tilde{X}$ as in Proposition 5.24. Then $\gamma_* \tilde{X} \in \mathsf{PreOp}^G$ is fibrant by Theorem 4.5 and the adjoint map $X \to \gamma_* \tilde{X}$ has the following properties: (i) it is a monomorphism; (ii) it is an isomorphism on the η-level; (iii) it is a DK-equivalence when regarded as a map in sdSet^G (since $\gamma_* \gamma^* \tilde{X} \to \tilde{X}$ is tautologically a DK-equivalence); (iv) it is hence a trivial Reedy cofibration when regarded as a map in sdSet^G. But then the hypothesis that $\gamma^* X$ is a dendroidal Segal space yields a lift

$$\begin{tikzcd}
\gamma^* X \arrow[r, equal] \arrow[d] & \gamma^* X \\
\gamma^* \gamma_* X \arrow[ur, dashed] &
\end{tikzcd}$$

showing that X is a retract of $\gamma_* X$ and finishing the proof. $\square$

Remark 5.27. For any dendroidal Segal space $X \in \mathsf{sdSet}^G$ one hence has complete equivalences

$$\gamma_* X \to X \to \tilde{X}$$

where γ_* is a fibrant preoperad and $\tilde{X}$ a complete dendroidal Segal space.

$$u^* \colon \mathsf{dSet}_G \rightleftarrows \mathsf{dSet}^G \colon u_*$$

discuss characteristic horns

6 Scratchwork (to be folded into previous sections eventually)

`ANHYPER REM`

Remark 6.1. The smallest hypersaturated class containing the inner horns and the left horn inclusion $\Lambda^0[2] \to \Delta[2]$ in fact contains all left horn inclusions $\Lambda^0[n] \to \Delta[n]$ for $n \geq 2$. Indeed, this follows inductively from the following diagram since the bottom map is inner

$$\begin{tikzcd}
\Lambda^{0,1}[n] \arrow[r] \arrow[d] & \Lambda^0[n] \arrow[d] \\
\Lambda^1[n] \arrow[r] & \Delta[n]
\end{tikzcd}$$

and the top and left maps are given by following pushouts

$$\begin{tikzcd}
\Lambda^0[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Lambda^{0,1}[n] \arrow[d] \\
\Delta[n-1] \arrow[r, "d^1"'] & \Lambda^0[n]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Lambda^0[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Lambda^{0,1}[n] \arrow[d] \\
\Delta[n-1] \arrow[r, "d^0"'] & \Lambda^1[n]
\end{tikzcd}$$

`CONTGR REM`

Remark 6.2. Write $\widetilde{[n]}$ for the contractible groupoid on objects $\{0, 1, \cdots, n\}$. Note that the k-simplices of $\widetilde{[n]}$ are encoded as strings $a_0 a_1 \cdots a_k$ with $a_i \in \{0, 1, \cdots, n\}$, and that a simplex is non-degenerate iff $a_{i-1} \neq a_i, 1 \leq i \leq k$. Then the maps

$$\Delta[n] = N[n] \xrightarrow{012\cdots n} N\widetilde{[n]}, \quad n \geq 1$$

`{INVER EQ}`

are built cellularly out of left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 2$.

Indeed, we show a little more. Call subcomplex $A \subset N\widetilde{[n]}$ is 0-*stable* if a n-simplex $\underline{a}$ is in A iff the $n+1$-simplex $0\underline{a}$ is. We claim that any inclusion $A \to A'$ of 0-stable subcomplexes is built cellularly from left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 1$. Indeed, it suffices to check this when A' attaches as little as possible to A, and 0-simplicity guarantees that in that case the only two non-degenerate simplices in $A - A'$ have the form $\underline{a}$ and $0\underline{a}$ (note that $\underline{a}$ can not start with a 0). But then $A \to A'$ is a pushout of $\Lambda^0[k+1] \to \Delta[k+1]$ where k is the dimension of $\underline{a}$.

The desired claim follows by noting that both the domain and codomain of (6.2) are 0-stable and that the horns $\Lambda^0[1]$ are unneeded since (6.2) is an isomorphism on 0-simplices.

$$\begin{tikzcd}
W_1 \times_{W_0} W_1 \times_{W_0} W_1 \arrow[r, dashed, bend left] \arrow[d, dashed, bend right] & W_1 \times_{W_0} W_2 \arrow[l, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, dashed, bend right] & W_1 \times_{W_0} W_1 \arrow[d, dashed, bend right] \\
W_2 \times_{W_0} W_1 \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \arrow[r, dashed, bend left] & W_3 \arrow[l, two heads, "\sim"'] \arrow[u, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, two heads] & W_2 \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \\
W_1 \times_{W_0} W_1 \arrow[r, dashed, bend left] & W_2 \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & W_1
\end{tikzcd}$$

Remark 6.3. Note that $\mathsf{Sc}_{T[Ge]}$, $\mathsf{Sc}_{T[Gf]}$ in (5) are cover inclusions, and thus G-anodyne, relate to [Rez10, §6.2], [Rez01, §10].

Remark 6.4. Indexing systems are precisely the Segal sieves of Ω_G.

