Equivariant dendroidal Segal spaces and G-∞-operads

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Abstract

bla bla, generalizing [CM13a].

In an appendix, we discuss Reedy categories in the equivariant context.

Contents

1 Introduction

This paper follows [Per17] and [BP17] and is the third piece of a larger project aimed at understanding the homotopy theory of *equivariant operads with norm maps*. Informally, norm maps are a new piece of structure that must be considered when dealing with equivariant operads (and which has no analogue in the theory of equivariant categories). The need to understand norm maps, as well as their usefulness, was made clear by Hill, Hopkins and Ravenel in their solution of the Kervaire invariant one problem [HHR16].

The starting point of this project was the discovery by the authors, for each finite group G, of a category Ω_G of G-trees whose objects diagrammatically encode compositions of norms maps and whose arrows encode the necessary compatibilities between such compositions. Our categories Ω_G are a somewhat non-obvious equivariant generalization of the dendroidal category Ω of Cisinski-Moerdijk-Weiss, and indeed all the key combinatorial concepts in their work, such as faces, degeneracies, boundaries and horns, generalize to G-trees [Per17, §5,6]. As such, it is natural to ask whether the Cisinski-Moerdijk program [CM11],[CM13a],[CM13b] can also be generalized to the equivariant context.

We recall that the main result of their program is the existence of a Quillen equivalence

$$W_! \colon \mathsf{dSet} \rightleftarrows \mathsf{sOp} \colon N_{hc}$$

where $\mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ is the category of presheaves over Ω, called *dendroidal sets*, and sOp is the category of simplicial colored operads. Their program was carried out in three main steps: (i) [CM11] established the existence of the model structure on dSet (with some of the key combinatorial analysis based on Moerdijk and Weiss' work in [MW09]); (ii) [CM13a] established auxiliary model structures on the categories sdSet and PreOp of dendroidal spaces and pre-operads, and showed that all three of dSet, sdSet and PreOp are Quillen equivalent; (iii) lastly, [CM13b] established the existence of the model structure on sOp as well as the Quillen equivalence between sOp and PreOp, finishing the proof of the main result.

From the perspective of the Cisinski-Moerdijk program, [Per17] is then the equivariant analogue of the first step [CM11] (as well as [MW09]), while the present paper provides the equivariant analogue of the second step [CM13b]. More explicitly, in [Per17], and inspired by the category Ω_G of G-trees, the second author equipped the category dSet^G of G-equivariant dendroidal sets with a model structure whose fibrant objects are "equivariant operads with norm maps up to homotopy", called G-∞-operads. In the present paper, our main results are then the existence of suitable model structures on the categories sdSet^G and PreOp^G of G-dendroidal spaces and G-pre-operads, as well as the existence of Quillen equivalences between all three of dSet^G, sdSet^G and PreOp^G.

It is worth noting that, much as was the case with the work in [Per17], our results are not formal consequences of their non-equivariant analogues, due to the nature of norm maps[1]. Indeed, in [BP17], the second piece of our project, the authors introduced the notion of *genuine equivariant operads*, which are new algebraic objects motivated by the combinatorics of norm maps as encoded by the category Ω_G of G-trees. And while a priori the work in [BP17] is largely perpendicular to the Cisinski-Moerdijk program (the main result [BP17, Thm. III] is what one might call the "operadic Elmendorf-Piacenza theorem", which is an equivariant phenomenon), many of the new technical hurdles in this paper versus [CM13a] can be traced back to the fact that at many points in the discussion we are secretly dealing with colored genuine equivariant

[1] Recall that by using the inclusions of simplicial categories and simplicial sets into simplicial operads into dendroidal sets, the Cisinski-Moerdijk program recovers the Bergner-Joyal-Lurie-Rezk-Tierney program studying ∞-categories. As a point of contrast, we note the lack of norms in the categorical case causes the equivariant generalization of this latter program to indeed be formal.

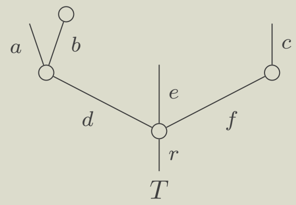
operads, which are the colored generalization of the structures discussed in [BP17], and the formal definition of which we prefer to postpone to a follow-up paper.

The organization of the paper is as follows.
Fill this

2 Preliminaries

2.1 The category of trees Ω

We start by recalling the key features of the category Ω of trees that will be used throughout. Our official model for Ω will Weiss' algebraic model of *broad posets* as discussed in [Per17, §5], hence we first recall some key notation and terminology. Given a tree diagram T such as



(2.1) FIRSTTREE EQ

and for each edge t of T topped by a vertex $\circ$, we write $t^{\uparrow}$ to denote the tuple of edges immediately above t. In our example, $r^{\uparrow} = def$, $d^{\uparrow} = ab$, $f^{\uparrow} = c$ and $b^{\uparrow} = \epsilon$, where ϵ is the empty tuple. Edges t for which: (i) $t^{\uparrow} \neq \epsilon$, such as r, d, f, are called *nodes*; (ii) $t^{\uparrow} = \epsilon$, such as b, are called *stumps*; (iii) $t^{\uparrow}$ is undefined, such as a, c, e, are called *leaves*. The vertices of T are then encoded symbolically as $t^{\uparrow} \leq t$, which we call a *generating broad relation*. This notation is meant to suggest a form of transitivity: for example, the generating relations $ab \leq d$ and $def \leq r$ generate, via *broad transitivity*, a relation $abef \leq r$ (we note that this is essentially compact notation for the operations and composition in the colored operad generated by T [MW07, §3]). The other broad relations obtained by broad transitivity are $dec \leq r$, $abec \leq r$, $aec \leq r$, $a \leq d$. The set of edges of T together with these broad relations (as well as identity relations $t \leq t$) form the *broad poset* associated to the tree, which is again denoted T.

Given a broad relation $t_0 \cdots t_n \leq t$, we further write $t_i \leq_d t$. Pictorially, this says that the edge t_i is above t, and it is thus clear that $\leq_d$ defines a partial order on edges of T. Trees always have a single $\leq_d$-maximal edge, called the *root*. Edges other than the root or the leaves are called *inner edges*. In our example r is the root, b, d, f are inner edges and a, e, c are leaves.

We denote the sets of edges (inner edges, vertices) of T by $E(T)$ (resp. $E^{\mathrm{i}}(T)$, $V(T)$).

The Cisinski-Moerdijk-Weiss category Ω of trees then has as objects tree diagrams as in (2.1) and as maps $\varphi : T \to S$ the monotone maps of broad posets (meaning that if $t_1 \cdots t_k \leq t$ then $\varphi(t_1) \cdots \varphi(t_k) \leq \varphi(t)$). In fact, Weiss further identified axioms characterizing those broad posets that are associated to trees (see [Per17, Defs. 5.1 and 5.9]).

Further, our discussion will be somewhat simplified by the assumption that Ω contains exactly one representative of each planarized tree. Informally, this means that trees $T \in \Omega$ come with a preferred planar representation, though this can also be formalized in purely algebraic terms, see [BP17, §3.1]. For our purposes, the main consequence is that any map $S \to T$ in Ω has a (strictly) unique factorization $S \simeq S' \to T$ as an isomorphism followed by a *planar map* [BP17, Prop. 3.21]. Roughly speaking, S' is obtained from S by pulling back the planarization of T.

We now recall the key classes of maps of Ω. A map $\varphi : S \to T$ which is injective on edges is called a *face map* while a map that is surjective on edges and preserves leaves is called a *degeneracy map* (the extra requirement ensures that leaves of S do not become stumps of T). Moreover, a face map is further called an *inner face map* if $\varphi(r_S) = r_T$ and $\varphi(\underline{l}_S) = \underline{l}_T$ (where $r_{(-)}$ denotes the root edge and $\underline{l}_{(-)}$ the leaf tuple) and called an *outer face map* if it does not factor through any non-identity inner face maps. The following result is [Per17, Cor. 3.32].

UNIQUEFACT PROP

Proposition 2.2. *A map $\varphi\colon S \to T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^o} T$$

as a degeneracy followed by an inner face map followed by an outer face map.

We now recall a more explicit characterization (and notation) for planar inner/outer faces (planar degeneracies are characterized by edge multiplicities, see [BP17, Prop. 3.47(ii)]). For any subset $D \subseteq E(T)$, there is a planar inner face $T - D$ which removes the inner edges in E but keeps all broad relations involving edges not in D (this is the hardest class of maps to visualize pictorially, as the vertices adjacent to each $d \in D$ are combined via broad transitivity/composition). For each broad relation $t_1 \cdots t_k = \underline{t} \leq t$ in T, there is a planar outer face $T_{\underline{t} \leq t}$ such that $r_{T_{\underline{t}\leq t}} = t$ and $\underline{l}_{T_{\underline{t}\leq t}} = \underline{t}$ (in fact, by Proposition 2.2 this is the maximal such face). Moreover, the edges s of $T_{\underline{t}\leq t}$ are the edges of T such that $s \leq_d t$ and $\forall_i s \nleq t_i$ while the vertices are the $s^\uparrow \leq s$ such that $s \leq_d t$ and $\forall_i s \nleq t_i$ (pictorially, $T_{\underline{t}\leq t}$ removes the parts of T not above t and above some t_i).

INNFULL REM

Remark 2.3. Inner faces $T - D \hookrightarrow T$ are always full, i.e. $T - D$ contains all broad relations of T whose edges are in $T - D$. By contrast, whenever T has stumps some of its outer faces $T_{\underline{t}\leq t}$ are not full, the main example being the maximal outer faces "removing stumps" [Per17, Not. 5.41].

DEGREE REM

Remark 2.4. Following [BM11, Ex. 2.8], one has a degree function $|-|\colon \Omega \to \mathbb{N}$ given by $|T| = |V(T)|$ such that non isomorphim face maps (resp. degeneracies) strictly increase (decrease) $|-|$. The category of face maps is thus denoted Ω^+ and that of degeneracies is denoted Ω^-.

We now collect a couple of useful lemmas concerning faces.

INNINT LEM

Lemma 2.5. *Consider a diagram of planar faces in Ω (implicitly regarded as inclusion maps)*

$$\begin{tikzcd} V \arrow[r, hook, "out"] \arrow[d, hook, "inn"'] & U \arrow[d, hook] \\ \bar{V} \arrow[r, hook, "out"] & \bar{U} \end{tikzcd}$$

such that the horizontal maps are outer face maps and the left vertical map is an inner face map. Then $E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(\bar{V})$.

Proof. Write r and $\underline{l} = l_1 \cdots l_n$ for the root and leaf tuple of V, or equivalently $\bar{V}$. Since the horizontal maps are outer, an edge $e \in E^{\mathsf{i}}(U)$ (resp. $e \in E^{\mathsf{i}}(\bar{U})$) is also in $E^{\mathsf{i}}(V)$ (resp. in $E^{\mathsf{i}}(\bar{V})$) iff $e <_d r$ and $\forall_i e \nleq l_i$. But then $E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(\bar{V})$. □

CUPCAP LEM

Lemma 2.6. *Let $\{U_i \hookrightarrow T\}$ be a collection of planar outer faces of T with a common root t. Then there are planar outer faces $U^\cup \hookrightarrow T$, $U^\cap \hookrightarrow T$, also with root t, such that*

$$E(U^\cup) = \bigcup_i E(U_i), \quad V(U^\cup) = \bigcup_i V(U_i), \qquad E(U^\cap) = \bigcap_i E(U_i), \quad V(U^\cap) = \bigcap_i V(U_i). \tag{2.7}$$

CUPCAP EQ

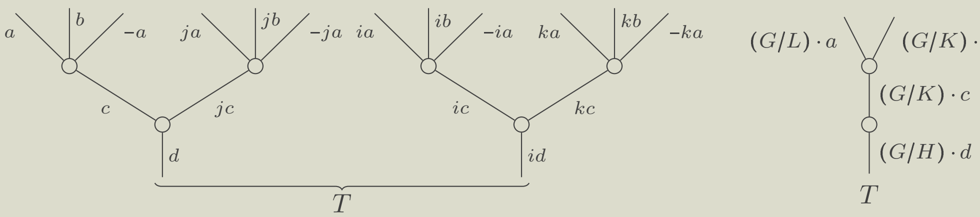
Moreover, these are the smallest (resp. largest) outer faces containing (contained in) all U_i.

Remark 2.8. One can check that it actually suffices to assume the U_i have a common edge.

Proof. (2.7) determines pre-broad posets (cf. [Per17, Rem. 5.2]) $U^\cup$ and $U^\cap$, hence we need only verify the axioms in [Per17, Defs. 5.1, 5.3, 5.9]. Antisymmetry and simplicity are inherited from T, the nodal axiom is obvious from (2.7), and the root axiom follows since the U_i have a common root (in $U^\cap$ case note that if s is in $U^\cap$, then so is any s' such that $s \leq_d s' \leq_d t$). □

2.2 The category of G-trees Ω_G

We next recall the category Ω_G of G-trees first defined in [Per17, §5.3]. We start with an explicit and representative example of a G-tree (for more examples, see [Per17, §4.3]). Letting $G = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternionic units and $G \geq H \geq K \geq L$ denote the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$, there is a G-tree T with *expanded representation* given by the two trees on the left below and *orbital representation* given by the (single) tree on the right.



(2.9) TWOREP EQ

Note that the edge labels on the expanded representation encode the action of G so that the edges a, b, c, d have stabilizers L, K, K, H.

The formal definition of Ω_G [Per17, Def. 5.44] is as follows. Firstly, given a non-equivariant forest diagram F (i.e. a finite collection of tree diagrams side by side), one can obtain an associated broad poset just as before, and thus a category Φ of forests. Letting Φ^G, the category of G-forests, denote G-objects on Φ, the category $\Omega_G \subset \Phi^G$ of G-trees is the full subcategory of those G-forests such that the G-action is transitive on tree components. We note that any G-tree T can then be written as $G \cdot_H T_*$, where T_* is some fixed tree component, $H \leq G$ is the subgroup sending that component to itself, and we regard $T_* \in \Omega^H$, i.e., as a tree with a H-action (where we caution that $\Omega^G \subsetneq \Omega_G$).

We note that we also assume G-trees (and forests in general) are planarized, meaning that they come with a total order of the tree components, which are themselves planarized.

Before discussing face maps in the equivariant context, it is worth commenting on the complementary roles of the expanded and orbital representations. On the one hand, the broad posets associated to G-trees are diagrammatically represented by the expanded representation, so that the arrows of Ω_G are best understood from that perspective. On the other hand, the diagrams encoding compositions of norm maps of an equivariant operad $\mathcal{O}$ are given by the orbital representations of G-trees (see [Per17, Ex. 4.9], [BP17, (1.10)]). As a result, different aspects of our discussion will be guided by different representations, and this will require us to discuss the different notions of face/boundary/horn suggested by the two representations. We start by recalling the notion of face discussed in [Per17], which is motivated by the expanded representation.

Definition 2.10. Let $T \in \Omega_G$ be a G-tree with non-equivariant tree components $T_1, T_2, \cdots, T_k$.

A *face* of T is an underlying face map $U \hookrightarrow T_i$ in Ω for some $1 \leq i \leq k$. Further, we abbreviate faces of T as $U \hookrightarrow T$, and call them *planar/outer* faces whenever so is the map $U \hookrightarrow T_i$.

Notation 2.11. Given $T \in \Omega_G$, we write $\mathsf{Face}(T)$ for the G-poset of *planar faces* $U \hookrightarrow T$. We note that the G-action is given by the unique factorization of the composite $U \hookrightarrow T \xrightarrow{g} T$ as $U \simeq gU \hookrightarrow T$ such that $gU \hookrightarrow T$ is planar.

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\simeq"'] & T \arrow[d, "g"] \\ gU \arrow[r, hook] & T \end{tikzcd} \tag{2.12}$$

FACEGACT EQ

Notation 2.13. Given $T \in \Omega_G$ and a planar face $U \hookrightarrow T$ we write $\bar{U}^T$, or just $\bar{U}$ when no confusion should arise, for the *outer closure of* U, i.e. the smallest planar outer face of T containing U.

PLFUNCTOR REM

Remark 2.14. Recalling that notation $\Omega^+ \subset \Omega$ (non-equivariant) subcategory of face maps, we write $\Omega^+ \downarrow T$ for the category of all faces of $T \in \Omega_G$. By pulling back the planarization of T one then obtains a *planarization functor*

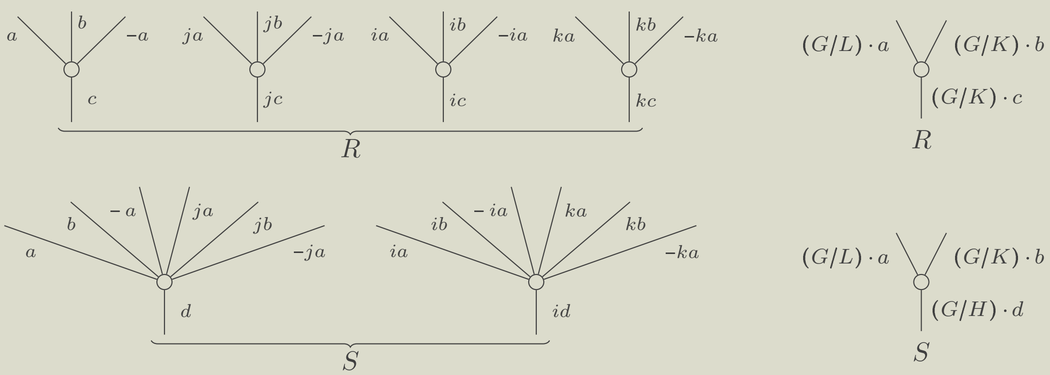
$$\Omega^+ \downarrow T \xrightarrow{pl} \mathsf{Face}(T)$$

which respects the G-actions on the two categories. Note, however, that the inclusion $\mathsf{Face}(T) \subset \Omega^+ \downarrow T$ (which is a section of pl) does not respect the G-actions, as displayed in (2.12).

We now introduce the notion of face suggested by the orbital representation.

Definition 2.15. Let $T \in \Omega_G$. An *orbital face* of T is a map $S \hookrightarrow T$ in Ω_G which is injective on edges. Further, an orbital face is called *planar/inner/outer* if any (and thus all) of its component maps is.

Example 2.16. The following are two planar orbital faces of the G-tree T in (2.9), with $R \hookrightarrow T$ an orbital outer face and $S \hookrightarrow T$ an orbital inner face.



These examples illustrate our motivation for the term "orbital face": the tree diagrams in the orbital representations of R, S look like faces of the tree in the orbital representation of T.

Adapting the notation for (non-equivariant) inner faces, we write $S = T - Gc = T - \{c, jc, ic, kc\}$ and analogously throughout the paper. We will need no analogous notation for orbital outer faces.

TREEDIFNOT NOT

Notation 2.17. In the remainder of the paper we sometimes need to consider (non-equivariant) and orbital faces simultaneously. As such, we reserve the letters U, V, W for trees in Ω and the letters R, S, T for G-trees in Ω_G.

INNOUTORB REM

Remark 2.18. It follows from Proposition 2.2 that any orbital face $S \hookrightarrow T$ has a factorization $S \hookrightarrow R \hookrightarrow T$, unique up to isomorphism, as an orbital inner face followed by an orbital outer face.

MINGFACT PROP

Proposition 2.19. *Let $T \in \Omega_G$. Any (non-equivariant) planar face $U \hookrightarrow T$ has a minimal factorization $U \hookrightarrow GU \hookrightarrow T$ through a planar orbital face GU.*

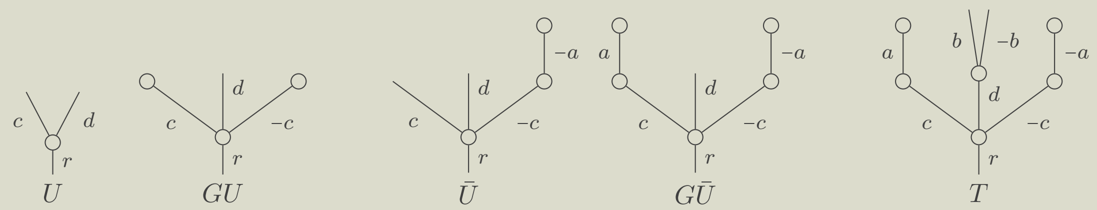
Proof. Assume first that $U = \bar{U}^T$ is outer and write $H \leq G$ for the isotropy of its root r_U. By Lemma 2.6 there exists a smallest outer face containing all $\{hU \hookrightarrow T\}_{h \in H}$, which we denote by HU. Moreover, HU inherits the H-action from T (by either its construction or its characterization). Moreover, the natural map $G \cdot_H HU \to T$ is then injective on edges (for a map of forests

$F \to F'$ the images of the tree components of F are pairwise $\leq_d$-incomparable iff so are the images of the roots) and we thus let GU be $G \cdot_H HU$ with the planar structure induced from T. Both the factorization $U \to GU \to T$ and its minimality are immediate from the description of HU.

Before tackling the general case, we collect some key observations. Firstly, if U is outer then so is the (non-equivariant) face HU and the orbital face GU. Secondly, the root tuple of GU is $G \cdot_H r_U$. Lastly, we need to characterize the leaf tuple of GU. We call a leaf l of U *orbital* if all the edges in $Hl \cap E(U)$ are leaves of U, and claim that the leaves of GU are the tuple $\underline{l}$ formed by the G-orbits of the orbital leaves of U. Indeed, a leaf l of U is also a leaf of HU iff $\forall_{h \in H}$ ($l \in E(hU)$ implies that l is a leaf of hU) iff $\forall_{h \in H}$ ($h^{-1} l \in E(U)$ implies that l is a leaf of U).

In the general case, we define GU as the orbital inner face of $G\bar{U}$ that removes all edge orbits not represented in U (that all such edge orbits are inner follows from the description of the roots and leaves of $G\bar{U}$ in the previous paragraph). It is now clear that $U \to G\bar{U} \to T$ is the minimal factorization with $G\bar{U}$ an *outer* orbital face, and thus the factorization $U \to GU \to T$ exists and is minimal since inner faces are full (Remark 2.3) together with the inner-outer factorization of orbital faces (Remark 2.18). □

Example 2.20. Much of the complexity in the previous proof is needed to handle the scenario of non outer faces $U \hookrightarrow T$ of G-trees T which have stumps, which is easily the subtlest case, as illustrated by the following example (where $G = \mathbb{Z}_{/2} = \{\pm 1\}$).



GINNER REM **Remark 2.21.** For any inner face $V - e$ of V one has that $G(V-e)$ is either $GV - Ge$ or GV. Indeed, the latter will happen iff $V - e$ contains either an inner edge of a leaf of the form ge.

Remark 2.22. Writing $\mathsf{Face}_o(T)$ for the poset of planar orbital faces, Proposition 2.19 gives a G-equivariant functor (note that G does not act on $\mathsf{Face}_o(T)$)

$$\mathsf{Face}(T) \xrightarrow{G(-)} \mathsf{Face}_o(T).$$

Moreover, there is a natural inclusion $\mathsf{Face}_o(T) \subseteq \mathsf{Face}(T)/G$ (sending an orbital face S to the class of components $[S_*]$) whose left adjoint is the induced functor $\mathsf{Face}(T)/G \to \mathsf{Face}_o(T)$.

