Equivariant dendroidal sets and simplicial operads

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Abstract

bla bla, generalizing CM13a [3].
Bla, also obtain an equivariant notion of Reedy category

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1 Equivariant dendroidal sets

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1.1 Preliminaries

Definition 1.1. A map $f : S_0 \to T_0$ in Ω is called a *face map* if it is injective on underlying sets. A face map is *inner* if it is of the form $T_0 \smallsetminus E \hookrightarrow T_0$, where E is a subset of the set of inner edges of T. Fixing such a subset E, let $\Phi^E_{\mathrm{Inn}}(T_0)$ denote the poset (under inclusion) of all inner face maps $S_0 \hookrightarrow T_0$ such that $E \subseteq T_0 \smallsetminus S_0$.

Definition 1.2. Given $S_0 \in \Omega$, $T \in \Omega_G$, and a map of forests $f : S_0 \to T$, let T_0 denote the component of T containing the image of S_0.

We say f is a (non-equivariant) *(inner) face map* if $f : S_0 \to T_0$ is a non-equivariant (inner) face map.

Given a subseteq E of inner edges, let $\Phi^E_{\mathrm{Inn}}(T)$ denote the subposet of faces such that miss all of E.

Definition 1.3. Fix $T \in \Omega_G$, and a component T_0 of T, with $H \coloneqq \mathrm{Stab}_G(T_0)$.

Let $\Phi(T)$ denote the poset (under inclusion) of face maps whose image is strictly containined in T_0. Given an inner edge $e \in T$, let $\Phi^{Ge}(T) = \Phi(T) \smallsetminus (T_0 \smallsetminus He)$. Define the *Ge-horn* of T to be the subdendroidal set

$$\Lambda^{Ge}[T] \coloneqq \operatorname*{colim}_{\Phi^{Ge}(T)} \Omega[G \cdot S_0]. \tag{1.4}$$

Definition 1.5. A face map $f : S_0 \to T$ is called *orbital* if $f(S_0) \subseteq T$ is K-closed, where $K = \mathrm{Stab}_G(f(r_s))$, for r_S the root of S_0.

Fixing a component T_0 of T, let $\Phi_o(T)$ denote the poset (under inclusion) of orbital face maps whose image is strictly contained in T_0. Define the *orbital boundary* of T to be the subdendroidal set

$$\partial_o \Omega[T] := \operatorname{colim}_{\Phi_{\mathrm{Orb}}(T)} \Omega[G \cdot S_0]. \tag{1.6}$$

Given an inner edge $e \in T$, let $\Phi^{Ge}_{\mathrm{Orb}}(T) := \Phi_o(T) \smallsetminus (T_0 \smallsetminus He)$ where $H = \mathrm{Stab}_G(T_0)$ (that is; those faces S_0 such that either $T \smallsetminus (G.f(S_0) \cup Ge) \neq \varnothing$, or outer faces removing a stump). Define the *Ge-orbital horn* to be the subdendroidal set

$$\Lambda^{Ge}_o[T] := \operatorname{colim}_{\Phi^{Ge}_o(T)} \Omega[G \cdot S_0]. \tag{1.7}$$

Definition 1.8. Given T, T_0, and H as above, let $U_0 \subseteq T_0$ be a (non-equivariant) subtree, and let $K = \mathrm{Stab}_H(U_0)$. Suppose we have another subdendroidal set $X \subseteq \Omega[T]$ which contains all outer faces of U, and an edge $e \in U_0$. We say that $Ke \subseteq U_0$ is a *characteristic edge orbit* of $X \subseteq \Omega[T] \supseteq \Omega[G \cdot U_0]$ if we have

$$e \in R_0 \in \Phi_{\mathrm{Inn}}(U) \cap X \text{ if and only if } R_0/\bar{K}e \in \Phi_{\mathrm{Inn}}(U) \cap X, \tag{1.9}$$

where $\bar{K} = \mathrm{Stab}_K(R_0)$.

DGE_ANODYNE_PROP

Proposition 1.10. *Given T, T_0, H, U_0, and X as above, we have that Ke is a characteristic edge orbit of $X \subseteq \Omega[T] \supseteq \Omega[G \cdot U]$ implies that $X \to \mathcal{X} \cup \Omega[G \cdot U]$ is inner G-anodyne.*

Proof. If $\Omega[U]$ is already contained in X, we are done. Assuming otherwise, it suffices to show that for all $C \subseteq C'$ K-closed concave subsets of $\Phi^{Ke}_{\mathrm{Inn}}(U_0) \smallsetminus X$, the map

$$X \cup G \cdot_K \left(\bigcup_{E \in C} \Omega[U_0 \smallsetminus E] \right) \to X \cup G \cdot_K \left(\bigcup_{E' \in C'} \Omega[U_0 \smallsetminus E'] \right) \tag{1.11}$$

is inner G-anodyne. Indeed, once $C = \Phi^{Ke}_{\mathrm{Inn}}(U_0) \smallsetminus X$, we have the pushout

$$\begin{tikzcd} G \cdot_K \left(\Lambda^{Ke}[U_0]\right) \arrow[r] \arrow[d] & X \cup G \cdot_K \left(\bigcup_{E \in \Phi^{Ke}_{\mathrm{Inn}}(U_0)} \Omega[U_0 \smallsetminus E] \right) \arrow[d] \\ G \cdot_K \Omega[U_0] \simeq \Omega[G \cdot U_0] \arrow[r] & X \cup \Omega[G \cdot U_0]. \end{tikzcd} \tag{1.12}$$

Moreover, it suffices to consider $C' = C \cup H.D$ for $D \subseteq U_0$, where without loss of generality $e \in D$ and $U_0 \smallsetminus D$ is not in the domain. Let $\bar{K} = \mathrm{Stab}_H(D)$.