Lemma 6.5. *A non-equivariant face $U \hookrightarrow T$ generates an equivariant face $G\cdot U \hookrightarrow T$ iff the G-isotropy of r_U matches the isotropy of U.*

Proposition 6.6. *Any non-equivariant face $U \hookrightarrow T$ has a minimal factorization $U \hookrightarrow GU \hookrightarrow T$ through an equivariant face GU.*

Proof. Our first task is that of building GU. The edges of GU will be given by combining the G-orbits of the edges in U, but describing the broad relations requires some caution.

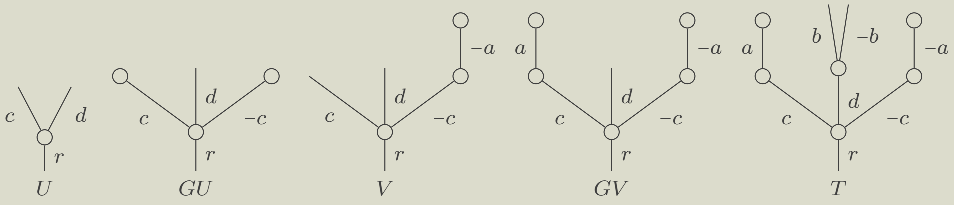
We first assume that U is an outer face and write $H \leq G$ for the isotropy of the root r_U in T. We form a (non-equivariant) subtree HU of T with edges (resp. generating broad relations) those edges hu (resp. relations $hu^{\uparrow} \leq hu$) in some hU for $h \in H$. Indeed, while a priori HU is merely a pre-broad poset of T, the antisymmetry and simplicity axioms in [Per17, Defs. 5.1 and 5.3] are inherited from T, and the nodal axiom in [Per17, Def. 5.9] follows since the hU, $h \in H$ have a common root. It is now clear that HU inherits a H-action and we thus set $GU = G\cdot_H HU$. Both the factorization $U \to GU \to T$ and its minimality are immediate from the description of HU.

Before tackling the general case, we collect some key observations. Firstly, the assumption that U is outer implies that so are the (non-equivariant) face HU and the equivariant face GH. Secondly, the root tuple of GU is $G\cdot_H r_U$. Lastly, we need a characterization of the leaf tuple of GU: we call a leaf l of U orbital if all the edges in $Hl \cap U$ are leaves of U, and claim that the leaves of GU are the tuple $\underline{l}$ formed by the G-orbits of the orbital leaves of U. Indeed, a leaf $l \in U$ will be a leaf of HU iff whenever $l \in hU$ for some $h \in H$ then l is a leaf of hU iff whenever $h^{-1}l \in U$ for some $h \in H$ then $h^{-1}l$ is a leaf of U.

In the general case, write $U \to V \to T$ for the factorization [BP17, Prop. 3.31] as a inner face followed by an outer face. We now define GU to be the equivariant inner face of GV that removes all edge orbits that are not represented in U (that all such edge orbits are inner follows from the description of the roots and leaves of GV in the previous paragraph). It is now clear that $U \to GV \to T$ is the minimal factorization with GV an *outer* equivariant face, so that the desired factorization $U \to GU \to T$ follows since inner faces are full while minimality follows from the equivariant analogue of the inner face followed by outer face factorization. □

Example 6.7. When T is open then GU is necessarily a full equivariant face, i.e. broad relations coincide with those in T, and this fact can be leveraged to obtain a simpler proof.

However, the presence of stumps can make the situation substantially more subtle, especially in the non-outer case, as illustrated by the following example (where $G = \mathbb{Z}_{/2} = \{\pm 1\}$).



GINNER REM

Remark 6.8. For any inner face $V - e$ of V one has that $G(V - e)$ is either $GV - Ge$ or GV. Indeed, the latter will happen iff $V - e$ contains either an inner edge of a leaf of the form ge.

6.1 Characteristic playground

INNINT LEM

Lemma 6.9. *Consider a diagram of faces in* Ω

$$\begin{tikzcd} V \arrow[r] \arrow[d] & U \arrow[d] \\ \bar{V} \arrow[r] & \bar{U} \end{tikzcd}$$

such that the horizontal maps are outer face maps and the left vertical map is an inner face map. Then $Inn(V) = Inn(U) \cap Inn(\bar{V})$.

Proof. Write r and $\underline{l} = l_1 \cdots l_n$ for the root and leaf tuple of V, or equivalently $\bar{V}$. Since the horizontal maps are outer, an edge $e \in Inn(U)$ (resp. $e \in Inn(\bar{U})$) is also in $Inn(V)$ (resp. in $Inn(\bar{V})$) iff $e <_d r$ and $\forall_i e \nleq l_i$. But then $Inn(V) = Inn(U) \cap Inn(V) = Inn(U) \cap Inn(\bar{V})$. ☐

Notation 6.10. Given a non-equivariant planar face $U \hookrightarrow T$ we write $\bar{U}^T$, or simply $\bar{U}$ when no confusion should arise, for the smallest planar outer face of T containing U.

