Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

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1 Introduction

No content yet.

2 Planar and tall maps

2.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

Our preferred model for trees will be that of broad posets first introduced by Weiss in We12 [2] and further worked out by the second author in Pe16b [1]. We now define planar structures in this context.

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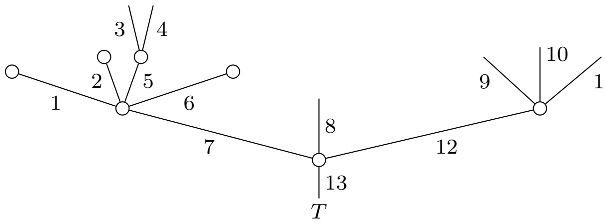
Definition 2.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\le_d$ to a total order $\le_p$ such that:

- *Planar*: if $e \le_p f$ and $e \not\le_d f$ then $g \le_d f$ implies $e \le_p g$.

Example 2.2. An example of a planar structure on a tree T follows, with $\le_r$ encoded by the number labels.



Intuitively, given a planar depiction of a tree T, $e \le_d f$ holds when the downward path from e passes through f and $e \le_p f$ holds if either $e \le_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 2.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT **Notation 2.3.** Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \le_d f\}$$

and refer to this poset as the *input path of* e.

We will repeatedly use the following, which is a consequence of [1, Cor. 5.26].

INCOMPNOTOP **Lemma 2.4.** *If $e \le_d f$, $e \le_d f'$, then f, f' are $\le_d$-comparable.*

INPUTPATHS PROP **Proposition 2.5.** *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \le_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 2.4. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee e_i$. □

Notation 2.6. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^\uparrow \in T$ the predecessor of b in $I(e)$.

ECESSORPROP PROP **Proposition 2.7.** *Suppose e, f are $\le_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^\uparrow \neq b_f^\uparrow$;

(b) $b_e^\uparrow, b_f^\uparrow \in b^\uparrow$. In fact $\{b_e^\uparrow\} = I(e) \cap b^\uparrow$, $\{b_f^\uparrow\} = I(f) \cap b^\uparrow$;

(c) if $e' \le_d e$, $f' \le_d f$ then $b = e' \vee f'$ and $b_{e'}^\uparrow = b_e^\uparrow$, $b_{f'}^\uparrow = b_f^\uparrow$.

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \le_d e$ (resp. $e \le_d f$) while the condition $b_e^\uparrow = b_f^\uparrow$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \le_d b_a^* <_d b$ for some unique $b_a^* \in b^\uparrow$, where uniqueness follows from Lemma 2.4. Choosing $a = e$ implies $I(e) \cap b^\uparrow = \{b_e^*\}$ and letting a range over edges such that $e \le_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \le_d a <_d f$ and $e \le_d a$, and this would imply $e \le_d f$, contradicting our hypothesis. □

TERNARYJOIN PROP **Proposition 2.8.** *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c_{e_i}^\uparrow \neq c_{e_j}^\uparrow$.*

Therefore, all ternary joins in $(T, \le_d)$ are binary, i.e.

$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{2.9}$$ TERNJOIN EQ

for some $1 \le i < j \le 3$, and (2.9) fails for at most one choice of $1 \le i < j \le 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$ then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 2.7(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

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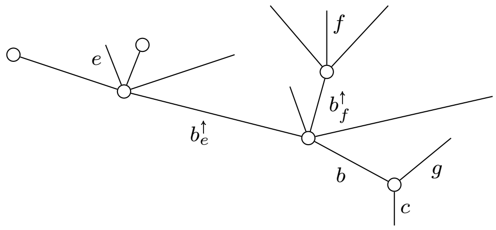
Example 2.10.



Notation 2.11. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will usually abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order in S.

Proposition 2.12. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\text{planar structures } (T, \leq_p)\} & \longrightarrow & \prod_{(a^{\uparrow} \leq a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p |_{a^{\uparrow}}) \end{array} \tag{2.13}$$

Proof. We will keep the setup of Proposition 2.7 throughout: e, f are $\leq_d$-incomparable edges and we write $b = e \vee f$.

We first show that (2.13) is injective, i.e. that the restrictions $\leq_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_e <_p b^{\uparrow}_f$, the relations $e \leq_d b^{\uparrow}_e <_p b^{\uparrow}_f \geq_d f$ and Definition 2.1 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_f <_p b^{\uparrow}_e$ then $f <_p e$. Thus $b^{\uparrow}_e <_p b^{\uparrow}_f \Leftrightarrow e <_p f$ and hence (2.13) is indeed injective.