2.3 Equivariant dendroidal sets

Recall [Per17, §5.4] that the category of *G-equivariant dendroidal sets* is the presheaf category $\mathsf{dSet}^G = \mathsf{Set}^{\Omega^{op} \times G}$. Given $T \in \Omega_G$ with non-equivariant tree components $T_1, \cdots, T_k$, we extend the usual notation for representable functors to obtain $\Omega[T] \in \mathsf{dSet}^G$ via

$$\Omega[T] = \Omega[T_1] \amalg \cdots \amalg \Omega[T_k]$$

regarded as a G-object in dSet. One further defines *boundaries* (in the union formula, the injection $\Omega[U] \to \Omega[T]$ is regarded as an inclusion; the equivalence between the colimit and union formulas follows from Proposition 2.2)

$$\partial\Omega[T] = \operatorname*{colim}_{U \in \mathsf{Face}(T), U \neq T_i} \Omega[U] = \bigcup_{U \in \mathsf{Face}(T), U \neq T_i} \Omega[U]$$

and, for $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$ a non-empty G-subset of inner edges (we abbreviate $E_i = E \cap E^{\mathsf{i}}(T_i)$), *$G$-inner horns*

$$\Lambda^E[T] = \underset{U \in \mathsf{Face}(T), (T_i - E_i) \not\hookrightarrow U}{\operatorname{colim}} \Omega[U] = \bigcup_{U \in \mathsf{Face}(T), (T_i - E_i) \not\hookrightarrow U} \Omega[U]$$

which, informally, are the subcomplexes of $\Omega[T]$ that remove the inner faces $T_i - D$ for $D \subseteq E_i$.

Lastly, letting $\mathsf{Face}_{sc}(T)$ denote those outer faces of T with no inner vertices (these are either single edges t or generated by single vertices $t^{\uparrow} \leq t$), we define the *Segal core of T*

$$Sc[T] = \underset{U \in \mathsf{Face}_{\mathsf{sc}}(T)}{\operatorname{colim}} \Omega[U] = \bigcup_{U \in \mathsf{Face}_{sc}(T)} \Omega[U].$$

Note that if $T \simeq G \cdot_H T_*$ for some $T_* \in \Omega^H$ then

$$\Omega[T] \simeq G \cdot_H \Omega[T_*], \quad \partial\Omega[T] \simeq G \cdot_H \partial\Omega[T_*], \quad \Lambda^E[T] \simeq G \cdot_H \Lambda^{E_*}[T_*], \quad Sc[T] \simeq G \cdot_H Sc[T_*].$$

As a cautionary note, we point out that though representable functors $\Omega[T]$ are defined for $T \in \Omega_G$, evaluations $X(U)$ of $X \in \mathsf{dSet}^G$ are defined only for $U \in \Omega$ (cf. Notation 2.17).

Remark 2.23. For $T \in \Omega_G$, a planar face $\varphi_U : U \to T$ can also be regarded as a dendrex $\varphi_U \in \Omega[T](U)$. However, the G-isotropy H of $U \in \mathsf{Face}(T)$ must not be confused with the G-isotropy of φ_U. Instead, $\Omega[T](U)$ has a larger $G \times \mathsf{Aut}(U)$-action, and the $G \times \mathsf{Aut}(U)$-isotropy of φ_U is a subgroup $\Gamma \leq G \times \mathsf{Aut}(U)$ which is the graph of a homomorphism $\phi : H \to \mathsf{Aut}(U)$. One readily checks that if $hU = U$ in $\mathsf{Face}(T)$ then $\phi(h)$ is the left isomorphism in (2.12), so that $U \in \Omega$ is equipped with a canonical H-action. We abuse notation by writing $U \in \Omega^H \subseteq \Omega_H$ to denote this.

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts.

The saturation of the boundary inclusions $\partial\Omega[T] \to \Omega[T]$ is the class of *G-normal monormorphisms*, i.e. those monomorphisms $X \to Y$ in dSet^G such that $Y(U) \smallsetminus X(U)$ has an $\mathsf{Aut}(U)$-free action for all $U \in \Omega$. Moreover, since this condition is actually independent of the G-action, we will usually call these simply *normal monomorphisms*.

The saturation of the G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ is called the class of *G-inner anodyne maps*, while those $X \in \mathsf{dSet}^G$ with the right lifting property against all G-inner horn inclusions are called *G-∞-operads*.

We can now recall [Per17, Thm 2.1], which was the main result therein.

Theorem 2.24. *There is a model structure on $\mathsf{dSet}^{\mathsf{G}}$ such that the cofibrations are the normal monomorphisms and the fibrant objects are the G-∞-operads.*

Remark 2.25. The definition G-∞-operads just given is a priori distinct from the original definition [Per17, Def. 6.12] which used only *generating G-inner horn inclusion*, i.e. those inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ with $E = Ge$ an inner edge orbit. The present definition has the technical advantages of being naturally compatible with restricting the G-action and of allowing for a simpler proof of Lemma 3.4, which is our main tool for showing that maps are G-inner anodyne. The equivalence between the two definitions follows from [Per17, Prop. 6.17], although we also independently recover this from Lemma 3.4 in Corollary 3.17.

In additional to the G-inner horns defined before, we now introduce a new kind of horn that, much like orbital faces, is naturally suggested by the orbital representation of G-trees. Given $E \subseteq E^{\mathsf{i}}(T)$ a G-equivariant set of inner edges, we define the associated *orbital G-inner horn* by

$$\Lambda_o^E[T] = \underset{S \in \mathsf{Face}_o(T), (T-E) \not\hookrightarrow S}{\operatorname{colim}} \Omega[S] = \bigcup_{S \in \mathsf{Face}_o(T), (T-E) \not\hookrightarrow S} \Omega[S]$$

where we note that the equivalence between the colimit and union formulas now follows from Proposition 2.19.

3 Equivariant inner anodyne maps

Much as in [CM13a, §2], it is essential for us to show that the inclusions $Sc[T] \to \Omega[T]$, $T \in \Omega_G$ are G-inner anodyne. In addition, some parts of the equivariant dendroidal story are naturally described in terms of orbital G-inner horns $\Lambda^E_o[T]$ (rather than G-inner horns $\Lambda^E[T]$), and one must hence also show that the inclusions $\Lambda^E_o[T] \to \Omega[T]$ are G-inner anodyne.

In practice, the proofs of such results are long and somewhat repetitive, as they share many technical arguments. Indeed, the case of orbital horns requires using many of the arguments in the long proof of [Per17, Thm 7.1]).

As such, we split our analysis into two parts. In §3.1 we prove Lemma 3.4 which we call the *characteristic edge lemma* and which abstractly identifies sufficient conditions for a map to be G-inner anodyne (see Remark 3.7 for a comparison with previous results in the literature). Then, in §3.2 we deduce that the desired maps are G-inner anodyne by applying Lemma 3.4.

3.1 The characteristic edge lemma

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CHAREDGE DEF **Definition 3.1.** Let $T \in \Omega_G$, $X \subseteq \Omega[T]$ a subdendroidal set, and $\{U_i\}_{i\in I} \subseteq \mathsf{Face}(T)$ a subset.

Given a set Ξ^i of inner edges of U_i and a subface $V \hookrightarrow U_i$, denote $\Xi^i_V = \Xi^i \cap E^{\mathsf{i}}(V)$.

Suppose further that the indexing set I is a finite G-poset. For each $i \in I$ denote

$$X_{<i} = X \cup \bigcup_{j:j<i} \Omega[U_j]$$

We say that $\{\Xi^i \subseteq E^{\mathsf{i}}(U_i)\}$ is a *characteristic inner edge collection of* $\{U_i\}$ *with respect to* X if:

(Ch0) X, $\{U_i\}$ and $\{\Xi^i\}$ are all G-equivariant, i.e. $gX = X$, $gU_i = U_{gi}$, $g\Xi^i = \Xi^{gi}$ as appropriate;

(Ch1) for all i, any outer face $V = \bar{V}^{U_i}$ of U_i such that $\Xi^i_V = \varnothing$ is contained in $X_{<i}$;

(Ch2) for all i, any face $V \hookrightarrow U_i$ such that $(V - \Xi^i_V) \in X$ is contained in $X_{<i}$;

(Ch3) for all $j \not\geq i$, all faces $V \hookrightarrow U_i$ such that $(V - \Xi^i_V) \hookrightarrow U_j$ are contained in $X_{<i}$.

XIIII REM **Remark 3.2.** If $gi \neq i$, then i, gi are incomparable in I. Indeed, otherwise $i < gi < g^2 i < g^3 i < \cdots$ would violate antisymmetry. Therefore, (Ch3) applies whenever $j = gi$ for $gi \neq i$.

In particular, we assume throughout that if $gi \neq i$ then $U_{gi} \neq U_i$, or else it would be $U_i \in X_{<i}$.

SOMEMAIN REM **Remark 3.3.** In some of the main examples (see Propositions 3.12 and 3.14), there exists a G-equivariant set Ξ of inner edges of T such that $\Xi^i = \Xi \cap E^{\mathsf{i}}(U_i)$.

We caution that, for fixed X and $\{U_i\}$, our characteristic conditions are *not* monotone on such Ξ since increasing Ξ makes (Ch1) more permissive while making (Ch2),(Ch3) more restrictive.

CHAREDGE LEM **Lemma 3.4.** *If* $\{\Xi^i\}_{i\in I}$ *is a characteristic inner edge collection of* $\{U_i\}_{i\in I}$ *with respect to* X, *then the map*

$$X \to X \cup \bigcup_{i\in I} \Omega[U_i] \tag{3.5}$$ CHARLEM EQ

is G*-inner anodyne. In fact, it is cellular on* G*-inner horn inclusions* $\Lambda^E[S] \to \Omega[S]$, $S \in \Omega_G$.

Proof. We start with the case of $I \simeq G/H$ orbital so that, abbreviating $U = U_{[e]}$, $\{U_i\}$ is the set of conjugates gU. Note that H is also the isotropy of U in $\mathsf{Face}(T)$. We likewise abbreviate $\Xi = \Xi^{[e]}$ and $\Xi_V = \Xi^{[e]}_V$ for $V \hookrightarrow U$. Moreover, in this case one has $X_{<[g]} = X$ in (Ch1),(Ch2),(Ch3).

We write $\mathsf{Face}^{lex}_{\Xi}(U)$ for the H-poset of planar faces $V \hookrightarrow U$ such that $\Xi_V \neq \varnothing$ and $\Xi_V = \Xi_{\bar{V}}$ ordered as follows: $V \leq V'$ if either (i) $\bar{V} \hookrightarrow \bar{V}'$ and $\bar{V} \neq \bar{V}'$ or (ii) $\bar{V} = \bar{V}'$ and $V \hookrightarrow V'$

(alternatively, this is the lexicographic order of pairs $(\bar{V}, V)$). We note that here and in the remainder of the proof all outer closures are implicitly taken in U (rather than T), i.e. $\bar{V} = \bar{V}^U$.

For any H-equivariant convex subset C of $\mathsf{Face}^{lex}_{\Xi}(U)$ we write

$$X_C = X \cup \bigcup_{g \in G, V \in C} \Omega[gV].$$

It now suffices to show that whenever $C \subseteq C'$ the map $X_C \to X_{C'}$ is built cellularly from G-inner horn inclusions (indeed, setting $C = \varnothing$ and $C' = \mathsf{Face}^{lex}_{\Xi}(U)$ recovers (CHARLEM EQ 3.5) when $I \simeq G/H$).

Without loss of generality we can assume that C' is obtained from C by adding the H-orbit of a single $W \hookrightarrow U$. Further, we may assume $W \notin X_C$ or else $X_C = X_{C'}$. Letting $K \leq H$ denote the isotropy of W in $\mathsf{Face}^{lex}_{\Xi}(U)$ and regarding $W \in \Omega^K \subseteq \Omega_K$, we claim there is a pushout diagram

$$\begin{tikzcd}
G \cdot_K \Lambda^{\Xi_W}[W] \arrow[r] \arrow[d] & X_C \arrow[d] \\
G \cdot_K \Omega[W] \arrow[r] & X_{C'}
\end{tikzcd}$$ FIRPUSH EQ

where we note that inner edge set Ξ_W is K-equivariant since $\Xi_W = \Xi \cap E^{\mathsf{i}}(W)$ and Ξ is H-equivariant by (Ch0). The desired pushout will follow once we establish the following claims:

(a) all proper outer faces V of W are in X_C;

(b) an inner face $W - D$ of W is in X_C iff $D \notin \Xi_W$;

(c) the G-isotropy (i.e. the isotropy in $\mathsf{Face}(T)$) of faces $W - D$, $D \subseteq \Xi_W$ is contained in K.

To check (a), writing $\bar{V}$ for the corresponding outer face of U, one has

$$\Xi_V = \Xi \cap E^{\mathsf{i}}(V) = \Xi \cap E^{\mathsf{i}}(W) \cap E^{\mathsf{i}}(\bar{V}) = \Xi \cap E^{\mathsf{i}}(\bar{W}) \cap E^{\mathsf{i}}(\bar{V}) = \Xi \cap E^{\mathsf{i}}(\bar{V}) = \Xi_{\bar{V}}$$

where the second step follows from Lemma INNINT LEM 2.5 (applied to $V \hookrightarrow W \hookrightarrow U$, $V \hookrightarrow \bar{V} \hookrightarrow U$) and the third since by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ it is $\Xi_W = \Xi_{\bar{W}}$. Thus either $\Xi_V = \Xi_{\bar{V}} = \varnothing$ so that $\bar{V} \in X$ by (Ch1), or $\Xi_V = \Xi_{\bar{V}} \neq \varnothing$ so that $V \in \mathsf{Face}^{lex}_{\Xi}(U)$ with $V < W$, and thus $V \in C$. In either case one has $V \in X_C$.

We now check the "if" direction of (b). If $D \notin \Xi_W$ then $W' = W - (D \smallsetminus \Xi_W)$ is in $\mathsf{Face}^{lex}_{\Xi}(U)$ (since $\bar{W}' = \bar{W}$ and $\Xi_{W'} = \Xi_W$) and $W' < W$, and thus $W' \in X_C$.

For the "only if" direction of (b), note first that it suffices to consider $D = \Xi_W$. The assumption $W \notin X_C$ together with (Ch2) imply that $W' = W - \Xi_W$ is not in X, and thus it remains to show that W' is not a face of any gV with $g \in G$, $V \in C$. Suppose otherwise, i.e. $W' \hookrightarrow gV$. If it were $g \notin H$, then it would be $W' \hookrightarrow gV \hookrightarrow gU \neq U$, and (Ch3) would imply $W \in X$. Thus we need only consider $g \in H$, and since C is H-equivariant, we can set $g = e$. It now suffices to show that if $W' \hookrightarrow V$ then it must be $W \leq V$ in $\mathsf{Face}^{lex}_{\Xi}(U)$, since by convexity of C this would contradict $W \notin C$. Since $W' \hookrightarrow V$ implies $\bar{W} = \bar{W}' \hookrightarrow \bar{V}$, the condition $W \leq V$ is automatic from the definition of $\leq$ unless $\bar{W} = \bar{V}$. In this latter case, by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ the face V must contain as inner edges all edges in $\Xi_V = \Xi_{\bar{V}} = \Xi_{\bar{W}} = \Xi_W$, so that not only $W - \Xi_W = W' \hookrightarrow V$ but also $W \hookrightarrow V$. But then it is $W \leq V$ in either case, establishing the desired contradiction.

We now show (c). If $g(W - D) = W - D$ then $g(W - \Xi_W) \hookrightarrow U$, and thus $W - \Xi_W \hookrightarrow g^{-1}U$, so that by (Ch3) it must be $g \in H$ or else it would be $W \in X$. Now suppose $h(W - D) = W - D$ with $h \in H$. Since Ξ is H-equivariant (by (Ch0)) and $\Xi_{W-D} = \Xi_W \smallsetminus D$ (due to $D \subseteq \Xi_W$) it follows that $h(W - \Xi_W) = W - \Xi_W$, so that we may assume $D = \Xi_W$. Now note that hW, $h(W - \Xi_W) = W - \Xi_W$, W are all faces of U with a common outer closure $\bar{W}$. Hence $h\Xi_W = \Xi_{hW} \subseteq \Xi_{\bar{W}} = \Xi_W$, where the last step follows since $W \in \mathsf{Face}^{lex}_{\Xi}(U)$, and by cardinality reasons it must in fact be $h\Xi_W = \Xi_W$. But then hW, W have the same outer closure and the same inner edges, and thus $hW = W$, establishing (c).

Lastly, we address the case of general I. For each G-equivariant convex subset J of I, set

$$X_J = X \cup \bigcup_{j \in J} \Omega[U_j].$$

As before, it suffices to check that for all convex subsets $J \subseteq J'$ the map $X_J \to X_{J'}$ is built cellularly from G-inner horns, and again we can assume that J' is obtained from J by adding a single G-orbit Gj of I. By the I orbital case, it now suffices to check that $\{\Xi^{gj}\}_{gj \in Gj}$ is also a characteristic inner edge collection of $\{U_{gj}\}_{gj \in Gj}$ with respect to X_J. (Ch0) is clear, and since by G-equivariance and convexity it is $X_{<gj} \subseteq X_J$, the new (Ch1),(Ch2),(Ch3) conditions follow from the original conditions. □

CHAREDGE2 REM

Remark 3.6. The requirement $X \subseteq \Omega[T]$ in Definition 3.1 can be relaxed. Given an inclusion $X \subseteq Y$, a set of non-degenerate dendrices $\{y_i \in Y(U_i)\}_{i \in I}$ and a collection of edges $\{\Xi^i \subset E^{\mathsf{i}}(U_i)\}_{i \in I}$, suppose that I is a finite G-poset and that:

(Ch0.1) the maps $y_i \colon \Omega[U_i] \to Y$ are monomorphisms;

(Ch0.2) X, $\{U_i\}$,$\{y_i\}$ and $\{\Xi^i\}$ are all G-equivariant in the sense that: (i) $gX = X$; (ii) there are associative and unital isomorphisms $U_i \xrightarrow{g} U_{gi}$; (iii) the composites $\Omega[U_i] \xrightarrow{y_i} Y \xrightarrow{g} Y$ and $\Omega[U_i] \xrightarrow{g} \Omega[U_{gi}] \xrightarrow{y_{gi}} Y$ coincide; (iv) $g\Xi^i = \Xi^{gi}$.

Under (Ch0.1), the $\Omega[U_i]$ are identified with subcomplexes of Y, and non-degenerate dendrices $y \in y_i(\Omega[U_i])(V)$ are identified with faces $V \hookrightarrow U_i$.

The original conditions (Ch1),(Ch2),(Ch3) can then be reinterpreted by, for each $V \hookrightarrow U_i$, regarding expressions such as $V \in X$, $(V - \Xi^i_V) \in U_j$ as $y_i(V) \in X$, $y_i(V - \Xi^i_V) \in y_j(\Omega[U_j])$.

The proof of Lemma 3.1 now carries out to show that

$$X \to X \cup \bigcup_{i \in I} y_i(\Omega[U_i])$$

is G-inner anodyne (again built cellularly from G-inner horn inclusions).

RECOVER REM

Remark 3.7. Lemma 3.4 readily recovers several arguments in the literature:

(i) In [Rez01, §10] (also [Rez10, §6.2]), Rezk introduces the notion of *covers*, which in our language are the subsets $Sc[n] \subseteq X \subseteq \Delta[n]$ such that if V is in X then so is the closure $\bar{V}^{[n]}$ (in words, X is generated by outer faces). Similarly, in the proof of [CM13a, Prop. 2.4] Cisinski and Moerdijk use subcomplexes S_j that can be regarded as *dendroidal covers*, i.e. subcomplexes $Sc[T] \subseteq X \subseteq \Delta[T]$ such that if V is in X then so is $\bar{V}^T$. Lastly, the subcomplexes $\Omega[T] \cup_l \Omega[S] \subseteq \Omega[T \circ_l S]$ in the grafting result [MW09, Lemma 5.2] (and similarly for the equivariant analogue [Per17, Prop. 6.19]) are also dendroidal covers.

Lemma 3.4 implies that any inclusion $X \to X'$ of G-equivariant (dendroidal) covers of $T \in \Omega_G$ is G-inner anodyne. Indeed, let $I = \mathsf{Face}^{out}_{X'}(T)$ be the G-poset of outer faces $V \hookrightarrow T$ contained in X', ordered by inclusion, $\Xi = E^{\mathsf{i}}(T)$ and $U_V = V$. (Ch0) is clear, (Ch1) follows since $Sc(T) \subseteq X$, (Ch2) follows since X is a cover and (Ch3) follows since the U_i are closed.

Alternatively, one can also use $I = \mathsf{Face}^{out}_{X',o}(T)$ for the G-trivial set of orbital outer faces $GV \to T$, together with an *arbitrary* total order (see Remark 3.15 for a similar example).

Lastly, we note that if $\{U_i\} = \{T\}$, $\Xi = E^{\mathsf{i}}(T)$ then (Ch1) says precisely that $Sc[T] \subseteq X$.

context for this sentence?

(ii) In [MW09, Lemma 9.7], Moerdijk and Weiss introduced a *characteristic edge* condition that can be regarded as a special case of our characteristic edge collection condition as generalized in Remark 3.6, and which served as one of our main inspirations.

Therein, they work in the case of $Y = \Omega[T] \otimes \Omega[S]$ a tensor product of (non-equivariant) representable dendroidal sets, in which case (Ch0.1) is easily verified (and (Ch0.2) is moot). In our notation, they then require that $I \simeq *$ (so that (Ch3) is also moot), the dendrex $y_* \in (\Omega[T] \otimes \Omega[S])(U_*)$ encodes a special type of subtree U_* of $\Omega[T] \otimes \Omega[S]$, which they call an *initial segment*, and they further require that $\Xi^* = \{\xi\}$ is a singleton, called the *characteristic edge*. Moreover, they then demand that X should contain all outer faces of the subtree U_*, from which (Ch1) follows, as well as the key characteristic condition [MW09, Lemma 9.7](ii), which coincides with (Ch2) in this specific setting.

Similarly, in [Per17, Lemma 7.39] the second author introduced a *characteristic edge orbit* condition that generalizes that in [MW09] to the equivariant context by letting $I \simeq G/H$ and the $\Xi^{[g]} = \Xi \cap E^{\mathrm{i}}(U_{[g]})$ be determined by a G-edge orbit $\Xi \simeq Gf$ (cf. Remark 3.3).

However, both of the lemmas in [MW09] and [Per17] have the drawback of needing to be used iteratively (so that much effort therein is spent showing that this can be done) while Lemma 3.4 is designed so that a single use suffices for the natural applications. Indeed, conditions (Ch1) and (Ch3), the first of which relaxes the requirement in [MW09],[Per17] that X should contain all outer faces, essentially provide abstract conditions under which the original characteristic edge arguments of [MW09],[Per17] can be iterated.

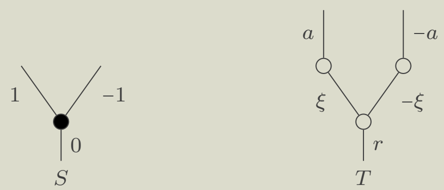
Example 3.8. As indicated above, Lemma 3.4 can be used to reorganize and streamline the rather long proofs of [Per17, Thms 7.1 and 7.2]. We illustrate this in the hardest case, that of [Per17, Thm. 7.1(i)], which states that if $S, T \in \Omega_G$ are open (i.e. have no stumps) and $G\xi$ is an inner edge orbit of T the maps

$$\partial\Omega[S] \otimes \Omega[T] \coprod_{\partial\Omega[S] \otimes \Lambda^{G\xi}[T]} \Omega[S] \otimes \Lambda^{G\xi}[T] \to \Omega[S] \otimes \Omega[T] \tag{3.9}$$

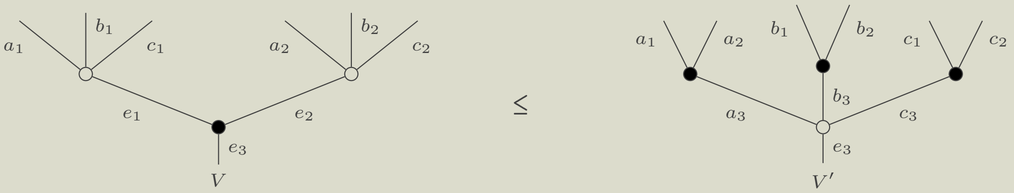
are G-inner anodyne.