CHAREDGE DEF

Definition 6.11. Let $T \in \Omega_G$, $X \subseteq \Omega[T]$ a subdendroidal set, and $\{U_i\}_{i \in I} \subseteq \mathsf{Face}(T)$ a subset.
Given a set Ξ of inner edges of T and a subface $V \hookrightarrow U$, denote $\Xi_V = \Xi \cap Inn(V)$.
We say that Ξ is *a characteristic inner edge set of* T *with respect to* X *and* $\{U_i\}$ if:

(Ch0) Ξ, X and $\{U_i\}$ are all G-equivariant;

(Ch1) for all i, all outer faces $V = \bar{V}^{U_i}$ of U_i such that $\Xi_V = \varnothing$ are contained in X;

(Ch2) for all i, all faces $V \hookrightarrow U_i$ such that $(V - \Xi_V) \in X$ are contained in X;

(Ch3) there exists a well-ordering of the G-cosets I/G such that for all $[j] \ngtr [i]$ all faces $V \hookrightarrow U_i$ such that $V \notin X$ and $(V - \Xi_V) \hookrightarrow U_j$ are contained in some U_k with $[k] \leq [j]$ or $[k] < [i]$.

Remark 6.12. In many natural examples, condition (Ch3) can be deduced from a much simpler condition. Namely, is suffices to check that:

(Ch3') for all i, and writing $\bar{U}_i = \bar{U}_i^T$ for the outer closure in T, one has $\Xi_{U_i} = \Xi_{\bar{U}_i}$.

Indeed, if (Ch3') holds then $V \hookrightarrow U_i$ iff $V - \Xi_V \hookrightarrow U_i$, hence (Ch3) holds for any well-ordering.

CHAREDGE LEM

Lemma 6.13. *If* Ξ *is a characteristic inner edge set of* T *with respect to* X *and* $\{U_i\}_{i \in I}$*, then the map*

$$X \to X \cup \bigcup_{i \in I} \Omega[U_i]$$

{CHARLEM EQ}

is G*-inner anodyne. In fact, it is built cellularly from* G*-horn inclusions* $\Lambda^E[S] \to \Omega[S]$, $S \in \Omega_G$.

Proof. We start with the case where $\{U_i\}$ consists of a single orbit, i.e. $\{U_i\}_{i \in I} = \{gU\}_{[g] \in G/H}$ for some $U \in \mathsf{Face}(T)$ with isotropy H. Note that in this case the condition (Ch3) is vacuous.

We write $\mathsf{Face}^{lex}_{\Xi}(U)$ for the H-poset of (planar) faces $V \hookrightarrow U$ such that $\Xi_V \neq \varnothing$ and $\Xi_V = \Xi_{\bar{V}}$ ordered as follows: $V \leq V'$ if either (i) $\bar{V} \hookrightarrow \bar{V}'$ and $\bar{V} \neq \bar{V}'$ or (ii) $\bar{V} = \bar{V}'$ and $V \hookrightarrow V'$ (alternatively, this is the lexicographic order of pairs $(\bar{V}, V)$). We note that here and in the remainder of the proof all outer closures are implicitly taken in U (rather than T), i.e. $\bar{V} = \bar{V}^U$.

For any H-equivariant convex subset C of $\mathsf{Face}^{lex}_{\Xi}(U)$ we write

$$X_C = X \cup \bigcup_{g \in G, V \in C} \Omega[gV].$$

It now suffices to show that whenever $C \subseteq C'$ the map $X_C \to X_{C'}$ is built cellularly from G-inner horn inclusions (indeed, setting $C = \varnothing$ and $C' = \mathsf{Face}^{lex}_{\Xi}(U)$ recovers (CHARLEM EQ 6.13)).

Without loss of generality we can assume that C' is obtained from C by adding the H-orbit of a single $V' \hookrightarrow U$. Further, we may assume $V' \notin X_C$ or else $X_C = X_{C'}$. Writing $H' \leq G$ for the isotropy of V' in $\mathsf{Face}(T)$ and regarding $V' \in \Omega^{H'} \subset \Omega_{H'}$, we claim that $X_C \to X_{C'}$ is then the pushout of the G-inner horn

$$G \cdot_{H'} \left(\Lambda^{\Xi_{V'}}[V'] \to \Omega[V'] \right),$$

where the inner edge set $\Xi_{V'}$ is H'-equivariant due to the G-equivariance of Ξ in (Ch0). The desired pushout will follow once we establish the following three claims: (a) all outer faces W of V' are in X_C; (b) an inner face $V' - D$ of V is in X_C iff $D \notin \Xi_{V'}$; (c) the G-isotropy of inner faces $V' - D$ with $D \subseteq \Xi_{V'}$ (i.e. their isotropy in $\mathsf{Face}(T)$) is contained in H'.

To check (a), writing $\bar{W}$ for the corresponding outer face of U, one has

$$\Xi_W = \Xi \cap Inn(W) = \Xi \cap Inn(V') \cap Inn(\bar{W}) = \Xi \cap Inn(\bar{V}') \cap Inn(\bar{W}) = \Xi \cap Inn(\bar{W}) = \Xi_{\bar{W}}$$

where the second step follows from Lemma INNINT LEM 6.9 and the third since by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ it is $\Xi_{V'} = \Xi_{\bar{V}'}$. Thus either $\Xi_W = \Xi_{\bar{W}} = \varnothing$ so that $\bar{W} \in X$ by (Ch1) or $\Xi_W = \Xi_{\bar{W}} \neq \varnothing$ so that $W \in \mathsf{Face}^{lex}_{\Xi}(U)$ with $W < V'$, and thus $W \in C$. In either case one has $W \in X_C$.

