Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

No content yet.

2 Basic definitions

In this section we recall some definitions that will be used throughout.

Recall that for a diagram category $\mathcal{D}$ and functor $\mathcal{I}_\bullet$

$$\begin{aligned} \mathcal{D} &\xrightarrow{\mathcal{I}_\bullet} \mathsf{Cat} \\ d &\longmapsto \mathcal{I}_d \end{aligned} \tag{2.1}$$

the (covariant) Grothendieck construction $\mathcal{D} \ltimes \mathcal{I}_\bullet$ has objects pairs (d,i) with $d \in \mathcal{D}$, $i \in \mathcal{I}_d$ and arrows $(d,i) \to (d',i')$ given by pairs

$$(f: d \to d', g: f_*(i) \to i'),$$

where $f_*: \mathcal{I}_d \to \mathcal{I}_{d'}$ is a shorthand for the functor $\mathcal{I}_\bullet(f)$.

We now discuss a basic property of over and under categories that will be used in §5.4. TRANSFSIMP SEC

Given $\mathcal{J}, \mathcal{C} \in \mathsf{Cat}$ and $j \in \mathcal{J}$ we will let $\mathcal{C}^{\downarrow j}$ denote the Grothendieck construction for the functor

$$\begin{aligned} \mathcal{J} &\longrightarrow \mathsf{Cat} \\ i &\longmapsto \mathcal{C}^{\mathcal{J}(i,j)} \end{aligned}$$

Explicitly, an object of $\mathcal{C}^{\downarrow j}$ is a pair $\left(i, \mathcal{J}(i,j) \xrightarrow{\varphi} \mathcal{C}\right)$ and an arrow $(i,\varphi) \to (i',\varphi')$ is a pair $(I: i \to i', \gamma: \varphi \circ I^* \to \varphi')$.

UNDERLEFTADJ LEM

Lemma 2.2. *Let $\mathcal{J} \in \mathsf{Cat}$ be a small category and $j \in \mathcal{J}$. One then has adjunctions*

$$(- \downarrow j): \mathsf{Cat}_{/\mathcal{J}} \rightleftarrows \mathsf{Cat}: (-)^{\downarrow j}, \qquad (j \downarrow -): \mathsf{Cat}_{/\mathcal{J}} \rightleftarrows \mathsf{Cat}: (-)^{j\downarrow}.$$

Proof. Since $j \downarrow \mathcal{I} = (\mathcal{I}^{op} \downarrow j)^{op}$ by defining $(\mathcal{C}^{j\downarrow}) = ((\mathcal{C}^{op})^{\downarrow j})^{op}$ one reduces to the leftmost adjuntion.

Given $\mathcal{I} \xrightarrow{\pi} \mathcal{J}$ and $\mathcal{C}$ we will show that functors $\mathcal{I} \downarrow j \xrightarrow{F} \mathcal{C}$ correspond to functors $\mathcal{I} \xrightarrow{G} \mathcal{C}^{\downarrow j}$ over $\mathcal{J}$.

On objects, F associates to each pair $(i, J: \pi(i) \to j)$ an object $F(i,J) \in \mathcal{C}$. One thus sets $G(i) = (\pi(i), F(i,-))$ and these are clearly inverse processes.

On arrows F associates to $(i, J' \circ \pi(I)) \xrightarrow{I} (i', J')$ an arrow $F(i, J' \circ \pi(I)) \xrightarrow{F(I)} F(i', J')$. One thus defines

$$G(I) = \left(\pi(i) \xrightarrow{\pi(I)} \pi(i'), F\left(i, (-) \circ \pi(i)\right) \xrightarrow{F(I)} F\left(i', -\right)\right)$$

and again it is clear that these are inverse processes. Finally, the fact that the associativity and unit conditions for F, G coincide is likewise clear. □

3 Planar and tall maps

3.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

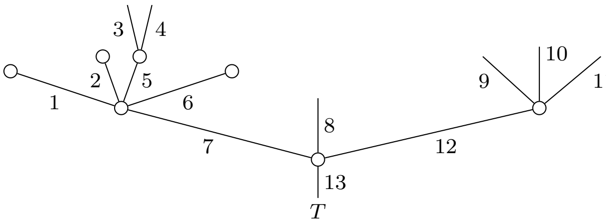
Our preferred model for trees will be that of broad posets first introduced by Weiss in [4] and further worked out by the second author in [3]. We now define planar structures in this context.

PLANARIZE DEF

Definition 3.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\le_d$ to a total order $\le_p$ such that:

- *Planar*: if $e \le_p f$ and $e \nleq_d f$ then $g \le_d f$ implies $e \le_p g$.

Example 3.2. An example of a planar structure on a tree T follows, with $\leq_r$ encoded by the number labels.



(3.3) PLANAREX EQ

Intuitively, given a planar depiction of a tree T, $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 3.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT **Notation 3.4.** Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path of* e.

We will repeatedly use the following, which is a consequence of [3, Cor. 5.26].

INCOMPNOTOP **Lemma 3.5.** *If $e \leq_d f$, $e \leq_d f'$, then f, f' are $\leq_d$-comparable.*

INPUTPATHS PROP **Proposition 3.6.** *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \leq_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 3.5. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee e_i$. ☐

Notation 3.7. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^{\uparrow} \in T$ the predecessor of b in $I(e)$.

ECESSORPROP PROP **Proposition 3.8.** *Suppose e, f are $\leq_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^{\uparrow} \neq b_f^{\uparrow}$;

(b) $b_e^{\uparrow}, b_f^{\uparrow} \in b^{\uparrow}$. In fact $\{b_e^{\uparrow}\} = I(e) \cap b^{\uparrow}$, $\{b_f^{\uparrow}\} = I(f) \cap b^{\uparrow}$;

(c) if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^{\uparrow} = b_e^{\uparrow}$, $b_{f'}^{\uparrow} = b_f^{\uparrow}$.

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^{\uparrow} = b_f^{\uparrow}$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^{\uparrow}$, where uniqueness follows from Lemma 3.5. Choosing $a = e$ implies $I(e) \cap b^{\uparrow} = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. ☐

Proposition 3.9. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$.*

Therefore, all ternary joins in $(T, \leq_d)$ are binary, i.e.

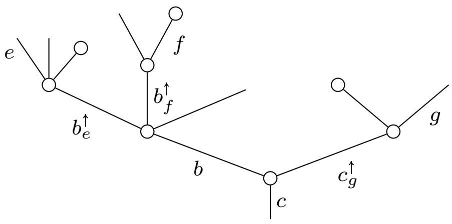
$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{3.10}$$

for some $1 \leq i < j \leq 3$, and (3.10) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$, then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 3.8(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

Example 3.11. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c^{\uparrow}_e = c^{\uparrow}_f = b$.



Notation 3.12. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will usually abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order in S.

Proposition 3.13. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \leq_p)\} & \longrightarrow & \prod_{(a^{\uparrow} \leq a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p |_{a^{\uparrow}}) \end{array} \tag{3.14}$$

Proof. We will keep the setup of Proposition 3.8 throughout: e, f are $\leq_d$-incomparable edges and we write $b = e \vee f$.

We first show that (3.14) is injective, i.e. that the restrictions $\leq_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_e <_p b^{\uparrow}_f$, the relations $e \leq_d b^{\uparrow}_e <_p b^{\uparrow}_f \geq_d f$ and Definition 3.1 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_f <_p b^{\uparrow}_e$ then $f <_p e$. Thus $b^{\uparrow}_e <_p b^{\uparrow}_f \Leftrightarrow e <_p f$ and hence (3.14) is indeed injective.

To check that (3.14) is surjective, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $b^{\uparrow}_e < b^{\uparrow}_f$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\leq_p$ in the case $e' \leq_d e <_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 3.8(c). Transitivity of $\leq_p$ in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are $\leq_d$-incomparable, in which case one can apply 3.8(c) with the roles of f, f' reversed.

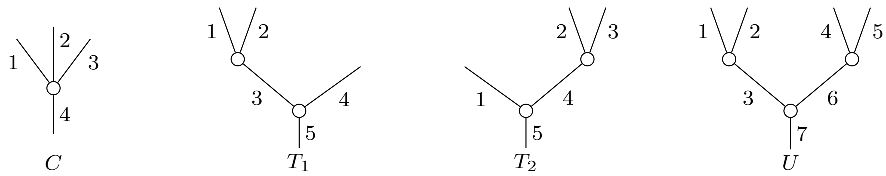
It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\leq_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 3.9, either (i) $e \vee f <_d c$, in which case Proposition 3.9 implies $c^{\uparrow}_e = c^{\uparrow}_f$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_e < c^{\uparrow}_f < c^{\uparrow}_g$ in $c^{\uparrow}$ so that $c^{\uparrow}_e \neq c^{\uparrow}_g$ and by Proposition 3.9 it is also $e \vee g = c$ and transitivity follows. □

Remark 3.15. Definition 3.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 3.13 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [3, Def. 7.44]), which adds a single edge η to F, serving as the (unique) root of $F \star \eta$.

When discussing the substitution procedure in §SUBS SEC 3.3 we will find it convenient to work with a model for the category Ω that possesses exactly one representative of each possible planar structure on each tree or, more precisely, such that the only isomorphisms preserving the planar structures are the identities. On the other hand, using such a model for Ω throughout would, among other issues, make the discussion of faces in §OUTTALL SEC 3.2 rather awkward. We now outline our conventions to address such issues.

Let Ω^p, the category of *planarized trees*, denote the category with objects pairs $T_{\leq_p} = (T, \leq_p)$ of trees together with a planar structure and morphisms the *underlying* maps of trees (so that the planar structures are ignored). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq_p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ and for which $\leq_p$ coincides with the canonical order.

STANDMODEL EX **Example 3.16.** Some examples of standard models, i.e. objects of Ω^s, follow (further, (PLANAREX EQ 3.3) can also be interpreted as such an example).



(3.17) PLANAROMEGAEX1 EQ

Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq_p} \in \Omega^p$ there is an obvious standard model $T^s_{\leq_p} \in \Omega^s$ given by replacing each edge by its order following $\leq_p$. Indeed, this defines a retraction $(-)^s : \Omega^p \to \Omega^s$ and a natural transformation $\sigma : id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is clearly unique).

PLANARCONV CON **Convention 3.18.** From now on, we will write simply Ω, Ω_G to denote the categories Ω^s, Ω^s_G of standard models (where planar structures are defined in the underlying forest as in Remark FORESTPLAN REM 3.15). Similarly O_G will denote the model O^s_G for the orbital category whose objects are the orbital G-sets whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$.

Therefore, whenever one of our constructions produces an object/diagram in Ω^p, Ω^p_G, O^p_G (of trees, G-trees, orbital G-sets with a planarization/total order) we will hence implicitly reinterpret it by using the standardization functor $(-)^s$.

Example 3.19. To illustrate our convention, we consider the trees in Example STANDMODEL EX 3.16.

One has subfaces $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures. Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

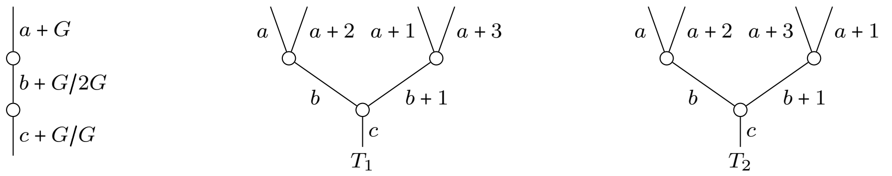
$$\begin{tikzcd} F_1 \arrow[rr, hook] \arrow[dr, hook] & & U \\ & F_2 \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & U \\ & T_1 \arrow[ur] & \end{tikzcd}$$

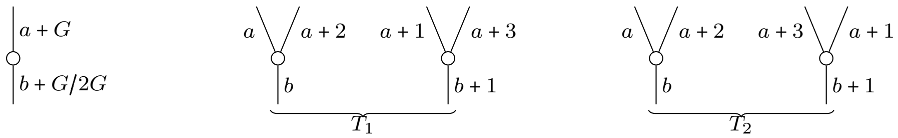
Similarly, let $\leq_{(12)}$ and $\leq_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs $1, 2$ and $4, 5$, so that one has objects $U_{\leq_{(12)}}$, $U_{\leq_{(45)}}$ in Ω^p. Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.

$$\begin{tikzcd} U \arrow[rr, "id"] \arrow[dr, "id"'] & & U_{\leq_{(45)}} \\ & U_{\leq_{(12)}} \arrow[ur, "id"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "(45)"] \arrow[dr, "(12)"'] & & U \\ & U \arrow[ur, "(12)(45)"'] & \end{tikzcd}$$

Example 3.20. An additional reason to leave the use of $(-)^s$ implicit is that when depicting G-trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, when $G = \mathbb{Z}_{/4}$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting of the two trees $T_1, T_2 \in \Omega_G$ on the right.





Definition 3.21. A morphism $S \xrightarrow{\varphi} T$ in Ω that is compatible with the planar structures $\leq_p$ is called a *planar map*.

More generally, a morphism $F \to G$ in the categories Φ, Φ^G, Ω^G of forests, G-forests, G-trees is called a *planar map* if it is an independent map (cf. [3, Def. 5.28]) compatible with the planar structures $\leq_p$.

Remark 3.22. The need for the independence condition is justified by [3, Lemma 5.33] and its converse, since non independent maps do not reflect $\leq_d$ inequalities.

We note that in the Ω_G case a map φ is independent iff φ does not factor through a non trivial quotient iff φ is injective on each edge orbit.

PLANARPULL EQ

Proposition 3.23. *Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization*

$$F \xrightarrow{\simeq} \bar{F} \to G$$

such that $F \xrightarrow{\simeq} \bar{F}$ is an isomorphism and $\bar{F} \to G$ is planar.

Proof. We need to show that there is a unique planar structure $\leq_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of G ensures that for any vertex $e^{\uparrow} \leq e$ of F the edges in $\varphi(e^{\uparrow})$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(\underline{r}_F)$ are distinct. The result now follows from (the forest version of) Proposition 3.13: one simply orders each set $e^{\uparrow}$ and $\underline{r}_F$ according to its image.

not quite complete... maybe that $\leq_p$ is the closure of $\leq_d$ and the vertex relations under transitivity and the planar condition □

PULLPLANAR REM

Remark 3.24. Proposition 3.23 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As an example, in the notation of (3.17), consider the map $C \to T_1$ defined by $1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 5$.

UNIQCOR REM

Remark 3.25. Given any tree $T \in \Omega$ there is a unique corolla $\mathsf{lr}(T) \in \Sigma$ and planar tall map $\mathsf{lr}(T) \to T$. Explicitly, the number of leaves of $\mathsf{lr}(T)$ matches that of T, together with the inherited order.

