Equivariant dendroidal Segal spaces and G-∞-operads

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Abstract

We introduce the analogues of complete Segal space and of Segal category in the context of equivariant operads with norm maps, and build suitable model categories with these as the fibrant objects. We then show that these model categories are Quillen equivalent to each other and to the model category for G-∞-operads built in a previous paper.

Moreover, we establish variants of these results for the Blumberg-Hill indexing systems.

In an appendix, we discuss Reedy categories in the equivariant context.

Contents

1 Introduction

This paper follows [Per17] and [BP17] and is the third piece of a larger project aimed at understanding the homotopy theory of *equivariant operads with norm maps*. Informally, norm maps are a new piece of structure that must be considered when dealing with equivariant operads (and which has no analogue in the theory of equivariant categories). The need to understand norm maps, particularly in the context of G-ring spectra, as well as their usefulness, was made clear by Hill, Hopkins and Ravenel in their solution of the Kervaire invariant one problem [HHR16].

The starting point of this project was the discovery by the authors of, for each finite group G, a category Ω_G of G-trees whose objects diagrammatically encode compositions of norms maps and whose arrows encode the necessary compatibilities between such compositions. Our categories Ω_G are a somewhat non-obvious generalization of the dendroidal category Ω of Cisinski-Moerdijk-Weiss, and indeed all the key combinatorial concepts in their work, such as faces, degeneracies, boundaries and horns, generalize to G-trees [Per17, §5,6]. As such, it is natural to ask whether the Cisinski-Moerdijk program [CM11],[CM13a],[CM13b] can also be generalized to the equivariant context.

Recall that the main result of their program is the existence of a Quillen equivalence

$$W_! \colon \mathsf{dSet} \rightleftarrows \mathsf{sOp} \colon N_{hc}$$

where $\mathsf{dSet} = \mathsf{Set}^{\Omega^{op}}$ is the category of presheaves over Ω, called *dendroidal sets*, and sOp is the category of simplicial colored operads. Their program was carried out in three main steps: (i) [CM11] established the existence of the model structure on dSet (with some of the key combinatorial analysis based on Moerdijk and Weiss' work in [MW09]); (ii) [CM13a] established auxiliary model structures on the categories sdSet and PreOp of dendroidal spaces and pre-operads, and showed that all three of dSet, sdSet and PreOp are Quillen equivalent; (iii) lastly, [CM13b] established the existence of the model structure on sOp as well as the Quillen equivalence between sOp and PreOp, finishing the proof of the main result.

From the perspective of the Cisinski-Moerdijk program, [Per17] is then the equivariant analogue of the first step [CM11] (as well as [MW09]), while the present paper provides the equivariant analogue of the second step [CM13a]. More explicitly, in [Per17], and inspired by the category Ω_G of G-trees, the second author equipped the category dSet^G of G-equivariant dendroidal sets with a model structure whose fibrant objects are "equivariant operads with norm maps up to homotopy", called G-∞-operads. Further, it was shown therein that whenever a G-operad $\mathcal{O} \in \mathsf{sOp}^G$ is suitably fibrant the homotopy coherent nerve $N_{hc}(\mathcal{O})$ is such a G-∞-operad (rather than just an "∞-operad with G-action"). In the present paper, our main results are then the existence of suitable model structures on the categories sdSet^G and PreOp^G of G-dendroidal spaces and G-pre-operads, as well as the existence of Quillen equivalences between all three of dSet^G, sdSet^G and PreOp^G.

It is worth noting that, much as was the case with the work in [Per17], our results are not formal consequences of their non-equivariant analogues, due to the nature of norm maps[1]. Indeed, in [BP17], the second piece of our project, the authors introduced the notion of *genuine equivariant operads*, which are new algebraic objects motivated by the combinatorics of norm maps as encoded by the category Ω_G of G-trees. And while a priori the work in [BP17] is largely perpendicular to the Cisinski-Moerdijk program (the main result [BP17, Thm. III] is what one might call the "operadic Elmendorf-Piacenza theorem", which is an equivariant phenomenon),

[1] Recall that by using the inclusions of simplicial categories and simplicial sets into simplicial operads and dendroidal sets (cf. the introduction to [CM13b]), the Cisinski-Moerdijk program recovers the Bergner-Joyal-Lurie-Rezk-Tierney program studying ∞-categories. As a point of contrast, we note that the lack of norms in the categorical case causes the equivariant generalization of this latter program to indeed be formal; see [Ste16, Ber17].

many of the new technical hurdles in this paper versus [CM13a] can be traced back to the fact that at many points in the discussion we are secretly dealing with colored genuine equivariant operads, which are the colored generalization of the structures discussed in [BP17], and the formal definition of which we prefer to postpone to a follow-up paper.

The organization of the paper is as follows.

§2 mostly recalls the necessary notions concerning the category Ω_G of G-trees and the category dSet^G of G-dendroidal sets that were introduced in [Per17]. However, some new notions and results can be found throughout, most notably the notion of *orbital face* in Definition 2.15 and the associated notion of *orbital horn* in §2.3.

The main goal of §3 is to establish Proposition 3.18, which roughly states that Segal core inclusions, horn inclusions and orbital horn inclusions can in some circumstances be used interchangeably. The bulk of the technical work takes place in §3.1 where Lemma 3.4, a powerful technical result we call the *characteristic edge lemma*, is established. §3.2 then shows Proposition 3.18 via a string of easy applications of Lemma 3.4. Lastly, §3.3 recasts the *genuine equivariant operads* of [BP17] in a different perspective more suitable for our purposes in §5.

§4 establishes the desired Quillen equivalences between dSet^G, sdSet^G, PreOp^G via largely abstract methods. Our approach is inspired by [CM13a, Thm. 6.6], which observes that the (complete) model structure on sdSet can be built via two distinct localization procedures. As such, in §4.1 we first discuss an abstract setting for such common localizations, which is then applied in §4.2 to obtain the Quillen equivalence $\mathsf{dSet}^G \rightleftarrows \mathsf{sdSet}^G$ in Theorem 4.19. §4.3 then uses purely formal techniques to induce the model structure on PreOp^G from the model structure on sdSet^G and to establish the Quillen equivalence $\mathsf{PreOp}^G \rightleftarrows \mathsf{sdSet}^G$ in Theorem 4.28.

Then in our last main section §5, and motivated by the fact that in our desired model structure on G-operads sOp^G (to be described in a follow-up paper) the weak equivalences are Dwyer Kan equivalences (i.e. characterized by fully faithfulness and essential surjectivity), we establish Theorem 5.30, which gives a Dwyer Kan type description of the weak equivalences between the fibrant objects in either of sdSet^G, PreOp^G.

§6, which is transversal to the remainder of the paper, outlines generalizations of all our results where the role of the category Ω_G of G-trees is played by a certain type of subcategory $\Omega_{\mathcal{F}} \subseteq \Omega_G$ which (almost exactly) corresponds to one of the *indexing systems* first identified by Blumberg and Hill in [BH15].

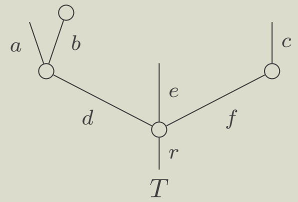
Lastly, Appendix A discusses an equivariant variation of the *generalized Reedy categories* of [BM11] which plays an essential role in §4.2 when describing the model structure on sdSet^G. The key to this appendix is the *Reedy-admissibility* condition in Definition A.2(iv), which is a rather non-obvious equivariant generalization of one of the Reedy category axioms in [BM11].

2 Preliminaries

2.1 The category of trees Ω

We start by recalling the key features of the category Ω of trees that will be used throughout. Our official model for Ω will be Weiss' algebraic model of *broad posets* as discussed in [Per17,

§5], hence we first recall some key notation and terminology. Given a tree diagram T such as



(2.1) FIRSTTREE EQ

and for each edge t of T topped by a vertex $\circ$, we write $t^{\uparrow}$ to denote the tuple of edges immediately above t. In our example, $r^{\uparrow} = def$, $d^{\uparrow} = ab$, $f^{\uparrow} = c$ and $b^{\uparrow} = \epsilon$, where ϵ is the empty tuple. Edges t for which: (i) $t^{\uparrow} \neq \epsilon$, such as r, d, f, are called *nodes*; (ii) $t^{\uparrow} = \epsilon$, such as b, are called *stumps*; (iii) $t^{\uparrow}$ is undefined, such as a, c, e, are called *leaves*. The vertices of T are then encoded symbolically as $t^{\uparrow} \leq t$, which we call a *generating broad relation*. This notation is meant to suggest a form of transitivity: for example, the generating relations $ab \leq d$ and $def \leq r$ generate, via *broad transitivity*, a relation $abef \leq r$ (we note that this is essentially compact notation for the operations and composition in the colored operad generated by T [MW07, §3]). The other broad relations obtained by broad transitivity are $dec \leq r$, $abec \leq r$, $aec \leq r$, $a \leq d$. The set of edges of T together with these broad relations (as well as identity relations $t \leq t$) form the *broad poset* associated to the tree, which is again denoted T.

Given a broad relation $t_0 \cdots t_n \leq t$, we further write $t_i \leq_d t$. Pictorially, this says that the edge t_i is above t, and it is thus clear that $\leq_d$ defines a partial order on edges of T. Trees always have a single $\leq_d$-maximal edge, called the *root*. Edges other than the root or the leaves are called *inner edges*. In our example r is the root, b, d, f are inner edges and a, e, c are leaves.

We denote the sets of edges (inner edges, vertices) of T by $E(T)$ (resp. $E^{\mathsf{i}}(T)$, $V(T)$).

The Cisinski-Moerdijk-Weiss category Ω of trees then has as objects tree diagrams as in (2.1) and as maps $\varphi: T \to S$ the monotone maps of broad posets (meaning that if $t_1 \cdots t_k \leq t$ then $\varphi(t_1) \cdots \varphi(t_k) \leq \varphi(t)$). In fact, Weiss further identified axioms characterizing those broad posets that are associated to trees (see [Per17, Defs. 5.1 and 5.9]).

Further, our discussion will be somewhat simplified by the assumption that Ω contains exactly one representative of each planarized tree. Informally, this means that trees $T \in \Omega$ come with a preferred planar representation, though this can also be formalized in purely algebraic terms, see [BP17, §3.1]. For our purposes, the main consequence is that any map $S \to T$ in Ω has a (strictly) unique factorization $S \simeq S' \to T$ as an isomorphism followed by a *planar map* [BP17, Prop. 3.21]. Roughly speaking, S' is obtained from S by pulling back the planarization of T.

We now recall the key classes of maps of Ω. A map $\varphi: S \to T$ which is injective on edges is called a *face map* while a map that is surjective on edges and preserves leaves is called a *degeneracy map* (the extra requirement ensures that leaves of S do not become stumps of T). Moreover, a face map is further called an *inner face map* if $\varphi(r_S) = r_T$ and $\varphi(\underline{l}_S) = \underline{l}_T$ (where $r_{(-)}$ denotes the root edge and $\underline{l}_{(-)}$ the leaf tuple) and called an *outer face map* if it does not factor through any non-identity inner face maps. The following result is [BP17, Cor. 3.32].

UNIQUEFACT PROP **Proposition 2.2.** *A map $\varphi: S \to T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^o} T$$

as a degeneracy followed by an inner face map followed by an outer face map.

We now recall a more explicit characterization (and notation) for planar inner/outer faces (planar degeneracies are characterized by edge multiplicities, see [BP17, Prop. 3.47(ii)]). For any subset $D \subseteq E(T)$, there is a planar inner face $T - D$ which removes the inner edges in E but keeps all broad relations involving edges not in D (this is the hardest class of maps to visualize pictorially, as the vertices adjacent to each $d \in D$ are combined via broad transitivity/composition). For each broad relation $t_1 \cdots t_k = \underline{t} \leq t$ in T, there is a planar outer face $T_{\underline{t} \leq t}$ such that $r_{T_{\underline{t}\leq t}} = t$ and $\underline{l}_{T_{\underline{t}\leq t}} = \underline{t}$ (in fact, by Proposition 2.2 this is the maximal such face). Moreover, the edges s of $T_{\underline{t}\leq t}$ are the edges of T such that $s \leq_d t$ and $\forall_i s \nleq t_i$ while the vertices are the $s^\uparrow \leq s$ such that $s \leq_d t$ and $\forall_i s \nleq t_i$ (pictorially, $T_{\underline{t}\leq t}$ removes the parts of T not above t and above some t_i).

INNFULL REM **Remark 2.3.** Inner faces $T - D \hookrightarrow T$ are always full, i.e. $T - D$ contains all broad relations of T whose edges are in $T - D$. By contrast, whenever T has stumps some of its outer faces $T_{\underline{t}\leq t}$ are not full, the main example being the maximal outer faces "removing stumps" [Per17, Not. 5.41].

DEGREE REM **Remark 2.4.** Following [BM11, Ex. 2.8], one has a degree function $|-|\colon \Omega \to \mathbb{N}$ given by $|T| = |V(T)|$ such that non isomorphim face maps (resp. degeneracies) strictly increase (decrease) $|-|$. The category of face maps is thus denoted Ω^+ and that of degeneracies is denoted Ω^-.

We now collect a couple of useful lemmas concerning faces.

INNINT LEM **Lemma 2.5.** *Consider a diagram of planar faces in Ω (implicitly regarded as inclusion maps)*

$$\begin{tikzcd} V \arrow[r, hook, "out"] \arrow[d, hook, "inn"'] & U \arrow[d, hook] \\ \bar{V} \arrow[r, hook, "out"] & \bar{U} \end{tikzcd}$$

such that the horizontal maps are outer face maps and the left vertical map is an inner face map. Then $E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(\bar{V})$.

Proof. Write r and $\underline{l} = l_1 \cdots l_n$ for the root and leaf tuple of V, or equivalently $\bar{V}$. Since the horizontal maps are outer, an edge $e \in E^{\mathsf{i}}(U)$ (resp. $e \in E^{\mathsf{i}}(\bar{U})$) is also in $E^{\mathsf{i}}(V)$ (resp. in $E^{\mathsf{i}}(\bar{V})$) iff $e <_d r$ and $\forall_i e \nleq l_i$. But then $E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(V) = E^{\mathsf{i}}(U) \cap E^{\mathsf{i}}(\bar{V})$. □

CUPCAP LEM **Lemma 2.6.** *Let $\{U_i \hookrightarrow T\}$ be a collection of planar outer faces of T with a common root t. Then there are planar outer faces $U^{\cup} \hookrightarrow T$, $U^{\cap} \hookrightarrow T$, also with root t, such that*

$$E(U^{\cup}) = \bigcup_i E(U_i), \quad V(U^{\cup}) = \bigcup_i V(U_i), \qquad E(U^{\cap}) = \bigcap_i E(U_i), \quad V(U^{\cap}) = \bigcap_i V(U_i). \tag{2.7}$$ CUPCAP EQ

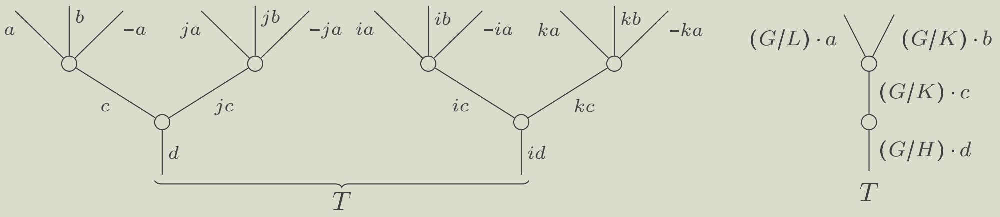
Moreover, these are the smallest (resp. largest) outer faces containing (contained in) all U_i.

Remark 2.8. One can check that it actually suffices to assume the U_i have a common edge.

Proof. (2.7) determines pre-broad posets (cf. [Per17, Rem. 5.2]) $U^{\cup}$ and $U^{\cap}$, hence we need only verify the axioms in [Per17, Defs. 5.1, 5.3, 5.9]. Antisymmetry and simplicity are inherited from T, the nodal axiom is obvious from (2.7), and the root axiom follows since the U_i have a common root (in $U^{\cap}$ case note that if s is in $U^{\cap}$, then so is any s' such that $s \leq_d s' \leq_d t$). □

2.2 The category of G-trees Ω_G

We next recall the category Ω_G of G-trees first defined in [Per17, §5.3]. We start with an explicit and representative example of a G-tree (for more examples, see [Per17, §4.3]). Letting $G = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternionic units and $G \geq H \geq K \geq L$ denote the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$, there is a G-tree T with *expanded representation* given by the two trees on the left below and *orbital representation* given by the (single) tree on the right.



(2.9) TWOREP EQ

Note that the edge labels on the expanded representation encode the action of G so that the edges a, b, c, d have stabilizers L, K, K, H.

Formally, the definition of Ω_G [Per17, Def. 5.44] is given as follows. Given a non-equivariant forest diagram F (i.e. a finite collection of tree diagrams side by side), there is an associated broad poset just as before, and thus one obtains a category Φ of forests. Letting Φ^G denote G-objects on Φ, referred to as G-forests, the category $\Omega_G \subset \Phi^G$ of G-trees is defined to be the full subcategory of those G-forests such that the G-action is transitive on tree components.

We note that any G-tree T can then be written as $G \cdot_H T_*$, where T_* is some fixed tree component, $H \leq G$ is the subgroup sending that component to itself, and we regard $T_* \in \Omega^H$, i.e., as a tree with a H-action (where we caution that $\Omega^G \subsetneq \Omega_G$).

We note that we also assume G-trees (and forests in general) are planarized, meaning that they come with a total order of the tree components, which are themselves planarized.

If $T \in \Omega_G$ has tree components $T_1, \cdots, T_k$, we write $E(T) = \amalg_i E(T_i)$, $E^{\mathsf{i}}(T) = \amalg_i E^{\mathsf{i}}(T_i)$, $V(T) = \amalg_i V(T_i)$ for its sets of edges, inner edges and vertices, as well as $E_G(T) = E(T)/G$, $E^{\mathsf{i}}_G(T) = E^{\mathsf{i}}(T)/G$, $V_G(T) = V(T)/G$ for its sets of *edge orbits*, *inner edge orbits* and *G-vertices*.

Before discussing face maps in the equivariant context, it is worth commenting on the complementary roles of the expanded and orbital representations. On the one hand, the G-broad posets associated to G-trees are diagrammatically represented by the expanded representation, so that the arrows of Ω_G are best understood from that perspective. On the other hand, the diagrams encoding compositions of norm maps of an equivariant operad $\mathcal{O}$ are given by the orbital representations of G-trees (see [Per17, Ex. 4.9], [BP17, (1.10)]). As a result, different aspects of our discussion will be guided by different representations, and this will require us to discuss the different notions of face/boundary/horn suggested by the two representations. We start by recalling the notion of face discussed in [Per17], which is motivated by the expanded representation.

Definition 2.10. Let $T \in \Omega_G$ be a G-tree with non-equivariant tree components $T_1, T_2, \cdots, T_k$.

A *face* of T is an underlying face map $U \hookrightarrow T_i$ in Ω for some $1 \leq i \leq k$. Further, we abbreviate faces of T as $U \hookrightarrow T$, and call them *planar/outer* faces whenever so is the map $U \hookrightarrow T_i$.

Notation 2.11. Given $T \in \Omega_G$, we write $\mathsf{Face}(T)$ for the G-poset of *planar faces* $U \hookrightarrow T$. We note that the G-action is given by the unique factorization of the composite $U \hookrightarrow T \xrightarrow{g} T$ as $U \simeq gU \hookrightarrow T$ such that $gU \hookrightarrow T$ is planar.

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\simeq"'] & T \arrow[d, "g"] \\ gU \arrow[r, hook] & T \end{tikzcd} \tag{2.12}$$

FACEGACT EQ

Notation 2.13. Given $T \in \Omega_G$ and a planar face $U \hookrightarrow T$ we write $\bar{U}^T$, or just $\bar{U}$ when no confusion should arise, for the *outer closure of* U, i.e. the smallest planar outer face of T containing U.

PLFUNCTOR REM **Remark 2.14.** Recalling that notation $\Omega^+ \subset \Omega$ (non-equivariant) subcategory of face maps, we write $\Omega^+ \downarrow T$ for the category of all faces of $T \in \Omega_G$. By pulling back the planarization of T one then obtains a *planarization functor*

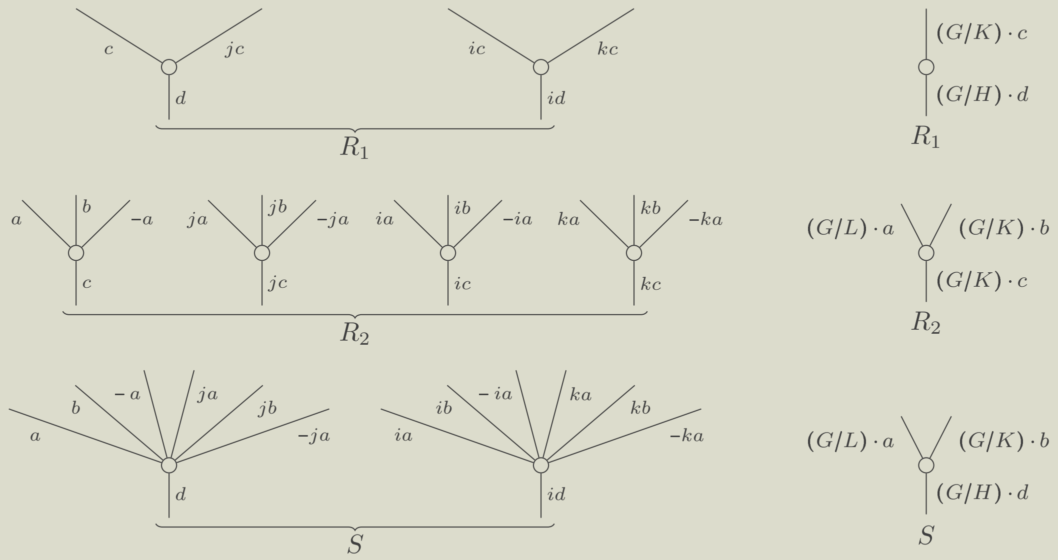
$$\Omega^+ \downarrow T \xrightarrow{pl} \mathsf{Face}(T)$$

which respects the G-actions on the two categories. Note, however, that the inclusion $\mathsf{Face}(T) \subset \Omega^+ \downarrow T$ (which is a section of pl) does not respect the G-actions, as displayed in (FACEGACT EQ 2.12).

We now introduce the notion of face suggested by the orbital representation.

ORBFACE DEF **Definition 2.15.** Let $T \in \Omega_G$. An *orbital face* of T is a map $S \hookrightarrow T$ in Ω_G which is injective on edges. Further, an orbital face is called *planar/inner/outer* if any (and thus all) of its component maps is.

ORBFACE EX **Example 2.16.** The following are three planar orbital faces of the G-tree T in (TWOREP EQ 2.9), with $R_1 \hookrightarrow T$, $R_2 \hookrightarrow T$ orbital outer faces and $S \hookrightarrow T$ an orbital inner face.



These examples illustrate our motivation for the term "orbital face": the tree diagrams in the orbital representations of R_1, R_2, S look like faces of the tree in the orbital representation of T.

Adapting the notation for (non-equivariant) inner faces, we write $S = T - Gc = T - \{c, jc, ic, kc\}$ and analogously throughout the paper. We will need no analogous notation for orbital outer faces.

TREEDIFNOT NOT **Notation 2.17.** In the remainder of the paper we sometimes need to consider (non-equivariant) faces and orbital faces simultaneously. As such, we reserve the letters U, V, W for trees in Ω and the letters R, S, T for G-trees in Ω_G.

INNOUTORB REM **Remark 2.18.** It follows from Proposition UNIQUEFACT PROP 2.2 that any orbital face $S \hookrightarrow T$ has a factorization $S \hookrightarrow R \hookrightarrow T$, unique up to isomorphism, as an orbital inner face followed by an orbital outer face.

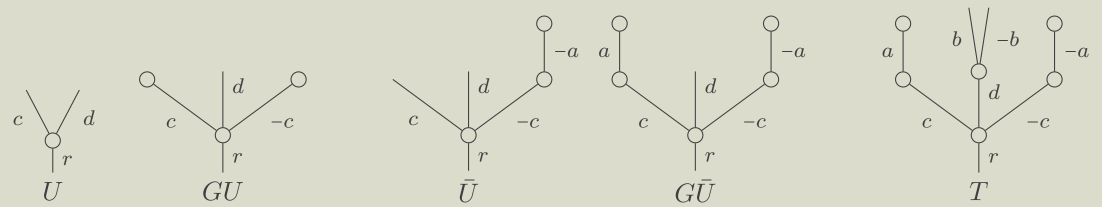
MINGFACT PROP **Proposition 2.19.** *Let $T \in \Omega_G$. Any (non-equivariant) planar face $U \hookrightarrow T$ has a minimal factorization $U \hookrightarrow GU \hookrightarrow T$ through a planar orbital face GU.*

Proof. Assume first that $U = \bar{U}^T$ is outer and write $H \le G$ for the isotropy of its root r_U. By Lemma 2.6 there exists a smallest outer face containing all $\{hU \hookrightarrow T\}_{h\in H}$, which we denote by HU. Moreover, HU inherits the H-action from T (by either its construction or its characterization). Moreover, the natural map $G \cdot_H HU \to T$ is then injective on edges (for a map of forests $F \to F'$ the images of the tree components of F are pairwise $\le_d$-incomparable iff so are the images of the roots) and we thus let GU be $G \cdot_H HU$ with the planar structure induced from T. Both the factorization $U \to GU \to T$ and its minimality are immediate from the description of HU.

