Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

No content yet.

2 Planar and tall maps

2.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

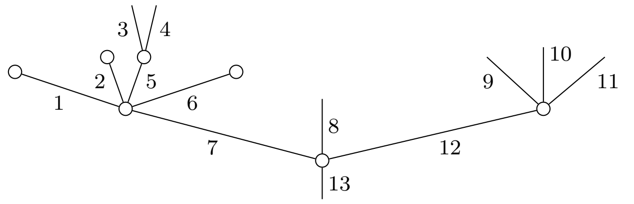
Our preferred model for trees will be that of broad posets first introduced by Weiss in We12 [2] and further worked out by the second author in Pe16b [1]. We now define planar structures in this context.

PLANARIZE DEF

Definition 2.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\leq_d$ to a total order $\leq_p$ such that:

- *Planar*: if $e \leq_p f$ and $e \nleq_d f$ then $g \leq_d f$ implies $e \leq_p g$.

Example 2.2. An example of a planar structure on a tree T follows, with $\leq_r$ encoded by the number labels.



T

Intuitively, given a planar depiction of a tree T, $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition PLANARIZE DEF 2.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT

Notation 2.3. Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path of* e.

INPUTPATHS PROP

Proposition 2.4. *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \leq_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. To prove (a), note that if $g, g' \in I(e)$ were $\leq_d$-incomparable, the relations $e \leq_d g$, $e \leq_d g'$ would contradict Pe16b [1, Cor. 5.26]. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee e_i$. ☐

Notation 2.5. Let $T \in \Omega$ be a tree and suppose that $e <_d g$. We will denote by $g_e^{\uparrow} \in T$ the predecessor of g in $I(e)$.

Proposition 2.6. *Suppose e, f are $\leq_d$-incomparable edges of T and write $g = e \vee f$. Then*

(a) $g_e^{\uparrow}, g_f^{\uparrow} \in g^{\uparrow}$. In fact $\{g_e^{\uparrow}\} = I(e) \cap g^{\uparrow}$, $\{g_f^{\uparrow}\} = I(f) \cap g^{\uparrow}$;

(b) $e <_d g$, $f <_d g$ and $g_e^{\uparrow} \neq g_f^{\uparrow}$;

(c) if $e' \leq_d e$, $f' \leq_d f$ then $g = e' \vee f'$ and $g_{e'}^{\uparrow} = g_e^{\uparrow}$, $g_{f'}^{\uparrow} = g_f^{\uparrow}$.

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Proof. kk ☐

TERNARYJOIN PROP

Proposition 2.7. *All ternary joins in $(T, \leq_d)$ are binary, i.e.*

$$e_1 \vee e_2 \vee e_3 = e_i \vee e_j$$

for some $1 \leq i < j \leq 3$. Further, $e_1 \vee e_2 \vee e_3 \neq e_i \vee e_j$ holds for at most one choice of $1 \leq i < j \leq 3$.

Proof. Writing $E = e_1 \vee e_2 \vee e_3$, by the characterization of joins in Proposition INPUTPATHS PROP 2.4(c) the edges $E_{e_1}^{\uparrow}$, $E_{e_2}^{\uparrow}$, $E_{e_3}^{\uparrow}$ can not all coincide. But if $E_{e_i}^{\uparrow} \neq E_{e_j}^{\uparrow}$ then $E = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$. The "further" claim follows since at most two of $E_{e_1}^{\uparrow}$, $E_{e_2}^{\uparrow}$, $E_{e_3}^{\uparrow}$ coincide. ☐

Notation 2.8. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will usually abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order in S.

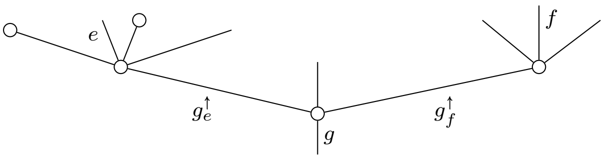
IZATIONCHAR PROP **Proposition 2.9.** *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \leq_p)\} & \longrightarrow & \prod_{(e^{\uparrow} \leq e) \in V(T)} \mathsf{Ord}(e^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p|_{e^{\uparrow}}) \end{array} \tag{2.10}$$

PLANAR EQ

Proof. Throughout we will let e, f be $\leq_d$-incomparable edges and write $g = e \vee f$. Note that one must have $e <_d g$, $f <_d g$ and $g^{\uparrow}_e \neq g^{\uparrow}_f$.

We first show that (2.10) is injective. If $g^{\uparrow}_e <_p g^{\uparrow}_f$, the relations $e \leq_d g^{\uparrow}_e <_p g^{\uparrow}_f \geq_d f$ and Definition 2.1 imply it must be $e <_p f$.



Dually, if $g^{\uparrow}_f <_p g^{\uparrow}_e$ it is $f <_p e$, i.e. $g^{\uparrow}_e <_p g^{\uparrow}_f \Leftrightarrow e <_p f$, and hence (2.10) is indeed injective.

To check that (2.10) is surjective, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $g^{\uparrow}_e < g^{\uparrow}_f$ holds in $g^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Suppose $e' \leq_d e$, $f' \leq_d f$.