OUTTALL SEC

3.2 Outer faces and tall maps

In preparation for our discussion of the substitution operation in §3.3, we now recall some basic notions and results concerning outer subtrees and tree grafting, as in [3, §5].

Definition 3.26. Let $T \in \Omega$ be a tree and $e_1 \cdots e_n = \underline{e} \leq e$ a broad relation in T.

We define the *planar outer face* $T_{\underline{e} \leq e}$ to be the subtree with underlying set those edges $f \in T$ such that

$$f \leq_d e, \quad \forall_i e_i \nleq_d f, \tag{3.27}$$ OUTERFACE EQ

generating broad relations the relations $f^{\uparrow} \leq f$ for f satisfying (3.27) and $\forall_i f \neq e_i$, and planar structure pulled back from T.

Remark 3.28. If one forgoes the requirement that $T_{\underline{e}\le e}$ be equipped with the pullback planar structure, the inclusion $T_{\underline{e}\le e} \to T$ is usually called simply an *outer face.*

We now recap some basic results.

Proposition 3.29. *Let $T \in \Omega$ be a tree.*

(a) $T_{\underline{e}\le e}$ is a tree with root e and edge tuple $\underline{e}$;

(b) there is a bijection

$$\{\textit{planar outer faces of } T\} \leftrightarrow \{\textit{broad relations of } T\};$$

(c) if $R \to S$ and $S \to T$ are outer face maps then so is $R \to T$;

(d) any pair of broad relations $\underline{g} \le v$, $\underline{f}v \le e$ induces a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "v"] \arrow[d, "v"'] & T_{\underline{g}\le v} \arrow[d] \\ T_{\underline{f}v\le e} \arrow[r] & T_{\underline{f}\underline{g}\le v} \end{tikzcd} \tag{3.30}$$ GRAPTPUSH EQ

Proof. We first show (a). That $T_{\underline{e}\le e}$ is indeed a tree is the content of [3, Prop. 5.20]: more precisely, $T_{\underline{e}\le e} = (T^{\le e})_{<\underline{e}}$ in the notation therein. That the root of $T_{\underline{e}\le e}$ is e is clear and that the root tuple is $\underline{e}$ follows from [3, Remark 5.23].

(b) follows from (a), which shows that $\underline{e} \le e$ can be recovered from $T_{\underline{e}\le e}$.

(c) follows from the definition of outer face together with [3, Lemma 5.33], which states that the $\le_d$ relations on S, T coincide.

Since by (c) both $T_{\underline{g}\le v}$ and $T_{\underline{f}v\le e}$ are outer faces of $T_{\underline{f}\underline{g}\le v}$, (d) is a restatement of [3, Prop. 5.15]. $\square$

Definition 3.31. A map $S \xrightarrow{\varphi} T$ in Ω is called a *tall map* if

$$\varphi(\underline{l}_S) = \underline{l}_T, \qquad \varphi(r_S) = r_T,$$

where $l_{(-)}$ denotes the leaf tuple and $r_{(-)}$ the root.

The following is a restatement of [3, Cor. 5.24]

TALLOUTERDEC COR **Proposition 3.32.** *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphism,*

$$S \xrightarrow{\varphi^t} U \xrightarrow{\varphi^u} T$$

as a tall map followed by an outer face (in fact, $U = T_{\varphi(\underline{l}_S)\le r_S}$).

We recall that a face $F \to T$ is called inner if is obtained by iteratively removing inner edges, i.e. edges other than the root or the leaves. In particular, it follows that a face is inner iff it is tall. The usual face-degeneracy decomposition thus combines with Corollary 3.32 to give the following.

Corollary 3.33. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^u} T \tag{3.34}$$ TRIPLEFACT EQ

as a degeneracy followed by an inner face followed by an outer face.

Proof. The factorization (3.34) can be built by first performing the degeneracy-face decomposition and then performing the tall-outer decomposition on the face map. $\square$

3.3 Substitution

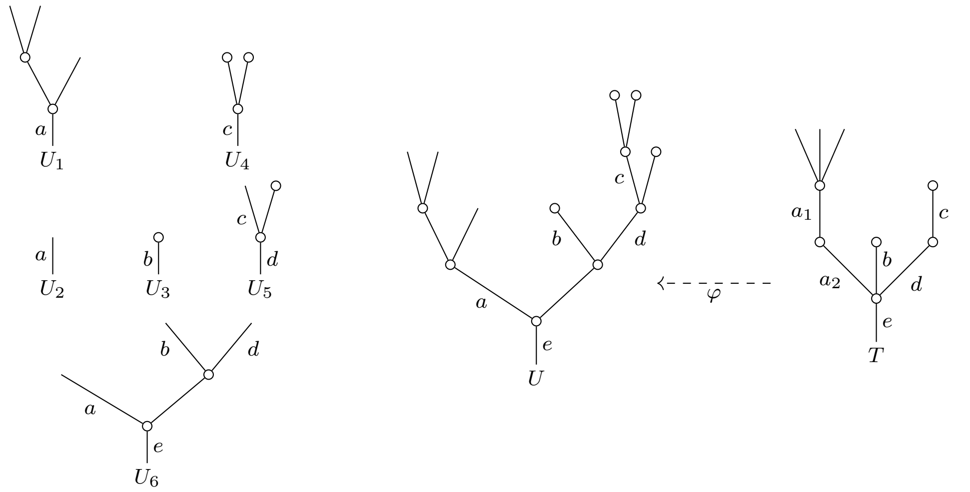
SUBS SEC

One of the key ideas needed to describe operads is that of substitution of tree nodes, a process that we will prefer to repackage in terms of maps of trees. We start by discussing an example, focusing on the related notion of iterated graftings of trees (as described in (3.30)).

Example 3.35. The trees $U_1, U_2, \cdots, U_6$ on the left below can be grafted into the tree U in the middle. More precisely (among other possible grafting orders), one has

$$U = (((((U_6 \amalg_a U_2)) \amalg_a U_1) \amalg_b U_3) \amalg_d U_5) \amalg_c U_4 \tag{3.36}$$

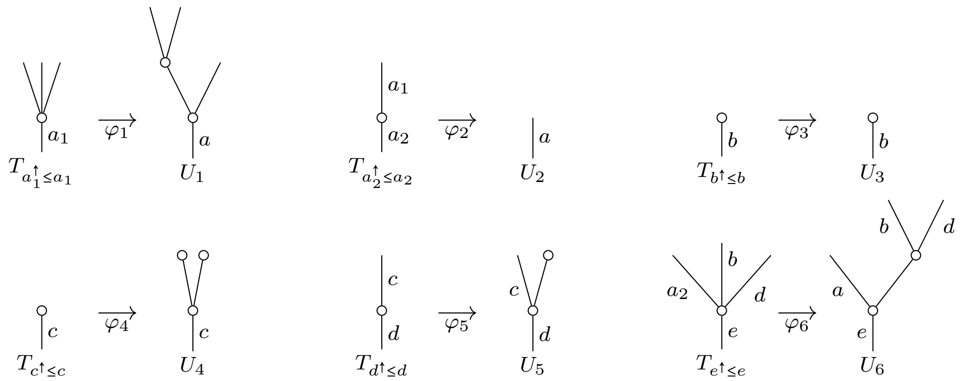
UFORMULA EQ



(3.37) SUBSDATUMTREES EQ

We now consider the tree T, which is built by converting each U_i into the corolla $\mathrm{lr}(U_i)$ (cf. Remark 3.25), and then performing the same grafting operations as in (3.36). T can then be regarded as encoding the combinatorics of the iterated grafting in (3.36), with alternative ways to reorder operations in (3.36) in bijection with ways to assemble T out of its nodes.

One can now therefore think of the iterated grafting (3.36) as being instead encoded by the tree T together with the (unique) planar tall maps φ_i below.



(3.38) SUBSDATUMTREES2 EQ

From this perspective, U can now be thought as obtained from T by substituting each of its nodes with the corresponding U_i. Moreover, the φ_i assemble to a planar tall map $\varphi: T \to U$ (such that $a_i \mapsto a, b \mapsto b, \cdots, e \mapsto e$), which likewise encodes the same information.

Our perspective will then be that data for substitution of tree nodes such as in (3.38) can equivalently be repackaged using planar tall maps.

UBSTITUTIONDATUM

Definition 3.39. Let $T \in \Omega$ be a tree.

A *T-substitution datum* is a tuple $\{U_{e^\uparrow \leq e}\}_{(e^\uparrow \leq e) \in V(T)}$ together with tall maps $T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}$.

Further, a map of planar T-substitution data $\{U_{e^\uparrow \leq e}\} \to \{V_{e^\uparrow \leq e}\}$ is a tuple of tall maps $\{U_{e^\uparrow \leq e} \to V_{e^\uparrow \leq e}\}$ compatible with the chosen maps.

Lastly, a substitution datum is called a *planar T-substitution datum* if the chosen maps are planar (so that $\mathsf{lr}(U_{e^\uparrow \leq e}) = T_{e^\uparrow \leq e}$) and a morphism of planar data is called a planar morphism if it consists of a tuple of planar maps.

Definition 3.40. Let $T \in \Omega$.

The *Segal core poset* $\mathsf{Sc}(T)$ is the poset with objects the edge subtrees η_e and vertex substrees $T_{e^\uparrow \leq e}$. The order relation is given by inclusion.

Remark 3.41. Note that the only maps in $\mathsf{Sc}(T)$ are inclusions of the form $\eta_a \subset T_{e^\uparrow \leq e}$. In particular, there are no pairs of composable non-identity relations in $\mathsf{Sc}(T)$.

Given a T-substitution datum $\{U_{\{e^\uparrow \leq e\}}\}$ we abuse notation by writing

$$U_{(-)} \colon \mathsf{Sc}(T) \to \Omega$$

for the functor $\eta_a \mapsto \eta$, $T_{e^\uparrow \leq e} \mapsto U_{e^\uparrow \leq e}$ and sending the inclusions $\eta_a \subset T_{e^\uparrow \leq e}$ to the composites

$$\eta \xrightarrow{a} T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}.$$

TAUNDERPLAN PROP

Proposition 3.42. *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\begin{aligned} \mathsf{Sub}_p(T) &\rightleftarrows \Omega^{\mathsf{pt}}_{T/} \\ \{U_{e^\uparrow \leq e}\} &\longmapsto \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \\ \{U_{\varphi(e^\uparrow) \leq \varphi(e)}\} &\longleftarrow\!\shortmid \left(T \xrightarrow{\varphi} U\right) \end{aligned} \tag{3.43}$$ SUBDATAUNDERPLAN EQ

where $\mathsf{Sub}_p(T)$ denotes the category of planar T-substitution data and $\Omega^{\mathsf{pt}}_{T/}$ the category of planar tall maps under T.

Proof. We first claim that (i) the $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ indeed exists; (ii) for the canonical datum $\{T_{e^\uparrow \leq e}\}$, it is $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$; (iii) the induced map $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall.

The argument is by induction on the number of vertices of T, with the base cases of T with 0 or 1 vertices being immediate, since then T is the terminal object of $\mathsf{Sc}(T)$. Otherwise, one can choose a non trivial grafting decomposition so as to write $T = R \amalg_e S$, resulting in identifications $\mathsf{Sc}(R) \subset \mathsf{Sc}(T)$, $\mathsf{Sc}(S) \subset \mathsf{Sc}(T)$ so that $\mathsf{Sc}(R) \cup \mathsf{Sc}(S) = \mathsf{Sc}(T)$ and $\mathsf{Sc}(R) \cap \mathsf{Sc}(S) = \{\eta_e\}$. The existence of $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is thus equivalent to the existence of the pushout below.

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)} \arrow[d, dashed] \\ \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)} \arrow[r, dashed] & \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)} \end{tikzcd} \tag{3.44}$$ ASSEMBLYGRAFT EQ

By induction, the top right and bottom left colimits exist for any $U_{(-)}$, equal R and S in the case $U_{(-)} = T_{(-)}$, and the maps $R \to \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)}$, $S \to \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)}$ are planar tall. But is now follows that (3.44) is a grafting pushout diagram, so that the pushout indeed exists. The conditions that $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$ and $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall follow.

The fact that the two functors in (3.43) are inverse to each other is clear by the same inductive argument. ∎

ATAUNDERPLAN COR

Corollary 3.45. *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\mathsf{Sub}(T) \rightleftarrows \Omega^{\mathsf{t}}_{T/} \tag{3.46}$$ SUBDATAUNDERNONPL EQ

where $\mathsf{Sub}(T)$ denotes the category of T-substitution data and $\Omega^{\mathsf{t}}_{T/}$ the category of tall maps under T.

Proof. This is a consequence of Proposition 3.23 together with the previous result, with the functor $\mathsf{Sub}(T) \to \Omega^{\mathsf{t}}_{T/}$ given by the same formula. Indeed, Proposition 3.13 can be restated as saying that isomorphisms $T \to T'$ are in bijection with substitution data consisting of isomorphisms, and thus bijectiveness reduces to that in the previous result. □

VERTEXDECOMP REM **Remark 3.47.** It follows from the previous proof that, writing $U = \mathrm{colim}_{\mathsf{Sc}(T)} U_{(-)}$, one has

$$V(U) = \coprod_{(e^{\uparrow} \le e) \in V(T)} V(U_{e^{\uparrow} \le e}). \tag{3.48}$$ VERTEXDECOMP EQ

Alternatively, (3.48) can be regarded as a map $f^*: V(U) \to V(T)$ induced by the planar tall map $f: T \to U$. Explicitly, $f^*(U_{u^{\uparrow} \le u})$ is the unique $T_{t^{\uparrow} \le t}$ such that $U_{u^{\uparrow} \le u} \subset U_{t^{\uparrow} \le t}$. We note that f^* is indeed contravariant in the tall planar map f.

The following is a converse of sorts to Proposition 3.42.

BUILDABLE PROP **Proposition 3.49.** *Let $U \in \Omega$ be a tree. Then:*

(i) given non stick outer subtrees U_i such that $V(U) = \coprod_i V(U_i)$ there is a unique tree T and planar tall map $T \to U$ such that $\{U_i\} = \{U_{e^{\uparrow} \le e}\}$;

(ii) given multiplicities $m_e \ge 1$ for each edge $e \in U$, there is a unique planar degeneracy $\rho: T \to U$ such that $\rho^{-1}(e)$ has m_e elements;

(iii) planar tall maps $T \to U$ are in bijection with collections $\{U_i\}$ of outer subtrees such that $V(U) = \coprod_i V(U_i)$ and U_j is not an inner edge of any U_i whenever $U_j \simeq \eta$ is a stick.

