

# Equivariant dendroidal Segal spaces and $G$ - $\infty$ -operads

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## Abstract

bla bla, generalizing [\[CM13a\]](#) [\[CMT13a\]](#).

In an appendix, we discuss Reedy categories in the equivariant context.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
<b>3</b>	<b>Segal cores, horns and orbital horns</b>	<b>4</b>
3.1	The characteristic edge argument . . . . .	4
3.2	Segal covers, horns and orbital horns . . . . .	10
<b>4</b>	<b>Other Stuff</b>	<b>12</b>
4.1	Preliminaries - to be combined with other Pre . . . . .	12
4.2	Proof of Proposition <a href="#">4.43</a> <a href="#">[HYPER PROP]</a> . . . . .	14
4.3	Weak Equivalences in $\mathbf{dSet}^G$ . . . . .	20
<b>5</b>	<b>Equivariant dendroidal spaces</b>	<b>23</b>
<b>6</b>	<b>Pre-operads</b>	<b>27</b>
<b>7</b>	<b>Equivariant dendroidal Segal spaces</b>	<b>29</b>
<b>8</b>	<b>Scratchwork (to be folded into previous sections eventually)</b>	<b>38</b>
<b>A</b>	<b>Equivariant Reedy model structures</b>	<b>40</b>

## 1 Introduction

This paper follows [\[Per17\]](#) and [\[BP17\]](#) and is the third piece of a larger project aimed at understanding the homotopy theory of *equivariant operads with norm maps*. Informally, norm maps are a new piece of structure that must be considered when dealing with equivariant operads (and which has no analogue in the theory of equivariant categories). The need to understand norm maps, as well as their usefulness, was made clear by Hill, Hopkins and Ravenel in their solution of the Kervaire invariant one problem [\[HHR16\]](#).

The starting point of this project was the discovery by the authors, for each finite group  $G$ , of a category  $\Omega_G$  of  $G$ -trees whose objects diagrammatically encode compositions of norms maps and whose arrows encode the necessary compatibilities between such compositions. Our categories  $\Omega_G$  are a somewhat non-obvious equivariant generalization of the dendroidal category  $\Omega$  of Cisinski-Moerdijk-Weiss, and indeed all the key combinatorial concepts in their work, such as faces, degeneracies, boundaries and horns, generalize to  $G$ -trees [Per17, §5.6]. As such, it is natural to ask whether the Cisinski-Moerdijk program [CM11], [CM13a], [CM13b] can also be generalized to the equivariant context.

We recall that the main result of their program is the existence of a Quillen equivalence

$$W_! : \mathbf{dSet} \rightleftarrows \mathbf{sOp} : N_{hc}$$

where  $\mathbf{dSet} = \mathbf{Set}^{\Omega^{op}}$  is the category of presheaves over  $\Omega$ , called *dendroidal sets*, and  $\mathbf{sOp}$  is the category of simplicial colored operads. Their program was carried out in three main steps: (i) [CM11] established the existence of the model structure on  $\mathbf{dSet}$  (with some of the key combinatorial analysis based on Moerdijk and Weiss' work in [MW09]); (ii) [CM13a] established auxiliary model structures on the categories  $\mathbf{sdSet}$  and  $\mathbf{PreOp}$  of dendroidal spaces and pre-operads, and showed that all three of  $\mathbf{dSet}$ ,  $\mathbf{sdSet}$  and  $\mathbf{PreOp}$  are Quillen equivalent; (iii) lastly, [CM13b] established the existence of the model structure on  $\mathbf{sOp}$  as well as the Quillen equivalence between  $\mathbf{sOp}$  and  $\mathbf{PreOp}$ , finishing the proof of the main result.

From the perspective of the Cisinski-Moerdijk program, [Per17] is then the equivariant analogue of the first step [CM11] (as well as [MW09]), while the present paper provides the equivariant analogue of the second step [CM13b]. More explicitly, in [Per17], and inspired by the category  $\Omega_G$  of  $G$ -trees, the second author equipped the category  $\mathbf{dSet}^G$  of  $G$ -equivariant dendroidal sets with a model structure whose fibrant objects are “equivariant operads with norm maps up to homotopy”, called  $G$ - $\infty$ -operads. In the present paper, our main results are then the existence of suitable model structures on the categories  $\mathbf{sdSet}^G$  and  $\mathbf{PreOp}^G$  of  $G$ -dendroidal spaces and  $G$ -pre-operads, as well as the existence of Quillen equivalences between all three of  $\mathbf{dSet}^G$ ,  $\mathbf{sdSet}^G$  and  $\mathbf{PreOp}^G$ .

It is worth noting that, much as was the case with the work in [Per17], our results are not formal consequences of their non-equivariant analogues, due to the nature of norm maps<sup>1</sup>. Indeed, in [BP17], the second piece of our project, the authors introduced the notion of *genuine equivariant operads*, which are new algebraic objects motivated by the combinatorics of norm maps as encoded by the category  $\Omega_G$  of  $G$ -trees. And while a priori the work in [BP17] is largely perpendicular to the Cisinski-Moerdijk program (the main result [BP17, Thm. III] is what one might call the “operadic Elmendorf-Piacenza theorem”, which is an equivariant phenomenon), many of the new technical hurdles in this paper versus [CM13a] can be traced back to the fact that at many points in the discussion we are secretly dealing with colored genuine equivariant operads, which are the colored generalization of the structures discussed in [BP17], and the formal definition of which we prefer to postpone to a follow-up paper.

The organization of the paper is as follows.

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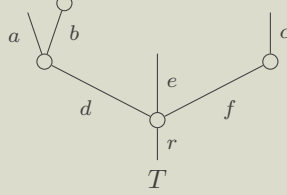
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<sup>1</sup>Recall that by using the inclusions of simplicial categories and simplicial sets into simplicial operads into dendroidal sets, the Cisinski-Moerdijk program recovers the Bergner-Joyal-Lurie-Rezk-Tierney program studying  $\infty$ -categories. As a point of contrast, we note the lack of norms in the categorical case causes the equivariant generalization of this latter program to indeed be formal.

## 2 Preliminaries

We start by recalling the key features of the category  $\Omega$  of trees that will be used throughout. Our official model for  $\Omega$  will be Weiss' algebraic model of *broad posets* as discussed in [Per17, §5], hence we first recall some key notation and terminology. Given a tree diagram  $T$  such as



(2.1) FIRSTTREE EQ

and for each edge  $t$  of  $T$  topped by a vertex  $\circ$ , we write  $t^\dagger$  to denote the tuple of edges immediately above  $t$ . In our example,  $r^\dagger = def$ ,  $d^\dagger = ab$ ,  $f^\dagger = c$  and  $b^\dagger = \epsilon$ , where  $\epsilon$  is the empty tuple. Edges  $t$  for which: (i)  $t^\dagger \neq \epsilon$ , such as  $r, d, f$ , are called *nodes*; (ii)  $t^\dagger = \epsilon$ , such as  $b$ , are called *stumps*; (iii)  $t^\dagger$  is undefined, such as  $a, c, e$ , are called *leaves*. The vertices of  $T$  are then encoded symbolically as  $t^\dagger \leq t$ , which we call a *generating broad relation*. This notation is meant to suggest a form of transitivity: for example, the generating relations  $ab \leq d$  and  $def \leq r$  generate, via *broad transitivity*, a relation  $abef \leq r$  (we note that this is essentially compact notation for the operations and composition in the colored operad generated by  $T$  [MW07, §3]). The other broad relations obtained by broad transitivity are  $dec \leq r$ ,  $abec \leq r$ ,  $aec \leq r$ ,  $a \leq d$ . The set of edges of  $T$  together with these broad relations (as well as identity relations  $t \leq t$ ) form the *broad poset* associated to the tree, which is again denoted  $T$ .

Given a broad relation  $t_0 \cdots t_n \leq t$ , we further write  $t_i \leq_d t$ . Pictorially, this says that the edge  $t_i$  is above  $t$ , and it is thus clear that  $\leq_d$  defines a partial order on edges of  $T$ . Trees always have a single  $\leq_d$ -maximal edge, called the *root*. Edges other than the root or the leaves are called *inner edges*. In our example  $r$  is the root,  $b, d, f$  are inner edges and  $a, e, c$  are leaves.

We denote the sets of edges (inner edges, vertices) of  $T$  by  $E(T)$  (resp.  $E^i(T)$ ,  $V(T)$ ).

The Cisinski-Moerdijk-Weiss category  $\Omega$  of trees then has as objects tree diagrams as in (2.1) and as maps  $\varphi: T \rightarrow S$  the monotone maps of broad posets (meaning that if  $t_1 \cdots t_k \leq t$  then  $\varphi(t_1) \cdots \varphi(t_k) \leq \varphi(t)$ ). In fact, Weiss further identified axioms characterizing those broad posets that are associated to trees (see [Per17, Defs. 5.1 and 5.9]).

Further, our discussion will be somewhat simplified by the assumption that  $\Omega$  contains exactly one representative of each planarized tree. Informally, this means that trees  $T \in \Omega$  come with a preferred planar representation, though this can also be formalized in purely algebraic terms, see [BP17, §3.1]. For our purposes, the main consequence is that any map  $S \rightarrow T$  in  $\Omega$  has a (strictly) unique factorization  $S \simeq S' \rightarrow T$  as an isomorphism followed by a *planar map* [BP17, Prop. 3.21]. Roughly speaking,  $S'$  is obtained from  $S$  by pulling back the planarization of  $T$ .

We now recall the key classes of maps of  $\Omega$ . A map  $\varphi: S \rightarrow T$  which is injective on edges is called a *face map* while a map that is surjective on edges and preserves leaves is called a *degeneracy map* (the extra requirement ensures that leaves of  $S$  do not become stumps of  $T$ ). Moreover, a face map is further called an *inner face map* if  $\varphi(r_S) = r_T$  and  $\varphi(l_S) = l_T$  (where  $r_{(-)}$  denotes the root edge and  $l_{(-)}$  the leaf tuple) and called an *outer face map* if it does not factor through any non-identity inner face maps. The following result is [Per17, Cor. 3.32].

**Proposition 2.2.** *A map  $\varphi: S \rightarrow T$  in  $\Omega$  has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^o} T$$

*as a degeneracy followed by an inner face map followed by an outer face map.*

HERE

All these maps have more explicit characterizations

**Notation 2.3.** Given  $T \in \Omega_G$ , we write  $\mathbf{Face}(T)$  for the  $G$ -poset of *non-equivariant planar faces*  $U \hookrightarrow T$ . We note that  $G$ -action is given by the unique factorization of the composite  $U \hookrightarrow T \xrightarrow{g} T$  as  $U \simeq gU \hookrightarrow T$  such that  $gU \hookrightarrow T$  is planar.

$$\begin{array}{ccc} U & \hookrightarrow & T \\ \simeq \downarrow & & \downarrow g \\ gU & \hookrightarrow & T \end{array}$$

FACEGACT EQ

FACEGACT REM

**Remark 2.4.** For  $T \in \Omega_G$ , any planar face  $\varphi_U: U \rightarrow T$  can also be regarded as a dendrex  $\varphi_U \in \Omega[T](U)$ . However, the  $G$ -isotropy  $H \leq G$  of  $U \in \mathbf{Face}(T)$  must not be confused with the  $G$ -isotropy of  $\varphi_U$ . Instead,  $\Omega[T](U)$  has a larger  $G \times \text{Aut}(U)$ -action, and the  $G \times \text{Aut}(U)$ -isotropy of  $\varphi_U$  is a  $G$ -graph subgroup  $\Gamma \leq G \times \text{Aut}(U)$  encoding a homomorphism  $\phi: H \rightarrow \text{Aut}(U)$ . One readily checks that if  $hU = U$  in  $\mathbf{Face}(T)$  then  $\phi(h)$  is the leftmost isomorphism in (2.3), and it hence follows that  $U \in \Omega$  is equipped with a canonical  $H$ -action, and we will abuse notation slightly by writing  $U \in \Omega^H \subseteq \Omega_H$  to denote this.

FACEGACT EQ

PLFUNCTOR REM

**Remark 2.5.** Recalling that the (non-equivariant) subcategory of face maps is sometimes denoted  $\Omega^+ \subset \Omega$ , we write  $\Omega^+ \downarrow T$  for the category of all non-equivariant faces of  $T \in \Omega_G$ . By pulling back the planarization of  $T$  one then obtains a *planarization functor*

$$\Omega^+ \downarrow T \xrightarrow{pl} \mathbf{Face}(T)$$

which respects the  $G$ -actions on the two categories. Note, however, that the inclusion  $\mathbf{Face}(T) \subset \Omega^+ \downarrow T$  (which is a section of  $pl$ ) does not respect the  $G$ -actions, as displayed in (2.3).

FACEGACT EQ

INNINT LEM

**Lemma 2.6.** Consider a diagram of faces in  $\Omega$

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ \bar{V} & \longrightarrow & \bar{U} \end{array}$$

such that the horizontal maps are outer face maps and the left vertical map is an inner face map. Then  $E^i(V) = E^i(U) \cap E^i(\bar{V})$ .

*Proof.* Write  $r$  and  $\underline{l} = l_1 \cdots l_n$  for the root and leaf tuple of  $V$ , or equivalently  $\bar{V}$ . Since the horizontal maps are outer, an edge  $e \in E^i(U)$  (resp.  $e \in E^i(\bar{U})$ ) is also in  $E^i(V)$  (resp. in  $E^i(\bar{V})$ ) iff  $e <_d r$  and  $\forall_i e \not\leq l_i$ . But then  $E^i(V) = E^i(U) \cap E^i(V) = E^i(U) \cap E^i(\bar{V})$ .  $\square$

**Notation 2.7.** Given  $T \in \Omega_G$  and a non-equivariant planar face  $U \hookrightarrow T$  we write  $\bar{U}^T$ , or simply  $\bar{U}$  when no confusion should arise, for the smallest planar outer face of  $T$  containing  $U$ .

### 3 Segal cores, horns and orbital horns

#### 3.1 The characteristic edge argument

CHAREGE DEF

**Definition 3.1.** Let  $T \in \Omega_G$ ,  $X \subseteq \Omega[T]$  a subdendroidal set, and  $\{U_i\}_{i \in I} \subseteq \mathbf{Face}(T)$  a subset.

Given a set  $\Xi^i$  of inner edges of  $U_i$  and a subface  $V \hookrightarrow U_i$ , denote  $\Xi_V^i = \Xi^i \cap E^i(V)$ . Suppose further that the indexing set  $I$  is a finite  $G$ -poset. For each  $i \in I$  denote

$$X_{<i} = X \cup \bigcup_{j:j < i} \Omega[U_j]$$

- We say that  $\{\Xi^i \subseteq E^i(U_i)\}$  is a *characteristic inner edge collection* of  $\{U_i\}$  with respect to  $X$  if:
- (Ch0)  $X$ ,  $\{U_i\}$  and  $\{\Xi^i\}$  are all  $G$ -equivariant, i.e.  $gX = X$ ,  $gU_i = U_{gi}$ ,  $g\Xi^i = \Xi^{gi}$  as appropriate;
  - (Ch1) for all  $i$ , any outer face  $V = \bar{V}^{U_i}$  of  $U_i$  such that  $\Xi_V^i = \emptyset$  is contained in  $X_{<i}$ ;
  - (Ch2) for all  $i$ , any face  $V \hookrightarrow U_i$  such that  $(V - \Xi_V^i) \in X$  is contained in  $X_{<i}$ ;
  - (Ch3) for all  $j \not\leq i$ , all faces  $V \hookrightarrow U_i$  such that  $(V - \Xi_V^i) \hookrightarrow U_j$  are contained in  $X_{<i}$ .

XIIII REM

**Remark 3.2.** If  $gi \neq i$ , then  $i, gi$  are incomparable in  $I$ . Indeed, otherwise  $i < gi < g^2i < g^3i < \dots$  would violate antisymmetry. Therefore, (Ch3) applies whenever  $j = gi$  for  $gi \neq i$ .

In particular, we assume throughout that if  $gi \neq i$  then  $U_{gi} \neq U_i$ , or else it would be  $U_i \in X_{<i}$ .

SOMEMAIN REM

**Remark 3.3.** In some of the main examples (see Propositions ORB\_HORN\_PRRB\_HORN\_PROP 3.12 and 3.14), there exists a  $G$ -equivariant set  $\Xi$  of inner edges of  $T$  such that  $\Xi^i = \Xi \cap E^i(U_i)$ .

We caution that, for fixed  $X$  and  $\{U_i\}$ , our characteristic conditions are *not* monotone on such  $\Xi$  since increasing  $\Xi$  makes (Ch1) more permissive while making (Ch2),(Ch3) more restrictive.

CHAREDGE LEM

**Lemma 3.4.** If  $\{\Xi^i\}_{i \in I}$  is a characteristic inner edge collection of  $\{U_i\}_{i \in I}$  with respect to  $X$ , then the map

$$X \rightarrow X \cup \bigcup_{i \in I} \Omega[U_i] \tag{3.5}$$

CHARLEM EQ

is  $G$ -inner anodyne. In fact, it is cellular on  $G$ -inner horn inclusions  $\Lambda^E[S] \rightarrow \Omega[S]$ ,  $S \in \Omega_G$ .

*Proof.* We start with the case of  $I \simeq G/H$  orbital so that, abbreviating  $U = U_{[e]}$ ,  $\{U_i\}$  is the set of conjugates  $gU$ . Note that  $H$  is also the isotropy of  $U$  in  $\text{Face}(T)$ . We likewise abbreviate  $\Xi = \Xi^{[e]}$  and  $\Xi_V = \Xi_V^{[e]}$  for  $V \hookrightarrow U$ . Moreover, in this case one has  $X_{<[g]} = X$  in (Ch1),(Ch2),(Ch3).

We write  $\text{Face}_{\Xi}^{lex}(U)$  for the  $H$ -poset of planar faces  $V \hookrightarrow U$  such that  $\Xi_V \neq \emptyset$  and  $\Xi_V = \Xi_{\bar{V}}$  ordered as follows:  $V \leq V'$  if either (i)  $\bar{V} \hookrightarrow \bar{V}'$  and  $\bar{V} \neq \bar{V}'$  or (ii)  $\bar{V} = \bar{V}'$  and  $V \hookrightarrow V'$  (alternatively, this is the lexicographic order of pairs  $(\bar{V}, V)$ ). We note that here and in the remainder of the proof all outer closures are implicitly taken in  $U$  (rather than  $T$ ), i.e.  $\bar{V} = \bar{V}^U$ .

For any  $H$ -equivariant convex subset  $C$  of  $\text{Face}_{\Xi}^{lex}(U)$  we write

$$X_C = X \cup \bigcup_{g \in G, V \in C} \Omega[gV].$$

It now suffices to show that whenever  $C \subseteq C'$  the map  $X_C \rightarrow X_{C'}$  is built cellularly from  $G$ -inner horn inclusions (indeed, setting  $C = \emptyset$  and  $C' = \text{Face}_{\Xi}^{lex}(U)$  recovers (3.5) when  $I \simeq G/H$ ).

Without loss of generality we can assume that  $C'$  is obtained from  $C$  by adding the  $H$ -orbit of a single  $W \hookrightarrow U$ . Further, we may assume  $W \notin X_C$  or else  $X_C = X_{C'}$ . Letting  $K \leq H$  denote the isotropy of  $W$  in  $\text{Face}_{\Xi}^{lex}(U)$  and regarding  $W \in \Omega^K \subseteq \Omega_K$ , we claim there is a pushout diagram

$$\begin{array}{ccc} G \cdot_K \Lambda^{\Xi_W}[W] & \longrightarrow & X_C \\ \downarrow & & \downarrow \\ G \cdot_K \Omega[W] & \longrightarrow & X_{C'} \end{array}$$

FIRPUSH EQ

where we note that inner edge set  $\Xi_W$  is  $K$ -equivariant since  $\Xi_W = \Xi \cap E^i(W)$  and  $\Xi$  is  $H$ -equivariant by (Ch0). The desired pushout will follow once we establish the following claims:

- (a) all proper outer faces  $V$  of  $W$  are in  $X_C$ ;
- (b) an inner face  $W - D$  of  $W$  is in  $X_C$  iff  $D \notin \Xi_W$ ;
- (c) the  $G$ -isotropy (i.e. the isotropy in  $\text{Face}(T)$ ) of faces  $W - D$ ,  $D \subseteq \Xi_W$  is contained in  $K$ .

To check (a), writing  $\bar{V}$  for the corresponding outer face of  $U$ , one has

$$\Xi_V = \Xi \cap E^i(V) = \Xi \cap E^i(W) \cap E^i(\bar{V}) = \Xi \cap E^i(\bar{W}) \cap E^i(\bar{V}) = \Xi \cap E^i(\bar{V}) = \Xi_{\bar{V}}$$

where the second step follows from Lemma INNINT LEM 2.6 (applied to  $V \hookrightarrow W \hookrightarrow U$ ,  $V \hookrightarrow \bar{V} \hookrightarrow U$ ) and the third since by definition of  $\text{Face}_{\Xi}^{lex}(U)$  it is  $\Xi_W = \Xi_{\bar{W}}$ . Thus either  $\Xi_V = \Xi_{\bar{V}} = \emptyset$  so that  $\bar{V} \in X$  by (Ch1), or  $\Xi_V = \Xi_{\bar{V}} \neq \emptyset$  so that  $V \in \text{Face}_{\Xi}^{lex}(U)$  with  $V < W$ , and thus  $V \in C$ . In either case one has  $V \in X_C$ .

We now check the “if” direction of (b). If  $D \notin \Xi_W$  then  $W' = W - (D \setminus \Xi_W)$  is in  $\text{Face}_{\Xi}^{lex}(U)$  (since  $\bar{W}' = \bar{W}$  and  $\Xi_{W'} = \Xi_W$ ) and  $W' < W$ , and thus  $W' \in X_C$ .

For the “only if” direction of (b), note first that it suffices to consider  $D = \Xi_W$ . The assumption  $W \notin X_C$  together with (Ch2) imply that  $W' = W - \Xi_W$  is not in  $X$ , and thus it remains to show that  $W'$  is not a face of any  $gV$  with  $g \in G$ ,  $V \in C$ . Suppose otherwise, i.e.  $W' \hookrightarrow gV$ . If it were  $g \notin H$ , then it would be  $W' \hookrightarrow gV \hookrightarrow gU \neq U$ , and (Ch3) would imply  $W \in X$ . Thus we need only consider  $g \in H$ , and since  $C$  is  $H$ -equivariant, we can set  $g = e$ . It now suffices to show that if  $W' \hookrightarrow V$  then it must be  $W \leq V$  in  $\text{Face}_{\Xi}^{lex}(U)$ , since by convexity of  $C$  this would contradict  $W \notin C$ . Since  $W' \hookrightarrow V$  implies  $\bar{W} = \bar{W}' \hookrightarrow \bar{V}$ , the condition  $W \leq V$  is automatic from the definition of  $\leq$  unless  $\bar{W} = \bar{V}$ . In this latter case, by definition of  $\text{Face}_{\Xi}^{lex}(U)$  the face  $V$  must contain as inner edges all edges in  $\Xi_V = \Xi_{\bar{V}} = \Xi_W = \Xi_{W'}$ , so that not only  $W - \Xi_W = W' \hookrightarrow V$  but also  $W \hookrightarrow V$ . But then it is  $W \leq V$  in either case, establishing the desired contradiction.