We now check the "if" direction of (b). If $D \notin \Xi_{V'}$ then $V'' = V' - (D \smallsetminus \Xi_{V'})$ is in $\mathsf{Face}^{lex}_{\Xi}(U)$ (since $\bar{V}'' = \bar{V}'$ and $\Xi_{V''} = \Xi_{V'}$) and $V'' < V'$, and thus indeed $V'' \in X_C$.

For the "only if" direction of (b), the assumption $V' \notin X_C$ together with (Ch2) imply that $V'' = V' - \Xi_{V'}$ is not in X, and hence it remains to show that V'' is not a face of any $W \in C$. Suppose otherwise. Then since $V'' \hookrightarrow W$ it is $\bar{V}' = \bar{V}'' \hookrightarrow \bar{W}$ and by convexity of C' it must in fact be $\bar{V}' = \bar{W}$. But by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ the face W must contain as inner edges all edges in $\Xi_W = \Xi_{\bar{W}} = \Xi_{\bar{V}'} = \Xi_{V'}$, so that it is not only $V' - \Xi_{V'} = V'' \hookrightarrow W$ but also $V' \hookrightarrow W$. This contradicts the assumption $V' \notin X_C$.

We now show (c). Write $V'' = V' - D$ and suppose $gV'' = V''$ for some $g \in G$. Then $g\bar{V}'' = \bar{V}''$ or, equivalently, $g\bar{V}' = \bar{V}'$. Further, the G-equivariance of Ξ in (Ch0) guarantees $g\Xi_{\bar{V}'} = \Xi_{\bar{V}'}$ or, by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$, $g\Xi_{V'} = \Xi_{V'}$. Therefore, since $D \subseteq \Xi_{V'}$, the assumption $gV'' = V''$, i.e. $g(V' - D) = V' - D$, implies $g(V' - \Xi_{V'}) = V' - \Xi_{V'}$ and thus also $gV' = V'$. Since H' was defined as the G-isotropy of V', we obtain $g \in H'$, finishing the proof when $\{U_i\}$ is orbital.

We now turn to the case of a general $\{U_i\}_{i \in I}$. For the well-ordering of the cosets I/G given by (Ch3) and each convex subset J of I/G, set

$$X_J = X \cup \bigcup_{g \in G, [j] \in J} \Omega[gU_j].$$

As before, it suffices to check that for all convex subsets $J \subseteq J'$ the map $X_J \to X_{J'}$ is built cellularly from G-inner horns, and again we can assume that J' is obtained from J by adding a single $[j'] \in I/G$. By the $\{U_i\}$ orbital case, it now suffices to check that Ξ is also a characteristic inner edge set with respect to X_J and $\{U_{gj'}\}_{gj' \in [j']}$. But (Ch0) and (Ch3) for this new characteristic condition are obvious, the new (Ch1) condition is a particular case of the original (Ch1) condition, and the new (Ch2) condition readily follows from the original (Ch2) and (Ch3) conditions (in fact, one can check that the original (Ch3) condition is equivalent to demanding that, in the case $V \notin X$, the new (Ch2) condition holds for the *any possible choice* of J). ☐

HERE

Remark 6.14. Lemma CHAREDGE LEM 6.13 readily recovers several arguments in the literature:

(i) In Rez01 [Rez01, §10], Rezk introduces the notion of *covers*, which in our language are the subsets $Sc[n] \subseteq X \subseteq \Delta[n]$ such that if V is in X then so is the closure $\bar{V}^{[n]}$ (in words, X is generated by outer faces). Similarly, in the proof of CM13a [CM13, Prop. 2.4] Cisinski and Moerdijk use subcomplexes S_j that can be regarded as *dendroidal covers*, i.e. subcomplexes $Sc[T] \subseteq X \subseteq \Delta[T]$ such that if V is in X then so is $\bar{V}^{T}$. Lastly, the subcomplexes $\Omega[T] \cup_l \Omega[S] \subseteq \Omega[T \circ_l S]$ in the grafting result MW09 [MW09, Lemma 5.2] (and similarly for the equivariant analogue Per17 [Per17, Prop. 6.19]) are also dendroidal covers.

It follows from Lemma CHAREDGE LEM 6.13 that any inclusion $X \to X'$ of G-equivariant (dendroidal) covers of $T \in \Omega_G$ is G-inner anodyne. Indeed, letting $\Xi = Inn(T)$ and $\{U_i\}$ be the set of closed faces $U = \bar{U}^T$ in X', (Ch0) is obvious, (Ch1) follows since $Sc(T) \subseteq X$, (Ch2) follows since X is a cover and (Ch3') follows since the U_i are closed.

- Moerdijk Weiss
- Pereira

HERE

Proposition 6.15. *Inclusions of orbital G-inner horns*

$$\Lambda_o^{E \amalg F}[T] \to \Lambda_o^{E}[T]$$

{ORBHORNINC EQ}

are G-inner anodyne.

Proof. We are free to assume that $T \in \Omega^G \hookrightarrow \Omega_G$. Indeed, otherwise writing $T = G \cdot_H T_*$, where $T_* \in \Omega^H$ is a fixed component and $E_* = E \cap Inn(T_*)$, $F_* = F \cap Inn(T_*)$, the map ORBHORNINC EQ (6.15) is $G \cdot_H \left(\Lambda_o^{E_* \amalg F_*}[T_*] \to \Lambda_o^{E_*}[T_*]\right)$.