Proof. We first show (i) by induction on the number of subtrees U_i. The base case $\{U_i\} = \{U\}$ is immediate, setting $T = \mathsf{lr}(U)$. Otherwise, letting e be edge that is both an inner edge of U and a root of some U_i, and one can form a pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "e"] \arrow[d, "e"'] & V \arrow[d] \\ W \arrow[r] & U \end{tikzcd} \tag{3.50}$$ DECOMPPROOF EQ

inducing a nontrivial partition $\{U_i\} = \{U_i | U_i \hookrightarrow V\} \amalg \{U_i | U_i \hookrightarrow W\}$. Existence of $T \to U$ now follows from the induction hypothesis. For uniqueness, the condition that no U_i is a stick guarantees that T possesses a single inner edge mapping to e, and thus admits a compatible decomposition as in (3.50), and thus uniqueness too follows by the induction hypothesis.

For (ii), we argue existence by nested induction on the number of vertices $|V(U)|$ and the sum of the multiplicities m_e. The base case $|V(U)| = 0$, i.e. $U = \eta$ is immediate. Otherwise, writing $m_e = m'_e + 1$, one can form a decomposition (3.50) where either $|V(V)|, |V(W)| < |V(U)|$ or one of V, W is η, so that $T \to U$ can be built via the induction hypothesis. For uniqueness, note first that by [3, Lemma 5.33] each pre-image $\rho^{-1}(e)$ is linearly ordered and by the "further" claim in [3, Cor. 5.39] the remaining broad relations are precisely the pre-image of the non-identity relations in U, showing that the tree T is uniquely determined.

(iii) follows by combining (i) and (ii). Indeed, any planar tall map $T \to U$ has a unique decomposition $T \twoheadrightarrow \bar{T} \hookrightarrow U$ as a planar degeneracy followed by a planar inner face, and each of these maps is classified by the data in (b) and (a). □

TERFACEUNION LEM **Lemma 3.51.** *Suppose $T_1, T_2 \hookrightarrow T$ are two outer faces with at least one common edge e. Then there exists an unique outer face $T_1 \cup T_2$ such that $V(T_1 \cup T_2) = V(T_1) \cup V(T_2)$.*

Proof. If either of T_1, T_2 is the root or a leaf the result is obvious. Otherwise, one can necessarily choose e to be an inner edge of T, in which case all of T_1, T_2, T admit compatible decompositions (3.50) and the result follows by induction on $|V(T)|$. □

4 The genuine equivariant operad monad

We now turn to the task of building the monad encoding genuine equivariant operads.

4.1 Wreath product over finite sets

In what follows we will let F denote the usual skeleton of the category of finite sets and all set maps. Explicitly, its objects are the finite sets $\{1, 2, \cdots, n\}$ for $n \geq 0$. However, much as in the discussion in Convention 3.18 we will often find it more convenient to regard the elements of F as equivalence classes of finite sets equipped with total orders.

Definition 4.1. For a category $\mathcal{C}$, we let $\mathsf{F} \wr \mathcal{C}$ denote the opposite of the Grothendieck construction for the functor

$$\begin{aligned} \mathsf{F}^{op} &\longrightarrow \mathsf{Cat} \\ I &\longmapsto \mathcal{C}^{I} \end{aligned}$$

Explicitly, the objects of $\mathsf{F} \wr \mathcal{C}$ are tuples $(c_i)_{i \in I}$ and a map $(c_i)_{i \in I} \to (d_j)_{j \in J}$ consists of a pair

$$(\phi \colon I \to J, (f_i \colon c_i \to d_{\phi(i)})_{i \in I}),$$

henceforth abbreviated as $(\phi, (f_i))$.

The following is immediate.

Proposition 4.2. *Suppose $\mathcal{C}$ has all finite coproducts. One then has a functor as on the left below. Dually, if $\mathcal{C}$ has all finite products, one has a functor as on the right below.*

$$\begin{aligned} \mathsf{F} \wr \mathcal{C} &\xrightarrow{\ \amalg\ } \mathcal{C} \\ (c_i)_{i \in I} &\longmapsto \textstyle\coprod_{i \in I} c_i \end{aligned} \qquad\qquad \begin{aligned} (\mathsf{F} \wr \mathcal{C}^{op})^{op} &\xrightarrow{\ \Pi\ } \mathcal{C} \\ (c_i)_{i \in I} &\longmapsto \textstyle\prod_{i \in I} c_i \end{aligned}$$

WREATPRODLIM LEM **Lemma 4.3.** *Suppose that $\mathcal{E}$ is a bicomplete category such that coproducts commute with limits in each variable. If the leftmost diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "k"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "G"'] \arrow[u, phantom, "\Uparrow \eta"] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{F} \wr \mathcal{C} \arrow[r, "\mathsf{F}\wr F"] \arrow[d, "\mathsf{F}\wr k"'] & \mathsf{F} \wr \mathcal{E} \arrow[r, "\amalg"] & \mathcal{E} \\ \mathsf{F} \wr \mathcal{D} \arrow[ur, "\mathsf{F}\wr G"'] \arrow[u, phantom, "\Uparrow \mathsf{F}\wr\eta"] \arrow[urr, bend right, "\amalg \circ \mathsf{F}\wr G"'] & & \end{tikzcd} \tag{4.4}$$ WRRAN EQ

is a right Kan extension diagram then so is the composite of the rightmost diagram.

Dually, if in $\mathcal{E}$ products commute with colimits in each variable, and the leftmost diagram

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "F"] \arrow[d, "k"'] & \mathcal{E} \\ \mathcal{D}^{op} \arrow[ur, "G"'] \arrow[u, phantom, "\Downarrow \epsilon"] & \end{tikzcd} \qquad \begin{tikzcd} (\mathsf{F} \wr \mathcal{C})^{op} \arrow[r, "(\mathsf{F}\wr F^{op})^{op}"] \arrow[d, "(\mathsf{F}\wr k^{op})^{op}"'] & (\mathsf{F} \wr \mathcal{E}^{op})^{op} \arrow[r, "\Pi"] & \mathcal{E} \\ (\mathsf{F} \wr \mathcal{D})^{op} \arrow[ur, "(\mathsf{F}\wr G^{op})^{op}"'] \arrow[u, phantom, "\Downarrow"] \arrow[urr, bend right, "\Pi \circ (\mathsf{F}\wr G^{op})^{op}"'] & & \end{tikzcd} \tag{4.5}$$ WRLAN EQ

is a left Kan extension diagram then so is the composite of the rightmost diagram.

Proof. Unpacking definitions using the pointwise formula for Kan extensions ([2, X.3.1]), the claim concerning (4.4) amounts to showing that for each $(d_i) \in \mathsf{F} \wr \mathcal{D}$ one has natural isomorphisms

$$\lim_{((d_i) \to (kc_j)) \in ((d_i) \downarrow \mathsf{F}\wr\mathcal{C})} \left(\coprod_j F(c_j) \right) \simeq \coprod_i \lim_{(d_i \to kc_i) \in d_i \downarrow \mathcal{C}} \left(F(c_i) \right). \tag{4.6}$$ POINTKAN EQ

Noting that the canonical factorizations of each $(\varphi, (f_i)) \colon (d_i)_{i \in I} \to (kc_j)_{j \in J}$ as

$$(d_i)_{i \in I} \to (c_{\phi(i)})_{i \in I} \to (kc_j)_{j \in J}$$

exhibit $\prod_i (d_i \downarrow \mathcal{C})$ as a coreflexive subcategory of $(d_i) \downarrow \mathsf{F} \wr \mathcal{C}$, we see that it is an initial subcategory. Therefore

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow\mathsf{F}\wr\mathcal{C})} \left(\coprod_j F(c_j)\right) \simeq \lim_{((d_i)\to(kc_i))\in\prod_i(d_i\downarrow\mathcal{D})} \left(\coprod_i F(c_i)\right)$$

POINTKAN EQ
and hence (4.6) now follows from the assumption that coproducts commute with limits in each variable. □

Notation 4.7. Using the coproduct functor $\mathsf{F}^{\wr 2} = \mathsf{F}^{\wr\{0,1\}} = \mathsf{F}\wr\mathsf{F} \xrightarrow{\amalg} \mathsf{F}$ (where $\coprod_{i\in I} J_i$ is ordered lexicographically) and the simpleton $\{1\} \in \mathsf{F}$ one can regard the collection of categories $\mathsf{F}^{\wr\{0,\cdots,n\}} \wr \mathcal{C} = \mathsf{F}^{\wr\underline{n}} \wr \mathcal{C}$ as a coaugmented cosimplicial object in Cat. As such, we will denote by

$$\delta^i \colon \mathsf{F}^{\wr\underline{n-1}} \wr \mathcal{C} \to \mathsf{F}^{\wr\underline{n}} \wr \mathcal{C}, \qquad 0 \le i \le n$$

the cofaces obtained by inserting simpletons $\{1\} \in \mathsf{F}$ and by

$$\sigma^i \colon \mathsf{F}^{\wr\underline{n+1}} \wr \mathcal{C} \to \mathsf{F}^{\wr\underline{n}} \wr \mathcal{C}, \qquad 0 \le i \le n$$

the codegeneracies obtained by applying the coproduct $\mathsf{F}^{\wr 2} \xrightarrow{\amalg} \mathsf{F}$ to adjacent F coordinates.

4.2 Equivariant leaf-root and vertex functors

Definition 4.8. A morphism $T \xrightarrow{\varphi} S$ in Ω_G is called a *quotient* if the underlying morphism of forests

$$\coprod_{[g]\in G/H} T_{[g]} \to \coprod_{[h]\in G/K} S_{[h]}$$

maps each tree component (or, equivalently, some tree component) isomorphically onto its image component.

We denote the subcategory of G-trees and quotients by Ω_G^q.

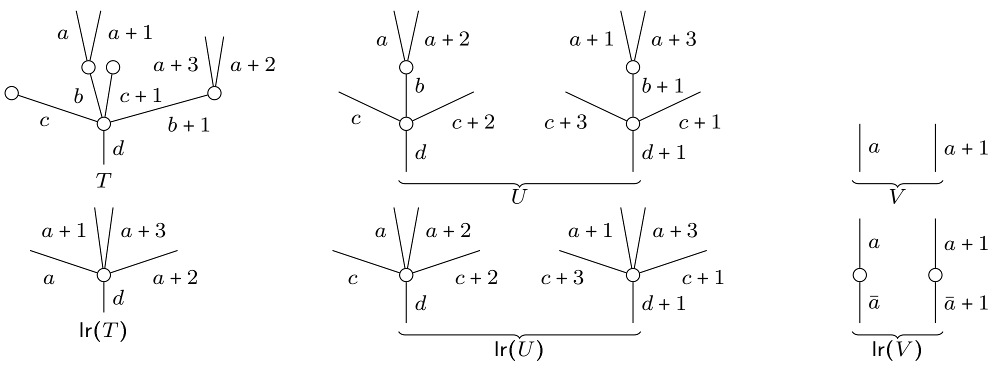
Definition 4.9. The *G-symmetric category*, which we will also call the *category of G-corollas*, is the full subcategory $\Sigma_G \subset \Omega_G^q$ of those G-trees that are corollas, i.e. G-trees such that each edge is either a root or a leaf (but not both).

Definition 4.10. The *leaf-root functor* is the functor $\Omega_G^q \xrightarrow{\mathsf{lr}} \Sigma_G$ defined by

$$\mathsf{lr}(T) = \{\text{leaves of } T\} \amalg \{\text{roots of } T\}$$

with a broad relation $l_1 \cdots l_n \le r$ holding in $\mathsf{lr}(T)$ iff its image holds in T and similarly for the planar structure $\le_p$.

EAFROOTEXAMP REM UNIQCOR REM
Remark 4.11. Generalizing Remark 3.25, $\mathsf{lr}(T)$ can alternatively be characterized as being the *unique* G-corolla which admits an also unique (tree-wise) tall planar map $\mathsf{lr}(T) \to T$. Moreover, $\mathsf{lr}(T)$ can usually be regarded as the "smallest inner face" of T, obtained by removing all the inner edges, although this characterization fails when $T = G \cdot_H \eta$ is a stick G-tree. Some examples with $G = \mathbb{Z}_{/4}$ follow.



Remark 4.12. One consequence of the fact that planarizations can not be pushed forward along tree maps (cf. Remark PULLPLANAR REM 3.24) is that $\mathsf{lr}\colon \Omega_G^q \to \Sigma_G$ is not a categorical fibration. maybe add to this.

VG DEF

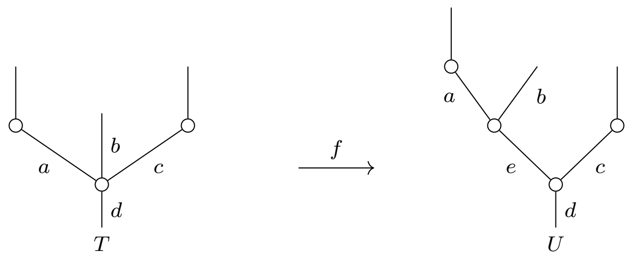
Definition 4.13. Given $T \in \Omega_G$ we define the set $V_G(T)$ of *G-vertices* of T to be the orbit set $V(T)/G$, i.e. the quotient of the vertex set $V(T)$ by its G-action.

Furthermore, we will regard $V_G(T)$ as an object in F by equipping it with its lexicographic order: i.e. vertex equivalence classes $[e^{\uparrow} \leq e]$ are ordered according to the planar order $\leq_p$ of the smallest representative ge, $g \in G$.

ERTEXDECOMPG REM

Remark 4.14. Following Remark VERTEXDECOMP REM 3.47, a planar tall map $f\colon T \to U$ of G-trees induces a G-equivariant map $f^*\colon V(U) \to V(T)$ and thus also a map of orbits $f^*\colon V_G(U) \to V_G(T)$. We note, however, that f^* is not in general compatible with the order on V_G, as is indeed the case even in the non-equivariant case.

A minimal example follows.



In $V(T)$ the vertices are ordered as $a < c < d$ while in $V(U)$ they are ordered as $a < e < c < d$ but the map $f^*\colon V(U) \to V(T)$ is given by $a \mapsto a, c \mapsto c, d \mapsto d, e \mapsto d$.