We first claim that $\Lambda^{Ke}[U_0 \smallsetminus D]$ is in the domain. If S_0 is an outer face of $U_0 \smallsetminus D$, then S_0 factors through an outer face of U_0, and so S_0 is in X. Further, if $S_0 = U_0 \smallsetminus (D \cup E)$ with $E \cap Ke = \varnothing$, then concavity implies that S_0 is in the domain, as required.

Second, we claim that no face $S_0 = U_0 \smallsetminus D \cup \bar{e}$, with $\bar{e} \subseteq Ke$, is in the domain. Suppose $U_0 \smallsetminus D \cup \bar{e}$ is contained in some $U_0 \smallsetminus E$ already attached. Then, since $E \cap Ke = \varnothing$¡ we have $U_0 \smallsetminus D \subseteq U_0 \smallsetminus E$, so $U_0 \smallsetminus D$ is also in the domain, a contradiction. Further, if $U_0 \smallsetminus D \cup \bar{e}$ is in X, then so is $U_0 \smallsetminus D \cup Ke$, and hence so is $U_0 \smallsetminus D$ (by definition of characteristic edge orbit), also a contradiction.

Now, all faces $U_0 \smallsetminus D \cup \bar{e}$ with $\varnothing \subseteq \bar{e} \subseteq Ke$ have stabilizer $\bar{K}$ (else $D \cap Ke \neq \varnothing$). Thus the desired map is the pushout of

$$G \cdot_{\bar{K}} \left(\Lambda^{\bar{K}e}[U_0 \smallsetminus D] \to \Omega[U_0 \smallsetminus D] \right), \tag{1.13}$$

and hence is anodyne, as required. ∎

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Lemma 1.14. *Suppose U_0 is a minimal outer face of T not in $\Lambda^{Ge}_0[T]$, and suppose $e \in U_0$. Then Ke is a characteristic edge orbit for $\Lambda^{Ge}_0[T] \subseteq \Omega[T] \supseteq [G \cdot U_0]$.*

Proof. Consider an inner face $U_0 \smallsetminus D$ of U_0, with $\bar{K} = \mathrm{Stab}_H(U_0 \smallsetminus D) \le K$, and suppose $U_0 \smallsetminus D \cup \bar{K}e \in \Lambda_0^{Ge}[T]$. Then

$$K_r = \mathrm{Stab}_H(r_U) \le \mathrm{Stab}_H(U_0 \smallsetminus D \cup \bar{K}e) \le \mathrm{Stab}_H(U_0) \le Kr, \tag{1.15}$$

so the outer face U_0 is K_r-closed and not in $\Lambda_0^{Ge}[T]$, and hence we conclude that U_0 must be T_0. However, if $\Lambda_0^{Ge}[T] \neq \Lambda^{Ge}[T]$, T_0 is not a minimal outer face not in $\Lambda_0^{Ge}[T]$. Thus, in these cases, Ke is a characteristic edge orbit vacuously. If in fact $\Lambda_o^{Ge}[T] = \Lambda^{Ge}[T]$, then $T \simeq G \cdot T_0$, so $H = \bar{K} = K_r = \{e\}$. Now, since $U_0 \smallsetminus D \cup e \in \Lambda_0^{Ge}[T]$, $D \smallsetminus e \neq \varnothing$, so $U_0 \smallsetminus D \in \Lambda_0^{Ge}[T]$. ☐

Proposition 1.16. *$\Lambda_o^{Ge}[T] \to \Omega[T]$ is inner G-anodyne, and hence the (hyper)saturated class of the orbital horn inclusions is contained in the (hyper)saturated class of the horn inclusions.*

Proof. Let $\mathrm{Out}_o(T)$ be the poset of outer faces U_0 of T which are not in $\Lambda_0^{Ge}[T]$. It suffices to show that for any G-closed convex subsets $B \subseteq B'$ of $\mathrm{Out}_o(T)$, the map

$$\Lambda_o^{Ge}[T] \cup \bigcup_{R_0 \in B} \Omega[G \cdot R_0] \to \Lambda_o^{Ge}[T] \cup \bigcup_{R_0' \in B'} \Omega[G \cdot R_0'] \tag{1.17}$$

is inner G-anodyne. Again, suffices to show when $B' = B \cup \{U_0\}$. Let $K = \mathrm{Stab}_G(U_0)$, and without loss of generality assume $e \in U_0$. The case $B = \varnothing$ is the previous lemma. For general B, we know the domain contains all outer faces of U_0 by convexity. Now, let $U_0 \smallsetminus D$ be an inner face, with stabilizer $\bar{K}$. Then $U_0 \smallsetminus D \cup \bar{K}e$ is in the domain if either $U_0 \smallsetminus D \cup \bar{K}e$ is contained in some $\Omega[G \cdot R_0]$, or $U_0 \smallsetminus D \cup \bar{K}e \in \Lambda_o^{Ge}[T]$. But then U_0 is contained in R_0 since both are outer faces, or again we apply the previous lemma. Both lead to contradictions, and thus we may conclude that Ke is a characteristic edge orbit for the domain relative to $\Omega[T] \supseteq \Omega[G \cdot U_0]$. Thus, the result holds by Proposition CHAR_EDGE_ANODYNE_PROP 1.10. ☐

1.2 Actual Stuff

Notation 1.18. Given subgroups $H_i \le G$, $0 \le i \le k$ such that $H_0 \ge H_i$, $1 \le i \le k$ we write $C_{\amalg_i H_0/H_i}$ for the G-corolla encoding the H_0-set $H_0/H_1 \amalg \cdots \amalg H_0/H_k$.