Further, we are clearly free to assume $F \neq \varnothing$. There are two main cases, $E = \varnothing$ and $E \neq \varnothing$.

If $E = \varnothing$ then $\Lambda_o^E[T] = \Omega[T]$ and we apply Lemma CHAREDGE LEM 6.13 with

$$\Xi = F, \qquad X = \Lambda_o^F[T], \qquad \{U_i\} = \{T\}.$$

It remains to check the characteristic conditions in Definition CHAREDGE DEF 6.11. (Ch0) and (Ch3) are clear.

For (Ch1), the condition $\Xi_V = \varnothing$ says that none of the inner edges of V are in F, and thus that the equivariant outer face GV contains none of the edge orbits in F. Since $F \neq \varnothing$, the equivariant outer face GV is not T itself, and hence $X = \Lambda_o^F[T]$ contains GV and thus also V.

For (Ch2), note first that V will be in $X = \Lambda_o^F[T]$ unless GV has the form $T - F'$ for some G-equivariant subset $F' \subseteq F$. But hence by Remark GINNER REM 6.8 it is $G(V - \Xi_V) = T - F''$ for $F' \subseteq F'' \subseteq F$ and thus $(V - \Xi_V)$ is not in $X = \Lambda_o^F[T]$.

In the case $E \neq \varnothing$ we instead apply Lemma CHAREDGE LEM 6.13 with

$$\Xi = E, \qquad X = \Lambda_o^{E \amalg F}[T], \qquad \{U_i\} = \{T - Gf\}_{Gf \subseteq F}.$$

Note that the U_i are the orbital inner faces $T - Gf$ for $Gf \subseteq F$, and thus the map in Lemma CHAREDGE LEM 6.13 is indeed ORBHORNINC EQ (6.15). We must again check the characteristic conditions. (Ch0) and (Ch3') are clear.

For (Ch1), note that for an outer face $V \hookrightarrow U_i$, and writing $\bar{V} = \bar{V}^T$, Lemma INNINT LEM 6.9 implies $Inn(V) = Inn(U_i) \cap Inn(\bar{V})$ and hence since $\Xi_{U_i} = \Xi$ the hypothesis $\Xi_V = \varnothing$ in (Ch1) implies it is also $\Xi_V = \varnothing$. Hence just as before $G\bar{V}$ is an equivariant outer face other than T, hence V is in $X = \Lambda_o^{E \amalg F}[T]$. The argument for (Ch2) is identical to the one in the case $E = \varnothing$. □

Proposition 6.16. *Inclusions of (regular) G-inner horns*

$$\Lambda^{E \amalg F}[T] \to \Lambda^{E}[T]$$

are G-inner anodyne.

Proof. The case $E = \varnothing$ is tautological.

HERE $\square$

A Equivariant Reedy model structures

Bla bla one of the axioms in [BM11] is different from the others point of view

In [BM11] Berger and Moerdijk extend the notion of Reedy category so as to allow for categories $\mathbb{R}$ with non-trivial automorphism groups $\mathsf{Aut}(r)$ for $r \in \mathbb{R}$. For such $\mathbb{R}$ and suitable model category $\mathcal{C}$ they then show that there is a *Reedy model structure* on $\mathcal{C}^{\mathbb{R}}$ that is defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see [BM11, Thm. 1.6] or Theorem A.7 below) to be determined by the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ for each $r \in \mathbb{R}$.

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ with the more general $\mathcal{C}^{\mathsf{Aut}(r)}_{\mathcal{F}_r}$ model structures for $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a nice collection of families of subgroups of each $\mathsf{Aut}(r)$.

To do so, we first need some essential notation. For each map $r \to r'$ in a category $\mathbb{R}$ we will write $\mathsf{Aut}(r \to r')$ for its automorphim group in the arrow category and write

$$\mathsf{Aut}(r) \xleftarrow{\pi_r} \mathsf{Aut}(r \to r') \xrightarrow{\pi_{r'}} \mathsf{Aut}(r')$$

{PIDEFR EQ}

for the obvious projections. We now introduce our equivariant generalization of the "generalized Reedy categories" of [BM11, Def. 1.1].

GENRED DEF

Definition A.1. A *generalized Reedy category structure* on a small category $\mathbb{R}$ consists of wide subcategories $\mathbb{R}^+$, $\mathbb{R}^-$ and a degree function $|-|\colon ob(\mathbb{R}) \to \mathbb{N}$ such that:

(i) non-invertible maps in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) raise (lower) degree; isomorphisms preserve degree;

(ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathsf{Iso}(\mathbb{R})$;

(iii) every map f in $\mathbb{R}$ factors as $f = f^+ \circ f^-$ with $f^+ \in \mathbb{R}^+$, $f^- \in \mathbb{R}^-$, and this factorization is unique up to isomorphism.

Let $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ be a collection of families of subgroups of the groups $\mathsf{Aut}(r)$. The collection $\{\mathcal{F}_r\}$ is called *Reedy-admissible* if:

(iv) for all maps $r \twoheadrightarrow r'$ in $\mathbb{R}^-$ one has $\pi_{r'}\left(\pi_r^{-1}(H)\right) \in \mathcal{F}_{r'}$ for all $H \in \mathcal{F}_r$.