We now show (c). If  $g(W - D) = W - D$  then  $g(W - \Xi_W) \hookrightarrow U$ , and thus  $W - \Xi_W \hookrightarrow g^{-1}U$ , so that by (Ch3) it must be  $g \in H$  or else it would be  $W \in X$ . Now suppose  $h(W - D) = W - D$  with  $h \in H$ . Since  $\Xi$  is  $H$ -equivariant (by (Ch0)) and  $\Xi_{W-D} = \Xi_W \setminus D$  (due to  $D \subseteq \Xi_W$ ) it follows that  $h(W - \Xi_W) = W - \Xi_W$ , so that we may assume  $D = \Xi_W$ . Now note that  $hW$ ,  $h(W - \Xi_W) = W - \Xi_W$ ,  $W$  are all faces of  $U$  with a common outer closure  $\bar{W}$ . Hence  $h\Xi_W = \Xi_{hW} \subseteq \Xi_{\bar{W}} = \Xi_W$ , where the last step follows since  $W \in \text{Face}_{\Xi}^{lex}(U)$ , and by cardinality reasons it must in fact be  $h\Xi_W = \Xi_W$ . But then  $hW, W$  have the same outer closure and the same inner edges, and thus  $hW = W$ , establishing (c).

Lastly, we address the case of general  $I$ . For each  $G$ -equivariant convex subset  $J$  of  $I$ , set

$$X_J = X \cup \bigcup_{j \in J} \Omega[U_j].$$

As before, it suffices to check that for all convex subsets  $J \subseteq J'$  the map  $X_J \rightarrow X_{J'}$  is built cellularly from  $G$ -inner horns, and again we can assume that  $J'$  is obtained from  $J$  by adding a single  $G$ -orbit  $Gj$  of  $I$ . By the  $I$  orbital case, it now suffices to check that  $\{\Xi^{gj}\}_{gj \in Gj}$  is also a characteristic inner edge collection of  $\{U_{gj}\}_{gj \in Gj}$  with respect to  $X_J$ . (Ch0) is clear, and since by  $G$ -equivariance and convexity it is  $X_{<gj} \subseteq X_J$ , the new (Ch1),(Ch2),(Ch3) conditions follow from the original conditions.  $\square$

CHAREGE2 REM

**Remark 3.6.** The requirement  $X \subseteq \Omega[T]$  in Definition CHAREGE DEF 3.1 can be relaxed. Given an inclusion  $X \subseteq Y$ , a set of non-degenerate dendrices  $\{y_i \in Y(U_i)\}_{i \in I}$  and a collection of edges  $\{\Xi^i \subset E^i(U_i)\}_{i \in I}$ , suppose that  $I$  is a finite  $G$ -poset and that:

- (Ch0.1) the maps  $y_i: \Omega[U_i] \rightarrow Y$  are monomorphisms;

(Ch0.2)  $X$ ,  $\{U_i\}$ ,  $\{y_i\}$  and  $\{\Xi^i\}$  are all  $G$ -equivariant in the sense that: (i)  $gX = X$ ; (ii) there are associative and unital isomorphisms  $U_i \xrightarrow{g} U_{gi}$ ; (iii) the composites  $\Omega[U_i] \xrightarrow{y_i} Y \xrightarrow{g} Y$  and  $\Omega[U_i] \xrightarrow{g} \Omega[U_{gi}] \xrightarrow{y_{gi}} Y$  coincide; (iv)  $g\Xi^i = \Xi^{gi}$ .

Under (Ch0.1), the  $\Omega[U_i]$  are identified with subcomplexes of  $Y$ , and non-degenerate dendrices  $y \in y_i(\Omega[U_i])(V)$  are identified with faces  $V \hookrightarrow U_i$ .

The original conditions (Ch1),(Ch2),(Ch3) can then be reinterpreted by, for each  $V \hookrightarrow U_i$ , regarding expressions such as  $V \in X$ ,  $(V - \Xi_V^i) \in U_j$  as  $y_i(V) \in X$ ,  $y_i(V - \Xi_V^i) \in y_j(\Omega[U_j])$ .

The proof of Lemma 3.1 now carries out to show that

$$X \rightarrow X \cup \bigcup_{i \in I} y_i(\Omega[U_i])$$

is  $G$ -inner anodyne (again built cellularly from  $G$ -inner horn inclusions).

**Remark 3.7.** Lemma 3.4 readily recovers several arguments in the literature:

- (i) In [Rez01, §10] (also [Rez10, §6.2]), Rezk introduces the notion of *covers*, which in our language are the subsets  $Sc[n] \subseteq X \subseteq \Delta[n]$  such that if  $V$  is in  $X$  then so is the closure  $\bar{V}^{[n]}$  (in words,  $X$  is generated by outer faces). Similarly, in the proof of [CM13a, Prop. 2.4] Cisinski and Moerdijk use subcomplexes  $S_j$  that can be regarded as *dendroidal covers*, i.e. subcomplexes  $Sc[T] \subseteq X \subseteq \Delta[T]$  such that if  $V$  is in  $X$  then so is  $\bar{V}^T$ . Lastly, the subcomplexes  $\Omega[T] \cup_l \Omega[S] \subseteq \Omega[T \circ_l S]$  in the grafting result [MW09, Lemma 5.2] (and similarly for the equivariant analogue [Per17, Prop. 6.19]) are also dendroidal covers.

Lemma 3.4 implies that any inclusion  $X \rightarrow X'$  of  $G$ -equivariant (dendroidal) covers of  $T \in \Omega_G$  is  $G$ -inner anodyne. Indeed, let  $I = \text{Face}_{X'}^{\text{out}}(T)$  be the  $G$ -poset of outer faces  $V \hookrightarrow T$  contained in  $X'$ , ordered by inclusion,  $\Xi = E^i(T)$  and  $U_V = V$ . (Ch0) is clear, (Ch1) follows since  $Sc(T) \subseteq X$ , (Ch2) follows since  $X$  is a cover and (Ch3) follows since the  $U_i$  are closed.

Alternatively, one can also use  $I = \text{Face}_{X',o}^{\text{out}}(T)$  for the  $G$ -trivial set of orbital outer faces  $GV \rightarrow T$ , together with an *arbitrary* total order (see Remark 3.15 for a similar example).

Lastly, we note that if  $\{U_i\} = \{T\}$ ,  $\Xi = E^i(T)$  then (Ch1) says precisely that  $Sc(T) \subseteq X$ .

- (ii) In [MW09, Lemma 9.7], Moerdijk and Weiss introduced a *characteristic edge* condition that can be regarded as a special case of our characteristic edge collection condition as generalized in Remark 3.6, and which served as one of our main inspirations.

Therein, they work in the case of  $Y = \Omega[T] \otimes \Omega[S]$  a tensor product of (non-equivariant) representable dendroidal sets, in which case (Ch0.1) is easily verified (and (Ch0.2) is moot). In our notation, they then require that  $I \simeq *$  (so that (Ch3) is also moot), the dendrex  $y_* \in (\Omega[T] \otimes \Omega[S])(U_*)$  encodes a special type of subtree  $U_*$  of  $\Omega[T] \otimes \Omega[S]$ , which they call an *initial segment*, and they further require that  $\Xi^* = \{\xi\}$  is a singleton, called the *characteristic edge*. Moreover, they then demand that  $X$  should contain all outer faces of the subtree  $U_*$ , from which (Ch1) follows, as well as the key characteristic condition [MW09, Lemma 9.7](ii), which coincides with (Ch2) in this specific setting.

Similarly, in [Per17, Lemma 7.39] the second author introduced a *characteristic edge orbit* condition that generalizes that in [MW09] to the equivariant context by letting  $I \simeq G/H$  and the  $\Xi^{[g]} = \Xi \cap E^i(U_{[g]})$  be determined by a  $G$ -edge orbit  $\Xi \simeq Gf$  (cf. Remark 3.3).

However, both of the lemmas in [MW09] and [Per17] have the drawback of needing to be used iteratively (so that much effort therein is spent showing that this can be done) while Lemma 3.4 is designed so that a single use suffices for the natural applications. Indeed,

context  
for this  
sentence?



conditions (Ch1) and (Ch3), the first of which relaxes the requirement in [MW09], [Per17] that  $X$  should contain all outer faces, essentially provide abstract conditions under which the original characteristic edge arguments of [MW09], [Per17] can be iterated.

**Example 3.8.** As indicated above, Lemma 3.4 can be used to reorganize and streamline the rather long proofs of [Per17, Thms 7.1 and 7.2]. We illustrate this in the hardest case, that of [Per17, Thm. 7.1(i)], which states that if  $S, T \in \Omega_G$  are open (i.e. have no stumps) and  $G\xi$  is an inner edge orbit of  $T$  the maps

$$\partial\Omega[S] \otimes \Omega[T] \coprod_{\partial\Omega[S] \otimes \Lambda^{G\xi}[T]} \Omega[S] \otimes \Lambda^{G\xi}[T] \rightarrow \Omega[S] \otimes \Omega[T] \quad (3.9)$$

THM71 EQ

are  $G$ -inner anodyne.

As detailed in [Per17, §5.1], and originally due to Weiss in [Wei12], there is an algebraically flavoured model for  $\Omega$  as certain types of *broad posets*. Given  $S, T \in \Omega_G$ , it is then possible [Per17, §7.1] to define a  $G$ -equivariant broad poset  $S \otimes T$  so that  $(\Omega[S] \otimes \Omega[T])(V) = \text{Hom}(V, S \otimes T)$  where the Hom set is taken in broad posets. Intuitively  $S \otimes T$  is an object with edge set  $E(S) \times E(T)$  and where each edge  $(s, t) \in S \otimes T$  may, depending on whether  $s \in S$ ,  $t \in T$  are leaves or not, admit two *distinct* vertices: a  $S$ -vertex  $(s, t)^{\uparrow S} = s^{\uparrow} \times t \leq (s, t)$  and a  $T$ -vertex  $(s, t)^{\uparrow T} = s \times t^{\uparrow} \leq (s, t)$ .

To recover [Per17, Thm. 7.1(i)] from Lemma 3.4, we first let  $I = \text{Max}(S \otimes T)$  be the  $G$ -poset of maximal subtrees  $U \hookrightarrow S \otimes T$  (these are called *percolation schemes* in [MW09, §9]), ordered lexicographically [Per17, Def. 7.29]. As an example, let  $\mathbb{Z}/2 = \{\pm 1\}$  and consider the  $\mathbb{Z}/2$ -trees

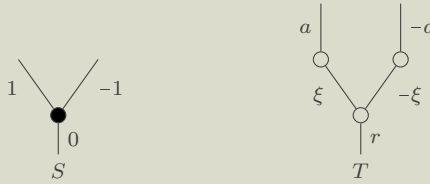
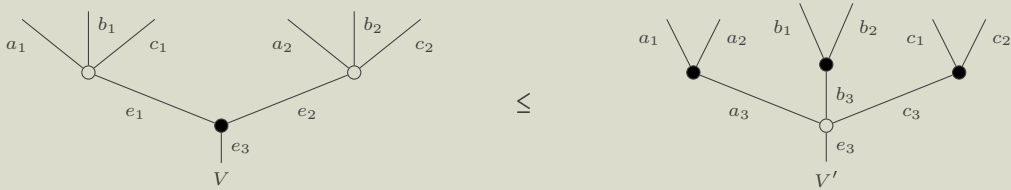


FIGURE 3.1

We depict the  $\mathbb{Z}/2$ -poset  $\text{Max}(S \otimes T)$  in Figure 3.1 (note that  $(s, t)$  is abbreviated as  $t_s$ ). In words, the maximal subtrees are built by starting with the “double root”  $r_0$  and iteratively choosing between the available  $S$  and  $T$  vertices (along all upward paths) until the “double leaves” are reached. The generating relations  $U \leq U'$  in  $\text{Max}(S \otimes T)$  occur whenever  $U$  contains an outer face  $V$  shaped as on the left below and, by “replacing”  $V$  with  $V'$  as on the right, one obtains  $U'$ .



(3.10)

GENLEXREL EQ

The claim that  $\leq$  is indeed a partial order (at least if one of  $S, T$  is open) is [Per17, Prop. 7.31]. As an aside, we note that  $V, V'$  above have a common inner face  $V - \{e_1, e_2\} = V' - \{a_3, b_3, c_3\}$ , which encodes an (universal!) example of a Boardman-Vogt relation (see [MW07, §5.1]).

Returning to the task of proving that (3.9) is  $G$ -inner anodyne, we define  $\Xi^U$ , for each maximal subtree  $U \hookrightarrow S \otimes T$ , to be the set of inner edges of  $U$  of the form  $(g\xi)_s$  such that the vertex  $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$  in  $U$  is a  $T$ -vertex (see Figure 3.1). We now verify (Ch1), (Ch2), (Ch3). We



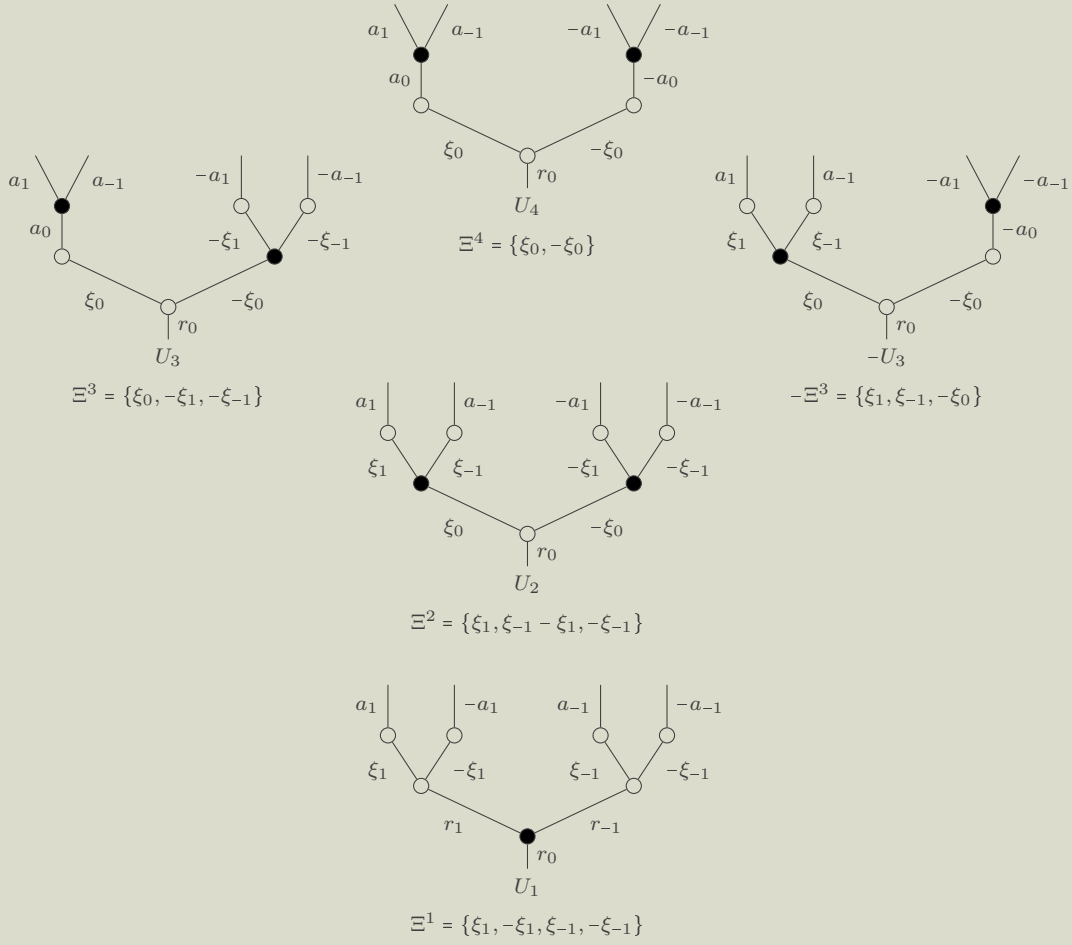


Figure 3.1: The  $\mathbb{Z}/2$ -poset  $\text{Max}(S \otimes T)$  and characteristic edges  $\Xi^i$

FIGURE

recall that, since  $S, T$  are assumed open, [Per17, Lemma 7.19] guarantees that, for faces  $S' \hookrightarrow S$ ,  $T' \hookrightarrow T$ , a factorization  $V \hookrightarrow S' \otimes T' \hookrightarrow S \otimes T$  exists iff the edges of  $V$  are in  $E(S') \times E(T')$ .

For (Ch1), note first that there is an equivariant grafting decomposition  $T = T_{\not\leq G\xi} \sqcup_{G\xi} T^{\leq G\xi}$ , where  $T_{\not\leq G\xi}$  contains the edges  $t \in T$  such that  $\forall_{g \in G} t \not\leq g\xi$  (pictorially, this is a lower equivariant outer face of  $T$ ) while  $T^{\leq G\xi}$  contains the edges  $t \in T$  such that  $\exists_{g \in G} t \leq g\xi$  (an upper equivariant outer face of  $T$ ). But one now readily checks that if  $V \hookrightarrow U$  is an *outer* face such that  $\Xi_V^U = \emptyset$ , then either  $V \hookrightarrow S \otimes T_{\not\leq G\xi}$  or  $V \hookrightarrow S \otimes T^{\leq G\xi}$ , and thus  $V \in X$ .

For (Ch3), suppose  $U_j \not\leq U_i$ ,  $V \hookrightarrow U_i$  and  $(V - \Xi_V^{U_i}) \hookrightarrow U_j$ . Then it follows from [Per17, Lemma 7.37] that there exists a generating relation  $U_k < U_i$  such that  $(V - \Xi_V^{U_i}) \hookrightarrow U_k$  (indeed, [Per17, Lemma 7.37] makes the slightly stronger claim that such a relation can be performed on the outer closure  $\tilde{V}^{U_i}$ ). But then, as one sees from (3.10), all edges  $e \in U_i$  that are not in  $U_k$  are topped by the  $S$ -vertex  $e^{\uparrow S} \leq e$ , and thus it is  $e \notin \Xi_V^{U_i}$ . Therefore  $V \hookrightarrow U_k$ , as desired.

Lastly, for (Ch2), suppose  $V \hookrightarrow U$  and  $(V - \Xi_V^U) \in X$ . If it were  $(V - \Xi_V^U) \hookrightarrow S \otimes \Lambda^{G\xi}[T]$ , then it would also be  $V \hookrightarrow S \otimes \Lambda^{G\xi}[T]$  since all edges of  $\Xi_V^U$  have  $T$ -coordinate in  $G\xi$ . Now

consider the more interesting case  $(V - \Xi_V^U) \hookrightarrow S' \otimes T$  for some face  $S' \hookrightarrow S$ . Then it will also be  $V \hookrightarrow S' \otimes T$  unless there is at least one edge  $(g\xi)_s \in \Xi_V^U$  such that  $s \notin S'$ . But then since the outer closure  $\bar{V}^U$  can have no leaf with  $S$ -coordinate  $s$  (this would contradict  $s \notin S'$ ), there exists some minimal outer face  $U_{(g\xi)_s}^{<s}$  of  $U$  with root  $(g\xi)_s$  and such that its leaves have  $S$ -coordinate  $<_d s$ . By minimality, one has that  $U_{(g\xi)_s}^{<s} \hookrightarrow \bar{V}^U$  and that all inner edges of  $U_{(g\xi)_s}^{<s}$  have  $S$ -coordinate  $s$ . Further, note that  $U_{(g\xi)_s}^{<s}$  has at least one inner edge (since by definition of  $\Xi^U$  the vertex  $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$  is a  $T$ -vertex) and that  $V$  contains none of those inner edges (or else it would be  $s \in S'$ ). Thus by applying [Per17, Lemma 7.34] to  $U_{(g\xi)_s}^{<s}$  one obtains a maximal subtree  $U' < U$  containing all edges of  $U$  that are not inner edges of  $U_{(g\xi)_s}^{<s}$ . But then  $V \hookrightarrow U'$  and (Ch2) follows.

**Remark 3.11.** We briefly outline how to modify the previous example to prove [Per17, Thm 7.1(ii)], in which case some notable subtleties arise. The result again states that (3.9) is  $G$ -inner anodyne, but now with one of  $S, T$  allowed to have stumps while the other is required to be linear.

One again sets  $I = \text{Max}(S \otimes T)$ , where maximal trees are defined just as before, but some care is now needed. To see why, note that if the black nodes  $\bullet$  in (3.10) are replaced with stumps then  $V'$  is actually a subtree of  $V$ .

When  $S$  has stumps and  $T$  is linear this causes no issues and the proof above holds (though we note that it can now be  $\Xi^U = \emptyset$ , in which case the argument for (Ch1) shows  $U \in X$ ).

However, when  $S$  is linear and  $T$  has stumps the proof above breaks down (more precisely, the tree  $U_{(g\xi)_s}^{<s}$  that appears when arguing (Ch2) may now fail to have inner edges). The solution is then to *reverse* the poset structure on  $\text{Max}(S \otimes T)$  and to modify the  $\Xi^U$  to be those inner edges  $(g\xi)_s$  such that  $(g_x i)_s \in t_s^{\uparrow T}$  for some  $t_s$  (pictorially, this says that these are the lowermost edges with  $T$ -coordinate in  $G\xi$ , whereas before they were the uppermost ones). The arguments for (Ch1), (Ch3) then hold. For (Ch2), only the argument for the interesting case of  $V - \Xi_V^U \hookrightarrow S' \otimes T$ ,  $s \notin S'$  changes. In this case, there is then a maximal edge  $t'_s$  such that  $(g\xi)_s < t'_s$ , where  $s$  can not be the root of  $S$  (or else it would be  $s \in S'$ ). Pictorially,  $t'_s$  looks like the edge  $e_1 \in V$  in (3.10) in the case where the  $\bullet$  node is unary (since  $S$  is assumed linear). But then since  $V$  can not contain  $t'_s$  there exists a maximal subtree  $U' > U$  such that  $V \hookrightarrow U'$ , and (Ch2) follows.

Lastly, we note that [Per17, Thm. 7.2] follows from a minor variant of the argument for [Per17, Thm. 7.1(ii)] when  $S$  is linear.

## 3.2 Segal covers, horns and orbital horns

**Proposition 3.12.** *Inclusions of orbital  $G$ -inner horns*

$$\Lambda_o^{EuF}[T] \rightarrow \Lambda_o^E[T] \quad (3.13)$$

are  $G$ -inner anodyne.

*Proof.* We are free to assume that  $T \in \Omega^G \hookrightarrow \Omega_G$ . Indeed, otherwise writing  $T = G \cdot_H T_*$  where  $T_* \in \Omega^H$  is a fixed component and  $E_* = E \cap E^i(T_*)$ ,  $F_* = F \cap E^i(T_*)$ , the map (3.13) is  $G \cdot_H (\Lambda_o^{EuF_*}[T_*] \rightarrow \Lambda_o^{E_*}[T_*])$ .

Further, we are clearly free to assume  $F \neq \emptyset$ . There are two main cases,  $E = \emptyset$  and  $E \neq \emptyset$ .