As detailed in [Per17, §5.1], and originally due to Weiss in [Wei12], there is an algebraically flavoured model for Ω as certain types of *broad posets*. Given $S, T \in \Omega_G$, it is then possible [Per17, §7.1] to define a G-equivariant broad poset $S \otimes T$ so that $(\Omega[S] \otimes \Omega[T])(V) = Hom(V, S \otimes T)$ where the Hom set is taken in broad posets. Intuitively $S \otimes T$ is an object with edge set $E(S) \times E(T)$ and where each edge $(s,t) \in S \otimes T$ may, depending on whether $s \in S$, $t \in T$ are leaves or not, admit two *distinct* vertices: a S-vertex $(s,t)^{\uparrow_S} = s^{\uparrow} \times t \le (s,t)$ and a T-vertex $(s,t)^{\uparrow_T} = s \times t^{\uparrow} \le (s,t)$.

To recover [Per17, Thm. 7.1(i)] from Lemma 3.4, we first let $I = \mathsf{Max}(S \otimes T)$ be the G-poset of maximal subtrees $U \hookrightarrow S \otimes T$ (these are called *percolation schemes* in [MW09, §9]), ordered lexicographically [Per17, Def. 7.29]. As an example, let $\mathbb{Z}_{/2} = \{\pm 1\}$ and consider the $\mathbb{Z}_{/2}$-trees



We depict the $\mathbb{Z}_{/2}$-poset $\mathsf{Max}(S \otimes T)$ in Figure 3.1 (note that (s,t) is abbreviated as t_s). In words, the maximal subtrees are built by starting with the "double root" r_0 and iteratively choosing between the available S and T vertices (along all upward paths) until the "double leaves" are reached. The generating relations $U \leq U'$ in $\mathsf{Max}(S \otimes T)$ occur whenever U contains an outer face V shaped as on the left below and, by "replacing" V with V' as on the right, one obtains U'.



(3.10)

The claim that $\leq$ is indeed a partial order (at least if one of S, T is open) is [Per17, Prop. 7.31]. As an aside, we note that V, V' above have a common inner face $V - \{e_1, e_2\} = V' - \{a_3, b_3, c_3\}$, which encodes an (universal!) example of a Boardman-Vogt relation (see [MW07, §5.1]).

Returning to the task of proving that (3.9) is G-inner anodyne, we define Ξ^U, for each maximal subtree $U \hookrightarrow S \otimes T$, to be the set of inner edges of U of the form $(g\xi)_s$ *such that* the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ in U is a T-vertex (see Figure 3.1). We now verify (Ch1),(Ch2),(Ch3). We recall that, since S, T are assumed open, [Per17, Lemma 7.19] guarantees that, for faces $S' \hookrightarrow S$, $T' \hookrightarrow T$, a factorization $V \hookrightarrow S' \otimes T' \hookrightarrow S \otimes T$ exists iff the edges of V are in $E(S') \times E(T')$.

For (Ch1), note first that there is an equivariant grafting decomposition $T = T_{\nleq G\xi} \amalg_{G\xi} T^{\leq G\xi}$, where $T_{\nleq G\xi}$ contains the edges $t \in T$ such that $\forall_{g \in G} t \nleq g\xi$ (pictorially, this is a lower equivariant outer face of T) while $T^{\leq G\xi}$ contains the edges $t \in T$ such that $\exists_{g \in G} t \leq g\xi$ (an upper equivariant outer face of T). But one now readily checks that if $V \hookrightarrow U$ is an *outer* face such that $\Xi_V^U = \varnothing$, then either $V \hookrightarrow S \otimes T_{\nleq G\xi}$ or $V \hookrightarrow S \otimes T^{\leq G\xi}$, and thus $V \in X$.

For (Ch3), suppose $U_j \nleq U_i$, $V \hookrightarrow U_i$ and $(V - \Xi_V^{U_i}) \hookrightarrow U_j$. Then it follows from [Per17, Lemma 7.37] that there exists a generating relation $U_k < U_i$ such that $(V - \Xi_V^{U_i}) \hookrightarrow U_k$ (indeed, [Per17, Lemma 7.37] makes the slightly stronger claim that such a relation can be performed on the outer closure $\bar{V}^{U_i}$). But then, as one sees from (3.10), all edges $e \in U_i$ that are not in U_k are topped by the S-vertex $e^{\uparrow S} \leq e$, and thus it is $e \notin \Xi^{U_i}$. Therefore $V \hookrightarrow U_k$, as desired.

Lastly, for (Ch2), suppose $V \hookrightarrow U$ and $(V - \Xi_V^U) \in X$. If it were $(V - \Xi_V^U) \hookrightarrow S \otimes \Lambda^{G\xi}[T]$, then it would also be $V \hookrightarrow S \otimes \Lambda^{G\xi}[T]$ since all edges of Ξ_V^U have T-coordinate in $G\xi$. Now consider the more interesting case $(V - \Xi_V^U) \hookrightarrow S' \otimes T$ for some face $S' \hookrightarrow S$. Then it will also be $V \hookrightarrow S' \otimes T$ unless there is at least one edge $(g\xi)_s \in \Xi_V^U$ such that $s \notin S'$. But then since the outer closure $\bar{V}^U$ can have no leaf with S-coordinate s (this would contradict $s \notin S'$), there exists some minimal outer face $U_{(g\xi)_s}^{<s}$ of U with root $(g\xi_s)$ and such that its leaves have S-coordinate $<_d s$. By minimality, one has that $U_{(g\xi)_s}^{<s} \hookrightarrow \bar{V}^U$ and that all inner edges of $U_{(g\xi)_s}^{<s}$ have S-coordinate s. Further, note that $U_{(g\xi)_s}^{<s}$ has at least one inner edge (since by definition of Ξ^U the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ is a T-vertex) and that V contains none of those inner edges (or else it would be $s \in S'$). Thus by applying [Per17, Lemma 7.34] to $U_{(g\xi)_s}^{<s}$ one obtains a maximal subtree $U' < U$ containing all edges of U that are not inner edges of $U_{(g\xi)_s}^{<s}$. But then $V \hookrightarrow U'$ and (Ch2) follows.

Remark 3.11. We briefly outline how to modify the previous example to prove [Per17, Thm 7.1(ii)], in which case some notable subtleties arise. The result again states that (3.9) is G-inner anodyne, but now with one of S, T allowed to have stumps while the other is required to be linear.

One again sets $I = \mathsf{Max}(S \otimes T)$, where maximal trees are defined just as before, but some care is now needed. To see why, note that if the black nodes $\bullet$ in (3.10) are replaced with stumps then V' is actually a subtree of V.

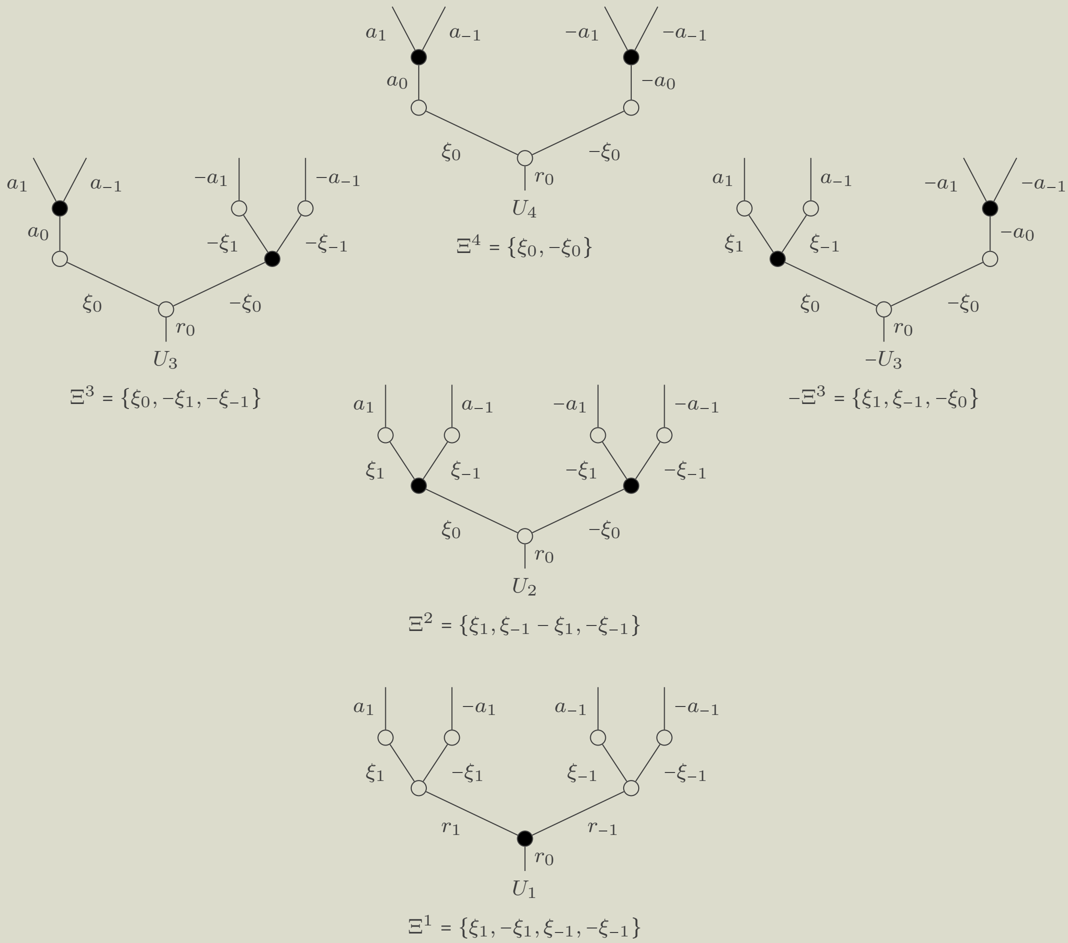


Figure 3.1: The $\mathbb{Z}_{/2}$-poset $\mathsf{Max}(S \otimes T)$ and characteristic edges Ξ^i FIGURE

When S has stumps and T is linear this causes no issues and the proof above holds (though we note that it can now be $\Xi^U = \varnothing$, in which case the argument for (Ch1) shows $U \in X$).

However, when S is linear and T has stumps the proof above breaks down (more precisely, the tree $U^{<s}_{(g\xi)_s}$ that appears when arguing (Ch2) may now fail to have inner edges). The solution is then to *reverse* the poset structure on $\mathsf{Max}(S \otimes T)$ and to modify the Ξ^U to be those inner edges $(g\xi)_s$ such that $(g_x i)_s \in t_s^{\uparrow T}$ for some t_s (pictorially, this says that these are the lowermost edges with T-coordinate in $G\xi$, whereas before they were the uppermost ones). The arguments for (Ch1),(Ch3) then hold. For (Ch2), only the argument for the interesting case of $V - \Xi^U_V \hookrightarrow S' \otimes T$, $s \notin S'$ changes. In this case, there is then a maximal edge t'_s such that $(g\xi)_s < t'_s$, where s can not be the root of S (or else it would be $s \in S'$). Pictorially, t'_s looks like the edge $e_1 \in V$ in (3.10) in the case where the $\bullet$ node is unary (since S is assumed linear). But then since V can not contain t'_s there exists a maximal subtree $U' > U$ such that $V \hookrightarrow U'$, and (Ch2) follows.

Lastly, we note that [Per17, Thm. 7.2] follows from a minor variant of the argument for [Per17, Thm. 7.1(ii)] when S is linear.

HYPERSAT SEC

3.2 Segal covers, horns and orbital horns

ORB_HORN_PROP

Proposition 3.12. *For G-subsets $\varnothing \neq F \subseteq E \subseteq \boldsymbol{E}^{\mathsf{i}}(T)$ the inclusions*

$$\Lambda_o^E[T] \to \Omega[T], \qquad \Lambda_o^E[T] \to \Lambda_o^F[T] \tag{3.13}$$

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are G-inner anodyne.

Proof. We are free to assume that $T \in \Omega^G \subseteq \Omega_G$. Indeed, otherwise writing $T = G \cdot_H T_*$ where $T_* \in \Omega^H$ is a fixed component and $E_* = E \cap \boldsymbol{E}^{\mathsf{i}}(T_*)$, $F_* = F \cap \boldsymbol{E}^{\mathsf{i}}(T_*)$, the maps in (3.13) are $G \cdot_H \left(\Lambda_o^{E_*}[T_*] \to \Omega[T_*]\right)$ and $G \cdot_H \left(\Lambda_o^{E_*}[T_*] \to \Lambda_o^{F_*}[T_*]\right)$.

In the $\Lambda_o^E[T] \to \Omega[T]$ case we apply Lemma 3.4 with $I = \{*\}$ a singleton and

$$\Xi^* = E, \qquad U_* = T, \qquad X = \Lambda_o^E[T].$$

It remains to check the characteristic conditions in Definition 3.1. (Ch0) and (Ch3) are clear.

Note that for $V \hookrightarrow T$ it is $V \notin X$ iff $GV = T - E'$ for some G-subset $E' \subseteq E$.

For (Ch1), the condition $\Xi_V = \varnothing$ says that none of the inner edges of V are in E, and thus that the orbital outer face GV contains none of the edge orbits in E as inner edge orbits. Since $E \neq \varnothing$, the orbital outer face GV is not T itself, and hence $X = \Lambda_o^E[T]$ contains V.

For (Ch2), note that if $V \notin X$, i.e., $GV = T - E'$, then Remark 2.21 implies that $G(V - \Xi_V) = T - E''$ for $E' \subseteq E'' \subseteq E$, and thus also $(V - \Xi_V) \notin X$.

In the $\Lambda_o^E[T] \to \Lambda_o^F[T]$ case we instead apply Lemma 3.4 with $I = (E \smallsetminus F)/G$, with an *arbitrary* choice of total order, and (writing elements of $(E \smallsetminus F)/G$ as orbits $Ge \subseteq E \smallsetminus F$)

$$\Xi^{Ge} = F, \qquad U_{Ge} = T - Ge, \qquad X = \Lambda_o^E[T].$$

Note that the U_{Ge} are the orbital inner faces $T - Ge$ for $Ge \subseteq E \smallsetminus F$, and thus the map in Lemma 3.4 is indeed $\Lambda_o^E[T] \to \Lambda_o^F[T]$. Further, we are free to abbreviate $\Xi = \Xi^{Ge}$ and $\Xi_V = \Xi_V^{Ge}$, since Ξ^{Ge} is independent of Ge. We again check the characteristic conditions. (Ch0) is clear.

For (Ch1), note that for an outer face $V \hookrightarrow U_i$, and writing $\bar{V} = \bar{V}^T$, Lemma 2.5 implies $\boldsymbol{E}^{\mathsf{i}}(V) = \boldsymbol{E}^{\mathsf{i}}(U_i) \cap \boldsymbol{E}^{\mathsf{i}}(\bar{V})$ and hence since $\Xi_{U_i} = F = \Xi$ the hypothesis $\Xi_V = \varnothing$ in (Ch1) implies it is also $\Xi_{\bar{V}} = \varnothing$. Hence just as before $G\bar{V}$ is an orbital outer face other than T, hence V is in $X = \Lambda_o^E[T]$. The argument for (Ch2) is identical to the one in the $\Lambda_o^E[T] \to \Omega[T]$ case. Lastly, (Ch3) follows since if $V \notin X$, so that $GV = T - E'$ and $G(V - \Xi_V) = T - E' - F'$ with $E' \subseteq E$, $F' \subseteq F$, then $GV \hookrightarrow T - Ge$ iff $G(V - \Xi_V) \hookrightarrow T - Ge$ and thus $V \hookrightarrow T - Ge$ iff $V - \Xi_V \hookrightarrow T - Ge$. $\square$

REG_HORN_PROP

Proposition 3.14. *For G-equivariant $\varnothing \neq F \subseteq E \subseteq \boldsymbol{E}^{\mathsf{i}}(T)$ the inclusions*

$$\Lambda^E[T] \to \Lambda^F[T]$$

are G-inner anodyne.

Proof. We now apply Lemma 3.4 with $I = \mathcal{P}_0(E \smallsetminus F)$ the poset of non-empty subsets $\varnothing \neq E' \subseteq (E \smallsetminus F)$, ordered by *reverse inclusion*, and

$$\Xi^{E'} = F, \qquad U_{E'} = T - E', \qquad X = \Lambda^E[T].$$

We again need to verify the characteristic conditions, and as in the previous result we abbreviate $\Xi = \Xi^{E'}$, $\Xi_V = \Xi_V^{E'}$. (Ch0) is clear. (Ch1) follows from an easier version of the argument in the previous proof. (Ch2) follows since $V \in X$ iff $V - \Xi_V \in X$. Similarly, (Ch3) follows since $V \hookrightarrow T - E'$ iff $(V - \Xi_V) \hookrightarrow T - E'$ and since if $V \hookrightarrow T - E', V \hookrightarrow T - E''$ then $V \hookrightarrow T - (E' \cup E'')$. $\square$

TWOPROOF REM

Remark 3.15. By specifying to the non-equivariant case $G = *$ the previous results yield two distinct proofs that inclusions of non-equivariant horns $\Lambda^E[T] \to \Lambda^F[T]$ are inner anodyne, with the first proof using $I = E \setminus F$ (with any total order) and the second using $I = \mathcal{P}_0(E \setminus F)$.

The discrepancy is explained as follows: when T, E, F are G-equivariant, showing that $\Lambda^E[T] \to \Lambda^F[T]$ is G-inner anodyne requires a control of isotropies not needed when showing that the underlying map is non-equivariant inner anodyne, and since this control is given by (Ch3), it is necessary to include in the $\{U_i\}$ the "intersections" of $T - e$ and $T - ge$ for $e \in E \setminus F$.

FACCES REM

Remark 3.16. All horn inclusions attached in the proof of Lemma 3.4 correspond to G-trees whose non-equivariant components are faces of the U_i. Moreover, when I is orbital the last horn inclusion attached (corresponding to the maximum of $\mathsf{Face}^{lex}_{\Xi}(U)$) is $G \cdot_H \left(\Lambda^{\Xi}[U] \to \Omega[U]\right)$.

REGGENHORN COR

Corollary 3.17. *G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ are built cellularly from generating horn inclusions $\Lambda^{Ge}[S] \to \Omega[S]$.*

Proof. The proof is by induction on $|T_*|$ for $T_* \in \Omega$ a tree component (cf. Remark 2.4). As before one is free to assume $T \in \Omega^G$. A choice of edge orbit Ge in E yields a factorization $\Lambda^E[T] \to \Lambda^{Ge}[T] \to \Omega[T]$, hence we need only show that $\Lambda^E[T] \to \Lambda^{Ge}[T]$ is built cellularly from generating horns. But this is immediate from the induction hypothesis, Remark 3.16, and the proof of Proposition 3.14 since all U_i therein satisfy $|U_i| < |T|$. □

Following the discussion preceding [HHM16, Prop. 3.6.8], a class of normal monomorphisms of dSet^G (or, more generally, a subclass of the cofibrations in a model category) is called *hypersaturated* if it is closed under pushouts, transfinite composition, retracts, as well as the following additional cancellation property: if f, g are normal monomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{3.18}$$

CANCEL_EQ

such that both f and gf are in the class, then so is g.

The following is an equivariant generalization of [CM13a, Props. 2.4 and 2.5].

HYPER PROP

Proposition 3.19. *The following sets of maps generate the same hypersaturated class:*

- *the G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ for $T \in \Omega_G$ and G-equivariant $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$;*
- *the orbital G-inner horn inclusions $\Lambda^E_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and G-equivariant $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$;*
- *the G-Segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.*

In the following proof we refer to the hypersaturation of the orbital horn (resp. Segal core) inclusions as the orbital (resp. Segal) hypersaturation.

Proof. The fact that G-inner horn inclusions generate the orbital and Segal hypersaturations has been established in Proposition 3.12 and Remark 3.7(i).

To see that the G-inner horn inclusions are in the orbital hypersaturation, we again argue by induction on $|T_*|$, with the base cases those where $\Lambda^E[T] = \Lambda^E_o[T]$. Recalling that in the proof of Proposition 3.12 one sets $I = *$, $U_* = T$ and $\Xi^* = E$, Remark 3.16 implies that in the factorization $\Lambda^E_o[T] \to \Lambda^E F[T] \to \Omega[T]$ the first map $\Lambda^E_o[T] \to \Lambda^E F[T]$ is built cellularly out of G-horns with $|S_*| < |T_*|$. But then the induction hypothesis says that $\Lambda^E_o[T] \to \Lambda^E[T]$ is in the orbital hypersaturation, and by the cancellation property so is $\Lambda^E[T] \to \Omega[T]$.

For the claim that the G-inner inclusions are in the Segal hypersaturation, note that $Sc[T] \to \Omega[T]$ can be shown to be G-inner anodyne by setting $I = *$, $U_* = T$, $\Xi^* = E^{\mathsf{i}}(T)$ (this differs from

Remark 3.7(i), but the arguments therein still hold). Therefore, arguing exactly as above for the factorization $Sc[T] \to \Lambda^{E^{\mathrm{i}}(T)}[T] \to \Omega[T]$, one obtains by induction on $|T_*|$ that $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Omega[T]$ is in the Segal hypersaturation. But now letting $E \subseteq E^{\mathrm{i}}(T)$ be any G-equivariant subset and considering the factorization $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Lambda^{E}[T] \to \Omega[T]$ the induction hypothesis applies to the cells of $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Lambda^{E}[T]$ (just as in Corollary 3.17), which is thus also in the Segal hypersaturation. But by the cancellation property, so is $\Lambda^{E}[T] \to \Omega[T]$, finishing the proof. $\square$

SLICE REM

Remark 3.20. Setting $G = e$ and slicing over the stick tree η, the previous result recovers the well known claim that the hypersaturation of the simplicial inner horns $\{\Lambda^i[n] \to \Delta[n] : 0 < i < n\}$ coincides with the hypersaturation of the simplicial Segal core inclusions $\{Sc[n] \to \Delta[n] : n \geq 0\}$.

HYPERSATKAN REM

Remark 3.21. We will make use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns $\{\Lambda^i[n] \to \Delta[n] : 0 \leq i \leq n\}$ coincides with the hypersaturation of all vertex inclusion maps $\{\Delta[0] \to \Delta[n]\}$. Call the latter hypersaturation S. One easily checks that the maps $\{0\} \to Sc[T]$ are in S, so that by cancellation so are the Segal core inclusions $\{Sc[n] \to \Delta[n]\}$ and hence by Remark 3.20 so are all inner horn inclusions. Moreover, for left horns $\Lambda^0[n]$ the maps $\{0\} \to \Lambda^0[n]$ are built cellularly from left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k < n$ (in join notation (see [Lur09, §1.2.8] or [Per17, §7.4]), $\{0\} \to \Lambda^0[n]$ is $\Delta[0] \star (\varnothing \to \partial\Delta[n-1])$, and the filtration follows from the cellular filtration of $\partial\Delta[n-1]$). But hence by induction and the cancellation property all left horn inclusions $\Lambda^0[n] \to \Delta[n]$ are in S. The case of right horn inclusions $\Lambda^n[n] \to \Delta[n]$ is dual.

Given a class $\mathcal{C}$ of normal monomorphisms in dSet^G, let $\mathcal{C}^{\boxslash!}$ denote the class of maps satisfying the *strict right lifting property* (i.e the usual lifts exist and are *unique*) against all maps in $\mathcal{C}$.

HYPER_LP_PROP

Proposition 3.22. *If two classes $\mathcal{C}, \mathcal{D}$ of normal monomorphisms of dSet^G have the same hypersaturation then $\mathcal{C}^{\boxslash!} = \mathcal{D}^{\boxslash!}$.*

Proof. It suffices to check that the hypersaturation closure conditions are compatible with strict right lifting properties. The claims concerning pushouts, transfinite compositions and retracts follow by the easy observation that the proofs of the analogue claims for the usual right lifting property [Rie14, Lemma 11.1.4] are compatible with the uniqueness requirement.