We note that condition (iv) above should be thought as of a constraint on the pair $(\mathbb{R}, \{\mathcal{F}_r\})$. The original setup of [BM11] then deals with the case where $\{\mathcal{F}_r\} = \{\{e\}\}$ is the collection of trivial families. Indeed, our setup recovers the setup in [BM11], as follows.

Example A.2. When $\{\mathcal{F}_r\} = \{\{e\}\}$, Reedy-admissibility coincides with axiom (iv) in [BM11, Def. 1.1], stating that if $\theta \circ f^- = f^-$ for some $f^- \in \mathbb{R}^-$ and $\theta \in \mathsf{Iso}(\mathbb{R})$ then θ is an identity.

Example A.3. For any generalized Reedy category $\mathbb{R}$, the collection $\{\mathcal{F}_{\text{all}}\}$ of the families of all subgroups of $\mathsf{Aut}(r)$ is Reedy-admissible.

Example A.4. Let G be a group and set $\mathbb{R} = G \times (0 \to 1)$ with $\mathbb{R} = \mathbb{R}^+$. Then any pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible.

Similarly, set $\mathbb{S} = G \times (0 \leftarrow 1)$ with $\mathbb{S} = \mathbb{S}^-$. Then a pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible iff $\mathcal{F}_0 \supset \mathcal{F}_1$.

GGRAPHREEDY EX

Example A.5. Letting $\mathbb{S}$ denote any generalized Reedy category in the sense of [BM11, Def. 1.1] and G a group, we set $\mathbb{R} = G \times \mathbb{S}$ with $\mathbb{R}^+ = G \times \mathbb{S}^+$ and $\mathbb{R}^- = G \times \mathbb{S}^+$. Further, for each $s \in \mathbb{S}$ we write $\mathcal{F}^{\Gamma}_s$ for the family of G-graph subgroups of $G \times \mathsf{Aut}_{\mathbb{S}}(s)$, i.e., those subgroups $K \leq G \times \mathsf{Aut}_{\mathbb{S}}(s)$ such that $K \cap \mathsf{Aut}_{\mathbb{S}}(s) = \{e\}$.

Reedy admissibility of $\{\mathcal{F}^{\Gamma}_s\}$ follows since for every degeneracy map $s \twoheadrightarrow s'$ in $\mathbb{S}^-$ one has that the homomorphism $\pi_s \colon \mathsf{Aut}_{\mathbb{S}}(s \twoheadrightarrow s') \to \mathsf{Aut}_{\Omega}(s)$ is injective (we note that this is equivalent to axiom (iv) in [BM11, Def. 1.1] for $\mathbb{S}$).

Our primary example of interest will come by setting $\mathbb{S} = \Omega^{op}$ in the previous example. In fact, in this case we will also be interested in certain subfamilies $\{\mathcal{F}_U\}_{U\in\Omega} \subset \{\mathcal{F}^{\Gamma}_U\}_{U\in\Omega}$.

Example A.6. Let $\mathbb{R} = G \times \Omega^{op}$ and let $\{\mathcal{F}_U\}_{U\in\Omega}$ be the family of graph subgroups determined by a weak indexing system $\mathcal{F}$. Then $\{\mathcal{F}_U\}$ is Reedy-admissible. To see this, recall first that each $K \in \mathcal{F}_U$ encodes an H-action on $U \in \Omega$ for some $H \leq G$ so that $G \cdot_H U$ is a $\mathcal{F}$-tree. Given a face map $f \colon U' \hookrightarrow U$, the subgroup $\pi^{-1}_U(K)$ is then determined by the largest subgroup $\bar{H} \leq H$ such that U' inherits the $\bar{H}$-action from U along f (so that f becomes a $\bar{H}$-map), so that $\pi_{U'}(\pi^{-1}_U(K))$ encodes the $\bar{H}$-action on U'. Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of G-trees $G \cdot_{\bar{H}} U' \to G \cdot_H U$.

We now state the main result. We will assume throughout that $\mathcal{C}$ is a model category such that for any group G and family of subgroups $\mathcal{F}$, the category $\mathcal{C}^G$ admits the $\mathcal{F}$-model structure (for example, this is the case whenever $\mathcal{C}$ is a cofibrantly generated cellular model category in the sense of [Ste16]).

REEDYADM THM

Theorem A.7. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible collection of families. Then there is a $\{\mathcal{F}_r\}$-**Reedy model structure** on $\mathcal{C}^{\mathbb{R}}$ such that a map $A \to B$ is*

- *a (trivial) cofibration if $A_r \coprod_{L_r A} L_r B \to B_r$ is a (trivial) $\mathcal{F}_r$-cofibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a weak equivalence if $A_r \to B_r$ is a $\mathcal{F}_r$-weak equivalence in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a (trivial) fibration if $A_r \to B_r \underset{M_r B}{\times} M_r A$ is a (trivial) $\mathcal{F}_r$-fibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$.*

The proof of this result is given at the end of the section after establishing some routine generalizations of the key lemmas in [BM11] (indeed, the true novelty in this appendix is the Reedy-admissibility condition in part (iv) of Definition A.1).

We first recall the following, cf. [BP17, Props. 6.5 and 6.6] (we note that [BP17, Prop. 6.6] can be proven in terms of fibrations, and thus does not depend on special assumptions on $\mathcal{C}$).