If  $E = \emptyset$  then  $\Lambda_o^E[T] = \Omega[T]$  and we apply Lemma 3.4 with  $T = \{*\}$  a singleton and

$$\Xi^* = F, \quad U_* = T, \quad X = \Lambda_o^F[T].$$

It remains to check the characteristic conditions in Definition 3.1. (Ch0) and (Ch3) are clear.

Note for  $V \hookrightarrow T$  it is  $V \notin X$  iff  $GV$  has the form  $T - F'$  for some  $G$ -equivariant subset  $F' \subseteq F$ .

For (Ch1), the condition  $\Xi_V = \emptyset$  says that none of the inner edges of  $V$  are in  $F$ , and thus that the equivariant outer face  $GV$  contains none of the edge orbits in  $F$  as inner edge orbits. Since  $F \neq \emptyset$ , the equivariant outer face  $GV$  is not  $T$  itself, and hence  $X = \Lambda_o^F[T]$  contains  $V$ .

For (Ch2), note that if  $V \notin X$ , i.e.,  $GV = T - F'$ , then Remark 8.10 implies that  $G(V - \Xi_V) = T - F''$  for  $F' \subseteq F'' \subseteq F$ , and thus also  $(V - \Xi_V) \notin X$ .

In the case  $E \neq \emptyset$  we instead apply Lemma 3.4 with  $I = F/G$ , with an *arbitrary* choice of total order, and (writing elements of  $F/G$  as subsets  $Gf \subseteq F$ )

$$\Xi^{Gf} = E, \quad U_{Gf} = T - Gf, \quad X = \Lambda_o^{EuF}[T].$$

Note that the  $U_i$  are the orbital inner faces  $T - Gf$  for  $Gf \subseteq F$ , and thus the map in Lemma 3.4 is indeed (3.13). Note that we are free to abbreviate  $\Xi_V = \Xi_V^{Gf}$ , since  $\Xi$  is independent of  $Gf$ . We again check the characteristic conditions. (Ch0) is clear.

For (Ch1), note that for an outer face  $V \hookrightarrow U_i$ , and writing  $\bar{V} = \bar{V}^T$ , Lemma 2.6 implies  $E^i(V) = E^i(U_i) \cap E^i(\bar{V})$  and hence since  $\Xi_{U_i} = \Xi$  the hypothesis  $\Xi_V = \emptyset$  in (Ch1) implies it is also  $\Xi_{\bar{V}} = \emptyset$ . Hence just as before  $G\bar{V}$  is an equivariant outer face other than  $T$ , hence  $V$  is in  $X = \Lambda_o^{EuF}[T]$ . The argument for (Ch2) is identical to the one in the case  $E = \emptyset$ . Lastly, (Ch3) follows since if  $V \notin X$ , so that  $GV = T - F' - E'$  and  $G(V - \Xi_V) = T - F' - E''$  with  $F' \subseteq F$ ,  $E' \subseteq E'' \subseteq E$ , then  $GV \hookrightarrow T - Gf$  iff  $G(V - \Xi_V) \hookrightarrow T - Gf$  and thus  $V \hookrightarrow T - Gf$  iff  $V - \Xi_V \hookrightarrow T - Gf$ .  $\square$

REG\_HORN\_PROP

**Proposition 3.14.** *Inclusions of (regular)  $G$ -inner horns*

$$\Lambda^{EuF}[T] \rightarrow \Lambda^E[T]$$

are  $G$ -inner anodyne.

*Proof.* The case  $E = \emptyset$  is now tautological, hence we can now assume that both  $E, F \neq \emptyset$ .

We now apply Lemma 3.4 with  $I = \mathcal{P}_0(F)$  the poset of non-empty subsets  $\emptyset \neq F' \subseteq F$ , ordered by *reverse inclusion*, and

$$\Xi^{F'} = E, \quad U_{F'} = T - F', \quad X = \Lambda^{EuF}[T].$$

We again need to verify the characteristic conditions, and as in the previous result we can abbreviate  $\Xi_V = \Xi_V^{F'}$ . (Ch0) is clear. (Ch1) follows from an easier version of the argument in the previous proof. (Ch2) follows since  $V \in X$  iff  $V - \Xi_V \in X$ . Similarly, (Ch3) follows since  $V \hookrightarrow T - F'$  iff  $(V - \Xi_V) \hookrightarrow T - F'$  and since if  $V \hookrightarrow T - F'$ ,  $V \hookrightarrow T - F''$  then  $V \hookrightarrow T - (F' \cup F'')$ .  $\square$

TWOPROOF REM

**Remark 3.15.** By specifying to the non-equivariant case  $G = *$  in the case  $E, F \neq \emptyset$ , the previous results yield two distinct proofs that inclusions of non-equivariant horns  $\Lambda^{EuF}[T] \rightarrow \Lambda^E[T]$  are inner anodyne, with the first proof using  $I = F$  (with any order) and a second using  $I = \mathcal{P}_0(F)$ .

The discrepancy is explained as follows: when  $T, E, F$  are  $G$ -equivariant, showing that  $\Lambda^{EuF}[T] \rightarrow \Lambda^E[T]$  is  $G$ -inner anodyne requires a control of isotropies not needed when showing that the underlying map is non-equivariant inner anodyne, and since this control is given by (Ch3), it is necessary to include in the  $\{U_i\}$  the “intersections” of  $T - f$  and  $T - gf$  for  $f \in F$ .

## 4 Other Stuff

### 4.1 Preliminaries - to be combined with other Pre

**Definition 4.1.** A map  $f : S_0 \rightarrow T_0$  in  $\Omega$  is called a *face map* if it is planar and injective on underlying sets;  $S_0$  is called a *face* of  $T_0$ .

**Remark 4.2.** In particular, we may assume that for any face map the  $S_0$  is a *subset* of  $T_0$ .

**Definition 4.3.** A face  $S_0 \hookrightarrow T_0$  is called

- *proper* if  $S_0 \neq T_0$ .
- *inner* if it is of the form  $T_0 \setminus E \hookrightarrow T_0$ , where  $E$  is a subset of the set of inner edges of  $T$ .
- *outer* if it is a composite of *leaf vertex outer faces*  $T_{\downarrow e} \hookrightarrow T$ , *stump outer faces*  $T_{\downarrow e} \hookrightarrow T$ , and *root vertex outer faces*  $T^{\geq e} \hookrightarrow T$ ; see [Per17, Notation 5.41] for more details.

**Remark 4.4.** We may think of inner faces as *composing* some generating broad relations, and outer faces as *removing* some.

**Definition 4.5.** Given  $S_0 \in \Omega$ ,  $T \in \Omega_G$ , and a map of forests  $f : S_0 \rightarrow T$ , let  $T_0$  denote the component of  $T$  containing the image of  $S_0$ . We say  $f$  is a (*proper, inner, outer*) *face map* if  $f : S_0 \rightarrow T_0$  is a (proper, inner, outer) face map on underlying trees.

**Definition 4.6.** Fixing  $T \in \Omega_G$ , let  $\mathcal{F}(T)$  denote the poset (under inclusion) of face maps. Given a  $G$ -closed subset  $E$  of inner edges, let  $\mathcal{F}^E(T)$  denote the subposet removing those faces of the form  $T_0 - \bar{E}$  where  $T_0$  is a component of  $T$  and  $\bar{E} \subseteq E \cap T_0$ . Define the  $E$ -horn of  $T$  to be the subdendroidal set

$$\Lambda^E[T] := \operatorname{colim}_{\mathcal{F}^E(T)} \Omega[U_0] \simeq \bigcup_{\mathcal{F}^E(T)/G} \bigcup_G \Omega[gU_0].$$

**Remark 4.7.** Equivalently, if we have a decomposition  $T \simeq G \cdot_H T_0$ , then

$$\Lambda^E[T] \simeq G \cdot_H \Lambda^{E \cap T_0}[T_0] \simeq G \cdot_H \operatorname{colim}_{\mathcal{F}^{E \cap T_0}(T_0)} \Omega[U_0].$$

**Remark 4.8.** If  $E$  is the entire set of inner edges, we denote  $\Lambda^E[T]$  by  $\partial^{\text{out}}\Omega[T]$ , and call it the *outer boundary*.<sup>2</sup>

**Definition 4.9.** For  $T \in \Omega_G$ , let  $\mathcal{F}_{\text{SC}}(T) \subseteq \text{Out}(T)$  denote the poset of outer faces with precisely one (non-equivariant) vertex; that is, every element records precisely one generating broad relation  $e^\uparrow \leq e$  from  $T$ . We define the *Segal core* of  $T$  to be the subdendroidal set

$$\text{Sc}[T] := \operatorname{colim}_{\mathcal{F}_{\text{SC}}(T)} \Omega[C_0] = \bigcup_{\mathcal{F}_{\text{SC}}(T)/G} \bigcup_{g \in G} \Omega[gC_0].$$

**Remark 4.10.** Explicitly, a map  $\text{Sc}[T] \rightarrow X$  is given by elements in  $X(T_v)$  for all  $v \in V(T_0)$ , which are compatible on overlapping edges and under the action of  $G$ . Equivalently, a map  $\text{Sc}[T] \rightarrow X$  is given by an element in  $X(T_{Gv})$  for each  $Gv \in V_G(T)$  which are compatible on overlapping edges.

**Remark 4.11.** If  $T \simeq G \cdot_H T_0$ , then  $\text{Sc}[T] \simeq G \cdot_H \text{Sc}[T_0]$ .

<sup>2</sup> in [CM13a], this was called the *external* boundary.

**Definition 4.12.** A face map  $f : S_0 \hookrightarrow T$  is called *orbital* if  $f(S_0) \subseteq T$  is  $K_r$ -closed, where  $K_r = \text{Stab}_G(f(r_s))$  for  $r_s$  the root of  $S_0$ .

ORB\_REM

**Remark 4.13.**  $U_0$  is orbital iff the (non-equivariant) subdendroidal sets  $\Omega[gU_0]$  of  $\Omega[T]$  are either disjoint or equal, with  $\Omega[U_0] = \Omega[gU_0]$  iff  $g \in K = \text{Stab}_G(U_0)$ .

ORB\_INJ\_LEM

**Lemma 4.14.** Let  $U_0$  be a face of  $T$  with isotropy  $K$ . Then  $\Omega[G \cdot_K U_0]$  is a subdendroidal set of  $\Omega[T]$  iff  $U_0$  is orbital.

*Proof.* This follows immediately from Remark 4.13. In particular, if  $U_0$  is not orbital, then there exist  $g \in G \setminus K$  such that  $R_0 := U_0 \cap gU_0$  is a proper, non-empty subface of  $U_0$ , so  $R_0 \in \Omega[U_0]$  and  $\Omega[gU_0]$ . Thus  $R_0 \in \Omega[T]$  is hit at least twice.  $\square$

**Remark 4.15.** The data of an orbital face  $U_0 \hookrightarrow T$  is equivalent to both

1. the data of a map of  $G$ -trees  $U \rightarrow T$  which is planar and injective; and
2. the data of a face map  $U/G \rightarrow T/G$  of the *orbital representation* of  $T$ .

expand/contrast/combine with later writing

**Definition 4.16.** Fixing a component  $T_0$  of  $T$ , let  $\mathcal{F}_o(T)$  denote the poset (under inclusion) of orbital face maps. Define the *orbital boundary* of  $T$  to be the subdendroidal set

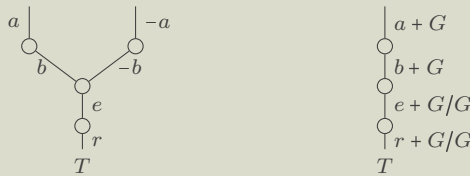
$$\partial_o \Omega[T] := \text{colim}_{\mathcal{F}_o(T)} \Omega[G \cdot S_0].$$

Given an inner edge  $e \in T$ , let  $\mathcal{F}_o^{Ge}(T) := \mathcal{F}_o(T) \cap \mathcal{F}^E(T)$ , and define the *Ge-orbital horn* to be the subdendroidal set

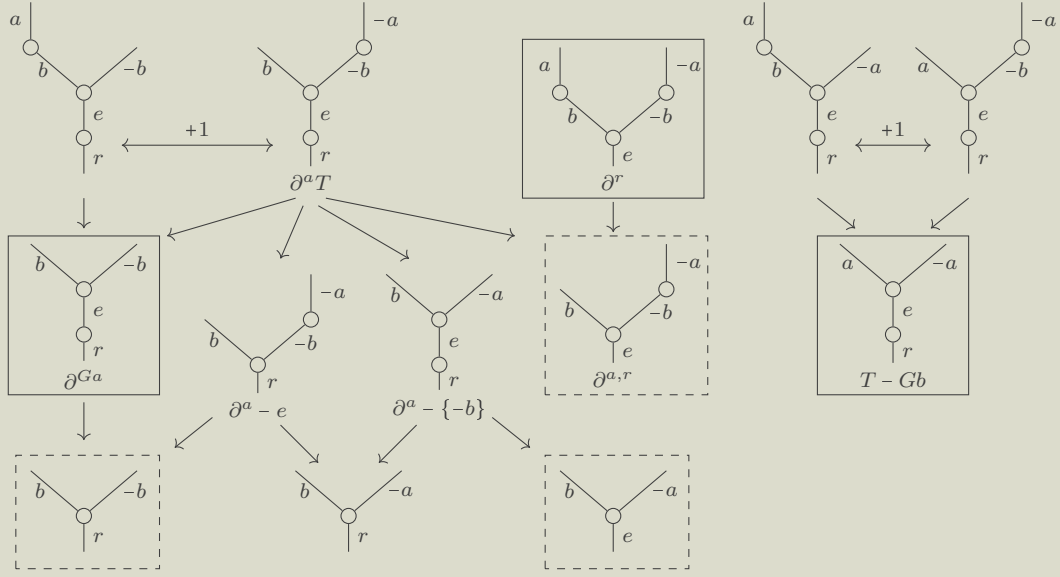
$$\Lambda_o^{Ge}[T] := \text{colim}_{\mathcal{F}_o^{Ge}(T)} \Omega[U_0] = \text{colim}_{\mathcal{F}_o^{Ge}(T)/G} \Omega[G \cdot_K U_0] = \bigcup_{\mathcal{F}_o^{Ge}(T)/G} \bigcup_{g \in G} \Omega[gU_0].$$

**Remark 4.17.** The poset  $\mathcal{F}_o(T)/G$  and  $\mathcal{F}_o^{Ge}(T)/G$  can be interpreted as the poset of faces of the *orbital representation* of the  $G$ -tree  $T$ .

**Example 4.18.** let  $G = C_2$  be the cyclic group with two elements, and consider the tree  $T \in \Omega^G \subset \Omega_G$  below.



The horn  $\Lambda^{Ge}[T]$  is only missing the faces  $T$  and  $T - r$ , and has maximal faces given by the top row below; the orbital horn  $\Lambda_o^{Ge}[T]$  has maximal faces  $\partial^r$ ,  $\partial^{Ga}$ , and  $\partial^{Gb}$ , boxed below. We have also included some of the subfaces of  $\partial^a$ ; those included in the orbital horn are (dashed) boxed. Note that  $\partial^a$  and its maximal subfaces outside the orbital horn themselves have faces both inside and outside the orbital horn.



## 4.2 Proof of Proposition <sup>HYPER PROP</sup> 4.43

need to think about hypersaturation for non-monomorphisms

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts. Moreover, following the discussion preceding [HHM16, Prop. 3.6.8], we will call a class of maps of  $\mathbf{dSet}^G$  *hypersaturated* if it is further closed under the following cancellation property: if in

$$A \xrightarrow{f} B \xrightarrow{g} C$$

both  $f$  and  $gf$  are in the class, then so is  $g$ .

**Notation 4.19.** Given a class of morphisms  $\mathcal{C}$  in  $\mathcal{V}$ , let  $W(\mathcal{C})$  and  $\hat{W}(\mathcal{C})$  denote the saturation and hypersaturation of  $\mathcal{C}$ .

change  $W$  notation

**Remark 4.20.** If  $L$  is a left adjoint, then  $L(\hat{W}(\mathcal{C})) \subseteq \hat{W}(L(\mathcal{C}))$ . In particular, this applies to the “free  $G$ -object” functor  $G \cdot (-)$ .

The following lemma identifies some of the utility of hypersaturations.

**Notation 4.21.** Given a class of maps  $\mathcal{C}$  in  $\mathcal{V}$ , let  $\mathcal{C}^\square$  (respectively  $\mathcal{C}^{\square!}$ ) denote the class of maps with the (*strict*) *right lifting property* (abbreviated (S)RLP) against  $\mathcal{C}^3$ .

HYPER\_LP\_LEM

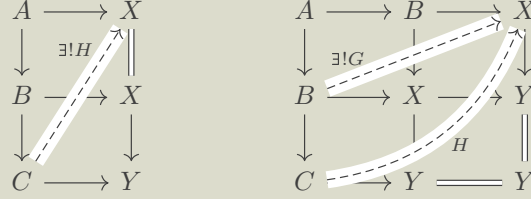
**Lemma 4.22.** Let  $\mathcal{C}$  be a class of maps in  $\mathcal{V}$ . Then  $\mathcal{C}^\square = W(\mathcal{C})^\square$  and  $\mathcal{C}^{\square!} = \hat{W}(\mathcal{C})^{\square!}$ .

*Proof.* It is a straightforward exercise to show that if  $X \rightarrow Y$  has the (S)RLP with respect to a map  $A \rightarrow B$  (resp. maps  $A_\alpha \rightarrow A_{\alpha+1}$ ), then  $X \rightarrow Y$  has the (S)RLP with respect to any pushout or retract of  $A \rightarrow B$  (resp. the transfinite composite).

<sup>3</sup> In many sources (e.g. [Hov99]),  $\mathcal{C}^\square$  is denoted  $\mathcal{C} - \text{inj}$ .

- For the pushout, the map  $C \rightarrow X$  always provides one component of the new lift, while  $B \rightarrow D \rightarrow X$  is always a lift over  $A \rightarrow B$ ; hence strictness implies strictness.
- For the retract, again we have that  $B \rightarrow D \rightarrow X$  is a lift over  $A \rightarrow B$ , and hence strictness implies this composite is uniquely determined; composing with the section map yields that  $D \rightarrow X$  is then also uniquely determined.
- Both the definition of the lift for the composite and its uniqueness are immediate.

Now, suppose  $X \rightarrow Y$  has the SRLP against  $A \rightarrow B$  and the composite  $A \rightarrow B \rightarrow C$ . Consider the diagrams below.



By assumption, there exist unique lifts  $H$  and  $G$  as denoted by the dashed arrows. It is immediate that both  $B \rightarrow X$  and  $B \rightarrow C \xrightarrow{H} X$  are candidates for lift  $G$ , and hence by strictness all three are equal. Thus  $H$  is also a lift of the bottom-left square. Since any lift of the bottom square is also a lift of the rectangle, strictness for  $A \rightarrow B$  implies strictness for  $B \rightarrow C$ .  $\square$

In the vast majority of cases of interest (in particular when dealing with cofibrantly generated model categories), the hypersaturation itself is characterized by a lifting condition using the *small object argument*.

**Lemma 4.23** ([Hov99, Corollary 2.1.15]). *If  $\mathcal{C}$  is a set of maps in  $\mathcal{V}$  such that the domains of maps in  $\mathcal{C}$  are small relative to the closure of  $\mathcal{C}$  under pushouts and transfinite compositions. Then*

$$W(\mathcal{C}) = \square(\mathcal{C}^{\square}).$$

In this subsection, we will prove the following equivariant generalization of [CM13a, Props. 2.4 and 2.5].

**Proposition 4.24.** *The following sets of maps generate the same hypersaturated class:*

- the  $G$ -inner horn inclusions  $\Lambda^{Ge}[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$  and  $Ge$  an inner edge orbit;
- the orbital  $G$ -inner horn inclusions  $\Lambda_o^{Ge}[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$  and  $Ge$  an inner edge orbit;
- the  $G$ -segal core inclusions  $Sc[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$ .

Moreover, one also has the following:

- orbital  $G$ -inner horn inclusions are in the saturation of  $G$ -inner horn inclusions;
- $G$ -segal core inclusions are in the saturation of both orbital  $G$ -inner horn and  $G$ -inner horn inclusions.

**Remark 4.25.** Setting  $G = e$  and slicing over the stick tree  $\eta$  in the previous result one recovers the well known claim that the hypersaturation of the simplicial inner horns  $\{\Lambda^i[n] \rightarrow \Delta[n] : 0 < i < n\}$  coincides with the hypersaturation of the simplicial Segal core inclusions  $\{Sc[n] \rightarrow \Delta[n] : n \geq 2\}$ .



HYPERSATKAN REM

**Remark 4.26.** We will also make use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns  $\{\Lambda^i[n] \rightarrow \Delta[n] : 0 \leq i \leq n\}$  coincides with the hypersaturation of all vertex inclusion maps  $\{\Delta[0] \rightarrow \Delta[n]\}$ . Indeed, call the latter hypersaturation  $S$ . An easy argument shows that the Segal core inclusions  $\{Sc[n] \rightarrow \Delta[n]\}$  are in  $S$  and thus so are all inner horn inclusions. On the other hand, the skeletal filtration of the left horns  $\Lambda^0[n]$  is built exclusively out of left horn inclusions, and thus since  $\Delta[0] = \Lambda^0[1] \rightarrow \Delta[1]$  is in  $S$  so are all left horn inclusions  $\Lambda^0[n] \rightarrow \Delta[n]$ . The case of right horn inclusions  $\Lambda^n[n] \rightarrow \Delta[n]$  is dual.

The observation below will be useful in the proofs which follow.

Combine with description of  $GU$ .

FOF\_OHORN\_LEM

**Lemma 4.27.** Fix  $T \in \Omega_G$ , an edge  $e \in T$ , and a face  $U_0 \hookrightarrow T$  containing  $e$ . Then, for any  $\bar{e} \subseteq Ge \cap U_0$ ,  $U_0 \in \Lambda_o^{Ge}[T]$  iff  $U_0 - \bar{e} \in \Lambda_o^{Ge}[T]$ .

*Proof.* We have a commuting diagram

$$\begin{array}{ccc} U_0 - \bar{e} & \longrightarrow & G(U_0 - \bar{e}) \\ \downarrow & & \downarrow \\ U_0 & \longrightarrow & GU_0 \end{array}$$

where  $GU_0$  is the minimal orbital face containing  $U_0$ , and  $G(U_0 - \bar{e})$  is either  $GU_0$  or  $GU_0 - Ge$  by Remark 8.10. Since it is immediate that for any face  $V_0$  of  $T$ , we have

- $V_0 \in \Lambda_o^{Ge}[T]$  iff  $GV_0$  is, and
- $GV_0 \in \Lambda_o^{Ge}[T]$  iff  $GV_0 - Ge$  is,

we may conclude that  $V_0 - \bar{e}$  is in the orbital horn  $\Lambda_o^{Ge}[T]$  iff  $V_0$  is as well.  $\square$

We break up the proof of Proposition 4.43 into its constituent pieces.

**Proposition 4.28.** Proposition 3.12.

The reverse inclusion is only true on the level of hypersaturations.

**Definition 4.29.** Let  $W_o$  (respectively  $\hat{W}_o$ ) denote the (hyper)saturation of the orbital horn inclusions.