Following the discussion preceding HHM16 [4, Prop. 3.6.8], we will call a class of maps of dSet^G *hypersaturated* if is closed under pushouts, transfinite composition, retracts, and satisfies the following cancellation property: if in

$$A \xrightarrow{f} B \xrightarrow{g} C$$

both f and gf are in the class, then so is g.

The following is an equivariant generalization of CM13a [3, Props. 2.4 and 2.5].

Proposition 1.19. *The following sets of maps generate the same hypersaturated class:*

- *the G-inner horn inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the G-inner orbital horn inclusions $\Lambda_o^{Ge}[T] \to \Omega[T]$ for $T \in \Omega_G$ and Ge an inner edge orbit;*
- *the G-segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.*

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Remark 1.20. Setting $G = e$ and slicing over the stick tree η in the previous result one recovers the more well known claim that the hypersaturation (in fact, saturation) of the simplicial inner horns $\{\Lambda^i[n] \to \Delta[n] : 0 < i < n\}$ coincides with the hypersaturation of the simplicial Segal core inclusions $\{Sc[n] \to \Delta[n]\}$.

HYPERSATKAN REM

Remark 1.21. We will also make use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns $\{\Lambda^i[n] \to \Delta[n]: 0 \leq i \leq n\}$ coincides with the hypersaturation of all vertex inclusion maps $\{\Delta[0] \to \Delta[n]\}$. Indeed, call the latter hypersaturation S. An easy argument shows that the Segal core inclusions $\{Sc[n] \to \Delta[n]\}$ are in S and thus so are all inner horn inclusions. On the other hand, the skeletal filtration of the left horns $\Lambda^0[n]$ is built exclusively out of left horn inclusions, and thus since $\Delta[0] = \Lambda^0[1] \to \Delta[1]$ is in S so are all left horn inclusions $\Lambda^0[n] \to \Delta[n]$. The case of right horn inclusions $\Lambda^n[n] \to \Delta[n]$ is dual.

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The following is the equivariant generalization of CM13a [3, Thm. 3.5].

TFAE PROP

Proposition 1.22. *Let $X \to Y$ be a map between G-∞-operads. The following are equivalent:*

(a) for all G-corollas C_A and $H \leq G$ the maps

$$k(\Omega[C_A], X) \to k(\Omega[C_A], Y), \qquad k(\Omega[G/H \cdot \eta], X) \to k(\Omega[G/H \cdot \eta], Y)$$

are weak equivalences in sSet*;*

(b) for all G-trees T the maps

$$k(\Omega[T], X) \to k(\Omega[T], Y)$$

are weak equivalences in sSet*;*

(c) for all normal G-dendroidal sets A, the maps

$$k(A, X) \to k(A, Y)$$

are weak equivalences in sSet*;*

(d) $f: X \to Y$ is a weak equivalence in dSet^G.

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Definition 1.23. Let X be a G-∞-operad. A G*-profile* on X is a map

$$\partial\Omega[C] \to X$$

for some G-corolla $C \in \Sigma_G$.

More explicitly, a G-profile consists of:

- subgroups $H_i \leq G$, $0 \leq i \leq k$ such that $H_0 \geq H_i$ for $1 \leq i \leq k$;
- objects $x_i \in X(\eta)^{H_i}$ for $0 \leq i \leq k$.

To simplify notation, we will prefer to denote a G-profile as $(x_1, \cdots, x_k; x_0)$, and refer to it as a C*-profile.*

Definition 1.24. Given a G-∞-operad and a C-profile $(x_1, \cdots, x_k; x_0)$ we define the space of maps $X(x_1, \cdots, x_k; x_0)$ to be given by the pullback

$$\begin{tikzcd} X(x_1, \cdots, x_k; x_0) \arrow[r] \arrow[d] & Hom(\Omega[C], X) \arrow[d] \\ G \cdot \eta \arrow[r, "(x_1, \cdots, x_k; x_0)"'] & \prod_{0 \leq i \leq k} X(\eta)^{H_i} \end{tikzcd}$$

Noting that there are equivalences of categories (the first of which is an isomorphism)

$$(\mathsf{dSet}_G)/G \cdot \eta \simeq \mathsf{sSet}^{B_G} \simeq \mathsf{sSet},$$

one sees that $X(x_1, \cdots, x_k; x_0)$ can indeed be regarded as a simplicial set (in fact, this is a Kan complex).

Definition 1.25. Let $f\colon X \to Y$ be a map of G-∞-operads.

The map f is called *fully faithful* if, for each C-profile $(x_1, \cdots, x_k; x_0)$ one has that

$$X(x_1, \cdots, x_k; x_0) \to Y\left(f(x_1), \cdots, f(x_k); f(x_0)\right)$$

is weak equivalence in sSet.

The map f is called *essentially surjective* if for each subgroup $H \le G$ the map of categories $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ are essentially surjective.

The following is the equivariant generalization of [3, Thm. 3.11 and Remark 3.12].