It then must be $I(e') \cap I(f') = I(e) \cap I(f)$

Then g' must be $\leq_d$-comparable with both e and f (since the relevant pairs lie in either $I(e')$ or $I(f')$) and this HERE

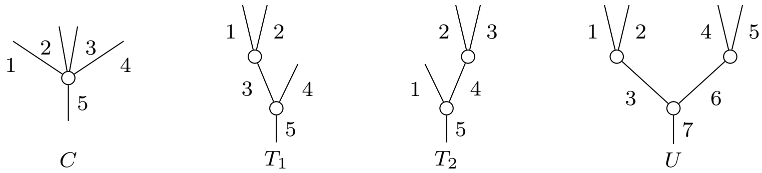
Noting that $e' \vee f'$ must be $\leq_d$-comparable with both e and f (since the relevant pairs lie in either $I(e')$ or $I(f')$), one sees that it must be $e, f <_d e' \vee f'$ (since otherwise all three would lie in either $I(e')$ or $I(f')$) and that hence $e \vee f = e' \vee f'$ and $(e \vee f)^{\uparrow}_e = (e \vee f)^{\uparrow}_{e'}$, $(e \vee f)^{\uparrow}_f = (e \vee f)^{\uparrow}_{f'}$. Therefore, in the conditions of (??) one has $e <_p f$ iff $e' <_p f'$. The planar condition and the non-trivial instances of transitivity thus follow, with the single exception of the $e <_p f <_p g$ case. To check this last case, note that by Proposition 2.7 either: (i) both $e \vee f$, $f \vee g$ equal $e \vee f \vee g$, in which case $(e \vee f \vee g)^{\uparrow}_e <_p (e \vee f \vee g)^{\uparrow}_f <_p (e \vee f \vee g)^{\uparrow}_g$ implies that $e \vee g$ must also equal $e \vee f \vee g$ and transitivity follows; (ii) $e \vee f <_d e \vee f \vee g$, in which case transitivity follows from noting that $(e \vee f \vee g)^{\uparrow}_e = (e \vee f \vee g)^{\uparrow}_f$; (iii) $f \vee g <_d e \vee f \vee g$, which follows just as the previous case. ☐

Remark 2.11. Definition 2.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 2.9 then states that the data of a planarization is equivalent to total orderings of the vertices of F together with a total ordering of the roots of F. The interested reader may wish to suitably modify the proof of Proposition 2.1 to obtain this result. Instead, we simply note that planarizations of F are clearly in bijection with planarizations of the join tree $F \star \eta$ (cf. Equiv. dend. sets).

PLANARCONV CON **Convention 2.12.** From now on, we will write Ω (resp. Ω_G) to denote a model for the category of trees (resp. G-trees) where

- each object (i.e. tree) is equipped with a planar structure;
- morphisms ignore the planar structure;
- there is exactly one representative of each planarization, i.e. the identities are the only isomorphisms that preserve the planar structure.

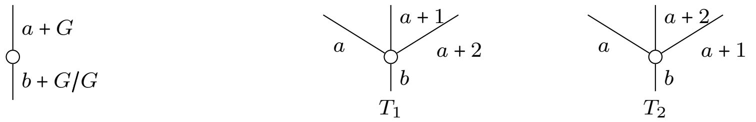
CONCRETETREE REM **Remark 2.13.** The reader desiring extra concreteness is welcome to think of the objects of Ω, Ω_G as consisting of planarized tree structures on one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ such that the planarization $\leq_d$ is the canonical total order. Some trees depicted in this convention follow.



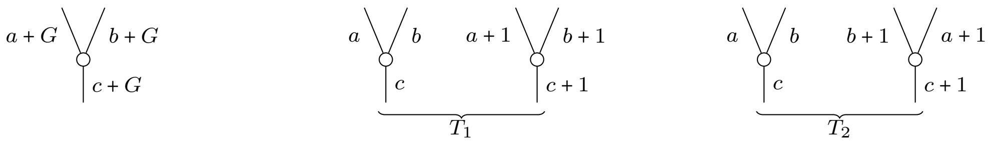
(2.14) PLANAROMEGAEX1 EQ

We note that T_1 and T_2 are isomorphic and, moreover, they encode the only two isomorphism classes of planar structures on the their underlying dendroidal set, so that no other object of Ω is isomorphic to them. C and U, on the other hand, are isomorphic to no other object of Ω, since the planarizations of the underlying broad posets sets are unique up to isomorphism.

One drawback of the concrete convention illustrated in (2.14), however, is that discussion of subfaces of trees becomes awkward, since one can not then technically regard them as subobjects. To avoid this issue, we will often regard the objects of Ω as equivalence classes of trees with planarizations (with no ambiguity resulting since representatives are related via unique isomorphisms). Moreover, this is particularly convenient when discussing G-trees, as it otherwise the task of depicting the G-action becomes cumbersome. For some examples, (and recalling that the numbering of the edges as in (2.14) is superfluous, in the sense that it is already encoded in the planar picture itself), we note that for $G = \mathbb{Z}_{/3}$ the orbital representation on the left below encodes the two isomorphic objects of Ω_G on the right (which are isomorphic to no other object of Ω_G).



Similarly, for $G = \mathbb{Z}_{/2}$, the orbital representation on the left represents the two G-trees presented.



References

Pe16b [1] L. A. Pereira. Equivariant dendroidal sets. Available at: `http://www.faculty.virginia.edu/luisalex/`, 2016.

We12 [2] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: `https://arxiv.org/abs/1201.3987`, 2012.