HORN\_ORB\_PROP

**Proposition 4.30.** For any  $T \in \Omega_G$  and  $G$ -set of inner edges  $E$ , the generalized inner horn inclusion  $\Lambda^E[T] \rightarrow \Omega[T]$  is in  $\hat{W}_o$ .

*Proof.* We go by induction on  $|H| \times |V_G(T)|$ , ordered lexicographically, where  $T \simeq G \cdot_H T_*$  is any decomposition (as  $|H|$  is independent of choice of  $T_*$ ).

Can this be replaced with an “We may assume  $T \in \Omega_G \subseteq \Omega_G$ ” argument?

When  $G = \{e\}$ , the orbital horns are regular inner horns, and so the result holds by Proposition 6.17.

Now, the proof of Lemma 3.4 says that if  $\{\Xi^i\}_{i \in I}$  is a characteristic edge collection of  $\{U_i\}_{i \in I}$  with respect to  $X$ , the map

$$X \rightarrow X \cup \bigcup_{i \in I} \Omega[U_i] \quad (4.31)$$

CHAR\_ATTACH\_EQ

is cellular on inner horn inclusions of inner faces of the  $U_i$ , and thus such maps (4.31) are in  $W_o$  by induction.

With that in mind, we apply Lemma 3.4 with both  $I$  and  $U_i$  the subposet of

$$(\Lambda^E[T] \setminus \Lambda_o^E[T]) \cap \text{Face}_E^{lex}(T)$$

such that  $E_i := U_i \cap E$  is maximal, with

$$X = \Lambda_o^E[T], \quad \{\Xi^i\} = \{E^i(U_i) \cap E\}.$$

(Ch0) is immediate, while (Ch2) follows from Lemma 4.27.

For (Ch1), let  $V$  be an outer face of some  $U_i$ , such that  $\Xi_V^i = E^i(V) \cap E = \emptyset$ . If  $U_i$  factors through an outer face of  $T$ , then  $V$  has a minimal orbital outer face  $GV$  also with  $\Xi_{GV}^i = \emptyset$ , and hence both  $GV$  and  $V$  are in  $\Lambda_o^E[T] = X \subseteq X_{<i}$ . If  $U_i$  is an inner face, then  $\bar{U}_i = T_*$  and hence  $E^i(U_i) \cap E \neq \emptyset$ . Moreover, Lemma 2.6 implies that  $E^i(V) = E^i(U_i) \cap E^i(\bar{V})$ , and since by hypothesis  $E^i(V) \cap E = \emptyset$ , we must have  $E^i(\bar{V}) \cap E = \emptyset$ . Thus we have  $G\bar{V}$  is an orbital outer face not equal to  $T$ , so again we may conclude that  $V \in X$ .

For (Ch3), we first note that if  $(V - \Xi_V^i) \hookrightarrow U_j$ , then so must  $V \hookrightarrow U_j$  since, by maximality,  $\Xi_V^i \subseteq U_j$ . Thus for any  $V \hookrightarrow U_i$  with  $(V - \Xi_V^i) \hookrightarrow U_j$ ,  $V \hookrightarrow (U_i \cap U_j)$ , and since  $\Xi^i = \Xi^j$  and  $E_i = E_j$ , we have  $U_i \cap U_j = U_k$  with  $k < i, j$ . Hence  $V \in X_{<i}$ .

Thus  $\Lambda_o^{Ge}[T] \rightarrow \Lambda^{Ge}[T]$  is in  $\hat{W}_o$ , and hence, by the cancelation property of hypersaturations, so is  $\Lambda^{Ge}[T] \rightarrow \Omega[T]$ .  $\square$

We move now to comparing Segal core inclusions with the inner horn inclusions.

**Proposition 4.32.** *For all  $T \in \Omega_G$ ,  $Sc[T] \rightarrow \Omega[T]$  is inner  $G$ -anodyne.*

*Proof.* Fix some decomposition  $T \simeq G \cdot_H T_0$ , and let  $I_0$  denote the set of inner edges of  $T_0$ . Then it is immediate that  $I_0$  is an characteristic inner edge set for  $T_0$  with respect to  $Sc[T]$ , and the result then follows from Lemma 3.4.

also see Remark 2.12 (i)

also holds  
by maxi-  
mality

**Proposition 4.33.** *For any  $T \in \Omega_G$  and  $G$ -sets of edges  $E$  and  $F$ , the maps*

$$\Lambda_o^{E \sqcup F}[T] \rightarrow \Lambda^E[T]$$

are in  $W_o$ .

*Proof.* It suffices to show that, for any edge orbit  $Ge \subseteq E$ , the map  $\Lambda_o^{E-Ge}[T] \rightarrow \Lambda_o^E[T]$  is in  $W_o$ . But the following commuting square is a pushout,

$$\begin{array}{ccc} \Lambda_o^{E-Ge}[T - Ge] & \longrightarrow & \Lambda_o^E[T] \\ \downarrow & & \downarrow \\ \Omega[T - Ge] & \longrightarrow & \Lambda_o^{E-Ge}[T] \end{array}$$

and thus the result follows by induction on  $|T/G| \times |E/G|$  ordered lexicographically.  $\square$

**Proposition 4.34.** *For all  $T \in \Omega_G$ ,  $Sc[T] \rightarrow \Omega[T]$  is in  $W_o$ .*

*Proof.* Define  $\mathcal{F}'_o(T)$  denote the poset  $\mathcal{F}_o^{Ge}(T) \setminus \mathcal{F}_{SC}(T)$ . It suffices to show that for all  $G$ -convex  $B \subseteq B' \subseteq \mathcal{F}'_o(T)$ , the map

$$Sc[T] \cup \bigcup_{B/G} \Omega[G \cdot_K U_0] \rightarrow Sc[T] \cup \bigcup_{B'/G} \Omega[G \cdot_K U_0]$$

is in  $W_o$ ; in particular, we may assume  $B' - B = G \cdot_K \{U_0\}$  where  $\text{Stab}_G(U_0) = K$ .

Now, since  $B$  is convex, every proper orbital outer face of  $U_0$  is in the domain. Moreover, for any face  $V_0 \in \Omega[G \cdot_K U_0]$ ,  $V_0 \in Sc[T]$  implies  $V_0 \in Sc[G \cdot_K U_0] \subseteq \partial_o^{out} \Omega[G \cdot_K U_0]$ , while on the other hand, if  $V_0 \in \Omega[G \cdot_L R_0]$  for some orbital face  $R_0$  already attached, then in fact  $V_0$  is a face of some intersection  $U_0 \cap gR_0$ , which is also orbital. Thus, the above map is the pushout over

$$\partial_o^{out} \Omega[G \cdot_K U_0] \rightarrow \Omega[G \cdot_K U_0],$$

and by Proposition [GORB\\_OHORN\\_PROP 4.33](#), the result is proven.  $\square$

The above combined with Proposition [ORB\\_HORN\\_PROP 3.12](#) immediately implies the following.

**Corollary 4.35.** *Segal core inclusions are inner  $G$ -anodyne.*

**Definition 4.36.** Let  $\hat{W}_{SC} = \hat{W}(SCI_G)$  denote the hypersaturation of the Segal core inclusions.

**Proposition 4.37.** *Any inner horn inclusion*

$$\Lambda^{Ge}[T] \rightarrow \Omega[T]$$

is in  $\hat{W}_{SC}$ .

*Proof.* It suffices to show that  $\Lambda^{Ge}[T] \rightarrow \Omega[T]$  is in  $\hat{W}_{SC}$  for any  $T \in \Omega_G$  and inner edge  $e$  of  $T$ .

Suppose we have a decomposition  $T \simeq G \cdot_H T_0$ , where  $H = \text{Stab}(T_0)$ . We go by induction on  $|H| \times |V_H(T_0)|$ .

replace with  $T \in \Omega^G \subset \Omega_G$  argument

If  $H = e$ , then, since  $G \cdot SCI \subseteq SCI_G$ , and by [CM13a](#), 2.5] we know  $IHI \subseteq \hat{W}(SCI)$ , we conclude that

$$G \cdot IHI \subseteq G \cdot (\hat{W}(SCI)) \subseteq \hat{W}(G \cdot SCI) \subseteq \hat{W}(SCI_G) = \hat{W}_{SC}.$$

If  $H \neq e$ , and  $V_H(T_0) = 2$ , then there is only one inner edge orbit, so the Segal core is an inner horn. Thus, we may assume that  $V_H(T_0) > 2$ .

We first apply Lemma [3.4](#) with  $\hat{T} = \{U_i\} = \text{Out}_p^{>1}(T)$  of proper outer faces of  $T$  not in  $Sc[T]$ ,  $X = Sc[T]$ , and  $\Xi = T$ . (Ch0) is immediate. (Ch1) follows since  $\Xi_V = \emptyset$  iff  $V \in Sc[T]$ , while (Ch2) follows since  $(V - \Xi_V) \in Sc[T]$  iff  $Ksi_V = \emptyset$ . Finally, (Ch3) follows since if  $V - D \hookrightarrow U_j$  with  $U_j$  an outer face of  $T$ , then  $V$  itself must be a face of  $U_j$ , and hence  $V \hookrightarrow U_i \cap U_j$  where this intersection is either in  $X$  or is less than  $U_i$ . Thus, following the logic in the proof of Proposition [ORB\\_HORN\\_PROP 3.12](#), we have that  $Sc[T] \rightarrow \partial_o^{out} \Omega[T]$  is in  $\hat{W}_o$  by induction.

Now, we note by induction and Proposition [3.14](#) that  $\partial_o^{out} \Omega[T] \rightarrow \Lambda^E[T]$  is also in  $\hat{W}_o$ , and hence the result follows by the cancellation closure property of hypersaturations.  $\square$

come back - clean up after here

As a corollary of the above, we can fully characterize the image of the nerve functor  $N : \text{Op}^G \rightarrow \text{dSet}^G$ .

**Definition 4.38.**  $X \in \text{dSet}^G$  is called a (strict) inner  $G$ -Kan complex if  $X$  has the strict right lifting property against inner  $G$ -horn inclusions (that is,  $(X \rightarrow *) \in (IHI_G)^{\square(1)}$ ). Let  $(S)\text{Kan}_G \subseteq \text{dSet}^G$  denote the full subcategory.

Similarly, let  $(S)\text{Kan}$  denote the category of (strict) inner Kan complexes in  $\text{dSet}$  (that is,  $(X \rightarrow *) \in (IHI)^{\square(1)}$ ). We will denote their image under  $\iota_!$  by  $(S)\text{Kan}^G = G \cdot (S)\text{Kan}$ .

make sure we say  $(\iota_!) \Lambda^E[T_0] = \Lambda^{GE}[G \cdot T_0]$  somewhere

**Remark 4.39.** If  $IHI$  denotes the set of inner horn inclusions in  $\text{dSet}$ , with image  $\iota_! IHI = G \cdot IHI$  in  $\text{dSet}^G$ , then it is immediate that  $(G \cdot IHI)^{\square} = \text{Kan}^G$  and  $(G \cdot IHI)^{\square!} = S\text{Kan}^G$ .

We will in fact show that  $(G \cdot IHI)^{\square!} = (IHI_G)^{\square!}$ . First, a lemma.

**Lemma 4.40.** Let  $GIHI$  denote the class of generalized inner horn inclusions  $\Lambda^E[T_0] \rightarrow \Omega[T]$ . If  $X \in (G \cdot GIHI)^{\square!}$ , then  $X \in (G \cdot IHI)^{\square!}$ .

*Proof.* We go by induction on  $|V(T_0)| \times |E|$ , where the base case is immediate. Now, given  $e \in E$ , the square below is a pushout.

$$\begin{array}{ccc} \iota_! \Lambda^{E \setminus e}[T_0/e] & \xrightarrow{\quad} & \iota_! \Lambda^E[T_0] \xrightarrow{f} X \\ \downarrow & \nearrow \exists! F & \downarrow \exists! F' \\ \iota_! \Omega[T_0/e] & \xrightarrow{\quad} & \iota_! \Lambda^{E \setminus e}[T_0] \end{array}$$

Moreover, by induction, any map  $f$  induces a unique lift  $F$ . The universal property of the pushout then implies that we have a unique lift  $F'$ .

Now, we have the following diagram

$$\begin{array}{ccc} \iota_! \Lambda^E[T_0] & \xrightarrow{f} & X \\ \downarrow i & \nearrow \exists \psi & \parallel \\ (i_G)_! \Lambda^{E \setminus e}[T_0] & \xrightarrow{\quad} & X \\ \downarrow j & \nearrow \exists! \phi & \\ \iota_! \Omega[T_0] & \xrightarrow{\quad} & X \end{array}$$

where the dotted arrows are lifts:  $F'$  is the unique lift of  $f$  against  $i$ ,  $\phi$  is the unique lift of  $F'$  against  $j$ , and  $\psi$  is some lift of  $f$  over  $ji$ . This implies that  $\psi j$  is a lift of  $f$  over  $i$ , so by uniqueness  $\psi j = F'$ , but then  $\psi$  is a lift of  $F'$  over  $j$ , and hence  $\psi = \phi$ . Thus the lift of  $f$  over  $ji$  is uniquely determined.  $\square$

SRLP\_IHI\_PROP

**Proposition 4.41.**  $(IHI_G)^{\square!} = (G \cdot IHI)^{\square!}$ . Thus  $\text{Kan}_G = \text{Kan}^G$

*Proof.* Any inner  $G$ -horn inclusion  $i : \Lambda^{Ge}[T] \rightarrow \Omega[T]$  has a decomposition description as

$$j/N : (\iota_! \Lambda^{He}[T_0])/N \rightarrow (\iota_! \Omega[T_0])/N,$$

where  $e \in T_0$ ,  $T \simeq G \cdot_H T_0 \simeq (G \cdot T_0)/N$ ,  $N$  a graph subgroup of  $G \times \text{Aut}(T_0)$ , and  $j \in GIHI$ . Now, for any span  $\Omega[T] \xleftarrow{j/N} \Lambda^{Ge}[T] \xrightarrow{f}$ , consider the following diagram, where  $n \in N$ , and the dashed arrows are lifts.

$$\begin{array}{ccccc}
 \iota_! \Lambda^{He}[T_0] & \xrightarrow{\quad} & \iota \Lambda^{H.e}[T_0]/N \simeq \Lambda^{Ge}[T] & \xrightarrow{f} & X \\
 \downarrow j & \searrow n & \nearrow & \downarrow \exists! \phi & \nearrow \\
 & \iota_! \Lambda^{H.e}[T_0] & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \iota_! \Omega[T_0] & \xrightarrow{\quad} & \iota_! \Omega[T_0]/N \simeq \Omega[T] & \xrightarrow{\quad} & X \\
 \downarrow n & \searrow & \nearrow & \downarrow \exists! \phi & \nearrow \\
 & \iota_! \Omega[T_0] & & & 
 \end{array}$$

Uniqueness implies that  $\phi = n \cdot \phi$  for all  $n \in N$ , and hence factors through  $\Omega[T]$ , finishing the proof.  $\square$

**Corollary 4.42.** *Fix  $X \in \mathbf{dSet}^G$ . Then  $X$  is a strict inner  $G$ -Kan complex iff  $X \simeq N^G(\mathcal{O})$  for some  $\mathcal{O} \in \mathbf{Op}^G$ .*

*Proof.*  $N^G(\mathcal{O})(T)$  is given by maps  $\mathbf{Op}^G(\Omega(T), \mathcal{O})$ , and since  $\Omega(T)$  is a free coloured operad on it's vertices, we may conclude that

$$N^G(\mathcal{O})(T) \simeq \mathbf{Op}^G(\Omega(T), \mathcal{O}) \simeq \mathbf{dSet}^G(Sc[T], N^G(\mathcal{O})).$$

Thus  $N^G(\mathcal{O})$  is in  $(SCI_G)^{\mathbb{Z}^1}$ , and hence is in  $(IHI_G)^{\mathbb{Z}^1}$  by Lemma 4.22 and Proposition 4.37. HYPER\_LP\_LEM

Conversely, by Proposition 4.41 and [MW09, Theorem 6.1, Proposition 6.10], we have that  $X \simeq N_d \circ \tau_d \circ X$  (where we consider  $X$  as a functor  $G \rightarrow \mathbf{Kan}$ ). GHORN\_IN\_SC\_PROP

check this  
ref

### 4.3 Weak Equivalences in $\mathbf{dSet}^G$

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts. Moreover, following the discussion preceding [HHM16, Prop. 3.6.8], we will call a class of maps of  $\mathbf{dSet}^G$  *hypersaturated* if it is further closed under the following cancellation property: if in

$$A \xrightarrow{f} B \xrightarrow{g} C$$

both  $f$  and  $gf$  are in the class, then so is  $g$ .

The following is an equivariant generalization of [CM13a, Props. 2.4 and 2.5]. CM13a

HYPER PROP

**Proposition 4.43.** *The following sets of maps generate the same hypersaturated class:*

- the  $G$ -inner horn inclusions  $\Lambda^{Ge}[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$  and  $Ge$  an inner edge orbit;
- the orbital  $G$ -inner horn inclusions  $\Lambda_o^{Ge}[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$  and  $Ge$  an inner edge orbit;
- the  $G$ -segal core inclusions  $Sc[T] \rightarrow \Omega[T]$  for  $T \in \Omega_G$ .

Moreover, one also has the following:

- (a) orbital  $G$ -inner horn inclusions are in the saturation of  $G$ -inner horn inclusions;

(b) *G*-segal core inclusions are in the saturation of both orbital *G*-inner horn and *G*-inner horn inclusions.

**Remark 4.44.** Setting  $G = e$  and slicing over the stick tree  $\eta$  in the previous result one recovers the well known claim that the hypersaturation of the simplicial inner horns  $\{\Lambda^i[n] \rightarrow \Delta[n]: 0 < i < n\}$  coincides with the hypersaturation of the simplicial Segal core inclusions  $\{Sc[n] \rightarrow \Delta[n]: n \geq 2\}$ .

HYPERSATKAN REM

**Remark 4.45.** We will also make use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns  $\{\Lambda^i[n] \rightarrow \Delta[n]: 0 \leq i \leq n\}$  coincides with the hypersaturation of all vertex inclusion maps  $\{\Delta[0] \rightarrow \Delta[n]\}$ . Indeed, call the latter hypersaturation  $S$ . An easy argument shows that the Segal core inclusions  $\{Sc[n] \rightarrow \Delta[n]\}$  are in  $S$  and thus so are all inner horn inclusions. On the other hand, the skeletal filtration of the left horns  $\Lambda^0[n]$  is built exclusively out of left horn inclusions, and thus since  $\Delta[0] = \Lambda^0[1] \rightarrow \Delta[1]$  is in  $S$  so are all left horn inclusions  $\Lambda^0[n] \rightarrow \Delta[n]$ . The case of right horn inclusions  $\Lambda^n[n] \rightarrow \Delta[n]$  is dual.

**Notation 4.46.** Given subgroups  $H_i \leq G$ ,  $0 \leq i \leq k$  such that  $H_0 \geq H_i$ ,  $1 \leq i \leq k$  we write  $C_{\sqcup_i H_0/H_i}$  for the *G*-corolla encoding the  $H_0$ -set  $H_0/H_1 \sqcup \dots \sqcup H_0/H_k$ .

The following is the equivariant generalization of [CM13a, Thm. 3.5].

TFAE PROP

**Proposition 4.47.** *Let  $X \rightarrow Y$  be a map between  $G$ - $\infty$ -operads. The following are equivalent:*

(a) *for all  $G$ -corollas  $C_A$  and  $H \leq G$  the maps*

$$k(\Omega[C_A], X) \rightarrow k(\Omega[C_A], Y), \quad k(\Omega[G/H \cdot \eta], X) \rightarrow k(\Omega[G/H \cdot \eta], Y)$$

*are weak equivalences in  $\mathbf{sSet}$ ;*

(b) *for all  $G$ -trees  $T$  the maps*

$$k(\Omega[T], X) \rightarrow k(\Omega[T], Y)$$

*are weak equivalences in  $\mathbf{sSet}$ ;*

(c) *for all normal  $G$ -dendroidal sets  $A$ , the maps*

$$k(A, X) \rightarrow k(A, Y)$$

*are weak equivalences in  $\mathbf{sSet}$ ;*

(d)  *$f: X \rightarrow Y$  is a weak equivalence in  $\mathbf{dSet}^G$ .*

PROF DEF

**Definition 4.48.** Let  $X$  be a  $G$ - $\infty$ -operad. A *G*-profile on  $X$  is a map

$$\partial\Omega[C] \rightarrow X$$

for some  $G$ -corolla  $C \in \Sigma_G$ .

More explicitly, a *G*-profile consists of:

- subgroups  $H_i \leq G$ ,  $0 \leq i \leq k$  such that  $H_0 \geq H_i$  for  $1 \leq i \leq k$ ;
- objects  $x_i \in X(\eta)^{H_i}$  for  $0 \leq i \leq k$ .

To simplify notation, we will prefer to denote a *G*-profile as  $(x_1, \dots, x_k; x_0)$ , and refer to it as a *C*-profile.

MAPSPACE DEF

**Definition 4.49.** Given a  $G$ - $\infty$ -operad and a  $C$ -profile  $(x_1, \dots, x_k; x_0)$  we define the space of maps  $X(x_1, \dots, x_k; x_0)$  to be given by the pullback

$$\begin{array}{ccc} X(x_1, \dots, x_k; x_0) & \longrightarrow & \text{Hom}(\Omega[C], X) \\ \downarrow & & \downarrow \\ G \cdot \eta & \xrightarrow{(x_1, \dots, x_k; x_0)} & \prod_{0 \leq i \leq k} X^{H_i} \end{array}$$

Noting that there are equivalences of categories (the first of which is an isomorphism)

$$(\mathbf{dSet}_G)/G \cdot \eta \simeq \mathbf{sSet}^{B_G} \simeq \mathbf{sSet},$$

one sees that  $X(x_1, \dots, x_k; x_0)$  can indeed be regarded as a simplicial set (in fact, this is a Kan complex).

**Definition 4.50.** Let  $f: X \rightarrow Y$  be a map of  $G$ - $\infty$ -operads.

The map  $f$  is called *fully faithful* if, for each  $C$ -profile  $(x_1, \dots, x_k; x_0)$  one has that

$$X(x_1, \dots, x_k; x_0) \rightarrow Y(f(x_1), \dots, f(x_k); f(x_0))$$

is weak equivalence in  $\mathbf{sSet}$ .

The map  $f$  is called *essentially surjective* if for each subgroup  $H \leq G$  the map of categories  $\tau(\iota^*(X^H)) \rightarrow \tau(\iota^*(Y^H))$  are essentially surjective.

The following is the equivariant generalization of [CM13a, Thm. 3.11 and Remark 3.12].

COMSQ THM

**Theorem 4.51.** A map  $f: X \rightarrow Y$  of  $G$ - $\infty$ -operads is fully faithful iff for all  $G$ -corollas  $C \in \Sigma_G$  the commutative squares of Kan complexes

$$\begin{array}{ccc} k(\Omega[C], X) & \longrightarrow & k(\Omega[C], Y) \\ p \downarrow & & \downarrow q \\ k(\partial\Omega[C], X) & \xrightarrow{f_*} & k(\partial\Omega[C], Y) \end{array} \quad (4.52) \quad \text{COMSQ EQ}$$

are homotopy pullback squares.