Note that each element of $V_G(T)$ corresponds to an unique edge orbit Ge for e not a leaf. As such, we will represent the corresponding G-vertex by $v_{Ge} = (Ge)^{\uparrow} \leq Ge$ (which we interpret as the concatenation of the relations $f^{\uparrow} \leq f$ for $f \in Ge$) and write

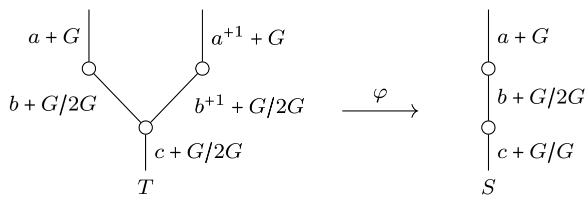
$$T_{v_{Ge}} = T_{(Ge)^{\uparrow} \leq Ge} = \coprod_{f \in Ge} T_{f^{\uparrow} \leq f}.$$

We note that $T_{v_{Ge}}$ is always a G-corolla. Indeed, noting that a quotient map $\varphi\colon T \to S$ induces quotient maps $T_{v_{ge}} \to S_{v_{G\varphi(e)}}$ one obtains a functor

$$\begin{aligned} \Omega_G^q &\xrightarrow{V_G} \mathsf{F} \wr \Sigma_G \\ T &\longmapsto (T_{v_{Ge}})_{v_{Ge} \in V_G(T)}. \end{aligned} \tag{4.15}$$

VFUNCTOR EQ

Remark 4.16. The need to introduce the $\mathsf{F} \wr \mathcal{C}$ categories comes from the fact that general quotient maps do not preserve the number of G-vertices. For a simple example, let $G = \mathbb{Z}_{/4}$ and consider the quotient map



sending edges labeled a, b, c to the edges with the same name and the edges a^{+1}, b^{+1} to the edges $a+1$, $b+1$. We note that T has three G-vertices v_{Gc}, v_{Gb}, $v_{Gb^{+1}}$ while S has only two G-vertices v_{Gc} and v_{Gb}. $V(\phi)$ then maps the two corollas $T_{v_{Gb}}$ and $T_{v_{Gb^{+1}}}$ isomorphically onto $T_{S_{Gb}}$ and the corolla $T_{v_{Gc}}$ non-isomorphically onto $S_{v_{Gc}}$.

Definition 3.39 now immediately generalizes. Here a map is called *rooted* if it induces an ordered isomorphism on the root orbit.

Definition 4.17. Let $T \in \Omega_G$ be a G-tree.

A *rooted (resp. planar) T-substitution datum* is a tuple $\{U_{v_{Ge}}\}_{v_{Ge} \in V_G(T)}$ together with rooted (resp. planar) tall maps $T_{v_{Ge}} \to U_{v_{Ge}} = T_{v_{Ge}}$.

Further, a map of rooted (resp. planar) T-substitution data $\{U_{v_{Ge}}\} \to \{V_{v_{Ge}}\}$ is a tuple of rooted (resp. planar) tall maps $\{U_{v_{Ge}} \to V_{v_{Ge}}\}$.

Remark 4.18. To establish the equivariant analogue of Proposition 3.42 we will prefer to repackage equivariant substitution data in terms of non-equivariant terms.

Noting that there are decompositions $U_{v_{Ge}} = \coprod_{ge \in Ge} U_{ge^{\uparrow} \le ge}$ and letting $G \ltimes V(T)$ denote the Grothendieck construction for the action of G on the non-equivariant vertices $V(T)$ (often called the action groupoid), it is immediate that an equivariant T-substitution datum is the same as a functor $G \ltimes V(T) \to \Omega$ whose restriction to $V(T) \subset G \ltimes V(T)$ is a (non-equivariant) substitution datum.

Proposition 4.19. *Let $T \in \Omega_G$ be a G-tree. There are isomorphisms of categories*

$$\begin{array}{ccccccc} \mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & \Omega^{\mathsf{pt}}_{G,T/} & \qquad & \mathsf{Sub}_{\mathsf{r}}(T) & \rightleftarrows & \Omega^{\mathsf{rt}}_{G,T/} \\ \{U_{v_{Ge}}\} & \longmapsto & \big(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\big) & & \{U_{v_{Ge}}\} & \longmapsto & \big(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\big) \end{array} \tag{4.20}$$

Proof. This is a minor adaptation of the non-equivariant analogues Proposition 4.19 and Corollary 3.45. Since $\mathsf{Sc}(T)$ inherits a G action, one can form the Grothendieck construction $G \ltimes \mathsf{Sc}(T)$ and by Remark 4.18 equivariant substitution data $\{U_{v_{Ge}}\}$ therefore induce functors $U_{(-)} \colon G \ltimes \mathsf{Sc}(T) \to \Omega$. It is then immediate that $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ inherits a G-action, provided it exists. The key observation is then that, since $\mathsf{Sc}(T)$ is now a disconnected poset, this colimit is to be interpreted as taken in the category Φ of forests rather than in Ω.

Additionally, we note that the need to use rooted data comes from the fact that rooted isomorphisms $T \to T'$ are in bijection with rooted substitution data that are given by isomorphisms, a statement that fails in the absence of the rooted condition. $\square$

Remark 4.21. We will need to know that in the planar case each of the maps

$$U_{v_{Ge}} \to U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$$

induced by the previous proof is a planar map of G-trees. This requires two observations: (i) the restrictions to each of the constituent non-equivariant trees $U_{ge^{\uparrow} \le ge}$ is planar by Proposition 4.19; (ii) the restriction to the roots of $U_{v_{Ge}}$ is injective and order preserving since it matches the inclusion of the roots of $T_{v_{Ge}}$, and the map $T \to U$ is a planar map of G-trees.

Remark 4.22. The isomorphisms in Proposition 4.19 are compatible with root pullback of trees. More concretely, any pullback $\pi \colon S = \varphi^* T \to T$ induces pullbacks $\pi_{Ge} \colon S_{v_{Ge}} \to T_{v_{Ge}}$ for $v_{Ge} \in V_G(S)$ and one has commutative diagrams

$$\begin{tikzcd} \mathsf{Sub}_{\mathsf{p}}(S) \arrow[r, shift left] & \Omega^{\mathsf{pt}}_{G,S/} \arrow[l, shift left] & \mathsf{Sub}_{\mathsf{r}}(S) \arrow[r, shift left] & \Omega^{\mathsf{rt}}_{G,S/} \arrow[l, shift left] \\ \mathsf{Sub}_{\mathsf{p}}(T) \arrow[u, "(\pi_{Ge})"] \arrow[r, shift left] & \Omega^{\mathsf{pt}}_{G,T/} \arrow[u, "\pi^*"'] \arrow[l, shift left] & \mathsf{Sub}_{\mathsf{r}}(T) \arrow[u, "(\pi_{Ge})"] \arrow[r, shift left] & \Omega^{\mathsf{rt}}_{G,T/} \arrow[u, "\pi^*"'] \arrow[l, shift left] \end{tikzcd} \tag{4.23}$$

4.3 Planar strings

The leaf-root and vertex functors will allow us to reinterpret our results concerning substitution.

Definition 4.24. The category $\Omega_{G,n}$ of *substitution n-strings* is the category whose objects are strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} T_n$$

where $T_i \in \Omega_G$ and the f_i are tall planar maps, and arrows are commutative diagrams

$$\begin{tikzcd} T_0 \arrow[r, "f_1"] \arrow[d, "q_0"'] & T_1 \arrow[r, "f_2"] \arrow[d, "q_1"'] & \cdots \arrow[r, "f_n"] & T_n \arrow[d, "q_n"'] \\ T_0' \arrow[r, "f_1'"'] & T_1' \arrow[r, "f_2'"'] & \cdots \arrow[r, "f_n'"'] & T_n' \end{tikzcd} \tag{4.25}$$ PTNARROW EQ

where each q_i is a quotient map.

IMPOPERATORS NOT **Notation 4.26.** Since compositions of planar tall arrows are planar tall and identity arrows are planar tall it follows that $\Omega_{G,\bullet}$ forms a simplicial object in Cat, with faces given by composing and degeneracies by inserting identities.

Noting that $\Omega_{G,0} = \Omega_G^q$ and setting $\Omega_{G,-1} = \Sigma_G$, the leaf-root functor $\Omega_G^q \xrightarrow{\mathsf{lr}} \Sigma_G$ makes $\Omega_{G,\bullet}^q$ into an augmented simplicial object and, furthermore, the maps $s_{-1}: \Omega_{G,n}^q \to \Omega_{G,n+1}^q$ sending $T_0 \to T_1 \to \cdots \to T_n$ to $\mathsf{lr}(T_0) \to T_0 \to T_1 \to \cdots \to T_n$ equip it with extra degeneracies.

Notation 4.27. We extend the vertex functor to a functor $V_G: \Omega_{G,n+1} \to \mathsf{F} \wr \Omega_{G,n}$ by

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)} \tag{4.28}$$ VGDEF EQ

where we abuse notation by writing $T_{i,v_{Ge}}$ for $T_{i,(f_i \circ \cdots \circ f_1)(v_{Ge})}$.

The following is a reinterpretation of Proposition 4.19.

SUBSASPULL PROP **Proposition 4.29.** *The diagram*

$$\begin{tikzcd} \Omega_{G,n+1} \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n+1}"'] & \mathsf{F} \wr \Omega_{G,n} \arrow[d, "\mathsf{F} \wr d_{0,\cdots,n}"] \\ \Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd} \tag{4.30}$$ PTPULL EQ

is a pullback diagram in Cat.

Proof. An object in the pullback (4.30) over $T \in \Omega_{G,0} = \Omega_G^q$ is precisely the same as a n-string in $\mathsf{Sub}(T)$, and thus by Proposition 4.19 equivalent to a $n+1$ planar tall string starting at T.

The case of arrows is slightly more subtle. A quotient map $\pi: T \to T'$ induces a G-equivariant poset map $\pi_*: \mathsf{Sc}(T) \to \mathsf{Sc}(T')$ (or equivalently, a map of Grothendieck constructions $G \ltimes \mathsf{Sc}(T) \to G \ltimes \mathsf{Sc}(T')$) and diagrams as on the left below (where v_{Ge} ranges over $V_G(T)$ and $e' = \varphi(e)$) induce diagrams (of functors $\mathsf{Sc}(T) \to \Omega$) as on the right below.

$$\begin{tikzcd} T_{v_{Ge}} \arrow[r] \arrow[d] & T_{1,v_{Ge}} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_{n,v_{Ge}} \arrow[d] \\ T'_{v_{Ge'}} \arrow[r] & T'_{1,v_{Ge'}} \arrow[r] & \cdots \arrow[r] & T'_{n,v_{Ge'}} \end{tikzcd} \qquad \begin{tikzcd} T_{(-)} \arrow[r] \arrow[d] & T_{1,(-)} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_{n,(-)} \arrow[d] \\ T'_{(-)} \circ \pi_* \arrow[r] & T'_{1,(-)} \circ \pi_* \arrow[r] & \cdots \arrow[r] & T'_{n,(-)} \circ \pi_* \end{tikzcd} \tag{4.31}$$ PTNARROWLOC EQ

Passing to colimits then gives the desired commutative diagram (4.25). Moreover, diagrams of the form (4.25) clearly induce diagrams as in (4.31) and it is straightforward to check that these are inverse processes. $\square$

DSCOM REM **Remark 4.32.** The diagrams (with back and lower slanted faces instances of (4.30)) PTPULL EQ

$$\begin{tikzcd}
\Omega_{G,n+2} \arrow[rr] \arrow[dr, "d_{i+1}"] \arrow[dd] & & \mathsf{F} \wr \Omega_{G,n+1} \arrow[dr, "\mathsf{F}\wr d_i"] \arrow[dd] & \\
& \Omega_{G,n+1} \arrow[rr] \arrow[dl] & & \mathsf{F} \wr \Omega_{G,n} \arrow[dl] \\
\Omega_{G,0} \arrow[rr] & & \mathsf{F} \wr \Sigma_G &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{G,n+1} \arrow[rr] \arrow[dr, "s_i"] \arrow[dd] & & \mathsf{F} \wr \Omega_{G,n} \arrow[dr, "\mathsf{F}\wr s_{i-1}"] \arrow[dd] & \\
& \Omega_{G,n+2} \arrow[rr] \arrow[dl] & & \mathsf{F} \wr \Omega_{G,n+1} \arrow[dl] \\
\Omega_{G,0} \arrow[rr] & & \mathsf{F} \wr \Sigma_G &
\end{tikzcd}$$

commute whenever defined (i.e. $0 \le i \le n+1$).

INDVNG NOT **Notation 4.33.** We will let

$$V_{G,n} \colon \Omega_{G,n} \to \mathsf{F} \wr \Sigma_G$$

be inductively defined by $V_{G,n} = \sigma_0 \circ V_{G,n-1} \circ V_G$.

Remark 4.34. When $n = 2$, $V_{G,2}$ is thus the composite

$$\Omega_{G,2} \xrightarrow{V_G} \mathsf{F} \wr \Omega_{G,1} \xrightarrow{V_G} \mathsf{F} \wr \mathsf{F} \wr \Omega_{G,0} \xrightarrow{V_G} \mathsf{F} \wr \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F} \wr \Sigma_G$$

In light of Remarks 3.47 and 4.14, VERTEXDECOMP REM VERTEXDECOMPG REM $V_{G,n}(T_0 \to \cdots \to T_n)$ is identified with the tuple

$$(T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_n)}, \tag{4.35}$$ VGNISO EQ

though this requires changing the total order in $V_G(T_n)$. Rather than using the order induced by T_n, one instead equips $V_G(T_n)$ with the order induced lexicographically from the maps $V_G(T_n) \to V_G(T_{n-1}) \to \cdots \to V_G(T_0)$, i.e., for $v, w \in V_G(T_n)$ the condition $v < w$ is determined by the lowest i such that the images of $v, w \in V_G(T_i)$ are distinct.

4.4 A monad on spans

WSPAN DEF **Definition 4.36.** We will write $\mathsf{WSpan}^l(\mathcal{C}, \mathcal{D})$ (resp. $\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})$), which we call the category of *left weak spans* (resp. *right weak spans*), to denote the category with objects the spans

$$\mathcal{C} \xleftarrow{k} A \xrightarrow{F} \mathcal{D},$$

arrows the diagrams as on the left (resp. right) below

$$\begin{tikzcd}
& A_1 \arrow[dl, "k_1"'] \arrow[dr, "F_1"] \arrow[dd, "i"] \arrow[dd, phantom, shift left=6, "\Swarrow\varphi"] & \\
\mathcal{C} & & \mathcal{D} \\
& A_2 \arrow[ul, "k_2"] \arrow[ur, "F_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[dl, "k_1"'] \arrow[dr, "F_1"] \arrow[dd, "i"] \arrow[dd, phantom, shift left=6, "\Nearrow\varphi"] & \\
\mathcal{C} & & \mathcal{D} \\
& A_2 \arrow[ul, "k_2"] \arrow[ur, "F_2"'] &
\end{tikzcd} \tag{4.37}$$ TWISTEDARROWRIGHT EQ

which we write as $(i, \varphi) \colon (k_1, F_1) \to (k_2, F_2)$, and composition given in the obvious way.

Remark 4.38. There are natural isomorphisms

$$\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D}) \simeq \mathsf{WSpan}^l(\mathcal{C}^{op}, \mathcal{D}^{op}). \tag{4.39}$$ LRSPANISO EQ

RANLANADJ REM **Remark 4.40.** The terms *left/right* are motivated by the existence of adjunctions (which are seen to be equivalent by using (4.39)) LRSPANISO EQ

$$\mathsf{Lan} \colon \mathsf{WSpan}^l(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \colon \iota$$

$$\iota \colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})^{op} \colon \mathsf{Ran}$$

where the functors ι denote the obvious inclusions (note the need for the $(-)^{op}$ in the second adjunction) and $\mathsf{Lan}/\mathsf{Ran}$ denote the left/right Kan extension functors.