To check that (2.13) is surjective, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $b^{\uparrow}_e < b^{\uparrow}_f$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\leq_p$ in the case $e' \leq_d e \leq_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 2.7(c). Transitivity of $\leq_p$ in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are $\leq_d$-incomparable, in which case one can apply 2.7(c) with the roles of f, f' reversed.

It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\leq_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 2.8, either (i) $e \vee f <_d c$, in which case Proposition 2.8 implies $c^{\uparrow}_e = c^{\uparrow}_f$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_e < c^{\uparrow}_f < c^{\uparrow}_g$ in $c^{\uparrow}$ so that $c^{\uparrow}_e \neq c^{\uparrow}_g$ and by Proposition 2.8 it is also $e \vee g = c$ and transitivity follows. □

Remark 2.14. Definition 2.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 2.12 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [1, Def. 7.44]), which adds a single edge η to F, serving as the (unique) root of $F \star \eta$.

Convention 2.15. From now on, we will write Ω (resp. Ω_G) to denote a model for the category of trees (resp. G-trees) where

- each object (i.e. tree) is equipped with a planar structure;
- morphisms ignore the planar structure;

- there is exactly one representative of each planar structure, i.e. the identities are the only isomorphisms that preserve the planar structure.

Similarly, O_G will denote a model for the G-orbit category where each orbital G-set X is given a total order, and with exactly one representative for each order.

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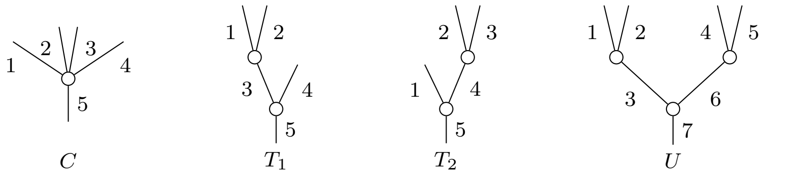
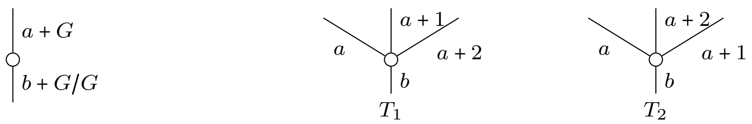
Remark 2.16. The reader desiring extra concreteness is welcome to think of the objects of Ω, Ω_G as consisting of planarized tree structures on one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ such that the planarization $\leq_d$ is the canonical total order. Some trees depicted in this convention follow.



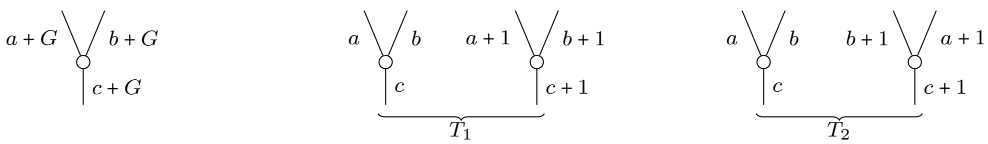
(2.17) PLANAROMEGAEX1 EQ

We note that T_1 and T_2 are isomorphic and, moreover, they encode the only two isomorphism classes of planar structures on the their underlying dendroidal set, so that no other object of Ω is isomorphic to them. C and U, on the other hand, are isomorphic to no other object of Ω, since the planarizations of the underlying broad posets sets are unique up to isomorphism.

One drawback of the concrete convention illustrated in (2.17), however, is that discussion of subfaces of trees becomes awkward, since one can not then technically regard them as subobjects. To avoid this issue, we will often regard the objects of Ω as equivalence classes of trees with planarizations (with no ambiguity resulting since representatives are related via unique isomorphisms). Moreover, this is particularly convenient when discussing G-trees, as it otherwise the task of depicting the G-action becomes cumbersome. For some examples, (and recalling that the numbering of the edges as in (2.17) is superfluous, in the sense that it is already encoded in the planar picture itself), we note that for $G = \mathbb{Z}_{/3}$ the orbital representation on the left below encodes the two isomorphic objects of Ω_G on the right (which are isomorphic to no other object of Ω_G).



Similarly, for $G = \mathbb{Z}_{/2}$, the orbital representation on the left represents the two G-trees presented.



References

Pe16b [1] L. A. Pereira. Equivariant dendroidal sets. Available at: http://www.faculty.virginia.edu/luisalex/, 2016.

We12 [2] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: https://arxiv.org/abs/1201.3987, 2012.