Hence,  $f$  is a weak equivalence in  $\mathbf{dSet}^G$  iff  $f$  is both fully faithful and essentially surjective.

*Proof.* Noting that the 0-simplices of  $k(\partial\Omega[C], X)$  are precisely the  $C$ -profiles  $(x_1, \dots, x_k, x_0)$ , fully faithfulness can be reinterpreted as saying that all fiber maps  $p^{-1}(x_1, \dots, x_k, x_0) \rightarrow q^{-1}(f(x_1), \dots, f(x_k), f(x_0))$  are weak equivalences. But since  $p, q$  are Kan fibrations, this is equivalent to the condition that (4.52) is a homotopy pullback (see [CM13a, Lemma 3.9]), and the first half follows.

For the second half, note first that the bottom map in (4.52) can be rewritten as

$$\prod_{0 \leq i \leq k} k(G/H_i \cdot \eta, X) \rightarrow \prod_{0 \leq i \leq k} k(G/H_i \cdot \eta, Y).$$

Assume first that  $f$  is a weak equivalence. Proposition 4.47 then implies that the horizontal maps in (4.52) are weak equivalences, so that the square is a pull back square, and thus  $f$  is fully faithful. That  $f$  is essentially surjective follows from the identity  $k(G/H \cdot \eta, X) = k(\iota^*(Z^H))$ , so that  $\tau(\iota^*(X^H)) \rightarrow \tau(\iota^*(Y^H))$  is essentially surjective at the level of maximal groupoids, and this suffices for essential surjectivity.



Assume now that  $f$  is fully faithful and essentially surjective. Since  $(4.52)$ , Proposition 4.47 <sup>COMSQ EQ</sup> <sup>TFAE PROP</sup> now implies that one needs only check that the maps of Kan complexes

$$k(G/H \cdot \eta, X) \rightarrow k(G/H \cdot \eta, Y) \quad \text{or} \quad k(\iota^*(X^H)) \rightarrow k(\iota^*(Y^H)) \quad (4.53)$$

KANMAP EQ

are weak equivalences. As before, essential surjectivity is equivalent to the fact that the maps  $(4.53)$  induce surjections on connected components. Hence, it now suffices to show that for each 0-simplex  $x \in X^H$  the top map of loop spaces in

$$\begin{array}{ccc} \Omega(k(\iota^* X^H), x) & \longrightarrow & \Omega(k(\iota^* Y^H), f(x)) \\ \downarrow & & \downarrow \\ X(x; x) & \longrightarrow & Y(f(x); f(x)) \end{array} \quad (4.54)$$

OMEGASQ EQ

is a weak equivalence. Noting that the bottom map in  $(4.54)$  is a weak equivalence since  $F$  is fully faithful and that the vertical maps are the inclusion of the connected components corresponding to automorphisms of  $x$  in  $\tau(\iota^* X^H)$ . It thus suffices to check that the top map in  $(4.54)$  is an isomorphism on  $\pi_0$ , and this follows since the map of categories  $\tau(\iota^*(X^H)) \rightarrow \tau(\iota^*(Y^H))$  is fully faithful.  $\square$

## 5 Equivariant dendroidal spaces

The results in [CM13a, §4] concerning the simplicial Reedy model structure all generalized mutatis mutandis.

**Proposition 5.1.** *Suppose  $\mathcal{C}$  admits two model structures  $(C, W_1, F_1)$  and  $(C, W_2, F_2)$  with the same class of cofibrations, and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

*Then there is a smallest common left Bousfield localization  $(C, W, F)$  and for any  $(C, W, F)$ -local  $c, d$  objects one has that  $c \rightarrow d$  is in  $W$  iff it is in  $W_1$  iff it is in  $W_2$ .*

*Moreover, an object  $X$  is local in the common left Bousfield localization iff it is simultaneously fibrant in both of the two initial model structures.*

*Proof.* The model structure  $(C, W, F)$  can be obtained by either localizing  $(C, W_1, F_1)$  with regards to the generating trivial cofibrations of  $(C, W_2, F_2)$  or vice-versa. That the two processes yield the same model structure follows from the universal property of left Bousfield localizations [Hir03, Prop. 3.4.18]. The claim concerning local  $c, d$  follows from the local Whitehead theorem [Hir03, Thm. 3.3.8], stating that the local equivalences between local objects match the initial weak equivalences.

That local objects are fibrant in both model structures follows since  $C \cap W$  contains both  $C \cap W_1$  and  $C \cap W_2$  (in fact, this shows that local fibrations are fibrations in both model structures). The converse claim follows from the observation that fibrant objects in any model structure are local with respect to the weak equivalences in that same model structure.  $\square$

The prototypical example of Proposition 5.1 is given by the category  $\mathbf{ssSet}$  of bisimplicial sets together with the two possible Reedy structures (over the Kan model structure on  $\mathbf{sSet}$ ).

Explicitly, writing the levels of  $X \in \mathbf{ssSet}$  as  $X_{n,m}$  one can either form a Reedy model structure with respect to the *horizontal index*  $n$  or with respect to the *vertical index*  $m$ . In either case, the generating cofibrations are then given by the maps

$$(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (\partial\Delta[m] \rightarrow \Delta[m]), \quad n, m \geq 0.$$

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (\Lambda^j[m] \rightarrow \Delta[m]), \quad n \geq 0, m \geq j \geq 0. \quad (5.2)$$

GTRCOHOR EQ

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$(\Lambda^i[n] \rightarrow \Delta[n]) \sqcup (\partial\Delta[m] \rightarrow \Delta[m]), \quad n \geq i \geq 0, m \geq 0. \quad (5.3)$$

GTRCOVER EQ

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, i.e. maps inducing Kan equivalences of simplicial sets  $X_{n,\bullet} \rightarrow Y_{n,\bullet}$  for each  $n \geq 0$ , and dually for the vertical Reedy structure.

In the next result we refer to the localized model structure given by Proposition 5.1 as the *joint Reedy model structure*.

CUMMODSTR PROP

SSSETJREE PROP

**Proposition 5.4.** *Suppose that  $X, Y \in \mathbf{ssSet}$  are horizontal Reedy fibrant. Then:*

- (i) *for each fixed  $m$  all vertex maps  $X_{\bullet,m} \rightarrow X_{\bullet,0}$  are trivial Kan fibrations;*
- (ii) *any vertical Reedy fibrant replacement  $\tilde{X}$  of  $X$  is in fact fibrant in the joint Reedy model structure;*
- (iii) *a map  $X \rightarrow Y$  is a horizontal weak equivalence iff it is a joint weak equivalence;*
- (iv) *the canonical map  $X_{n,0} \rightarrow X_{n,n}$  is a Kan equivalence.*

*Proof.* (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form  $(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (\Delta[0] \rightarrow \Delta[m])$ .

(ii) follows since by (i)  $\tilde{X}$  is then local (with the vertical Reedy model structure as the initial model structure) with respect to all maps of the form  $\Delta[0] \times (\Delta[0] \rightarrow \Delta[m])$ , and thus by Remark 4.45 it is fibrant in the joint Reedy model structure (add a remark about this).

(iii) follows from (ii) since the localizing maps  $X \rightarrow \tilde{X}$ ,  $Y \rightarrow \tilde{Y}$  are horizontal equivalences.

For (iv), note first that the diagonal functor  $\delta^*: \mathbf{ssSet} \rightarrow \mathbf{sSet}$  is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure). But noting that all objects are cofibrant, and regarding  $X_{n,0}$  as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map  $X_{n_0} \rightarrow X$  is a horizontal weak equivalence in  $\mathbf{ssSet}$ .  $\square$

WEAKDIAG COR

**Corollary 5.5.** *A map  $f: X \rightarrow Y$  in  $\mathbf{ssSet}$  is a joint equivalence iff it induces a Kan equivalence on diagonals  $\delta^*(X) \rightarrow \delta^*(Y)$  in  $\mathbf{sSet}$ .*

*Proof.* Since horizontal Reedy fibrant replacement maps  $X \rightarrow \tilde{X}$  are diagonal equivalences, one reduces to the case of  $X, Y$  horizontal Reedy fibrant.

But Proposition 5.4 (i) and (iii) then combine to say that  $X \rightarrow Y$  is a joint equivalence iff  $X_{\bullet,0} \rightarrow Y_{\bullet,0}$  is a Kan equivalence, so that the result follows from Proposition 5.4 (iv).  $\square$

SSSETJREE PROP

SSSETJREE PROP

SSETSETADJ COR

**Corollary 5.6.** *The adjunction*

$$\delta_! : \mathbf{sSet} \rightleftarrows \mathbf{ssSet} : \delta^*$$

*is a Quillen equivalence.*

*Moreover if a map  $f: X \rightarrow Y$  in  $\mathbf{ssSet}$  has the right lifting property against both sets of maps in (5.2) and (5.3), then  $\delta^*(f)$  is a Kan fibration in  $\mathbf{sSet}$ .*

GTRCOHOR EQ

GTRCOVER EQ

Note that the “moreover” claim in this result is not quite formal, since the maps in (5.2), (5.3) are not known to be generating trivial cofibrations for the joint model structure in  $\mathbf{ssSet}$ .

GTRCOVER EQ

*Proof.* Recall that  $\delta_!$  can be characterized as the unique colimit preserving functor such that  $\delta_!(\Delta[n]) = \Delta[n] \times \Delta[n]$ .

The claim that  $\delta_!$  preserves cofibrations thus follows provided that  $\delta_!(\partial\Delta[n] \rightarrow \Delta[n])$  is a monomorphism for all  $n \geq 0$ . This holds since: (i) any two face inclusions  $F_1 \rightarrow \Delta[n]$ ,  $F_2 \rightarrow \Delta[n]$  factor through a minimal face inclusion  $F \rightarrow \Delta[n]$  (indeed, faces are indexed by subsets of  $\{0, 1, \dots, n\}$ ); (ii) for any face inclusion one has  $\delta_!(F \rightarrow \Delta[n]) = (F^{\times 2} \rightarrow \Delta[n]^{\times 2})$ , which is a monomorphism.

The claim that  $\delta_!$  preserves trivial cofibrations easily follows from Remark 4.45 together with Corollary 5.5, but here we give a harder argument needed to establish the stronger “moreover” claim. Namely, we will argue that the maps  $\delta_!(\Lambda^i[n] \rightarrow \Delta[n])$  are built cellularly out of the maps in (5.2), (5.3). One has a factorization

$$\delta_!\Lambda^i[n] \rightarrow \Lambda^i[n] \times \Delta[n] \rightarrow \Delta[n]^{\times 2}$$

where the second map is clearly built cellularly out of maps in (5.3), and we claim that the first map is likewise built cellularly out of maps in (5.2). Indeed, this first map be built by iteratively attaching the maps

$$(\Lambda^F[n] \rightarrow \Lambda^i[n]) \sqcup (\Lambda^i F \rightarrow F)$$

where  $F$  ranges over the poset  $\mathbf{Face}_{\supseteq \{i\}}$  of faces of  $\Delta[n]$  strictly containing  $\{i\}$  (note that when  $F = \Delta[n]$  then  $\Lambda^F[n] = \emptyset$ , so that these maps can not in general be built out of the maps in (5.3)).

Lastly, the Quillen equivalence condition is that for all  $X \in \mathbf{sSet}$  and point fibrant  $Y \in \mathbf{ssSet}$  a map  $X \rightarrow \delta^* Y$  is a weak equivalence iff  $\delta_! X \rightarrow Y$  is. But by Corollary 5.5 this reduces to showing that the unit maps  $X \rightarrow \delta^* \delta_! X$  are weak equivalences, and this latter claim easily follows by cellular induction on  $X$  (recall that since  $\mathbf{sSet}$  is left proper the pushouts attaching cells are homotopy pushouts).  $\square$

**Remark 5.7.** Hypersaturations can be used to simplify the lifting condition required in the previous result. Indeed, note first that  $X \rightarrow Y$  will have the lifting condition against the maps in (5.2) iff all maps  $X^L \rightarrow X^K \times_{Y^K} Y^L$  are Kan fibrations in  $\mathbf{sSet}$  for  $K \rightarrow L$  a monomorphism in  $\mathbf{sSet}$ . But hence  $X \rightarrow Y$  will also have the lifting condition against the maps in (5.3) provided  $X^L \rightarrow X^K \times_{Y^K} Y^L$  is also a *trivial* Kan fibration when  $K \rightarrow L$  is an horn inclusion. But one readily checks that horn inclusions can now be replaced with any set with the same hypersaturation. In particular, by Remark 4.45 it suffices to check that the maps  $X_n \rightarrow X_0 \times_{Y_0} Y_n$  induced by the vertex maps  $[0] \rightarrow [n]$  are trivial Kan fibrations.

**Remark 5.8.** The adjunction  $\delta^*: \mathbf{ssSet} \rightleftarrows \mathbf{sSet}: \delta_*$  is also a Quillen equivalence, but we will not need this fact.

We now turn to our main application of Proposition 5.1, the category  $\mathbf{sdSet}^G = \mathbf{Set}^{\Delta^{op} \times \Omega^{op} \times G}$  of  $G$ -equivariant simplicial dendroidal sets.

Using the fact that  $\Delta$  is a (usual) Reedy category and the model structure on  $\mathbf{dSet}^G$  given by [Per17, Thm. 2.1] yields a model structure on  $\mathbf{sdSet}^G$  that we will refer to as the *simplicial Reedy model structure*.

On the other hand, in the context of Definition A.2,  $\Omega^{op} \times G$  is a generalized Reedy category such that the families  $\{\mathcal{F}_U^\Gamma\}_{U \in \Omega}$  of  $G$ -graph subgroups are Reedy-admissible (see Example A.6), and hence using the underlying Kan model structure on  $\mathbf{sSet}$ , Theorem A.8 yields a model structure on  $\mathbf{sdSet}^G$  that we will refer to as the *equivariant dendroidal Reedy model structure*, or simply as *dendroidal Reedy model structure* for the sake of brevity.

**Proposition 5.9.** *Both the simplicial and dendroidal Reedy model structures on  $\mathbf{sdSet}^G$  have generating cofibrations given by the maps*

$$(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq 0, T \in \Omega_G.$$

Further, the dendroidal Reedy structure has generating trivial cofibrations the maps

$$(\Lambda^i[n] \rightarrow \Delta[n]) \sqcup (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq i \geq 0, T \in \Omega_G. \quad (5.10) \quad \text{DENDTRIVCOF EQ}$$

while the simplicial Reedy structure has generating trivial cofibrations the maps

$$(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (A \rightarrow B), \quad n \geq 0 \quad (5.11) \quad \text{SIMPTRIVCOF EQ}$$

for  $\{A \rightarrow B\}$  a set of generating trivial cofibrations of  $\mathbf{dSet}^G$ .

*Proof.* For the claims concerning the dendroidal Reedy structure, note that the presheaves  $\Omega[T] \in \mathbf{dSet}^G$  are precisely the quotients  $(G \cdot \Omega[U])/K$  for  $U \in \Omega$  and  $K \leq G \times \Sigma_U$  a  $G$ -graph subgroup, so that  $\partial\Omega[T] \rightarrow \Omega[T]$  represents the maps  $X_U^K \rightarrow (M_U X)^K$  for  $X \in \mathbf{dSet}^G$ .

The claims concerning the simplicial Reedy structure are immediate.  $\square$

JOINTFIBCHAR COR

**Corollary 5.12.** *The joint fibrant objects  $X \in \mathbf{sdSet}^G$  have the following equivalent characterizations:*

- (i)  $X$  is both simplicial Reedy fibrant and dendroidal Reedy fibrant;
- (ii)  $X$  is simplicial Reedy fibrant and all maps  $X_0 \rightarrow X_n$  are equivalences in  $\mathbf{dSet}^G$ ;
- (iii)  $X$  is dendroidal Reedy fibrant and all maps

$$X^{\Omega[T]} \rightarrow X^{Sc[T]} \quad \text{and} \quad X^{\Omega[T]} \rightarrow X^{\Omega[T] \otimes J_d}$$

for  $T \in \Omega_G$  are Kan equivalences in  $\mathbf{sSet}$ .

*Proof.* (i) simply repeats the last part of Proposition 5.1. In the remainder we write  $K \rightarrow L$  for a generic monomorphism in  $\mathbf{sSet}$  and  $A \rightarrow B$  a generic normal monomorphism in  $\mathbf{dSet}^G$ .

For (ii), note first that  $X$  is simplicial fibrant iff  $X^L \rightarrow X^K$  is always a fibration in  $\mathbf{dSet}^G$ . Hence, such  $X$  will have the right lifting property against all maps in (5.10) iff  $X^L \rightarrow X^K$  is a trivial fibration whenever  $K \rightarrow L$  is anodyne. But Remark 4.45 implies that it suffices to verify this for the vertex inclusions  $\Delta[0] \rightarrow \Delta[n]$ .

For (iii), note first that  $X$  is dendroidal fibrant iff  $X^B \rightarrow X^A$  is always a Kan fibration in  $\mathbf{sSet}$ . Therefore,  $X$  will have the right lifting property against all maps (5.11) iff  $X^B \rightarrow X^A$  is a trivial Kan fibration whenever  $A \rightarrow B$  is a generating trivial of  $\mathbf{dSet}^G$ . By adjunction, this is equivalent to showing that  $X^L \rightarrow X^K$  is a fibration in  $\mathbf{dSet}^G$  for any monomorphism  $K \rightarrow L$  in  $\mathbf{sSet}$ . Moreover, by the fibration between fibrant objects part of [Per17, Prop. 8.8] (see also the beginning of [Per17, §8.1]) it suffices to verify that the maps  $X^L \rightarrow X^K$  have the right lifting property against the maps

$$\Lambda^{Ge}\Omega[T] \rightarrow \Omega[T], \quad T \in \Omega_G, e \in \text{Inn}(T) \quad \text{and} \quad \Omega[T] \otimes (\{i\} \rightarrow J_d), \quad T \in \Omega_G, i = \{0, 1\}$$

and it thus suffices to check that  $X^B \rightarrow X^A$  is a trivial Kan fibration whenever  $A \rightarrow B$  is one of these maps. Proposition 4.43 now finishes the proof.  $\square$

We now obtain the following partial analogue of Proposition 5.4. Note that the equivalences in the simplicial Reedy model structure are the dendroidal equivalences and vice versa.

SDSETG COR

**Corollary 5.13.** Suppose that  $X, Y \in \mathbf{sdSet}^G$  are dendroidal Reedy fibrant. Then:

- (i) for each fixed  $m$  all vertex maps  $X_{\bullet, m} \rightarrow X_{\bullet, 0}$  are trivial fibrations in  $\mathbf{dSet}^G$ ;
- (ii) any simplicial Reedy fibrant replacement  $\tilde{X}$  of  $X$  is in fact fibrant in the joint Reedy model structure;
- (iii) a map  $X \rightarrow Y$  is a dendroidal weak equivalence iff it is a joint weak equivalence;
- (iv) regarding  $X_0$  as a simplicially constant object in  $\mathbf{sdSet}^G$ , the map  $X_0 \rightarrow X$  is a dendroidal equivalence, and thus a joint equivalence. (iv) follows from (i).

*Proof.* The proof adapts that of Proposition 5.4. (i) follows since  $X$  then has the right lifting property with respect to all maps  $(\Delta[0] \rightarrow \Delta[m]) \sqcup (\partial\Omega[T] \rightarrow \Omega[T])$ . (ii) follows from (i) and the characterization in Corollary 5.12 (ii). (iii) follows from (ii) since the simplicial fibrant replacement maps  $X \rightarrow \tilde{X}$  are dendroidal equivalences.  $\square$

SSSETJREE PROP

JOINTFIBCHAR COR

INCOAGJ THM

**Theorem 5.14.** The inclusion/0-th level adjunction

$$\iota: \mathbf{dSet}^G \rightleftarrows \mathbf{sdSet}^G: (-)_0,$$

where  $\mathbf{sdSet}^G$  is given the joint Reedy model structure, is a Quillen equivalence.

*Proof.* It is clear that the inclusion preserves both normal monomorphisms and all weak equivalences, hence the adjunction is Quillen. Consider any map  $\iota(A) \rightarrow X$  with  $X$  joint fibrant and perform a trivial cofibration followed by fibration factorization on the left

$$\iota(A) \xrightarrow{\sim} \widetilde{\iota(A)} \rightarrow X \quad A \xrightarrow{\sim} \widetilde{\iota(A)}_0 \rightarrow X_0$$

for the simplicial Reedy model structure. Corollary 5.12 (ii) now implies that  $\widetilde{\iota(A)}$  is in fact joint fibrant and thus that the leftmost composite above is a joint equivalence iff  $\widetilde{\iota(A)} \rightarrow X$  is a dendroidal equivalence in  $\mathbf{sdSet}^G$  iff  $\widetilde{\iota(A)}_0 \rightarrow X_0$  is an equivalence in  $\mathbf{dSet}^G$  iff the rightmost composite is an equivalence in  $\mathbf{dSet}^G$ .  $\square$

JOINTFIBCHAR COR

CONCRECOM REM

**Remark 5.15.** Given a  $G$ - $\infty$ -operad  $X$ , a joint fibrant model in  $\mathbf{sdSet}^G$  is given by  $X^{J_d(m)}$ . Moreover, note that it follows from [Per17, Cor. 8.21] that  $u_*(X^{J_d(m)}) = X^{(J_d(m))} \rightarrow X^{(\Delta_m)}$  is a trivial fibration in  $\mathbf{sdSet}_G$ , so that  $X^{J_d(m)}(T) \sim k(\Omega[T], X)$ .

Per17

## 6 Pre-operads

Recall that the category  $\mathbf{PreOp}$  of *pre-operads* is the full subcategory  $\mathbf{PreOp} \subset \mathbf{sdSet}$  of those  $X$  such that  $X(\eta)$  is a discrete simplicial set. Writing  $\gamma^*$  for the inclusion one has left and right adjoints  $\gamma_!$  and  $\gamma_*$

$$\begin{array}{ccc} & \xleftarrow{\gamma_!} & \\ \mathbf{PreOp}^G & \xrightarrow{\quad} & \mathbf{sdSet}^G \\ & \xleftarrow{\gamma_*} & \end{array}$$

described as follows [CM13a, §7]:  $\gamma_! X(T) = X(T)$  if  $T \notin \Delta$  while  $\gamma_! X([n])$  for  $[n] \in \Delta$  is given by the pushout on the left below;  $\gamma_* X(T)$  is given by the pullback on the right below.

$$\begin{array}{ccc} X(\eta) & \xrightarrow{\quad r \quad} & \pi_0 X(\eta) \\ \downarrow & & \downarrow \\ X([n]) & \longrightarrow & \gamma_! X([n]) \end{array} \qquad \begin{array}{ccc} \gamma_* X(T) & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ \coprod_{\text{Edges}(T)} X(\eta)_0 & \longrightarrow & \coprod_{\text{Edges}(T)} X(\eta) \end{array}$$

BOUNDED REM

**Remark 6.1.** Let  $A \rightarrow B$  denote a monomorphism in  $\mathbf{sdSet}^G$  of the form

$$(\partial\Delta[n] \rightarrow \Delta[n]) \sqcup (\partial\Omega[T] \rightarrow \Omega[T]), \quad n \geq 0, T \in \Omega_G, T \neq G/H \cdot \eta. \quad (6.2)$$

BOUNDED EQ

Then one has an isomorphism  $A(\eta) \simeq B(\eta)$  so that the square below is a pushout square.

$$\begin{array}{ccc} A & \xrightarrow{\quad r \quad} & \gamma_! A \\ \downarrow & & \downarrow \\ B & \longrightarrow & \gamma_! B \end{array} \quad (6.3)$$

GAMMASH EQ

In the next results we write  $I'$  for the set of maps in  $(\text{BOUNDED EQ } 6.2)$ .