We will mainly be interested in the span categories $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \simeq \mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$.

OMEGAGNA NOT

Notation 4.41. Given a functor $\pi\colon A \to \Sigma_G$, we let $\Omega_{G,n}^{(A)}$ denote the pullback (in Cat)

$$\begin{tikzcd}
\Omega_{G,n}^{(A)} \arrow[r, "V_{G,n}^{(A)}"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_{G,n} \arrow[r, "V_{G,n}"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}$$

Explicitly, the objects of $\Omega_{G,n}^{(A)}$ are pairs

$$(T_0 \to \cdots \to T_n, (a_{e^{\uparrow} \leq e})_{(e^{\uparrow} \leq e) \in V_G(T_n)}) \tag{4.42}$$ OMEGAGNA EQ

such that $\pi(a_{e^{\uparrow}\leq e}) = T_{n,e^{\uparrow}\leq e}$.

Remark 4.43. Our primary interest here will be in the $\Omega_{G,0}^{(A)}$ construction. Importantly, the composite maps $\Omega_{G,0}^{(A)} \to \Omega_{G,0} \to \Sigma_G$ allow us to iterate the $\Omega_{G,0}^{(-)}$ construction. In practice, the role of higher strings $\Omega_{G,n}^{(A)}$ will then be to provide more convenient models for iterated $\Omega_{G,0}^{(-)}$ constructions.

Indeed, the content of Proposition SUBSASPULL PROP 4.29 is then that there are compatible identifications $\Omega_{G,0}^{(\Omega_{G,n})} \simeq \Omega_{G,n+1}$ which identify $V_G^{(\Omega_{G,n})}$ with V_G.

Moreover, since all squares in the diagram

$$\begin{tikzcd}
\Omega_{G,n+1}^{(A)} \arrow[r, "V_G^{(A)}"] \arrow[d] & \mathsf{F} \wr \Omega_{G,n}^{(A)} \arrow[r, "\mathsf{F}\wr V_{G,n}^{(A)}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_{G,n+1} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \Omega_{G,n} \arrow[r, "\mathsf{F}\wr V_{G,n}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \arrow[r, "\sigma^0"] & \mathsf{F} \wr \Sigma_G \\
\Omega_{G,0} \arrow[r] & \mathsf{F} \wr \Sigma_G & &
\end{tikzcd} \tag{4.44}$$ ALLSQUARES EQ

are pullback squares (the top center square is so by induction, the top right square by direct verification, the total top square by definition of $\Omega_{G,n+1}^{(A)}$ and the bottom left square by Proposition SUBSASPULL PROP 4.29), we likewise obtain identifications $\Omega_G^{\left(\Omega_{G,n}^{(A)}\right)} \simeq \Omega_{G,n+1}^{(A)}$.

Proposition 4.45. *For any $A \to \Sigma_G$ there are functors $d_0^{(A)}\colon \Omega_{G,1}^{(A)} \to \Omega_G^{(A)}$ and natural isomorphisms*

$$\begin{tikzcd}
\Omega_{G,1}^{(A)} \arrow[r, "V_G"] \arrow[d, "d_0^{(A)}"'] & \mathsf{F} \wr \Omega_G^{(A)} \arrow[r, "\mathsf{F}\wr V_G"] & \mathsf{F} \wr \mathsf{F} \wr A \arrow[d, "\sigma^0"] \arrow[dll, Rightarrow, "\pi^{(A)}"] \\
\Omega_G^{(A)} \arrow[rr, "V_G"'] & & \mathsf{F} \wr A,
\end{tikzcd} \tag{4.46}$$ SHUFFLEPERMA EQ

both natural in $A \to \Sigma$. Here naturality of $\pi^{(-)}$ means that for a functor $H\colon A \to B$ with *corresponding diagram*

$$
\begin{tikzcd}
\Omega^{(A)}_{G,1} \arrow[rr, "V^{(A)}_G"] \arrow[dr, "d^{(A)}_0"] \arrow[dd, "\Omega^{(H)}_{G,1}"'] & & \mathsf{F}\wr\Omega^{(A)}_{G,0} \arrow[r, "\mathsf{F}\wr V^{(A)}_G"] \arrow[dd, dashed] & \mathsf{F}\wr\mathsf{F}\wr A \arrow[dr, "\sigma^0"] \arrow[dd, dashed] \arrow[dll, Rightarrow, "\pi^{(A)}"] & \\
& \Omega^{(A)}_{G,0} \arrow[rrr, "V^{(A)}_G"'] \arrow[dd] & & & \mathsf{F}\wr A \arrow[dd, "\mathsf{F}\wr H"] \\
\Omega^{(B)}_{G,1} \arrow[rr, dashed] \arrow[dr, "d^{(B)}_0"'] & & \mathsf{F}\wr\Omega^{(B)}_{G,0} \arrow[r, dashed] & \mathsf{F}\wr\mathsf{F}\wr B \arrow[dr, dashed] \arrow[dll, Rightarrow, dashed] & \\
& \Omega^{(B)}_{G,0} \arrow[rrr, "V^{(B)}_G"'] & & & \mathsf{F}\wr B
\end{tikzcd}
\tag{4.47}
$$

PICUBOIDAB EQ

one has an equality

$$(\mathsf{F}\wr H)\pi^{(A)} = \pi^{(B)}\Omega^{(H)}_{G,1}$$

(i.e. the two natural isomorphisms between the two distinct functors $\Omega^{(A)}_{G,1} \rightrightarrows \mathsf{F}\wr B$ coincide).

Proof. Informally, using the object description in (4.42), $d^{(A)}_0$ is simply given by the formula

$$d^{(A)}_0\left(T_0 \to T_1, (a_{e\uparrow\leq e})_{(e\uparrow\leq e)\in V_G(T_1)}\right) = \left(T_1, (a_{e\uparrow\leq e})_{(e\uparrow\leq e)\in V_G(T_1)}\right), \tag{4.48}$$

GENDO EQ

though one must note that since in (4.42) the order in $V_G(T_1)$ is induced lexicographically from the string, the two orders for $V_G(T_1)$ in each side of (4.48) do not coincide.

It now follows that the composites $\sigma^0 \circ (\mathsf{F}\wr V^{(A)}_G) \circ V^{(A)}_G$ and $V^{(A)}_G \circ d^{(A)}_0$ differ by the natural automorphism $\pi^{(A)}$ given by the tuple permutations interchanging the two orders in $V_G(T_1)$ for each $T_0 \to T_1$.

The commutativity of (4.47) is clear. □

Definition 4.49. Suppose $\mathcal{V}$ has finite products.

We define an endofunctor N of $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$ by letting $N(\Sigma_G \leftarrow A \to \mathcal{V}^{op})$ be the span $\Sigma_G \leftarrow \Omega^{(A)}_G \to \mathcal{V}^{op}$ given composition along the diagram

$$
\begin{tikzcd}
\Omega^{(A)}_{G,0} \arrow[r] \arrow[d] & \mathsf{F}\wr A \arrow[r] \arrow[d] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r, "\Pi^{op}"] & \mathcal{V}^{op} \\
\Omega_{G,0} \arrow[r] \arrow[d] & \mathsf{F}\wr\Sigma_G & & \\
\Sigma_G & & &
\end{tikzcd}
$$

and defined on maps of spans in the obvious way.

One has a multiplication $\mu\colon N\circ N \Rightarrow N$ given by the natural isomorphisms

$$
\begin{tikzcd}
\Sigma \arrow[d, equal] & \Omega^{(A)}_{G,1} \arrow[l] \arrow[r, "V_G"] \arrow[d, "d^{(A)}_0"'] & \mathsf{F}\wr\Omega^{(A)}_G \arrow[r, "\mathsf{F}\wr V_G"] & \mathsf{F}\wr\mathsf{F}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dll, Rightarrow, "\pi^{(A)}"] & \mathsf{F}\wr\mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dll, Rightarrow, "\alpha"] \\
\Sigma & \Omega^{(A)}_G \arrow[l] \arrow[rr, "V_G"'] & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd}
\tag{4.50}
$$

MULTDEFSPAN EQ

where α is an associativity isomorphism for the product Π. We we note that naturality of μ follows from the commutativity of (4.47).

Lastly, there is a unit $\eta\colon id \Rightarrow N$ given by the strictly commutative diagrams

$$
\begin{tikzcd}
\Sigma \arrow[d, equal] & A \arrow[l] \arrow[r, equal] \arrow[d, "s^{(A)}_{-1}"'] & A \arrow[r] \arrow[d] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma & \Omega^{(A)}_G \arrow[l] \arrow[r, "V_G"'] & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}.
\end{tikzcd}
\tag{4.51}
$$

UNITSPAN EQ

MONSPAN PROP

Proposition 4.52. (N, μ, η) *form a monad on* $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$.

Proof. The natural transformation component of $\mu \circ (N\mu)$ is given by the composite diagram

$$\begin{tikzcd}
\Omega^{(A)}_{G,2} \arrow[r] \arrow[d, "d^{(A)}_1"'] & \mathsf{F} \wr \Omega^{(A)}_{G,1} \arrow[r] \arrow[d] & \mathsf{F}^{\wr 2} \wr \Omega^{(A)}_G \arrow[r] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_{G,1} \arrow[r] \arrow[d, "d^{(A)}_0"'] & \mathsf{F} \wr \Omega^{(A)}_G \arrow[rr] \arrow[r, phantom, "\mathsf{F}\wr\pi^{(A)}"] & & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[rr] \arrow[d, "\sigma^0"] \arrow[r, phantom, "\mathsf{F}\wr\alpha"] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_G \arrow[rrr] \arrow[rr, phantom, "\pi^{(A)}"] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] \arrow[rr, phantom, "\alpha"] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.53}$$

ASSOCSPAN1 EQ

whereas the natural transformation component of $\mu \circ (\mu N)$ is given by

$$\begin{tikzcd}
\Omega^{(A)}_{G,2} \arrow[r] \arrow[d, "d^{(A)}_0"'] & \mathsf{F} \wr \Omega^{(A)}_{G,1} \arrow[r] & \mathsf{F}^{\wr 2} \wr \Omega^{(A)}_G \arrow[r] \arrow[d] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d] & \mathsf{F}^{\wr} \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_{G,1} \arrow[rr, "\pi\left(\Omega^{(A)}_G\right)"'] \arrow[d, "d^{(A)}_0"'] & & \mathsf{F} \wr \Omega^{(A)}_G \arrow[r] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_G \arrow[rrr] \arrow[rr, phantom, "\pi^{(A)}"] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] \arrow[rr, phantom, "\alpha"] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.54}$$

ASSOCSPAN2 EQ

That the rightmost sections of (4.53) and (4.54) coincide follows from compatibility of the associativity isomorphisms for Π^{op}.

For the leftmost sections, note first that, in either diagram, the top right and bottom left paths $\Omega^{(A)}_{G,2} \to \mathsf{F} \wr A$ differ only by the induced order on $V_G(T_2)$ for each string $T_0 \to T_1 \to T_2$. More explicitly, the top right paths use the order induced lexicographically from the string $T_0 \to T_1 \to T_2$ while the bottom left paths use the order induced exclusively by T_2. The two left sections then coincide since are both given by the permutation interchanging these orders, the only difference being that the intermediate stage of (4.53) uses the order induced lexicographically from $T_0 \to T_2$ while (4.54) uses the order induced lexicographically from $T_1 \to T_2$.

As for unit conditions, $\mu \circ (N\eta)$ is represented by

$$\begin{tikzcd}
\Omega^{(A)}_G \arrow[r] \arrow[d, "s^{(A)}_0"'] & \mathsf{F} \wr A \arrow[r, equal] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_{G,1} \arrow[r] \arrow[d, "d^{(A)}_0"'] & \mathsf{F} \wr \Omega^{(A)}_G \arrow[r] \arrow[d, phantom, "\pi^{(A)}"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_G \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.55}$$

UNITSPAN1 EQ

while $\mu \circ (\eta N)$ is represented by

$$\begin{tikzcd}
\Omega^{(A)}_G \arrow[r, equal] \arrow[d, "s^{(A)}_{-1}"'] & \Omega^{(A)}_G \arrow[r] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_{G,1} \arrow[r] \arrow[d, "d^{(A)}_0"'] & \mathsf{F} \wr \Omega^{(A)}_G \arrow[r] \arrow[d, phantom, "\pi^{(A)}"] & \mathsf{F} \wr \mathsf{F} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_G \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.56}$$

UNITSPAN2 EQ

It is straightforward to check that the composites of the left and right sections of both (4.55) and (4.56) are strictly commutative diagrams, and thus that (4.55) and (4.56) coincide. $\square$

4.5 The free genuine operad monad

Recalling that $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op}) \simeq \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$, Proposition 4.52 and Remark 4.40 give an adjuntion

$$\mathsf{Lan}\colon \mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})\colon \iota \tag{4.57}$$ LANIOTAADJ EQ

together with a monad N in the leftmost category $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$. We now turn to showing that, under reasonable hypothesis on $\mathcal{V}$, the composite $\mathsf{Lan} \circ N \circ \iota$ inherits a monad structure from N. The key will be to show that under such conditions the map $\mathsf{Lan} \circ N \Rightarrow \mathsf{Lan} \circ N \circ \iota \circ \mathsf{Lan}$ is a natural isomorphism.

Recall that following Convention 3.18 our model for O_G consists of totally ordered sets. One therefore has *root functors*

$$\Omega_G^q \xrightarrow{\mathsf{r}} \mathsf{O}_G, \qquad \Sigma_G \xrightarrow{\mathsf{r}} \mathsf{O}_G$$

sending each planar G-tree to its ordered orbital G-set of roots.

Root functors are compatible with the leaf-root functor and the inclusion, i.e. the following commute.

$$\begin{tikzcd} \Omega_G^q \arrow[r, "\mathsf{lr}"] \arrow[dr, "\mathsf{r}"'] & \Sigma_G \arrow[d, "\mathsf{r}"] \\ & \mathsf{O}_G \end{tikzcd} \qquad \begin{tikzcd} \Sigma_G \arrow[r, hook] \arrow[dr, "\mathsf{r}"'] & \Omega_G^q \arrow[d, "\mathsf{r}"] \\ & \mathsf{O}_G \end{tikzcd} \tag{4.58}$$ ROOTLEAFTROOTCOM EQ

Moreover, the diagrams (4.58) possess some extra structure we will need to make use of. Indeed, both functors are split Grothendieck fibrations: given a map $\varphi\colon A \to B$ in O_G and G-tree T such that $\mathsf{r}(T) = B$ we can build a cartesian arrow $\varphi^*(T) \to T$ by letting $\varphi^*(T)$ to be the pullback G-tree together with the planar structure on roots given by A and on non-equivariant nodes given by their image via $\varphi^*(T) \to T$.