Before tackling the general case, we collect some key observations. Firstly, if U is outer then so is the (non-equivariant) face HU and the orbital face GU. Secondly, the root tuple of GU is $G \cdot_H r_U$. Lastly, we need to characterize the leaf tuple of GU. We call a leaf l of U *orbital* if all the edges in $Hl \cap E(U)$ are leaves of U, and claim that the leaves of GU are the tuple $\underline{l}$ formed by the G-orbits of the orbital leaves of U. Indeed, a leaf l of U is also a leaf of HU iff $\forall_{h\in H}$ ($l \in E(hU)$ implies that l is a leaf of hU) iff $\forall_{h\in H}$ ($h^{-1}l \in E(U)$ implies that l is a leaf of U).

In the general case, we define GU as the orbital inner face of $G\bar{U}$ that removes all edge orbits not represented in U (that all such edge orbits are inner follows from the description of the roots and leaves of $G\bar{U}$ in the previous paragraph). It is now clear that $U \to G\bar{U} \to T$ is the minimal factorization with $G\bar{U}$ an *outer* orbital face, and thus the factorization $U \to GU \to T$ exists and is minimal since inner faces are full (Remark 2.3) together with the inner-outer factorization of orbital faces (Remark 2.18). □

Example 2.20. Much of the complexity in the previous proof is needed to handle the scenario of non outer faces $U \hookrightarrow T$ of G-trees T which have stumps, which is easily the subtlest case, as illustrated by the following example (where $G = \mathbb{Z}_{/2} = \{\pm 1\}$).



Remark 2.21. It is clear from the proof of Proposition 2.19 that, if $U \in \mathsf{Face}(T)$ has isotropy H, the induced map $G \cdot_H U \to T$ is a monomorphism iff H is also the isotropy of the root r_U.

GINNER REM **Remark 2.22.** For any inner face $V - e$ of V one has that $G(V - e)$ is either $GV - Ge$ or GV. Indeed, the latter will happen iff $V - e$ contains either an inner edge of a leaf of the form ge.

Remark 2.23. Writing $\mathsf{Face}_o(T)$ for the poset of planar orbital faces, Proposition 2.19 gives a G-equivariant functor (note that G does not act on $\mathsf{Face}_o(T)$)

$$\mathsf{Face}(T) \xrightarrow{G(-)} \mathsf{Face}_o(T).$$

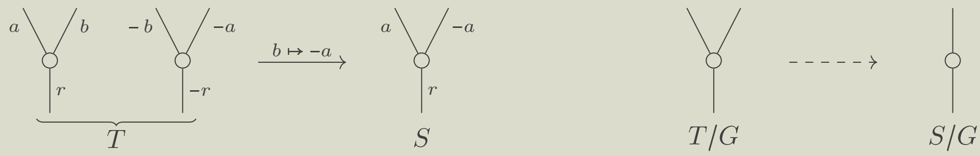
Moreover, there is a natural inclusion $\mathsf{Face}_o(T) \subseteq \mathsf{Face}(T)/G$ (sending an orbital face S to the class of components $[S_*]$) whose left adjoint is the induced functor $\mathsf{Face}(T)/G \to \mathsf{Face}_o(T)$.

ORB_FACE_REM **Remark 2.24.** In fact, there is an isomorphism of posets

$$\mathsf{Face}_o(T) \xrightarrow{\simeq} \mathsf{Face}(T/G), \qquad S \mapsto S/G,$$

where T/G denotes the underlying tree in the orbital representation of T.

However, we caution that though this claim is intuitive, care is needed when formalizing it. For example, the broad poset of T/G is in general *not* the quotient of the broad poset of T, as that may fail the simplicity axiom in [Per17, Def. 5.9]. In fact, the assignment $T \mapsto T/G$ is *not* a functor $\Omega_G \to \Omega$, as shown by the following (for $G = \mathbb{Z}_{/2} = \{\pm 1\}$), since no dashed arrow exists.



(2.25) QUOTMAP EQ

We now outline the formal construction of T/G, starting with some preliminary notation.

Given $\underline{e}, \underline{f}$ tuples of edges of T, write $\underline{f} \leq \underline{e}$ if $\underline{e} = e_1 e_2 \cdots e_k$ and there is a tuple decomposition $\underline{f} = \underline{f}_1 \underline{f}_2 \cdots \underline{f}_k$ such that $\underline{f}_i \leq e_i$. When the e_i are $\leq_d$-incomparable, [Per17, Prop. 5.30] says that such decomposition is unique, so that $\underline{e}, \underline{f}$ consist of distinct edges and we can regard $\underline{e}, \underline{f}$ as subsets $\underline{e}, \underline{f} \subseteq E(T)$.

We now say that a relation $\underline{f} \leq \underline{e}$ is an *orbital relation* if $\underline{e} \subseteq E(T)$ is a orbital G-subset and $\underline{f} \subseteq E(T)$ is a G-subset. Reinterpreting the orbital relations of T as broad relations on the set $E_G(T)$ of edge orbits, one readily checks that this defines a dendroidally ordered set [Per17, Def. 5.9], i.e. a tree, that we call T/G. Note that one hence has a functor $(-)/G \colon \Omega_G^+ \to \Omega$, where Ω_G^+ is the subcategory of orbital face maps, and planarizations of the T/G are chosen arbitrarily.

Lastly, we observe that, much as in the non-equivariant case, the orbital outer faces of T are indexed by orbital relations.

2.3 Equivariant dendroidal sets

EQDENDSETS SEC

Recall [Per17, §5.4] that the category of *G-equivariant dendroidal sets* is the presheaf category $\mathsf{dSet}^G = \mathsf{Set}^{\Omega^{op} \times G}$. Given $T \in \Omega_G$ with non-equivariant tree components $T_1, \cdots, T_k$, we extend the usual notation for representable functors to obtain $\Omega[T] \in \mathsf{dSet}^G$ via

$$\Omega[T] = \Omega[T_1] \amalg \cdots \amalg \Omega[T_k]$$

regarded as a G-object in dSet. One further defines *boundaries* (in the union formula, the injection $\Omega[U] \to \Omega[T]$ is regarded as an inclusion; the equivalence between the colimit and union formulas follows from Proposition 2.2)

$$\partial\Omega[T] = \operatorname*{colim}_{U \in \mathsf{Face}(T), U \neq T_i} \Omega[U] = \bigcup_{U \in \mathsf{Face}(T), U \neq T_i} \Omega[U]$$

and, for $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$ a non-empty G-subset of inner edges (we abbreviate $E_i = E \cap E^{\mathsf{i}}(T_i)$), *$G$-inner horns*

$$\Lambda^E[T] = \operatorname*{colim}_{U \in \mathsf{Face}(T), (T_i - E_i) \not\hookrightarrow U} \Omega[U] = \bigcup_{U \in \mathsf{Face}(T), (T_i - E_i) \not\hookrightarrow U} \Omega[U]$$

which, informally, are the subcomplexes of $\Omega[T]$ that remove the inner faces $T_i - D$ for $D \subseteq E_i$.

Lastly, letting $\mathsf{Face}_{sc}(T)$ denote those outer faces of T with no inner vertices (these are either single edges t or generated by single vertices $t^\uparrow \leq t$), we define the *Segal core of T*

$$Sc[T] = \operatorname*{colim}_{U \in \mathsf{Face}_{sc}(T)} \Omega[U] = \bigcup_{U \in \mathsf{Face}_{sc}(T)} \Omega[U].$$

Note that if $T \simeq G \cdot_H T_*$ for some $T_* \in \Omega^H$ then

$$\Omega[T] \simeq G \cdot_H \Omega[T_*], \quad \partial\Omega[T] \simeq G \cdot_H \partial\Omega[T_*], \quad \Lambda^E[T] \simeq G \cdot_H \Lambda^{E_*}[T_*], \quad Sc[T] \simeq G \cdot_H Sc[T_*]. \tag{2.26}$$

T_DECOMP_EQ

As a cautionary note, we point out that though representable functors $\Omega[T]$ are defined for $T \in \Omega_G$, evaluations $X(U)$ of $X \in \mathsf{dSet}^G$ are defined only for $U \in \Omega$ (cf. Notation 2.17).

FACEGACT REM

Remark 2.27. For $T \in \Omega_G$, a planar face $\varphi_U : U \to T$ can also be regarded as a dendrex $\varphi_U \in \Omega[T](U)$. However, the G-isotropy H of $U \in \mathsf{Face}(T)$ must not be confused with the G-isotropy of φ_U. Instead, $\Omega[T](U)$ has a larger $G \times \mathsf{Aut}(U)$-action, and the $G \times \mathsf{Aut}(U)$-isotropy of φ_U is a subgroup $\Gamma \leq G \times \mathsf{Aut}(U)$ which is the graph of a homomorphism $\phi: H \to \mathsf{Aut}(U)$. One readily checks that if $hU = U$ in $\mathsf{Face}(T)$ then $\phi(h)$ is the left isomorphism in (2.12), so that $U \in \Omega$ is equipped with a canonical H-action. We abuse notation by writing $U \in \Omega^H \subseteq \Omega_H$ to denote this.

Recall that a class of maps is called *saturated* if it is closed under pushouts, transfinite composition and retracts.

The saturation of the boundary inclusions $\partial\Omega[T] \to \Omega[T]$ is the class of *G-normal monormorphisms*, i.e. those monomorphisms $X \to Y$ in dSet^G such that $Y(U) \smallsetminus X(U)$ has an $\mathsf{Aut}(U)$-free action for all $U \in \Omega$. Moreover, since this condition is actually independent of the G-action, we will usually call these simply *normal monomorphisms*.

The saturation of the G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ is called the class of *G-inner anodyne maps*, while those $X \in \mathsf{dSet}^G$ with the right lifting property against all G-inner horn inclusions are called *G-∞-operads*.

We can now recall [Per17, Thm 2.1], which was the main result therein.

Theorem 2.28. *There is a model structure on* $\mathsf{dSet^G}$ *such that the cofibrations are the normal monomorphisms and the fibrant objects are the G-∞-operads.*

Remark 2.29. The definition G-∞-operads just given is a priori distinct from the original definition [Per17, Def. 6.12] which used only *generating G-inner horn inclusion*, i.e. those inclusions $\Lambda^{Ge}[T] \to \Omega[T]$ with $E = Ge$ an inner edge orbit. The present definition has the technical advantages of being naturally compatible with restricting the G-action and of allowing for a simpler proof of Lemma 3.4, which is our main tool for showing that maps are G-inner anodyne. The equivalence between the two definitions follows from [Per17, Prop. 6.17], although we also independently recover this from Lemma 3.4 in Corollary 3.17.

In additional to the G-inner horns defined before, we now introduce a new kind of horn that, much like orbital faces, is naturally suggested by the orbital representation of G-trees. Given $E \subseteq E^{\mathrm{i}}(T)$ a G-equivariant set of inner edges, we define the associated *orbital G-inner horn* by

$$\Lambda^E_o[T] = \operatorname*{colim}_{S \in \mathsf{Face}_o(T), (T-E) \not\to S} \Omega[S] = \bigcup_{S \in \mathsf{Face}_o(T), (T-E) \not\to S} \Omega[S]$$

where we note that the equivalence between the colimit and union formulas now follows from Proposition 2.19.

ORB_HORN_REM

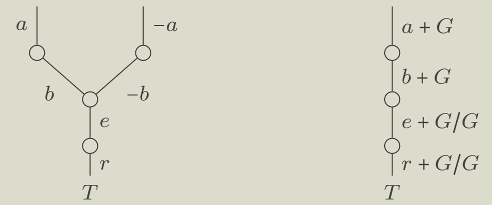
Remark 2.30. We now extend the identification $\mathsf{Face}_o(T) \simeq \mathsf{Face}(T/G)$ in Remark 2.24.

Say a subcomplex $A \subseteq \Omega[T]$ is *orbital* if it is the union of orbital faces $\Omega[S]$, $S \in \mathsf{Face}_o(T)$. Equivalently, by Proposition 2.19 this means that for $U \in \mathsf{Face}(T)$ one has $\Omega[U] \subseteq A$ iff $\Omega[GU] \subseteq A$. There is then a natural bijection of posets (under inclusion)

$$\left\{\text{orbital subcomplexes } \bigcup_i \Omega[S_i] \text{ of } \Omega[T]\right\} \leftrightarrow \left\{\text{subcomplexes } \bigcup_i \Omega[S_i/G] \text{ of } \Omega[T/G]\right\}.$$

In particular, note that $\Lambda^{Ge}_o[T]$ corresponds to $\Lambda^{[e]}[T/G]$ and $Sc[T]$ corresponds to $Sc[T/G]$.

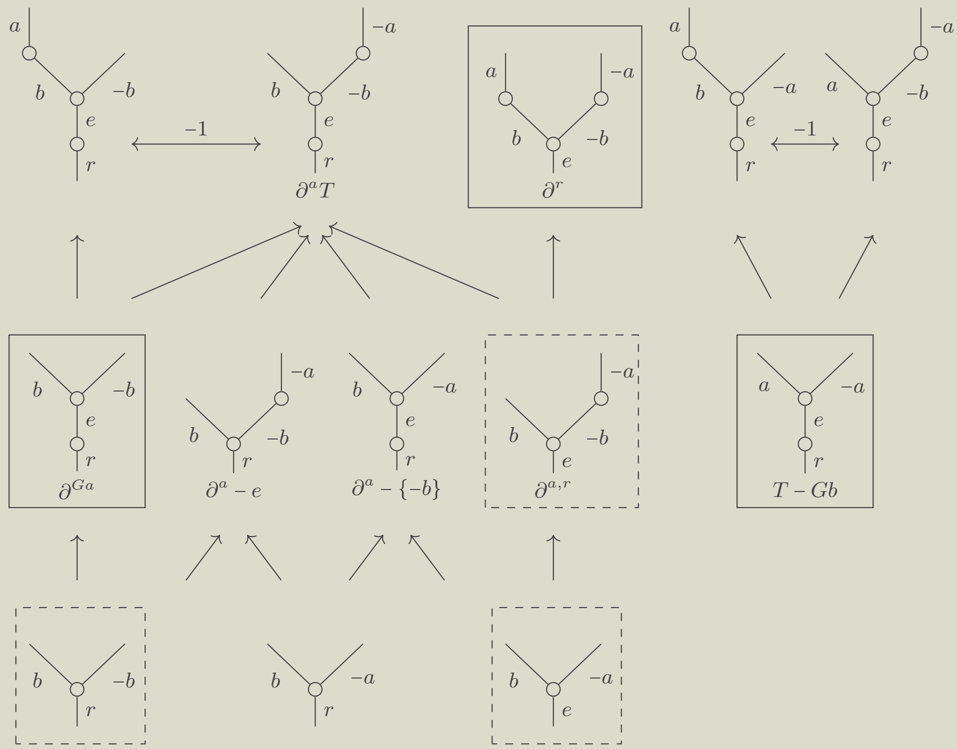
Example 2.31. Let $G = \mathbb{Z}_{/2} = \{\pm 1\}$, and consider the tree $T \in \Omega^G \subset \Omega_G$ below.



We note that, in this case, the subposet in $\mathsf{Face}_o(T)$ of faces with root orbit G/G (which includes all maximal faces) is isomorphic to a subposet of $\mathsf{Face}(T)$. Thus we may compare the horn and the orbital horn simply by comparing which elements of $\mathsf{Face}(T)$ each are missing.

Consider the subposet of $\mathsf{Face}(T)$ is displayed in Figure HORN_EX_FIG 2.31 below, where the orbital faces have been boxed. The horn $\Lambda^{Ge}[T]$ is only missing the faces T and $T - e$, and has as maximal faces the five trees in the top row. In contrast, the orbital horn $\Lambda^{Ge}_o[T]$ has as maximal faces those three trees in the first two rows which are solidly boxed. Figure HORN_EX_FIG 2.31 also includes a string of faces of the non-orbital face $\partial^a T$, and we highlight that each $S \hookrightarrow \partial^a T$ has at least one face contained in the orbital horn.

HORN_EX_FIG



EQINNERAN SEC

3 Equivariant inner anodyne maps

Much as in CM13a [CM13a, §2], it is essential for us to show that the inclusions $Sc[T] \to \Omega[T]$, $T \in \Omega_G$ are G-inner anodyne. In addition, some parts of the equivariant dendroidal story are naturally described in terms of orbital G-inner horns $\Lambda^E_o[T]$ (rather than G-inner horns $\Lambda^E[T]$), and one must hence also show that the inclusions $\Lambda^E_o[T] \to \Omega[T]$ are G-inner anodyne.

In practice, the proofs of such results are long and somewhat repetitive, as they share many technical arguments. Indeed, the case of orbital horns requires using many of the arguments in the long proof of [Per17, Thm 7.1]).

As such, we split our analysis into two parts. In §3.1 we prove Lemma 3.4 which we call the *characteristic edge lemma* and which abstractly identifies sufficient conditions for a map to be G-inner anodyne (see Remark 3.7 for a comparison with previous results in the literature). Then, in §3.2 we deduce that the desired maps are G-inner anodyne by applying Lemma 3.4.

3.1 The characteristic edge lemma

CHAREDGE SEC
CHAREDGE DEF

Definition 3.1. Let $T \in \Omega_G$, $A \subseteq \Omega[T]$ a subdendroidal set, and $\{U_i\}_{i\in I} \subseteq \mathsf{Face}(T)$ a subset.

Given a set Ξ^i of inner edges of U_i and a subface $V \hookrightarrow U_i$, denote $\Xi^i_V = \Xi^i \cap E^{\mathsf{i}}(V)$.

Suppose further that the indexing set I is a finite G-poset. For each $i \in I$ denote

$$A_{<i} = A \cup \bigcup_{j:j<i} \Omega[U_j]$$

We say that $\{\Xi^i \subseteq E^{\mathsf{i}}(U_i)\}$ is a *characteristic inner edge collection of* $\{U_i\}$ *with respect to* A if:

(Ch0) A, $\{U_i\}$ and $\{\Xi^i\}$ are all G-equivariant, i.e. $gA = A$, $gU_i = U_{gi}$, $g\Xi^i = \Xi^{gi}$ as appropriate;

(Ch1) for all i, any outer face $V = \bar{V}^{U_i}$ of U_i such that $\Xi^i_V = \varnothing$ is contained in $A_{<i}$;

(Ch2) for all i, any face $V \hookrightarrow U_i$ such that $(V - \Xi^i_V) \in A$ is contained in $A_{<i}$;

(Ch3) for all $j \not\geq i$, all faces $V \hookrightarrow U_i$ such that $(V - \Xi^i_V) \hookrightarrow U_j$ are contained in $A_{<i}$.

XIIII REM

Remark 3.2. If $gi \neq i$, then i, gi are incomparable in I. Indeed, otherwise $i < gi < g^2 i < g^3 i < \cdots$ would violate antisymmetry. Therefore, (Ch3) applies whenever $j = gi$ for $gi \neq i$.

In particular, we assume throughout that if $gi \neq i$ then $U_{gi} \neq U_i$, or else it would be $U_i \in A_{<i}$.

SOMEMAIN REM

Remark 3.3. In some of the main examples (see Propositions 3.12 and 3.14), there exists a G-equivariant set Ξ of inner edges of T such that $\Xi^i = \Xi \cap E^{\mathsf{i}}(U_i)$.

We caution that, for fixed A and $\{U_i\}$, our characteristic conditions are *not* monotone on such Ξ since increasing Ξ makes (Ch1) more permissive while making (Ch2),(Ch3) more restrictive.

CHAREDGE LEM

Lemma 3.4. *If* $\{\Xi^i\}_{i\in I}$ *is a characteristic inner edge collection of* $\{U_i\}_{i\in I}$ *with respect to* A, *then the map*

$$A \to A \cup \bigcup_{i\in I} \Omega[U_i] \qquad (3.5)$$ CHARLEM EQ

is G*-inner anodyne. In fact, it is cellular on* G*-inner horn inclusions* $\Lambda^E[S] \to \Omega[S]$, $S \in \Omega_G$.

Proof. We start with the case of $I \simeq G/H$ transitive so that, abbreviating $U = U_{[e]}$, $\{U_i\}$ is the set of conjugates gU. Note that H is also the isotropy of U in $\mathsf{Face}(T)$. We likewise abbreviate $\Xi = \Xi^{[e]}$ and $\Xi_V = \Xi^{[e]}_V$ for $V \hookrightarrow U$. Moreover, in this case one has $A_{<[g]} = A$ in (Ch1),(Ch2),(Ch3).

We write $\mathsf{Face}^{lex}_{\Xi}(U)$ for the H-poset of planar faces $V \hookrightarrow U$ such that $\Xi_V \neq \varnothing$ and $\Xi_V = \Xi_{\bar{V}}$ ordered as follows: $V \leq V'$ if either (i) $\bar{V} \hookrightarrow \bar{V}'$ and $\bar{V} \neq \bar{V}'$ or (ii) $\bar{V} = \bar{V}'$ and $V \hookrightarrow V'$ (alternatively, this is the lexicographic order of pairs $(\bar{V}, V)$). We note that here and in the remainder of the proof all outer closures are implicitly taken in U (rather than T), i.e. $\bar{V} = \bar{V}^U$.

For any H-equivariant convex subset C of $\mathsf{Face}^{lex}_{\Xi}(U)$ we write

$$A_C = A \cup \bigcup_{g \in G, V \in C} \Omega[gV].$$

It now suffices to show that whenever $C \subseteq C'$ the map $A_C \to A_{C'}$ is built cellularly from G-inner horn inclusions (indeed, setting $C = \varnothing$ and $C' = \mathsf{Face}^{lex}_{\Xi}(U)$ recovers (CHARLEM EQ 3.5) when $I \simeq G/H$).

Without loss of generality we can assume that C' is obtained from C by adding the H-orbit of a single $W \hookrightarrow U$. Further, we may assume $W \notin A_C$ or else $A_C = A_{C'}$. Letting $K \leq H$ denote the isotropy of W in $\mathsf{Face}^{lex}_{\Xi}(U)$ and regarding $W \in \Omega^K \subseteq \Omega_K$, we claim there is a pushout diagram

$$\begin{tikzcd}
G \cdot_K \Lambda^{\Xi_W}[W] \arrow[r] \arrow[d] & A_C \arrow[d] \\
G \cdot_K \Omega[W] \arrow[r] & A_{C'}
\end{tikzcd}$$

FIRPUSH EQ

where we note that inner edge set Ξ_W is K-equivariant since $\Xi_W = \Xi \cap E^{\mathrm{i}}(W)$ and Ξ is H-equivariant by (Ch0). The desired pushout will follow once we establish the following claims:

(a) all proper outer faces V of W are in A_C;

(b) an inner face $W - D$ of W is in A_C iff $D \notin \Xi_W$;

(c) the G-isotropy (i.e. the isotropy in $\mathsf{Face}(T)$) of faces $W - D$, $D \subseteq \Xi_W$ is contained in K.

To check (a), writing $\bar{V}$ for the corresponding outer face of U, one has

$$\Xi_V = \Xi \cap E^{\mathrm{i}}(V) = \Xi \cap E^{\mathrm{i}}(W) \cap E^{\mathrm{i}}(\bar{V}) = \Xi \cap E^{\mathrm{i}}(\bar{W}) \cap E^{\mathrm{i}}(\bar{V}) = \Xi \cap E^{\mathrm{i}}(\bar{V}) = \Xi_{\bar{V}}$$

where the second step follows from Lemma INNINT LEM 2.5 (applied to $V \hookrightarrow W \hookrightarrow U$, $V \hookrightarrow \bar{V} \hookrightarrow U$) and the third since by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ it is $\Xi_W = \Xi_{\bar{W}}$. Thus either $\Xi_V = \Xi_{\bar{V}} = \varnothing$ so that $\bar{V} \in A$ by (Ch1), or $\Xi_V = \Xi_{\bar{V}} \neq \varnothing$ so that $V \in \mathsf{Face}^{lex}_{\Xi}(U)$ with $V < W$, and thus $V \in C$. In either case one has $V \in A_C$.