GENSET LEM

**Lemma 6.4.** *The normal monomorphisms of  $\mathbf{PreOp}^G$  are the saturation of the set of maps  $\{\emptyset \rightarrow G/H \cdot \eta \mid H \leq G\} \cup \gamma_!(I')$ .*

*Proof.* Using the cellular filtration in  $\mathbf{sdSet}^G$ , any monomorphism  $A \rightarrow B$  in  $\mathbf{PreOp}^G$  can be written as a transfinite composition of pushouts of maps in  $\{\emptyset \rightarrow G/H \cdot \eta\} \cup I'$ . But, since the squares  $(\text{GAMMASH EQ } 6.3)$  are pushouts, the same also holds for the maps  $\{\emptyset \rightarrow G/H \cdot \eta\} \cup \gamma_!(I')$ .  $\square$

TRIVFIB LEM

**Lemma 6.5.** *Any map in  $\mathbf{PreOp}^G$  which has the right lifting property against all normal monomorphisms in  $\mathbf{PreOp}^G$  is a joint equivalence in  $\mathbf{sdSet}^G$ .*

*Proof.* We need simply adapt the proof of  $(\text{CM13a, Lemma 8.12})$  mutatis mutandis.

Choose a normalization  $E_\infty$  of  $*$  in  $\mathbf{sdSet}^G$ , i.e. a normal object such that  $E_\infty \rightarrow *$  is a trivial fibration. Regarding  $E_\infty$  as a simplicially constant object in  $\mathbf{sdSet}^G$ , a map  $X \rightarrow Y$  in  $\mathbf{PreOp}^G$  will have the right lifting property against all monomorphisms iff so does  $E_\infty \times (X \rightarrow Y)$ , so that one is free to assume  $X, Y$  are normal.

One is thus free to pick a section  $s: Y \rightarrow X$  of  $p: X \rightarrow Y$  and regarding  $J_d \in \mathbf{dSet}^G$  as a simplicially constant object of  $\mathbf{sdSet}^G$  our assumption yields the lift below, so that  $p$  is a homotopy equivalence.

$$\begin{array}{ccc} X \amalg X & \xrightarrow{(id_X, sp)} & X \\ \downarrow & \nearrow \text{dashed} & \downarrow p \\ X \otimes J_d & \longrightarrow & Y \end{array}$$

$\square$

**Theorem 6.6.** *The category  $\mathbf{Preop}^G$  of  $G$ -preoperads has a model structure such that*

- *the cofibrations are the normal monomorphisms;*
- *the weak equivalences are the maps that become joint equivalences when regarded as maps on  $\mathbf{sdSet}^G$ .*

*Proof.* One repeats the proof of the non-equivariant analogue  $(\text{CM13a, Thm. 8.13})$ , applying J. Smith's theorem  $(\text{Bek00, Thm. 1.7})$  with the required set of generating cofibrations the set  $\{\emptyset \rightarrow G/H \cdot \eta \mid H \leq G\} \cup \gamma_!(I')$  given by Lemma 6.4. Indeed, conditions c0 and c2 in  $(\text{Bek00})$  are inherited from  $\mathbf{sdSet}^G$  and c1 follows from Lemma 6.5. The technical “solution set” condition c3 follows from  $(\text{Bek00, Prop. 1.15})$  since weak equivalences are accessible, being the preimage by  $\gamma^*$  of the weak equivalences in  $\mathbf{sdSet}^G$  (see  $(\text{Lur09, Cor. A.2.6.5})$  and  $(\text{Lur09, Cor. A.2.6.6})$ ).  $\square$

**Theorem 6.7.** *The adjunction*

$$\gamma^*: \mathbf{PreOp}^G \rightleftarrows \mathbf{sdSet}^G: \gamma_*$$

*is a Quillen equivalence.*

*Proof.* It is tautological that the left adjoint  $\gamma^*$  preserves and detects cofibrations and weak equivalences, so it suffices to show that for all fibrant  $X \in \mathbf{sdSet}^G$  there exists  $Y \in \mathbf{PreOp}^G$  and a weak equivalence  $\gamma^*Y \rightarrow X$ . But by Corollary 5.13 it suffices to take  $Y = X_0$ , regarded as a simplicially trivial object.  $\square$

We will find it useful to also have a characterization of the fibrant objects in  $\mathbf{PreOp}^G$ . In doing so, it becomes useful to consider a fourth model structure on the category  $\mathbf{sdSet}^G$  whose fibrant objects “interpolate” between the fibrant objects in the two model structures in Theorem 6.7. This is the model structure of equivariant dendroidal Segal spaces, that we discuss in the next section.

## 7 Equivariant dendroidal Segal spaces

**Definition 7.1.** The *equivariant Segal space model structure* on the category  $\mathbf{sdSet}^G$ , which we denote  $\mathbf{sdSet}_S^G$ , is the left Bousfield localization of the dendroidal Reedy model structure with respect to the equivariant Segal core inclusions

$$Sc[T] \rightarrow \Omega[T], \quad T \in \Omega_G.$$

**Remark 7.2.** By Proposition 4.43 this model structure can equivalently be obtained by localizing with respect to the  $G$ -inner horn inclusions  $\Lambda^{Ge}[T] \rightarrow \Omega[T]$ .

**Notation 7.3.** We will refer to the fibrant objects in  $\mathbf{sdSet}_S^G$  as *equivariant dendroidal Segal spaces*, or just *dendroidal Segal spaces*. Further, a pre-operad  $X \in \mathbf{PreOp}^G$  is called *fibrant* if  $\gamma^*X$  is a dendroidal Segal space.

**Remark 7.4.** If  $X \in \mathbf{sdSet}^G$  is a dendroidal Segal space, then  $\gamma_*X \in \mathbf{PreOp}^G$  is fibrant. Indeed, the equality  $Sc[T](\eta) = \Omega[T](\eta)$  implies that the square below is a pullback square.

$$\begin{array}{ccc} \gamma_*X(T) & \longrightarrow & X(T) \\ \downarrow & & \downarrow \\ (\gamma_*X)^{Sc[T]} & \xrightarrow{j} & X^{Sc[T]} \end{array}$$

The following is a variation on Definition 4.49

**Definition 7.5.** Given a dendroidal Segal space  $X \in \mathbf{sdSet}_S^G$  and a  $C$ -profile  $(x_1, \dots, x_n; x_0)$  on  $X$  (defined exactly as in Definition 4.48) we define the space of maps  $X(x_1, \dots, x_n; x_0)$  via the pullback

$$\begin{array}{ccc} X(x_1, \dots, x_k; x_0) & \longrightarrow & X(C) \\ \downarrow & & \downarrow \\ \Delta[0] & \xrightarrow{(x_1, \dots, x_k; x_0)} & \prod_{0 \leq i \leq k} X(\eta)^{H_i} \end{array}$$



**Definition 7.6.** Let  $X \in \mathbf{sdSet}^G$  be a dendroidal Segal space. The *homotopy genuine operad*  $ho(X) \in \mathbf{dSet}_G$  is defined by

$$ho(X) = \pi_0(u_*(\gamma_*X)).$$

**Remark 7.7.** It is immediate that  $X(x_1, \dots, x_k; x_0) = \gamma_*X(x_1, \dots, x_k; x_0)$ , so that both of the previous definitions depend only on the fibrant pre-operad  $\gamma_*X$ .

**Remark 7.8.** It is important not to confuse Definitions 4.49 and 7.5. Indeed, when  $X$  is a dendroidal Segal, its 0-th level  $X_0$  is a  $G$ - $\infty$ -operad, and one can thus form two “spaces of maps”  $X_0(x_1, \dots, x_k; x_0)$  (cf. Definition 4.49) and  $X(x_1, \dots, x_k; x_0)$  (cf. Definition 7.5). The constructions leading to these spaces are quite different. When  $X$  is complete Segal, the fact that these two spaces are homotopic follows from Remark 5.15, since  $X$  must then be both dendroidally and simplicially equivalent to  $X_0^{Ja(m)}$ . The claim that this holds without completeness is harder, with the rest of the section devoted to establishing this.

**Remark 7.9.** Writing  $\iota$  for the inclusion  $\Delta \rightarrow \Omega$  and  $\iota_G$  for the composite inclusion  $\Delta \times \mathbf{O}_G \rightarrow \Omega \times \mathbf{O}_G \rightarrow \Omega_G$ , one has that  $\iota_G^*ho(X)$  is the  $G$ -coefficient system of categories formed by the homotopy categories  $ho(\iota^*(X^H)) = \pi_0(\iota^*\gamma_*X^H)$ .

**Definition 7.10.** A map  $f: X \rightarrow Y$  of equivariant dendroidal Segal spaces is called

- *fully faithful* if for all  $C \in \Sigma_G$  and  $C$ -profile  $(x_1, \dots, x_n; x_0)$  on  $X$  the map

$$X(x_1, \dots, x_k; x_0) \rightarrow Y(f(x_1), \dots, f(x_k); f(x_0))$$

is a Kan equivalence in  $\mathbf{sSet}$ ;

- *essentially surjective* if the map  $\iota_G^*ho(X) \rightarrow \iota_G^*ho(Y)$  is essentially surjective on all category levels of the  $G$ -coefficient system;
- a *DK-equivalence*<sup>4</sup> if it is both fully faithful and essentially surjective.

**Remark 7.11.** This definition depends only on the map  $\gamma_*X \rightarrow \gamma_*Y$  of fibrant pre-operads.

**Definition 7.12.** Let  $X \in \mathbf{sdSet}^G$  be a dendroidal Segal space. We call a point  $f \in X(C_{H/H})_0$  a  $H$ -equivalence if the corresponding class

$$[f] \in ho(X)(C_{H/H}) = ho(\iota^*(X^H)) = \pi_0(\iota^*\gamma_*X^H)$$

is an isomorphism.

DK-equivalences will provide an explicit description of complete/joint equivalences between dendroidal Segal objects (and thus also between fibrant pre-operads), as we will prove in **whatever below**. We now introduce some auxiliary notions.

Suppose  $C, D \in \Sigma_G$  are  $G$ -corollas that can be grafted, i.e. that  $C$  has a leaf orbit and  $D$  a root orbit both isomorphic to  $G/H$ . Denote this orbit as  $Ge$  and write  $T = C \sqcup_{Ge} D$  for the grafted  $G$ -tree. For any dendroidal Segal space  $X$  one then has  $X^{Sc[T]} \simeq X(C) \times_{X(\eta)^H} X(D)$  and one can hence form the section in the middle row below

$$\begin{array}{ccc}
\{\varphi\} \times X(z_1, \dots, z_l; e) & \xrightarrow{\varphi_{Ge}(-)} & X(z_1, \dots, z_l, y_2, \dots, y_k; x) \\
\downarrow & \dashrightarrow & \downarrow \\
X(C) \times_{X(\eta)^H} X(D) & \xleftarrow{\sim} X(T) \xrightarrow{\quad} X(T - Ge) & \\
\uparrow & & \uparrow \\
X(e, y_2, \dots, y_k; x) \times \{\psi\} & \xrightarrow{(-) \circ_{Ge} \psi} & X(z_1, \dots, z_l, y_2, \dots, y_k; x)
\end{array} \tag{7.13}$$

HOMOTCIRC EQ

<sup>4</sup>Here DK stands for Dwyer and Kan.

thus defining maps  $\varphi \circ_{Ge} (-)$  (resp.  $(-) \circ_{Ge} \psi$ ) for any choice of  $\varphi \in X(e, y_2, \dots, y_k; x)$  (resp.  $\psi \in X(z_1, \dots, z_l; e)$ ).

GENOPHO PROP

**Proposition 7.14.** *The maps  $\varphi \circ_{Ge} (-)$ ,  $(-) \circ_{Ge} \psi$  are well defined up to homotopy. Further, if  $\varphi, \bar{\varphi} \in X(e, y_2, \dots, y_k; x)$  are homotopic (i.e. in the same connected component), then the maps  $\varphi \circ_{Ge} (-)$ ,  $\bar{\varphi} \circ_{Ge} (-)$  are homotopic, and likewise for  $\psi$ .*

*In particular,  $\varphi \circ_{Ge} \psi \in X(z_1, \dots, z_l, y_2, \dots, y_k; x)$  is well defined up to homotopy.*

*Lastly, the maps  $\varphi \circ_{Ge} (-)$ ,  $(-) \circ_{Ge} \psi$  are functorial in maps  $f: X \rightarrow Y$  between dendroidal Segal spaces.*

*Proof.* This is immediate once one notes that, writing  $E = T - Ge$  for the “composite  $G$ -corolla”, all solid maps in (7.13) are compatible with the projections to  $X^{\partial\Omega[E]}$ .

For functoriality, one simply notes that either type of section as on the leftmost diagram

$$\begin{array}{ccc} X(T) & \longrightarrow & Y(T) \\ \downarrow \sim & & \downarrow \sim \\ X(C) \times_{X(\eta)^H} X(D) & \longrightarrow & Y(C) \times_{Y(\eta)^H} Y(D) \end{array} \quad \begin{array}{ccc} & & Y(T) \\ & \nearrow & \downarrow \sim \\ X(C) \times_{X(\eta)^H} X(D) & \longrightarrow & Y(C) \times_{Y(\eta)^H} Y(D) \end{array}$$

induces lifts as on the rightmost diagram. However, a standard argument shows that all such lifts are homotopic over  $X^{\partial\Omega[E]}$ .  $\square$

We will now show that the operations  $\varphi \circ_{Ge} (-)$ ,  $(-) \circ_{Ge} \psi$  satisfy the obvious compatibilities one expects, but we will find it convenient to first package these compatibilities into a common format. In the categorical case (corresponding to linear trees), there are three types of compatibilities, corresponding to homotopies

$$\varphi \circ (\psi \circ (-)) \sim (\varphi \circ \psi) \circ (-) \quad \varphi \circ ((-) \circ \psi) \sim (\varphi \circ (-)) \circ \psi \quad ((-) \circ \varphi) \circ \psi \sim (-) \circ (\varphi \circ \psi)$$

but in the operadic case there are instead five cases, corresponding to the different possible roles of the nodes in  $G$ -trees with exactly three  $G$ -nodes, whose *orbital* representation falls into one of the two cases illustrated below.



Since all these compatibilities can be simultaneously encoded in terms of such trees, we will simply refer to all types of compatibility as “associativity”.

In the next result, note that a  $G$ -tree  $T$  with three  $G$ -nodes contains precisely two inner edge orbits  $Ge$  and  $Gf$ . We will write  $T[Ge]$  (resp.  $T[Gf]$ ) for the unique orbital outer face of  $T$  with  $Ge$  (resp.  $Gf$ ) as its single inner edge orbit.

**Proposition 7.15.** *The operations  $\varphi \circ_{Ge} (-)$ ,  $(-) \circ_{Ge} \psi$  satisfy all associativity conditions with respect to 3-nodal  $G$ -trees.*

*Proof.* For any 3-nodal  $G$ -tree  $T$  with inner edge orbits  $Ge$ ,  $Gf$ , consider the following diagram, where all solid maps are fibrations, and the maps labelled  $\sim$  are trivial fibrations.

$$\begin{array}{ccccc}
X^{\text{Sc}[T]} & \xleftarrow{\sim} & X^{\text{Sc}_{T[Ge]}[T]} & \xrightarrow{\sim} & X^{\text{Sc}[T-Ge]} \\
\uparrow \sim & & \uparrow \sim & & \uparrow \sim \\
X^{\text{Sc}_{T[Gf]}[T]} & \xleftarrow{\sim} & X^{\Omega[T]} & \xrightarrow{\sim} & X^{\Omega[T-Ge]} \\
\downarrow & & \downarrow & & \downarrow \\
X^{\text{Sc}[T-Gf]} & \xleftarrow{\sim} & X^{\Omega[T-Gf]} & \xrightarrow{\sim} & X^{\Omega[T-Ge-Gf]}
\end{array} \tag{7.16} \quad \boxed{\text{FOURSQ EQ}}$$

Noting that the following three diagrams, where  $\Lambda_o^{Ge, Gf}[T]$  is a generalized orbital  $G$ -horn and  $\Lambda_{o,c}^{Ge}[T]$ ,  $\Lambda_{o,c}^{Gf}[T]$  are characteristic orbital  $G$ -horns (cf. [ref](#)), are pullbacks

$$\begin{array}{ccccc}
X^{\text{Sc}[T]} & \xleftarrow{\sim} & X^{\text{Sc}_{T[Ge]}[T]} & \xrightarrow{\sim} & X^{\text{Sc}[T-Ge]} & X^{\text{Sc}_{T[Gf]}[T]} & \xleftarrow{\sim} & X^{\Lambda_{o,c}^{Ge}[T]} \\
\uparrow \sim & & \uparrow \sim & & \uparrow \sim & \downarrow & & \downarrow \\
X^{\text{Sc}_{T[Gf]}[T]} & \xleftarrow{\sim} & X^{\Lambda_{o,c}^{Ge, Gf}[T]} & \xrightarrow{\sim} & X^{\Lambda_{o,c}^{Gf}[T]} & \xrightarrow{\sim} & X^{\Omega[T-Ge]} & X^{\text{Sc}[T-Gf]} & \xleftarrow{\sim} & X^{\Omega[T-Gf]}
\end{array}$$

one sees that: (i) sections in the top left square of [\(7.16\)](#) can be chosen to be compatible in the sense that the two composites  $X^{\text{Sc}[T]} \rightarrow X^{\Omega[T]}$  coincide; (ii) sections in the top right and bottom left squares of [\(7.16\)](#) can be chosen to be compatible in the sense that the two composites  $X^{\text{Sc}[T-Ge]} \rightarrow X^{\Omega[T]}$  and  $X^{\text{Sc}[T-Gf]} \rightarrow X^{\Omega[T]}$  coincide. Note that we do not claim (or need) that (i) and (ii) hold simultaneously. We thus conclude that the possible choices of maps  $X^{\text{Sc}[T]} \rightarrow X^{\Omega[T-Ge-Gf]}$  given by outer paths in [\(7.16\)](#) are homotopic. All desired forms of associativity follow from taking fibers of these maps over the objects  $X^{\partial\Omega[T-Ge-Gf]}$ . [FOURSQ EQ](#)

Actually, I may just be able to do without these arguments. □

**Remark 7.17.** While in the non-equivariant case the associativity conditions in the previous result capture all the key compatibilities of the  $\varphi \circ_e (-)$ ,  $(-) \circ_e \psi$  operations, in the equivariant case there are further “compatibilities with pullback of  $G$ -trees”, which are closely related to the genuine equivariant operads introduced in [BP17](#). However, describing these extra compatibilities would require using  $G$ -trees with more than 3  $G$ -nodes, and since such compatibilities are not needed for our current goals, we omit their discussion.

**Corollary 7.18.** *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-3.*

$$\begin{array}{ccc}
X & \xrightarrow{gf} & Z \\
& \searrow f & \nearrow g \\
& Y &
\end{array}$$

*Proof.* The non trivial claim is that when  $f$  and  $gf$  are DK-equivalences then so is  $g$ , or more precisely, that the maps

$$Y(y_1, \dots, y_n; y_0) \rightarrow Z(g(y_1), \dots, g(y_n); g(y_0))$$

are weak equivalences even if the  $y_i$  are not in the image of  $f$ . But this follows from the functoriality in [Proposition 7.14](#), essential surjectivity (note that when  $y_i \in Y(\eta)^{H_i}$  one needs to use  $H_i$ -equivalences), and the fact that by the previous corollary the maps  $f \circ_{Ge} (-)$ ,  $(-) \circ_{Ge} f$  are weak equivalences whenever  $f$  is a  $H$ -equivalence. □

**Corollary 7.19.** *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-6, i.e. whenever  $gf$  and  $hg$  are DK-equivalences then so are  $f$ ,  $g$ ,  $h$ ,  $hgf$ .*

*Proof.* The hypothesis implies together with the 2-out-of-6 condition for the Kan model structure in  $\mathbf{sSet}$  imply that  $g$  is fully faithful for objects in the image of  $f$ . But this easily implies that  $f$  is a DK equivalence, and thus 2-out-of-3 concludes the proof.  $\square$

We now recover the following from [Rez01] (add commentary)

**Proposition 7.20.** *Let  $X \in \mathbf{ssSet}$  be a Segal space. Then:*

(i) *equivalences define a subset of connected components  $X_1^h \subset X_1$ ;*

(ii) *the pullbacks*

$$\begin{array}{ccc} X_n^h & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ X_1^h \times_{X_0} \cdots \times_{X_0} X_1^h & \longrightarrow & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \end{array} \quad (7.21) \quad \text{XHDEF EQ}$$

*define a Segal space  $X^h \subset X$ , consisting of a union of connected components at each level;*

(iii) *the maps  $X_2^h \xrightarrow{(d_2, d_1)} X_1^h \times_{X_0} X_1^h$ ,  $X_2^h \xrightarrow{(d_0, d_1)} X_1^h \times_{X_0} X_1^h$  are trivial fibrations;*

(iv) *the map  $X^J \rightarrow X^{\Delta^1} = X_1$  factors through a weak equivalence  $X^J \rightarrow X_1^h$ .*

*Proof.* For (i), note first that given  $f: x \rightarrow y$  in  $X_{1,0}$ , then  $[f]$  has a left inverse iff one can find a lift  $p$  as on the leftmost diagram below. But for any path  $H$  between  $f$  and  $f'$  in  $X_1$

$$\begin{array}{ccc} & X_2 & \\ p \nearrow & \downarrow (d_2, d_1) & \\ \Delta[0] & \xrightarrow{(f, s_0(x))} X_1 \times_{X_0} X_1 & \end{array} \quad \begin{array}{ccc} \Delta[0] & \xrightarrow{p} & X_2 \\ 0 \downarrow & \nearrow & \downarrow (d_2, d_1) \\ \Delta[1] & \xrightarrow{(H, s_0 d_1(H))} & X_1 \times_{X_0} X_1 \end{array}$$

one can form the lift in the rightmost diagram, showing that  $f'$  is also left-invertible. The situation for right inverses is identical, thus (i) follows.