It now follows that (4.58) are diagrams of split Grothendieck fibrations.

Definition 4.59. A split Grothendieck fibration $A \xrightarrow{\mathsf{r}} \mathsf{O}_G$ is called a *root fibration* and a split Grothendieck fibration diagram

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\mathsf{r}"'] & B \arrow[d, "\mathsf{r}"] \\ & \mathsf{O}_G \end{tikzcd}$$

is called a *root fibration functor*.

The relevance of root fibrations is given by the following couple of lemmas.

ROOTFIBPULL LEM **Lemma 4.60.** *If $A \to \Sigma_G$ is a root fibration functor then so is $\Omega_G^{(A)} \to \Omega_G$, naturally in A.*

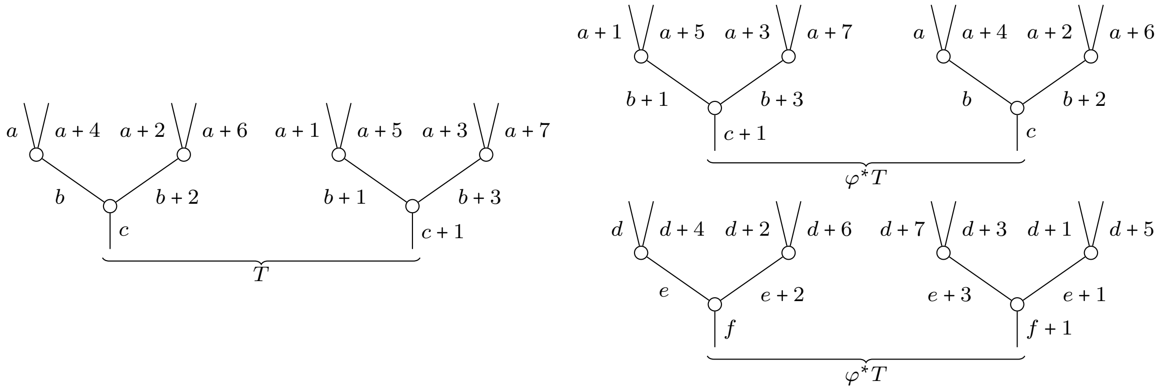
Proof. We consider the pullback diagram below.

$$\begin{tikzcd} \Omega_{G,0}^{(A)} \arrow[r, "V_G^{(A)}"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\ \Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd} \tag{4.61}$$ ROOTIMPLIESROOT EQ

The hypothesis that $A \to \Sigma_G$ is root fibration implies that the rightmost map in (4.61) is a map of split Grothendieck fibrations over $\mathsf{F} \wr \mathsf{O}_G$.

Since the map V_G sends the chosen cartesian arrows in $\Omega_{G,0}$ (over O_G) to chosen cartesian arrows of $\mathsf{F} \wr \Sigma_G$ (over $\mathsf{F} \wr \mathsf{O}_G$), the result follows. $\square$

Example 4.62. Let $G = \mathbb{Z}_{/8}$. The following exemplifies a pull back along the twist map $\varphi: G/2G \to G/2G$ (i.e., accounting for order, φ is the permutation (12)), with the topmost representation of $\varphi^* T$ maintaining the chosen generators for each edge orbit from T and the bottom representation choosing instead the generators to be minimal with regard to the planar structure.



(4.63)

We note that $(\varphi^*(T))_{v_{Ge}} = \psi^*(T_{v_{Gb}})$ for ψ the permutation $(13)(24)$ encoded by the composite identifications $\{1,2,3,4\} \simeq \{e, e+2, e+3, e+1\} \simeq \{b+1, b+3, b, b+2\} \simeq \{3,4,1,2\}$.

LANPULLCOMA LEM

Lemma 4.64. *Suppose that $\mathcal{V}$ is complete and that $A \to \Sigma_G$ is a root fibration. If the rightmost triangle in*

$$\begin{tikzcd} \Omega^{(A)}_{G,0} \arrow[r, "V^{(A)}_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathcal{V} \\ \Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \arrow[ur] \arrow[u, phantom, shift left=4, "\Uparrow"] & \end{tikzcd} \tag{4.65}$$

is a right Kan extension diagram then so is the composite diagram.

Proof. Unpacking definitions using the pointwise formula for right Kan extensions (McL [2, X.3.1]), it suffices to check that for each $T \in \Omega_{G,0}$ the functor

$$T \downarrow \Omega^{(A)}_{G,0} \to V_G(T) \downarrow \mathsf{F} \wr A \tag{4.66}$$

LANPULLCOMA EQ

is initial. In the course of the proof of Lemma FINWREATPRODLIM LEM 4.3 it was shown that the subcategory

$$\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow A$$

is initial in the $V_G(T) \downarrow \mathsf{F} \wr A$.

On the other hand, since $\Omega^{(A)}_G \to \Omega_G$ is a root fibration functor, $T \downarrow \Omega^{(A)}_G$ has an initial subcategory $T \downarrow_{\mathsf{r},\simeq} \Omega^{(A)}_G$ with objects $(S \in \Omega^{(A)}_G, T \to u(S))$ such that $T \to u(S)$ is a quotient map that induces an ordered isomorphism on roots. Note that this can be restated as saying that $T \to u(S)$ is an isomorphism preserving the order of the roots.

The result now follows from the natural isomorphism

$$T \downarrow_{\mathsf{r},\simeq} \Omega^{(A)}_G \simeq \prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r},\simeq} A. \tag{4.67}$$

TDOWNISOA EQ

To see this, we focus first on the case $A = \Sigma_G$. In that case, the left hand side of (TDOWNISOA EQ 4.67) encodes replanarizations of T that preserve the root order. On the other hand, the right hand side encodes replanarizations of all the G-vertices that preserve the order of their roots, or, equivalently, replanarizations of the non-equivariant vertices of T. That these are equivalent is the content of Proposition 3.13.

Note that $(T \to S) \in (T \downarrow_{\mathsf{r},\simeq} \Omega_G)$ is then encoded by a tuple $\left(T_{v_{Ge}} \to \varphi^*_{v_{Ge}} S_{v_{Ge}}\right)_{v_{Ge} \in V_G(T)}$ where the pullbacks $\varphi^*_{v_{Ge}}$ are needed to correct the root order.

The case of general A follows likewise, using the corresponding pullbacks $\varphi^*_{v_{Ge}}$.

Note: an addendum is needed to show that (4.67) suffices, since $T \downarrow_{\mathsf{r},\simeq} \Omega_G^{(A)}$ is not sent directly to $\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r},\simeq} A$ □

Lemma 4.60 can be interpreted as saying that, if one defines a category $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$ of *rooted spans*

$$\Sigma_G^{op} \leftarrow A^{op} \to \mathcal{V}$$

where $A \to \Sigma_G$ is a root fibration functor, the monad N built in Proposition 4.52 lifts to a monad N_{r} in $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$, and likewise for the adjunction (4.57).

Corollary 4.68. *Suppose that finite products in $\mathcal{V}$ commute with colimits in each variable. The functors*

$$\mathsf{Lan} \circ N_{\mathsf{r}} \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan}, \qquad \mathsf{Lan} \circ \iota \Rightarrow id$$

are natural isomorphisms.

Proof. This follows by combining Lemma 4.64 with Lemma 4.3. □

THEMONAD DEF

Definition 4.69. The *genuine equivariant operad monad* is the monad $\mathbb{F}_G$ on $\mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$ given by

$$\mathbb{F}_G = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

and with multiplication and unit given by the composites

$$\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ N_{\mathsf{r}} \circ N_{\mathsf{r}} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

$$id \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota.$$

Remark 4.70. The functor $\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$ is isomorphic to $\mathsf{Lan} \circ N \circ \iota$, and this isomorphism is compatible with the multiplication and unit in Definition 4.69, and we will henceforth simply write N rather than N_{r}.

From this point of view, the role of root fibrations is to guarantee that $\mathsf{Lan} \circ N \circ \iota$ is indeed a monad, but unnecessary to describe the monad structure itself.

REPACKAGERES REM

Remark 4.71. Since a map

$$\mathbb{F}_G X = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota X \to X$$

is adjoint to a map

$$N_{\mathsf{r}} \circ \iota X \to \iota X$$

one easily verifies that X is a genuine equivariant operad, i.e. a $\mathbb{F}_G$-algebra, iff ιX is a N-algebra. Moreover, the bar resolution

$$\mathbb{F}_G^{\bullet+1} X$$

is isomorphic to

$$\mathsf{Lan}\left(N^{\bullet+1} \iota X\right).$$

5 Free extensions

Our overall goal in this section will be to produce a description of free genuine operad pushouts, i.e. pushouts of the form

$$\begin{tikzcd} \mathbb{F}_G A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{F}_G B \arrow[r] & Y \end{tikzcd}$$

in the category Op_G of genuine equivariant operads.

5.1 Extensions over general monads

EXTGENMON SEC

Any monad T on $\mathcal{C}$ one obtains induced monads $T^{\times l}$ on $\mathcal{C}^{\times l}$, and we will make use of several standard relations between these. In particular, any map $\alpha\colon \underline{l} \to \underline{m}$ induces a forgetful functor such that for the forgetul functor $\alpha^*\colon \mathcal{C}^{\times l} \to \mathcal{C}^{\times n}$ one has $T^{\times l}\alpha^* \simeq \alpha^* T^{\times m}$.

Indeed, we will need to make use of a slightly more general setup. Letting I denote the identity monad on $\mathcal{C}$, and $K \subset \underline{m}$ be a subset, there is a monad $T^{\times K} \times I^{\times(\underline{m}-K)}$ on $\mathcal{C}^{\times m}$, which we abusively denote simply as $T^{\times K}$. Identities then determine maps of monads $T^J \to T^{\times K}$ whenever $J \subset K$ and, moreover, there are identifications $T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$. One then has the following.

MONADICFUN PROP

Proposition 5.1. *The functor*

$$T^{\times \alpha^{-1}(K)} \Rightarrow \alpha^* T^{\times K} \alpha_! \tag{5.2}$$

MONADFUNCTORALPHA EQ

adjoint to the identification $T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ is a map of monads on $\mathcal{C}^{\times n}$.

Proof. We first note that there are identifications of functors $(FG)^{\times K} \simeq F^{\times K} G^{\times K}$ which are compatible with the identifications $F^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* F^{\times K}$ in the sense that the identification $(FG)^{\times \alpha^{-1}(K)} \circ \alpha^* \simeq \alpha^*(FG)^{\times K}$ matches the composite identification $F^{\times \alpha^{-1}(K)} G^{\times \alpha^{-1}(K)} \alpha^* \simeq F^{\times \alpha^{-1}(K)} \alpha^* G^{\times K} \simeq \alpha^* F^{\times K} G^{\times K}$.

Letting η, ϵ denote the unit and counit for the $(\alpha_!, \alpha^*)$ adjunction, (5.2) [MONADFUNCTORALPHA EQ] is then the composite

$$T^{\times \alpha^{-1}(K)} \xrightarrow{\eta} T^{\times \alpha^{-1}(K)} \alpha^* \alpha_! \simeq \alpha^* T^{\times K} \alpha_!.$$

That this is a monad map is the condition that the following multiplication and unit diagrams commute.

$$\begin{tikzcd}
T^{\times \alpha^{-1}(K)} \circ T^{\times \alpha^{-1}(K)} \arrow[r] \arrow[d] & \alpha^* T^{\times K} \alpha_! \circ \alpha^* T^{\times K} \alpha_! \arrow[d] \\
T^{\times \alpha^{-1}(K)} \arrow[r] & \alpha^* T^{\times K} \alpha_!
\end{tikzcd}
\qquad
\begin{tikzcd}
I^{\times n} \arrow[d] \arrow[dr] & \\
T^{\times \alpha^{-1}(K)} \arrow[r] & \alpha^* T^{\times K} \alpha_!
\end{tikzcd}$$

We argue only the case of the leftmost multiplication diagram, with commutativity of the unit diagram following by a similar but simpler argument. Since the precomposition $(-) \circ \alpha^*$ is the left adjoint to the precomposition $(-) \circ \alpha_!$ this follows from the following diagram.

$$\begin{tikzcd}
T^{\times \alpha^{-1}(K)} T^{\times \alpha^{-1}(K)} \alpha^* \arrow[r, "\simeq"] \arrow[dd] & T^{\times \alpha^{-1}(K)} \alpha^* T^{\times K} \arrow[r, "\eta"] \arrow[dr, equal] & T^{\times \alpha^{-1}(K)} \alpha^* \alpha_! \alpha^* T^{\times K} \arrow[r, "\simeq"] \arrow[d, "\epsilon"] & \alpha^* T^{\times K} \alpha_! \alpha^* T^{\times K} \arrow[d, "\epsilon"] \\
 & & T^{\times \alpha^{-1}(K)} \alpha^* T^{\times K} \arrow[r, "\simeq"] & \alpha^* T^{\times K} T^{\times K} \arrow[d] \\
T^{\times \alpha^{-1}(K)} \alpha^* \arrow[rrr, "\simeq"] & & & \alpha^* T^{\times K}
\end{tikzcd}$$

□

TALPHAKMOD REM

Remark 5.3. Since $T^{\times K}\alpha_!$ is a right $\alpha^* T^{\times K} \alpha_!$-module, Proposition 5.1 [MONADICFUN PROP] implies that it is also a right $T^{\times \alpha^{-1}(K)}$-module or, moreover, a right $T^{\times J}$-module whenever $\alpha(J) \subset K$.

COMPPOSTCOMP REM

Remark 5.4. Combining the precomposition and postcomposition adjunctions, the identification $T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ is then adjoint to a functor $\alpha_! T^{\times \alpha^{-1}(K)} \to T^{\times K}\alpha_!$ which is readily checked to be a map of right $T^{\times \alpha^{-1}(K)}$-modules.