We now check the "if" direction of (b). If $D \notin \Xi_W$ then $W' = W - (D \setminus \Xi_W)$ is in $\mathsf{Face}^{lex}_{\Xi}(U)$ (since $\bar{W}' = \bar{W}$ and $\Xi_{W'} = \Xi_W$) and $W' < W$, and thus $W' \in A_C$.

For the "only if" direction of (b), note first that it suffices to consider $D = \Xi_W$. The assumption $W \notin A_C$ together with (Ch2) imply that $W' = W - \Xi_W$ is not in A, and thus it remains to show that W' is not a face of any gV with $g \in G$, $V \in C$. Suppose otherwise, i.e. $W' \hookrightarrow gV$. If it were $g \notin H$, then it would be $W' \hookrightarrow gV \hookrightarrow gU \neq U$, and (Ch3) would imply $W \in A$. Thus we need only consider $g \in H$, and since C is H-equivariant, we can set $g = e$. It now suffices to show that if $W' \hookrightarrow V$ then it must be $W \leq V$ in $\mathsf{Face}^{lex}_{\Xi}(U)$, since by convexity of C this would contradict $W \notin C$. Since $W' \hookrightarrow V$ implies $\bar{W} = \bar{W}' \hookrightarrow \bar{V}$, the condition $W \leq V$ is automatic from the definition of $\leq$ unless $\bar{W} = \bar{V}$. In this latter case, by definition of $\mathsf{Face}^{lex}_{\Xi}(U)$ the face V must contain as inner edges all edges in $\Xi_V = \Xi_{\bar{V}} = \Xi_{\bar{W}} = \Xi_W$, so that not only $W - \Xi_W = W' \hookrightarrow V$ but also $W \hookrightarrow V$. But then it is $W \leq V$ in either case, establishing the desired contradiction.

We now show (c). If $g(W - D) = W - D$ then $g(W - \Xi_W) \hookrightarrow U$, and thus $W - \Xi_W \hookrightarrow g^{-1}U$, so that by (Ch3) it must be $g \in H$ or else it would be $W \in A$. Now suppose $h(W - D) = W - D$ with $h \in H$. Since Ξ is H-equivariant (by (Ch0)) and $\Xi_{W-D} = \Xi_W \setminus D$ (due to $D \subseteq \Xi_W$) it follows that $h(W - \Xi_W) = W - \Xi_W$, so that we may assume $D = \Xi_W$. Now note that hW, $h(W - \Xi_W) = W - \Xi_W$, W are all faces of U with a common outer closure $\bar{W}$. Hence $h\Xi_W = \Xi_{hW} \subseteq \Xi_{\bar{W}} = \Xi_W$, where the last step follows since $W \in \mathsf{Face}^{lex}_{\Xi}(U)$, and by cardinality reasons it must in fact be $h\Xi_W = \Xi_W$. But then hW, W have the same outer closure and the same inner edges, and thus $hW = W$, establishing (c).

Lastly, we address the case of general I. For each G-equivariant convex subset J of I, set

$$A_J = A \cup \bigcup_{j \in J} \Omega[U_j].$$

As before, it suffices to check that for all convex subsets $J \subseteq J'$ the map $A_J \to A_{J'}$ is built cellularly from G-inner horns, and again we can assume that J' is obtained from J by adding a single G-orbit Gj of I. By the I transitive case, it now suffices to check that $\{\Xi^{gj}\}_{gj \in Gj}$ is also a characteristic inner edge collection of $\{U_{gj}\}_{gj \in Gj}$ with respect to A_J. (Ch0) is clear, and since by G-equivariance and convexity it is $A_{<gj} \subseteq A_J$, the new (Ch1),(Ch2),(Ch3) conditions follow from the original conditions. □

CHAREDGE2 REM **Remark 3.6.** The requirement $A \subseteq \Omega[T]$ in Definition 3.1 can be relaxed. Given an inclusion $A \subseteq B$, a set of non-degenerate dendrices $\{b_i \in B(U_i)\}_{i \in I}$ and a collection of edges $\{\Xi^i \subset E^{\mathrm{i}}(U_i)\}_{i \in I}$, suppose that I is a finite G-poset and that:

(Ch0.1) the maps $b_i \colon \Omega[U_i] \to B$ are monomorphisms;

(Ch0.2) A, $\{U_i\}$,$\{b_i\}$ and $\{\Xi^i\}$ are all G-equivariant in the sense that: (i) $gA = A$; (ii) there are associative and unital isomorphisms $U_i \xrightarrow{g} U_{g_i}$; (iii) the composites $\Omega[U_i] \xrightarrow{b_i} Y \xrightarrow{g} Y$ and $\Omega[U_i] \xrightarrow{g} \Omega[U_{gi}] \xrightarrow{b_{gi}} Y$ coincide; (iv) $g\Xi^i = \Xi^{gi}$.

Under (Ch0.1), the $\Omega[U_i]$ are identified with subcomplexes of B, and non-degenerate dendrices $b \in b_i(\Omega[U_i])(V)$ are identified with faces $V \hookrightarrow U_i$.

The original conditions (Ch1),(Ch2),(Ch3) can then be reinterpreted by, for each $V \hookrightarrow U_i$, regarding expressions such as $V \in A$, $(V - \Xi^i_V) \in U_j$ as $b_i(V) \in A$, $b_i(V - \Xi^i_V) \in b_j(\Omega[U_j])$.

The proof of Lemma 3.1 now carries through to show that

$$A \to A \cup \bigcup_{i \in I} b_i(\Omega[U_i])$$

is G-inner anodyne (again built cellularly from G-inner horn inclusions).

RECOVER REM **Remark 3.7.** Lemma 3.4 readily recovers several arguments in the literature:

(i) In [Rez01, §10] (also [Rez10, §6.2]), Rezk introduces the notion of *covers*, which in our language are the subsets $Sc[n] \subseteq A \subseteq \Delta[n]$ such that if V is in A then so is the closure $\bar{V}^{[n]}$ (in words, A is generated by outer faces). Similarly, in the proof of [CM13a, Prop. 2.4] Cisinski and Moerdijk use subcomplexes S_j that can be regarded as *dendroidal covers*, i.e. subcomplexes $Sc[T] \subseteq A \subseteq \Omega[T]$ such that if V is in A then so is $\bar{V}^T$. Lastly, the subcomplexes $\Omega[T] \cup_l \Omega[S] \subseteq \Omega[T \circ_l S]$ in the grafting result [MW09, Lemma 5.2] (and similarly for the equivariant analogue [Per17, Prop. 6.19]) are also dendroidal covers.

Lemma 3.4 implies that any inclusion $A \to A'$ of G-equivariant (dendroidal) covers of $T \in \Omega_G$ is G-inner anodyne. Indeed, let $I = \mathsf{Face}^{out}_{A'}(T)$ be the G-poset of outer faces $V \hookrightarrow T$ contained in A', ordered by inclusion, $\Xi = E^{\mathrm{i}}(T)$ and $U_V = V$. (Ch0) is clear, (Ch1) follows since $Sc(T) \subseteq A$, (Ch2) follows since A is a cover and (Ch3) follows since the U_i are closed.

Alternatively, one can also use $I = \mathsf{Face}^{out}_{A',o}(T)$ for the G-trivial set of orbital outer faces $GV \hookrightarrow T$, together with an *arbitrary* total order (see Remark 3.15 for a similar example).

Lastly, note that in the special case $\{U_i\} = \{T\}$, $\Xi = E^{\mathrm{i}}(T)$, (Ch1) says precisely $Sc[T] \subseteq A$.

(ii) In [MW09, Lemma 9.7], Moerdijk and Weiss introduced a *characteristic edge* condition that can be regarded as a special case of our characteristic edge collection condition as generalized in Remark 3.6, and which served as one of our main inspirations.

Therein, they work in the case of $B = \Omega[T] \otimes \Omega[S]$ a tensor product of (non-equivariant) representable dendroidal sets, in which case (Ch0.1) is easily verified (and (Ch0.2) is moot). In our notation, they then require that $I \simeq *$ (so that (Ch3) is also moot), the dendrex $b_* \in (\Omega[T] \otimes \Omega[S])(U_*)$ encodes a special type of subtree U_* of $\Omega[T] \otimes \Omega[S]$, which they call an *initial segment*, and they further require that $\Xi^* = \{\xi\}$ is a singleton, called the *characteristic edge*. Moreover, they then demand that A should contain all outer faces of the subtree U_*, from which (Ch1) follows, as well as the key characteristic condition [MW09, Lemma 9.7](ii), which coincides with (Ch2) in this specific setting.

Similarly, in [Per17, Lemma 7.39] the second author introduced a *characteristic edge orbit* condition that generalizes that in [MW09] to the equivariant context by letting $I \simeq G/H$ and the $\Xi^{[g]} = \Xi \cap E^{\mathrm{i}}(U_{[g]})$ be determined by a G-edge orbit $\Xi \simeq Gf$ (cf. Remark 3.3).

However, both of the lemmas in [MW09] and [Per17] have the drawback of needing to be used iteratively (so that much effort therein is spent showing that this can be done) while Lemma 3.4 is designed so that a single use suffices for the natural applications. Indeed, conditions (Ch1) and (Ch3), the first of which relaxes the requirement in [MW09],[Per17] that A should contain all outer faces, essentially provide abstract conditions under which the original characteristic edge arguments of [MW09],[Per17] can be iterated.

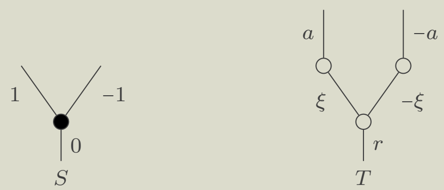
THM71 EX **Example 3.8.** As indicated above, Lemma 3.4 can be used to reorganize and streamline the rather long proofs of [Per17, Thms 7.1 and 7.2]. We illustrate this in the hardest case, that of [Per17, Thm. 7.1(i)], which states that if $S, T \in \Omega_G$ are open (i.e. have no stumps) and $G\xi$ is an inner edge orbit of T the maps

$$\partial\Omega[S] \otimes \Omega[T] \coprod_{\partial\Omega[S]\otimes\Lambda^{G\xi}[T]} \Omega[S] \otimes \Lambda^{G\xi}[T] \to \Omega[S] \otimes \Omega[T] \tag{3.9}$$ THM71 EQ

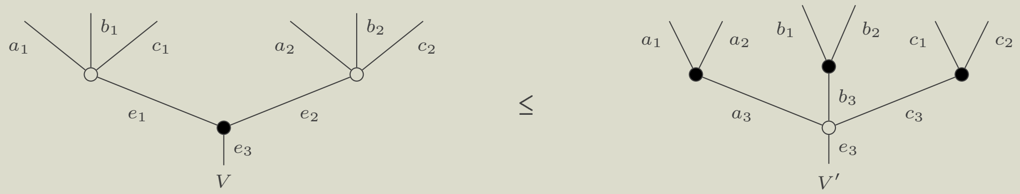
are G-inner anodyne.

Given $S, T \in \Omega_G$, it is possible [Per17, §7.1] to define a G-equivariant broad poset $S \otimes T$ so that $(\Omega[S] \otimes \Omega[T])(V) = Hom(V, S\otimes T)$ where the Hom set is taken in broad posets. Intuitively $S\otimes T$ is an object with edge set $E(S)\times E(T)$ and where each edge $(s,t) \in S\otimes T$ may, depending on whether $s \in S$, $t \in T$ are leaves or not, admit two *distinct* vertices: a S-vertex $(s,t)^{\uparrow S} = s^{\uparrow}\times t \leq (s,t)$ and a T-vertex $(s,t)^{\uparrow T} = s \times t^{\uparrow} \leq (s,t)$.

To recover [Per17, Thm. 7.1(i)] from Lemma 3.4, we first let $I = \mathsf{Max}(S \otimes T)$ be the G-poset of maximal subtrees $U \hookrightarrow S \otimes T$ (these are called *percolation schemes* in [MW09, §9]), ordered lexicographically [Per17, Def. 7.29]. As an example, let $\mathbb{Z}_{/2} = \{\pm 1\}$ and consider the $\mathbb{Z}_{/2}$-trees



We depict the $\mathbb{Z}_{/2}$-poset $\mathsf{Max}(S\otimes T)$ in Figure 3.1 (note that (s,t) is abbreviated as t_s). In words, the maximal subtrees are built by starting with the "double root" r_0 and iteratively choosing between the available S and T vertices (along all upward paths) until the "double leaves" are reached. The generating relations $U \leq U'$ in $\mathsf{Max}(S \otimes T)$ occur whenever U contains an outer face V shaped as on the left below and, by "replacing" V with V' as on the right, one obtains U'.



(3.10) GENLEXREL EQ

The claim that $\leq$ is indeed a partial order (at least if one of S, T is open) is [Per17, Prop. 7.31]. As an aside, we note that V, V' above have a common inner face $V - \{e_1, e_2\} = V' - \{a_3, b_3, c_3\}$, which encodes an (universal!) example of a Boardman-Vogt relation (see [MW07, §5.1]).

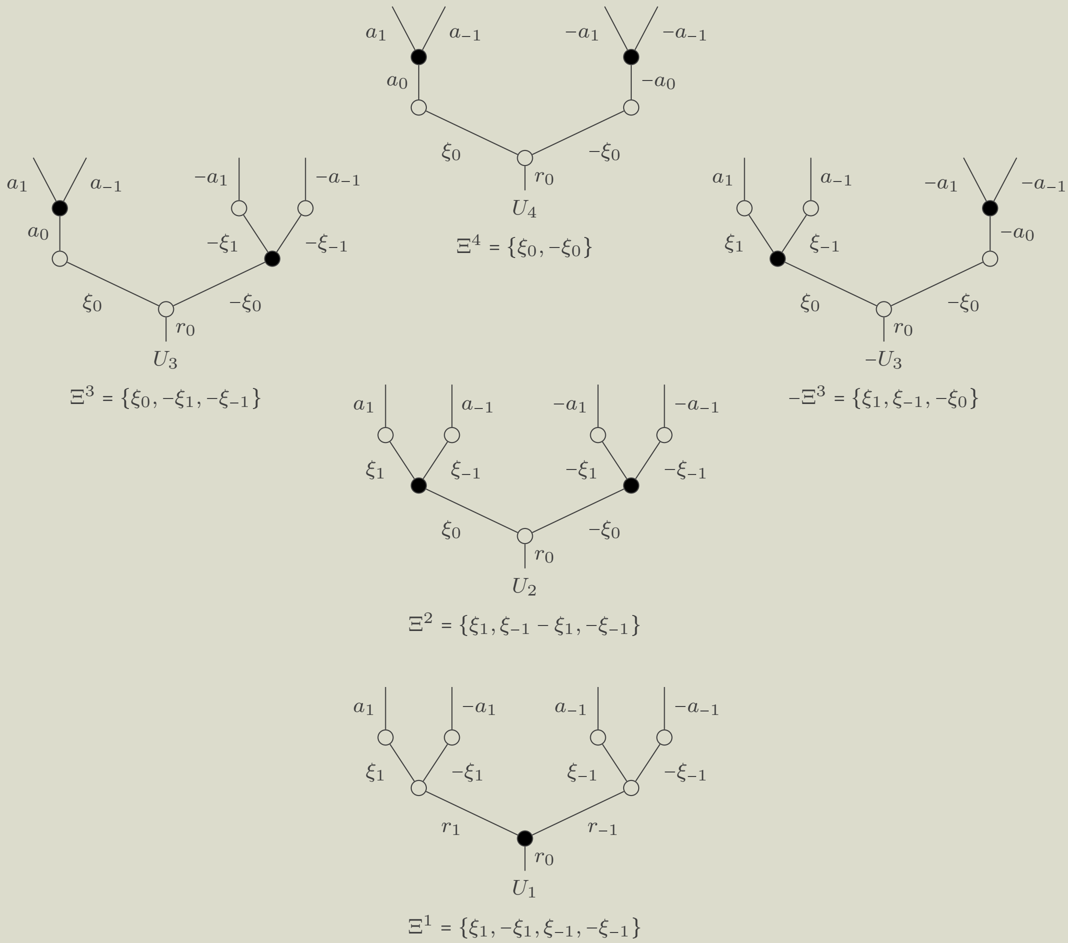


Figure 3.1: The $\mathbb{Z}_{/2}$-poset $\mathsf{Max}(S \otimes T)$ and characteristic edges Ξ^i FIGURE

Returning to the task of proving that (3.9) is G-inner anodyne, we define Ξ^U, for each maximal subtree $U \hookrightarrow S \otimes T$, to be the set of inner edges of U of the form $(g\xi)_s$ *such that* the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ in U is a T-vertex (see Figure 3.1). We now verify (Ch1),(Ch2),(Ch3). We recall that, since S, T are assumed open, [Per17, Lemma 7.19] guarantees that, for faces $S' \hookrightarrow S$, $T' \hookrightarrow T$, a factorization $V \hookrightarrow S' \otimes T' \hookrightarrow S \otimes T$ exists iff the edges of V are in $E(S') \times E(T')$.

For (Ch1), note first that there is an equivariant grafting decomposition $T = T_{\not\leq G\xi} \amalg_{G\xi} T^{\leq G\xi}$, where $T_{\not\leq G\xi}$ contains the edges $t \in T$ such that $\forall_{g\in G} t \not\leq g\xi$ (pictorially, this is a lower equivariant outer face of T) while $T^{\leq G\xi}$ contains the edges $t \in T$ such that $\exists_{g\in G} t \leq g\xi$ (an upper equivariant outer face of T). But one now readily checks that if $V \hookrightarrow U$ is an *outer* face such that $\Xi^U_V = \varnothing$, then either $V \hookrightarrow S \otimes T_{\not\leq G\xi}$ or $V \hookrightarrow S \otimes T^{\leq G\xi}$, and thus $V \in A$.

For (Ch3), suppose $U_j \not\leq U_i$, $V \hookrightarrow U_i$ and $(V - \Xi^{U_i}_V) \hookrightarrow U_j$. Then it follows from [Per17, Lemma 7.37] that there exists a generating relation $U_k < U_i$ such that $(V - \Xi^{U_i}_V) \hookrightarrow U_k$ (indeed, [Per17, Lemma 7.37] makes the slightly stronger claim that such a relation can be performed on the outer closure $\bar{V}^{U_i}$). But then, as one sees from (3.10), all edges $e \in U_i$ that are not in U_k are topped by the S-vertex $e^{\uparrow S} \leq e$, and thus it is $e \notin \Xi^{U_i}$. Therefore $V \hookrightarrow U_k$, as desired.

Lastly, for (Ch2), suppose $V \hookrightarrow U$ and $(V - \Xi^U_V) \in A$. If it were $(V - \Xi^U_V) \hookrightarrow S \otimes \Lambda^{G\xi}[T]$, then it would also be $V \hookrightarrow S \otimes \Lambda^{G\xi}[T]$ since all edges of Ξ^U_V have T-coordinate in $G\xi$. Now consider the more interesting case $(V - \Xi^U_V) \hookrightarrow S' \otimes T$ for some face $S' \hookrightarrow S$. Then it will also be $V \hookrightarrow S' \otimes T$ unless there is at least one edge $(g\xi)_s \in \Xi^U_V$ such that $s \notin S'$. But then since the outer closure $\bar{V}^U$ can have no leaf with S-coordinate s (this would contradict $s \notin S'$), there exists some minimal outer face $U^{<s}_{(g\xi)_s}$ of U with root $(g\xi_s)$ and such that its leaves have S-coordinate $<_d s$. By minimality, one has that $U^{<s}_{(g\xi)_s} \hookrightarrow \bar{V}^U$ and that all inner edges of $U^{<s}_{(g\xi)_s}$ have S-coordinate s. Further, note that $U^{<s}_{(g\xi)_s}$ has at least one inner edge (since by definition of Ξ^U the vertex $(g\xi)_s^{\uparrow U} \leq (g\xi)_s$ is a T-vertex) and that V contains none of those inner edges (or else it would be $s \in S'$). Thus by applying [Per17, Lemma 7.34] to $U^{<s}_{(g\xi)_s}$ one obtains a maximal subtree $U' < U$ containing all edges of U that are not inner edges of $U^{<s}_{(g\xi)_s}$. But then $V \hookrightarrow U'$ and (Ch2) follows.

Remark 3.11. We briefly outline how to modify the previous example to prove [Per17, Thm 7.1(ii)], in which case some notable subtleties arise. The result again states that (3.9) is G-inner anodyne, but now with one of S, T allowed to have stumps while the other is required to be linear.

One again sets $I = \mathsf{Max}(S \otimes T)$, where maximal trees are defined just as before, but some care is now needed. To see why, note that if the black nodes $\bullet$ in (3.10) are replaced with stumps then V' is actually a subtree of V.

When S has stumps and T is linear this causes no issues and the proof above holds (though we note that it can now be $\Xi^U = \varnothing$, in which case the argument for (Ch1) shows $U \in A$).

However, when S is linear and T has stumps the proof above breaks down (more precisely, the tree $U^{<s}_{(g\xi)_s}$ that appears when arguing (Ch2) may now fail to have inner edges). The solution is then to *reverse* the poset structure on $\mathsf{Max}(S \otimes T)$ and to modify the Ξ^U to be those inner edges $(g\xi)_s$ such that $(g_x i)_s \in t_s^{\uparrow T}$ for some t_s (pictorially, this says that these are the lowermost edges with T-coordinate in $G\xi$, whereas before they were the uppermost ones). The arguments for (Ch1),(Ch3) then hold. For (Ch2), only the argument for the interesting case of $V - \Xi^U_V \hookrightarrow S' \otimes T$, $s \notin S'$ changes. In this case, there is then a maximal edge t'_s such that $(g\xi)_s < t'_s$, where s can not be the root of S (or else it would be $s \in S'$). Pictorially, t'_s looks like the edge $e_1 \in V$ in (3.10) in the case where the $\bullet$ node is unary (since S is assumed linear). But then since V can not contain t'_s there exists a maximal subtree $U' > U$ such that $V \hookrightarrow U'$, and (Ch2) follows.

Lastly, we note that [Per17, Thm. 7.2] follows from a minor variant of the argument for [Per17, Thm. 7.1(ii)] when S is linear.

3.2 Segal cores, horns and orbital horns

HYPERSAT SEC

ORB_HORN_PROP **Proposition 3.12.** *For G-subsets $\varnothing \neq F \subseteq E \subseteq \boldsymbol{E}^{\mathsf{i}}(T)$ the inclusions*

$$\Lambda_o^E[T] \to \Omega[T], \qquad \Lambda_o^E[T] \to \Lambda_o^F[T] \tag{3.13}$$

ORBHORNINC EQ

are G-inner anodyne.

Proof. We are free to assume that $T \in \Omega^G \subseteq \Omega_G$. Indeed, otherwise writing $T = G \cdot_H T_*$ where $T_* \in \Omega^H$ is a fixed component and $E_* = E \cap \boldsymbol{E}^{\mathsf{i}}(T_*)$, $F_* = F \cap \boldsymbol{E}^{\mathsf{i}}(T_*)$, the maps in ORBHORNINC EQ (3.13) are $G \cdot_H \left(\Lambda_o^{E_*}[T_*] \to \Omega[T_*]\right)$ and $G \cdot_H \left(\Lambda_o^{E_*}[T_*] \to \Lambda_o^{F_*}[T_*]\right)$.

In the $\Lambda_o^E[T] \to \Omega[T]$ case we apply Lemma CHAREDGE LEM 3.4 with $I = \{*\}$ a singleton and

$$\Xi^* = E, \qquad U_* = T, \qquad A = \Lambda_o^E[T].$$

It remains to check the characteristic conditions in Definition CHAREDGE DEF 3.1. (Ch0) and (Ch3) are clear.

Note that for $V \hookrightarrow T$ it is $V \notin A$ iff $GV = T - E'$ for some G-subset $E' \subseteq E$.

For (Ch1), the condition $\Xi_V = \varnothing$ says that none of the inner edges of V are in E, and thus that the orbital outer face GV contains none of the edge orbits in E as inner edge orbits. Since $E \neq \varnothing$, the orbital outer face GV is not T itself, and hence $A = \Lambda_o^E[T]$ contains V.

For (Ch2), note that if $V \notin A$, i.e., $GV = T - E'$, then Remark GINNER REM 2.22 implies that $G(V - \Xi_V) = T - E''$ for $E' \subseteq E'' \subseteq E$, and thus also $(V - \Xi_V) \notin A$.

In the $\Lambda_o^E[T] \to \Lambda_o^F[T]$ case we instead apply Lemma CHAREDGE LEM 3.4 with $I = (E \smallsetminus F)/G$, with an *arbitrary* choice of total order, and (writing elements of $(E \smallsetminus F)/G$ as orbits $Ge \subseteq E \smallsetminus F$)

$$\Xi^{Ge} = F, \qquad U_{Ge} = T - Ge, \qquad A = \Lambda_o^E[T].$$

Note that the U_{Ge} are the orbital inner faces $T - Ge$ for $Ge \subseteq E \smallsetminus F$, and thus the map in Lemma CHAREDGE LEM 3.4 is indeed $\Lambda_o^E[T] \to \Lambda_o^F[T]$. Further, we are free to abbreviate $\Xi = \Xi^{Ge}$ and $\Xi_V = \Xi_V^{Ge}$, since Ξ^{Ge} is independent of Ge. We again check the characteristic conditions. (Ch0) is clear.