We thus address only the cancellation property, which is the most interesting case. Suppose then that r has the strict right lifting property against normal monomorphisms f and gf, and consider a lifting problem as on the left below.

$$
\begin{tikzcd}
B \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "r"] \\
C \arrow[r, "q"'] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "r"] \\
B \arrow[r] \arrow[ur, "p"] \arrow[d, "g"'] & Y \arrow[d, equal] \\
C \arrow[r, "q"'] \arrow[uur, dashed, "\exists! H"'] & Y
\end{tikzcd}
$$

By assumption, one can find a unique lift H for the outer square on the right. We claim that H is also the unique lift for the left square. Noting that $pf = Hgf$ and $rp = qg = rHg$ it follows that both p and Hg are lifts for the top square in the right diagram, so that by the uniqueness assumption it is $p = Hg$. This shows that H is indeed a lift for the left square. Uniqueness follows since any lift of the left square induces a lift of the outer right square. $\square$

4 Other Stuff

4.1 Preliminaries - to be combined with other Pre

Definition 4.1. A map $f : S_0 \to T_0$ in Ω is called a *face map* if it is planar and injective on underlying sets; S_0 is called a *face* of T_0.

Remark 4.2. In particular, we may assume that for any face map the S_0 is a *subset* of T_0.

Definition 4.3. A face $S_0 \hookrightarrow T_0$ is called

- *proper* if $S_0 \neq T_0$.
- *inner* if it is of the form $T_0 - E \hookrightarrow T_0$, where E is a subset of the set of inner edges $E^i(T)$ of T.
- *outer* if it is a composite of *leaf vertex outer faces* $T_{\not\leq e} \hookrightarrow T$, *stump outer faces* $T_{\not\leq e} \hookrightarrow T$, and *root vertex outer faces* $T^{\geq e} \hookrightarrow T$; see [Per17, Notation 5.41] for more details.

Remark 4.4. In terms of the underlying broad posets, the generating relations for an inner face (respectively, outer face) of T are given by *compositions* (respectively, a *subset*) of the generating relations for T,

We now recall the generalizations of these definitions to the category Ω_G of G-trees.

Definition 4.5. Given $S_0 \in \Omega$, $T \in \Omega_G$, and a map of forests $f : S_0 \to T$, let T_0 denote the component of T containing the image of S_0. We say f is a *(proper, inner, outer) face map* if $f : S_0 \to T_0$ is a (proper, inner, outer) face map on underlying trees.

Definition 4.6. Fixing $T \in \Omega_G$, let $\mathscr{F}(T)$ denote the G-poset (under inclusion) of face maps. Given a G-closed subset E of inner edges, let $\mathscr{F}^E(T)$ denote the subposet removing those faces of the form $T_0 - \bar{E}$ where T_0 is a component of T and $\bar{E} \subseteq E \cap T_0$. Define the *$E$-horn* of T to be the subdendroidal set

$$\Lambda^E[T] := \operatorname*{colim}_{\mathscr{F}^E(T)} \Omega[U_0] \simeq \bigcup_{\mathscr{F}^E(T)/G} \bigcup_{G} \Omega[gU_0].$$

Remark 4.7. Equivalently, if we have a decomposition $T \simeq G \cdot_H T_0$, then

$$\Lambda^E[T] \simeq G \cdot_H \Lambda^{E \cap T_0}[T_0] \simeq G \cdot_H \operatorname*{colim}_{\mathscr{F}^{E \cap T_0}(T_0)} \Omega[U_0].$$

Remark 4.8. If E is the entire set of inner edges, we denote $\Lambda^E[T]$ by $\partial^{out}\Omega[T]$, and call it the *outer boundary.*[2]

Definition 4.9. For $T \in \Omega_G$, let $\mathscr{F}_{\mathrm{SC}}(T) \subseteq \mathrm{Out}(T)$ denote the poset of outer faces with precisely one (non-equiviariant) vertex; that is, every element records precisely one generating broad relation $e^{\uparrow} \leq e$ from T. We define the *Segal core* of T to be the subdendroidal set

$$Sc[T] := \operatorname*{colim}_{\mathscr{F}_{SC}(T)} \Omega[C_0] = \bigcup_{\mathscr{F}_{\mathrm{SC}}(T)/G} \bigcup_{g \in G} \Omega[gC_0].$$

Remark 4.10. Explicitly, a map $Sc[T] \to X$ is given by elements in $X(T_v)$ for all $v \in V(T_0)$, which are compatible on overlapping edges and under the action of G. Equivalently, a map $Sc[T] \to X$ is given by an element in $X(T_{Gv})$ for each $Gv \in V_G(T)$ which are compatible on overlapping edges.

[2] in [CM13a], this was called the *external* boundary.

Remark 4.11. If $T \simeq G \cdot_H T_0$, then $Sc[T] \simeq G \cdot_H Sc[T_0]$.

Definition 4.12. A face map $f : S_0 \hookrightarrow T$ is called *orbital* if $f(S_0) \subseteq T$ is K_r-closed, where $K_r = \mathrm{Stab}_G(f(r_s))$ for r_S the root of S_0.

ORB_REM **Remark 4.13.** U_0 is orbital iff the (non-equivariant) subdendroidal sets $\Omega[gU_0]$ of $\Omega[T]$ are either disjoint or equal, with $\Omega[U_0] = \Omega[gU_0]$ iff $g \in K = \mathrm{Stab}_G(U_0)$.

ORB_INJ_LEM **Lemma 4.14.** *Let U_0 be a face of T with isotropy K. Then $\Omega[G \cdot_K U_0]$ is a subdendroidal set of $\Omega[T]$ iff U_0 is orbital.*

Proof. This follows immediately from Remark 4.13. In particular, if U_0 is not orbital, then there exist $g \in G \setminus K$ such that $R_0 \coloneqq U_0 \cap gU_0$ is a proper, non-empty subface of U_0, so $R_0 \in \Omega[U_0]$ and $\Omega[gU_0]$. Thus $R_0 \in \Omega[T]$ is hit at least twice. □

ORB_FACE_REM **Remark 4.15.** The data of an orbital face $U_0 \hookrightarrow T$ is equivalent to both

1. the data of a map of G-trees $U \to T$ which is planar and injective (that is, an *eqivariant face*); and

2. the data of a face map $U/G \to T/G$ of the *orbital representation* of T.

expand/contrast/combine with later writing/make this precise

Definition 4.16. Let $\mathscr{F}_o(T)$ denote the G-poset (under inclusion) of orbital face maps.

Given a G-set E of inner edges, let $\mathscr{F}^E_o(T) \coloneqq \mathscr{F}_o(T) \cap \mathscr{F}^E(T)$, and define the *Ge-orbital horn* to be the subdendroidal set

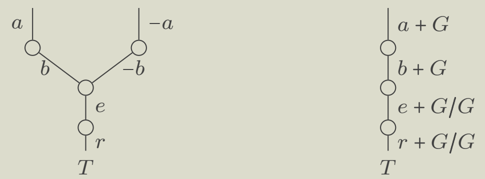
$$\Lambda^E_o[T] \coloneqq \operatorname*{colim}_{\mathscr{F}_o E(T)} \Omega[U_0] = \operatorname*{colim}_{\mathscr{F}^E_o(T)/G} \Omega[G \cdot_K U_0] = \bigcup_{\mathscr{F}^E_o(T)/G} \bigcup_{g \in G} \Omega[gU_0],$$

where $K = \mathsf{Stab}_G(U_0)$.

If $E = E^i(T)$, we denote $\Lambda^E_o[T]$ by $\partial^{out}_o[T]$, referred to as the *orbital outer boundary.*

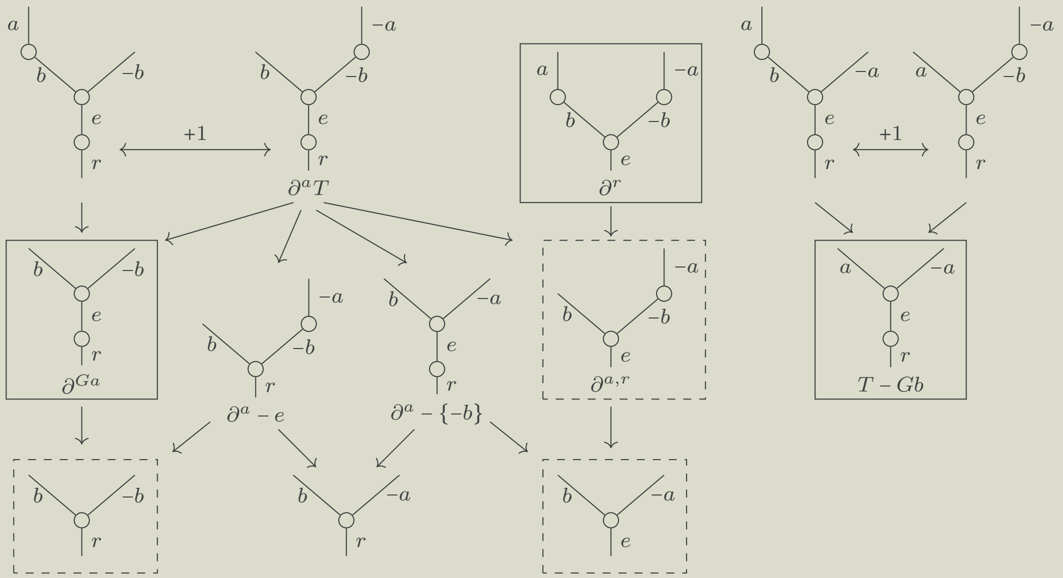
Remark 4.17. Following Remark 4.15, we note that the poset $\mathscr{F}_o(T)/G$ is isomorphic to the poset of faces of the (non-equivariant) tree corresponding to the *orbital representation* of the G-tree T, and similarly for $\mathscr{F}^E_o(T)/G$.

Example 4.18. Let $G = C_2$ be the cyclic group with two elements, and consider the tree $T \in \Omega^G \subset \Omega_G$ below.



We compare the two horns discussed above by considering the subposet of $\mathsf{Face}(T)$ displayed below in Figure 4.18. The horn $\Lambda^{Ge}[T]$ is only missing the faces T and $T - r$, with maximal faces in $\Lambda^{Ge}[T]$ given by the five faces in the top row; the maximal faces of the orbital horn $\Lambda^{Ge}_o[T]$ are those in the first two rows which are boxed. We have also included some of the subfaces of $S = \partial^a T$; those included in the orbital horn are (dashed) boxed. In particular, we note that S and all of its maximal subfaces each have at least one face contained in the orbital horn.

HORN_EX_FIG



4.2 Proof of Proposition HYPER_PROP 3.19

Notation 4.19. Given a class of morphisms $\mathcal{C}$ in $\mathcal{V}$, let $W(\mathcal{C})$ and $\hat{W}(\mathcal{C})$ denote the saturation and hypersaturation of $\mathcal{C}$.

change W notation

Remark 4.20. If L is a left adjoint, then $L(\hat{W}(\mathcal{C})) \subseteq \hat{W}(L(\mathcal{C}))$. In particular, this applies to the "free G-object" functor $G \cdot (-)$.

The following lemma identifies some of the utility of hypersaturations.

Notation 4.21. Given a class of maps $\mathcal{C}$ in $\mathcal{V}$, let $\mathcal{C}^{\boxslash}$ (respectively $\mathcal{C}^{\boxslash !}$) denote the class of maps with the *(strict) right lifting property* (abreviated (S)RLP) against $\mathcal{C}$[3].

HYPER_LP_LEM **Lemma 4.22.** *Let $\mathcal{C}$ be a class of maps in $\mathcal{V}$. Then $\mathcal{C}^{\boxslash} = W(\mathcal{C})^{\boxslash}$ and $\mathcal{C}^{\boxslash !} = \hat{W}(\mathcal{C})^{\boxslash !}$.*

Proof. It is a straightforward exercise to show that if $X \to Y$ has the (S)RLP with respect to a map $A \to B$ (resp. maps $A_\alpha \to A_{\alpha+1}$), then $X \to Y$ has the (S)RLP with respect to any pushout or retract of $A \to B$ (resp. the transfinite composite).

Indeed, for any pushout (respectively (dashed) retract)

$$\begin{tikzcd} C \arrow[r, dashed] \arrow[d, dashed] & A \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & X \arrow[d] \\ D \arrow[r, dashed] & B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$ LIFTING_EQ

the lift $B \to X$ and the given map $C \to X$ define a compatible map from the pushout to X (resp. the composite $D \to B \to X$ is a lift); moreover, for any lift $D \to X$, the composite $B \to D \to X$ is a lift over $A \to B$, and hence strictness implies strictness by the universal property of the pushout (resp. by composing with the section map $D \to B$).

[3] In many sources (e.g. Hov99 [Hov99]), $\mathcal{C}^{\boxslash}$ is denoted $\mathcal{C} - \mathrm{inj}$.

For transfinite composites, both the definition of the lift for the composite and its uniqueness are immediate.

Now, suppose $X \to Y$ has the SRLP against $A \to B$ and the composite $A \to B \to C$, and suppose we are given the commuting square in the bottom of the leftmost diagram below. Then, the following two diagrams commute.

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X \arrow[d, equal] \\
B \arrow[r] \arrow[d] & X \arrow[d] \\
C \arrow[r] \arrow[uur, dashed, "\exists ! H"] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & X \arrow[d] \\
B \arrow[r] \arrow[d] \arrow[urr, dashed, "\exists ! G"] & X \arrow[r] \arrow[d] & Y \arrow[d, equal] \\
C \arrow[r] \arrow[uurr, dashed, bend right, "H"'] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

Further, by assumption, there exist unique lifts H and G as denoted by the dashed arrows. It is immediate that both $B \to X$ and $B \to C \xrightarrow{H} X$ are candidates for lift G, and hence by strictness all three are equal. Thus H is also a lift of the bottom-left square. Since any lift of the bottom square is also a lift of the rectangle, strictness for $A \to B$ implies strictness for $B \to C$. $\square$

In the vast majority of cases of interest (in particular when dealing with cofibrantly generated model categories), the hypersaturation itself is characterized by a lifting condition using the *small object argument*.

Lemma 4.23 ([Hov99, Corollary 2.1.15])**.** *Suppose $\mathcal{C}$ is a set of maps in $\mathcal{V}$ such that the domains of maps in $\mathcal{C}$ are small relative to the closure of $\mathcal{C}$ under pushouts and transfinite compositions. Then*

$$W(\mathcal{C}) = {}^{\boxslash}(\mathcal{C}^{\boxslash}).$$

In this subsection, we will prove the following equivariant generalization of [CM13a, Props. 2.4 and 2.5].

Proposition 4.24. *The following sets of maps generate the same hypersaturated class:*

(i.1) the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;

(i.2) the generalized G-inner horn inclusions $\Lambda^{E}[T] \to \Omega[T]$ for $T \in \Omega_G$ and $E \subseteq E^{i}(T)$ a G-set;

(ii.1) the orbital G-inner horn inclusions $\Lambda^{Ge}_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;

(ii.2) the generalized orbital G-inner horn inclusions $\Lambda^{E}_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and $E \subseteq E^{i}(T)$ a G-set;

(iii) the G-segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.

Moreover, one also has the following:

(a) orbital G-inner horn inclusions are in the saturation of G-inner horn inclusions;

(b) G-segal core inclusions are in the saturation of both orbital G-inner horn and G-inner horn inclusions.

Proof of Proposition 3.19. The equality of (i.1) and (i.2) is given by [Per17, Proposition 6.17].

The equality of (ii.1) and (ii.2) is given by Proposition 4.30.

The equality of (i) and (ii), and (a), is given by Propositions 3.12 and 4.27.

The equality of (i) and (iii) is given by Propositions 4.29 (or 4.31, which also yields (b)) and 4.34. $\square$

We will now prove the above cited results.

The observation below will be useful in the proofs which follow.

Combine with description of GU.

FOF_OHORN_LEM

Lemma 4.25. *Fix $T \in \Omega_G$, a G-set $E \subseteq E^i(T)$, and a face $U \hookrightarrow T$. Then, for any $\bar{E} \subseteq E \cap U$, $U \in \Lambda^E_o[T]$ iff $U - \bar{E} \in \Lambda^E_o[T]$.*

Proof. We have a commuting diagram

$$\begin{tikzcd} U - \bar{E} \arrow[r] \arrow[d] & G(U - \bar{E}) \arrow[d] \\ U \arrow[r] & GU \end{tikzcd}$$

where GV is the minimal orbital face containing V. By Remark GINNER_REM 2.21, $G(U - \bar{E})$ is of the form $GU - E'$ for some sub G-set $E' \subseteq E$. It is immediate that for any face V of T we have

- $V \in \Lambda^E_o[T]$ iff GV is, and
- $GV \in \Lambda^E_o[T]$ iff $GV - E'$ is,

we may conclude that $U - \bar{E}$ is in the orbital horn $\Lambda^E_o[T]$ iff U is as well. □

We break up the proof of Proposition HYPER_PROP 3.19 into its constituent pieces.

The reverse inclusion is only true on the level of hypersaturations.

Definition 4.26. Let W_o (respectively $\hat{W}_o$) denote the (hyper)saturation of the orbital horn inclusions.

HORN_ORB_PROP

Proposition 4.27. *For any $T \in \Omega_G$ and G-set of inner edges E, the generalized inner horn inclusion $\Lambda^E[T] \to \Omega[T]$ is in $\hat{W}_o$.*

Proof. We go by induction on $|H| \times |V_G(T)|$, ordered lexicographically, where $T \simeq G \cdot_H T_*$ is any decomposition (as $|H|$ is independent of choice of T_*).

Can this be replaced with an "We may assume $T \in \Omega^G \subseteq \Omega_G$" argument?

When $G = \{e\}$, the orbital horns are regular inner horns, and so the result holds by Per17 [Per17, Proposition 6.17].

Now, the proof of Lemma CHAREDGE_LEM 3.4 says that if $\{\Xi^i\}_{i \in I}$ is a characteristic edge collection of $\{U_i\}_{i \in I}$ with respect to X, the map

$$X \to X \cup \bigcup_{i \in I} \Omega[U_i] \tag{4.28}$$ CHAR_ATTACH_EQ

is cellular on inner horn inclusions of inner faces of the U_i, and thus such maps CHAR_ATTACH_EQ (4.28) are in W_o by induction.

With that in mind, we apply Lemma CHAREDGE_LEM 3.4 with both I and U_i the subposet of

$$\Lambda^E[T] \smallsetminus \Lambda^E_o[T]$$

such that (i) $U_i \cap E \neq \varnothing$, (ii) $U_i \cap E = \bar{U}_i \cap E$,[4] and (iii) $U_i \cap E$ is maximal, with

$$X = \Lambda^E_o[T], \qquad \{\Xi^i\} = \{E^i(U_i) \cap E\}.$$

[4] This does not quite say that $U_i \in \mathsf{Face}^{lex}_E(T)$, as we are allowing U_i to intersect E on external edges as well.

(Ch0) is immediate, while (Ch2) follows from Lemma FOF_OHORN_LEM 4.25.

For (Ch1), let V be an outer face of some U_i, such that $\Xi^i_V = E^i(V) \cap E = \varnothing$. If U_i factors through an outer face of T, then V has a minimal orbital outer face GV also with $\Xi^i_{GV} = \varnothing$, and hence both GV and V are in $\Lambda^E_o[T] = X \subseteq X_{<i}$. If U_i is an inner face, then $\bar{U}_i = T_*$ and hence $E^i(U_i) \cap E \neq \varnothing$. Moreover, Lemma INNINT_LEM 2.5 implies that $E^i(V) = E^i(U_i) \cap E^i(\bar{V})$, and since by hypothesis $E^i(V) \cap E = \varnothing$, we must have $E^i(\bar{V}) \cap E = \varnothing$. Thus we have $G\bar{V}$ is an orbital outer face not equal to T, so again we may conclude that $V \in X$.

also holds by maximality

For (Ch3), we first note that if $(V - \Xi^i_V) \hookrightarrow U_j$, then so must $V \hookrightarrow U_j$ since, by maximality, $\Xi^i_V \subseteq U_j$. Thus for any $V \hookrightarrow U_i$ with $(V - \Xi^i_V) \hookrightarrow U_j$, $V \hookrightarrow (U_i \cap U_j)$, and since $\Xi^i = \Xi^j$ and $E_i = E_j$, we have $U_i \cap U_j = U_k$ with $k < i, j$. Hence $V \in X_{<i}$.

Thus $\Lambda^{Ge}_o[T] \to \Lambda^{Ge}[T]$ is in $\hat{W}_o$, and hence, by the cancelation closure property (CANCEL_EQ 3.18) of hypersaturations, so is $\Lambda^{Ge}[T] \to \Omega[T]$. □

We move now to comparing Segal core inclusions with the inner horn inclusions.

SC_IN_GHORN_PROP **Proposition 4.29.** *For all $T \in \Omega_G$, $Sc[T] \to \Omega[T]$ is inner G-anodyne.*

Proof. It is immediate that we may apply Lemma CHAREDGE_LEM 3.4 with $X = Sc[T]$, $I = \{*\}$, $\{U_i\} = T$, and $\Xi = E^i(T)$.

also see Remark 2.12 (i)

□

GORB_OHORN_PROP **Proposition 4.30.** *For any $T \in \Omega_G$ and G-sets of edges E and F, the maps*

$$\Lambda^{E \amalg F}_o[T] \to \Lambda^E_o[T]$$

are in W_o.

Proof. It suffices to show that, for any edge orbit $Ge \subseteq E$, the map $\Lambda^{E-Ge}_o[T] \to \Lambda^E_o[T]$ is in W_o. But the following commuting square is a pushout,

$$\begin{tikzcd}
\Lambda^{E-Ge}_o[T - Ge] \arrow[r] \arrow[d] & \Lambda^E_o[T] \arrow[d] \\
\Omega[T - Ge] \arrow[r] & \Lambda^{E-Ge}_o[T]
\end{tikzcd}$$

and thus the result follows by induction on $|T/G| \times |E/G|$ ordered lexicographically. □

SC_IN_OHORN_PROP **Proposition 4.31.** *For all $T \in \Omega_G$, $Sc[T] \to \Omega[T]$ is in W_o.*

Proof. Define $\mathscr{F}'_o(T)$ denote the poset $\mathscr{F}^{Ge}_o(T) \setminus \mathscr{F}_{SC}(T)$. It suffices to show that for all G-convex $B \subseteq B' \subseteq \mathscr{F}'_o(T)$, the map

$$Sc[T] \cup \bigcup_{B/G} \Omega[G \cdot_K U] \to Sc[T] \cup \bigcup_{B'/G} \Omega[G \cdot_K U]$$

is in W_o; in particular, we may assume $B' - B = G \cdot_K \{U\}$ where $\mathrm{Stab}_G(U) = K$.

Now, since B is convex, every proper orbital outer face of U is in the domain. Moreover, for any face $V \in \Omega[G \cdot_K U]$, $V \in Sc[T]$ implies $V \in Sc[G \cdot_K U] \subseteq \partial^{out}_o \Omega[G \cdot_K U]$, while on the other hand, if $V \in \Omega[G \cdot_L R]$ for some orbital face R already attached, then in fact V is a face of some intersection $U \cap gR$, which is also orbital. Thus, the above map is the pushout over

$$\partial^{out}_o \Omega[G \cdot_K U] \to \Omega[G \cdot_K U],$$

and by Proposition GORB_OHORN_PROP 4.30, the result is proven. □

The above combined with Proposition 3.12 immediately implies the following.

Corollary 4.32. *Segal core inclusions are inner G-anodyne.*

Notation 4.33. Let Ω_{SCI} and Ω_G^{SCI} denote the classes of maps in Ω and Ω_G of Segal core inclusions, and let $\hat{W}_{SC} = \hat{W}(\Omega_G^{SCI})$ denote the hypersaturation of the Segal core inclusions.

We have a natural map $G \cdot (-) : \Omega_{SCI} \to \Omega_G^{SCI}$.