Theorem 1.26. *A map $f\colon X \to Y$ of G-∞-operads is fully faithful iff for all G-corollas $C \in \Sigma_G$ the commutative squares of Kan complexes*

$$\begin{tikzcd} k(\Omega[C], X) \arrow[r] \arrow[d, "p"'] & k(\Omega[C], Y) \arrow[d, "q"] \\ k(\partial\Omega[C], X) \arrow[r, "f_*"'] & k(\partial\Omega[C], Y) \end{tikzcd} \tag{1.27}$$

COMSQ EQ

are homotopy pullback squares.

Hence, f is a weak equivalence in dSet^G iff f is both fully faithful and essentially surjective.

Proof. Noting that the 0-simplices of $k(\partial\Omega[C], X)$ are precisely the C-profiles $(x_1, \cdots, x_k, x_0)$, fully faithfulness can be reinterpreted as saying that all fiber maps $p^{-1}(x_1, \cdots, x_k, x_0) \to q^{-1}(f(x_1), \cdots, f(x_k), f(x_0))$ are weak equivalences. But since p, q are Kan fibrations, this is equivalent to the condition that (1.27) is a homotopy pullback (see [3, Lemma 3.9]), and the first half follows.

For the second half, note first that the bottom map in (1.27) can be rewritten as

$$\prod_{0 \le i \le k} k\left(G/H_i \cdot \eta, X\right) \to \prod_{0 \le i \le k} k\left(G/H_i \cdot \eta, Y\right).$$

Assume first that f is a weak equivalence. Proposition 1.22 then implies that the horizontal maps in (1.27) are weak equivalences, so that the square is a pull back square, and thus f is fully faithful. That f is essentially surjective follows from the identity $k(G/H \cdot \eta, X) = k(\iota^*(Z^H))$, so that $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ is essentially surjective at the level of maximal groupoids, and this suffices for essential surjectivity.

Assume now that f is fully faithful and essentially surjective. Since (1.27), Proposition 1.22 now implies that one needs only check that the maps of Kan complexes

$$k\left(G/H \cdot \eta, X\right) \to k\left(G/H \cdot \eta, Y\right) \qquad \text{or} \qquad k\left(\iota^*\left(X^H\right)\right) \to k\left(\iota^*\left(Y^H\right)\right) \tag{1.28}$$

KANMAP EQ

are weak equivalences. As before, essential surjectivity is equivalent to the fact that the maps (1.28) induce surjections on connected components. Hence, it now suffices to show that for each 0-simplex $x \in X^H$ the top map of loop spaces in

$$\begin{tikzcd} \Omega(k(\iota^* X^H), x) \arrow[r] \arrow[d] & \Omega(k(\iota^* Y^H), f(x)) \arrow[d] \\ X(x; x) \arrow[r] & Y(f(x); f(x)) \end{tikzcd} \tag{1.29}$$

OMEGASQ EQ

is a weak equivalence. Noting that the bottom map in (1.29) is a weak equivalence since F is fully faithful and that the vertical maps are the inclusion of the connected components corresponding to automorphisms of x in $\tau(\iota^* X^H)$. It thus suffices to check that the top map in (1.29) is an isomorphism on π_0, and this follows since the map of categories $\tau(\iota^*(X^H)) \to \tau(\iota^*(Y^H))$ is fully faithful. $\square$

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2 Equivariant simplicial dendroidal sets

The results in CM13a [3, §4] concerning the simplicial Reedy model structure all generalized mutatis mutandis.

COMBMODSTR PROP

Proposition 2.1. *Suppose $\mathcal{C}$ admits two model structures (C, W_1, F_1) and (C, W_2, F_2) with the same class of cofibrations, and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

Then there is a smallest common left Bousfield localization (C, W, F) and for any (C, W, F)-local c, d objects one has that $c \to d$ is in W iff it is in W_1 iff it is in W_2.

Proof. The model structure (C, W, F) can be obtained by either localizing (C, W_1, F_1) with regards to the generating trivial cofibrations of (C, W_2, F_2) or vice-versa. That the two processes yield the same model structure follows from from the universal property of left Bousfield localizations Hi03 [5, Prop. 3.4.18]. The claim concerning local c, d follows from the local Whitehead theorem Hi03 [5, Thm. 3.3.8], stating that the local equivalences between local objects match the initial weak equivalences. □

The prototypical example of Proposition COMBMODSTR PROP 2.1 is given by the category ssSet of bisimplicial sets together with the two possible Reedy structures (over the Kan model structure on sSet).

Explicitly, writing the levels of $X \in$ ssSet as $X_{n,m}$ one can either form a Reedy model structure with respect to the *horizontal index* n or with respect to the *vertical index* m. In either case, the generating cofibrations are then given by the maps

$$(\partial\Delta[n] \to \Delta[n]) \square (\partial\Delta[m] \to \Delta[m]), \qquad n, m \geq 0.$$

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$(\partial\Delta[n] \to \Delta[n]) \square \left(\Lambda^j[m] \to \Delta[m]\right), \qquad n \geq 0, m \geq j \geq 0.$$

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square (\partial\Delta[m] \to \Delta[m]), \qquad n \geq i \geq 0, m \geq 0.$$

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, i.e. maps inducing Kan equivalences of simplicial sets $X_{n,\bullet} \to Y_{n,\bullet}$ for each $n \geq 0$, and dually for the vertical Reedy structure.

In the next result we refer to the localized model structure given by Proposition COMBMODSTR PROP 2.1 as the *joint Reedy model structure*.