For (ii), the fact that  $X_\bullet^h$  is closed under the simplicial operators follows since the composite of equivalences is an equivalence. In fact, this further implies that the bottom row in the pullback (7.21) could have been replaced with  $(X^h)^{sk_1 \Delta[n]} \rightarrow X^{sk_1 \Delta[n]}$ , from which it follows that the squares

$$\begin{array}{ccc} (X^h)^K & \longrightarrow & X^K \\ \downarrow & & \downarrow \\ (X^h)^{sk_1 K} & \longrightarrow & X^{sk_1 K} \end{array}$$

are pullbacks. Since  $sk_1(\partial \Delta[n]) = sk_1 \Delta[n]$  if  $n \geq 2$  it follows that the maps  $X_n^h \rightarrow (X^h)^{\partial \Delta[n]}$ ,  $n \geq 2$  are Kan fibrations, and since the map  $X_1^h \rightarrow X_0 \times X_0$  is certainly a Kan fibration,  $X^h$  is indeed Reedy fibrant. The Segal condition is obvious from the pullback (7.21).  $\square$

For (iii), it suffices by symmetry to show the first claim. Moreover, one reduces to showing show that for any choice of section in the following diagram the top composite is a weak

equivalence.

$$\begin{array}{ccccc}
X_1^h \times_{X_0} X_1^h & \xleftarrow[\sim]{(d_2, d_0)} & X_2^h & \xrightarrow{(d_2, d_1)} & X_1^h \times_{X_0} X_1^h \\
& \searrow (id, d_0) & & \swarrow (id, d_0) & \\
& & X_1^h \times X_0 & & 
\end{array}$$

But this composite is a map of fibrations over  $X_1^h \times X_0$  with the map between the fibers over  $(f: x \rightarrow y, z)$  computing the map  $(-) \circ f: X^h(y; z) \rightarrow X^h(x; z)$ , which is a Kan equivalence since  $f \in X_1^h$  is an equivalence. Thus the composite is an equivalence, establishing (iii).

Lastly, for (iv) we first note that (iii) can be restated as saying that  $X^h$  is local with respect to the outer horn inclusions  $\Lambda^0[2] \rightarrow \Delta[2]$  and  $\Lambda^2[2] \rightarrow \Delta[2]$ , and that hence by [Remarks 8.1](#) and [8.2](#) the map  $(X^h)^J \rightarrow X_1^h$  is a Kan equivalence. Hence, the only remaining claim is that  $(X^h)^J = X^J$ , which is clear.  $\square$

**Remark 7.22.** The inclusion  $X^h \rightarrow X$  is a Reedy fibration. [justify](#)

JDDK PROP

**Proposition 7.23.** Let  $X \in \mathbf{sdSet}^G$  be a dendroidal Segal space. Then the map  $X \rightarrow X^{J_d}$  is a  $DK$ -equivalence.

*Proof.* Note first that for any  $T \in \Omega_G$  the map  $X^{J_d}(T) \rightarrow X^{\Omega[1]}(T)$  can be rewritten as  $(X^{\Omega[T]})^{J_d} \rightarrow (X^{\Omega[T]})^{\Omega[1]} = (X^{\Omega[T]})_1$ , and since  $X^{\Omega[C]}$  is a (simplicial) Segal space the previous result implies that this map is a weak equivalence onto a subset of components (such maps are also called homotopy monomorphisms). It thus follows that for any  $G$ -corolla  $C_{u_i H_0 / H_i}$  the horizontal maps in the rightmost square below are homotopy monomorphisms.

$$\begin{array}{ccccc}
X(C) & \longrightarrow & X^{J_d}(C) & \longrightarrow & X^{\Omega[1]}(C) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{0 \leq i \leq k} X(\eta)^{H_i} & \longrightarrow & \prod_{0 \leq i \leq k} X^{J_d}(\eta)^{H_i} & \longrightarrow & \prod_{0 \leq i \leq k} X^{\Omega[1]}(\eta)^{H_i}
\end{array}$$

BIGSQ EQ

Since the claim that  $X \rightarrow X^{J_d}$  is a  $DK$ -equivalence is the statement that the leftmost square induces weak equivalences on fibers, it thus suffices to show that so does the composite square.

Now note that we can rewrite  $X^{\Omega[1]}(C) = X^{\Omega[1] \otimes \Omega[C]}$  and that there is a pullback diagram

$$\begin{array}{ccc}
X^{\Omega[1] \otimes \Omega[C]} & \longrightarrow & X(C \star \eta) \\
\downarrow & & \downarrow \\
X(\eta \star C) & \longrightarrow & X(C)
\end{array}$$

Noting that the required cube is projective fibrant, one reduces to checking that the following squares induce weak equivalences on fibers, and this claim is clear from the top right vertical

trivial fibrations (which are instances of the Segal condition).

$$\begin{array}{ccc}
X(C) & \xrightarrow{s_\eta} & X(C \star \eta) \\
\downarrow & & \downarrow \sim \\
& & X(C) \times_{X(\eta)^{H_0}} X([1])^{H_0} \\
& & \downarrow \\
\prod_{0 \leq i \leq n} X(\eta)^{H_i} & \xrightarrow{s_0} & \left( \prod_{1 \leq i \leq n} X(\eta)^{H_i} \right) \times X([1])^{H_0}
\end{array}
\quad
\begin{array}{ccc}
X(C) & \xrightarrow{s_\eta} & X(C \star \eta) \\
\downarrow & & \downarrow \sim \\
& & \prod_{1 \leq i \leq n} X([1])^{H_i} \times_{\prod_{1 \leq i \leq n} X(\eta)^{H_i}} X(C) \\
& & \downarrow \\
\prod_{0 \leq i \leq n} X(\eta)^{H_i} & \xrightarrow{s_0} & \left( \prod_{1 \leq i \leq n} X([1])^{H_i} \right) \times X(\eta)^{H_0}
\end{array}$$

□

**Definition 7.24.** Two maps  $f, f': A \rightrightarrows B$  between dendroidal Segal spaces are called  $J$ -homotopic, written  $f \sim_J f'$ , if there is a  $H$  such that the two composites  $A \xrightarrow{H} B^J \rightrightarrows B$  are  $f, f'$ .

Further, a map  $f: X \rightarrow Y$  of dendroidal Segal spaces is called a  $J$ -homotopy equivalence if there is  $g: Y \rightarrow X$  such that  $gf \sim_J id$ ,  $fg \sim_J id$ .

**Remark 7.25.** It follows from Proposition [JDDK PROP 7.23](#) and 2-out-of-3 that if  $f \sim_J f'$  then  $f$  is a DK-equivalence iff  $f'$  is. Thus by 2-out-of-6 all  $J$ -homotopy equivalences are DK-equivalences.

ALLXJK REM

**Remark 7.26.** Let  $X$  be a Segal space. All simplicial operators  $X^{J_n} \rightarrow X^{J_m}$  are induced from equivalences of groupoids  $[m] \rightarrow [n]$ , and one easily checks that these operators are thus  $J$ -homotopy equivalences and thus also DK-equivalences.

COMPLETE PROP

**Proposition 7.27.** Let  $X \in \mathbf{sdSet}^G$  be a dendroidal Segal space. Then there is a complete dendroidal Segal space  $\tilde{X}$  and complete equivalence  $X \rightarrow \tilde{X}$  such that

- (i)  $X \rightarrow \tilde{X}$  is a monomorphism and a DK-equivalence;
- (ii)  $X_0(\eta) \rightarrow \tilde{X}_0(\eta)$  is an isomorphism.

*Proof.* Our argument will mostly adapt the construction of the completion functor in [Rez01, §10.4](#).

Firstly, we let  $X^{J_\bullet} \in (\mathbf{sdSet}^G)^{\Delta^{op}} = \mathbf{ssdSet}^G$  be the object whose  $k$ -th level (in the new simplicial direction) is  $X^{J_k}$ . Since  $J_\bullet$  is a Reedy cofibrant cosimplicial object,  $X^{J_\bullet}$  is Reedy fibrant with respect to the dendroidal Space model structure on  $\mathbf{sdSet}^G$ . In particular, it follows that  $X^{J_\bullet} \rightarrow \mathbf{csk}_\eta X^{J_\bullet}$  is a fibration in  $\mathbf{ssdSet}^G$ .

In particular, this implies that for each  $T \in \Omega_G$  and vertex map  $[0] \rightarrow [k]$  the induced square

$$\begin{array}{ccc}
X^{J_k}(T) & \longrightarrow & X(T) \\
\downarrow & & \downarrow \\
\prod_{e_i \in E_G(T)} (X^{J_k}(\eta))^{H_i} & \longrightarrow & \prod_{e_i \in E_G(T)} (X(\eta))^{H_i}
\end{array}$$

is an (injective) fibrant square, which by Remark [ALLXJK REM 7.26](#) induces weak equivalences on fibers, so that the map from  $X^{J_k}(T)$  to the pullback of the remaining diagram is a trivial Kan fibration. By Remark [HYPERSIMPL REM 5.7](#) we have just shown that for each fixed  $T \in \Omega_G$  the map

$$X^{J_\bullet}(T) \rightarrow (\mathbf{csk}_\eta X^{J_\bullet})(T) \tag{7.28}$$

MOREOVER EQ

satisfies the “moreover” condition in Corollary 5.6. Therefore, applying  $\delta^*$  to (7.28) yields a Kan fibration, so that all fibers of this map are in fact homotopy fibers.

We now write  $\tilde{X}$  for any dendroidal Reedy fibrant replacement of the diagonal  $\delta^*(X^{J_\bullet})$ , which we note can always be chosen so that  $\delta^*(X^{J_\bullet}) \rightarrow \tilde{X}$  is a monomorphism and  $\tilde{X}_0(\eta) = (\delta^*(X^{J_\bullet}))_0(\eta) = X_0(\eta)$  (this follows since fibrant replacements in the Kan model structure in  $\mathbf{sSet}$  can be chosen to preserve 0-simplices, since existence of lifts against the horn inclusions  $\Delta[0] = \Lambda^0[1] \rightarrow \Delta[1]$ ,  $\Delta[0] = \Lambda^1[1] \rightarrow \Delta[1]$  is automatic).

To see that  $\tilde{X}$  is a complete Segal space, note that there is a composite  $X^{J_\bullet} \xrightarrow{\delta^*} \delta^*(X^{J_\bullet}) \rightarrow \tilde{X}$  where the first map is a dendroidal Reedy equivalence by Proposition 5.4(iv) and the second by definition of  $\tilde{X}$ . But since  $X_0^{J_\bullet}$  is a complete Segal space, so is  $\tilde{X}$ .

For the remaining claim that the composite  $X = X^{J_0} \rightarrow \delta^*(X^{J_\bullet}) \rightarrow \tilde{X}$  is a DK equivalence, though the first map is no longer a dendroidal Reedy equivalence, it is nonetheless an equivalence on fibers over  $\prod_{e_i \in E_G(T)} (X(\eta))^{H_i}$  for each  $T \in \Omega_G$ . And since we established above that the fibers of  $\delta^*(X^{J_\bullet})(T)$  are homotopy fibers, these are equivalent to the fibers of  $\tilde{X}(T)$  (since Reedy replacement does not change the homotopy fibers), and thus  $X \rightarrow \tilde{X}$  is indeed fully faithful. Essential surjectivity is trivial since the objects coincide. The monomorphism condition is clear.  $\square$

**Corollary 7.29.** *A map of  $X \rightarrow Y$  of dendroidal Segal spaces is a joint equivalence iff it is a DK equivalence.*

*Proof.* By the (proof of) previous result one is free to, via a zigzag, replace  $X, Y$  with  $X_0^{J_\bullet}, Y_0^{J_\bullet}$ . But by Theorem 5.14 this map is a joint equivalence iff  $X_0 \rightarrow Y_0$  is an equivalence in  $\mathbf{dSet}^G$ , which by Theorem 4.51 holds iff this is a fully faithful and essentially surjective map of  $G$ - $\infty$ -operads. But it is easy to check that  $X_0 \rightarrow Y_0$  is a fully faithful and essentially surjective map of  $G$ - $\infty$ -operads iff  $X_0^{J_\bullet} \rightarrow Y_0^{J_\bullet}$  is a fully faithful and essentially surjective map of dendroidal Segal spaces.  $\square$

**Corollary 7.30.** *A pre-operad  $X \in \mathbf{PreOp}^G$  is fibrant iff  $\gamma^*(X)$  is fibrant in the Segal space model structure on  $\mathbf{sdSet}^G$ .*

*Proof.* We start with the “only if” direction. Recall that  $\gamma^*X$  is a dendroidal Segal space if it has the right lifting property against the maps of the form

$$(\Lambda^i[n] \rightarrow \Delta[n]) \sqcap (\partial\Omega[T] \rightarrow \Omega[T]) \quad (\partial\Delta[n] \rightarrow \Delta[n]) \sqcap (Sc[T] \rightarrow \Omega[T]). \quad (7.31)$$

With the exception of the first type of maps when  $T = \eta$ , in which case the lifting condition is automatic since  $\gamma^*X(\eta)$  is discrete, all other maps induce isomorphisms at the  $\eta$ -level, so that by (6.3) applying  $\gamma_!$  to these maps yields trivial cofibrations in  $\mathbf{PreOp}^G$ . Thus, if  $X \in \mathbf{PreOp}^G$  is fibrant, an adjunction argument shows that  $\gamma^*(X)$  indeed has the lifting property against all maps (7.31), i.e.  $\gamma^*(X)$  is a dendroidal Segal space.

For the “if” direction, form the completion  $\gamma^*X \rightarrow \tilde{X}$  as in Proposition 7.27. Then  $\gamma_*\tilde{X} \in \mathbf{PreOp}^G$  is fibrant by Theorem 6.7 and the adjoint map  $X \rightarrow \gamma_*\tilde{X}$  has the following properties: (i) it is a monomorphism; (ii) it is an isomorphism on the  $\eta$ -level; (iii) it is a DK-equivalence when regarded as a map in  $\mathbf{sdSet}^G$  (since  $\gamma_*\gamma^*\tilde{X} \rightarrow \tilde{X}$  is tautologically a DK-equivalence); (iv) it is hence a trivial Reedy cofibration when regarded as a map in  $\mathbf{sdSet}^G$ . But then the hypothesis that  $\gamma^*X$  is a dendroidal Segal space yields a lift

$$\begin{array}{ccc} \gamma^*X & \xlongequal{\quad} & \gamma^*X \\ \downarrow & \nearrow & \\ \gamma^*\gamma_*X & & \end{array}$$



showing that  $X$  is a retract of  $\gamma_* X$  and finishing the proof.  $\square$

**Remark 7.32.** For any dendroidal Segal space  $X \in \mathbf{sdSet}^G$  one hence has complete equivalences

$$\gamma_* X \rightarrow X \rightarrow \tilde{X}$$

where  $\gamma_*$  is a fibrant preoperad and  $\tilde{X}$  a complete dendroidal Segal space.

$$u^*: \mathbf{dSet}_G \rightleftarrows \mathbf{dSet}^G: u_*$$

discuss characteristic horns

## 8 Scratchwork (to be folded into previous sections eventually)

ANHYPER REM

**Remark 8.1.** The smallest hypersaturated class containing the inner horns and the left horn inclusion  $\Lambda^0[2] \rightarrow \Delta[2]$  in fact contains all left horn inclusions  $\Lambda^0[n] \rightarrow \Delta[n]$  for  $n \geq 2$ . Indeed, this follows inductively from the following diagram since the bottom map is inner

$$\begin{array}{ccc} \Lambda^{0,1}[n] & \longrightarrow & \Lambda^0[n] \\ \downarrow & & \downarrow \\ \Lambda^1[n] & \longrightarrow & \Delta[n] \end{array}$$

and the top and left maps are given by following pushouts

$$\begin{array}{ccc} \Lambda^0[n-1]_r & \longrightarrow & \Lambda^{0,1}[n] \\ \downarrow & & \downarrow \\ \Delta[n-1] & \xrightarrow{d^1} & \Lambda^0[n] \end{array} \quad \begin{array}{ccc} \Lambda^0[n-1]_r & \longrightarrow & \Lambda^{0,1}[n] \\ \downarrow & & \downarrow \\ \Delta[n-1] & \xrightarrow{d^0} & \Lambda^1[n] \end{array}$$

CONTGR REM

**Remark 8.2.** Write  $\widetilde{[n]}$  for the contractible groupoid on objects  $\{0, 1, \dots, n\}$ . Note that the  $k$ -simplices of  $\widetilde{[n]}$  are encoded as strings  $a_0 a_1 \dots a_k$  with  $a_i \in \{0, 1, \dots, n\}$ , and that a simplex is non-degenerate iff  $a_{i-1} \neq a_i, 1 \leq i \leq k$ . Then the maps

$$\Delta[n] = N[n] \xrightarrow{012 \dots n} N[\widetilde{[n]}], \quad n \geq 1 \quad (8.3)$$

INVER EQ

are built cellularly out of left horn inclusions  $\Lambda^0[k] \rightarrow \Delta[k]$  with  $k \geq 2$ .

Indeed, we show a little more. Call subcomplex  $A \subset N[\widetilde{[n]}]$  is 0-stable if a  $n$ -simplex  $\underline{a}$  is in  $A$  iff the  $n+1$ -simplex  $0\underline{a}$  is. We claim that any inclusion  $A \rightarrow A'$  of 0-stable subcomplexes is built cellularly from left horn inclusions  $\Lambda^0[k] \rightarrow \Delta[k]$  with  $k \geq 1$ . Indeed, it suffices to check this when  $A'$  attaches as little as possible to  $A$ , and 0-simplicity guarantees that in that case the only two non-degenerate simplices in  $A - A'$  have the form  $\underline{a}$  and  $0\underline{a}$  (note that  $\underline{a}$  can not start with a 0). But then  $A \rightarrow A'$  is a pushout of  $\Lambda^0[k+1] \rightarrow \Delta[k+1]$  where  $k$  is the dimension of  $\underline{a}$ .

The desired claim follows by noting that both the domain and codomain of (8.3) are 0-stable and that the horns  $\Lambda^0[1]$  are unneeded since (8.3) is an isomorphism on 0-simplices.

$$\begin{array}{ccccc} W_1 \times_{W_0} W_1 & \times_{W_0} W_1 & \xleftarrow{\sim} & W_1 \times_{W_0} W_2 & \twoheadrightarrow W_1 \times_{W_0} W_1 \\ \uparrow \sim & & & \uparrow \sim & \uparrow \sim \\ W_2 \times_{W_0} W_1 & \xleftarrow{\sim} & W_3 & \twoheadrightarrow & W_2 \\ \downarrow & & \downarrow & & \downarrow \\ W_1 \times_{W_0} W_1 & \xleftarrow{\sim} & W_2 & \twoheadrightarrow & W_1 \end{array}$$

**Remark 8.4.** Note that  $\text{Sc}_{T[Ge]}$ ,  $\text{Sc}_{T[Gf]}$  in (7.16) are cover inclusions, and thus  $G$ -anodyne, relate to [Rez10, §6.2], [Rez01, §10].

**Remark 8.5.** Indexing systems are precisely the Segal sieves of  $\Omega_G$ .

**Remark 8.6.** [bla bla](#) the diagrams for compositions of norm maps are given by orbital representations, but the category  $\Omega_G$  is better described in terms of the expanded representation.

**Lemma 8.7.** *A non-equivariant face  $U \hookrightarrow T$  generates an equivariant face  $G \cdot U \hookrightarrow T$  iff the  $G$ -isotropy of  $r_U$  matches the isotropy of  $U$ .*

**Proposition 8.8.** *Any non-equivariant face  $U \hookrightarrow T$  has a minimal factorization  $U \hookrightarrow GU \hookrightarrow T$  through an equivariant face  $GU$ .*

*Proof.* Our first task is that of building  $GU$ . The edges of  $GU$  will be given by combining the  $G$ -orbits of the edges in  $U$ , but describing the broad relations requires some caution.

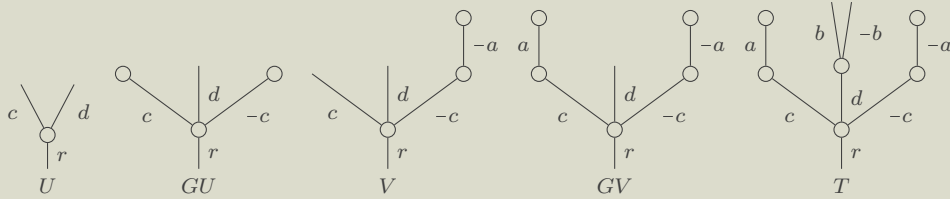
We first assume that  $U$  is an outer face and write  $H \leq G$  for the isotropy of the root  $r_U$  in  $T$ . We form a (non-equivariant) subtree  $HU$  of  $T$  with edges (resp. generating broad relations) those edges  $hu$  (resp. relations  $hu^\dagger \leq hu$ ) in some  $hU$  for  $h \in H$ . Indeed, while a priori  $HU$  is merely a pre-broad poset of  $T$ , the antisymmetry and simplicity axioms in [Per17, Defs. 5.1 and 5.3] are inherited from  $T$ , and the nodal axiom in [Per17, Def. 5.9] follows since the  $hU$ ,  $h \in H$  have a common root. It is now clear that  $HU$  inherits a  $H$ -action and we thus set  $GU = G \cdot_H HU$ . Both the factorization  $U \rightarrow GU \rightarrow T$  and its minimality are immediate from the description of  $HU$ .

Before tackling the general case, we collect some key observations. Firstly, the assumption that  $U$  is outer implies that so are the (non-equivariant) face  $HU$  and the equivariant face  $GH$ . Secondly, the root tuple of  $GU$  is  $G \cdot_H r_U$ . Lastly, we need a characterization of the leaf tuple of  $GU$ : we call a leaf  $l$  of  $U$  orbital if all the edges in  $Hl \cap U$  are leaves of  $U$ , and claim that the leaves of  $GU$  are the tuple  $\underline{l}$  formed by the  $G$ -orbits of the orbital leaves of  $U$ . Indeed, a leaf  $l \in U$  will be a leaf of  $HU$  iff whenever  $l \in hU$  for some  $h \in H$  then  $l$  is a leaf of  $hU$  iff whenever  $h^{-1}l \in U$  for some  $h \in H$  then  $h^{-1}l$  is a leaf of  $U$ .

In the general case, write  $U \rightarrow V \rightarrow T$  for the factorization [BP17, Prop. 3.31] as an inner face followed by an outer face. We now define  $GU$  to be the equivariant inner face of  $GV$  that removes all edge orbits that are not represented in  $U$  (that all such edge orbits are inner follows from the description of the roots and leaves of  $GV$  in the previous paragraph). It is now clear that  $U \rightarrow GV \rightarrow T$  is the minimal factorization with  $GV$  an *outer* equivariant face, so that the desired factorization  $U \rightarrow GU \rightarrow T$  follows since inner faces are full while minimality follows from the equivariant analogue of the inner face followed by outer face factorization.  $\square$

**Example 8.9.** When  $T$  is open then  $GU$  is necessarily a full equivariant face, i.e. broad relations coincide with those in  $T$ , and this fact can be leveraged to obtain a simpler proof.

However, the presence of stumps can make the situation substantially more subtle, especially in the non outer case, as illustrated by the following example (where  $G = \mathbb{Z}_{/2} = \{\pm 1\}$ ).



GINNER REM

**Remark 8.10.** For any inner face  $V - e$  of  $V$  one has that  $G(V - e)$  is either  $GV - Ge$  or  $GV$ . Indeed, the latter will happen iff  $V - e$  contains either an inner edge of a leaf of the form  $ge$ .