For (Ch1), note that for an outer face $V \hookrightarrow U_i$, and writing $\bar{V} = \bar{V}^T$, Lemma INNINT LEM 2.5 implies $\boldsymbol{E}^{\mathsf{i}}(V) = \boldsymbol{E}^{\mathsf{i}}(U_i) \cap \boldsymbol{E}^{\mathsf{i}}(\bar{V})$ and hence since $\Xi_{U_i} = F = \Xi$ the hypothesis $\Xi_V = \varnothing$ in (Ch1) implies it is also $\Xi_{\bar{V}} = \varnothing$. Hence just as before $G\bar{V}$ is an orbital outer face other than T, hence V is in $A = \Lambda_o^E[T]$. The argument for (Ch2) is identical to the one in the $\Lambda_o^E[T] \to \Omega[T]$ case. Lastly, (Ch3) follows since if $V \notin A$, so that $GV = T - E'$ and $G(V - \Xi_V) = T - E' - F'$ with $E' \subseteq E$, $F' \subseteq F$, then $GV \hookrightarrow T - Ge$ iff $G(V - \Xi_V) \hookrightarrow T - Ge$ and thus $V \hookrightarrow T - Ge$ iff $V - \Xi_V \hookrightarrow T - Ge$. $\square$

REG_HORN_PROP **Proposition 3.14.** *For G-equivariant $\varnothing \neq F \subseteq E \subseteq \boldsymbol{E}^{\mathsf{i}}(T)$ the inclusions*

$$\Lambda^E[T] \to \Lambda^F[T]$$

are G-inner anodyne.

Proof. We now apply Lemma CHAREDGE LEM 3.4 with $I = \mathcal{P}_0(E \smallsetminus F)$ the poset of non-empty subsets $\varnothing \neq E' \subseteq (E \smallsetminus F)$, ordered by *reverse inclusion*, and

$$\Xi^{E'} = F, \qquad U_{E'} = T - E', \qquad A = \Lambda^E[T].$$

We again need to verify the characteristic conditions, and as in the previous result we abbreviate $\Xi = \Xi^{E'}$, $\Xi_V = \Xi_V^{E'}$. (Ch0) is clear. (Ch1) follows from an easier version of the argument in the previous proof. (Ch2) follows since $V \in A$ iff $V - \Xi_V \in A$. Similarly, (Ch3) follows since $V \hookrightarrow T - E'$ iff $(V - \Xi_V) \hookrightarrow T - E'$ and since if $V \hookrightarrow T - E', V \hookrightarrow T - E''$ then $V \hookrightarrow T - (E' \cup E'')$. $\square$

TWOPROOF REM **Remark 3.15.** By specifying to the non-equivariant case $G = *$ the previous results yield two distinct proofs that inclusions of non-equivariant horns $\Lambda^E[T] \to \Lambda^F[T]$ are inner anodyne, with the first proof using $I = E \smallsetminus F$ (with any total order) and the second using $I = \mathcal{P}_0(E \smallsetminus F)$.

The discrepancy is explained as follows: when T, E, F are G-equivariant, showing that $\Lambda^E[T] \to \Lambda^F[T]$ is G-inner anodyne requires a control of isotropies not needed when showing that the underlying map is non-equivariant inner anodyne, and since this control is given by (Ch3), it is necessary to include in the $\{U_i\}$ the "intersections" of $T - e$ and $T - ge$ for $e \in E \smallsetminus F$.

FACCES REM **Remark 3.16.** All G-inner horn inclusions attached in the proof of the characteristic edge lemma, Lemma 3.4, correspond to G-trees whose non-equivariant components are faces of the U_i. Moreover, when I has a transitive G-action, the last horn inclusion attached (corresponding to the maximum of $\mathsf{Face}^{lex}_{\Xi}(U)$) is $G \cdot_H \left(\Lambda^{\Xi}[U] \to \Omega[U]\right)$.

REGGENHORN COR **Corollary 3.17.** *G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ are built cellularly from generating horn inclusions $\Lambda^{Ge}[S] \to \Omega[S]$.*

Proof. The proof is by induction on $|T_*|$ for $T_* \in \Omega$ a tree component (cf. Remark 2.4). As before one is free to assume $T \in \Omega^G$. A choice of edge orbit Ge in E yields a factorization $\Lambda^E[T] \to \Lambda^{Ge}[T] \to \Omega[T]$, hence we need only show that $\Lambda^E[T] \to \Lambda^{Ge}[T]$ is built cellularly from generating horns. But this is immediate from the induction hypothesis, Remark 3.16, and the proof of Proposition 3.14 since all U_i therein satisfy $|U_i| < |T|$. □

Following the discussion preceding [HHM16, Prop. 3.6.8], a class of normal monomorphisms of dSet^G (or, more generally, a subclass of the cofibrations in a model category) is called *hypersaturated* if it is closed under pushouts, transfinite composition, retracts, as well as the following additional cancellation property: if f, g are normal monomorphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \qquad \text{CANCEL\_EQ}$$

such that both f and gf are in the class, then so is g.

The following is an equivariant generalization of [CM13a, Props. 2.4 and 2.5].

HYPER PROP **Proposition 3.18.** *The following sets of maps generate the same hypersaturated class:*

- *the G-inner horn inclusions $\Lambda^E[T] \to \Omega[T]$ for $T \in \Omega_G$ and G-subset $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$;*
- *the orbital G-inner horn inclusions $\Lambda^E_o[T] \to \Omega[T]$ for $T \in \Omega_G$ and G-subset $\varnothing \neq E \subseteq E^{\mathsf{i}}(T)$;*
- *the G-Segal core inclusions $Sc[T] \to \Omega[T]$ for $T \in \Omega_G$.*

In the following proof we refer to the hypersaturation of the orbital horn (resp. Segal core) inclusions as the orbital (resp. Segal) hypersaturation.

Proof. The fact that G-inner horn inclusions generate the orbital and Segal hypersaturations has been established in Proposition 3.12 and Remark 3.7(i).

To see that the G-inner horn inclusions are in the orbital hypersaturation, we again argue by induction on $|T_*|$, with the base cases those where $\Lambda^E[T] = \Lambda^E_o[T]$. Recalling that in the proof of Proposition 3.12 one sets $I = *$, $U_* = T$ and $\Xi^* = E$, Remark 3.16 implies that in the factorization $\Lambda^E_o[T] \to \Lambda^E F[T] \to \Omega[T]$ the first map $\Lambda^E_o[T] \to \Lambda^E F[T]$ is built cellularly out of G-horns with $|S_*| < |T_*|$. But then the induction hypothesis says that $\Lambda^E_o[T] \to \Lambda^E[T]$ is in the orbital hypersaturation, and by the cancellation property so is $\Lambda^E[T] \to \Omega[T]$.

For the claim that the G-inner inclusions are in the Segal hypersaturation, note that $Sc[T] \to \Omega[T]$ can be shown to be G-inner anodyne by setting $I = *$, $U_* = T$, $\Xi^* = E^{\mathsf{i}}(T)$ (this differs from

Remark 3.7(i), but the arguments therein still hold). Therefore, arguing exactly as above for the factorization $Sc[T] \to \Lambda^{E^{\mathrm{i}}(T)}[T] \to \Omega[T]$, one obtains by induction on $|T_*|$ that $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Omega[T]$ is in the Segal hypersaturation. But now letting $E \subseteq E^{\mathrm{i}}(T)$ be any G-subset and considering the factorization $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Lambda^{E}[T] \to \Omega[T]$ the induction hypothesis applies to the cells of $\Lambda^{E^{\mathrm{i}}(T)}[T] \to \Lambda^{E}[T]$ (just as in Corollary 3.17), which is thus also in the Segal hypersaturation. But by the cancellation property, so is $\Lambda^{E}[T] \to \Omega[T]$, finishing the proof. $\square$

Remark 3.19. The identification between orbital subcomplexes $\bigcup_i \Omega[S_i] \subseteq \Omega[T]$ and subcomplexes of $\bigcup_i \Omega[S_i/G] \subseteq \Omega[T/G]$ described in Remark 2.30 is compatible with attaching horn inclusions. As such, non-equivariant results concerning horns in Ω imply the analogue results for orbital horns in Ω_G. For example, mimicking [MW09, Lemma 5.1], one has pushouts

$$\begin{tikzcd}
\Lambda_o^{E-Ge}[T-Ge] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Lambda_o^{E}[T] \arrow[d] \\
\Omega[T-Ge] \arrow[r] & \Lambda_o^{E-Ge}[T]
\end{tikzcd} \tag{3.20}$$

ORBHORNPUSH EQ

which imply the orbital horn analogue of Corollary 3.17. It is worth noting that while setting $G = *$ in Corollary 3.17 does recover [MW09, Lemma 5.1], the analogue of the pushouts (3.20) does not hold for (non-orbital) G-inner horns, so that the proof of Corollary 3.17 (see also the original proof in [Per17, Prop. 6.17]) is intrinsically harder when $G \neq *$, due to isotropy concerns.

Similarly, [CM13a, Props. 2.4 and 2.5] (or Remark 3.7(i) and Proposition 3.18 when $G = *$) imply that the Segal core inclusions $Sc[T] \to \Omega[T]$ are built cellularly from the orbital horn inclusions $\Lambda_o^{E}[T] \to \Omega[T]$, and that the two classes have the same hypersaturation.

Indeed, this observation indicates an alternate route to the proof of Proposition 3.18 (which the authors considered in early versions of this work) without making direct use of the characteristic edge lemma machinery. Namely, following the considerations above, the main missing claim is the first part of Proposition 3.12, stating that the inclusions $\Lambda_o^{E}[T] \to \Omega[T]$ are G-inner anodyne. This latter claim is not too hard to prove directly. Indeed, while the proof does require some of the ideas in the proof of Lemma 3.4, many of the subtler arguments in the proof of Lemma 3.4 become trivial when $I = *$ is a singleton, as is the case in Proposition 3.12.

We end this section with some necessary remarks about hypersaturations of simplicial horns.

SLICE REM **Remark 3.21.** Setting $G = e$ and restricting to the overcategory $\mathsf{dSet}_{/\eta} \simeq \mathsf{sSet}$, Proposition 3.18 recovers the well known claim that the hypersaturation of the simplicial inner horn inclusions $\{\Lambda^i[m] \to \Delta[m] : 0 < i < m\}$ coincides with the hypersaturation of the simplicial Segal core inclusions $\{Sc[m] \to \Delta[m] : m \geq 0\}$.

HYPERSATKAN REM **Remark 3.22.** We will use of a variant of the previous remark for the hypersaturation of *all* simplicial horns. Namely, we claim that the hypersaturation of all simplicial horns $\{\Lambda^i[m] \to \Delta[m] : 0 \leq i \leq m, 0 < m\}$ matches the hypersaturation of all vertex inclusion maps $\{\Delta[0] \to \Delta[m]\}$.

Call the latter hypersaturation S. One easily checks that the maps $\{0\} \to Sc[m]$ are in S, so that by cancellation so are the maps $Sc[m] \to \Delta[m]$ and hence by Remark 3.21 so are all inner horn inclusions. Moreover, for left horns $\Lambda^0[m]$ the maps $\{0\} \to \Lambda^0[m]$ are built cellularly from left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k < m$ (in join notation (see [Lur09, §1.2.8] or [Per17, §7.4]), $\{0\} \to \Lambda^0[m]$ is $\Delta[0] \star (\varnothing \to \partial\Delta[m-1])$, and the filtration follows from the cellular filtration of $\partial\Delta[m-1]$). But hence by induction and the cancellation property all left horn inclusions $\Lambda^0[m] \to \Delta[m]$ are in S. The case of right horn inclusions $\Lambda^m[m] \to \Delta[m]$ is dual.

ANHYPER REM

Remark 3.23. The smallest hypersaturated class containing the inner horn inclusions and the left horn inclusion $\Lambda^0[2] \to \Delta[2]$ in fact contains all left horn inclusions $\Lambda^0[m] \to \Delta[m]$ for $m \geq 2$. Indeed, this follows inductively from the left diagram below since the bottom map is inner while the top and left maps are given by the center and right pushout diagrams.

$$\begin{tikzcd} \Lambda^{0,1}[m] \arrow[r] \arrow[d] & \Lambda^0[m] \arrow[d] \\ \Lambda^1[m] \arrow[r] & \Delta[m] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^0[m-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Lambda^{0,1}[m] \arrow[d] \\ \Delta[m-1] \arrow[r, "d^1"'] & \Lambda^0[m] \end{tikzcd} \qquad \begin{tikzcd} \Lambda^0[m-1] \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Lambda^{0,1}[m] \arrow[d] \\ \Delta[m-1] \arrow[r, "d^0"'] & \Lambda^1[m] \end{tikzcd}$$

The case of right horn inclusions is dual.

CONTGR REM

Remark 3.24. Write $\widetilde{[m]} = (0 \rightleftarrows 1 \rightleftarrows \cdots \rightleftarrows m)$ for the contractible groupoid on objects $0, 1, \cdots, m$. Note that the k-simplices of $\widetilde{[m]}$ are encoded as strings $a_0 a_1 \cdots a_k$ with $a_i \in \{0, 1, \cdots, m\}$, and that a simplex is non-degenerate iff $a_{i-1} \neq a_i$, $1 \leq i \leq k$. We claim that the maps

$$\Delta[n] = N[m] \xrightarrow{012\cdots m} N\widetilde{[m]}, \quad m \geq 1 \tag{3.25}$$ INVER EQ

are built cellularly out of left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 2$.

Indeed, we show a little more. We say a subcomplex $A \subseteq N\widetilde{[m]}$ is 0-*stable* if a m-simplex $\underline{a}$ is in A iff the $(m+1)$-simplex $0\underline{a}$ is. We claim that any inclusion $A \to A'$ of 0-stable subcomplexes is built cellularly from left horn inclusions $\Lambda^0[k] \to \Delta[k]$ with $k \geq 1$. Indeed, it suffices to check this when A' attaches as little as possible to A, and 0-stability guarantees that in that case the only two non-degenerate simplices in $A \setminus A'$ have the form $\underline{a}$ and $0\underline{a}$ (note that $\underline{a}$ can not start with a 0). But then $A \to A'$ is a pushout of $\Lambda^0[k+1] \to \Delta[k+1]$ where k is the dimension of $\underline{a}$.

The desired claim follows by noting that both the domain and codomain of (3.25) are 0-stable and that the inclusions $\Lambda^0[1] \to \Delta[1]$ are unneeded since (3.25) is an isomorphism on 0-simplices.

GENEQOP SEC

3.3 Genuine equivariant operads

Recall that categories can be identified with their nerves, since the *nerve functor* $N\colon \mathsf{Cat} \to \mathsf{sSet}$, given by $N\mathcal{C}(n) = \mathsf{Cat}([n], \mathcal{C})$, is fully faithful. Moreover, the essential image of the nerve is characterized as those simplicial sets with the strict right lifting property (where *strict* means that the lifts are unique) against the inner horn inclusions [Lur09, Prop. 1.1.2.2].

More generally, there is a similar operadic story. Any tree $U \in \Omega$ is naturally associated to a colored operad $\Omega(U) \in \mathsf{Op}$ [MW07, §3], and [MW09, Prop. 5.3 and Thm. 6.1] show that the operadic nerve $N\colon \mathsf{Op} \to \mathsf{sdet}$, given by $N\mathcal{O}(U) = \mathsf{Op}(\Omega(U), \mathcal{O})$, is again fully faithful with essential image the dendroidal sets with the strict right lifting property against (dendroidal) inner horn inclusions. Moreover, [CM13a, Cor. 2.6] provides an alternate characterization via strict lifts against Segal core inclusions. The equivalence between these two characterizations is an observation concerning the notion of hypersaturation in the previous section, as follows.

HYPERLP PROP

Proposition 3.26. *If two classes $\mathcal{C}, \mathcal{D}$ of cofibrations in a model category have the same hypersaturation, then the two classes of maps with the strict right lifting property against $\mathcal{C}$ and $\mathcal{D}$ coincide.*

Proof. It suffices to check that the hypersaturation closure conditions are compatible with strict right lifting properties. The claims concerning pushouts, transfinite compositions and retracts follow from the easy observation that the proofs of the analogue claims for the usual right lifting property [Rie14, Lemma 11.1.4] are compatible with the uniqueness requirement.

We thus address only the cancellation property. Suppose then that r has the strict right lifting property against cofibrations f and gf, and consider a lifting problem as on the left below.

$$\begin{tikzcd}
B \arrow[r, "p"] \arrow[d, "g"'] & X \arrow[d, "r"] \\
C \arrow[r, "q"'] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "r"] \\
B \arrow[r] \arrow[ur, "p"] \arrow[d, "g"'] & Y \arrow[d, equal] \\
C \arrow[r, "q"'] \arrow[uur, dashed, "\exists ! H"'] & Y
\end{tikzcd}$$

By assumption, there is a unique lift H for the outer square on the right, and we claim that H is also the unique lift for the left square. Noting that $pf = Hgf$ and $rp = qg = rHg$ it follows that both p and Hg are lifts for the top square in the right diagram, so that by the uniqueness assumption it is $p = Hg$. This shows that H is also in fact a lift for the left square. Uniqueness follows since any lift of the left square induces a lift of the outer right square. □

Roughly speaking, our goal in this section is that of describing those presheaves with the strict right lifting property against any of the classes of maps in Proposition 3.18, which we call genuine equivariant operads. However, some care is needed. Namely, it is essential to work with the category $\mathsf{dSet}_G = \mathsf{Set}^{\Omega_G^{op}}$ of *genuine G-dendroidal sets* rather than with the category $\mathsf{dSet}^G = \mathsf{Set}^{\Omega^{op}\times G}$ of G-dendroidal sets, i.e. it is essential to work with presheaves that are evaluated on G-trees $T \in \Omega_G$ rather than non-equivariant trees $U \in \Omega$ (the motivation for this is given in Remark 3.35 below). To relate these presheaf categories, note that the fully faithful inclusion $\upsilon\colon \Omega \times G^{op} \to \Omega_G$ given by $U \mapsto G \cdot U$ induces an adjunction

$$\upsilon^* : \mathsf{dSet}_G \rightleftarrows \mathsf{dSet}^G : \upsilon_*$$ DSETG_EQ

Remark 3.27. Mimicking the notation in §2.3, we write $\Omega_G[-]\colon \Omega_G \to \mathsf{dSet}_G$ for the Yoneda embedding. On the other hand, in §2.3 we extended the notation $\Omega[-]$ to obtain a functor $\Omega[-]\colon \Omega_G \to \mathsf{dSet}^G$. These two "representable functors" are related by $\Omega_G[T] = \upsilon_*\Omega[T]$.

The following definition is then the main purpose of this section.

GEN_OP_DEF **Definition 3.28.** $\mathcal{P} \in \mathsf{dSet}_G$ is called a *genuine equivariant operad* if $\mathcal{P}$ has the strict right lifting property against the images under υ_* of the Segal core inclusions, i.e. against the maps

$$\upsilon_*\left(Sc[T] \to \Omega[T]\right), \qquad T \in \Omega_G. \tag{3.29}$$ GGIOP_EQ

Equivalently, by Propositions 3.18 and 3.26, one may replace Segal core inclusions with either orbital G-inner horn inclusions or G-inner horn inclusions.

Example 3.30. To illustrate the role of the strict lifting condition against the maps in (3.29), consider the G-tree T in (2.9), along with the subgroup $K = \{-1\}$ therein and the orbital faces R_1, R_2, S in Example 2.16. The strict lifting condition then says that the left map in

$$\mathcal{P}(R_1) \times_{\mathcal{P}(G/K\cdot\eta)} \mathcal{P}(R_2) \xleftarrow{\simeq} \mathcal{P}(T) \longrightarrow \mathcal{P}(S) \tag{3.31}$$ NORM_COMP_GEN_EQ

is an isomorphism, so that T induces a *composition map* $\mathcal{P}(R_1) \times_{\mathcal{P}(G/K\cdot\eta)} \mathcal{P}(R_2) \to \mathcal{P}(S)$. Here we note that R_1, R_2, S are *G-corollas* (i.e. G-trees with a single G-vertex). Informally, one then thinks of the $\mathcal{P}(C)$, where C ranges over the G-corollas, as the mapping sets of the genuine equivariant operad $\mathcal{P}$, so that the strict lifting conditions equip these mapping sets with associative and unital composition maps.

We caution, however, that this is not quite the whole story, since the composition maps need also be compatible with the presheaf structure, which is more complex in the equivariant context. More explicitly, non-equivariantly one needs only compatibility with the symmetric group actions, reflecting the fact that (almost) all maps between corollas are symmetry isomorphisms. But in the equivariant context G-corollas are also related via *quotient maps* (such as the map in (2.25)) which induce subtler compatibility conditions. Nonetheless, our intended application in §5 will not require an explicit discussion of these additional compatibilities.

Remark 3.32. Consider a single colored G-operad $\mathcal{O}$ (i.e. an operad with a G-action commuting with all structure) and a finite H-set A for some subgroup $H \leq G$, and write $\Gamma_A \leq G \times \Sigma_{|A|}$ for the graph of the homomorphism $H \to \Sigma_{|A|}$ encoding A. We then abbreviate $\mathcal{O}(A)^H = \mathcal{O}(|A|)^{\Gamma_A}$, and call this the *set of A-norm maps of* $\mathcal{O}$ (this is because, for each $\mathcal{O}$-algebra R, $\mathcal{O}(A)^H$ indexes operations $N^A R \to R$, where the *norm* $N^A R$ denotes $R^{\times|A|}$ with a suitably twisted H-action [BH15, Def. 6.1].

Letting $T, R_1, R_2, S \in \Omega_G$ and $H, K, L \leq G$ again be as in (2.9) and Example 2.16, the diagram of hom sets

$$\mathsf{Op}^G(\Omega(R_1),\mathcal{O}) \times \mathsf{Op}^G(\Omega(R_2),\mathcal{O}) \xleftarrow{\simeq} \mathsf{Op}^G(\Omega(T),\mathcal{O}) \longrightarrow \mathsf{Op}^G(\Omega(S),\mathcal{O}) \tag{3.33}$$ NORM_COMP_OP_EQ

can be interpreted, after unpacking notation, as giving a composition of norm maps

$$\mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \to \mathcal{O}(H/L \amalg H/K)^H \tag{3.34}$$ NORM_COMP_EQ

The diagrams (3.31) for genuine operads $\mathcal{P}$ can then be regarded as abstracting the diagrams (3.33) for G-operads $\mathcal{O}$, though with two key differences. The more obvious difference is the fact that (3.33) features no analogue of the $\mathcal{P}(G/K \cdot \eta)$ term, though this is simply since we chose $\mathcal{O}$ to be single colored. The subtler, and more crucial, difference is the fact that the terms in (3.31) need not be described by fixed point sets as in (3.34).

Therefore, one can regard genuine equivariant operads as objects that mimic the composition combinatorics of the norm maps in a (regular) equivariant operad, while relaxing the fixed point conditions. In fact, the reader of [BP17] may recognize this as the informal description of genuine equivariant operads given in the introduction to that work, though our current formal setting is rather different. Moreover, the connection between the two settings is as follows. There is a nerve functor

$$N_G \colon \mathsf{Op}_G \to \mathsf{dSet}_G, \qquad N_G\mathcal{P}(T) = \mathsf{Op}_G(\Omega_G(T),\mathcal{P})$$ NG EQ

where Op_G denotes a colored generalization of the genuine equivariant operads of [BP17]. Moreover, N_G is fully faithful and its essential image are the genuine equivariant operads in the sense of Definition 3.28. However, we will not require these facts, and thus delay their proof to a sequel.

We end this section by explaining why genuine G-dendroidal sets dSet_G, rather than G-dendroidal sets dSet_G, must be used in Definition 3.28.