SSSETJREE PROP

Proposition 2.2. *Suppose that $X, Y \in$ ssSet are horizontal Reedy fibrant. Then:*

(i) for each fixed m all vertex maps $X_{\bullet,m} \to X_{\bullet,0}$ are trivial Kan fibrations;

(ii) any vertical Reedy fibrant replacement $\tilde{X}$ of X is in fact fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a horizontal weak equivalence iff it is a joint weak equivalence;

(iv) the canonical map $X_{n,0} \to X_{n,n}$ is a Kan equivalence.

Proof. (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form $(\partial\Delta[n] \to \Delta[n]) \square (\Delta[0] \to \Delta[m])$.

(ii) follows since by (i) $\tilde{X}$ is then local (with the vertical Reedy model structure as the initial model structure) with respect to all maps of the form $\Delta[0] \times (\Delta[0] \to \Delta[m])$, and thus by Remark HYPERSATKAN REM 1.21 it is fibrant in the joint Reedy model structure (add a remark about this).

(iii) follows from (ii) since the localizing maps $X \to \tilde{X}$, $Y \to \tilde{Y}$ are horizontal equivalences.

For (iv), note first that the diagonal functor Δ: ssSet $\to$ sSet is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure).

But noting that all objects are cofibrant, and regarding $X_{n,0}$ as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map $X_{n_0} \to X$ is a horizontal weak equivalence in ssSet. $\square$

Corollary 2.3. *A map $f\colon X \to Y$ in ssSet is a joint equivalence iff it induces a Kan equivalence on diagonals $\Delta(X) \to \Delta(Y)$ in sSet.*

Proof. Since horizontal Reedy fibrant replacement maps $X \to \tilde{X}$ are diagonal equivalences, one reduces to the case of X, Y horizontal Reedy fibrant.

But Proposition SSSETJREE_PROP ?? (i) and (iii) then combine to say that $X \to Y$ is a joint equivalence iff $X_{\bullet,0} \to Y_{\bullet,0}$ is a Kan equivalence, so that the result follows from Proposition SSSETJREE_PROP ?? (iv). $\square$

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Proposition 2.4. *There is a model structure on sdSet such that*

- *cofibrations are the Reedy cofibrations*
- *weak equivalences are generated by the levelwise dendroidal equivalences and the levelwise topological equivalences.*

Proof. Since both Δ^{op} and Ω^{op} are generalized Reedy, one can attempt to build such a model structure in two ways.

The first way, regarding the category as $\mathsf{sSet}^{\Omega^{op}}$, starts with generating cofibrations

$$(\partial\Omega[T] \to \Omega[T]) \,\square\, (\partial\Delta^n \to \Delta^n)$$

and generating trivial cofibrations

$$(\partial\Omega[T] \to \Omega[T]) \,\square\, (\Lambda^n_i \to \Delta^n)$$

and then localizes at the maps

$$\Lambda^e\Omega[T] \to \Omega[T].$$

The second way, regarding the category as $\mathsf{dSet}^{\Delta^{op}}$, starts with the same generating cofibrations and generating trivial cofibrations

$$J_{\mathsf{dSet}} \,\square\, (\partial\Delta^n \to \Delta^n),$$

where J_{dSet} is a set of generating trivial cofibrations for dSet, and then localizes at

$$(\partial\Omega[T] \to \Omega[T]) \,\square\, (\Lambda^n_i \to \Delta^n).$$

It suffices to check that the two procedures produce the same model structure. Since the cofibrations coincide, and clearly the latter procedure has at least as many weak equivalences as the former, it suffices to check that local objects in the former are in fact also local in the latter. For this the non trivial claim (using the characterization of fibrations between fibrant objects in dSet) is that the former model structure also localized the maps $(* \to J) \,\square\, (\partial\Delta^n \to \Delta^n)$. But the original result easily implies that we have in fact also localized the maps $(* \to J) \times \Delta^n$ and a 2-out-of-3 argument then finishes the proof. $\square$

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A Equivariant Reedy model structures

In [1] Berger and Moerdijk extend the notion of Reedy category so as to allow for categories $\mathbb{R}$ with non-trivial automorphism groups $\mathsf{Aut}(r)$ for $r \in \mathbb{R}$. For such $\mathbb{R}$ and suitable model category $\mathcal{C}$ they then show that there is a *Reedy model structure* on $\mathcal{C}^{\mathbb{R}}$ that is defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see [1, Thm. 1.6] or Theorem A.8 below) to be determined by the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ for each $r \in \mathbb{R}$.

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ with the more general $\mathcal{C}^{\mathsf{Aut}(r)}_{\mathcal{F}_r}$ model structures for $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a nice collection of families of subgroups of each $\mathsf{Aut}(r)$.

To do so, we first need some essential notation. For each map $r \to r'$ in a category $\mathbb{R}$ we will write $\mathsf{Aut}(r \to r')$ for its automorphim group in the arrow category and write

$$\mathsf{Aut}(r) \xleftarrow{\pi_r} \mathsf{Aut}(r \to r') \xrightarrow{\pi_{r'}} \mathsf{Aut}(r') \tag{A.1}$$

PIDEFR EQ

for the obvious projections. We now introduce our equivariant generalization of the "generalized Reedy categories" of [1, Def. 1.1].

GENRED DEF

Definition A.2. A *generalized Reedy category structure* on a small category $\mathbb{R}$ consists of wide subcategories $\mathbb{R}^+$, $\mathbb{R}^-$ and a degree function $|-|: ob(\mathbb{R}) \to \mathbb{N}$ such that:

(i) non-invertible maps in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) raise (lower) degree; isomorphisms preserve degree;

(ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathsf{Iso}(\mathbb{R})$;

(iii) every map f in $\mathbb{R}$ factors as $f = f^+ \circ f^-$ with $f^+ \in \mathbb{R}^+$, $f^- \in \mathbb{R}^-$, and this factorization is unique up to isomorphism.