Proposition A.8. *Let $\phi \colon G \to \bar{G}$ be a homomorphism and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of subgroups of $G, \bar{G}$. Then the leftmost (resp. rightmost) adjunction below is a Quillen adjunction*

$$\bar{G} \cdot_G (-) \colon \mathcal{C}^G_{\mathcal{F}} \rightleftarrows \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \colon \mathsf{res}^{\bar{G}}_G \qquad \mathsf{res}^{\bar{G}}_G \colon \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{C}^G_{\mathcal{F}} \colon \mathsf{Hom}_G(\bar{G}, -)$$

provided that for $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$ (resp. for $\bar{H} \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$).

RESGEN COR

Corollary A.9. *For any homomorphism $\phi \colon G \to \bar{G}$, the functor $\mathsf{res}^{\bar{G}}_G \colon \mathcal{C}^{\bar{G}} \to \mathcal{C}^G$ preserves all four classes of genuine cofibrations, trivial cofibrations, fibrations and trivial fibrations.*

The following formalizes an argument implicit in the proof of [BM11, Lemma 5.2]).

Definition A.10. Consider a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \tag{A.10}$$

{BLA EQ}

in $\mathcal{C}^{\mathbb{R}}$. A collection of maps $f_s \colon B_s \to X_s$ for $|s| \leq n$ that induce a lift of the restriction of (A.10) to $\mathcal{C}^{\mathbb{R}_{\leq n}}$ will be called a *n-partial lift*.

BLALIFT LEM

Lemma A.11. *Let $\mathcal{C}$ be any bicomplete category, and consider a commutative diagram as in (A.10). Then any $(n-1)$-partial lift uniquely induces commutative diagrams*

$$\begin{tikzcd} A_r \amalg_{L_r A} L_r B \arrow[r] \arrow[d] & X_r \arrow[d] \\ B_r \arrow[r] \arrow[ur, dashed] & Y_r \times_{M_r Y} M_r X \end{tikzcd} \qquad \{\text{BLALIFT EQ}\}$$

in $\mathcal{C}^{\mathsf{Aut}(r)}$ for each r such that $|r| = n$. Furthermore, extensions of the $(n-1)$-partial lift to a n-partial lift are in bijection with choices of $\mathsf{Aut}(r)$-equivariant lifts of the diagrams (A.11) for r ranging over representatives of the isomorphism classes of r with $|r| = n$.

In the next result, by $\{\mathcal{F}_r\}$-cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in Theorem A.7, regardless of whether such a model structure exists.

BLALIFT COR

Corollary A.12. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}$ an arbitrary family of subgroups of $\mathsf{Aut}(r)$, $r \in \mathbb{R}$. Then a map in $\mathcal{C}^{\mathbb{R}}$ is a $\{\mathcal{F}_r\}$-cofibration (resp. trivial cofibration) iff it has the left lifting property with respect to all $\{\mathcal{F}_r\}$-trivial fibrations (resp. fibrations), and vice-versa for the right lifting property.*

GINJ LEM

Lemma A.13. *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^+$, K a group, and $\pi\colon \mathbb{S} \to K$ a functor.*

Then if a map $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $A_s \amalg_{L_s A} L_s B \to B_s$ are (resp. trivial) $\mathsf{Aut}(s)$-cofibrations one has that $\mathsf{Lan}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-cofibration.

Proof. By adjunction, one needs only show that for any K-fibration $X \to Y$ in $\mathcal{C}^K$, the map $\pi^*(X \to Y)$ has the right lifting property against all maps $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ as in the statement. By Corollary A.12, it thus suffices to check that the maps

$$(\pi^* X)_s \to (\pi^* Y)_s \times_{M_s \pi^* Y} M_s \pi^* X$$

are $\mathsf{Aut}(s)$-fibrations. But since $M_s Z = *$ (recall $\mathbb{S} = \mathbb{S}^+$) this map is just $X \to Y$ with the $\mathsf{Aut}(s)$-action induced by $\pi\colon \mathsf{Aut}(s) \to K$, hence Corollary A.9 finishes the proof. ☐

GINJMIN LEM

Lemma A.14. *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^-$, K a group, and $\pi\colon \mathbb{S} \to K$ a functor.*

Then if a map $X \to Y$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $X_s \to Y_s \times_{M_s Y} M_s X$ are (resp. trivial) $\mathsf{Aut}(s)$-fibrations one has that $\mathsf{Ran}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-fibration.

Proof. This follows dually to the previous proof. ☐

Remark A.15. Lemmas A.13 and A.14 generalize key parts of the proofs of [BM11, Lemmas 5.3 and 5.5]. The duality of their proofs reflects the duality in Corollary A.9.

Remark A.16. Lemma A.13 will be applied when $K \leq \mathsf{Aut}_{\mathbb{R}}(r)$ and $\mathbb{S} = K \ltimes \mathbb{R}^+(r)$ for $\mathbb{R}$ a given generalized Reedy category and $r \in \mathbb{R}$. Similarly, Lemma A.14 will be applied when $\mathbb{S} = K \ltimes \mathbb{R}^-(r)$. It is straightforward to check that in the $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) case maps in $\mathbb{S}$ can be identified with squares as on the left (right)

$$\begin{tikzcd} r' \arrow[r, "+"] \arrow[d, "+"'] & r \arrow[d, "\simeq"] \\ r'' \arrow[r, "+"'] & r \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, "-"] \arrow[d, "\simeq"'] & r' \arrow[d, "-"] \\ r \arrow[r, "-"'] & r'' \end{tikzcd}$$

such that the maps labelled $+$ are in $\mathbb{R}^+$, maps labelled $-$ are in $\mathbb{R}^-$, the horizontal maps are non-invertible, and the maps labeled $\simeq$ are automorphisms in K.