DESTIAL REM **Remark 3.35.** Suppose $X \in \mathsf{dSet}^G$ has the strict right lifting property against all Segal core inclusions $Sc[T] \to \Omega[T]$, $T \in \Omega_G$. By specifying to the case of $T = G \cdot T_*$ a free G-tree, (2.26) implies that, after forgetting the G-action, $X \in \mathsf{dSet}$ has the strict lifting property against the inclusions $Sc[T_*] \to \Omega[T_*]$, T_*. But the strict lifting properties with respect to all other G-trees

$T \in \Omega_G$ are now automatic. Indeed, writing $T \simeq G \cdot_H T_*$ for some $T_* \in \Omega^H$ one has that by (T_DECOMP_EQ 2.26) G-equivariant lifts against $G \cdot_H (Sc[T_*] \to \Omega[T_*])$ are the same as H-equivariant lifts against $Sc[T_*] \to \Omega[T_*]$. Consider now the following diagram, where Φ is the unique non-equivariant lift.

$$\begin{tikzcd}
Sc[T_*] \arrow[r, "h"] \arrow[d] & Sc[T_*] \arrow[r, "\phi"] \arrow[d] & X \arrow[r, "h^{-1}"] & X \\
\Omega[T_*] \arrow[r, "h"'] & \Omega[T_*] \arrow[ur, dashed, "\Phi"'] & &
\end{tikzcd}$$

Then $h^{-1}\Phi h$ is a lift for the composite lifting problem, but since $h^{-1}\phi h = \phi$, that composite lifting problem in fact coincides with the middle problem, so that strictness implies it is also $h^{-1}\Phi h = \Phi$. In other words, Φ is also the unique H-equivariant lift.

In summary, we have shown that if we had instead used dSet^G in Definition GEN_OP_DEF 3.28, then non-free G-trees would be superfluous, so that by MW09 [MW09, Theorem 6.1] the $X \in \mathsf{dSet}^G$ with such a lifting property would be simply the nerves of G-operads. To see why this is an unsatisfactory situation we recall a fundamental basic example. The category Top^G of G-spaces admits two main equivariant notions of weak equivalence: the genuine equivalences, which care about all fixed point spaces, and the naive equivalences, which care only about the total spaces. However, this distinction vanishes when working in the discrete setting of G-sets Set^G, unless one instead works with G-coefficient systems $\mathsf{Set}^{\mathsf{O}_G^{op}}$. Similarly, the category sOp^G of G-simplicial operads admits two natural notions of weak equivalence, one which cares about the spaces of norm maps for all H-sets A and one which cares only about the spaces of norm maps for trivial H-sets. However, this distinction vanishes when working in the discrete setting of G-dendroidal sets dSet^G, unless one works instead with genuine G-dendroidal sets dSet_G.

4 Quillen equivalences

QUIEQ SEC

Our main goal in this section is to prove Theorems 4.19 and 4.28, which jointly establish the Quillen equivalence of three model categories: the category of equivariant dendroidal sets dSet^G with the "G-∞-operad" model structure of [Per17, Thm 2.1]; the category of equivariant dendroidal spaces sdSet^G with the "complete equivariant dendroidal Segal space" model structure in §4.2 and; the category of equivariant preoperads PreOp^G with the "equivariant Segal operad" model structure in §4.3.

Our perspective will be that these Quillen equivalences are best understood in light of the equivariant analogue of [CM13a, Thm. 6.6], which says that the complete dendroidal space model structure on $\mathsf{sdSet} = \mathsf{dSet}^{\Delta^{op}}$ can be obtained via two distinct left Bousfield localization procedures. As such, we will find it helpful to first focus on the abstract properties of such "joint left Bousfield localizations".

4.1 Joint left Bousfield localizations

JOINBOUS SEC

Throughout we assume familiarity with the theory of left Bousfield localizations as in [Hir03].

COMBMODSTR PROP

Proposition 4.1. *Suppose that the category $\mathcal{C}$ admits two model structures (C, W_1, F_1) and (C, W_2, F_2) with a common class of cofibrations C, and assume further that both model structures are cofibrantly generated and admit left Bousfield localizations with respect to any set of maps.*

Then (C, W_1, F_1), (C, W_2, F_2) have a smallest joint left Bousfield localization (C, W, F) and:

(i) $c \in \mathcal{C}$ is (C, W, F)-fibrant iff it is simultaneously (C, W_1, F_1)-fibrant and (C, W_2, F_2)-fibrant;

(ii) for (C, W, F)-fibrant $c, d \in \mathcal{C}$ one has that $c \to d$ is in W iff it is in W_1 iff it is in W_2.

Proof. The joint model structure (C, W, F) can be obtained by either left Bousfield localizing (C, W_1, F_1) with regards to the generating trivial cofibrations of (C, W_2, F_2) or vice-versa. That the two processes yield the same model structure follows from from the universal property of left Bousfield localizations [Hir03, Prop. 3.4.18].

(ii) follows from the local Whitehead theorem [Hir03, Thm. 3.3.8], stating that the local equivalences between local objects match the initial weak equivalences.

For (i), the claim that joint fibrant objects are fibrant in both of the original model structures follows since $C \cap W$ contains both $C \cap W_1$ and $C \cap W_2$ (in fact, this shows that $F \subseteq F_1 \cap F_2$). The converse claim follows from the observation that fibrant objects in any model structure are already local with respect to the weak equivalences in that same model structure. $\square$

The prototypical example of Proposition 4.1 is given by the category ssSet of bisimplicial sets together with the two possible Reedy structures (over the Kan model structure on sSet). Explicitly, writing the levels of $X \in \mathsf{ssSet}$ as $X_n(m)$ one can either form a Reedy model structure with respect to the *horizontal index* m or with respect to the *vertical index* n.

In either case, the generating cofibrations are then given by the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Delta[m] \to \Delta[m]), \qquad n, m \geq 0.$$

Further, in the horizontal Reedy model structure the generating trivial cofibrations are the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square\, (\partial\Delta[m] \to \Delta[m]), \qquad n \geq i \geq 0, m \geq 0. \tag{4.2}$$ GTRCOHOR EQ

while for the vertical Reedy model structure the generating trivial cofibrations are the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, \left(\Lambda^j[m] \to \Delta[m]\right), \qquad n \geq 0, m \geq j \geq 0. \tag{4.3}$$ GTRCOVER EQ

We caution the reader about a possible hiccup with the terminology: the weak equivalences for the horizontal Reedy structure are the *vertical equivalences*, i.e. maps inducing Kan equivalences of simplicial sets $X_\bullet(m) \to Y_\bullet(m)$ for each $m \geq 0$, and dually for the vertical Reedy structure.

UNIQUELIM NOT **Notation 4.4.** Given a fixed $X \in \mathsf{ssSet}$ we will also write $X_{(-)} \colon \mathsf{sSet}^{op} \to \mathsf{sSet}$ for the unique limit preserving functor such that $X_{\Delta[n]} = X_n$.

In the next result we refer to the localized model structure given by Proposition 4.1 as the *joint Reedy model structure* and we write $\delta^* \colon \mathsf{ssSet} \to \mathsf{sSet}$ for the diagonal functor.

SSSETJREE PROP **Proposition 4.5.** *Suppose that $X, Y \in \mathsf{ssSet}$ are horizontal Reedy fibrant. Then:*

(i) for each fixed n all vertex maps $X_n \to X_0$ are trivial Kan fibrations in sSet;

(ii) any vertical Reedy fibrant replacement $\tilde{X}$ of X is fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a joint weak equivalence iff it is a horizontal weak equivalence iff $X_0 \to Y_0$ is a Kan equivalence in sSet;

(iv) the canonical map $X_0 \to \delta^*(X)$ (with levels $X_0(n) \to X_n(n)$ induced by degeneracies) is a Kan equivalence in sSet.

Proof. (i) follows since the trivial cofibrations for the horizontal Reedy structure include all the maps of the form $(\Delta[0] \to \Delta[n]) \square (\partial\Delta[m] \to \Delta[m])$.

For (ii), the fact that $\tilde{X}$ is vertical fibrant implies that for any monomorphism $K \to L$ in sSet the induced map $\tilde{X}_L \to \tilde{X}_K$ is a Kan fibration. Therefore, (i) implies that all vertex maps $\tilde{X}_n \to \tilde{X}_0$ are trivial Kan fibrations, so that by Remark 3.22 one has that $\tilde{X}_L \to \tilde{X}_K$ is a trivial Kan fibration whenever $K \to L$ is anodyne. Therefore, $\tilde{X}$ is horizontal fibrant, as desired.

The first "iff" in (iii) follows from (ii) since the localizing maps $X \to \tilde{X}$, $Y \to \tilde{Y}$ are horizontal equivalences while the second "iff" in (iii) follows from (i).

For (iv), note first that $\delta^* \colon \mathsf{ssSet} \to \mathsf{sSet}$ is left Quillen for either the horizontal or vertical Reedy structures (and thus also for the joint Reedy structure). But noting that all objects are cofibrant, and regarding X_0 as a bisimplicial set that is vertically constant, the claim follows by noting that by (i) the map $X_0 \to X$ is a horizontal weak equivalence in ssSet. □

WEAKDIAG COR **Corollary 4.6.** *A map $f \colon X \to Y$ in ssSet is a joint equivalence iff it induces a Kan equivalence on diagonals $\delta^*(X) \to \delta^*(Y)$ in sSet.*

Proof. Since horizontal Reedy fibrant replacement maps $X \to \tilde{X}$ are diagonal equivalences, one reduces to the case of X, Y horizontal Reedy fibrant. The result now follows by combining Proposition 4.5 (iii) and (iv). □

SSETSSETADJ COR **Corollary 4.7.** *The adjunction*

$$\delta_! \colon \mathsf{sSet} \rightleftarrows \mathsf{ssSet} \colon \delta^*$$

is a Quillen equivalence. Moreover, if a map $f \colon X \to Y$ in ssSet has the right lifting property against both sets of maps in (4.2) and (4.3), then $\delta^*(f)$ is a Kan fibration in sSet.

Note that the "moreover" claim in this result is not quite formal, since the maps in (4.2),(4.3) are not known to be generating trivial cofibrations for the joint model structure in ssSet.

Proof. Recall that $\delta_!$ is the unique colimit preserving functor such that $\delta_!(\Delta[n]) = \Delta[n] \times \Delta[n]$.

To see that $\delta_!$ preserves cofibrations it is enough to show that $\delta_!\,(\partial\Delta[n] \to \Delta[n])$ is a monomorphism for all $n \geq 0$. This holds since: (i) any two face inclusions $F_1 \to \Delta[n]$, $F_2 \to \Delta[n]$ factor through a minimal face inclusion $F \to \Delta[n]$ (indeed, faces are indexed by subsets of $\{0,1,\cdots,n\}$); (ii) for any face inclusion one has $\delta_!\left(F \to \Delta[n]\right) = \left(F^{\times 2} \to \Delta[n]^{\times 2}\right)$, which is a monomorphism.

The claim that $\delta_!$ preserves trivial cofibrations easily follows from Remark 3.22 together with Corollary 4.6, but here we give a harder argument needed to establish the stronger "moreover" claim. Namely, we will argue that the maps $\delta_!\left(\Lambda^i[n] \to \Delta[n]\right)$ are built cellularly out of the maps in (4.2), (4.3). One has a factorization

$$\delta_!\Lambda^i[n] \to \Lambda^i[n] \times \Delta[n] \to \Delta[n]^{\times 2}$$

where the second map is clearly built cellularly out of the maps in (4.2), and we claim that the first map is likewise built cellularly out of the maps in (4.3). Indeed, this first map be built by iteratively attaching the maps

$$\left(\Lambda^F[n] \to \Lambda^i[n]\right) \square \left(\Lambda^i F \to F\right)$$

where F ranges over the poset $\mathsf{Face}_{\supsetneq\{i\}}$ of faces of $\Delta[n]$ strictly containing $\{i\}$ (note that for $F = \Delta[n]$ it is $\Lambda^F[n] = \varnothing$, so that these maps are not in general built out of the maps in (4.2)).

Lastly, the Quillen equivalence condition is that for all $X \in \mathsf{sSet}$ and joint fibrant $Y \in \mathsf{ssSet}$ a map $X \to \delta^* Y$ is a weak equivalence iff $\delta_! X \to Y$ is. But by Corollary 4.6 this reduces to showing that the unit maps $X \to \delta^*\delta_! X$ are weak equivalences. This latter claim follows by cellular induction on X, since those pushouts attaching cells are homotopy pushouts (due to sSet being left proper). □

HYPERSIMPL REM **Remark 4.8.** Just as in the proof of Proposition 4.5, one can use hypersaturations to simplify the lifting condition in the previous result. Namely, $X \to Y$ is a vertical fibration (i.e. it has the lifting property against (4.3)) iff, for each monomorphism $K \to L$ in sSet, $X_L \to X_K \times_{Y_K} Y_L$ is a Kan fibration. The lifting property against (4.3) is then the condition that $X_L \to X_K \times_{Y_K} Y_L$ is a trivial Kan fibration when $K \to L$ is a horn inclusion. But then a straightforward hypersaturation argument together with Remark 3.22 show that it suffices to check that the maps $X_n \to X_0 \times_{Y_0} Y_n$, induced by the vertex maps $[0] \to [n]$, are trivial Kan fibrations.

Remark 4.9. The adjunction $\delta^* \colon \mathsf{ssSet} \rightleftarrows \mathsf{sSet} \colon \delta_*$ can also be shown to be a Quillen equivalence.

4.2 Complete equivariant dendroidal Segal spaces

CEDSS SEC

We now turn to our main application of Proposition 4.1, the category $\mathsf{sdSet}^G = \mathsf{Set}^{\Delta^{op}\times\Omega^{op}\times G}$ of G-equivariant simplicial dendroidal sets.

Since Δ is a (usual) Reedy category the model structure on dSet^G in [Per17, Thm. 2.1] induces a model structure on sdSet^G that we will refer to as the *simplicial Reedy model structure*.

On the other hand, in the context of Definition A.2, $\Omega^{op} \times G$ is a generalized Reedy category such that the families $\{\mathcal{F}_U^\Gamma\}_{U\in\Omega}$ of G-graph subgroups are Reedy-admissible (see Example A.6) and hence, using the underlying Kan model structure on sSet, Theorem A.8 yields a model structure on sdSet^G that we will refer to as the *equivariant dendroidal Reedy model structure*, or simply as the *dendroidal Reedy model structure* for the sake of brevity.

Throughout, we will write the levels of $X \in \mathsf{sdSet}^G$ as $X_n(U)$. We now extend Notation 4.4. Note that the representable functor of $U \in \Omega \times G^{op}$ is given by $\Omega[G \cdot U] = G \cdot \Omega[U]$.

UNILIMDEN NOT **Notation 4.10.** Given a fixed $X \in \mathsf{sdSet}^G$ we will also write

$$X(-)\colon \left(\mathsf{dSet}^G\right)^{op} \to \mathsf{sSet}, \qquad X_{(-)}\colon \mathsf{sSet}^{op} \to \mathsf{dSet}^G$$

for the unique limit preserving functors such that $X(\Omega[G\cdot U]) = X(U)$, $X_{\Delta[n]} = X_n$.

Moreover, for fixed $J \in \mathsf{dSet}^G$ we define $X^J \in \mathsf{sdSet}^G$ by $X^J(U) = X\left(\Omega[G\cdot U] \otimes J\right)$.

JM NOT **Notation 4.11.** Writing $\widetilde{[m]} = (0 \rightleftarrows 1 \rightleftarrows \cdots \rightleftarrows m)$ for the contractible groupoid with objects $0, 1, \cdots, m$ (cf. Remark 3.24), we denote

$$J^m = \iota_!(N\widetilde{[m]})$$

where N is the nerve functor. J^m is regarded as equipped with the trivial G-action. Further, we abbreviate $J = J^1$.

Proposition 4.12. *Both the simplicial and dendroidal Reedy model structures on* sdSet^G *have generating cofibrations given by the maps*

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq 0, T \in \Omega_G. \tag{4.13}$$ JOINTCOF EQ

Further, the dendroidal Reedy structure has as generating trivial cofibrations the maps

$$\left(\Lambda^i[n] \to \Delta[n]\right) \square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq i \geq 0, T \in \Omega_G. \tag{4.14}$$ DENDTRIVCOF EQ

while the simplicial Reedy structure has as generating trivial cofibrations the maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (A \to B), \qquad n \geq 0 \tag{4.15}$$ SIMPTRIVCOF EQ

for $\{A \to B\}$ *a set of generating trivial cofibrations of* dSet^G*.*

Proof. For the claims concerning the dendroidal Reedy structure, note that the presheaves $\Omega[T] \in \mathsf{dSet}^G$ are precisely the quotients $(G \cdot \Omega[U])/K$ for $U \in \Omega$ and $K \leq G \times \mathsf{Aut}(U)$ a G-graph subgroup, so that $\partial\Omega[T] \to \Omega[T]$ represents the maps $X(U)^K \to (M_U X)^K$ for $X \in \mathsf{dSet}^G$.

The claims concerning the simplicial Reedy structure are immediate. □

We call the saturation of the maps in (4.13) the class of *normal monomorphisms* of sdSet^G.

JOINTFIBCHAR COR **Corollary 4.16.** *The joint fibrant objects* $X \in \mathsf{sdSet}^G$ *have the following equivalent characterizations:*

(i) X *is both simplicial Reedy fibrant and dendroidal Reedy fibrant;*

(ii) X *is simplicial Reedy fibrant and all maps* $X_0 \to X_n$ *are equivalences in* dSet^G*;*

(iii) X *is dendroidal Reedy fibrant and all maps*

$$X(\Omega[T]) \to X(Sc[T]) \qquad \text{and} \qquad X(\Omega[T]) \to X(\Omega[T] \otimes J)$$

for $T \in \Omega_G$ *are Kan equivalences in* sSet*.*

Proof. (i) simply repeats Proposition 4.1(i). In the remainder we write $K \to L$ for a generic monomorphism in sSet and $A \to B$ a generic normal monomorphism in dSet^G.

For (ii), note that X is simplicial fibrant iff $X_L \to X_K$ is always a fibration in dSet^G. Thus, X will also have the right lifting property against (4.14) iff $X_L \to X_K$ is a trivial fibration whenever $K \to L$ is anodyne. But by Remark 3.22 it suffices to consider the vertex inclusions $\Delta[0] \to \Delta[n]$. The claim now follows from 2-out-of-3 applied to the composites $X_0 \to X_n \to X_0$.

For (iii), note first that X is dendroidal fibrant iff $X(B) \to X(A)$ is always a Kan fibration. Therefore, X will have the right lifting property against (4.15) iff $X(B) \to X(A)$ is a trivial Kan fibration whenever $A \to B$ is a generating trivial cofibration of dSet^G. By adjunction, this is equivalent to showing that $X_L \to X_K$ is a fibration in dSet^G for any monomorphism $K \to L$ in sSet. Moreover, by the fibration between fibrant objects part of [Per17, Prop. 8.8] (see also the beginning of [Per17, §8.1]) it suffices to verify that the maps $X_L \to X_K$ have the right lifting property against the maps

$$\Lambda^{Ge}[T] \to \Omega[T], \quad T \in \Omega_G, e \in E^{\mathrm{i}}(T) \qquad \text{and} \qquad \Omega[T] \otimes (\{i\} \to J_d), \quad T \in \Omega_G, i = \{0,1\}$$

and it thus suffices to check that $X(B) \to X(A)$ is a trivial Kan fibration when $A \to B$ is one of these maps. Proposition 3.18 now finishes the proof. $\square$

Remark 4.17. We will use the terms "joint fibrant" and "complete dendroidal Segal space" interchangeably; see §5 for further discussion and historical context.

We now obtain the following partial analogue of Proposition 4.5. Note that the equivalences in the simplicial Reedy model structure are the dendroidal equivalences and vice versa.

SDSETG COR **Corollary 4.18.** *Suppose that $X, Y \in \mathsf{sdSet}^G$ are dendroidal Reedy fibrant. Then:*

(i) for all n the vertex maps $X_n \to X_0$ are trivial fibrations in dSet^G;

(ii) any simplicial Reedy fibrant replacement $\tilde{X}$ of X is in fact fibrant in the joint Reedy model structure;

(iii) a map $X \to Y$ is a joint weak equivalence iff it is a dendroidal weak equivalence iff $X_0 \to Y_0$ is an equivalence in dSet^G;

(iv) regarding X_0 as a simplicially constant object in sdSet^G, the map $X_0 \to X$ is a dendroidal equivalence, and thus a joint equivalence.

Proof. The proof adapts that of Proposition 4.5. (i) follows since X then has the right lifting property with respect to all maps $(\Delta[0] \to \Delta[m]) \square (\partial\Omega[T] \to \Omega[T])$. (ii) follows from (i) and the characterization in Corollary 4.16 (ii). The first "iff" in (iii) follows from (ii) since the simplicial fibrant replacement maps $X \to \tilde{X}$ are dendroidal equivalences and the second "iff" in (iii) follows from (i). (iv) follows from (i). $\square$

INCOAGJ THM **Theorem 4.19.** *The constant/0-th level adjunction*

$$c_!\colon \mathsf{dSet}^G \rightleftarrows \mathsf{sdSet}^G \colon (-)_0$$

where sdSet^G is given the joint Reedy model structure, is a Quillen equivalence.

Proof. It is clear that the constant functor $c_!$ preserves both normal monomorphisms and all weak equivalences, hence the adjunction is Quillen. Consider any map $c_!(A) \to X$ with X joint fibrant and perform a "trivial cofibration followed by fibration" factorization as on the left

$$c_!(A) \overset{\sim}{\rightarrowtail} \widetilde{c_!(A)} \twoheadrightarrow X \qquad A \xrightarrow{\sim} \widetilde{c_!(A)}_0 \to X_0$$

for the simplicial Reedy model structure. Corollary 4.16(ii) now implies that $\widetilde{c_!(A)}$ is in fact joint fibrant and thus that the leftmost composite is a joint equivalence iff $\widetilde{c_!(A)} \to X$ is a dendroidal equivalence in sdSet^G iff $\widetilde{c_!(A)}_0 \to X_0$ is an equivalence in dSet^G iff the rightmost composite is an equivalence in dSet^G. $\square$

CONCRECOM REM **Remark 4.20.** Given a G-∞-operad $X \in \mathsf{dSet}^G$, one can obtain an explicit model for $\widetilde{c_!(X)}$ as the object $X^{J^\bullet} \in \mathsf{sdSet}^G$. Indeed, since $J^\bullet$ is a Reedy cofibrant cosimplicial object in dSet^G, one has that $X^{J^\bullet} \in \mathsf{sdSet}^G$ is simplicial fibrant. Hence, by Corollary 4.16(ii) $c_!(X) \to X^{J^\bullet}$ will indeed be a joint fibrant replacement provided that it is a dendroidal equivalence. But this follows from [Per17, Cor. 8.21], which implies that the maps $X^{J^m} \to X^{J^0} = X$ are trivial fibrations in dSet^G (formally, [Per17, Cor. 8.21] says that $\upsilon_*(X^{J^m}) = X^{(J^m)} \to \upsilon_*(X)$ is a trivial fibration in dSet_G, which is an equivalent statement, as noted at the end of the proof of [Per17, Thm. 8.22]).

PREOP SEC

4.3 Equivariant Segal operads

Recall that the category PreOp of *pre-operads* is the full subcategory $\mathsf{PreOp} \subset \mathsf{sdSet}$ of those X such that $X(\eta)$ is a discrete simplicial set. Writing γ^* for the inclusion one has left and right adjoints $\gamma_!$ and γ_*

$$\begin{tikzcd} \mathsf{PreOp}^G \arrow[r, "\gamma^*"'] & \mathsf{sdSet}^G \arrow[l, bend right, "\gamma_!"'] \arrow[l, bend left, "\gamma_*"] \end{tikzcd}$$

described as follows CM13a [CM13a, §7]: $\gamma_! X(U) = X(U)$ if $U \notin \Delta$ while $\gamma_! X([n])$ for $[n] \in \Delta$ is given by the pushout on the left below; $\gamma_* X(U)$ is given by the pullback on the right below.

$$\begin{tikzcd} X(\eta) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \pi_0 X(\eta) \arrow[d] \\ X([n]) \arrow[r] & \gamma_! X([n]) \end{tikzcd} \qquad \begin{tikzcd} \gamma_* X(U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X(U) \arrow[d] \\ \prod_{E(U)} X_0(\eta) \arrow[r] & \prod_{E(U)} X(\eta) \end{tikzcd}$$

GAMMASH REM **Remark 4.21.** Any monomorphism $A \to B$ in sdSet^G such that $A(\eta) \to B(\eta)$ is an isomorphism induces a pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near end] & \gamma^*\gamma_! A \arrow[d] \\ B \arrow[r] & \gamma^*\gamma_! B \end{tikzcd} \tag{4.22}$$ GAMMASH EQ

Noting that the assignment $U \mapsto \prod_{E(U)} Y(\eta)$ is the coskeleton $\mathsf{csk}_\eta Y$ leads to the following.