Let $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ be a collection of families of subgroups of the groups $\mathsf{Aut}(r)$. The collection $\{\mathcal{F}_r\}$ is called *Reedy-admissible* if:

(iv) for all maps $r \twoheadrightarrow r'$ in $\mathbb{R}^-$ one has $\pi_{r'}\left(\pi_r^{-1}(H)\right) \in \mathcal{F}_{r'}$ for all $H \in \mathcal{F}_r$.

We note that condition (iv) above should be thought as of a constraint on the pair $(\mathbb{R}, \{\mathcal{F}_r\})$. The original setup of [1] then deals with the case where $\{\mathcal{F}_r\} = \{\{e\}\}$ is the collection of trivial families. Indeed, our setup recovers the setup in [1], as follows.

Example A.3. When $\{\mathcal{F}_r\} = \{\{e\}\}$, Reedy-admissibility coincides with axiom (iv) in [1, Def. 1.1], stating that if $\theta \circ f^- = f^-$ for some $f^- \in \mathbb{R}^-$ and $\theta \in \mathsf{Iso}(\mathbb{R})$ then θ is an identity.

Example A.4. For any generalized Reedy category $\mathbb{R}$, the collection $\{\mathcal{F}_{\text{all}}\}$ of the families of all subgroups of $\mathsf{Aut}(r)$ is Reedy-admissible.

Example A.5. Let G be a group and set $\mathbb{R} = G \times (0 \to 1)$ with $\mathbb{R} = \mathbb{R}^+$. Then any pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible.

Similarly, set $\mathbb{S} = G \times (0 \leftarrow 1)$ with $\mathbb{S} = \mathbb{S}^-$. Then a pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible iff $\mathcal{F}_0 \supset \mathcal{F}_1$.

Example A.6. Letting $\mathbb{S}$ denote any generalized Reedy category in the sense of [1, Def. 1.1] and G a group, we set $\mathbb{R} = G \times \mathbb{S}$ with $\mathbb{R}^+ = G \times \mathbb{S}^+$ and $\mathbb{R}^- = G \times \mathbb{S}^+$. Further, for each $s \in \mathbb{S}$ we write $\mathcal{F}^{\Gamma}_s$ for the family of G-graph subgroups of $G \times \mathsf{Aut}_{\mathbb{S}}(s)$, i.e., those subgroups $K \leq G \times \mathsf{Aut}_{\mathbb{S}}(s)$ such that $K \cap \mathsf{Aut}_{\mathbb{S}}(s) = \{e\}$.

Reedy admissibility of $\{\mathcal{F}^{\Gamma}_s\}$ follows since for every degeneracy map $s \twoheadrightarrow s'$ in $\mathbb{S}^-$ one has that the homomorphism $\pi_s: \mathsf{Aut}_{\mathbb{S}}(s \twoheadrightarrow s') \to \mathsf{Aut}_{\Omega}(s)$ is injective (we note that this is equivalent to axiom (iv) in [1, Def. 1.1] for $\mathbb{S}$).

Our primary example of interest will come by setting $\mathbb{S} = \Omega^{op}$ in the previous example. In fact, in this case we will also be interested in certain subfamilies $\{\mathcal{F}_T\}_{T\in\Omega} \subset \{\mathcal{F}^{\Gamma}_T\}_{T\in\Omega}$.

Example A.7. Let $\mathbb{R} = G \times \Omega^{op}$ and let $\{\mathcal{F}_T\}_{T\in\Omega}$ be the family of graph subgroups determined by a weak indexing system $\mathcal{F}$. Then $\{\mathcal{F}_T\}$ is Reedy-admissible. To see this, recall first that each $K \in \mathcal{F}_T$ encodes an H-action on $T \in \Omega$ for some $H \leq G$ so that $G \cdot_H T$ is a $\mathcal{F}$-tree. Given a face map $f: T' \hookrightarrow T$, the subgroup $\pi_T^{-1}(K)$ is then determined by the largest subgroup $\bar{H} \leq H$ such that T' inherits the $\bar{H}$-action from T along f (so that f becomes a $\bar{H}$-map), so that $\pi_{T'}(\pi_T^{-1}(K))$ encodes the $\bar{H}$-action on T'. Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of G-trees $G \cdot_{\bar{H}} T' \to G \cdot_H T$.

We now state the main result. We will assume throughout that $\mathcal{C}$ is a model category such that for any group G and family of subgroups $\mathcal{F}$, the category $\mathcal{C}^G$ admits the $\mathcal{F}$-model structure (for example, this is the case whenever $\mathcal{C}$ is a cofibrantly generated cellular model category in the sense of [7]).

REEDYADM THM

Theorem A.8. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible collection of families. Then there is a $\{\mathcal{F}_r\}$-**Reedy model structure** on $\mathcal{C}^{\mathbb{R}}$ such that a map $A \to B$ is*

- *a (trivial) cofibration if $A_r \coprod_{L_r A} L_r B \to B_r$ is a (trivial) $\mathcal{F}_r$-cofibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a weak equivalence if $A_r \to B_r$ is a $\mathcal{F}_r$-weak equivalence in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a (trivial) fibration if $A_r \to B_r \underset{M_r B}{\times} M_r A$ is a (trivial) $\mathcal{F}_r$-fibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$.*

The proof of this result is given at the end of the section after establishing some routine generalizations of the key lemmas in [1] (indeed, the true novelty in this appendix is the Reedy-admissibility condition in part (iv) of Definition A.2).