Similarly, let Ω_{IHI} denote the class of inner horn inclusions in Ω.

Proposition 4.34. *Any inner horn inclusion*

$$\Lambda^{Ge}[T] \to \Omega[T]$$

is in $\hat{W}_{SC}$.

Proof. We go by induction on $|H| \times \mathcal{V}_G(T)|$ ordered lexicographically, where $T \simeq G \cdot_H T_*$ is any decomposition.

If $H = e$, then, since $G \cdot \Omega^{SCI} \subset \Omega_{G,SCI}$, and by [CM13a, 2.5] we know $\Omega_{IHI} \subset \hat{W}(\Omega_{SCI})$, we conclude that

$$G \cdot \Omega_{IHI} \subset G \cdot (\hat{W}(\Omega_{SCI})) \subset \hat{W}(G \cdot \Omega_{SCI}) \subset \hat{W}(\Omega_{G,SCI}) = \hat{W}_{SC}.$$

If $H \neq e$, and $V_G(T) = 2$, then there is only one inner edge orbit, so the Segal core is an inner horn. Thus, we may assume that $V_G(T) > 2$.

We first apply Lemma 3.4 with $I = \{U_i\} = \mathsf{Out}_p^{>1}(T)$ of proper outer faces of T not in $Sc[T]$, $X = Sc[T]$, and $\Xi = T$. (Ch0) is immediate. (Ch1) follows since $\Xi_V = \varnothing$ iff $V \in Sc[T]$, while (Ch2) follows since $(V - \Xi_V) \in Sc[T]$ iff $\Xi_V = \varnothing$. For (Ch3), if $(V - D) \hookrightarrow U_j$ with U_j an outer face of T, then V itself must be a face of U_j. Hence $V \hookrightarrow (U_i \cap U_j)$, where this intersection is either in X or is less than U_i, and thus, following the logic in the proof of Proposition 3.12, we have that $Sc[T] \to \partial^{out}\Omega[T]$ is in $\hat{W}_o$ by induction.

Now, we note by induction and Proposition 3.14 that $\partial^{out}\Omega[T] \to \Lambda^E[T]$ is also in $\hat{W}_o$, and hence the result follows by the cancellation closure property (3.18) of hypersaturations. □

As a corollary of Proposition 3.19, we will fully characterize the image of the nerve functor $N : \mathsf{Op}^G \to \mathsf{dSet}^G$.

Definition 4.35. $X \in \mathsf{dSet}^G$ is called a *(strict) inner G-Kan complex* if X has the strict right lifting property against inner G-horn inclusions. Let $\mathsf{SKan}_G \subseteq \mathsf{Kan}_G \subseteq \mathsf{dSet}^G$ denote the respective full subcategories.

SKAN_REM **Remark 4.36.** Recall that we have an adjunction

$$\begin{tikzcd} \mathsf{dSet}^G \arrow[r, "\iota^*"] & \mathsf{dSet} \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"] \end{tikzcd}$$

Denoting the image of SKan and Kan under $\iota_!$ by SKan^G and Kan^G, it is immediate that $(G \cdot \Omega_{IHI})^{\boxslash} = \mathsf{Kan}^G$ and $(G \cdot \Omega_{IHI})^{\boxslash !} = \mathsf{SKan}^G$.

SRLP_IHI_PROP **Proposition 4.37.** $(\Omega_{G,IHI})^{\boxslash !} = (G \cdot \Omega_{IHI})^{\boxslash !}$*; or equivalently,* $\mathsf{Kan}_G = \mathsf{Kan}^G$.

Proof. We recall that any G-tree $T \in \Omega_G$ has a decomposition $T \simeq (G \cdot T_*)/N$ where T_* is a component of T and N is a graph subgroup of $G \times \mathrm{Aut}(T_*)$. Similarly, any inner G-horn inclusion $\Lambda^{Ge}[T] \to \Omega[T]$ is isomorphic to a map of the form

$$(\iota_! j)/N : \iota_! \left(\Lambda^{He}[T_*] \to \Omega[T_*]\right)/N,$$

where $e \in T_*$, $H = \mathrm{Stab}_G(T_*)$, and j is a generalized inner horn inclusion in Ω. Now, for any span $\Omega[T] \xleftarrow{\iota_! j/N} \Lambda^{Ge}[T] \xrightarrow{f} X$, consider the following diagram, where $n \in N$.

$$\begin{tikzcd}
\iota_!\Lambda^{He}[T_*] \arrow[rr] \arrow[dd, "\iota_! j"'] \arrow[dr, "n"] & & \iota\Lambda^{H.e}[T_*]/N \simeq \Lambda^{Ge}[T] \arrow[r, "f"] \arrow[dd, "\iota_! j/N"] & X \\
& \iota_!\Lambda^{H.e}[T_*] \arrow[ur, two heads] \arrow[dd] & & \\
\iota_!\Omega[T_*] \arrow[rr] \arrow[dr, "n"] \arrow[urrr, dotted, "\exists ! \phi"] & & \iota_!\Omega[T_*]/N \simeq \Omega[T] & \\
& \iota_!\Omega[T_*] \arrow[ur, two heads] \arrow[uuurr, dotted, "\exists ! \phi"'] & &
\end{tikzcd}$$

Since $X \in \mathsf{SKan}^G$, and the hypersaturations (in Ω) of the inner horn inclusions and the generalized inner horn inclusions coincide by [MW09, Lemma 5.1] (or Proposition 3.19), we have unique lifts ϕ as written above by Lemma 4.22 and Remark 4.36. Uniqueness then implies that $\phi = n \cdot \phi$ for all $n \in N$, and hence factors through $\Omega[T]$, finishing the proof. ☐

Corollary 4.38. *Fix $X \in \mathsf{dSet}^G$. Then $X \in \mathsf{SKan}_G$ iff $X \simeq N^G(\mathcal{O})$ for some $\mathcal{O} \in \mathsf{Op}^G$.*

Proof. $N^G(\mathcal{O})(T)$ is given by maps $\mathsf{Op}^G(\Omega(T), \mathcal{O})$, and since $\Omega(T)$ is a free coloured operad on it's vertices, we may conclude that

$$N^G(\mathcal{O})(T) = \mathsf{Op}^G(\Omega(T), \mathcal{O}) \simeq \mathsf{dSet}^G(Sc[T], N^G\mathcal{O}).$$

Thus $N^G(\mathcal{O})$ is in $(\Omega_{G,SCI})^{\boxslash !}$, and hence is in $(\Omega_{G,IHI})^{\boxslash !}$ by Lemma 4.22 and Proposition 4.29.

Conversely, by Proposition 4.37 and [MW09, Theorem 6.1, Proposition 6.10], we have that $X \simeq N_d \circ \tau_d \circ X$ (where we consider X as a functor $G \to \mathsf{Kan}$). ☐

5 Quillen equivalences

Our main goal in this section is to prove Theorems 5.18 and 5.27, which jointly establish the Quillen equivalence of three model categories: the category of equivariant dendroidal sets dSet^G with the "G-∞-operad" model structure of [Per17, Thm 2.1]; the category of equivariant dendroidal spaces sdSet^G with the "complete equivariant dendroidal Segal space" model structure in §5.2 and; the category of equivariant preoperads PreOp^G with the "equivariant Segal operad" model structure in §5.3.

Our perspective will be that these Quillen equivalences are best understood in light of the equivariant analogue of [CM13a, Thm. 6.6], which says that the complete dendroidal space model structure on $\mathsf{sdSet} = \mathsf{dSet}^{\Delta^{op}}$ can be obtained via two distinct left Bousfield localization procedures. As such, we will find it helpful to first focus on the abstract properties of such "joint left Bousfield localizations".

5.1 Joint left Bousfield localizations

JOINBOUS SEC

Throughout we assume familiarity with the theory of left Bousfield localizations as in [Hir03].

COMBMODSTR PROP

Proposition 5.1. *Suppose that the category $\mathcal{C}$ admits two model structures (C, W_1, F_1) and (C, W_2, F_2) with a common class of cofibrations C, and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

Then (C, W_1, F_1), (C, W_2, F_2) have a smallest joint left Bousfield localization (C, W, F) and:

(i) $c \in \mathcal{C}$ is (C, W, F)-fibrant iff it is simultaneously (C, W_1, F_1)-fibrant and (C, W_2, F_2)-fibrant;

(ii) for (C, W, F)-fibrant $c, d \in \mathcal{C}$ one has that $c \to d$ is in W iff it is in W_1 iff it is in W_2.

Proof. The joint model structure (C, W, F) can be obtained by either left Bousfield localizing (C, W_1, F_1) with regards to the generating trivial cofibrations of (C, W_2, F_2) or vice-versa. That the two processes yield the same model structure follows from from the universal property of left Bousfield localizations [Hir03, Prop. 3.4.18].

(ii) follows from the local Whitehead theorem [Hir03, Thm. 3.3.8], stating that the local equivalences between local objects match the initial weak equivalences.

For (i), the claim that joint fibrant objects are fibrant in both of the original model structures follows since $C \cap W$ contains both $C \cap W_1$ and $C \cap W_2$ (in fact, this shows that $F \subseteq F_1 \cap F_2$). The converse claim follows from the observation that fibrant objects in any model structure are already local with respect to the weak equivalences in that same model structure. □

The prototypical example of Proposition 5.1 is given by the category ssSet of bisimplicial sets together with the two possible Reedy structures (over the Kan model structure on sSet). Explicitly, writing the levels of $X \in \mathsf{ssSet}$ as $X_n(m)$ one can either form a Reedy model structure with respect to the *horizontal index* m or with respect to the *vertical index* n.

In either case, the generating cofibrations are then given by the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Delta[m] \to \Delta[m]), \qquad n, m \geq 0.$$

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square\, (\partial\Delta[m] \to \Delta[m]), \qquad n \geq i \geq 0, m \geq 0. \tag{5.2}$$ GTRCOHOR EQ

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square \left(\Lambda^j[m] \to \Delta[m]\right), \qquad n \geq 0, m \geq j \geq 0. \tag{5.3}$$ GTRCOVER EQ

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, i.e. maps inducing Kan equivalences of simplicial sets $X_\bullet(m) \to Y_\bullet(m)$ for each $m \geq 0$, and dually for the vertical Reedy structure.

UNIQUELIM NOT **Notation 5.4.** Given a fixed $X \in \mathsf{ssSet}$ we will also write $X_{(-)}\colon \mathsf{sSet}^{op} \to \mathsf{sSet}$ for the unique limit preserving functor such that $X_{\Delta[n]} = X_n$.

In the next result we refer to the localized model structure given by Proposition 5.1 as the *joint Reedy model structure* and we write $\delta^*\colon \mathsf{ssSet} \to \mathsf{sSet}$ for the diagonal functor.

SSSETJREE PROP **Proposition 5.5.** *Suppose that $X, Y \in \mathsf{ssSet}$ are horizontal Reedy fibrant. Then:*

(i) for each fixed n all vertex maps $X_n \to X_0$ are trivial Kan fibrations in sSet;

(ii) any vertical Reedy fibrant replacement $\tilde{X}$ of X is fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a joint weak equivalence iff it is a horizontal weak equivalence iff $X_0 \to Y_0$ is a Kan equivalence in sSet;

(iv) the canonical map $X_0 \to \delta^*(X)$ (with levels $X_0(n) \to X_n(n)$ induced by degeneracies) is a Kan equivalence in sSet.

Proof. (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form $(\Delta[0] \to \Delta[n]) \,\square\, (\partial\Delta[m] \to \Delta[m])$.

For (ii), the fact that $\tilde{X}$ is vertical fibrant implies that for any monomorphism $K \to L$ in sSet the induced map $\tilde{X}_L \to \tilde{X}_K$ with K is a Kan fibration. Therefore, (i) implies that all vertex maps $\tilde{X}_n \to \tilde{X}_0$ are trivial Kan fibrations, so that by Remark 3.21 one has that $\tilde{X}_L \to \tilde{X}_K$ is a trivial Kan fibration whenever $K \to L$ is anodyne. Therefore, $\tilde{X}$ is horizontal fibrant, as desired.

The first "iff" in (iii) follows from (ii) since the localizing maps $X \to \tilde{X}$, $Y \to \tilde{Y}$ are horizontal equivalences while the second "iff" in (iii) follows from (i).

For (iv), note first that $\delta^*\colon \mathsf{ssSet} \to \mathsf{sSet}$ is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure). But noting that all objects are cofibrant, and regarding X_0 as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map $X_0 \to X$ is a horizontal weak equivalence in ssSet. □

WEAKDIAG COR **Corollary 5.6.** *A map $f\colon X \to Y$ in ssSet is a joint equivalence iff it induces a Kan equivalence on diagonals $\delta^*(X) \to \delta^*(Y)$ in sSet.*

Proof. Since horizontal Reedy fibrant replacement maps $X \to \tilde{X}$ are diagonal equivalences, one reduces to the case of X, Y horizontal Reedy fibrant. The result now follows by combining Proposition 5.5 (iii) and (iv). □

SSETSSETADJ COR **Corollary 5.7.** *The adjunction*

$$\delta_!\colon \mathsf{sSet} \rightleftarrows \mathsf{ssSet}\colon \delta^*$$

is a Quillen equivalence. Moreover, if a map $f\colon X \to Y$ in ssSet has the right lifting property against both sets of maps in (5.2) and (5.3), then $\delta^*(f)$ is a Kan fibration in sSet.

Note that the "moreover" claim in this result is not quite formal, since the maps in (5.2),(5.3) are not known to be generating trivial cofibrations for the joint model structure in ssSet.

Proof. Recall that $\delta_!$ is the unique colimit preserving functor such that $\delta_!(\Delta[n]) = \Delta[n] \times \Delta[n]$.

To see that $\delta_!$ preserves cofibrations it is enough to show that $\delta_!\,(\partial\Delta[n] \to \Delta[n])$ is a monomorphism for all $n \geq 0$. This holds since: (i) any two face inclusions $F_1 \to \Delta[n]$, $F_2 \to \Delta[n]$ factor through a minimal face inclusion $F \to \Delta[n]$ (indeed, faces are indexed by subsets of $\{0, 1, \cdots, n\}$); (ii) for any face inclusion one has $\delta_!\left(F \to \Delta[n]\right) = \left(F^{\times 2} \to \Delta[n]^{\times 2}\right)$, which is a monormorphism.

The claim that $\delta_!$ preserves trivial cofibrations easily follows from Remark 3.21 together with Corollary 5.6, but here we give a harder argument needed to establish the stronger "moreover" claim. Namely, we will argue that the maps $\delta_!\left(\Lambda^i[n] \to \Delta[n]\right)$ are built cellularly out of the maps in (5.2), (5.3). One has a factorization

$$\delta_! \Lambda^i[n] \to \Lambda^i[n] \times \Delta[n] \to \Delta[n]^{\times 2}$$

where the second map is clearly built cellularly out of the maps in (5.2), and we claim that the first map is likewise built cellularly out of the maps in (5.3). Indeed, this first map be built by iteratively attaching the maps

$$\left(\Lambda^F[n] \to \Lambda^i[n]\right) \square \left(\Lambda^i F \to F\right)$$

where F ranges over the poset $\mathsf{Face}_{\supsetneq\{i\}}$ of faces of $\Delta[n]$ strictly containing $\{i\}$ (note that for $F = \Delta[n]$ it is $\Lambda^F[n] = \varnothing$, so that these maps are not in general built out of the maps in (5.2)).

Lastly, the Quillen equivalence condition is that for all $X \in \mathsf{sSet}$ and joint fibrant $Y \in \mathsf{ssSet}$ a map $X \to \delta^* Y$ is a weak equivalence iff $\delta_! X \to Y$ is. But by Corollary 5.6 this reduces to showing that the unit maps $X \to \delta^* \delta_! X$ are weak equivalences. This latter claim follows by cellular induction on X, since those pushouts attaching cells are homotopy pushouts (due to sSet being left proper). □

HYPERSIMPL REM **Remark 5.8.** Just as in the proof of Proposition 5.5, one can use hypersaturations to simplify the lifting condition in the previous result. Namely, $X \to Y$ is a vertical fibration (i.e. it has the lifting property against (5.3)) iff, for each monomorphism $K \to L$ in sSet, $X_L \to X_K \times_{Y_K} Y_L$ is a Kan fibration. The lifting property against (5.3) is then the condition that $X_L \to X_K \times_{Y_K} Y_L$ is a trivial Kan fibration when $K \to L$ is a horn inclusion. But then a straightforward hypersaturation argument together with Remark 3.21 show that it suffices to check that the maps $X_n \to X_0 \times_{Y_0} Y_n$, induced by the vertex maps $[0] \to [n]$, are trivial Kan fibrations.

Remark 5.9. The adjunction $\delta^* \colon \mathsf{ssSet} \rightleftarrows \mathsf{sSet} \colon \delta_*$ can also be shown to be a Quillen equivalence.

5.2 Complete equivariant dendroidal Segal spaces

CEDSS SEC

We now turn to our main application of Proposition 5.1, the category $\mathsf{sdSet}^G = \mathsf{Set}^{\Delta^{op} \times \Omega^{op} \times G}$ of G-equivariant simplicial dendroidal sets.

Since Δ is a (usual) Reedy category the model structure on dSet^G in [Per17, Thm. 2.1] induces a model structure on sdSet^G that we will refer to as the *simplicial Reedy model structure*.

On the other hand, in the context of Definition A.2, $\Omega^{op} \times G$ is a generalized Reedy category such that the families $\{\mathcal{F}^\Gamma_U\}_{U \in \Omega}$ of G-graph subgroups are Reedy-admissible (see Example A.6) and hence, using the underlying Kan model structure on sSet, Theorem A.8 yields a model structure on sdSet^G that we will refer to as the *equivariant dendroidal Reedy model structure*, or simply as the *dendroidal Reedy model structure* for the sake of brevity.

Throughout, we will write the levels of $X \in \mathsf{sdSet}^G$ as $X_n(U)$. We now extend Notation 5.4. Note that the representable functor of $U \in \Omega \times G^{op}$ is given by $\Omega[G \cdot U] = G \cdot \Omega[U]$.

UNILIMDEN NOT **Notation 5.10.** Given a fixed $X \in \mathsf{sdSet}^G$ we will also write

$$X(-) \colon \left(\mathsf{dSet}^G\right)^{op} \to \mathsf{sSet}, \qquad X_{(-)} \colon \mathsf{sSet}^{op} \to \mathsf{dSet}^G$$

for the unique limit preserving functors such that $X(\Omega[G \cdot U]) = X(U)$, $X_{\Delta[n]} = X_n$.

Moreover, for fixed $J \in \mathsf{dSet}^G$ we define $X^J \in \mathsf{sdSet}^G$ by $X^J(U) = X\left(\Omega[G \cdot U] \otimes J\right)$.

JM NOT **Notation 5.11.** Writing $\widetilde{[m]} = (0 \rightleftarrows 1 \rightleftarrows \cdots \rightleftarrows m)$ for the contractible groupoid with objects $0, 1, \cdots, m$, we denote

$$J^m = \iota_!(N\widetilde{[m]})$$

where N is the nerve functor. J^m is regarded as equipped with the trivial G-action. Further, we abbreviate $J = J^1$.

Proposition 5.12. *Both the simplicial and dendroidal Reedy model structures on* sdSet^G *have generating cofibrations given by the maps*

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq 0, T \in \Omega_G. \tag{5.13}$$ JOINTCOF EQ

Further, the dendroidal Reedy structure has as generating trivial cofibrations the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq i \geq 0, T \in \Omega_G. \tag{5.14}$$ DENDTRIVCOF EQ

while the simplicial Reedy structure has as generating trivial cofibrations the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (A \to B), \qquad n \geq 0 \tag{5.15}$$ SIMPTRIVCOF EQ

for $\{A \to B\}$ *a set of generating trivial cofibrations of* dSet^G.

Proof. For the claims concerning the dendroidal Reedy structure, note that the presheaves $\Omega[T] \in \mathsf{dSet}^G$ are precisely the quotients $(G\cdot\Omega[U])/K$ for $U \in \Omega$ and $K \leq G \times \mathsf{Aut}(U)$ a G-graph subgroup, so that $\partial\Omega[T] \to \Omega[T]$ represents the maps $X(U)^K \to (M_U X)^K$ for $X \in \mathsf{dSet}^G$.

The claims concerning the simplicial Reedy structure are immediate. □

We call the saturation of the maps in (5.13) the class of *normal monomorphisms* of sdSet^G.

JOINTFIBCHAR COR **Corollary 5.16.** *The joint fibrant objects* $X \in \mathsf{sdSet}^G$ *have the following equivalent characterizations:*

(i) X *is both simplicial Reedy fibrant and dendroidal Reedy fibrant;*

(ii) X *is simplicial Reedy fibrant and all maps* $X_0 \to X_n$ *are equivalences in* dSet^G*;*

(iii) X *is dendroidal Reedy fibrant and all maps*

$$X(\Omega[T]) \to X(Sc[T]) \qquad \text{and} \qquad X(\Omega[T]) \to X(\Omega[T] \otimes J)$$

for $T \in \Omega_G$ *are Kan equivalences in* sSet.

Proof. (i) simply repeats Proposition 5.1(i). In the remainder we write $K \to L$ for a generic monomorphism in sSet and $A \to B$ a generic normal monomorphism in dSet^G.

For (ii), note that X is simplicial fibrant iff $X_L \to X_K$ is always a fibration in dSet^G. Thus, X will also have the right lifting property against (5.14) iff $X_L \to X_K$ is a trivial fibration whenever $K \to L$ is anodyne. But by Remark 3.21 it suffices to consider the vertex inclusions $\Delta[0] \to \Delta[n]$. The claim now follows from 2-out-of-3 applied to the composites $X_0 \to X_n \to X_0$.

For (iii), note first that X is dendroidal fibrant iff $X(B) \to X(A)$ is always a Kan fibration. Therefore, X will have the right lifting property against (5.15) iff $X(B) \to X(A)$ is a trivial Kan fibration whenever $A \to B$ is a generating trivial cofibration of dSet^G. By adjunction, this is equivalent to showing that $X_L \to X_K$ is a fibration in dSet^G for any monomorphism $K \to L$ in sSet. Moreover, by the fibration between fibrant objects part of [Per17, Prop. 8.8] (see also the beginning of [Per17, §8.1]) it suffices to verify that the maps $X_L \to X_K$ have the right lifting property against the maps

$$\Lambda^{Ge}[T] \to \Omega[T], \quad T \in \Omega_G, e \in E^{\mathrm{i}}(T) \qquad \text{and} \qquad \Omega[T] \otimes (\{i\} \to J_d), \quad T \in \Omega_G, i = \{0,1\}$$

and it thus suffices to check that $X(B) \to X(A)$ is a trivial Kan fibration when $A \to B$ is one of these maps. Proposition 3.19 now finishes the proof. □

We now obtain the following partial analogue of Proposition 5.5. Note that the equivalences in the simplicial Reedy model structure are the dendroidal equivalences and vice versa.

SDSETG COR **Corollary 5.17.** *Suppose that $X, Y \in \mathsf{sdSet}^G$ are dendroidal Reedy fibrant. Then:*

(i) for all n the vertex maps $X_n \to X_0$ are trivial fibrations in dSet^G;

(ii) any simplicial Reedy fibrant replacement $\tilde{X}$ of X is in fact fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a joint weak equivalence iff it is a dendroidal weak equivalence iff $X_0 \to Y_0$ is an equivalence in dSet^G;

(iv) regarding X_0 as a simplicially constant object in sdSet^G, the map $X_0 \to X$ is a dendroidal equivalence, and thus a joint equivalence.