CSKETALT PROP **Proposition 4.23.** *Let $X \in \mathsf{sdSet}^G$. Then:*

(i) if $X \in \mathsf{sdSet}^G$ is dendroidal Reedy fibrant then so is $\gamma^*\gamma_* X$;

(ii) regarding X_0 as a simplicially constant object of sdSet^G, the left square below is a pullback;

(iii) if $A \to A'$ is a map in dSet^G such that $A(\eta) \simeq A'(\eta)$, the right square below is a pullback.

$$\begin{tikzcd} \gamma^*\gamma_* X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X \arrow[d] \\ \mathsf{csk}_\eta X_0 \arrow[r] & \mathsf{csk}_\eta X \end{tikzcd} \qquad \begin{tikzcd} \gamma^*\gamma_* X(A') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near end] & X(A') \arrow[d] \\ \gamma^*\gamma_* X(A) \arrow[r] & X(A) \end{tikzcd}$$

Proof. (ii) is immediate from the observation that $\mathsf{csk}_\eta Y = \prod_{E(-)} Y(\eta)$. Moreover, it readily follows that for $B \in \mathsf{dSet}^G$ it is $(\mathsf{csk}_\eta Y)(B) = \prod_{B(\eta)} Y(\eta)$ and thus (iii) follows from (ii).

For (i), formal considerations imply that if X is dendroidal fibrant then the map $X \to \mathsf{csk}_\eta X$ is a dendroidal fibration (and that $\mathsf{csk}_\eta X$ is dendroidal fibrant). Hence, the result will follow provided that $\mathsf{csk}_\eta X_0$ is also dendroidal fibrant. But since $\mathsf{csk}_\eta X_0$ is η-coskeletal, it suffices to check that the η-matching map $(\mathsf{csk}_\eta X_0)(\eta) \to M_\eta(\mathsf{csk}_\eta X_0)$ is a G-fibration in sSet^G. But this is simply $X_0(\eta) \to *$ regarded as a map of constant simplicial sets, and the result follows. □

Notation 4.24. In the remainder of the section we write $\mathcal{I}'$ for the set of maps

$$(\partial\Delta[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]), \qquad n \geq 0, T \in \Omega_G, T \neq G/H \cdot \eta.$$ BOUNDRED EQ

Further, we note that Remark GAMMASH REM 4.21 applies to these maps.

GENSET LEM

Lemma 4.25. *The maps in* PreOp^G *that are normal monomorphisms in* sdSet^G *are the saturation of the set of maps* $\{\varnothing \to G/H \cdot \eta : H \le G\} \cup \gamma_!(\mathcal{I}')$.

Proof. Using the cellular filtration in sdSet^G, any normal monomorphism $A \to B$ in PreOp^G can (upon inclusion) be written as a transfinite composition of pushouts of maps in $\{\varnothing \to G/H \cdot \eta\} \cup \mathcal{I}'$. But since the squares (4.22) are pushouts the same also holds for $\{\varnothing \to G/H \cdot \eta\} \cup \gamma_!(\mathcal{I}')$. □

TRIVFIB LEM

Lemma 4.26. *Any map in* PreOp^G *which has the right lifting property against all normal monomorphisms in* PreOp^G *is a joint equivalence in* sdSet^G.

Proof. We simply adapt the proof of [CM13a, Lemma 8.12] mutatis mutandis.

Choose a normalization E_∞ of $*$ in dSet^G, i.e. a normal object such that $E_\infty \to *$ is a trivial fibration. Regarding E_∞ as a simplicially constant object in sdSet^G, a map $X \to Y$ in PreOp^G will have the right lifting property against all monomorphisms iff so does $E_\infty \times (X \to Y)$, so that one is free to assume that X, Y are normal.

One is thus free to pick a section $s : Y \to X$ of $p : X \to Y$ and regarding $J \in \mathsf{dSet}^G$ as a simplicially constant object of sdSet^G our assumption yields the lift below, so that p is a homotopy equivalence.

$$\begin{tikzcd}
X \amalg X \arrow[r, "(id_X, sp)"] \arrow[d] & X \arrow[d, "p"] \\
X \otimes J \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

□

PREOPMOD THM

Theorem 4.27. *The category* Preop^G *of* G*-preoperads has a model structure such that*

- *the cofibrations are the normal monomorphisms;*
- *the weak equivalences are the maps that become joint equivalences when regarded as maps in* sdSet^G.

Proof. One repeats the proof of the non-equivariant analogue [CM13a, Thm. 8.13], applying J. Smith's theorem [Bek00, Thm. 1.7] with the required set of generating cofibrations the set $\{\varnothing \to G/H \cdot \eta | H \le G\} \cup \gamma_!(\mathcal{I}')$ given by Lemma 4.25. Indeed, conditions c0 and c2 in [Bek00] are inherited from sdSet^G and c1 follows from Lemma 4.26. The technical "solution set" condition c3 follows from [Bek00, Prop. 1.15] since weak equivalences are accessible, being the preimage by γ^* of the weak equivalences in sdSet^G (see [Lur09, Cor. A.2.6.5] and [Lur09, Cor. A.2.6.6]). □

ANOQUEQUIV THM

Theorem 4.28. *The adjunction*

$$\gamma^* : \mathsf{PreOp}^G \rightleftarrows \mathsf{sdSet}^G : \gamma_*$$

is a Quillen equivalence.

Proof. It is tautological that the left adjoint γ^* preserves and detects cofibrations and weak equivalences, so it suffices to show that for all fibrant $X \in \mathsf{sdSet}^G$ the counit map $\gamma^* \gamma_* X \to X$ is a weak equivalence. But by Proposition 4.23(i) both $\gamma^* \gamma_* X$ and X are dendroidal fibrant, so that the result follows from Corollary 4.18(iii) together with the observation that $(\gamma^* \gamma_* X)_0 = X_0$. □

5 Equivariant dendroidal Segal spaces

EDSS_SEC

As mentioned in the introduction, we conjecture that the model categories sdSet^G and PreOp^G in §4.2 and §4.3 are equivalent to a suitable model structure on the category sOp^G of (coloured) G-operads. However, our present description of the weak equivalences in sdSet^G and PreOp^G is rather different from the description of the desired weak equivalences in sOp^G, which are the Dwyer Kan equivalences, characterized by fully faithfulness and essential surjectivity requirements.

As such, our goal in this final section is to prove Theorem 5.30, which states that weak equivalences between fibrant objects in either of sdSet^G, PreOp^G do indeed admit a Dwyer Kan type description. Moreover, in Corollary 5.33 we also characterize the fibrant objects of PreOp^G.

To do so, it is useful to consider yet another model structure on the category sdSet^G, whose fibrant objects are the so called *equivariant dendroidal Segal spaces*, and which "interpolate" between the fibrant objects in the categories sdSet^G and PreOp^G. See Remark 5.35 for a precise statement.

5.1 The homotopy genuine operad and Dwyer Kan equivalences

HMPTYGEN_SEC

Definition 5.1. The *equivariant Segal space model structure* on the category sdSet^G, which we denote sdSet^G_S, is the left Bousfield localization of the dendroidal Reedy model structure with respect to the equivariant Segal core inclusions (regarded as simplicially constant)

$$Sc[T] \to \Omega[T], \qquad T \in \Omega_G.$$

FIB_PREOP_NOT

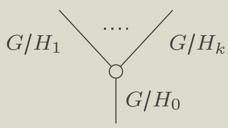
Notation 5.2. We will refer to the fibrant objects in sdSet^G_S as *equivariant dendroidal Segal spaces*, or just *dendroidal Segal spaces*. Further, a pre-operad $X \in \mathsf{PreOp}^G$ is called *fibrant* if $\gamma^* X$ is a dendroidal Segal space.

Proposition 5.3. *If $X \in \mathsf{sdSet}^G$ is a dendroidal Segal space, then $\gamma_* X \in \mathsf{PreOp}^G$ is fibrant.*

Proof. By Proposition 4.23(i) $\gamma_* X$ is dendroidal fibrant. And, since $Sc[T](\eta) = \Omega[T](\eta)$, Proposition 4.23(iii) shows that $(\gamma^*\gamma_* X)(\Omega[T]) \to (\gamma^*\gamma_* X)(Sc[T])$ is fact a trivial Kan fibration. □

GCOR_NOT

Notation 5.4. Given subgroups $H_i \le G$, $0 \le i \le k$ such that $H_0 \ge H_i$, $1 \le i \le k$ we write $C_{\amalg_i H_0/H_i}$ for the G-corolla (well defined up to isomorphism) whose orbital representation is



Writing C_n for the non-equivariant corolla with n leaves, we note that $C_{\amalg_i H_0/H_i} \simeq G \cdot_{H_0} C_{\Sigma_i |H_0/H_i|}$, where $C_{\Sigma_i |H_0/H_i|}$ regarded as a (non-equivariant) corolla together with the obvious H_0-action.

PROF_DEF

Definition 5.5. Let X be a dendroidal Segal space. A *G-profile* on X is a map

$$c_! \partial\Omega[C] \to X$$

for some G-corolla $C \in \Sigma_G$. More explicitly, a G-profile is described by the following data:

- subgroups $H_i \le G$, $0 \le i \le k$ such that $H_0 \ge H_i$ for $1 \le i \le k$;
- objects $x_i \in X(\eta)^{H_i}$ for $0 \le i \le k$.

To simplify notation, we denote a G-profile as $(x_1, \cdots, x_k; x_0)$, and refer to it as a C*-profile* on X.

MAPSPACESEG DEF

Definition 5.6. Given a dendroidal Segal space $X \in \mathsf{sdSet}^G_S$ and a C-profile $(x_1, \cdots, x_n; x_0)$ on X we define the space of maps $X(x_1, \cdots, x_n; x_0) \in \mathsf{sSet}$ via the pullback square

$$\begin{tikzcd}
X(x_1, \cdots, x_k; x_0) \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near end] & X(\Omega[C]) \arrow[d, two heads] \\
\Delta[0] \arrow[r, "(x_1, \cdots, x_k; x_0)"'] & \prod_{0 \leq i \leq k} X(\eta)^{H_i}
\end{tikzcd}$$

HMTPYGEN DEF

Definition 5.7. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. The *homotopy genuine operad* $ho(X) \in \mathsf{dSet}_G$ is defined by

$$ho(X) = \pi_0 \left(\upsilon_* \left(\gamma_* X \right) \right).$$

Remark 5.8. Writing ι for the inclusion $\Delta \to \Omega$ and ι_G for the composite inclusion $\Delta \times \mathsf{O}_G \to \Omega \times \mathsf{O}_G \to \Omega_G$, one has that $\iota_G^* ho(X)$ is the G-coefficient system of categories formed by the homotopy categories $ho\left(\iota^* \left(X^H\right)\right) = \pi_0 \left(\iota^* \gamma_* X^H\right)$ for $H \leq G$.

DKEQUIV DEF

Definition 5.9. A map $f \colon X \to Y$ of equivariant dendroidal Segal spaces is called

- *fully faithful* if for all $C \in \Sigma_G$ and C-profile $(x_1, \cdots, x_n; x_0)$ on X the map

  $$X(x_1, \cdots, x_k; x_0) \to Y\left(f(x_1), \cdots, f(x_k); f(x_0)\right)$$

  is a Kan equivalence in sSet;

- *essentially surjective* if the map $\iota_G^* ho(X) \to \iota_G^* ho(Y)$ is essentially surjective on all category levels of the G-coefficient system;

- a *DK-equivalence* if it is both fully faithful and essentially surjective.

ONLYPREOP REM

Remark 5.10. Definitions 5.6, 5.7 and 5.9 depend only on the fibrant pre-operads $\gamma_* X, \gamma_* Y$, since $X(x_1, \cdots, x_k; x_0) = \gamma_* X(x_1, \cdots, x_k; x_0)$. In fact, for each G-corolla C one has a decomposition

$$ho(X)(C) = \coprod_{C\text{-profiles } (x_1, \cdots, x_k; x_0)} \pi_0 \left(X(x_1, \cdots, x_k; x_0)\right)$$

so that, given $\varphi \in X_0(x_1, \cdots, x_k; x_0)$ we will write $[\varphi] \in ho(X)(C)$ for the corresponding class.

Remark 5.11. One can extend the previous definitions to G-∞-operads $X, Y \in \mathsf{dSet}^G$ by applying them to the dendroidal Segal spaces $X^{J^\bullet}, Y^{J^\bullet} \in \mathsf{sdSet}^G$ (cf. Remark 4.20).

HOMOLIFTS REM

Remark 5.12. As remarked in the proof of Corollary 4.16, $X(\Omega[T]) \to X(Sc[T])$ are trivial Kan fibrations for all $T \in \Omega_G$. Moreover, in what follows, we will repeatedly use the observation that, for $X \to Y$ a trivial Kan fibration, any two lifts of the form below are homotopic.

$$\begin{tikzcd}
 & X \arrow[d, two heads, "\sim"] \\
A \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

In particular, this implies that $ho(X)$ is a genuine operad in the sense of Definition 3.28

HEQUIV DEF

Definition 5.13. Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. For $H \leq G$, we call $f \in X_0(\Omega[C_{H/H}]) = X_0([1])^H$ a H*-equivalence* if $[f]$ is an isomorphism in the category $ho\left(\iota^* (X^H)\right)$.

add context

Suppose $C, D \in \Sigma_G$ are G-corollas that can be grafted, i.e. that C has a leaf orbit and D a root orbit both isomorphic to G/H. Denote this orbit as Ge and write $T = C \amalg_{Ge} D$ for the grafted G-tree. For any dendroidal Segal space X one then has $X(Sc[T]) \simeq X(\Omega[C]) \times_{X(\eta)^H} X(\Omega[D])$ and one can hence choose a section in the middle row below

$$\begin{tikzcd}
\{\varphi\} \times X(z_1, \cdots, z_l; e) \arrow[rr, dashed, "\varphi \circ_{Ge} (-)"] \arrow[d, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[d, hook] \\
X(\Omega[C]) \times_{X(\eta)^H} X(\Omega[D]) \arrow[r, bend left, dashed] & X(\Omega[T]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & X(\Omega[T - Ge]) \\
X(e, y_2, \cdots, y_k; x) \times \{\psi\} \arrow[rr, dashed, "(-) \circ_{Ge} \psi"'] \arrow[u, hook] & & X(z_1, \cdots, z_l, y_2, \cdots, y_k; x) \arrow[u, hook]
\end{tikzcd} \tag{5.14}$$

HOMOTCIRC EQ

thus defining maps $\varphi \circ_{Ge} (-)$ (resp. $(-) \circ_{Ge} \psi$) for any choice of $\varphi \in X_0(e, y_2, \cdots, y_k; x)$ (resp. $\psi \in X_0(z_1, \cdots, z_l; e)$).

GENOPHO PROP

Proposition 5.15. *(i) the maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are well defined up to homotopy;*

(ii) if $[\varphi] = [\bar{\varphi}]$ then the maps $\varphi \circ_{Ge} (-)$, $\bar{\varphi} \circ_{Ge} (-)$ are homotopic, and likewise for $[\psi] = [\bar{\psi}]$;

(iii) $[\varphi \circ_{Ge} \psi]$ depends only on $[\varphi]$, $[\psi]$;

(iv) the homotopy classes of the maps $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ are natural with respect to maps $f : X \to Y$ between dendroidal Segal spaces.

Proof. Noting that all possible middle row sections in (5.14) (and homotopies between them) are necessarily compatible with the projections to $X(\partial\Omega[T - Ge])$, (i) follows from Remark 5.12. The middle row in (5.14) gives the necessary homotopies for (ii). (iii) is immediate from (ii). Lastly, (iv) follows from Remark 5.12 applied to the two $X(\Omega[C]) \underset{X(\eta)^H}{\times} X(\Omega[D]) \to Y(\Omega[T])$ paths in

$$\begin{tikzcd}
X(\Omega[T]) \arrow[r] \arrow[d, two heads, "\sim"] & Y(\Omega[T]) \arrow[d, two heads, "\sim"] \\
X(\Omega[C]) \underset{X(\eta)^H}{\times} X(\Omega[D]) \arrow[r] \arrow[u, bend left, dashed] & Y(\Omega[C]) \underset{Y(\eta)^H}{\times} Y(\Omega[D]) \arrow[u, bend left, dashed]
\end{tikzcd}$$

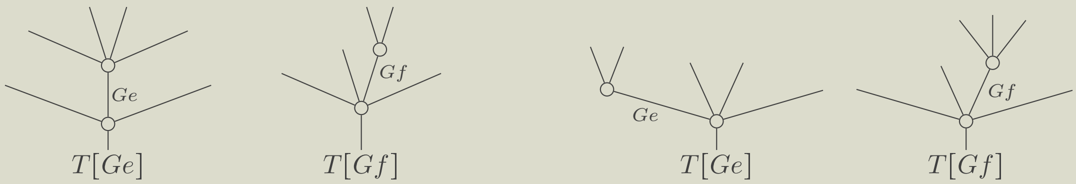
□

We will now show that the operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy the obvious compatibilities one expects, but we will find it convenient to first package these compatibilities into a common format. In the categorical case (corresponding to linear trees), there are three types of "associativity" compatibilities, corresponding to homotopies

$$\varphi \circ (\psi \circ (-)) \sim (\varphi \circ \psi) \circ (-) \qquad \varphi \circ ((-) \circ \psi) \sim (\varphi \circ (-)) \circ \psi \qquad ((-) \circ \varphi) \circ \psi \sim (-) \circ (\varphi \circ \psi)$$

but in the operadic case there are instead five cases, corresponding to the different possible roles of the nodes in G-trees T with exactly three G-vertices, whose *orbital* representation falls into one of the two cases illustrated below.

Since all these compatibilities can be simultaneously encoded in terms of such trees, we will simply refer to all types of compatibility simply as *associativity*. As noted pictorially above, such a G-tree T has exactly two inner edge orbits Ge and Gf. In the next result, we write $T[Ge]$ (resp. $T[Gf]$) for the orbital outer face of T with Ge (resp. Gf) as its single inner edge orbit.



ASSOC PROP **Proposition 5.16.** *The operations $\varphi \circ_{Ge} (-)$, $(-) \circ_{Ge} \psi$ satisfy all associativity conditions with respect to G-trees with three G-vertices. Further, if $C = C_{H/H}$ and $\varphi = s(e)$ is the degeneracy on e, then $\varphi \circ_{Ge} (-)$ is homotopic to the identity, and similarly for $D = C_{H/H}$ and $\varphi = s(e)$.*

Proof. We abbreviate $Sc_{T[Ge]}[T] = Sc[T] \amalg_{Sc[T[Ge]]} \Omega[T[Ge]] = Sc[T] \amalg_{\Lambda^{Ge}_o[T[Ge]]} \Omega[T[Ge]]$, which can be regarded as the union $Sc[T] \cup \Omega[T[Ge]]$ of subcomplexes of $\Omega[T]$. We now consider the following diagram, where all solid maps are Kan fibrations, and the maps labelled $\sim$ are trivial Kan fibrations ($Sc_{T[Ge]}[T]$ is a cover in the sense of Remark RECOVER REM 3.7(i), hence both maps $Sc[T] \to Sc_{T[Ge]}[T] \to \Omega[T]$ are G-inner anodyne), so that one can choose the indicated sections.

$$\begin{tikzcd}
X(Sc[T]) \arrow[r, dashed, bend left] \arrow[d, dashed, bend right] & X(Sc_{T[Ge]}[T]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] \arrow[d, dashed, bend right] & X(Sc[T-Ge]) \arrow[d, dashed, bend right] \\
X(Sc_{T[Gf]}[T]) \arrow[u, two heads, "\sim"'] \arrow[r, dashed, bend left] \arrow[d, two heads] & X(\Omega[T]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] \arrow[u, two heads, "\sim"'] \arrow[d, two heads] & X(\Omega[T-Ge]) \arrow[u, two heads, "\sim"'] \arrow[d, two heads] \\
X(Sc[T-Gf]) \arrow[r, dashed, bend left] & X(\Omega[T-Gf]) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & X(\Omega[T-Ge-Gf])
\end{tikzcd}$$
FOURSQ EQ

But since the desired associativity conditions amount to the claim that the top right and left bottom composites $X(Sc[T]) \to X(\Omega[T-Ge-Gf])$ are homotopic, the associativity result follows from Remark HOMOLIFTS REM 5.12 and a diagram chase. For the "further" claim, note that by Remark HOMOLIFTS REM 5.12 one is free to modify (HOMOTCIRC EQ 5.14) so as to use any lift of the form below. But then since the G-tree T is degenerate on the G-corolla D, such a lift is given by the degeneracy operator and the result follows.

$$\begin{tikzcd}
 & X(\Omega[T]) \arrow[d, two heads, "\sim"] \\
\{s(e)\} \times X(z_1, \cdots, z_l; e) \arrow[r] \arrow[ur, dashed] & X(Sc[T])
\end{tikzcd}$$

$\square$

Remark 5.17. In the non-equivariant case the associativity and unit conditions in the previous result capture all the key compatibilities of the $\varphi \circ_e (-)$, $(-) \circ_e \psi$ operations. However, in the equivariant case there are further "compatibilities with pullback of G-trees", which are closely related to the genuine equivariant operads introduced in [BP17]. Nonetheless, describing these extra compatibilities would require using G-trees with more than three G-vertices, and since such compatibilites are not needed for our present goals, we omit their discussion.

26COR **Corollary 5.18.** *DK-equivalences between dendroidal Segal spaces satisfy 2-out-of-6, i.e. when in* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$ *the maps* gf *and* hg *are DK-equivalences then so are* f, g, h, hgf.

Proof. Applying the 2-out-of-6 properties in sSet and Cat to mapping spaces and homotopy categories $\iota_G^* ho$, the only non obvious conditions are the fully faithfulness of g, h for C-profiles not in the image of f. But since by Proposition 5.16 the maps $f \circ_{Ge} (-)$, $(-) \circ_{Ge} f$ are weak equivalences when f is a H-equivalence, this last claim follows from essential surjectivity. □

The following roughly summarizes (and slightly refines) [Rez01, Lemma 5.8, Theorem 6.2, Prop. 11.1, Lemma 11.10] in our setup.

SESP PROP **Proposition 5.19.** *Let* $X \in \mathsf{ssSet}$ *be a Segal space. Then:*

(i) equivalences of X *define a subset of connected components* $X^h(1) \subset X(1)$*;*

(ii) the pullbacks

$$\begin{tikzcd} X^h(n) \arrow[r] \arrow[d] & X(n) \arrow[d] \\ X^h(1) \times_{X(0)} \cdots \times_{X(0)} X^h(1) \arrow[r] & X(1) \times_{X(0)} \cdots \times_{X(0)} X(1) \end{tikzcd} \tag{5.20}$$ XHDEF EQ

define a Segal space $X^h \subset X$*, consisting of a union of connected components at each level;*

(iii) the maps $X^h(2) \xrightarrow{(d_2,d_1)} X^h(\Lambda^0[2])$, $X^h(2) \xrightarrow{(d_0,d_1)} X^h(\Lambda^2[2])$ *are trivial fibrations;*

(iv) the map $X(J) \to X(\Delta[1]) = X(1)$ *factors through a weak equivalence* $X(J) \to X^h(1)$.

Proof. For (i), given $f : x \to y$ in $X_0(1)$ then $[f]$ has a left inverse iff there exists p as on the left diagram below. But for any path H between f and f' in $X(1)$, by Remark 3.22 there is a lift in the right diagram

$$\begin{tikzcd} & X(2) \arrow[d, "{(d_2,d_1)}"] \\ \Delta[0] \arrow[ur, "p"] \arrow[r, "{(f,s_0(x))}"'] & X(1) \times_{X(0)} X(1) \end{tikzcd} \qquad \begin{tikzcd} \Delta[0] \arrow[r, "p"] \arrow[d, "0"'] & X(2) \arrow[d, "{(d_2,d_1)}"] \\ \Delta[1] \arrow[ur, dashed] \arrow[r, "{(H,s_0d_1(H))}"'] & X(1) \times_{X(0)} X(1) \end{tikzcd}$$

showing that f' is also left-invertible. The situation for right inverses is identical, thus (i) follows.

For (ii), that X^h is closed under the simplicial operators follows equivalences are closed under composition. Moreover, noting that (5.20) can be reinterpreted as on the left below, cellular induction induces the right pullbacks for all $K \in \mathsf{sSet}$.

$$\begin{tikzcd} X^h(\Delta[n]) \arrow[r] \arrow[d] & X(\Delta[n]) \arrow[d] \\ X^h(\mathsf{sk}_1\Delta[n]) \arrow[r] & X(\mathsf{sk}_1\Delta[n]) \end{tikzcd} \qquad \begin{tikzcd} X^h(K) \arrow[r] \arrow[d] & X(K) \arrow[d] \\ X^h(\mathsf{sk}_1 K) \arrow[r] & X(\mathsf{sk}_1 K) \end{tikzcd}$$

Since $\mathsf{sk}_1(\partial\Delta[n]) = \mathsf{sk}_1\Delta[n]$ if $n \geq 2$ it follows that the maps $X^h(n) \to X^h(\partial\Delta[n])$, $n \geq 2$ are Kan fibrations, and since the map $X^h(1) \to X(0) \times X(0)$ is certainly a Kan fibration, X^h is indeed Reedy fibrant. The Segal condition is obvious from the pullback (5.20).