More generally, for $\alpha(J) \subset K$, the composite $T^{\times J}\alpha^* \to T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ is thus adjoint to a map of right $T^{\times J}$-modules

$$\alpha_! T^{\times J} \to T^{\times K} \alpha_!. \tag{5.5}$$

RIGHTMODULETMAP EQ

We now unpack the content of (RIGHTMODULETMAP EQ 5.5) when $\alpha\colon \underline{l} \to *$ is the unique map to the simpleton $* = \underline{1}$. In this case we can instead write $\alpha_! = \coprod$, $\alpha^* = \Delta$, and we thus have commutative diagrams

$$\begin{tikzcd}
\coprod_J TTA_j \amalg \coprod_{\underline{n}-J} A_j \arrow[r] \arrow[d] & T\left(\coprod_J TA_j \amalg \coprod_{\underline{n}-J} A_j\right) \arrow[d] \\
\coprod_J TA_j \amalg \coprod_{\underline{n}-J} A_j \arrow[r] & T\left(\coprod_J A_j \amalg \coprod_{\underline{n}-J} A_j\right)
\end{tikzcd} \tag{5.6}$$

RIGHTMODULETMAPAUX EQ

where the vertical maps come from the right $T^{\times J}$-module structure. Writing $\amalg^a$ for the coproduct of T-algebras and recalling the canonical identifications $\coprod^a_K (TA_k) \simeq T\left(\coprod_K A_k\right)$, (RIGHTMODULETMAPAUX EQ 5.6) in fact shows that the right $T^{\times J}$-module structure on $T \circ \coprod$ in fact codifies the multiplication maps

$$\coprod^a_J TTA_j \amalg^a \coprod^a_{\underline{l}-J} TA_j \to \coprod^a_J TA_j \amalg^a \coprod^a_{\underline{l}-J} TA_j.$$

5.2 Labeled planar strings

We now translate the results in the previous section to the context of the monad N on $\mathsf{WSpan}^l(\Sigma^{op}, \mathcal{V})$. In analogy to the planar string models $\Omega^{(A)}_{G,n}$ for iterations $N^{\circ n+1}$ of the monad N, we will find it convenient to build similar string models $\Omega^{(A_J)}_{G,n}$ for $N \circ \coprod \circ (N^{\times J})^{\circ n}$.

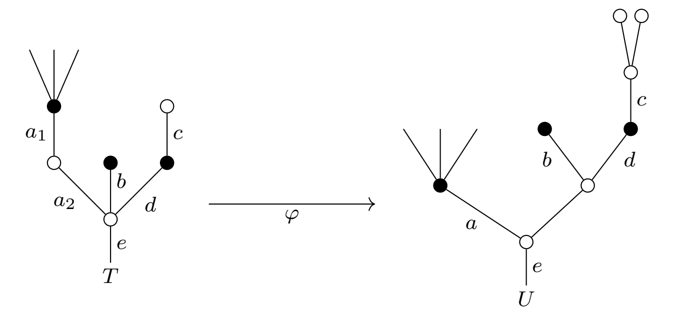
Definition 5.7. A *l-node labeled G-tree* (or just *l-labeled G-tree*) G-tree is a pair $(T, V_G(T) \to \{1, \cdots, l\})$ with $T \in \Omega_G$, which we think of as a G-tree together with G-vertices labels in $1, \cdots, l$.

Further, a tall map $\varphi\colon T \to S$ between l-labeled trees is called a *label map* if for each G-vertex v_{Ge} of T with label j, the vertices of the subtree $S_{v_{Ge}}$ are all labeled by j.

Lastly, given a subset $J \subset \underline{l}$, a planar label map $\varphi\colon T \to S$ is said to be J-inert if for every G-vertex v_{Ge} of T with label $j \in J$ it is $S_{v_{Ge}} = T_{v_{Ge}}$.

LABELEDTREES EX

Example 5.8. Consider the 2-labeled trees below (for $G = *$ the trivial group), with black nodes ($\bullet$) denoting labels by the number 1 and white nodes ($\circ$) labels by the number 2. The planar map φ (sending $a_i \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto d$, $e \mapsto e$) is a label map which is $\{1\}$-inert.



(5.9) SUBSDATUMTREESLAB EQ

Definition 5.10. Let $0 \le s \le n$ and $J \subset \underline{l}$ be a subset.

We define $\Omega^J_{G,n,s}$ to have as objects n-planar strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s \xrightarrow{f_{s+1}} T_{s+1} \xrightarrow{f_{s+2}} \cdots \xrightarrow{f_n} T_n \tag{5.11}$$

NSTRINGLAB EQ

together with l-labelings of $T_s, T_{s+1}, \cdots, T_n$ such that the $f_r, r > s$ are $(\underline{l} - J)$-inert label maps.

Arrows in $\Omega^J_{G,n,s}$ are quotients of strings $(q_r\colon T_r \to T'_r)$ such that $q_r, r \le s$ are label maps.

Informally, $\Omega^l_{G,n,s}$ consists of n-strings such that trees and maps after T_s are l-labeled.

Remark 5.12. Our main case of interest will that of $s = 0$, in which case we abbreviate $\Omega^J_{G,n} = \Omega^J_{G,n,0}$. Indeed, such strings will suffice to build models for $N \circ \coprod \circ (N^{\times J})^{\circ n}$.

However, to unpack the right $N^{\times J}$-module structure as in Remark TALPHAKMOD REM 5.3 one further needs to encode composites $NN \circ \coprod \circ (N^{\times J})^{\circ n-1}$, a role played by strings $\Omega^J_{G,n,1}$.

Notation 5.13. We will further write

$$\Omega^J_{G,n,-1} = \coprod_J \Omega_{G,n} \amalg \coprod_{\underline{l}-J} \Sigma_G, \qquad \Omega^J_{G,n,n+1} = \Omega_{G,n} \tag{5.14}$$

OMEGANMINUSONE EQ

To justify this convention, we note that a string as in (5.11) can be extended by prepending to it the map $\mathsf{lr}(T_0) = T_{-1} \xrightarrow{f_0} T_0$. If one then attempts to define $\Omega^J_{G,n,-1}$ by insisting that T_{-1} also be labeled, it follows that all node labels in each string must coincide, resulting in the coproduct decomposition in (5.14).

There are a number of obvious functors relating the $\Omega^J_{G,n,s}$ categories, which we now make explicit. Given $s \le s'$ or $J \subset J'$ there are forgetful functors

$$\Omega^J_{G,n,s} \to \Omega^J_{G,n,s'} \qquad \Omega^J_{G,n,s} \to \Omega^{J'}_{G,n,s} \tag{5.15}$$

NKNFGT EQ

The simplicial operators in Notation 4.26 generalize to operators (where $0 \le i \le n$, $-1 \le j \le n$)

$$\begin{array}{llll} d_i \colon \Omega^J_{G,n,s} \to \Omega^J_{G,n-1,s-1} & i < s & s_j \colon \Omega^J_{G,n,s} \to \Omega^J_{G,n+1,s+1} & j < s \\ d_i \colon \Omega^J_{G,n,s} \longrightarrow \Omega^J_{G,n-1,s} & s \le i & s_j \colon \Omega^J_{G,n,s} \longrightarrow \Omega^J_{G,n+1,s} & s \le j \end{array}$$

which are compatible with the forgetful functors in the obvious way.

Remark 5.16. For $J \subset J'$ the forgetful functor in (5.15) is a fully faithful inclusion. However, and somewhat subtly, this is not the case the for the $s \le s'$ forgetful functors. Indeed, regarding $T \to U$ in Examples 5.8 as an object in $\Omega^2_{*,n,0}$, changing the label of the $a_1 \le a_2$ vertex of T from a $\circ$-label to a $\bullet$-label yields an alternate object $\bar{T} \to U$ of $\Omega^2_{*,n,0}$ forgetting to the same object of $\Omega^2_{*,n,1}$, yet $T \to U$ and $\bar{T} \to U$ are not isomorphic.

We note that this is a consequence of the fact that substitution data can replace unary nodes by stumps, which have no nodes.

Generalizing Notation 4.33 there is a commutative diagram

$$\begin{tikzcd} \Omega^J_{G,n,s} \arrow[r, "V_{G,n}"] \arrow[d] & \mathsf{F} \wr \Sigma_G^{\amalg l} \arrow[d] \\ \Omega_{G,n} \arrow[r, "V_{G,n}"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd}$$

where for a labeled string it is $V_G(T_0 \to \cdots \to T_n) = (T_{n,v_{Ge}})_{V_G(T_n)}$, where we regard $T_{n,v_{Ge}} \in \Sigma_G^{\amalg l} \simeq \Omega^l_{G,-1,-1}$ by using the label in $1, \cdots, l$.

We now expand Notation 4.41.

Notation 5.17. Let $\underline{A}$ denote a $\underline{l}$-tuple $(\pi_j \colon A_j \to \Sigma_G)_{\underline{l}}$ of categories over Σ_G. We define $\Omega^{(\underline{A}),J}_{G,n,s}$ by the pullback diagram

$$\begin{tikzcd} \Omega^{(\underline{A}),J}_{G,n,s} \arrow[r, "V^{(\underline{A})}_{G,n}"] \arrow[d] & \mathsf{F} \wr \coprod A_j \arrow[d] \\ \Omega^J_{G,n,s} \arrow[r, "V_{G,n}"'] & \mathsf{F} \wr \Sigma_G^{\amalg l} \end{tikzcd} \tag{5.18}$$

LTUPLEAPULL EQ

Explicitly, an object of $\Omega^{(\underline{A}),J}_{G,n,s}$ consists of a labeled string $T_0 \to \cdots T_n$ as in (5.11) together with a tuple $(a_{v_{Ge}})_{V_G(T_n)}$ such that $a_{v_{Ge}} \in A_j$ if v_{Ge} has label j and $\pi_j(a_{v_{Ge}}) = T_{n,v_{Ge}}$.

The reader may have noticed a certain asymmetry between our definition of the $V_{G,n}$ functors here versus their analogues in §4.3, where they were defined iteratively in terms of simpler functors V_G. This is because of the possibility that $s = -1$, in which case (5.14) applies and some caution is needed in that the following result fails.

ALLSQUARESJ PROP

Proposition 5.19. ALLSQUARES EQ *Suppose $0 \le s \le n$. One has a diagram of pullback squares (generalizing (4.44))*

$$\begin{tikzcd}
\Omega^{(\underline{A}),J}_{G,n,s} \arrow[r, "V_G^{(A)}"] \arrow[d] & \mathsf{F} \wr \Omega^{(\underline{A}),J}_{G,n-1,s-1} \arrow[r, "\mathsf{F}\wr V^{(A)}_{G,n}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr \coprod A_j \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr \coprod A_j \arrow[d] \\
\Omega^{J}_{G,n,s} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \Omega^{J}_{G,n-1,s-1} \arrow[r, "\mathsf{F}\wr V_{G,n}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr \Sigma_G^{\amalg l} \arrow[r, "\sigma^0"] & \mathsf{F} \wr \Sigma_G^{\amalg l} \\
\Omega_{G,0} \arrow[r, "V_G"] & \mathsf{F} \wr \Sigma_G & &
\end{tikzcd} \tag{5.20}$$

ALLSQUARESJ EQ

such that the composite of the top squares is LTUPLEAPULL EQ *(5.18).*

Proof. The V_G functors are defined just as in VGDEF EQ (4.28) via the formula

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)}$$

with the strings $T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}}$ inheriting the extra structure in the obvious way.

Since the top composite square, top center square and top right square are all pullback squares, it remains only to show that the bottom left square is a pullback. This last claim is simply a variation of Proposition SUBSASPULL PROP 4.29, and follows from the same proof, since both labels and inertness conditions are inherited when assembling substitution data into trees via Proposition SUBDATAUNDERPLAN PROP 3.42. □

5.3 Bar constructions on spans

We use the results in the previous sections to obtain a string description of the bar constructions

$$\coprod_J^a N^{\bullet+1} A_j \amalg^a \coprod_{\underline{l}-J}^a N A_j .$$

For simplicity, we discuss first the particular case $\coprod^a N^{\bullet+1} A$. Writing the span as $\Sigma_G \leftarrow A \xrightarrow{F} \mathcal{V}$ the identifications $\Omega_{G,0}^{\left(\Omega^{(A)}_{G,n}\right)} \simeq \Omega^{(A)}_{G,n+1}$ iteratively identify the operator in the bar construction $N^{\bullet+1}A$ as follows.

The top boundaries d_n have natural transformation given by

$$\begin{tikzcd}
\Omega^{(A)}_{G,n} \arrow[r, "V_G^{\circ n}"] \arrow[d] & \mathsf{F}^{\wr n} \wr \Omega^{(A)}_{G,0} \arrow[r, "\mathsf{F}^{\wr n}\wr F_1"] \arrow[d] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"] \arrow[d, equal] \arrow[dl, phantom, "\Swarrow \mathsf{F}^{\wr n}\wr m"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(A)}_{G,n-1} \arrow[r, "V_G^{\circ n}"'] & \mathsf{F}^{\wr n} \wr A \arrow[r, "\mathsf{F}^{\wr n}\wr F"'] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.21}$$

where m is the natural transformation component of the multiplication $NA \to A$, and the remaining differentials d_i for $0 \le i < n$ are given by

$$\begin{tikzcd}
\Omega^{(A)}_{G,n} \arrow[r, "V_G^{\circ n+1}"] \arrow[d, "d_i^{(A)}"'] & \mathsf{F}^{\wr n+1} \wr A \arrow[r, "F"] \arrow[d, "\sigma^i"] \arrow[dl, phantom, "\Swarrow \pi_i^{(A)}"] & \mathsf{F}^{\wr n+1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n+1}"] \arrow[d, "\sigma^i"] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dl, phantom, "\Swarrow \alpha_i"] \\
\Omega^{(A)}_{G,n-1} \arrow[r, "V_G^{\circ n}"'] & \mathsf{F}^{\wr n} \wr A \arrow[r, "F"'] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.22}$$

REMAINDIFF EQ

where $\pi_i^{(A)}$ interchanges lexicographic orders on the i-th F coordinate of $F^{\wr n}$ and α_i is the natural associativity isomorphism.