Proof. The proof adapts that of Proposition 5.5. (i) follows since X then has the right lifting property with respect to all maps $(\Delta[0] \to \Delta[m]) \square (\partial\Omega[T] \to \Omega[T])$. (ii) follows from (i) and the characterization in Corollary 5.16 (ii). The first "iff" in (iii) follows from (ii) since the simplicial fibrant replacement maps $X \to \tilde{X}$ are dendroidal equivalences and the second "iff" in (iii) follows from (i). (iv) follows from (i). □

INCOAGJ THM **Theorem 5.18.** *The constant/0-th level adjunction*

$$c_! \colon \mathsf{dSet}^G \rightleftarrows \mathsf{sdSet}^G \colon (-)_0$$

where sdSet^G is given the joint Reedy model structure, is a Quillen equivalence.

Proof. It is clear that the constant functor $c_!$ preserves both normal monomorphisms and all weak equivalences, hence the adjunction is Quillen. Consider any map $c_!(A) \to X$ with X joint fibrant and perform a "trivial cofibration followed by fibration" factorization as on the left

$$c_!(A) \overset{\sim}{\rightarrowtail} \widetilde{c_!(A)} \twoheadrightarrow X \qquad A \xrightarrow{\sim} \widetilde{c_!(A)}_0 \to X_0$$

for the simplicial Reedy model structure. Corollary 5.16(ii) now implies that $\widetilde{c_!(A)}$ is in fact joint fibrant and thus that the leftmost composite is a joint equivalence iff $\widetilde{c_!(A)} \to X$ is a dendroidal equivalence in sdSet^G iff $\widetilde{c_!(A)}_0 \to X_0$ is an equivalence in dSet^G iff the rightmost composite is an equivalence in dSet^G. □

CONCRECOM REM **Remark 5.19.** Given a G-∞-operad $X \in \mathsf{dSet}^G$, one can obtain an explicit model for $\widetilde{c_!(X)}$ as the object $X^{J^\bullet} \in \mathsf{sdSet}^G$. Indeed, since $J^\bullet$ is a Reedy cofibrant cosimplicial object in dSet^G, one has that $X^{J^\bullet} \in \mathsf{sdSet}^G$ is simplicial fibrant. Hence, by Corollary 5.16(ii) $c_!(X) \to X^{J^\bullet}$ will indeed be a joint fibrant replacement provided that it is a dendroidal equivalence. But this follows from [Per17, Cor. 8.21], which implies that the maps $X^{J^m} \to X^{J^0} = X$ are trivial fibrations in dSet^G (formally, [Per17, Cor. 8.21] says that $\upsilon_*(X^{J^m}) = X^{(J^m)} \to \upsilon_*(X)$ is a trivial fibration in dSet_G, which is an equivalent statement, as noted at the end of the proof of [Per17, Thm. 8.22]).

5.3 Equivariant Segal operads

PREOP SEC

Recall that the category PreOp of *pre-operads* is the full subcategory $\mathsf{PreOp} \subset \mathsf{sdSet}$ of those X such that $X(\eta)$ is a discrete simplicial set. Writing γ^* for the inclusion one has left and right adjoints $\gamma_!$ and γ_*

$$\begin{tikzcd} \mathsf{PreOp}^G \arrow[r, "\gamma^*"'] & \mathsf{sdSet}^G \arrow[l, bend right, "\gamma_!"'] \arrow[l, bend left, "\gamma_*"] \end{tikzcd}$$

described as follows [CM13a, §7]: $\gamma_! X(U) = X(U)$ if $U \notin \Delta$ while $\gamma_! X([n])$ for $[n] \in \Delta$ is given by the pushout on the left below; $\gamma_* X(U)$ is given by the pullback on the right below.

$$\begin{tikzcd} X(\eta) \arrow[r] \arrow[d] & \pi_0 X(\eta) \arrow[d] \\ X([n]) \arrow[r] & \gamma_! X([n]) \end{tikzcd} \qquad \begin{tikzcd} \gamma_* X(U) \arrow[r] \arrow[d] & X(U) \arrow[d] \\ \prod_{E(U)} X_0(\eta) \arrow[r] & \prod_{E(U)} X(\eta) \end{tikzcd}$$

GAMMASH REM

Remark 5.20. Any monomorphism $A \to B$ in sdSet^G such that $A(\eta) \to B(\eta)$ is an isomorphism induces a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \gamma^*\gamma_! A \arrow[d] \\ B \arrow[r] & \gamma^*\gamma_! B \end{tikzcd} \tag{5.21}$$

GAMMASH EQ

Noting that the assignment $U \mapsto \prod_{E(U)} Y(\eta)$ is the coskeleton $\mathsf{csk}_\eta Y$ leads to the following.

CSKETALT PROP

Proposition 5.22. *Let $X \in \mathsf{sdSet}^G$. Then:*

(i) if $X \in \mathsf{sdSet}^G$ is dendroidal Reedy fibrant then so is $\gamma^*\gamma_* X$;

(ii) regarding X_0 as a simplicially constant object of sdSet^G, the left square below is a pullback;

(iii) if $A \to A'$ is a map in dSet^G such that $A(\eta) \simeq A'(\eta)$, the right square below is a pullback.

$$\begin{tikzcd} \gamma^*\gamma_* X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathsf{csk}_\eta X_0 \arrow[r] & \mathsf{csk}_\eta X \end{tikzcd} \qquad \begin{tikzcd} \gamma^*\gamma_* X(A') \arrow[r] \arrow[d] & X(A') \arrow[d] \\ \gamma^*\gamma_* X(A) \arrow[r] & X(A) \end{tikzcd}$$

Proof. (ii) is immediate from the observation that $\mathsf{csk}_\eta Y = \prod_{E(-)} Y(\eta)$. Moreover, it readily follows that for $B \in \mathsf{dSet}^G$ it is $(\mathsf{csk}_\eta Y)(B) = \prod_{B(\eta)} Y(\eta)$ and thus (iii) follows from (ii).

For (i), formal considerations imply that if X is dendroidal fibrant then the map $X \to \mathsf{csk}_\eta X$ is a dendroidal fibration (and that $\mathsf{csk}_\eta X$ is dendroidal fibrant). Hence, the result will follow provided that $\mathsf{csk}_\eta X_0$ is also dendroidal fibrant. But since $\mathsf{csk}_\eta X_0$ is η-coskeletal, it suffices to check that the η-matching map $(\mathsf{csk}_\eta X_0)(\eta) \to M_\eta(\mathsf{csk}_\eta X_0)$ is a G-fibration in sSet^G. But this is simply $X_0(\eta) \to *$ regarded as a map of constant simplicial sets, and the result follows. ☐

Notation 5.23. In the remainder of the section we write $\mathcal{I}'$ for the set of maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq 0, T \in \Omega_G, T \neq G/H \cdot \eta.$$

BOUNDRED EQ

Further, we note that Remark 5.20 applies to these maps.

GENSET LEM **Lemma 5.24.** *The maps in* PreOp^G *that are normal monomorphisms in* sdSet^G *are the saturation of the set of maps* $\{\varnothing \to G/H \cdot \eta : H \leq G\} \cup \gamma_!(\mathcal{I}')$.

Proof. Using the cellular filtration in sdSet^G, any normal monomorphism $A \to B$ in PreOp^G can (upon inclusion) be written as a transfinite composition of pushouts of maps in $\{\varnothing \to G/H \cdot \eta\} \cup \mathcal{I}'$. But since the squares (5.21) are pushouts the same also holds for $\{\varnothing \to G/H \cdot \eta\} \cup \gamma_!(\mathcal{I}')$. □

TRIVFIB LEM **Lemma 5.25.** *Any map in* PreOp^G *which has the right lifting property against all normal monomorphisms in* PreOp^G *is a joint equivalence in* sdSet^G.

Proof. We simply adapt the proof of [CM13a, Lemma 8.12] mutatis mutandis.

Choose a normalization E_∞ of $*$ in dSet^G, i.e. a normal object such that $E_\infty \to *$ is a trivial fibration. Regarding E_∞ as a simplicially constant object in sdSet^G, a map $X \to Y$ in PreOp^G will have the right lifting property against all monomorphisms iff so does $E_\infty \times (X \to Y)$, so that one is free to assume that X, Y are normal.

One is thus free to pick a section $s\colon Y \to X$ of $p\colon X \to Y$ and regarding $J \in \mathsf{dSet}^G$ as a simplicially constant object of sdSet^G our assumption yields the lift below, so that p is a homotopy equivalence.

$$\begin{tikzcd}
X \amalg X \arrow[r, "(id_X, sp)"] \arrow[d] & X \arrow[d, "p"] \\
X \otimes J \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

□

Theorem 5.26. *The category* Preop^G *of* G*-preoperads has a model structure such that*

- *the cofibrations are the normal monomorphisms;*
- *the weak equivalences are the maps that become joint equivalences when regarded as maps in* sdSet^G.

Proof. One repeats the proof of the non-equivariant analogue [CM13a, Thm. 8.13], applying J. Smith's theorem [Bek00, Thm. 1.7] with the required set of generating cofibrations the set $\{\varnothing \to G/H \cdot \eta | H \leq G\} \cup \gamma_!(\mathcal{I}')$ given by Lemma 5.24. Indeed, conditions c0 and c2 in [Bek00] are inherited from sdSet^G and c1 follows from Lemma 5.25. The technical "solution set" condition c3 follows from [Bek00, Prop. 1.15] since weak equivalences are accessible, being the preimage by γ^* of the weak equivalences in sdSet^G (see [Lur09, Cor. A.2.6.5] and [Lur09, Cor. A.2.6.6]). □

ANOQUEQUIV THM **Theorem 5.27.** *The adjunction*

$$\gamma^*\colon \mathsf{PreOp}^G \rightleftarrows \mathsf{sdSet}^G \colon \gamma_*$$

is a Quillen equivalence.

Proof. It is tautological that the left adjoint γ^* preserves and detects cofibrations and weak equivalences, so it suffices to show that for all fibrant $X \in \mathsf{sdSet}^G$ the counit map $\gamma^*\gamma_* X \to X$ is a weak equivalence. But by Proposition 5.22(i) both $\gamma^*\gamma_* X$ and X are dendroidal fibrant, so that the result follows from Corollary 5.17(iii) together with the observation that $(\gamma^*\gamma_* X)_0 = X_0$. □

We will find it useful to also have a characterization of the fibrant objects in PreOp^G. In doing so, it becomes useful to consider a fourth model structure on the category sdSet^G whose fibrant objects "interpolate" between the fibrant objects in the two model structures in Theorem 5.27. This is the model structure of equivariant dendroidal Segal spaces, that we discuss in the next section.

6 Equivariant dendroidal Segal spaces and DK-equivalences

HERE

6.1 Equivariant dendroidal Segal spaces

Definition 6.1. The *equivariant Segal space model structure* on the category sdSet^G, which we denote sdSet^G_S, is the left Bousfield localization of the dendroidal Reedy model structure with respect to the equivariant Segal core inclusions

$$Sc[T] \to \Omega[T], \qquad T \in \Omega_G.$$

Remark 6.2. By Proposition 3.19 this model structure can equivalently be obtained by localizing with respect to the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$.

FIB_PREOP_NOT **Notation 6.3.** We will refer to the fibrant objects in sdSet^G_S as *equivariant dendroidal Segal spaces*, or just *dendroidal Segal spaces*. Further, a pre-operad $X \in \mathsf{PreOp}^G$ is called *fibrant* if $\gamma^* X$ is a dendroidal Segal space.

Remark 6.4. If $X \in \mathsf{sdSet}^G$ is a dendroidal Segal space, then $\gamma_* X \in \mathsf{PreOp}^G$ is fibrant. Indeed, Proposition ?? ensures that $\gamma_* X$ is dendroidal fibrant while the Segal condition follows since the equality $Sc[T](\eta) = \Omega[T](\eta)$ implies that the square below is a pullback square.

$$\begin{tikzcd}
(\gamma_* X)(\Omega[T]) \arrow[r] \arrow[d] & X(\Omega[T]) \arrow[d] \\
(\gamma_* X)(Sc[T]) \arrow[r] & X(Sc[T])
\end{tikzcd}$$

The following is a variation on Definition 6.37.

MAPSPACESEG DEF **Definition 6.5.** Given a dendroidal Segal space $X \in \mathsf{sdSet}^G_S$ and a C-profile $(x_1, \cdots, x_n; x_0)$ on X (defined exactly as in Definition 6.36) we define the space of maps $X(x_1, \cdots, x_n; x_0)$ via the pullback

$$\begin{tikzcd}
X(x_1, \cdots, x_k; x_0) \arrow[r] \arrow[d] & X(\Omega[C]) \arrow[d] \\
\Delta[0] \arrow[r, "{(x_1, \cdots, x_k; x_0)}"'] & \prod_{0 \le i \le k} X(\eta)^{H_i}
\end{tikzcd}$$

Definition 6.6. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. The *homotopy genuine operad* $ho(X) \in \mathsf{dSet}_G$ is defined by

$$ho(X) = \pi_0 \left(\upsilon_* \left(\gamma_* X \right) \right).$$

Remark 6.7. It is immediate that $X(x_1, \cdots, x_k; x_0) = \gamma_* X(x_1, \cdots, x_k; x_0)$, so that both of the previous definitions depend only on the fibrant pre-operad $\gamma_* X$.

Remark 6.8. It is important not to confuse Definitions 6.37 and 6.5. Indeed, when X is a dendroidal Segal, its 0-th level X_0 is a G-∞-operad, and one can thus form two "spaces of maps" $X_0(x_1, \cdots, x_k; x_0)$ (cf. Definition 6.37) and $X(x_1, \cdots, x_k; x_0)$ (cf. Definition 6.5). The constructions leading to these spaces are quite different. When X is complete Segal, the fact that these two spaces are homotopic follows from Remark 5.19, since X must then be both dendroidally and simplicially equivalent to $X_0^{J_d(m)}$. The claim that this holds without completeness is harder, with the rest of the section devoted to establishing this.

Remark 6.9. Writing ι for the inclusion $\Delta \to \Omega$ and ι_G for the composite inclusion $\Delta \times \mathsf{O}_G \to \Omega \times \mathsf{O}_G \to \Omega_G$, one has that $\iota_G^* ho(X)$ is the G-coefficient system of categories formed by the homotopy categories $ho\left(\iota^*\left(X^H\right)\right) = \pi_0\left(\iota^*\gamma_* X^H\right)$.

Definition 6.10. A map $f \colon X \to Y$ of equivariant dendroidal Segal spaces is called

- *fully faithful* is for all $C \in \Sigma_G$ and C-profile $(x_1, \cdots, x_n; x_0)$ on X the map
$$X(x_1, \cdots, x_k; x_0) \to Y\left(f(x_1), \cdots, f(x_k); f(x_0)\right)$$
is a Kan equivalence in sSet;
- *essentially surjective* if the map $\iota_G^* ho(X) \to \iota_G^* ho(Y)$ is essentially surjective on all category levels of the G-coefficient system;
- a *DK-equivalence*[5] if it is both fully faithful and essentially surjective.

Remark 6.11. This definition depends only on the map $\gamma_* X \to \gamma_* Y$ of fibrant pre-operads.

Definition 6.12. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. We call a point $f \in X(\Omega[C_{H/H}])_0$ a H-equivalence if the corresponding class

$$[f] \in ho(X)(C_{H/H}) = ho\left(\iota^*\left(X^H\right)\right)(1) = \pi_0\left(\iota^*\gamma_* X^H\right)(1)$$

is an isomorphism.

DK-equivalences will provide an explicit description of complete/joint equivalences between dendroidal Segal objects (and thus also between fibrant pre-operads), as we will prove in whatever below. We now introduce some auxiliary notions.

Suppose $C, D \in \Sigma_G$ are G-corollas that can be grafted, i.e. that C has a leaf orbit and D a root orbit both isomorphic to G/H. Denote this orbit as Ge and write $T = C \amalg_{Ge} D$ for the grafted G-tree. For any dendroidal Segal space X one then has $X(Sc[T]) \simeq X(\Omega[C]) \times_{X(\eta)^H} X(\Omega[D])$ and one can hence form the section in the middle row below

$$\begin{tikzcd}
\{\varphi\} \times X(z_1, \cdots, z_l; e) \arrow[rr, dashed, "\varphi \circ_{Ge} (-)"] \arrow[d, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[d, hook] \\
X(\Omega[C]) \times_{X(\eta)^H} X(\Omega[D]) \arrow[r, dashed, bend left] & X(\Omega[T]) \arrow[l, two heads, "\sim"] \arrow[r, two heads] & X(\Omega[T - Ge]) \\
X(e, y_2, \cdots, y_k; x) \times \{\psi\} \arrow[rr, dashed, "(-) \circ_{Ge} \psi"'] \arrow[u, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[u, hook]
\end{tikzcd} \tag{6.13}$$

HOMOTCIRC EQ

thus defining maps $\varphi \circ_{Ge} (-)$ (resp. $(-) \circ_{Ge} \psi$) for any choice of $\varphi \in X(e, y_2, \cdots, y_k; x)$ (resp. $\psi \in X(z_1, \cdots, z_l; e)$).

GENOPHO PROP

Proposition 6.14. *The maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are well defined up to homotopy. Further, if $\varphi, \bar{\varphi} \in X(e, y_2, \cdots, y_k; x)$ are homotopic (i.e. in the same connected component), then the maps $\varphi \circ_{Ge} (-)$, $\bar{\varphi} \circ_{Ge} (-)$ are homotopic, and likewise for ψ.*

In particular, $\varphi \circ_{Ge} \psi \in X(z_1, \cdots, z_l, y_2, \cdots, y_k; x)$ is well defined up to homotopy.

Lastly, the maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are functorial in maps $f \colon X \to Y$ between dendroidal Segal spaces.

[5] Here DK stands for Dwyer and Kan.

Proof. This is immediate once one notes that, writing $E = T - Ge$ for the "composite G-corolla", all solid maps in (6.13) are compatible with the projections to $X(\partial\Omega[E])$.

For functoriality, one simply notes that either type of section as on the leftmost diagram

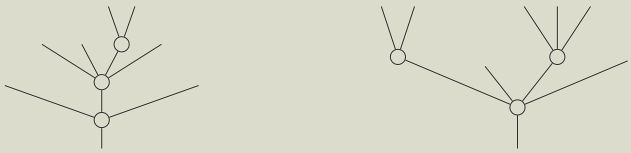
$$
\begin{tikzcd}
X(\Omega[T]) \arrow[r] \arrow[d, two heads, "\sim"] & Y(\Omega[T]) \arrow[d, two heads, "\sim"] \\
X(\Omega[C]) \underset{X(\eta)^H}{\times} X(\Omega[D]) \arrow[u, dashed, bend left] \arrow[r] & Y(\Omega[C]) \underset{Y(\eta)^H}{\times} Y(\Omega[D]) \arrow[u, dashed, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & Y(\Omega[T]) \arrow[d, two heads, "\sim"] \\
X(\Omega[C]) \underset{X(\eta)^H}{\times} X(\Omega[D]) \arrow[r] \arrow[ur, dashed] & Y(\Omega[C]) \underset{Y(\eta)^H}{\times} Y(\Omega[D])
\end{tikzcd}
$$

induces lifts as on the rightmost diagram. However, a standard argument shows that all such lifts are homotopic over $X(\partial\Omega[E])$. □

We will now show that the operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy the obvious compatibilities one expects, but we will find it convenient to first package these compatibilities into a common format. In the categorical case (corresponding to linear trees), there are three types of compatibilities, corresponding to homotopies

$$\varphi \circ (\psi \circ (-)) \sim (\varphi \circ \psi) \circ (-) \qquad \varphi \circ ((-) \circ \psi) \sim (\varphi \circ (-)) \circ \psi \qquad ((-) \circ \varphi) \circ \psi \sim (-) \circ (\varphi \circ \psi)$$

but in the operadic case there are instead five cases, corresponding to the different possible roles of the nodes in G-trees with exactly three G-nodes, whose *orbital* representation falls into one of the two cases illustrated below.



Since all these compatibilities can be simultaneously encoded in terms of such trees, we will simply refer to all types of compatibility as "associativity".

In the next result, note that a G-tree T with three G-nodes contains precisely two inner edge orbits Ge and Gf. We will write $T[Ge]$ (resp. $T[Gf]$) for the unique orbital outer face of T with Ge (resp. Gf) has its single inner edge orbit.

ASSOC PROP **Proposition 6.15.** *The operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy all associativity conditions with respect to 3-nodal G-trees.*

Proof. For any 3-nodal G-tree T with inner edge orbits Ge, Gf, consider the following diagram, where all solid maps are fibrations, and the maps labelled $\sim$ are trivial fibrations.

$$
\begin{tikzcd}
X(\mathsf{Sc}[T]) \arrow[d, dashed, bend right] \arrow[r, dashed, bend left] & X(\mathsf{Sc}_{T[Ge]}[T]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, dashed, bend right] & X(\mathsf{Sc}[T - Ge]) \arrow[d, dashed, bend right] \\
X(\mathsf{Sc}_{T[Gf]}[T]) \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \arrow[r, dashed, bend left] & X(\Omega[T]) \arrow[l, two heads, "\sim"'] \arrow[u, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, two heads] & X(\Omega[T - Ge]) \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \\
X(\mathsf{Sc}[T - Gf]) \arrow[r, dashed, bend left] & X(\Omega[T - Gf]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & X(\Omega[T - Ge - Gf])
\end{tikzcd}
\tag{6.16}
$$

FOURSQ EQ

Noting that the following three diagrams, where $\Lambda_o^{Ge,Gf}[T]$ is a generalized orbital G-horn and $\Lambda_{o,c}^{Ge}[T]$, $\Lambda_{o,c}^{Gf}[T]$ are characteristic orbital G-horns (cf. ref), are pullbacks

$$\begin{tikzcd}
X(\mathsf{Sc}[T]) & X(\mathsf{Sc}_{T[Ge]}[T]) \arrow[l, two heads, "\sim"'] \\
X(\mathsf{Sc}_{T[Gf]}[T]) \arrow[u, two heads, "\sim"'] & X(\Lambda_o^{Ge,Gf}[T]) \arrow[l, two heads, "\sim"'] \arrow[u, two heads, "\sim"'] \arrow[ul, phantom, very near end, "\ulcorner"]
\end{tikzcd}
\quad
\begin{tikzcd}
X(\mathsf{Sc}_{T[Ge]}[T]) \arrow[r, two heads] & X(\mathsf{Sc}[T-Ge]) \\
X(\Lambda_{o,c}^{Gf}[T]) \arrow[u, two heads, "\sim"'] \arrow[r, two heads] & X(\Omega[T-Ge]) \arrow[u, two heads, "\sim"'] \arrow[ul, phantom, very near end, "\urcorner"]
\end{tikzcd}
\quad
\begin{tikzcd}
X(\mathsf{Sc}_{T[Gf]}[T]) \arrow[d, two heads] \arrow[dr, phantom, very near end, "\llcorner"] & X(\Lambda_{o,c}^{Ge}[T]) \arrow[l, two heads, "\sim"'] \arrow[d, two heads] \\
X(\mathsf{Sc}[T-Gf]) & X(\Omega[T-Gf]) \arrow[l, two heads, "\sim"']
\end{tikzcd}$$

one sees that: (i) sections in the top left square of (FOURSQ EQ 6.16) can be chosen to be compatible in the sense that the two composites $X(Sc[T]) \to X(\Omega[T])$ coincide; (ii) sections in the top right and bottom left squares of (FOURSQ EQ 6.16) can be chosen to be compatible in the sense that the two composites $X(Sc[T-Ge]) \to X(\Omega[T])$ and $X(Sc[T-Gf]) \to X(\Omega[T])$ coincide. Note that we do not claim (or need) that (i) and (ii) hold simultaneously. We thus conclude that the possible choices of maps $X(\mathsf{Sc}[T]) \to X(\Omega[T-Ge-Gf])$ given by outer paths in (FOURSQ EQ 6.16) are homotopic. All desired forms of associativity follow from taking fibers of these maps over the objects $X(\partial\Omega[T-Ge-Gf])$.