For (iii), it suffices by symmetry to establish the first claim. It is then enough to show that for any choice of section in the following diagram the top composite is a Kan equivalence.

$$\begin{tikzcd}
X^h(1) \times_{X_0} X^h(1) \arrow[rr, dashed, bend left] \arrow[dr, "(id,d_0)"', two heads] & & X^h(2) \arrow[ll, "\sim", "(d_2,d_0)"', two heads] \arrow[rr, "(d_2,d_1)"', two heads] & & X^h(1) \times_{X_0} X^h(1) \arrow[dll, "(id,d_0)", two heads] \\
& X^h(1) \times X(0) & & &
\end{tikzcd}$$

But this composite is a map of fibrations over $X^h(1) \times X(0)$ with the map between the fibers over $(f\colon x \to y, z)$ computing the map $(-) \circ f\colon X^h(y;z) \to X^h(x;z)$, which is a Kan equivalence since $f \in X^h(1)$ is an equivalence. Thus the composite is a Kan equivalence, establishing (iii).

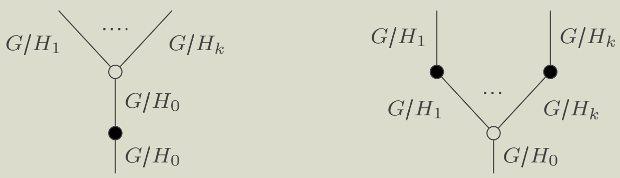
Lastly, for (iv) note that (iii) says that X^h is local with respect to the outer horn inclusions $\Lambda^0[2] \to \Delta[2]$ and $\Lambda^2[2] \to \Delta[2]$, and hence by Remarks 3.23 and 3.24 the map $X^h(J) \to X^h(1)$ is a Kan equivalence. The only remaining claim is that $X^h(J) = X(J)$, which is clear. □

Remark 5.21. The proof of (ii) shows that the inclusion $X^h \to X$ is a Reedy fibration.

REZKCOMP SEC

5.2 Rezk completion and fibrant Segal operads

In the next result we make use of a decomposition of the tensor product $[1] \otimes C$, where $[1]$ is the 1-simplex regarded as a G-trivial G-tree, and C is a G-corolla (see Notation 5.4). Slightly adapting the discussion in Example 3.8, $[1] \otimes C$ is the union of two maximal G-subtrees $C \star \eta$ and $\eta \star C$, whose orbital representations are depicted below.



$C \star \eta$ $\qquad\qquad$ $\eta \star C$

Moreover, much as in (3.10), $C \star \eta$ and $\eta \star C$ have a common orbital face, which we denote simply as C (this face is canonically isomorphic to the original C), leading to a decomposition

$$\Omega[1] \otimes \Omega[C] \simeq \Omega[C \star \eta] \otimes_{\Omega[C]} \Omega[\eta \star C]$$

We note that this holds even if $k = 0$, which is an exceptional case since then $[1] \otimes C = C \star \eta$.

JDDK PROP

Proposition 5.22. *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then the map $X \to X^J$ is a DK-equivalence.*

Proof. Note first that for any $T \in \Omega_G$ the map $X^J(T) \to X^{\Omega[1]}(T)$ can be rewritten as $\left(X^{\Omega[T]}\right)(J) \to \left(X^{\Omega[T]}\right)(\Omega[1])$, and since $\iota^*\left(X^{\Omega[T]}\right)$ is a (simplicial) Segal space Proposition 5.19(iv) says that this map is a weak equivalence onto a subset of components, i.e. a *homotopy monomorphism.*

Hence, for any G-corolla $C = C_{\amalg_i H_0/H_i}$ the horizontal maps in the right square below are homotopy monomorphisms.

$$\begin{tikzcd}
X(C) \arrow[r] \arrow[d, two heads] & X^J(C) \arrow[r] \arrow[d, two heads] & X^{\Omega[1]}(C) \arrow[d, two heads] \\
\prod_{0\leq i\leq k} X(\eta)^{H_i} \arrow[r] & \prod_{0\leq i\leq k}\left(X^J(\eta)\right)^{H_i} \arrow[r] & \prod_{0\leq i\leq k}\left(X^{\Omega[1]}(\eta)\right)^{H_i}
\end{tikzcd} \tag{5.23}$$

BIGSQ EQ

Since fully faithfulness of $X \to X^J$ is the statement that the leftmost square in (5.23) induces weak equivalences on fibers, it suffices to show that so does the composite square.

Now note that $X^{\Omega[1]}(C) = X(\Omega[1] \otimes \Omega[C])$ and that there is a pullback diagram as on the left below. Moreover, both squares below are pullback squares which are injective fibrant squares and the natural map of squares between them is an injective fibration of squares (alternatively, the fibrancy claims state that the resulting cube is an injective fibrant cube).

$$\begin{tikzcd}
X(\Omega[1]\otimes\Omega[C]) \arrow[r, two heads] \arrow[d, two heads] & X(\Omega[\eta\star C]) \arrow[d, two heads] \\
X(\Omega[C\star\eta]) \arrow[r, two heads] & X(\Omega[C])
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_i X([1])^{H_i} \arrow[r, two heads] \arrow[d, two heads] & \left(\prod_{i\neq 0} X([1])^{H_i}\right)\times X(\eta)^{H_0} \arrow[d, two heads] \\
\left(\prod_{i\neq 0} X(\eta)^{H_i}\right)\times X([1])^{H_0} \arrow[r, two heads] & \prod_i X(\eta)^{H_i}
\end{tikzcd} \tag{5.24}$$

OTBIGSQ EQ

Since the top left corner of the map of squares (5.24) is the right vertical map in (5.23), and noting that fibers (in the category of square diagrams) of a fibration between pullback squares are a fibrant pullback square, the desire claim that the total diagram in (5.23) induces equivalences on fibers will follow provided that the same holds for the following squares (which adapt [Rez01, Lemma 12.4]).

$$\begin{tikzcd}
X(\Omega[C]) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(\Omega[C\star\eta]) \arrow[d, two heads, "\sim"] \\
& X(\Omega[C]) \underset{X(\eta)^{H_0}}{\times} X([1])^{H_0} \arrow[d, two heads] \\
\prod_i X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod_{i\neq 0} X(\eta)^{H_i}\right)\times X([1])^{H_0}
\end{tikzcd}
\qquad
\begin{tikzcd}
X(\Omega[C]) \arrow[r, "s_\eta"] \arrow[dd, two heads] & X(\Omega[\eta\star C]) \arrow[d, two heads, "\sim"] \\
& \prod_{i\neq 0} X([1])^{H_i} \underset{\prod_{i\neq 0} X(\eta)^{H_i}}{\times} X(\Omega[C]) \arrow[d, two heads] \\
\prod_i X(\eta)^{H_i} \arrow[r, "s_0"'] & \left(\prod_{i\neq 0} X([1])^{H_i}\right)\times X(\eta)^{H_0}
\end{tikzcd}$$

But this is clear from the fact that the top right vertical maps in these diagrams are trivial Kan fibrations thanks to the Segal condition.

Lastly, to check essential surjectivity, since G acts trivially on J one has $\left(\iota_G^*\left(X^J\right)\right)(G/H) = \left(\iota_G^*(X)(G/H)\right)^J$ so that we reduce to the case of X a (simplicial) Segal space. Noting that J is a contractible Kan complex, one has a map $H\colon J\times J \to \{0\}\times J$ such that $H|_{\{0\}\times J} = id_{\{0\}\times J}$ and $H|_{\{1\}\times J} = (0,0)$. But noting that $X(J\times J)\to X(\{0\}\times J)$ can be written as $X^J(0)\to X^J(J)$, the composite below shows that any object in $\left(X^J\right)_0$ is indeed equivalent to a degenerate object, which is thus in the image of $X \to X^J$.

$$X^J(0) \xrightarrow{X(H)} X^J(J) \to X^J(1) \rightrightarrows X^J(0)$$

$\square$

Definition 5.25. Two maps $f, f' \colon A \rightrightarrows B$ between dendroidal Segal spaces are called J-homotopic, written $f \sim_J f'$, if there is a H such that the two composites $A \xrightarrow{H} B^J \rightrightarrows B$ are f, f'.

Further, a map $f \colon X \to Y$ of dendroidal Segal spaces is called a J-homotopy equivalence if there exists $g \colon Y \to X$ such that $gf \sim_J id_X$, $fg \sim_J id_Y$.

Remark 5.26. For $f \sim_J f'$, Proposition 5.22 and 2-out-of-3 applied to $A \xrightarrow{H} B^J \rightrightarrows B$ imply that f is a DK-equivalence iff f' is. Thus by 2-out-of-6 J-homotopy equivalences are DK-equivalences.

ALLXJK REM **Remark 5.27.** Let X be a dendroidal Segal space. All simplicial operators $X^{J^m} \to X^{J^{m'}}$ are induced by equivalences of groupoids $\widetilde{[m']} \to \widetilde{[m]}$, and are thus J-homotopy equivalences and thus also DK-equivalences.

COMPLE PROP **Proposition 5.28.** *Let $X \in \mathsf{sdSet}^G$ be a dendroidal Segal space. Then there is a complete dendroidal Segal space $\tilde{X}$ and complete equivalence $X \to \tilde{X}$ such that*

(i) $X \to \tilde{X}$ is a monomorphism and a DK-equivalence;

(ii) $X_0(\eta) \to \tilde{X}_0(\eta)$ is an isomorphism.

Proof. Our proof mostly adapts the construction of the completion functor in [Rez01, §10.4].

Firstly, we let $X^{J^\bullet} \in (\mathsf{sdSet}^G)^{\Delta^{op}} = \mathsf{ssdSet}^G$ be the object whose m-th level is X^{J^m}.

We will regard the new simplicial direction in $\mathsf{ssdSet}^G = (\mathsf{sdSet}^G)^{\Delta^{op}}$ as horizontal and the old one as vertical, and abbreviate the Reedy model structure on $(\mathsf{sdSet}^G)^{\Delta^{op}}$ with respect to the dendroidal Reedy model structure on sdSet^G as the *horizontal Reedy model structure* on ssdSet^G.

Noting that $J^\bullet$ is a Reedy cofibrant cosimplicial object, it follows that $X^{J^\bullet}$ is horizontal Reedy fibrant, and it is thus formal that $X^{J^\bullet} \to \mathsf{csk}_\eta X^{J^\bullet}$ is a horizontal fibration in $(\mathsf{sdSet}^G)^{\Delta^{op}}$ as well. As such, for $T \in \Omega_G$ the evaluations

$$X^{J^\bullet}(\Omega[T]) \to \left(\mathsf{csk}_\eta X^{J^\bullet}\right)(\Omega[T]) \tag{5.29}$$ MOREOVER EQ

are horizontal Reedy fibrations in ssSet in the sense of in §4.1. In particular, for each vertex map $[0] \to [m]$ the induced square

$$\begin{tikzcd}
X^{J^m}(\Omega[T]) \arrow[r, two heads] \arrow[d, two heads] & X(\Omega[T]) \arrow[d, two heads] \\
\prod_{e_i \in E_G(T)} \left(X^{J^m}(\eta)\right)^{H_i} \arrow[r, two heads] & \prod_{e_i \in E_G(T)} \left(X(\eta)\right)^{H_i}
\end{tikzcd}$$

is an (injective) fibrant square, which by Remark 5.27 induces weak equivalences on fibers, so that the map from $X^{J^m}(\Omega[T])$ to the pullback of the remaining diagram is a trivial Kan fibration.

By (the dual of) Remark 4.8 we have just shown that (5.29) satisfies the "moreover" condition in Corollary 4.7. Therefore, applying δ^* to (5.29) yields a Kan fibration, so that all fibers of $\delta^*\left(X^{J^\bullet}(\Omega[T])\right) \to \delta^*\left(\left(\mathsf{csk}_\eta X^{J^\bullet}\right)(\Omega[T])\right)$ are in fact homotopy fibers.

We now write $\tilde{X}$ for a dendroidal Reedy fibrant replacement of the diagonal $\delta^*\left(X^{J^\bullet}\right) \in \mathsf{sdSet}^G$, which we note can always be chosen so that $\delta^*\left(X^{J^\bullet}\right) \to \tilde{X}$ is a monomorphism and $\tilde{X}_0(\eta) = \left(\delta^*\left(X^{J^\bullet}\right)\right)_0(\eta) = X_0(\eta)$ (this follows since fibrant replacements in the Kan model structure in sSet can be chosen to preserve 0-simplices, since existence of lifts against the horn inclusions $\Delta[0] = \Lambda^0[1] \to \Delta[1]$, $\Delta[0] = \Lambda^1[1] \to \Delta[1]$ is automatic).

To see that $\tilde{X}$ is a complete Segal space, note that in the composite $X_0^{J^\bullet} \to \delta^*\left(X^{J^\bullet}\right) \to \tilde{X}$ the first map is a dendroidal Reedy equivalence by Proposition SSSETJREE PROP 4.5(iv) and the second by definition of $\tilde{X}$. But since $X_0^{J^\bullet}$ is a complete Segal space by Remark CONCRECOM REM 4.20, so is $\tilde{X}$.

For the remaining claim that the composite $X = X^{J^0} \to \delta^*\left(X^{J_\bullet}\right) \to \tilde{X}$ is a DK equivalence, note that though the first map is no longer a dendroidal Reedy equivalence, it is nonetheless an equivalence on fibers over $\prod_{e_i \in E_G(T)} (X(\eta))^{H_i}$ for each $T \in \Omega_G$. And since we have shown that the fibers of $\delta^*\left(X^{J^\bullet}\right)(T)$ are homotopy fibers, these are equivalent to the fibers of $\tilde{X}(T)$ (since Reedy replacement does not change the homotopy fibers), and thus $X \to \tilde{X}$ is indeed fully faithful. Essential surjectivity is trivial since the objects coincide. The monomorphism condition is clear. □

COMPIFFDK THM

Theorem 5.30. *A map of $X \to Y$ of dendroidal Segal spaces is a complete equivalence iff it is a DK-equivalence.*

Proof. By the previous result one is free to assume that X, Y are complete, so that by Proposition COMBMODSTR PROP 4.1(ii) complete equivalences coincide with simplicial equivalences.

Assume first that $f\colon X \to Y$ is a DK-equivalence. We first show that $X(\eta)^H \to Y(\eta)^H$ is a Kan equivalence in sSet. Indeed, the completion condition states that $X(\eta)^H = X(G\cdot\Omega[\eta])^H \xrightarrow{\sim} X(G\cdot J)^H$ is a weak equivalence, so that the fibers of the left diagram are equivalent to the homotopy fibers of the right diagram, i.e. to the loop spaces $\Omega\left(X(\eta)^H, x\right)$.

$$\begin{tikzcd} & X(G\cdot J)^H \arrow[d, two heads] \\ \Delta[0] \arrow[r, "{(x,x)}"] & X(\eta)^H \times X(\eta)^H \end{tikzcd} \qquad \begin{tikzcd} & X(\eta)^H \arrow[d] \\ \Delta[0] \arrow[r, "{(x,x)}"] & X(\eta)^H \times X(\eta)^H \end{tikzcd} \tag{5.31}$$

WHATEV EQ

Therefore, since $X(G\cdot J)^H \to X(G\cdot\Omega[1])^H = X(C_{H/H})$ is a homotopy monomorphism, fully faithfulness implies that $X(\eta)^H \to Y(\eta)^H$ induces isomorphisms on homotopy groups. Injectivity on components is similar (x, x' are in the same component iff the (x,x') fiber in (WHATEV EQ 5.31) is non-empty) while essential surjectivity implies surjectivity on components, and thus $X(\eta)^H \to Y(\eta)^H$ is indeed a Kan equivalence.

We now show that $X(\Omega[T]) \to Y(\Omega[T])$ is a Kan equivalence for all $T \in \Omega_G$. Consider the diagram

$$\begin{tikzcd} X(\Omega[T]) \arrow[r] \arrow[d, two heads, "\sim"'] & Y(\Omega[T]) \arrow[d, two heads, "\sim"] \\ X(Sc[T]) \arrow[r] \arrow[d, two heads] & Y(Sc[T]) \arrow[d, two heads] \\ \prod\limits_{e_i \in E_G(T)} (X(\eta))^{H_i} \arrow[r] & \prod\limits_{e_i \in E_G(T)} (Y(\eta))^{H_i} \end{tikzcd} \tag{5.32}$$

DKCOM EQ

where the maps marked ~ are trivial fibrations by the Segal condition. Since the bottom horizontal map is already known to be a Kan equivalence, it suffices to note that by fully faithfulness the middle horizontal map induces Kan equivalences on fibers, and is thus a Kan equivalence itself. This finishes the proof that DK-equivalences $f\colon X \to Y$ are also simplicial equivalences.

Assuming now that $f\colon X \to Y$ is a simplicial equivalence, fully faithfulness follows from (DKCOM EQ 5.32) by considering maps of fibers and essential surjectivity follows since the maps $X(\eta)^H \to Y(\eta)^H$ are surjective on components. □

FIB_PREOP_COR **Corollary 5.33.** *A pre-operad $X \in \mathsf{PreOp}^G$ is fibrant iff $\gamma^*(X)$ is fibrant in the Segal space model structure on sdSet^G.*

Proof. We start with the "only if" direction. Recall that $\gamma^* X$ is a dendroidal Segal space if it has the right lifting property against the maps of the form

$$(\Lambda^i[n] \to \Delta[n]) \,\square\, (\partial\Omega[T] \to \Omega[T]) \qquad (\partial\Delta[n] \to \Delta[n]) \,\square\, (Sc[T] \to \Omega[T]). \tag{5.34}$$ SOMEMAPS EQ

With the exception of the first type of maps when $T = G \cdot_H \eta$, in which case the lifting condition is automatic since $\gamma^* X(\eta)$ is discrete, all other maps induce isomorphisms at the η-level, so that by Remark 4.21 applying $\gamma_!$ to these maps yields trivial cofibrations in PreOp^G. Thus, if $X \in \mathsf{PreOp}^G$ is fibrant, an adjunction argument shows that $\gamma^*(X)$ indeed has the lifting property against all maps (5.34), i.e. that $\gamma^*(X)$ is a dendroidal Segal space.

For the "if" direction, we form the completion $\gamma^* X \to \tilde{X}$ described in Proposition 5.28. Then $\gamma_* \tilde{X} \in \mathsf{PreOp}^G$ is fibrant by Theorem 4.28 and the adjoint map $X \to \gamma_* \tilde{X}$ has the following properties: (i) it is a monomorphism; (ii) it is an isomorphism at the η-level; (iii) it is a DK-equivalence when regarded as a map in sdSet^G (since by Remark 5.10 $\gamma^*\gamma_* \tilde{X} \to \tilde{X}$ is tautologically a DK-equivalence); (iv) it is hence a trivial dendroidal Reedy cofibration when regarded as a map in sdSet^G. But then the hypothesis that $\gamma^* X$ is a dendroidal Segal space yields a lift

$$\begin{tikzcd}
\gamma^* X \arrow[r, equal] \arrow[d] & \gamma^* X \\
\gamma^*\gamma_* \tilde{X} \arrow[ur, dashed] &
\end{tikzcd}$$

showing that X is a retract of $\gamma_* \tilde{X}$ and finishing the proof. □

INTERP REM **Remark 5.35.** For any dendroidal Segal space $X \in \mathsf{sdSet}^G$ one hence has complete equivalences

$$\gamma_* X \to X \to \tilde{X}$$

where $\gamma_* X$ is a fibrant preoperad and $\tilde{X}$ a complete dendroidal Segal space.

6 Indexing system analogue results

INDEX SEC

Just as in [Per17, §9], we dedicate our final section to outlining the generalizations of our results to the *indexing systems* of Blumberg and Hill [BH15]. Or more precisely, we will work with the *weak indexing systems* of [Per17, §9], [BP17, §4.4], which are a slight generalization of indexing systems, and were also independently identified by Gutierrez and White in [GW17].

We begin by recalling the key notion of sieve.

Definition 6.1. A *sieve* of a category $\mathcal{C}$ is a full subcategory $\mathcal{S} \subseteq \mathcal{C}$ such that for any arrow $c \to s$ in $\mathcal{C}$ such that $s \in \mathcal{S}$ it is also $c \in \mathcal{C}$.

Note that a sieve $\mathcal{S} \subseteq \mathcal{C}$ determines a presheaf $\delta_{\mathcal{S}} \in \mathsf{Set}^{\mathcal{C}^{op}}$ by $\delta_{\mathcal{S}}(c) = *$ if $c \in \mathcal{S}$ and $\delta_{\mathcal{S}}(c) = \varnothing$ if $c \notin \mathcal{S}$. In fact, there is a clear bijection between sieves and such *characteristic presheaves*, i.e. presheaves taking only the values $*$ and $\varnothing$, and we will hence often blur the distinction between the two concepts.

Sieves are prevalent in equivariant homotopy theory. Indeed, families $\mathcal{F}$ of subgroups of G are effectively the same as sieves $\mathsf{O}_{\mathcal{F}} \subseteq \mathsf{O}_G$ of the orbit category O_G (formed by the G-sets G/H).

Weak indexing systems can then be thought of as the operadic analogue of families. In particular, they are described by sieves $\Omega_{\mathcal{F}} \subseteq \Omega_G$, but additional conditions are needed to ensure compatibility with the operadic composition and unit. In the following, we abbreviate $\delta_{\mathcal{F}} = \delta_{\Omega_{\mathcal{F}}}$ and, for each G-vertex v of $T \in \Omega_G$, we write T_V for the orbital outer face whose only G-vertex is v.

Definition 6.2. A *weak indexing system* is a full subcategory $\Omega_{\mathcal{F}} \subseteq \Omega_G$ such that:

(i) $\Omega_{\mathcal{F}}$ is a sieve of Ω_G;

(ii) for each $T \in \Omega_G$ it is $T \in \Omega_{\mathcal{F}}$ iff $\forall_{v \in V_G(T)} T_v \in \Omega_{\mathcal{F}}$ or, equivalently, if

$$\delta_{\mathcal{F}}(T) = \prod_{v \in V_G(T)} \delta_{\mathcal{F}}(T_v). \tag{6.3}$$ SEGCOMB EQ

SEGCOMB REM **Remark 6.4.** Condition (ii) can be interpreted as combining the following two conditions:

(ii') the characteristic presheaf $\delta_{\mathcal{F}}$ is Segal, i.e. $\delta_{\mathcal{F}}(T) = \delta_{\mathcal{F}}(Sc[T])$ for all $T \in \Omega_G$;

(ii") $(G/G \cdot \eta) \in \Omega_{\mathcal{F}}$.

Here, (ii") reflects the existence of units in G-operads, which are encoded by the G-trivial 1-corolla $G/G \cdot [1]$ (note that by the sieve condition (i) it is $(G/G \cdot [1]) \in \Omega_{\mathcal{F}}$ iff $(G/G \cdot \eta) \in \Omega_{\mathcal{F}}$).

Similarly, (ii') reflects the composition in G-operads. Indeed, (i) and (ii") imply that all stick G-trees $G/H \cdot \eta$ are in $\Omega_{\mathcal{F}}$, so that the right hand side of (6.3) can be reinterpreted as $\delta_{\mathcal{F}}(Sc[T])$ (more formally, $\delta_{\mathcal{F}}(Sc[T])$ is defined via an analogue of Notation 4.4, so as to obtain a functor $\delta_{\mathcal{F}}(-) \colon (\mathsf{dSet}_G)^{op} \to \mathsf{sSet}$ and by reinterpreting $Sc[T]$ as an object in dSet_G via applying υ_*).

Remark 6.5. The original notion of indexing system in [BH15, Def. 3.22] is recovered by demanding that all G-trivial n-corollas $G/G \cdot C_n$ are in $\Omega_{\mathcal{F}}$.

WHYF REM **Remark 6.6.** The $\mathcal{F}$ in the notation $\Omega_{\mathcal{F}}$ is meant to suggest an alternate description of (weak) indexing systems in terms of families of subgroups.

Namely, given a weak indexing system $\Omega_{\mathcal{F}}$ and $n \geq 0$, we let $\mathcal{F}_n$ denote the family of those subgroups of $\Gamma \leq G \times \Sigma_n = G \times \mathsf{Aut}(C_n)$ which are graphs of partial homomorphisms $G \geq H \to \Sigma_n$ such that the associated G-corolla $G \cdot_H C_n$ is in $\Omega_{\mathcal{F}}$. $\mathcal{F}$ then stands for the collection $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$.

More generally, for each $U \in \Omega$, we similarly write $\mathcal{F}_U$ for the family of graph subgroups of $G \times \mathsf{Aut}(U)$ encoding partial homomorphisms $G \geq H \to \mathsf{Aut}(U)$ such that $G \cdot_H U \in \Omega_{\mathcal{F}}$.

The fact that each $\mathcal{F}_U$ is a family is a consequence of the sieve condition (i). On the other hand, (ii) imposes more complex conditions on $\{\mathcal{F}_n\}_{n \geq 0}$ which [BH15, Def. 3.22] makes explicit.