In particular, there is thus a *domain* (resp. *target*) functor $d\colon \mathbb{S} \to \mathbb{R}$ ($t\colon \mathbb{S} \to \mathbb{R}$), and our interest is in maps $d^*A \to d^*B$ ($t^*A \to t^*B$) in $\mathcal{C}^{\mathbb{S}}$ induced from maps $A \to B$ in $\mathcal{C}^{\mathbb{R}}$ so that

$$\mathsf{Lan}_\pi d^*(A \to B) = (L_rA \to L_rB) \qquad \mathsf{Ran}_\pi t^*(A \to B) = (M_rA \to M_rB)$$

We are now in a position to prove the following, which are the essence of Theorem A.7.

REEDYTRCOF LEM

Lemma A.17. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Suppose $A \to B$ be a $\{\mathcal{F}_r\}$-Reedy cofibration. Then the maps $A_r \to B_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $A_r \amalg_{L_rA} L_rB \to B_r$.

Proof. It suffices to check by induction on n that the analogous claim with the restriction $|r| \leq n$ also holds. The $n = 0$ case is obvious. Otherwise, letting r range over representatives of the isomorphism classes of r with $|r| = n$, it suffices to check that for each $H \in \mathcal{F}_r$ the map $A_r \to B_r$ is a H-genuine weak equivalence iff so is $A_r \amalg_{L_rA} L_rB \to B_r$.

One now applies Lemma A.13 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^+(r)$ to the map $d^*A \to d^*B$. Note that $\mathcal{F}$-trivial cofibrations are always genuine trivial cofibrations, for any family, so that the trivial cofibrancy requirements are immediate from Corollary A.9. It thus follows that the maps labelled ~

$$\begin{tikzcd}
L_rA \arrow[r, "\sim"] \arrow[d] & L_rB \arrow[d] & \\
A_r \arrow[r, "\sim"'] & L_TB \amalg_{L_TA} A_T \arrow[r] & B_r
\end{tikzcd}$$

are H-genuine trivial cofibrations, finishing the proof. □

REEDYTRFIB LEM

Lemma A.18. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Let $X \to Y$ be a $\{\mathcal{F}_r\}$-Reedy fibration. Then the maps $X_r \to Y_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $X_r \to Y_r \times_{M_rY} M_rX$.

Proof. One repeats the same induction argument on $|r|$. In the induction step, it suffices to verify that, for each r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $X_r \to Y_r$ is a H-genuine weak equivalence iff so is $X_r \to Y_r \times_{M_rY} M_rX$.

One now applies Lemma A.14 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^-(r)$ to the map $t^*A \to t^*B$. Note that for each $(r \twoheadrightarrow r') \in \mathbb{S}$ one has $\mathsf{Aut}_{\mathbb{S}}(r \twoheadrightarrow r') = \pi_r^{-1}(H)$ (where π_r is as in (A)), so that the trivial fibrancy requirement in Lemma A.14 follows from $\{\mathcal{F}_r\}$ being Reedy-admissible. It follows that the maps labelled ~

$$\begin{tikzcd}
X_r \arrow[r] & Y_r \times_{M_rY} M_rX \arrow[r, "\sim"] \arrow[d] & Y_r \arrow[d] \\
& M_rX \arrow[r, "\sim"'] & M_rY
\end{tikzcd}$$

are H-genuine trivial fibrations, finishing the proof. □

Remark A.19. The proofs of Lemmas A.17 and A.18 are similar, but not dual, since Lemma A.18 uses Reedy-admissibility while Lemma A.17 does not. This reflects the difference in the proofs of [BM11, Lemmas 5.3 and 5.5] as discussed in [BM11, Remark 5.6], albeit with a caveat.

Setting $K = \{e\}$ in Lemma A.13 yields that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a genuine Reedy cofibration, i.e. a Reedy cofibration for $\{\mathcal{F}_{\text{all}}\}$ the families of all subgroups. On the other hand, the proof of [BM11, Lemma 5.3] argues that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a projective Reedy cofibration, i.e. a Reedy cofibration for $\{\{e\}\}$ the trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim is more general). Since the cofibration half of the projective analogue of Corollary A.9 only holds if ϕ is a monormorphism, the argument in the proof of [BM11, Lemma 5.3] also includes an injectivity check that is not needed for our proof of Lemma A.17.

proof of Theorem A.7. Lemmas A.17 and A.18 say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem A.7 are correct, i.e. that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [Hov99, Def. 1.1.3]. Both 2-out-of-3 and the retract axioms are immediate (recall that retracts commute with limits/colimits). The lifting axiom follows from Corollary A.12 while the task of building factorizations $X \to A \to Y$ of a given map $X \to Y$ follows by a similar standard argument by iteratively factorizing the maps

$$X_r \amalg_{L_r X} L_r A \to Y_r \times_{M_r Y} M_r A$$

in $\mathcal{C}^{\mathsf{Aut}(r)}$, thus building both A and the factorization inductively (see, e.g., the proof of [BM11, Thm. 1.6]). □

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