## A Equivariant Reedy model structures

Bla bla, one of the axioms in <sup>BM11</sup>[BM11] is different from the others point of view

In <sup>BM11</sup>[BM11] Berger and Moerdijk extend the notion of Reedy category so as to allow for categories  $\mathbb{R}$  with non-trivial automorphism groups  $\text{Aut}(r)$  for  $r \in \mathbb{R}$ . For such  $\mathbb{R}$  and suitable model category  $\mathcal{C}$  they then show that there is a *Reedy model structure* on  $\mathcal{C}^{\mathbb{R}}$  that is defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see <sup>BM11</sup>[BM11, Thm. 1.6] or Theorem <sup>REEDYADM THM</sup>A.8 below) to be determined by the  $\text{Aut}(r)$ -projective model structures on  $\mathcal{C}^{\text{Aut}(r)}$  for each  $r \in \mathbb{R}$ .

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the  $\text{Aut}(r)$ -projective model structures on  $\mathcal{C}^{\text{Aut}(r)}$  with the more general  $\mathcal{C}_{\mathcal{F}_r}^{\text{Aut}(r)}$  model structures for  $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$  a nice collection of families of subgroups of each  $\text{Aut}(r)$ .

To do so, we first need some essential notation. For each map  $r \rightarrow r'$  in a category  $\mathbb{R}$  we will write  $\text{Aut}(r \rightarrow r')$  for its automorphism group in the arrow category and write

$$\text{Aut}(r) \xleftarrow{\pi_r} \text{Aut}(r \rightarrow r') \xrightarrow{\pi_{r'}} \text{Aut}(r') \quad (\text{A.1})$$

PIDEFR EQ

for the obvious projections. We now introduce our equivariant generalization of the “generalized Reedy categories” of <sup>BM11</sup>[BM11, Def. 1.1].

GENRED DEF

**Definition A.2.** A *generalized Reedy category structure* on a small category  $\mathbb{R}$  consists of wide subcategories  $\mathbb{R}^+$ ,  $\mathbb{R}^-$  and a degree function  $|\cdot|: \text{ob}(\mathbb{R}) \rightarrow \mathbb{N}$  such that:

- (i) non-invertible maps in  $\mathbb{R}^+$  (resp.  $\mathbb{R}^-$ ) raise (lower) degree; isomorphisms preserve degree;
- (ii)  $\mathbb{R}^+ \cap \mathbb{R}^- = \text{Iso}(\mathbb{R})$ ;
- (iii) every map  $f$  in  $\mathbb{R}$  factors as  $f = f^+ \circ f^-$  with  $f^+ \in \mathbb{R}^+$ ,  $f^- \in \mathbb{R}^-$ , and this factorization is unique up to isomorphism.

Let  $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$  be a collection of families of subgroups of the groups  $\text{Aut}(r)$ . The collection  $\{\mathcal{F}_r\}$  is called *Reedy-admissible* if:

- (iv) for all maps  $r \twoheadrightarrow r'$  in  $\mathbb{R}^-$  one has  $\pi_{r'}(\pi_r^{-1}(H)) \in \mathcal{F}_{r'}$  for all  $H \in \mathcal{F}_r$ .

We note that condition (iv) above should be thought as of a constraint on the pair  $(\mathbb{R}, \{\mathcal{F}_r\})$ . The original setup of <sup>BM11</sup>[BM11] then deals with the case where  $\{\mathcal{F}_r\} = \{\{e\}\}$  is the collection of trivial families. Indeed, our setup recovers the setup in <sup>BM11</sup>[BM11], as follows.

**Example A.3.** When  $\{\mathcal{F}_r\} = \{\{e\}\}$ , Reedy-admissibility coincides with axiom (iv) in <sup>BM11</sup>[BM11, Def. 1.1], stating that if  $\theta \circ f^- = f^-$  for some  $f^- \in \mathbb{R}^-$  and  $\theta \in \text{Iso}(\mathbb{R})$  then  $\theta$  is an identity.

**Example A.4.** For any generalized Reedy category  $\mathbb{R}$ , the collection  $\{\mathcal{F}_{\text{all}}\}$  of the families of all subgroups of  $\text{Aut}(r)$  is Reedy-admissible.

**Example A.5.** Let  $G$  be a group and set  $\mathbb{R} = G \times (0 \rightarrow 1)$  with  $\mathbb{R} = \mathbb{R}^+$ . Then any pair  $\{\mathcal{F}_0, \mathcal{F}_1\}$  of families of subgroups of  $G$  is Reddy-admissible.

Similarly, set  $\mathbb{S} = G \times (0 \leftarrow 1)$  with  $\mathbb{S} = \mathbb{S}^-$ . Then a pair  $\{\mathcal{F}_0, \mathcal{F}_1\}$  of families of subgroups of  $G$  is Reddy-admissible iff  $\mathcal{F}_0 \supset \mathcal{F}_1$ .

GGRAPHREEDY EX

**Example A.6.** Letting  $\mathbb{S}$  denote any generalized Reedy category in the sense of <sup>BM11</sup>[BM11, Def. 1.1] and  $G$  a group, we set  $\mathbb{R} = G \times \mathbb{S}$  with  $\mathbb{R}^+ = G \times \mathbb{S}^+$  and  $\mathbb{R}^- = G \times \mathbb{S}^-$ . Further, for each  $s \in \mathbb{S}$  we write  $\mathcal{F}_s^\Gamma$  for the family of  $G$ -graph subgroups of  $G \times \text{Aut}_{\mathbb{S}}(s)$ , i.e., those subgroups  $K \leq G \times \text{Aut}_{\mathbb{S}}(s)$  such that  $K \cap \text{Aut}_{\mathbb{S}}(s) = \{e\}$ .

Reedy admissibility of  $\{\mathcal{F}_s^\Gamma\}$  follows since for every degeneracy map  $s \twoheadrightarrow s'$  in  $\mathbb{S}^-$  one has that the homomorphism  $\pi_s: \text{Aut}_{\mathbb{S}}(s \twoheadrightarrow s') \rightarrow \text{Aut}_{\Omega}(s)$  is injective (we note that this is equivalent to axiom (iv) in [BM11, Def. 1.1] for  $\mathbb{S}$ ).

Our primary example of interest will come by setting  $\mathbb{S} = \Omega^{op}$  in the previous example. In fact, in this case we will also be interested in certain subfamilies  $\{\mathcal{F}_U\}_{U \in \Omega} \subset \{\mathcal{F}_U^\Gamma\}_{U \in \Omega}$ .

**Example A.7.** Let  $\mathbb{R} = G \times \Omega^{op}$  and let  $\{\mathcal{F}_U\}_{U \in \Omega}$  be the family of graph subgroups determined by a weak indexing system  $\mathcal{F}$ . Then  $\{\mathcal{F}_U\}$  is Reedy-admissible. To see this, recall first that each  $K \in \mathcal{F}_U$  encodes an  $H$ -action on  $U \in \Omega$  for some  $H \leq G$  so that  $G \cdot_H U$  is a  $\mathcal{F}$ -tree. Given a face map  $f: U' \hookrightarrow U$ , the subgroup  $\pi_U^{-1}(K)$  is then determined by the largest subgroup  $\bar{H} \leq H$  such that  $U'$  inherits the  $\bar{H}$ -action from  $U$  along  $f$  (so that  $f$  becomes a  $\bar{H}$ -map), so that  $\pi_{U'}(\pi_U^{-1}(K))$  encodes the  $\bar{H}$ -action on  $U'$ . Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of  $G$ -trees  $G \cdot_{\bar{H}} U' \rightarrow G \cdot_H U$ .

We now state the main result. We will assume throughout that  $\mathcal{C}$  is a model category such that for any group  $G$  and family of subgroups  $\mathcal{F}$ , the category  $\mathcal{C}^G$  admits the  $\mathcal{F}$ -model structure (for example, this is the case whenever  $\mathcal{C}$  is a cofibrantly generated cellular model category in the sense of [Ste16]).

REEDYADM THM

**Theorem A.8.** Let  $\mathbb{R}$  be generalized Reedy and  $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$  a Reedy-admissible collection of families. Then there is a  $\{\mathcal{F}_r\}$ -**Reedy model structure** on  $\mathcal{C}^{\mathbb{R}}$  such that a map  $A \rightarrow B$  is

- a (trivial) cofibration if  $A_r \sqcup_{L_r A} L_r B \rightarrow B_r$  is a (trivial)  $\mathcal{F}_r$ -cofibration in  $\mathcal{C}^{\text{Aut}(r)}$ ,  $\forall r \in \mathbb{R}$ ;
- a weak equivalence if  $A_r \rightarrow B_r$  is a  $\mathcal{F}_r$ -weak equivalence in  $\mathcal{C}^{\text{Aut}(r)}$ ,  $\forall r \in \mathbb{R}$ ;
- a (trivial) fibration if  $A_r \rightarrow B_r \times_{M_r B} M_r A$  is a (trivial)  $\mathcal{F}_r$ -fibration in  $\mathcal{C}^{\text{Aut}(r)}$ ,  $\forall r \in \mathbb{R}$ .

The proof of this result is given at the end of the appendix after establishing some routine generalizations of the key lemmas in [BM11] (indeed, the true novelty in this appendix is the Reedy-admissibility condition in part (iv) of Definition [A.2]).

We first recall the following, cf. [BP17, Props. 6.5 and 6.6] (we note that [BP17, Prop. 6.6] can be proven in terms of fibrations, and thus does not depend on special assumptions on  $\mathcal{C}$ ).

**Proposition A.9.** Let  $\phi: G \rightarrow \bar{G}$  be a homomorphism and  $\mathcal{F}, \bar{\mathcal{F}}$  families of subgroups of  $G, \bar{G}$ . Then the leftmost (resp. rightmost) adjunction below is a Quillen adjunction

$$\bar{G} \cdot_G (-): \mathcal{C}_{\mathcal{F}}^G \rightleftarrows \mathcal{C}_{\bar{\mathcal{F}}}^{\bar{G}}: \text{res}_{\bar{G}}^{\bar{G}} \quad \text{res}_{\bar{G}}^{\bar{G}}: \mathcal{C}_{\bar{\mathcal{F}}}^{\bar{G}} \rightleftarrows \mathcal{C}_{\mathcal{F}}^G: \text{Hom}_G(\bar{G}, -)$$

provided that for  $H \in \mathcal{F}$  it is  $\phi(H) \in \bar{\mathcal{F}}$  (resp. for  $\bar{H} \in \bar{\mathcal{F}}$  it is  $\phi^{-1}(\bar{H}) \in \mathcal{F}$ ).

RESGEN COR

**Corollary A.10.** For any homomorphism  $\phi: G \rightarrow \bar{G}$ , the functor  $\text{res}_{\bar{G}}^{\bar{G}}: \mathcal{C}_{\bar{\mathcal{F}}}^{\bar{G}} \rightarrow \mathcal{C}_{\mathcal{F}}^G$  preserves all four classes of genuine cofibrations, trivial cofibrations, fibrations and trivial fibrations.

The following formalizes an argument implicit in the proof of [BM11, Lemma 5.2]).

**Definition A.11.** Consider a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ B & \longrightarrow & Y \end{array}$$

(A.12) BLA EQ

in  $\mathcal{C}^{\mathbb{R}}$ . A collection of maps  $f_s: B_s \rightarrow X_s$  for  $|s| \leq n$  that induce a lift of the restriction of (A.12) to  $\mathcal{C}^{\mathbb{R}_{\leq n}}$  will be called a  $n$ -partial lift.

BLALIFT LEM

**Lemma A.13.** Let  $\mathcal{C}$  be any bicomplete category, and consider a commutative diagram as in (A.12). Then any  $(n-1)$ -partial lift uniquely induces commutative diagrams

$$\begin{array}{ccc} A_r \sqcup_{L_r A} L_r B & \xrightarrow{\quad} & X_r \\ \downarrow & \searrow \text{dashed} & \downarrow \\ B_r & \xrightarrow{\quad} & Y_r \times_{M_r Y} M_r X \end{array} \quad (\text{A.14})$$

BLALIFT EQ

in  $\mathcal{C}^{\text{Aut}(r)}$  for each  $r$  such that  $|r| = n$ . Furthermore, extensions of the  $(n-1)$ -partial lift to a  $n$ -partial lift are in bijection with choices of  $\text{Aut}(r)$ -equivariant lifts of the diagrams (A.14) for  $r$  ranging over representatives of the isomorphism classes of  $r$  with  $|r| = n$ .

In the next result, by  $\{\mathcal{F}_r\}$ -cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in Theorem A.8, regardless of whether such a model structure exists.

BLALIFT COR

**Corollary A.15.** Let  $\mathbb{R}$  be generalized Reedy and  $\{\mathcal{F}_r\}$  an arbitrary family of subgroups of  $\text{Aut}(r)$ ,  $r \in \mathbb{R}$ . Then a map in  $\mathcal{C}^{\mathbb{R}}$  is a  $\{\mathcal{F}_r\}$ -cofibration (resp. trivial cofibration) iff it has the left lifting property with respect to all  $\{\mathcal{F}_r\}$ -trivial fibrations (resp. fibrations), and vice-versa for the right lifting property.

GINJ LEM

**Lemma A.16.** Let  $\mathbb{S}$  be a generalized Reedy with  $\mathbb{S} = \mathbb{S}^+$ ,  $K$  a group, and  $\pi: \mathbb{S} \rightarrow K$  a functor.

Then if a map  $A \rightarrow B$  in  $\mathcal{C}^{\mathbb{S}}$  is such that for all  $s \in \mathbb{S}$  the maps  $A_s \sqcup_{L_s A} L_s B \rightarrow B_s$  are (resp. trivial)  $\text{Aut}(s)$ -cofibrations one has that  $\text{Lan}_{\pi: \mathbb{S} \rightarrow K}(A \rightarrow B)$  is a (trivial)  $K$ -cofibration.

*Proof.* By adjunction, one needs only show that for any  $K$ -fibration  $X \rightarrow Y$  in  $\mathcal{C}^K$ , the map  $\pi^*(X \rightarrow Y)$  has the right lifting property against all maps  $A \rightarrow B$  in  $\mathcal{C}^{\mathbb{S}}$  as in the statement. By Corollary A.15, it thus suffices to check that the maps

$$(\pi^* X)_s \rightarrow (\pi^* Y)_s \times_{M_s \pi^* Y} M_s \pi^* X$$

are  $\text{Aut}(s)$ -fibrations. But since  $M_s Z = *$  (recall  $\mathbb{S} = \mathbb{S}^+$ ), this map is just  $X \rightarrow Y$  with the  $\text{Aut}(s)$ -action induced by  $\pi: \text{Aut}(s) \rightarrow K$ , hence Corollary A.10 finishes the proof.  $\square$

GINJMIN LEM

**Lemma A.17.** Let  $\mathbb{S}$  be a generalized Reedy with  $\mathbb{S} = \mathbb{S}^-$ ,  $K$  a group, and  $\pi: \mathbb{S} \rightarrow K$  a functor.

Then if a map  $X \rightarrow Y$  in  $\mathcal{C}^{\mathbb{S}}$  is such that for all  $s \in \mathbb{S}$  the maps  $X_s \rightarrow Y_s \times_{M_s Y} M_s X$  are (resp. trivial)  $\text{Aut}(s)$ -fibrations one has that  $\text{Ran}_{\pi: \mathbb{S} \rightarrow K}(A \rightarrow B)$  is a (trivial)  $K$ -fibration.

*Proof.* This follows dually to the previous proof.  $\square$

**Remark A.18.** Lemmas A.16 and A.17 generalize key parts of the proofs of [BM11, Lemmas 5.3 and 5.5]. The duality of their proofs reflects the duality in Corollary A.10.

**Remark A.19.** Lemma A.16 will be applied when  $K \leq \text{Aut}_{\mathbb{R}}(r)$ , and  $\mathbb{S} = K \rtimes \mathbb{R}^+(r)$  for  $\mathbb{R}$  a given generalized Reedy category and  $r \in \mathbb{R}$ . Similarly, Lemma A.17 will be applied when  $\mathbb{S} = K \rtimes \mathbb{R}^-(r)$ . It is straightforward to check that in the  $\mathbb{R}^+$  (resp.  $\mathbb{R}^-$ ) case maps in  $\mathbb{S}$  can be identified with squares as on the left (right)

$$\begin{array}{ccc} r' & \xrightarrow{+} & r \\ + \downarrow & & \downarrow \simeq \\ r'' & \xrightarrow{+} & r \end{array} \quad \begin{array}{ccc} r & \xrightarrow{-} & r' \\ \simeq \downarrow & & \downarrow - \\ r & \xrightarrow{-} & r'' \end{array}$$

such that the maps labelled  $+$  are in  $\mathbb{R}^+$ , maps labelled  $-$  are in  $\mathbb{R}^-$ , the horizontal maps are non-invertible, and the maps labeled  $\simeq$  are automorphisms in  $K$ .

In particular, there is thus a *domain* (resp. *target*) functor  $d: \mathbb{S} \rightarrow \mathbb{R}$  ( $t: \mathbb{S} \rightarrow \mathbb{R}$ ), and our interest is in maps  $d^*A \rightarrow d^*B$  ( $t^*A \rightarrow t^*B$ ) in  $\mathcal{C}^{\mathbb{S}}$  induced from maps  $A \rightarrow B$  in  $\mathcal{C}^{\mathbb{R}}$  so that

$$\text{Lan}_{\pi} d^*(A \rightarrow B) = (L_r A \rightarrow L_r B) \quad \text{Ran}_{\pi} t^*(A \rightarrow B) = (M_r A \rightarrow M_r B)$$

We are now in a position to prove the following, which are the essence of Theorem REEDYADM THM  
A.8.

**Lemma A.20.** *Let  $\mathbb{R}$  be generalized Reedy and  $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$  a Reedy-admissible family.*

*Suppose  $A \rightarrow B$  be a  $\{\mathcal{F}_r\}$ -Reedy cofibration. Then the maps  $A_r \rightarrow B_r$  are all  $\{\mathcal{F}_r\}$ -weak equivalences iff so are the maps  $A_r \sqcup_{L_r A} L_r B \rightarrow B_r$ .*

*Proof.* It suffices to check by induction on  $n$  that the analogous claim with the restriction  $|r| \leq n$  also holds. The  $n = 0$  case is obvious. Otherwise, letting  $r$  range over representatives of the isomorphism classes of  $r$  with  $|r| = n$ , it suffices to check that for each  $H \in \mathcal{F}_r$  the map  $A_r \rightarrow B_r$  is a  $H$ -genuine weak equivalence iff so is  $A_r \sqcup_{L_r A} L_r B \rightarrow B_r$ .

One now applies Lemma GINJ LEM A.16 with  $K = H$  and  $\mathbb{S} = H \times \mathbb{R}^+(r)$  to the map  $d^*A \rightarrow d^*B$ . Note that  $\mathcal{F}$ -trivial cofibrations are always genuine trivial cofibrations for any family, so that the trivial cofibrancy requirements are immediate from Corollary RESGEN COR A.10. It thus follows that the maps labelled  $\sim$

$$\begin{array}{ccc} L_r A & \xrightarrow{\sim} & L_r B \\ \downarrow & & \downarrow \\ A_r & \xrightarrow{\sim} & L_r B \sqcup_{L_r A} A_r \longrightarrow B_r \end{array}$$

are  $H$ -genuine trivial cofibrations, finishing the proof. □

**Lemma A.21.** *Let  $\mathbb{R}$  be generalized Reedy and  $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$  a Reedy-admissible family.*

*Let  $X \rightarrow Y$  be a  $\{\mathcal{F}_r\}$ -Reedy fibration. Then the maps  $X_r \rightarrow Y_r$  are all  $\{\mathcal{F}_r\}$ -weak equivalences iff so are the maps  $X_r \rightarrow Y_r \times_{M_r Y} M_r X$ .*

*Proof.* One repeats the same induction argument on  $|r|$ . In the induction step, it suffices to verify that, for each  $r$  with  $|r| = n$  and  $H \in \mathcal{F}_r$ , the map  $X_r \rightarrow Y_r$  is a  $H$ -genuine weak equivalence iff so is  $X_r \rightarrow Y_r \times_{M_r Y} M_r X$ .

One now applies Lemma GINJMIN LEM A.17 with  $K = H$  and  $\mathbb{S} = H \times \mathbb{R}^-(r)$  to the map  $t^*A \rightarrow t^*B$ . Note that for each  $(r \twoheadrightarrow r') \in \mathbb{S}$  one has  $\text{Aut}_{\mathbb{S}}(r \twoheadrightarrow r') = \pi_r^{-1}(H)$  (where  $\pi_r$  is as in (A.1)), so that the trivial fibrancy requirement in Lemma GINJMIN LEM A.17 follows from  $\{\mathcal{F}_r\}$  being Reedy-admissible. It follows that the maps labelled  $\sim$

$$\begin{array}{ccc} X_r & \longrightarrow & Y_r \times_{M_r Y} M_r X \xrightarrow{\sim} Y_r \\ & & \downarrow \\ & & M_r X \xrightarrow{\sim} M_r Y \end{array}$$

are  $H$ -genuine trivial fibrations, finishing the proof. □

**Remark A.22.** The proofs of Lemmas REEDYTRCOF LEM A.20 and REEDYTRFIB LEM A.21 are similar, but not dual, since Lemma REEDYTRCOF LEM A.21 uses Reedy-admissibility while Lemma REEDYTRFIB LEM A.20 does not. This reflects the difference in the proofs of [BM11, Lemmas 5.3 and 5.5] as discussed in [BM11, Remark 5.6], albeit with a caveat.

Setting  $K = \{e\}$  in Lemma GINJ LEM A.16 yields that  $\lim_{\mathbb{S}}(A \rightarrow B)$  is a cofibration provided that  $A \rightarrow B$  is a genuine Reedy cofibration, i.e. a Reedy cofibration for  $\{\mathcal{F}_{\text{all}}\}$  the families of all subgroups. On the other hand, the proof of BM11 [BM11, Lemma 5.3] argues that  $\lim_{\mathbb{S}}(A \rightarrow B)$  is a cofibration provided that  $A \rightarrow B$  is a projective Reedy cofibration, i.e. a Reedy cofibration for  $\{\{e\}\}$  the trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim

is more general). Since the cofibration half of the projective analogue of Corollary [A.10](#) only holds if  $\phi$  is a monomorphism, the argument in the proof of [\[BM11, Lemma 5.3\]](#) also includes an injectivity check that is not needed for our proof of Lemma [A.20](#).

[proof of Theorem \[A.8\]\(#\)](#). Lemmas [A.20](#) and [A.21](#) say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem [A.8](#) are correct, i.e. that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [\[Hov99, Def. 1.1.3\]](#). Both 2-out-of-3 and the retract axioms are immediate (recall that retracts commute with limits/colimits). The lifting axiom follows from Corollary [A.15](#) while the task of building factorizations  $X \rightarrow A \rightarrow Y$  of a given map  $X \rightarrow Y$  follows by a similar standard argument by iteratively factorizing the maps

$$X_r \sqcup_{L_r X} L_r A \rightarrow Y_r \times_{M_r Y} M_r A$$

in  $\mathcal{C}^{\text{Aut}(r)}$ , thus building both  $A$  and the factorization inductively (see, e.g., the proof of [\[BM11, Thm. 1.6\]](#)).  $\square$

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