All results in the paper now extend to the context of a general weak indexing system $\Omega_{\mathcal{F}}$ essentially by replacing Ω_G by $\Omega_{\mathcal{F}}$ throughout. The following are some notable consequences:

- notions in dSet^G such as "G-normal monomorphism", "G-inner horn", G-inner anodyne, G-∞-operad are replaced with "$\mathcal{F}$-normal monomorphism", "$\mathcal{F}$-inner horn", $\mathcal{F}$-inner anodyne, $\mathcal{F}$-∞-operad;
- the model structure on dSet^G from [Per17, Thm. 2.1] is replaced with the model structure $\mathsf{dSet}^G_{\mathcal{F}}$ (on the *same* underlying category) from [Per17, Thm. 2.2], whose cofibrations are the $\mathcal{F}$-normal monomorphisms and whose fibrant objects are the $\mathcal{F}$-∞-operads;
- $\mathsf{dSet}_G = \mathsf{Set}^{\Omega_G^{op}}$ is replaced with $\mathsf{dSet}_{\mathcal{F}} = \mathsf{Set}^{\Omega_{\mathcal{F}}^{op}}$.

We briefly outline the main reasons why these substitutions do not affect our proofs.

Firstly, the characteristic edge lemma, Lemma 3.4, extends automatically. Indeed, if $T \in \Omega_{\mathcal{F}}$ the sieve condition implies that the filtrations produced by the original Lemma 3.4 must necessarily use only $\mathcal{F}$-inner horn inclusions. Therefore, all results in §3.2, most notably Proposition 3.18 concerning hypersaturations, extend to a general weak indexing system $\Omega_{\mathcal{F}}$ via the same proof.

Next, in §4.2, one can again consider two different Reedy model structures on sdSet^G. Firstly, using the fact that Δ is Reedy and the model structure $\mathsf{dSet}^G_{\mathcal{F}}$, one obtains a *$\mathcal{F}$-simplicial model structure* on dSet^G. Secondly, using the fact that $\Omega^{op} \times G$ is generalized Reedy such that the families $\{\mathcal{F}_U\}$ in Remark 6.6 are Reedy admissible (see Example A.7) together with the Kan model structure on sSet, Theorem A.8 yields a *$\mathcal{F}$-dendroidal Reedy model structure* on sdSet^G. Thus, by applying Proposition 4.1 to combine the two structures, one obtains a $\mathcal{F}$-joint/$\mathcal{F}$-complete model structure, which we denote $\mathsf{sdSet}^G_{\mathcal{F}}$. The remaining discussion in §4.2 then follows through to yield the analogue of Theorem 4.19.

FINCOAGJ THM

Theorem 6.7. *The constant/0-th level adjunction*

$$c_! \colon \mathsf{dSet}^G_{\mathcal{F}} \rightleftarrows \mathsf{sdSet}^G_{\mathcal{F}} \colon (-)_0$$

is a Quillen equivalence.

In §4.3 the modifications are entirely straightforward, with the model structure $\mathsf{sdSet}^G_{\mathcal{F}}$ inducing a model structure $\mathsf{PreOp}^G_{\mathcal{F}}$ via the obvious analogue of Theorem 4.27, and yielding the analogue of Theorem 4.28.

FANOQUEQUIV THM

Theorem 6.8. *The adjunction*

$$\gamma^* \colon \mathsf{PreOp}^G_{\mathcal{F}} \rightleftarrows \mathsf{sdSet}^G_{\mathcal{F}} \colon \gamma_*$$

is a Quillen equivalence.

In §5.1 $\mathcal{F}$-dendroidal Segal spaces are defined in the natural way. The most notable difference is then that one works only with $\mathcal{F}$-corollas, i.e. G-corollas $C \in \Omega_{\mathcal{F}}$, and thus only with $\mathcal{F}$-profiles, thus obtaining a notion of $\mathcal{F}$-fully faithfullness and of $\mathcal{F}$-DK-equivalence (essential surjectivity needs not be changed due to condition (ii") in Remark 6.4 implying that the stick G-trees $G/H \cdot \eta$ are all in $\Omega_{\mathcal{F}}$). Thus, noting that the Segal condition (ii') in Remark 6.4 ensures that the grafted G-trees $T = C \amalg_{G \cdot e} D$ in (5.14) are in $\Omega_{\mathcal{F}}$ whenever C, D are $\mathcal{F}$-corollas, the remaining discussion in §5.1, §5.2 generalizes to yield the analogue of Theorem 5.30.

FCOMPIFFDK THM

Theorem 6.9. *A map of $X \to Y$ of $\mathcal{F}$-dendroidal Segal spaces is a $\mathcal{F}$-complete equivalence iff it is a $\mathcal{F}$-DK-equivalence.*

7 Scratchwork

Remark 7.1. bla bla the diagrams for compositions of norm maps are given by orbital representations, but the category Ω_G is better described in terms of the expanded representation.

A Equivariant Reedy model structures

EQREED AP

Bla bla one of the axioms in [BM11] is different from the others point of view

In [BM11] Berger and Moerdijk extend the notion of Reedy category so as to allow for categories $\mathbb{R}$ with non-trivial automorphism groups $\mathsf{Aut}(r)$ for $r \in \mathbb{R}$. For such $\mathbb{R}$ and suitable model category $\mathcal{C}$ they then show that there is a *Reedy model structure* on $\mathcal{C}^{\mathbb{R}}$ that is defined by modifying the usual characterizations of Reedy cofibrations, weak equivalences and fibrations (see [BM11, Thm. 1.6] or Theorem A.8 below) to be determined by the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ for each $r \in \mathbb{R}$.

The purpose of this appendix is to show that, under suitable conditions, this can also be done by replacing the $\mathsf{Aut}(r)$-projective model structures on $\mathcal{C}^{\mathsf{Aut}(r)}$ with the more general $\mathcal{C}^{\mathsf{Aut}(r)}_{\mathcal{F}_r}$ model structures for $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a nice collection of families of subgroups of each $\mathsf{Aut}(r)$.

To do so, we first need some key notation. For each map $r \to r'$ in the category $\mathbb{R}$ we will write $\mathsf{Aut}(r \to r')$ for its automorphim group in the arrow category and write

$$\mathsf{Aut}(r) \xleftarrow{\pi_r} \mathsf{Aut}(r \to r') \xrightarrow{\pi_{r'}} \mathsf{Aut}(r') \tag{A.1}$$

PIDEFR EQ

for the obvious projections. We now introduce our equivariant generalization of the "generalized Reedy categories" of [BM11, Def. 1.1].

GENRED DEF

Definition A.2. A *generalized Reedy category structure* on a small category $\mathbb{R}$ consists of wide subcategories $\mathbb{R}^+$, $\mathbb{R}^-$ and a degree function $|-|: ob(\mathbb{R}) \to \mathbb{N}$ such that:

(i) non-invertible maps in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) raise (lower) degree; isomorphisms preserve degree;

(ii) $\mathbb{R}^+ \cap \mathbb{R}^- = \mathsf{Iso}(\mathbb{R})$;

(iii) every map f in $\mathbb{R}$ factors as $f = f^+ \circ f^-$ with $f^+ \in \mathbb{R}^+$, $f^- \in \mathbb{R}^-$, and this factorization is unique up to isomorphism.

Let $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ be a collection of families of subgroups of the groups $\mathsf{Aut}(r)$. The collection $\{\mathcal{F}_r\}$ is called *Reedy-admissible* if:

(iv) for all maps $r \twoheadrightarrow r'$ in $\mathbb{R}^-$ one has $\pi_{r'}\left(\pi_r^{-1}(H)\right) \in \mathcal{F}_{r'}$ for all $H \in \mathcal{F}_r$.

We note that condition (iv) above should be thought as of a constraint on the pair $(\mathbb{R}, \{\mathcal{F}_r\})$. The original setup of [BM11] then deals with the case where $\{\mathcal{F}_r\} = \{\{e\}\}$ is the collection of trivial families. Indeed, our setup recovers the setup in [BM11], as follows.

Example A.3. When $\{\mathcal{F}_r\} = \{\{e\}\}$, Reedy-admissibility coincides with axiom (iv) in [BM11, Def. 1.1], stating that if $\theta \circ f^- = f^-$ for some $f^- \in \mathbb{R}^-$ and $\theta \in \mathsf{Iso}(\mathbb{R})$ then θ is an identity.

Example A.4. For any generalized Reedy category $\mathbb{R}$, the collection $\{\mathcal{F}_{\text{all}}\}$ of the families of all subgroups of $\mathsf{Aut}(r)$ is Reedy-admissible.

Example A.5. Let G be a group and set $\mathbb{R} = G \times (0 \to 1)$ with $\mathbb{R} = \mathbb{R}^+$. Then any pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible.

Similarly, set $\mathbb{S} = G \times (0 \leftarrow 1)$ with $\mathbb{S} = \mathbb{S}^-$. Then a pair $\{\mathcal{F}_0, \mathcal{F}_1\}$ of families of subgroups of G is Reddy-admissible iff $\mathcal{F}_0 \supset \mathcal{F}_1$.

GGRAPHREEDY EX

Example A.6. Letting $\mathbb{S}$ denote any generalized Reedy category in the sense of [BM11, Def. 1.1] and G a group, we set $\mathbb{R} = G \times \mathbb{S}$ with $\mathbb{R}^+ = G \times \mathbb{S}^+$ and $\mathbb{R}^- = G \times \mathbb{S}^+$. Further, for each $s \in \mathbb{S}$ we write $\mathcal{F}^{\Gamma}_s$ for the family of G-graph subgroups of $G \times \mathsf{Aut}_{\mathbb{S}}(s)$, i.e., those subgroups $K \leq G \times \mathsf{Aut}_{\mathbb{S}}(s)$ such that $K \cap \mathsf{Aut}_{\mathbb{S}}(s) = \{e\}$.

Reedy admissibility of $\{\mathcal{F}^{\Gamma}_s\}$ follows since for every degeneracy map $s \twoheadrightarrow s'$ in $\mathbb{S}^-$ one has that the homomorphism $\pi_s\colon \mathsf{Aut}_{\mathbb{S}}(s \twoheadrightarrow s') \to \mathsf{Aut}_{\Omega}(s)$ is injective (we note that this is equivalent to axiom (iv) in [BM11, Def. 1.1] for $\mathbb{S}$).

Our primary example of interest will come by setting $\mathbb{S} = \Omega^{op}$ in the previous example. In fact, in this case we will also be interested in certain subfamilies $\{\mathcal{F}_U\}_{U\in\Omega} \subset \{\mathcal{F}^{\Gamma}_U\}_{U\in\Omega}$.

FGRAPHREEDY EX **Example A.7.** Let $\mathbb{R} = G \times \Omega^{op}$ and let $\{\mathcal{F}_U\}_{U\in\Omega}$ be the family of graph subgroups determined by a weak indexing system $\mathcal{F}$. Then $\{\mathcal{F}_U\}$ is Reedy-admissible. To see this, recall first that each $K \in \mathcal{F}_U$ encodes an H-action on $U \in \Omega$ for some $H \leq G$ so that $G \cdot_H U$ is a $\mathcal{F}$-tree. Given a face map $f\colon U' \hookrightarrow U$, the subgroup $\pi_U^{-1}(K)$ is then determined by the largest subgroup $\bar{H} \leq H$ such that U' inherits the $\bar{H}$-action from U along f (so that f becomes a $\bar{H}$-map), so that $\pi_{U'}(\pi_U^{-1}(K))$ encodes the $\bar{H}$-action on U'. Thus, we see that Reedy-admissibility is simply the sieve condition for the induced map of G-trees $G \cdot_{\bar{H}} U' \to G \cdot_H U$.

We now state the main result. We will assume throughout that $\mathcal{C}$ is a model category such that for any group G and family of subgroups $\mathcal{F}$, the category $\mathcal{C}^G$ admits the $\mathcal{F}$-model structure (for example, this is the case whenever $\mathcal{C}$ is a cofibrantly generated cellular model category in the sense of [Ste16]).

REEDYADM THM **Theorem A.8.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r\in\mathbb{R}}$ a Reedy-admissible collection of families. Then there is a $\{\mathcal{F}_r\}$-**Reedy model structure** on $\mathcal{C}^{\mathbb{R}}$ such that a map $A \to B$ is*

- *a (trivial) cofibration if $A_r \coprod_{L_r A} L_r B \to B_r$ is a (trivial) $\mathcal{F}_r$-cofibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a weak equivalence if $A_r \to B_r$ is a $\mathcal{F}_r$-weak equivalence in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$;*
- *a (trivial) fibration if $A_r \to B_r \times_{M_r B} M_r A$ is a (trivial) $\mathcal{F}_r$-fibration in $\mathcal{C}^{\mathsf{Aut}(r)}$, $\forall r \in \mathbb{R}$.*

The proof of this result is given at the end of the appendix after establishing some routine generalizations of the key lemmas in [BM11] (indeed, the true novelty in this appendix is the Reedy-admissibility condition in part (iv) of Definition A.2).

We first recall the following, cf. [BP17, Props. 6.5 and 6.6] (we note that [BP17, Prop. 6.6] can be proven in terms of fibrations, and thus does not depend on special assumptions on $\mathcal{C}$).

Proposition A.9. *Let $\phi\colon G \to \bar{G}$ be a homomorphism and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of subgroups of $G, \bar{G}$. Then the leftmost (resp. rightmost) adjunction below is a Quillen adjunction*

$$\bar{G} \cdot_G (-)\colon \mathcal{C}^G_{\mathcal{F}} \rightleftarrows \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \colon \mathsf{res}^{\bar{G}}_G \qquad \mathsf{res}^{\bar{G}}_G\colon \mathcal{C}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{C}^G_{\mathcal{F}} \colon \mathsf{Hom}_G(\bar{G}, -)$$

provided that for $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$ (resp. for $\bar{H} \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$).

RESGEN COR **Corollary A.10.** *For any homomorphism $\phi\colon G \to \bar{G}$, the functor $\mathsf{res}^{\bar{G}}_G\colon \mathcal{C}^{\bar{G}} \to \mathcal{C}^G$ preserves all four classes of genuine cofibrations, trivial cofibrations, fibrations and trivial fibrations.*

The following formalizes an argument implicit in the proof of [BM11, Lemma 5.2]).

Definition A.11. Consider a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{A.12}$$

BLA EQ

in $\mathcal{C}^{\mathbb{R}}$. A collection of maps $f_s\colon B_s \to X_s$ for $|s| \leq n$ that induce a lift of the restriction of (A.12) to $\mathcal{C}^{\mathbb{R}_{\leq n}}$ will be called a *n-partial lift.*

BLALIFT LEM **Lemma A.13.** *Let $\mathcal{C}$ be any bicomplete category, and consider a commutative diagram as in (A.12). Then any $(n-1)$-partial lift uniquely induces commutative diagrams*

$$\begin{tikzcd} A_r \amalg_{L_rA} L_rB \arrow[r] \arrow[d] & X_r \arrow[d] \\ B_r \arrow[r] \arrow[ur, dashed] & Y_r \times_{M_rY} M_rX \end{tikzcd} \tag{A.14}$$

BLALIFT EQ

in $\mathcal{C}^{\mathsf{Aut}(r)}$ for each r such that $|r| = n$. Furthermore, extensions of the $(n-1)$-partial lift to a n-partial lift are in bijection with choices of $\mathsf{Aut}(r)$-equivariant lifts of the diagrams (A.14) for r ranging over representatives of the isomorphism classes of r with $|r| = n$.

In the next result, by $\{\mathcal{F}_r\}$-cofibration/trivial cofibration/fibration/trivial fibration we mean a map as described in Theorem A.8, regardless of whether such a model structure exists.

BLALIFT COR **Corollary A.15.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}$ an arbitrary family of subgroups of $\mathsf{Aut}(r)$, $r \in \mathbb{R}$. Then a map in $\mathcal{C}^{\mathbb{R}}$ is a $\{\mathcal{F}_r\}$-cofibration (resp. trivial cofibration) iff it has the left lifting property with respect to all $\{\mathcal{F}_r\}$-trivial fibrations (resp. fibrations), and vice-versa for the right lifting property.*

GINJ LEM **Lemma A.16.** *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^+$, K a group, and $\pi\colon \mathbb{S} \to K$ a functor.*

Then if a map $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $A_s \amalg_{L_sA} L_sB \to B_s$ are (resp. trivial) $\mathsf{Aut}(s)$-cofibrations one has that $\mathsf{Lan}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-cofibration.

Proof. By adjunction, one needs only show that for any K-fibration $X \to Y$ in $\mathcal{C}^K$, the map $\pi^*(X \to Y)$ has the right lifting property against all maps $A \to B$ in $\mathcal{C}^{\mathbb{S}}$ as in the statement. By Corollary A.15, it thus suffices to check that the maps

$$(\pi^* X)_s \to (\pi^* Y)_s \times_{M_s\pi^*Y} M_s\pi^* X$$

are $\mathsf{Aut}(s)$-fibrations. But since $M_sZ = *$ (recall $\mathbb{S} = \mathbb{S}^+$) this map is just $X \to Y$ with the $\mathsf{Aut}(s)$-action induced by $\pi\colon \mathsf{Aut}(s) \to K$, hence Corollary A.10 finishes the proof. □

GINJMIN LEM **Lemma A.17.** *Let $\mathbb{S}$ be a generalized Reedy with $\mathbb{S} = \mathbb{S}^-$, K a group, and $\pi\colon \mathbb{S} \to K$ a functor.*

Then if a map $X \to Y$ in $\mathcal{C}^{\mathbb{S}}$ is such that for all $s \in \mathbb{S}$ the maps $X_s \to Y_s \times_{M_sY} M_sX$ are (resp. trivial) $\mathsf{Aut}(s)$-fibrations one has that $\mathsf{Ran}_{\pi:\mathbb{S}\to K}(A \to B)$ is a (trivial) K-fibration.

Proof. This follows dually to the previous proof. □

Remark A.18. Lemmas A.16 and A.17 generalize key parts of the proofs of [BM11, Lemmas 5.3 and 5.5]. The duality of their proofs reflects the duality in Corollary A.10.

Remark A.19. Lemma A.16 will be applied when $K \leq \mathsf{Aut}_{\mathbb{R}}(r)$ and $\mathbb{S} = K \ltimes \mathbb{R}^+(r)$ for $\mathbb{R}$ a given generalized Reedy category and $r \in \mathbb{R}$. Similarly, Lemma A.17 will be applied when $\mathbb{S} = K \ltimes \mathbb{R}^-(r)$. It is straightforward to check that in the $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) case maps in $\mathbb{S}$ can be identified with squares as on the left (right)

$$\begin{tikzcd} r' \arrow[r, "+"] \arrow[d, "+"'] & r \arrow[d, "\simeq"] \\ r'' \arrow[r, "+"'] & r \end{tikzcd} \qquad \begin{tikzcd} r \arrow[r, "-"] \arrow[d, "\simeq"'] & r' \arrow[d, "-"] \\ r \arrow[r, "-"'] & r'' \end{tikzcd}$$

such that the maps labelled $+$ are in $\mathbb{R}^+$, maps labelled $-$ are in $\mathbb{R}^-$, the horizontal maps are non-invertible, and the maps labeled $\simeq$ are automorphisms in K.

In particular, there is thus a *domain* (resp. *target*) functor $d: \mathbb{S} \to \mathbb{R}$ ($t: \mathbb{S} \to \mathbb{R}$), and our interest is in maps $d^* A \to d^* B$ ($t^* A \to t^* B$) in $\mathcal{C}^{\mathbb{S}}$ induced from maps $A \to B$ in $\mathcal{C}^{\mathbb{R}}$ so that

$$\mathsf{Lan}_{\pi} d^* (A \to B) = (L_r A \to L_r B) \qquad \mathsf{Ran}_{\pi} t^* (A \to B) = (M_r A \to M_r B)$$

We are now in a position to prove the following, which are the essence of Theorem A.8.

REEDYTRCOF LEM **Lemma A.20.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a Reedy-admissible family.*

Suppose $A \to B$ be a $\{\mathcal{F}_r\}$-Reedy cofibration. Then the maps $A_r \to B_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $A_r \amalg_{L_r A} L_r B \to B_r$.

Proof. It suffices to check by induction on n that the analogous claim with the restriction $|r| \le n$ also holds. The $n = 0$ case is obvious. Otherwise, letting r range over representatives of the isomorphism classes of r with $|r| = n$, it suffices to check that for each $H \in \mathcal{F}_r$ the map $A_r \to B_r$ is a H-genuine weak equivalence iff so is $A_r \amalg_{L_r A} L_r B \to B_r$.

One now applies Lemma A.16 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^+(r)$ to the map $d^* A \to d^* B$. Note that $\mathcal{F}$-trivial cofibrations are always genuine trivial cofibrations, for any family, so that the trivial cofibrancy requirements are immediate from Corollary A.10. It thus follows that the maps labelled $\sim$

$$\begin{tikzcd}
L_r A \arrow[r, "\sim"] \arrow[d] & L_r B \arrow[d] & \\
A_r \arrow[r, "\sim"'] & L_T B \amalg_{L_T A} A_T \arrow[r] & B_r
\end{tikzcd}$$

are H-genuine trivial cofibrations, finishing the proof. □

REEDYTRFIB LEM **Lemma A.21.** *Let $\mathbb{R}$ be generalized Reedy and $\{\mathcal{F}_r\}_{r \in \mathbb{R}}$ a Reedy-admissible family.*

Let $X \to Y$ be a $\{\mathcal{F}_r\}$-Reedy fibration. Then the maps $X_r \to Y_r$ are all $\{\mathcal{F}_r\}$-weak equivalences iff so are the maps $X_r \to Y_r \times_{M_r Y} M_r X$.

Proof. One repeats the same induction argument on $|r|$. In the induction step, it suffices to verify that, for each r with $|r| = n$ and $H \in \mathcal{F}_r$, the map $X_r \to Y_r$ is a H-genuine weak equivalence iff so is $X_r \to Y_r \times_{M_r Y} M_r X$.

One now applies Lemma A.17 with $K = H$ and $\mathbb{S} = H \ltimes \mathbb{R}^-(r)$ to the map $t^* A \to t^* B$. Note that for each $(r \twoheadrightarrow r') \in \mathbb{S}$ one has $\mathsf{Aut}_{\mathbb{S}}(r \twoheadrightarrow r') = \pi_r^{-1}(H)$ (where π_r is as in (A.1)), so that the trivial fibrancy requirement in Lemma A.17 follows from $\{\mathcal{F}_r\}$ being Reedy-admissible. It follows that the maps labelled $\sim$

$$\begin{tikzcd}
X_r \arrow[r] & Y_r \times_{M_r Y} M_r X \arrow[r, "\sim"] \arrow[d] & Y_r \arrow[d] \\
& M_r X \arrow[r, "\sim"'] & M_r Y
\end{tikzcd}$$

are H-genuine trivial fibrations, finishing the proof. □

Remark A.22. The proofs of Lemmas A.20 and A.21 are similar, but not dual, since Lemma A.21 uses Reedy-admissibility while Lemma A.20 does not. This reflects the difference in the proofs of [BM11, Lemmas 5.3 and 5.5] as discussed in [BM11, Remark 5.6], albeit with a caveat.

Setting $K = \{e\}$ in Lemma A.16 yields that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a genuine Reedy cofibration, i.e. a Reedy cofibration for $\{\mathcal{F}_{\text{all}}\}$ the families of all subgroups. On the other hand, the proof of [BM11, Lemma 5.3] argues that $\mathsf{lim}_{\mathbb{S}}(A \to B)$ is a cofibration provided that $A \to B$ is a projective Reedy cofibration, i.e. a Reedy cofibration for $\{\{e\}\}$ the trivial families (note that all projective cofibrations are genuine cofibrations, so that our claim is more general). Since the cofibration half of the projective analogue of Corollary A.10 only holds if ϕ is a monormorphism, the argument in the proof of [BM11, Lemma 5.3] also includes an injectivity check that is not needed for our proof of Lemma A.20.

proof of Theorem A.8. Lemmas A.20 and A.21 say that the characterizations of trivial cofibrations (resp. trivial fibrations) in the statement of Theorem A.8 are correct, i.e. that they describe the maps that are both cofibrations (resp. fibrations) and weak equivalences.

We refer to the model category axioms in [Hov99, Def. 1.1.3]. Both 2-out-of-3 and the retract axioms are immediate (recall that retracts commute with limits/colimits). The lifting axiom follows from Corollary A.15 while the task of building factorizations $X \to A \to Y$ of a given map $X \to Y$ follows by a similar standard argument by iteratively factorizing the maps

$$X_r \amalg_{L_r X} L_r A \to Y_r \times_{M_r Y} M_r A$$

in $\mathcal{C}^{\mathsf{Aut}(r)}$, thus building both A and the factorization inductively (see, e.g., the proof of [BM11, Thm. 1.6]). $\square$

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