Actually, I may just be able to do without these arguments. □

Remark 6.17. While in the non-equivariant case the associativity conditions in the previous result capture all the key compatibilities of the $\varphi \circ_e (-)$, $(-) \circ_e \psi$ operations, in the equivariant case there are further "compatibilities with pullback of G-trees", which are closely related to the genuine equivariant operads introduced in [BP17 BP17]. However, describing these extra compatibilities would require using G-trees with more than 3 G-nodes, and since such compatibilites are not needed for our current goals, we omit their discussion.

Corollary 6.18. *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-3.*

$$\begin{tikzcd}
X \arrow[rr, "gf"] \arrow[dr, "f"'] & & Z \\
& Y \arrow[ur, "g"'] &
\end{tikzcd}$$

Proof. The non trivial claim is that when f and gf are DK-equivalences then so is g, or more precisely, that the maps

$$Y(y_1,\cdots,y_n;y_0) \to Z(g(y_1),\cdots,g(y_n);g(y_0))$$

are weak equivalences even if the y_i are not in the image of f. But this follows from the functoriality in Proposition GENOPHO PROP 6.14, essential surjectivity (note that when $y_i \in Y(\eta)^{H_i}$ one needs to use H_i-equivalences), and the fact that by Proposition ASSOC PROP 6.15 the maps $f \circ_{Ge} (-)$, $(-) \circ_{Ge} f$ are weak equivalences whenever f is a H-equivalence. □

26COR **Corollary 6.19.** *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-6, i.e. whenever gf and hg are DK-equivalences then so are f, g, h, hgf.*

Proof. The hypothesis together with the 2-out-of-6 condition for the Kan model structure in sSet imply that g is fully faithful for objects in the image of f. But this easily implies that f is a DK-equivalence, and thus 2-out-of-3 concludes the proof. □

We now recover the following from [Rez01 Rez01] (add commentary)

Proposition 6.20. *Let $X \in \mathsf{ssSet}$ be a Segal space. Then:*

(i) equivalences define a subset of connected components $X^h(1) \subset X(1)$;

(ii) the pullbacks

$$
\begin{tikzcd}
X^h(n) \arrow[rr] \arrow[d] & & X(n) \arrow[d] \\
X^h(1) \times_{X(0)} \cdots \times_{X(0)} X^h(1) \arrow[r] & X(1) \times_{X(0)} \cdots \times_{X(0)} X(1)
\end{tikzcd}
\tag{6.21}
$$

XHDEF EQ

define a Segal space $X^h \subset X$, consisting of a union of connected components at each level;

(iii) the maps $X^h(2) \xrightarrow{(d_2,d_1)} X^h(1) \times_{X(0)} X^h(1)$, $X^h(2) \xrightarrow{(d_0,d_1)} X^h(1) \times_{X(0)} X^h(1)$ are trivial fibrations;

(iv) the map $X(J) \to X(\Delta[1]) = X(1)$ factors through a weak equivalence $X(J) \to X^h(1)$.

Proof. For (i), note first that given $f\colon x \to y$ in $X_0(1)$, then $[f]$ has a left inverse iff one can find a lift p as on the leftmost diagram below. But for any path H between f and f' in X_1

$$
\begin{tikzcd}
 & X(2) \arrow[d, "{(d_2,d_1)}"] \\
\Delta[0] \arrow[ur, "p"] \arrow[r, "{(f,s_0(x))}"'] & X(1) \times_{X(0)} X(1)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta[0] \arrow[r, "p"] \arrow[d, "0"'] & X(2) \arrow[d, "{(d_2,d_1)}"] \\
\Delta[1] \arrow[ur, dashed] \arrow[r, "{(H,s_0d_1(H))}"'] & X(1) \times_{X(0)} X(1)
\end{tikzcd}
$$

one can form the lift in the rightmost diagram, showing that f' is also left-invertible. The situation for right inverses is identical, thus (i) follows.

For (ii), the fact that $X^h(\bullet)$ is closed under the simplicial operators follows since the composite of equivalences is an equivalence. In fact, this further implies that the bottom row in the pullback (XHDEF EQ 6.21) could have been replaced with $X^h(\mathsf{sk}_1\Delta[n]) \to X(\mathsf{sk}_1\Delta[n])$, from which it follows that the squares

$$
\begin{tikzcd}
X^h(K) \arrow[r] \arrow[d] & X(K) \arrow[d] \\
X^h(\mathsf{sk}_1 K) \arrow[r] & X(\mathsf{sk}_1 K)
\end{tikzcd}
$$

are pullbacks. Since $\mathsf{sk}_1(\partial\Delta[n]) = \mathsf{sk}_1\Delta[n]$ if $n \geq 2$ it follows that the maps $X^h(n) \to X^h(\partial\Delta[n])$, $n \geq 2$ are Kan fibrations, and since the map $X^h(1) \to X(0) \times X(0)$ is certainly a Kan fibration, X^h is indeed Reedy fibrant. The Segal condition is obvious from the pullback (XHDEF EQ 6.21).

For (iii), it suffices by symmetry to show the first claim. Moreover, one reduces to showing show that for any choice of section in the following diagram the top composite is a weak equivalence.

$$
\begin{tikzcd}
X^h(1) \times_{X_0} X^h(1) \arrow[r, dashed, bend left] \arrow[dr, two heads, "{(id,d_0)}"'] & X^h(2) \arrow[l, two heads, "\sim", "{(d_2,d_0)}"'] \arrow[r, two heads, "{(d_2,d_1)}"'] & X^h(1) \times_{X_0} X^h(1) \arrow[dl, two heads, "{(id,d_0)}"] \\
 & X^h(1) \times X(0) &
\end{tikzcd}
$$

But this composite is a map of fibrations over $X^h(1) \times X(0)$ with the map between the fibers over $(f\colon x \to y, z)$ computing the map $(-) \circ f\colon X^h(y;z) \to X^h(x;z)$, which is a Kan equivalence since $f \in X^h(1)$ is an equivalence. Thus the composite is an equivalence, establishing (iii).

Lastly, for (iv) we first note that (iii) can be restated as saying that X^h is local with respect to the outer horn inclusions $\Lambda^0[2] \to \Delta[2]$ and $\Lambda^2[2] \to \Delta[2]$, and that hence by Remarks 8.1 ANHYPER REM and 8.2 CONTGR REM the map $X^h(J) \to X^h(1)$ is a Kan equivalence. Hence, the only remaining claim is that $X^h(J) = X(J)$, which is clear. □

Remark 6.22. The inclusion $X^h \to X$ is a Reedy fibration. justify

JDDK PROP **Proposition 6.23.** *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then the map $X \to X^J$ is a DK-equivalence.*

Proof. Note first that for any $T \in \Omega_G$ the map $X^J(T) \to X^{\Omega[1]}(T)$ can be rewritten as $\left(X^{\Omega[T]}\right)(J) \to \left(X^{\Omega[T]}\right)(\Omega[1]) = \left(X^{\Omega[T]}\right)(1)$, and since $X^{\Omega[T]}$ is a (simplicial) Segal space the previous result implies that this map is a weak equivalence onto a subset of components (such maps are also called homotopy monomorphisms). It thus follows that for any G-corolla $C = C_{\amalg_i H_0/H_i}$ the horizontal maps in the rightmost square below are homotopy monomorphisms.

$$\begin{tikzcd}
X(C) \arrow[r] \arrow[d, two heads] & X^J(C) \arrow[r] \arrow[d, two heads] & X^{\Omega[1]}(C) \arrow[d, two heads] \\
\prod_{0\leq i\leq k} X(\eta)^{H_i} \arrow[r] & \prod_{0\leq i\leq k} X^J(\eta)^{H_i} \arrow[r] & \prod_{0\leq i\leq k} X^{\Omega[1]}(\eta)^{H_i}
\end{tikzcd} \qquad \text{BIGSQ EQ}$$

Since the claim that $X \to X^J$ is a DK-equivalence is the statement that the leftmost square induces weak equivalences on fibers, it suffices to show that so does the composite square.

Now note that we can rewrite $X^{\Omega[1]}(C) = X(\Omega[1] \otimes \Omega[C])$ and that there is a pullback diagram

$$\begin{tikzcd}
X(\Omega[1] \otimes \Omega[C]) \arrow[r, two heads] \arrow[d, two heads] & X(\Omega[C \star \eta]) \arrow[d, two heads] \\
X(\Omega[\eta \star C]) \arrow[r, two heads] & X(\Omega[C])
\end{tikzcd}$$

Noting that the required cube is projective fibrant, one reduces to checking that the following squares induce weak equivalences on fibers, and this claim is clear from the top right vertical trivial fibrations (which are instances of the Segal condition).

$$\begin{tikzcd}
X(\Omega[C]) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(\Omega[C \star \eta]) \arrow[d, two heads, "\sim"] \\
 & X(\Omega[C]) \underset{X(\eta)^{H_0}}{\times} X([1])^{H_0} \arrow[d, two heads] \\
\prod\limits_{0\leq i\leq n} X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod\limits_{1\leq i\leq n} X(\eta)^{H_i}\right) \times X([1])^{H_0}
\end{tikzcd}
\qquad
\begin{tikzcd}
X(\Omega[C]) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(\Omega[C \star \eta]) \arrow[d, two heads, "\sim"] \\
 & \prod\limits_{1\leq i\leq n} X([1])^{H_i} \underset{\prod\limits_{1\leq i\leq n} X(\eta)^{H_i}}{\times} X(\Omega[C]) \arrow[d, two heads] \\
\prod\limits_{0\leq i\leq n} X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod\limits_{1\leq i\leq n} X([1])^{H_i}\right) \times X(\eta)^{H_0}
\end{tikzcd}$$

□

Definition 6.24. Two maps $f, f' : A \rightrightarrows B$ between dendroidal Segal spaces are called J-homotopic, written $f \sim_J f'$, if there is a H such that the two composites $A \xrightarrow{H} B^J \rightrightarrows B$ are f, f'.

Further, a map $f: X \to Y$ of dendroidal Segal spaces is called a J-homotopy equivalence if there is $g: Y \to X$ such that $gf \sim_J id_X$, $fg \sim_J id_Y$.

Remark 6.25. It follows from Proposition 6.23 and 2-out-of-3 that if $f \sim_J f'$ then f is a DK-equivalence iff $f'$is. Thus by 2-out-of-6 all J-homotopy equivalences are DK-equivalences.

ALLXJK REM **Remark 6.26.** Let X be a Segal space. All simplicial operators $X^{J^n} \to X^{J^m}$ are induced by equivalences of groupoids $\widetilde{[m]} \to \widetilde{[n]}$, and one easily checks that these operators are thus J-homotopy equivalences and thus also DK-equivalences.

COMPLE PROP **Proposition 6.27.** *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then there is a complete dendroidal Segal space $\tilde{X}$ and complete equivalence $X \to \tilde{X}$ such that*

(i) $X \to \tilde{X}$ is a monomorphism and a DK-equivalence;

(ii) $X_0(\eta) \to \tilde{X}_0(\eta)$ is an isomorphism.

Proof. Our argument will mostly adapt the construction of the completion functor in [Rez01, §10.4].

Firstly, we let $X^{J^\bullet} \in (\mathsf{sdSet}^G)^{\Delta^{op}} = \mathsf{ssdSet}^G$ be the object whose m-th level (in the new simplicial direction) is X^{J^m}. Since $J^\bullet$ is a Reedy cofibrant cosimplicial object, $X^{J^\bullet}$ is Reedy fibrant with respect to the dendroidal space model structure on sdSet^G. In particular, it follows that $X^{J^\bullet} \to \mathsf{csk}_\eta X^{J^\bullet}$ is a fibration in ssdSet^G.

In particular, this implies that for each $T \in \Omega_G$ and vertex map $[0] \to [m]$ the induced square

$$\begin{tikzcd}
X^{J^m}(\Omega[T]) \arrow[r, two heads] \arrow[d, two heads] & X(\Omega[T]) \arrow[d, two heads] \\
\prod_{e_i \in E_G(T)} \left(X^{J^m}(\eta)\right)^{H_i} \arrow[r, two heads] & \prod_{e_i \in E_G(T)} \left(X(\eta)\right)^{H_i}
\end{tikzcd}$$

is an (injective) fibrant square, which by Remark 6.26 induces weak equivalences on fibers, so that the map from $X^{J^m}(T)$ to the pullback of the remaining diagram is a trivial Kan fibration. By Remark 5.8 we have just shown that for each fixed $T \in \Omega_G$ the map

$$X^{J^\bullet}(T) \to \left(\mathsf{csk}_\eta X^{J^\bullet}\right)(T) \tag{6.28}$$ MOREOVER EQ

satisfies the "moreover" condition in Corollary 5.7. Therefore, applying δ^* to (6.28) yields a Kan fibration, so that all fibers of this map are in fact homotopy fibers.

We now write $\tilde{X}$ for any dendroidal Reedy fibrant replacement of the diagonal $\delta^*\left(X^{J^\bullet}\right)$, which we note can always be chosen so that $\delta^*\left(X^{J^\bullet}\right) \to \tilde{X}$ is a monomorphism and $\tilde{X}_0(\eta) = \left(\delta^*\left(X^{J^\bullet}\right)\right)_0(\eta) = X_0(\eta)$ (this follows since fibrant replacements in the Kan model structure in sSet can be chosen to preserve 0-simplices, since existence of lifts against the horn inclusions $\Delta[0] = \Lambda^0[1] \to \Delta[1]$, $\Delta[0] = \Lambda^1[1] \to \Delta[1]$ is automatic).

To see that $\tilde{X}$ is a complete Segal space, note that there is a composite $X_0^{J^\bullet} \to \delta^*\left(X^{J^\bullet}\right) \to \tilde{X}$ where the first map is a dendroidal Reedy equivalence by Proposition 5.5(iv) and the second by definition of $\tilde{X}$. But since $X_0^{J^\bullet}$ is a complete Segal space, so is $\tilde{X}$.

For the remaining claim that the composite $X = X^{J^0} \to \delta^*\left(X^{J_\bullet}\right) \to \tilde{X}$ is a DK equivalence, though the first map is no longer a dendroidal Reedy equivalence, it is nonetheless an equivalence on fibers over $\prod_{e_i \in E_G(T)} \left(X(\eta)\right)^{H_i}$ for each $T \in \Omega_G$. And since we established above that the fibers of $\delta^*\left(X^{J^\bullet}\right)(T)$ are homotopy fibers, these are equivalent to the fibers of $\tilde{X}(T)$ (since Reedy replacement does not change the homotopy fibers), and thus $X \to \tilde{X}$ is indeed fully faithful. Essential surjectivity is trivial since the objects coincide. The monomorphism condition is clear. $\square$

Corollary 6.29. *A map of $X \to Y$ of dendroidal Segal spaces is a joint equivalence iff it is a DK-equivalence.*

HERE

Proof. By the (proof of) previous result one is free to, via a zigzag, replace X, Y with $X_0^{J^\bullet}$, $Y_0^{J^\bullet}$. But by Theorem 5.18 this map is a joint equivalence iff $X_0 \to Y_0$ is an equivalence in dSet^G, which by Theorem 6.39 holds iff this is a fully faithful and essentially surjective map of G-∞-operads. But it is easy to check that $X_0 \to Y_0$ is a fully faithful and essentially surjective map of G-∞-operads iff $X_0^{J^\bullet} \to Y_0^{J^\bullet}$ is a fully faithful and essentially surjective map of dendroidal Segal spaces. □

FIB_PREOP_COR

Corollary 6.30. *A pre-operad $X \in \mathsf{PreOp}^G$ is fibrant iff $\gamma^*(X)$ is fibrant in the Segal space model structure on sdSet^G.*

Proof. We start with the "only if" direction. Recall that $\gamma^* X$ is a dendroidal Segal space if it has the right lifting property against the maps of the form

$$(\Lambda^i[n] \to \Delta[n]) \square (\partial\Omega[T] \to \Omega[T]) \qquad (\partial\Delta[n] \to \Delta[n]) \square (Sc[T] \to \Omega[T]). \tag{6.31}$$

SOMEMAPS_EQ

With the exception of the first type of maps when $T = \eta$, in which case the lifting condition is automatic since $\gamma^* X(\eta)$ is discrete, all other maps induce isomorphisms at the η-level, so that by (5.21) applying $\gamma_!$ to these maps yields trivial cofibrations in PreOp^G. Thus, if $X \in \mathsf{PreOp}^G$ is fibrant, an adjunction argument shows that $\gamma^*(X)$ indeed has the lifting property against all maps (6.31), i.e. $\gamma^*(X)$ is a dendroidal Segal space.

For the "if" direction, form the completion $\gamma^* X \to \tilde{X}$ as in Proposition 6.27. Then $\gamma_* \tilde{X} \in \mathsf{PreOp}^G$ is fibrant by Theorem 5.27 and the adjoint map $X \to \gamma_* \tilde{X}$ has the following properties: (i) it is a monomorphism; (ii) it is an isomorphism on the η-level; (iii) it is a DK-equivalence when regarded as a map in sdSet^G (since $\gamma_* \gamma^* \tilde{X} \to \tilde{X}$ is tautologically a DK-equivalence); (iv) it is hence a trivial Reedy cofibration when regarded as a map in sdSet^G. But then the hypothesis that $\gamma^* X$ is a dendroidal Segal space yields a lift

$$\begin{tikzcd}
\gamma^* X \arrow[r, equal] \arrow[d] & \gamma^* X \\
\gamma^* \gamma_* X \arrow[ur, dashed] &
\end{tikzcd}$$

showing that X is a retract of $\gamma_* X$ and finishing the proof. □

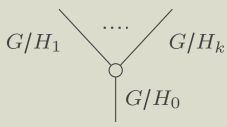
Remark 6.32. For any dendroidal Segal space $X \in \mathsf{sdSet}^G$ one hence has complete equivalences

$$\gamma_* X \to X \to \tilde{X}$$

where γ_* is a fibrant preoperad and $\tilde{X}$ a complete dendroidal Segal space.

6.2 Equivalences of G-∞-operads

Notation 6.33. Given subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$, $1 \leq i \leq k$ we write $C_{\amalg_i H_0/H_i}$ for the G-corolla encoding the H_0-set $H_0/H_1 \amalg \cdots \amalg H_0/H_k$.



We remind ourselves of the following notation (cf. [Per17, §8]).

HOM_NOTATIONS **Notation 6.34.** Fix $A, B, C \in \mathsf{dSet}^G$, $X \in \mathsf{dSet}^G$ a G-∞-operad, $K \in \mathsf{sSet}$.

- The *internal hom* $\mathrm{Hom}(A, B) \in \mathsf{dSet}^G$ is defined by

$$\mathrm{Hom}(A, B)(T) = \mathsf{dSet}^G(A \otimes \Omega[T], B),$$

and come equipped with canonical isomorphisms

$$\mathsf{dSet}^G(A \otimes C, B) = \mathsf{dSet}^G(C, \mathrm{Hom}(A, B)).$$

- The *simplicial mapping space* $B^A \in \mathsf{sSet}$ is given by $i^* \mathrm{Hom}(A, B)$, and comes with induced isomorphisms

$$\mathsf{sSet}(K, B^A) = \mathsf{dSet}^G(A, \mathrm{Hom}(i_! K, B)).$$

- Let $k(A, X) \in \mathsf{sSet}$ denote $k(X^A)$, where $k(-)$ is the maximal sub Kan complex.

reference X^A being Kan

- Let $X^{(K)} \in \mathsf{dSet}_G$ be defined to have T-dendrices given by

$$X^{(K)}(T) := k(i^* \mathrm{Hom}(i_! K \otimes \Omega[T], X),$$

and we have canonical isomorphisms

$$\mathsf{dSet}_G(u_* A, X^{(K)}) = \mathsf{sSet}(K, k(A, X)).$$

The following is the equivariant generalization of [CM13a, Thm. 3.5].

TFAE PROP **Proposition 6.35.** *Let $X \to Y$ be a map between G-∞-operads. The following are equivalent:*

(a) for all G-corollas $C \in \Sigma_G$ and $H \leq G$ the maps

$$k(\Omega[C], X) \to k(\Omega[C], Y), \qquad k(\Omega[G/H \cdot \eta], X) \to k(\Omega[G/H \cdot \eta], Y)$$

are Kan equivalences in sSet;

(b) for all G-trees $T \in \Omega_G$ the maps $k(\Omega[T], X) \to k(\Omega[T], Y)$ are Kan equivalences in sSet;

(c) for all normal $A \in \mathsf{dSet}^G$, the maps $k(A, X) \to k(A, Y)$ are Kan equivalences in sSet;

(d) $f\colon X \to Y$ is a weak equivalence in dSet^G.

PROF DEF **Definition 6.36.** Let X be a G-∞-operad. A *G-profile* on X is a map

$$\partial\Omega[C] \to X$$

for some G-corolla $C \in \Sigma_G$. More explicitly, a G-profile is described by the following data:

- subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$ for $1 \leq i \leq k$;
- objects $x_i \in X(\eta)^{H_i}$ for $0 \leq i \leq k$.

To simplify notation, we prefer to denote a G-profile as $(x_1, \cdots, x_k; x_0)$, and refer to it as a *C-profile.*

HERE

MAPSPACE DEF **Definition 6.37.** Given a G-∞-operad and a C-profile $(x_1,\cdots,x_k;x_0)$ we define the space of maps $X(x_1,\cdots,x_k;x_0)$ to be given by the pullback

$$\begin{tikzcd}
X(x_1,\cdots,x_k;x_0) \arrow[r] \arrow[d] & Hom^G(\Omega[C],X) \arrow[d] \\
\eta \arrow[r, "(x_1,\cdots,x_k;x_0)"'] & \prod_{0\le i\le k} X^{H_i}
\end{tikzcd}$$

one sees that $X(x_1,\cdots,x_k;x_0)$ can indeed be regarded as a simplicial set (in fact, this is a Kan complex).

Definition 6.38. Let $f\colon X\to Y$ be a map of G-∞-operads.

The map f is called *fully faithful* if, for each C-profile $(x_1,\cdots,x_k;x_0)$ one has that

$$X(x_1,\cdots,x_k;x_0)\to Y\left(f(x_1),\cdots,f(x_k);f(x_0)\right)$$

is a Kan equivalence in sSet.

The map f is called *essentially surjective* if for each subgroup $H\le G$ the map of categories $\tau(\iota^*(X^H))\to\tau(\iota^*(Y^H))$ is essentially surjective.

The following is the equivariant generalization of [CM13a, Thm. 3.11 and Remark 3.12].

COMSQ THM **Theorem 6.39.** *A map $f\colon X\to Y$ of G-∞-operads is fully faithful iff for all G-corollas $C\in\Sigma_G$ the commutative squares of Kan complexes*

$$\begin{tikzcd}
k(\Omega[C],X) \arrow[r] \arrow[d, "p"'] & k(\Omega[C],Y) \arrow[d, "q"] \\
k(\partial\Omega[C],X) \arrow[r, "f_*"'] & k(\partial\Omega[C],Y)
\end{tikzcd} \tag{6.40}$$ COMSQ EQ

are homotopy pullback squares.

Hence, f is a weak equivalence in dSet^G iff f is both fully faithful and essentially surjective.

Proof. Noting that the 0-simplices of $k(\partial\Omega[C],X)$ are precisely the C-profiles $(x_1,\cdots,x_k,x_0)$, fully faithfulness can be reinterpreted as saying that all fiber maps $p^{-1}(x_1,\cdots,x_k,x_0)\to q^{-1}(f(x_1),\cdots,f(x_k),f(x_0))$ are weak equivalences. But since p,q are Kan fibrations, this is equivalent to the condition that (6.40) is a homotopy pullback (see [CM13a, Lemma 3.9]), and the first half follows.

For the second half, note first that the bottom map in (6.40) can be rewritten as

$$\prod_{0\le i\le k} k\left(G/H_i\cdot\eta,X\right)\to\prod_{0\le i\le k} k\left(G/H_i\cdot\eta,Y\right).$$

Assume first that f is a weak equivalence. Proposition 6.35 then implies that the horizontal maps in (6.40) are weak equivalences, so that the square is a pull back square, and thus f is fully faithful. That f is essentially surjective follows from the identity $k\left(\Omega[G/H\cdot\eta],Z\right)=k(\iota^*(Z^H))$, so that $\tau(\iota^*(X^H))\to\tau(\iota^*(Y^H))$ is essentially surjective at the level of maximal groupoids, and this suffices for essential surjectivity.