We first recall the following, cf. [2, Props. 6.5 and 6.6] (we note that [2, Prop. 6.6] can be proven in terms of fibrations, and thus does not depend on special assumptions on $\mathcal{C}$).

Proposition A.9. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of subgroups of $G, \bar{G}$. Then the leftmost (resp. rightmost) adjunction below is a Quillen adjunction*

$$\bar{G} \cdot_G (-): \mathcal{C}^G_{\mathcal{F}} \rightleftarrows \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} : \mathsf{res}^{\bar{G}}_G \qquad \mathsf{res}^{\bar{G}}_G : \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{C}^G_{\mathcal{F}} : \mathsf{Hom}_G(\bar{G}, -)$$

provided that for $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$ (resp. for $\bar{H} \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$).

RESGEN COR

Corollary A.10. *For any homomorphism $\phi: G \to \bar{G}$, the functor $\mathsf{res}^{\bar{G}}_G : \mathcal{C}^{\bar{G}} \to \mathcal{C}^G$ preserves all four classes of genuine cofibrations, trivial cofibrations, fibrations and trivial fibrations.*

The following formalizes an argument implicit in the proof of [1, Lemma 5.2]).

Definition A.11. Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \text{(A.12)}$$

BLA EQ

in $\mathcal{C}^{\mathbb{R}}$. A collection of maps $f_s: B_s \to X_s$ for $|s| \leq n$ that induce a lift of the restriction of (A.12) to $\mathcal{C}^{\mathbb{R}_{\leq n}}$ will be called a *n-partial lift.*

BLALIFT LEM

Lemma A.13. *Let $\mathcal{C}$ be any bicomplete category, and consider a commutative diagram as in (A.12). Then any $(n-1)$-partial lift uniquely induces commutative diagrams*

$$\begin{tikzcd} A_r \amalg_{L_r A} L_r B \arrow[r] \arrow[d] & X_r \arrow[d] \\ B_r \arrow[r] \arrow[ur, dashed] & Y_r \times_{M_r Y} M_r X \end{tikzcd} \qquad \text{(A.14)}$$

BLALIFT EQ

in $\mathcal{C}^{\mathsf{Aut}(r)}$ for each r such that $|r| = n$. Furthermore, extensions of the $(n-1)$-partial lift to a n-partial lift are in bijection with choices of $\mathsf{Aut}(r)$-equivariant lifts of the diagrams (A.14) for r ranging over representatives of the isomorphism classes of r with $|r| = n$.

In the next result, by $\{\mathcal{F}_r\}$-cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in Theorem A.8, regardless of whether such a model structure exists.

BLALIFT COR

Corollary A.15. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}$ an arbitrary family of subgroups of $\mathsf{Aut}(r)$, $r \in \mathbb{R}$. Then a map in $\mathcal{C}^{\mathbb{R}}$ is a $\{\mathcal{F}_r\}$-cofibration (resp. trivial cofibration) iff it has the left lifting property with respect to all $\{\mathcal{F}_r\}$-trivial fibrations (resp. fibrations), and vice-versa for the right lifting property.*

GINJ LEM

Lemma A.16. *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^+$, K a group, and $\pi \colon \mathbb{S} \to K$ a functor.*

Then if a map $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $A_s \amalg_{L_s A} L_s B \to B_s$ are (resp. trivial) $\mathsf{Aut}(s)$-cofibrations one has that $\mathsf{Lan}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-cofibration.

Proof. By adjunction, one needs only show that for any K-fibration $X \to Y$ in $\mathcal{C}^K$, the map $\pi^*(X \to Y)$ has the right lifting property against all maps $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ as in the statement. By Corollary A.15, it thus suffices to check that the maps

$$(\pi^* X)_s \to (\pi^* Y)_s \times_{M_s \pi^* Y} M_s \pi^* X$$

are $\mathsf{Aut}(s)$-fibrations. But since $M_s Z = *$ (recall $\mathbb{S} = \mathbb{S}^+$) this map is just $X \to Y$ with the $\mathsf{Aut}(s)$-action induced by $\pi \colon \mathsf{Aut}(s) \to K$, hence Corollary A.10 finishes the proof. □

GINJMIN LEM

Lemma A.17. *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^-$, K a group, and $\pi \colon \mathbb{S} \to K$ a functor.*

Then if a map $X \to Y$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $X_s \to Y_s \times_{M_s Y} M_s X$ are (resp. trivial) $\mathsf{Aut}(s)$-fibrations one has that $\mathsf{Ran}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-fibration.

Proof. This follows dually to the previous proof. □

Remark A.18. Lemmas A.16 and A.17 generalize key parts of the proofs of [BM08, Lemmas 5.3 and 5.5]. The duality of their proofs reflects the duality in Corollary A.10.

Remark A.19. Lemma A.16 will be applied when $K \leq \mathsf{Aut}_{\mathbb{R}}(r)$ and $\mathbb{S} = K \ltimes \mathbb{R}^+(r)$ for $\mathbb{R}$ a given generalized Reedy category and $r \in \mathbb{R}$. Similarly, Lemma A.17 will be applied when $\mathbb{S} = K \ltimes \mathbb{R}^-(r)$. It is straightforward to check that in the $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) case maps in $\mathbb{S}$ can be identified with squares as on the left (right)

$$\begin{tikzcd} r' \arrow[r, "+"] \arrow[d, "+"'] & r \arrow[d, "\simeq"] \\ r'' \arrow[r, "+"'] & r \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, "-"] \arrow[d, "\simeq"'] & r' \arrow[d, "-"] \\ r \arrow[r, "-"'] & r'' \end{tikzcd} \tag{A.20}$$

such that the maps labelled + are in $\mathbb{R}^+$, maps labelled − are in $\mathbb{R}^-$, the horizontal maps are non-invertible, and the maps labeled $\simeq$ are automorphisms in K.

In particular, there is thus a *domain* (resp. *target*) functor $d \colon \mathbb{S} \to \mathbb{R}$ ($t \colon \mathbb{S} \to \mathbb{R}$), and our interest is in maps $d^* A \to d^* B$ ($t^* A \to t^* B$) in $\mathcal{C}^{\mathbb{S}}$ induced from maps $A \to B$ in $\mathcal{C}^{\mathbb{R}}$ so that

$$\mathsf{Lan}_\pi d^*(A \to B) = (L_r A \to L_r B) \qquad \mathsf{Ran}_\pi t^*(A \to B) = (M_r A \to M_r B)$$

We are now in a position to prove the following, which are the essence of Theorem A.8.

REEDYTRCOF LEM

Lemma A.21. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Suppose $A \to B$ be a $\{\mathcal{F}_r\}$-Reedy cofibration. Then the maps $A_r \to B_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $A_r \amalg_{L_r A} L_r B \to B_r$.

Proof. It suffices to check by induction on n that the analogous claim with the restriction $|r| \leq n$ also holds. The $n = 0$ case is obvious. Otherwise, letting r range over representatives of the isomorphism classes of r with $|r| = n$, it suffices to check that for each $H \in \mathcal{F}_r$ the map $A_r \to B_r$ is a H-genuine weak equivalence iff so is $A_r \amalg_{L_r A} L_r B \to B_r$.

One now applies Lemma A.16 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^+(r)$ to the map $d^* A \to d^* B$. Note that $\mathcal{F}$-trivial cofibrations are always genuine trivial cofibrations, for any family, so that the trivial cofibrancy requirements are immediate from Corollary A.10. It thus follows that the maps labelled ~

$$\begin{tikzcd} L_rA \arrow[r, "\sim"] \arrow[d] & L_rB \arrow[d] & \\ A_r \arrow[r, "\sim"'] & L_TB \amalg_{L_TA} A_T \arrow[r] & B_r \end{tikzcd}$$

are H-genuine trivial cofibrations, finishing the proof. □

REEDYTRFIB LEM

Lemma A.22. *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible family.*

Let $X \to Y$ be a $\{\mathcal{F}_r\}$-Reedy fibration. Then the maps $X_r \to Y_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $X_r \to Y_r \times_{M_rY} M_rX$.

Proof. One repeats the same induction argument on $|r|$. In the induction step, it suffices to verify that, for each r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $X_r \to Y_r$ is a H-genuine weak equivalence iff so is $X_r \to Y_r \times_{M_rY} M_rX$.

One now applies Lemma A.17 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^-(r)$ to the map $t^*A \to t^*B$. Note that for each $(r \twoheadrightarrow r') \in \mathbb{S}$ one has $\mathsf{Aut}_{\mathbb{S}}(r \twoheadrightarrow r') = \pi_r^{-1}(H)$ (where π_r is as in (A.1)), so that the trivial fibrancy requirement in Lemma A.17 follows from $\{\mathcal{F}_r\}$ being Reedy-admissible. It follows that the maps labelled ~

$$\begin{tikzcd} X_r \arrow[r] & Y_r \times_{M_rY} M_rX \arrow[r, "\sim"] \arrow[d] & Y_r \arrow[d] \\ & M_rX \arrow[r, "\sim"'] & M_rY \end{tikzcd}$$

are H-genuine trivial fibrations, finishing the proof. □

Remark A.23. The proofs of Lemmas A.21 and A.22 are similar, but not dual, since Lemma A.22 uses Reedy-admissibility while Lemma A.21 does not. This reflects the difference in the proofs of [1, Lemmas 5.3 and 5.5] as discussed in [1, Remark 5.6], albeit with a caveat.

Setting $K = \{e\}$ in Lemma A.16 yields that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a genuine Reedy cofibration, i.e. a Reedy cofibration for $\{\mathcal{F}_{\text{all}}\}$ the families of all subgroups. On the other hand, the proof of [1, Lemma 5.3] argues that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a projective Reedy cofibration, i.e. a Reedy cofibration for $\{\{e\}\}$ the trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim is more general). Since the cofibration half of the projective analogue of Corollary A.10 only holds if ϕ is a monormorphism, the argument in the proof of [1, Lemma 5.3] also includes an injectivity check that is not needed for our proof of Lemma A.21.

proof of Theorem A.8. Lemmas A.21 and A.22 say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem A.8 are correct, i.e. that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [6, Def. 1.1.3]. Both 2-out-of-3 and the retract axioms are immediate (recall that retracts commute with limits/colimits). The lifting axiom follows from Corollary A.15 while the task of building factorizations $X \to A \to Y$ of a given map $X \to Y$ follows by a similar standard argument by iteratively factorizing the maps

$$X_r \amalg_{L_rX} L_rA \to Y_r \times_{M_rY} M_rA$$

in $\mathcal{C}^{\mathsf{Aut}(r)}$, thus building both A and the factorization inductively (see, e.g., the proof of [1, Thm. 1.6]). □

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