Assume now that f is fully faithful and essentially surjective. Since (6.40) is a homotopy pullback, Proposition 6.35 implies that one needs only check that the maps of Kan complexes

$$k\left(\Omega[G/H\cdot\eta],X\right)\to k\left(\Omega[G/H\cdot\eta],Y\right)\qquad\text{or}\qquad k\left(\iota^*\left(X^H\right)\right)\to k\left(\iota^*\left(Y^H\right)\right) \tag{6.41}$$ KANMAP EQ

are weak equivalences. As before, essential surjectivity is equivalent to the fact that the maps (6.41) induce surjections on connected components. Hence, it now suffices to show that for each 0-simplex $x \in X^H$ the top map of loop spaces in

$$\begin{tikzcd} \Omega(k(\iota^* X^H), x) \arrow[r] \arrow[d] & \Omega(k(\iota^* Y^H), f(x)) \arrow[d] \\ X(x;x) \arrow[r] & Y(f(x);f(x)) \end{tikzcd} \tag{6.42}$$

OMEGASQ EQ

is a weak equivalence. Note that the bottom map in (6.42) is a weak equivalence since F is fully faithful and that the vertical maps are the inclusion of the connected components corresponding to automorphisms of x in $\tau(\iota^* X^H)$. It thus suffices to check that the top map in (6.42) is an isomorphism on π_0, and this follows since the map of categories $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ is fully faithful. □

7 Indexing system analogue results

FILL

8 Scratchwork (to be folded into previous sections eventually)

$$u^* \colon \mathsf{dSet}_G \rightleftarrows \mathsf{dSet}^G \colon u_*$$

ANHYPER REM **Remark 8.1.** The smallest hypersaturated class containing the inner horns and the left horn inclusion $\Lambda^0[2] \to \Delta[2]$ in fact contains all left horn inclusions $\Lambda^0[n] \to \Delta[n]$ for $n \geq 2$. Indeed, this follows inductively from the following diagram since the bottom map is inner

$$\begin{tikzcd}
\Lambda^{0,1}[n] \arrow[r] \arrow[d] & \Lambda^0[n] \arrow[d] \\
\Lambda^1[n] \arrow[r] & \Delta[n]
\end{tikzcd}$$

and the top and left maps are given by following pushouts

$$\begin{tikzcd}
\Lambda^0[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Lambda^{0,1}[n] \arrow[d] & \Lambda^0[n-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Lambda^{0,1}[n] \arrow[d] \\
\Delta[n-1] \arrow[r, "d^1"'] & \Lambda^0[n] & \Delta[n-1] \arrow[r, "d^0"'] & \Lambda^1[n]
\end{tikzcd}$$

CONTGR REM **Remark 8.2.** Write $\widetilde{[n]}$ for the contractible groupoid on objects $\{0,1,\cdots,n\}$. Note that the k-simplices of $\widetilde{[n]}$ are encoded as strings $a_0 a_1 \cdots a_k$ with $a_i \in \{0,1,\cdots,n\}$, and that a simplex is non-degenerate iff $a_{i-1} \neq a_i, 1 \leq i \leq k$. Then the maps

$$\Delta[n] = N[n] \xrightarrow{012\cdots n} N\widetilde{[n]}, \quad n \geq 1 \tag{8.3}$$ INVER EQ

are built cellularly out of left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 2$.

Indeed, we show a little more. Call subcomplex $A \subset N\widetilde{[n]}$ is 0-*stable* if a n-simplex $\underline{a}$ is in A iff the $n+1$-simplex $0\underline{a}$ is. We claim that any inclusion $A \to A'$ of 0-stable subcomplexes is built cellularly from left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 1$. Indeed, it suffices to check this when A' attaches as little as possible to A, and 0-simplicity guarantees that in that case the only two non-degenerate simplices in $A - A'$ have the form $\underline{a}$ and $0\underline{a}$ (note that $\underline{a}$ can not start with a 0). But then $A \to A'$ is a pushout of $\Lambda^0[k+1] \to \Delta[k+1]$ where k is the dimension of $\underline{a}$.

The desired claim follows by noting that both the domain and codomain of (8.3) are 0-stable and that the horns $\Lambda^0[1]$ are unneeded since (8.3) is an isomorphism on 0-simplices.

$$\begin{tikzcd}
W_1 \times_{W_0} W_1 \times_{W_0} W_1 \arrow[r, dashed, bend left] \arrow[d, dashed, bend right] & W_1 \times_{W_0} W_2 \arrow[l, two heads, "\sim"] \arrow[r, two heads] \arrow[d, dashed, bend right] & W_1 \times_{W_0} W_1 \arrow[d, dashed, bend right] \\
W_2 \times_{W_0} W_1 \arrow[u, two heads, "\sim"'] \arrow[r, dashed, bend left] \arrow[d, two heads] & W_3 \arrow[l, two heads, "\sim"] \arrow[r, two heads] \arrow[u, two heads, "\sim"'] \arrow[d, two heads] & W_2 \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \\
W_1 \times_{W_0} W_1 \arrow[r, dashed, bend left] & W_2 \arrow[l, two heads, "\sim"] \arrow[r, two heads] & W_1
\end{tikzcd}$$

Remark 8.4. Note that $\mathsf{Sc}_{T[Ge]}$, $\mathsf{Sc}_{T[Gf]}$ in (6.16) are cover inclusions, and thus G-anodyne, relate to [Rez10, §6.2], [Rez01, §10].

Remark 8.5. Indexing systems are precisely the Segal sieves of Ω_G.

Remark 8.6. bla bla the diagrams for compositions of norm maps are given by orbital representations, but the category Ω_G is better described in terms of the expanded representation.

A Equivariant Reedy model structures

Bla bla one of the axioms in [BM11] is different from the others point of view

In [BM11] Berger and Moerdijk extend the notion of Reedy category so as to allow for categories $\mathbb{R}$ with non-trivial automorphism groups $\mathsf{Aut}(r)$ for $r \in \mathbb{R}$. For such $\mathbb{R}$ and suitable model category $\mathcal{C}$ they then show that there is a *Reedy model structure* on $\mathcal{C}^{\mathbb{R}}$ that is defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see [BM11, Thm. 1.6] or Theorem A.8 below) to be determined by the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ for each $r \in \mathbb{R}$.

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ with the more general $\mathcal{C}^{\mathsf{Aut}(r)}_{\mathcal{F}_r}$ model structures for $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a nice collection of families of subgroups of each $\mathsf{Aut}(r)$.

To do so, we first need some essential notation. For each map $r \to r'$ in a category $\mathbb{R}$ we will write $\mathsf{Aut}(r \to r')$ for its automorphim group in the arrow category and write

$$\mathsf{Aut}(r) \xleftarrow{\pi_r} \mathsf{Aut}(r \to r') \xrightarrow{\pi_{r'}} \mathsf{Aut}(r') \tag{A.1}$$ PIDEFR EQ

for the obvious projections. We now introduce our equivariant generalization of the "generalized Reedy categories" of [BM11, Def. 1.1].

GENRED DEF **Definition A.2.** A *generalized Reedy category structure* on a small category $\mathbb{R}$ consists of wide subcategories $\mathbb{R}^+$, $\mathbb{R}^-$ and a degree function $|-|: ob(\mathbb{R}) \to \mathbb{N}$ such that:

(i) non-invertible maps in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) raise (lower) degree; isomorphisms preserve degree;

(ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathsf{Iso}(\mathbb{R})$;

(iii) every map f in $\mathbb{R}$ factors as $f = f^+ \circ f^-$ with $f^+ \in \mathbb{R}^+$, $f^- \in \mathbb{R}^-$, and this factorization is unique up to isomorphism.

Let $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ be a collection of families of subgroups of the groups $\mathsf{Aut}(r)$. The collection $\{\mathcal{F}_r\}$ is called *Reedy-admissible* if:

(iv) for all maps $r \twoheadrightarrow r'$ in $\mathbb{R}^-$ one has $\pi_{r'}\left(\pi_r^{-1}(H)\right) \in \mathcal{F}_{r'}$ for all $H \in \mathcal{F}_r$.

We note that condition (iv) above should be thought as of a constraint on the pair $(\mathbb{R}, \{\mathcal{F}_r\})$. The original setup of [BM11] then deals with the case where $\{\mathcal{F}_r\} = \{\{e\}\}$ is the collection of trivial families. Indeed, our setup recovers the setup in [BM11], as follows.

Example A.3. When $\{\mathcal{F}_r\} = \{\{e\}\}$, Reedy-admissibility coincides with axiom (iv) in [BM11, Def. 1.1], stating that if $\theta \circ f^- = f^-$ for some $f^- \in \mathbb{R}^-$ and $\theta \in \mathsf{Iso}(\mathbb{R})$ then θ is an identity.

Example A.4. For any generalized Reedy category $\mathbb{R}$, the collection $\{\mathcal{F}_{\text{all}}\}$ of the families of all subgroups of $\mathsf{Aut}(r)$ is Reedy-admissible.

Example A.5. Let G be a group and set $\mathbb{R} = G \times (0 \to 1)$ with $\mathbb{R} = \mathbb{R}^+$. Then any pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible.

Similarly, set $\mathbb{S} = G \times (0 \leftarrow 1)$ with $\mathbb{S} = \mathbb{S}^-$. Then a pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible iff $\mathcal{F}_0 \supset \mathcal{F}_1$.

GGRAPHREEDY EX **Example A.6.** Letting $\mathbb{S}$ denote any generalized Reedy category in the sense of [BM11, Def. 1.1] and G a group, we set $\mathbb{R} = G \times \mathbb{S}$ with $\mathbb{R}^+ = G \times \mathbb{S}^+$ and $\mathbb{R}^- = G \times \mathbb{S}^+$. Further, for each $s \in \mathbb{S}$ we write $\mathcal{F}^{\Gamma}_s$ for the family of G-graph subgroups of $G \times \mathsf{Aut}_{\mathbb{S}}(s)$, i.e., those subgroups $K \le G \times \mathsf{Aut}_{\mathbb{S}}(s)$ such that $K \cap \mathsf{Aut}_{\mathbb{S}}(s) = \{e\}$.

Reedy admissibility of $\{\mathcal{F}_s^\Gamma\}$ follows since for every degeneracy map $s \twoheadrightarrow s'$ in $\mathbb{S}^-$ one has that the homomorphism $\pi_s \colon \mathsf{Aut}_{\mathbb{S}}(s \twoheadrightarrow s') \to \mathsf{Aut}_\Omega(s)$ is injective (we note that this is equivalent to axiom (iv) in [BM11, Def. 1.1] for $\mathbb{S}$).

Our primary example of interest will come by setting $\mathbb{S} = \Omega^{op}$ in the previous example. In fact, in this case we will also be interested in certain subfamilies $\{\mathcal{F}_U\}_{U\in\Omega} \subset \{\mathcal{F}_U^\Gamma\}_{U\in\Omega}$.

Example A.7. Let $\mathbb{R} = G \times \Omega^{op}$ and let $\{\mathcal{F}_U\}_{U\in\Omega}$ be the family of graph subgroups determined by a weak indexing system $\mathcal{F}$. Then $\{\mathcal{F}_U\}$ is Reedy-admissible. To see this, recall first that each $K \in \mathcal{F}_U$ encodes an H-action on $U \in \Omega$ for some $H \leq G$ so that $G \cdot_H U$ is a $\mathcal{F}$-tree. Given a face map $f\colon U' \hookrightarrow U$, the subgroup $\pi_U^{-1}(K)$ is then determined by the largest subgroup $\bar{H} \leq H$ such that U' inherits the $\bar{H}$-action from U along f (so that f becomes a $\bar{H}$-map), so that $\pi_{U'}(\pi_U^{-1}(K))$ encodes the $\bar{H}$-action on U'. Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of G-trees $G \cdot_{\bar{H}} U' \to G \cdot_H U$.

We now state the main result. We will assume throughout that $\mathcal{C}$ is a model category such that for any group G and family of subgroups $\mathcal{F}$, the category $\mathcal{C}^G$ admits the $\mathcal{F}$-model structure (for example, this is the case whenever $\mathcal{C}$ is a cofibrantly generated cellular model category in the sense of [Ste16]).

REEDYADM THM **Theorem A.8.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible collection of families. Then there is a $\{\mathcal{F}_r\}$-**Reedy model structure** on $\mathcal{C}^{\mathbb{R}}$ such that a map $A \to B$ is*

- *a (trivial) cofibration if $A_r \coprod_{L_r A} L_r B \to B_r$ is a (trivial) $\mathcal{F}_r$-cofibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a weak equivalence if $A_r \to B_r$ is a $\mathcal{F}_r$-weak equivalence in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a (trivial) fibration if $A_r \to B_r \times_{M_r B} M_r A$ is a (trivial) $\mathcal{F}_r$-fibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$.*

The proof of this result is given at the end of the appendix after establishing some routine generalizations of the key lemmas in [BM11] (indeed, the true novelty in this appendix is the Reedy-admissibility condition in part (iv) of Definition A.2).

We first recall the following, cf. [BP17, Props. 6.5 and 6.6] (we note that [BP17, Prop. 6.6] can be proven in terms of fibrations, and thus does not depend on special assumptions on $\mathcal{C}$).

Proposition A.9. *Let $\phi\colon G \to \bar{G}$ be a homomorphism and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of subgroups of $G, \bar{G}$. Then the leftmost (resp. rightmost) adjunction below is a Quillen adjunction*

$$\bar{G} \cdot_G (-)\colon \mathcal{C}^G_{\mathcal{F}} \rightleftarrows \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \colon \mathsf{res}^{\bar{G}}_G \qquad \mathsf{res}^{\bar{G}}_G \colon \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{C}^G_{\mathcal{F}} \colon \mathsf{Hom}_G(\bar{G}, -)$$

provided that for $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$ (resp. for $\bar{H} \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$).

RESGEN COR **Corollary A.10.** *For any homomorphism $\phi\colon G \to \bar{G}$, the functor $\mathsf{res}^{\bar{G}}_G \colon \mathcal{C}^{\bar{G}} \to \mathcal{C}^G$ preserves all four classes of genuine cofibrations, trivial cofibrations, fibrations and trivial fibrations.*

The following formalizes an argument implicit in the proof of [BM11, Lemma 5.2]).

Definition A.11. Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.12}$$

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in $\mathcal{C}^{\mathbb{R}}$. A collection of maps $f_s\colon B_s \to X_s$ for $|s| \leq n$ that induce a lift of the restriction of (A.12) to $\mathcal{C}^{\mathbb{R}_{\leq n}}$ will be called a *n-partial lift*.

BLALIFT LEM **Lemma A.13.** *Let $\mathcal{C}$ be any bicomplete category, and consider a commutative diagram as in (A.12). Then any $(n-1)$-partial lift uniquely induces commutative diagrams*

$$\begin{tikzcd} A_r \amalg_{L_r A} L_r B \arrow[r] \arrow[d] & X_r \arrow[d] \\ B_r \arrow[r] \arrow[ur, dashed] & Y_r \times_{M_r Y} M_r X \end{tikzcd} \tag{A.14}$$

BLALIFT EQ

in $\mathcal{C}^{\mathsf{Aut}(r)}$ for each r such that $|r| = n$. Furthermore, extensions of the $(n-1)$-partial lift to a n-partial lift are in bijection with choices of $\mathsf{Aut}(r)$-equivariant lifts of the diagrams (A.14) for r ranging over representatives of the isomorphism classes of r with $|r| = n$.

In the next result, by $\{\mathcal{F}_r\}$-cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in Theorem A.8, regardless of whether such a model structure exists.

BLALIFT COR **Corollary A.15.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}$ an arbitrary family of subgroups of $\mathsf{Aut}(r)$, $r \in \mathbb{R}$. Then a map in $\mathcal{C}^{\mathbb{R}}$ is a $\{\mathcal{F}_r\}$-cofibration (resp. trivial cofibration) iff it has the left lifting property with respect to all $\{\mathcal{F}_r\}$-trivial fibrations (resp. fibrations), and vice-versa for the right lifting property.*

GINJ LEM **Lemma A.16.** *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^+$, K a group, and $\pi \colon \mathbb{S} \to K$ a functor.*

Then if a map $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $A_s \amalg_{L_s A} L_s B \to B_s$ are (resp. trivial) $\mathsf{Aut}(s)$-cofibrations one has that $\mathsf{Lan}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-cofibration.

Proof. By adjunction, one needs only show that for any K-fibration $X \to Y$ in $\mathcal{C}^K$, the map $\pi^*(X \to Y)$ has the right lifting property against all maps $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ as in the statement. By Corollary A.15, it thus suffices to check that the maps

$$(\pi^* X)_s \to (\pi^* Y)_s \times_{M_s \pi^* Y} M_s \pi^* X$$

are $\mathsf{Aut}(s)$-fibrations. But since $M_s Z = *$ (recall $\mathbb{S} = \mathbb{S}^+$) this map is just $X \to Y$ with the $\mathsf{Aut}(s)$-action induced by $\pi \colon \mathsf{Aut}(s) \to K$, hence Corollary A.10 finishes the proof. $\square$

GINJMIN LEM **Lemma A.17.** *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^-$, K a group, and $\pi \colon \mathbb{S} \to K$ a functor.*

Then if a map $X \to Y$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $X_s \to Y_s \times_{M_s Y} M_s X$ are (resp. trivial) $\mathsf{Aut}(s)$-fibrations one has that $\mathsf{Ran}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-fibration.

Proof. This follows dually to the previous proof. $\square$

Remark A.18. Lemmas A.16 and A.17 generalize key parts of the proofs of [BM11, Lemmas 5.3 and 5.5]. The duality of their proofs reflects the duality in Corollary A.10.

Remark A.19. Lemma A.16 will be applied when $K \le \mathsf{Aut}_{\mathbb{R}}(r)$ and $\mathbb{S} = K \ltimes \mathbb{R}^+(r)$ for $\mathbb{R}$ a given generalized Reedy category and $r \in \mathbb{R}$. Similarly, Lemma A.17 will be applied when $\mathbb{S} = K \ltimes \mathbb{R}^-(r)$. It is straightforward to check that in the $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) case maps in $\mathbb{S}$ can be identified with squares as on the left (right)

$$\begin{tikzcd} r' \arrow[r, "+"] \arrow[d, "+"'] & r \arrow[d, "\simeq"] \\ r'' \arrow[r, "+"'] & r \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, "-"] \arrow[d, "\simeq"'] & r' \arrow[d, "-"] \\ r \arrow[r, "-"'] & r'' \end{tikzcd}$$

such that the maps labelled $+$ are in $\mathbb{R}^+$, maps labelled $-$ are in $\mathbb{R}^-$, the horizontal maps are non-invertible, and the maps labeled $\simeq$ are automorphisms in K.

In particular, there is thus a *domain* (resp. *target*) functor $d\colon \mathbb{S} \to \mathbb{R}$ ($t\colon \mathbb{S} \to \mathbb{R}$), and our interest is in maps $d^* A \to d^* B$ ($t^* A \to t^* B$) in $\mathcal{C}^{\mathbb{S}}$ induced from maps $A \to B$ in $\mathcal{C}^{\mathbb{R}}$ so that

$$\mathsf{Lan}_\pi d^*(A \to B) = (L_r A \to L_r B) \qquad \mathsf{Ran}_\pi t^*(A \to B) = (M_r A \to M_r B)$$

We are now in a position to prove the following, which are the essence of Theorem A.8.

REEDYTRCOF LEM **Lemma A.20.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Suppose $A \to B$ be a $\{\mathcal{F}_r\}$-Reedy cofibration. Then the maps $A_r \to B_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $A_r \amalg_{L_r A} L_r B \to B_r$.

Proof. It suffices to check by induction on n that the analogous claim with the restriction $|r| \le n$ also holds. The $n = 0$ case is obvious. Otherwise, letting r range over representatives of the isomorphism classes of r with $|r| = n$, it suffices to check that for each $H \in \mathcal{F}_r$ the map $A_r \to B_r$ is a H-genuine weak equivalence iff so is $A_r \amalg_{L_r A} L_r B \to B_r$.

One now applies Lemma A.16 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^+(r)$ to the map $d^* A \to d^* B$. Note that $\mathcal{F}$-trivial cofibrations are always genuine trivial cofibrations, for any family, so that the trivial cofibrancy requirements are immediate from Corollary A.10. It thus follows that the maps labelled $\sim$

$$\begin{tikzcd}
L_r A \arrow[rr, "\sim"] \arrow[d] & & L_r B \arrow[d] & \\
A_r \arrow[r, "\sim"'] & L_T B \amalg_{L_T A} A_T \arrow[rr] & & B_r
\end{tikzcd}$$

are H-genuine trivial cofibrations, finishing the proof. ☐

REEDYTRFIB LEM **Lemma A.21.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Let $X \to Y$ be a $\{\mathcal{F}_r\}$-Reedy fibration. Then the maps $X_r \to Y_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $X_r \to Y_r \times_{M_r Y} M_r X$.

Proof. One repeats the same induction argument on $|r|$. In the induction step, it suffices to verify that, for each r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $X_r \to Y_r$ is a H-genuine weak equivalence iff so is $X_r \to Y_r \times_{M_r Y} M_r X$.

One now applies Lemma A.17 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^-(r)$ to the map $t^* A \to t^* B$. Note that for each $(r \twoheadrightarrow r') \in \mathbb{S}$ one has $\mathsf{Aut}_{\mathbb{S}}(r \twoheadrightarrow r') = \pi_r^{-1}(H)$ (where π_r is as in (A.1)), so that the trivial fibrancy requirement in Lemma A.17 follows from $\{\mathcal{F}_r\}$ being Reedy-admissible. It follows that the maps labelled $\sim$

$$\begin{tikzcd}
X_r \arrow[r] & Y_r \times_{M_r Y} M_r X \arrow[r, "\sim"] \arrow[d] & Y_r \arrow[d] \\
& M_r X \arrow[r, "\sim"'] & M_r Y
\end{tikzcd}$$

are H-genuine trivial fibrations, finishing the proof. ☐

Remark A.22. The proofs of Lemmas A.20 and A.21 are similar, but not dual, since Lemma A.21 uses Reedy-admissibility while Lemma A.20 does not. This reflects the difference in the proofs of [BM11, Lemmas 5.3 and 5.5] as discussed in [BM11, Remark 5.6], albeit with a caveat.

Setting $K = \{e\}$ in Lemma A.16 yields that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a genuine Reedy cofibration, i.e. a Reedy cofibration for $\{\mathcal{F}_{\text{all}}\}$ the families of all subgroups. On the other hand, the proof of [BM11, Lemma 5.3] argues that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a projective Reedy cofibration, i.e. a Reedy cofibration for $\{\{e\}\}$ the trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim is more general). Since the cofibration half of the projective analogue of Corollary A.10 only holds if ϕ is a monormorphism, the argument in the proof of [BM11, Lemma 5.3] also includes an injectivity check that is not needed for our proof of Lemma A.20.

proof of Theorem A.8. Lemmas A.20 and A.21 say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem A.8 are correct, i.e. that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [Hov99, Def. 1.1.3]. Both 2-out-of-3 and the retract axioms are immediate (recall that retracts commute with limits/colimits). The lifting axiom follows from Corollary A.15 while the task of building factorizations $X \to A \to Y$ of a given map $X \to Y$ follows by a similar standard argument by iteratively factorizing the maps

$$X_r \amalg_{L_r X} L_r A \to Y_r \times_{M_r Y} M_r A$$

in $\mathcal{C}^{\mathsf{Aut}(r)}$, thus building both A and the factorization inductively (see, e.g., the proof of [BM11, Thm. 1.6]). □

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