Maybe add degeneracies

Similarly, Proposition 5.19 shows that $\Omega^{(A)}_{G,n} \simeq \Omega_{G,0}^{\left(\amalg \Omega^{(A_j)}_{G,n-1}\right)}$ so that the top boundaries d_n in the bar construction $N \circ \amalg \circ (N^{\times l})^{\circ n} \underline{A}$ are given by

$$\begin{tikzcd}
\Omega^{(\underline{A})}_{G,n} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-1} \arrow[r, "V_G^{\circ n-1}"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \Omega^{(A_j)}_{G,0} \arrow[r, "F_1"] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"] \arrow[d, equal] \arrow[dl, Rightarrow, "\underline{m}"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n-1} \arrow[r, "V_G"'] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r, "V_G^{\circ n-1}"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr A_j \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"'] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.23}$$

where $\underline{m}$ stands for the functor induced by the tuple of multiplication maps $m_j \colon NA_j \to A_j$, and the other boundaries d_i for $0 \leq i < n$ are given by

$$\begin{tikzcd}
\Omega^{(\underline{A})}_{G,n} \arrow[r, "V_G"] \arrow[d, "d_i^{(\underline{A})}"'] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-1} \arrow[r, "V_G^{\circ n}"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr A \arrow[r, "F"] \arrow[d, "\sigma^i"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"] \arrow[d, "\sigma^i"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n-1} \arrow[r, "V_G"'] \arrow[urr, Leftrightarrow, "\pi_i^{(\underline{A})}"'] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r, "V_G^{\circ n-1}"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr A \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"'] \arrow[urr, Leftrightarrow, "\alpha_i"'] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.24}$$

DISJBARDN EQ

where again $\pi_i^{(\underline{A})}$ interchanges lexicographic orders on the i-th F coordinate and α_i is again an associativity isomorphism. We note that (5.24) follows directly from (5.22) for $0 < i < n$, but that the case $i = 0$, which uses the $N^{\times l}$ right action on $N \circ \amalg$ (cf. Remark 5.3), which after unpacked leads to the composite diagram below.

$$\begin{tikzcd}
\Omega^{(\underline{A})}_{G,n} \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-1} \arrow[rr] \arrow[d] & & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} A_j \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \mathcal{V}^{op} \arrow[rr] \arrow[d] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n,1} \arrow[r] \arrow[d, "d_0^{(\underline{A}}"'] & \mathsf{F} \wr \Omega^{(\underline{A})}_{G,n-1} \arrow[r] & \mathsf{F}^{\wr 2} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \coprod \mathsf{F}^{\wr n-2} A_j \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \coprod \mathsf{F}^{\wr n-2} \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n-1} \arrow[rr] \arrow[urr, Leftrightarrow, "\pi_0"'] & & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-2} A_j \arrow[r] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-2} \mathcal{V}^{op} \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] \arrow[ur, Leftrightarrow, "\alpha_0"'] & & \mathcal{V}^{op}
\end{tikzcd} \tag{5.25}$$

ASSOCSPANJ1 EQ

Finally, using the inclusions $\Omega^{(\underline{A}),J}_{G,n} \hookrightarrow \Omega^{(\underline{A})}_{G,n}$, one obtains analogous descriptions of the bar constructions $N \circ \amalg \circ (N^{\times J})^{\circ n} \underline{A}$, depicted below.

$$\begin{tikzcd}
\Omega^{(\underline{A}),J}_{G,n} \arrow[r] \arrow[d] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr \Omega^{(A_j)}_{G,0} \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F_1"] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] \arrow[dl, Rightarrow, "\underline{m}"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A}),J}_{G,n-1} \arrow[r] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.26}$$

$$\begin{tikzcd}
\Omega^{(\underline{A}),J}_{G,n} \arrow[r] \arrow[d, "d_i^{\underline{A}}"'] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"] \arrow[d, "\sigma^i"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^i"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A}),J}_{G,n-1} \arrow[r] \arrow[ur, Leftrightarrow, "\pi_i^{(\underline{A})}"'] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] \arrow[ur, Leftrightarrow, "\alpha_i"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.27}$$

DISJBARDNJ EQ

5.4 Transferring simplicial colimits of left Kan extensions

TRANSFSIMP SEC

Given genuine equivariant operads $X, Y \in \mathsf{Op}_G$ one has an isomorphism

$$X \amalg^a Y \simeq \operatorname{colim}_{\Delta^{op}} \left(\mathbb{F}_G^{\bullet+1} X \amalg^a \mathbb{F}_G^{\bullet+1} Y \right)$$

so that combining Remarks 4.71 and Remark 5.4 with the results in the previous section one obtains isomorphisms

$$\begin{aligned} X \amalg^a Y &\simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan} \left(N^{\bullet+1} \iota X \amalg^a N^{\bullet+1} \iota Y \right) \right) && (5.28) \\ &\simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan} \left(N \circ \amalg \circ (N^{\times 2})^{\bullet} (\iota X, \iota Y) \right) \right) && (5.29) \\ &\simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}_{\Omega^{2,op}_{G,\bullet} \to \Sigma_G^{op}} N_\bullet^{(X,Y)} \right) && (5.30) \end{aligned}$$

COLIMLAN EQ

where we write $N_\bullet^{(X,Y)} \colon \Omega^{2,op}_{G,\bullet} \to \mathcal{V}$ for the induced functor.

The purpose of this section will be show that one can repackage formulas such as (5.30) with a single left Kan extension over a category $\Omega^2_G = |\Omega^2_{G,\bullet}|$ obtained from $\Omega^2_{G,\bullet}$ via realization in Cat.

We note that $\Omega^2_{G,\bullet}$ together with the corresponding functors to Σ_G, $\mathcal{V}^{op}$ can be viewed as a simplicial object $\Delta^{op} \to \mathsf{WSpan}^l(\Sigma, G^{op}, \mathcal{V})$, and our first task will be to repackage such functors in terms of Grothendieck constructions.

MPSPANREIN LEMMA

Lemma 5.31. *Functors $F \colon \mathcal{D} \ltimes \mathcal{I}_\bullet \to \mathcal{C}$ are in bijection with lifts*

$$\begin{tikzcd} & \mathsf{WSpan}^l(*, \mathcal{C}) \arrow[d, "\mathsf{fgt}"] \\ \mathcal{D} \arrow[ur, dashed, "\mathcal{I}^F_\bullet"] \arrow[r, "\mathcal{I}_\bullet"'] & \mathsf{Cat}. \end{tikzcd}$$

where fgt *is the functor forgetting the maps to $*$ and $\mathcal{C}$.*

Proof. This is a matter of unpacking notation. The restrictions $F|_{\mathcal{I}_d}$ to the fibers $\mathcal{I}_d \subset D \ltimes \mathcal{I}_\bullet$ are precisely the functors $\mathcal{I}^F_d \colon \mathcal{I}_d \to \mathcal{C}$ describing $\mathcal{I}^F_\bullet(d)$.

Furthermore, the images $F\left((d,i) \to (d', f_*(i))\right)$ of the pushout arrows over a fixed arrow $f \colon d \to d'$ of $\mathcal{D}$ assemble to a natural transformation

$$\begin{tikzcd} \mathcal{I}_d \arrow[d, "f_*"'] \arrow[dr, "I^F_d"] \arrow[dr, phantom, shift right=4, "\Swarrow"] & \\ \mathcal{I}_{d'} \arrow[r, "I^F_{d'}"'] & \mathcal{C} \end{tikzcd} \tag{5.32}$$

which describes $\mathcal{I}^F_\bullet(f)$. It is straightforward to check that the associativity and unitality conditions coincide. □

In the cases of interest we will have $\mathcal{D} = \Delta^{op}$, so that $\mathcal{I}_\bullet$ can be interpreted as an object $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. By recalling the standard cosimplicial object $[\bullet] \in \mathsf{Cat}^{\Delta}$ given by $[n] = (0 \to 1 \to \cdots \to n)$ one obtains the following definition.

Definition 5.33. The left adjoint

$$|-| \colon \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat} \colon (-)^{[\bullet]}$$

will be called the *realization* functor.

Remark 5.34. More explicitly, one has

$$|\mathcal{I}_\bullet| = coeq \left(\coprod_{[n] \to [m]} [n] \times \mathcal{I}_m \rightrightarrows \coprod_{[n]} [n] \times \mathcal{I}_n \right). \tag{5.35}$$

REALDEF EQ

Example 5.36. Any $\mathcal{I} \in \mathsf{Cat}$ induces objects $\mathcal{I}, \mathcal{I}_\bullet, \mathcal{I}^{[\bullet]} \in \mathsf{Cat}^{\Delta^{op}}$ where $\mathcal{I}$ is the constant simplicial object and $\mathcal{I}_\bullet$ is the nerve $N\mathcal{I}$ with each level regarded as a discrete category. It is straightforward to check that $|\mathcal{I}| = |\mathcal{I}_\bullet| = |\mathcal{I}^{[\bullet]}| = \mathcal{I}$.

OBJGENREL LEMMA **Lemma 5.37.** *Given $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ one has an identification $ob(|\mathcal{I}_\bullet|) \simeq ob(\mathcal{I}_0)$. Furthermore, the arrows of $|\mathcal{I}_\bullet|$ are generated by the image of the arrows in $\mathcal{I}_0 \simeq \mathcal{I}_0 \times [0]$ and the image of the arrows in $[1] \times ob(\mathcal{I}_1)$.*

For each $i_1 \in \mathcal{I}_1$, we will denote the arrow of $|\mathcal{I}_\bullet|$ induced by the arrow in $[1] \times \{i_1\}$ by

$$d_1(i_1) \xrightarrow{i_1} d_0(i_1).$$

Proof. We write $d_{\hat{k}}$, $d_{\hat{k},\hat{l}}$ for the simplicial operators induced by the maps $[0] \xrightarrow{0 \mapsto k} [n]$, $[1] \xrightarrow{0 \mapsto k, 1 \mapsto l} [n]$ which can informally be thought of as the "composite of all faces other than d_k, d_l". Using (5.35) one has equivalence relations of objects

$$[n] \times \mathcal{I}_n \ni (k, i_n) \sim (0, d_{\hat{k}}(i_n)) \in [0] \times \mathcal{I}_0$$

and since for any generating relation $(k, i_n) \sim (l, i'_m)$ it is $d_{\hat{k}}(i_n) = d_{\hat{l}}(i'_m)$ the identification $ob(|\mathcal{I}_\bullet|) \simeq ob(\mathcal{I}_0)$ follows.

To verify the claim about generating arrows, note that any arrow of $[n] \times \mathcal{I}_n$ factors as

$$(k, i_n) \to (l, i_n) \xrightarrow{I_n} (l, i'_n) \tag{5.38}$$ FACTORIZATIONREAL EQ

for $I_n : i_n \to i'_n$ an arrow of $\mathcal{I}_n$. The $d_{\hat{l}}$ relation identifies the right arrow in (5.38) with $(0, d_{\hat{l}}(i_n)) \xrightarrow{d_{\hat{l}}(I_n)} (0, d_{\hat{l}}(i'_n))$ in $[0] \times \mathcal{I}_0$ while (if $k < l$) the $d_{\hat{k},\hat{l}}$ relation identifies the left arrow with $(0, d_{\hat{k},\hat{l}}(i_n)) \to (1, d_{\hat{k},\hat{l}}(i_n))$ in $[1] \times \mathcal{I}_1$. The result follows. $\square$

Remark 5.39. Given $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$, $\mathcal{C} \in \mathsf{Cat}$, the isomorphisms

$$Hom_{\mathsf{Cat}}(|\mathcal{I}_\bullet|, \mathcal{C}) \simeq Hom_{\mathsf{Cat}^{\Delta^{op}}}(\mathcal{I}_\bullet, \mathcal{C}^{[\bullet]})$$

together with the fact that $\mathcal{C}^{[\bullet]}$ is always 2-coskeletal show that $|\mathcal{I}_\bullet|$ is determined by the categories $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ and maps between them, i.e. by the truncated version of formula (5.35) with $n, m \leq 2$.

Indeed, it can be shown that a sufficient set of generating relations in $|\mathcal{I}_\bullet|$ is given by (i) the relations in $\mathcal{I}_0$ (including relations stating that identities of $\mathcal{I}_0$ are identities of $|\mathcal{I}_\bullet|$); (ii) relations stating that for each $i_0 \in \mathcal{I}_0$ the arrow $i_0 = d_1(s_0(i_0)) \xrightarrow{s_0(i_0)} d_1(s_0(i_0)) = i_0$ is an identity; (iii) for each arrow $I_1 : i_1 \to i'_1$ in $\mathcal{I}_1$ the relation that the square below commutes

$$\begin{tikzcd}
d_1(i_1) \arrow[r, "i_1"] \arrow[d, "d_1(I_1)"'] & d_0(i_1) \arrow[d, "d_0(I_1)"] \\
d_1(i'_1) \arrow[r, "i'_1"] & d_0(i'_1)
\end{tikzcd}$$

and (iv) for each object $i_2 \in \mathcal{I}_2$ the relation that the following triangle commutes.

$$\begin{tikzcd}
d_{1,2}(i_2) \arrow[rr, "d_1(i_2)"] \arrow[dr, "d_2(i_2)"'] & & d_{0,1}(i_2) \\
& d_{0,2}(i_2) \arrow[ur, "d_0(i_2)"'] &
\end{tikzcd}$$

PLANARSTRING EX **Example 5.40.** For $\Omega_{G,\bullet}$ the simplicial object of planar strings one has $|\Omega_{G,\bullet}| = \Omega^t_G$, the category of G-trees and tall maps. Indeed, arrows of $\Omega_{G,0}$ and objects of $\Omega_{G,1}$ are naturally identified with the quotient arrows and planar tall arrows of Ω^t_G, which are a generating set of arrows. And likewise, relations in $\Omega_{G,0}$, arrows in $\Omega_{G,1}$ and objects in $\Omega_{G,2}$ are identified with the relations of Ω^t_G.

Analogously, for $\Omega^J_{G,\bullet}$ the simplicial object of planar $\underline{l}$-labeled strings that are $(\{l\} - J)$-inert, one has $|\Omega^J_{G,\bullet}| = \Omega^{J,t}_G$, the category of $\underline{l}$-labeled G-trees and $(\{l\} - J)$-inert tall maps.

The following is the key result in this section.

SOURCEFINAL PROP

Proposition 5.41. *Let $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then there is a natural functor*

$$\Delta^{op} \ltimes \mathcal{I}_\bullet \xrightarrow{\ s\ } |\mathcal{I}_\bullet| . \tag{5.42}$$

Further, s is final.

Remark 5.43. The s in the result above stands for *source*. This is because, for any $\mathcal{I} \in \mathsf{Cat}$, the map $\Delta^{op} \ltimes \mathcal{I}^{[\bullet]} \to \left|\mathcal{I}^{[\bullet]}\right| \simeq \mathcal{I}$ is given by $s(i_0 \to \cdots \to i_n) = i_0$.

Proof. Recall that $|\mathcal{I}_\bullet|$ is the coequalizer (5.35). Given $(k, g_m) \in [n] \times \mathcal{I}_m$, we will write $[k, g_m]$ for the corresponding object in $|\mathcal{I}_\bullet|$. To simplify notation, we will write objects of $\mathcal{I}_n$ as i_n and implicitly assume that $[k, i_n]$ refers to the class of the object $(k, i_n) \in [n] \times \mathcal{I}_n$.

We define s on objects by $s([n], i_n) = [0, i_n]$ and on an arrow $(\phi, I_m)\colon (n, i_n) \to (m, i'_m)$ as the composite (note that $\phi\colon [m] \to [n]$ and $I_m\colon \phi^*(i_n) \to i_m$)

$$[0, i_n] \to [\phi(0), i_n] = [0, \phi^*(i_n)] \xrightarrow{I_m} [0, i'_m]. \tag{5.44}$$ TARGETDEFINITON EQ

To check associativity, the cases of of a pair of either two fiber arrows (i.e. arrows where ϕ is the identity) or two pushforward arrows (i.e. arrows where I_m is the identity) are immediate from (5.44), hence we are left with the case $([n], i_n) \xrightarrow{I_n} ([n], i'_n) \to ([m], \phi^*(i'_n))$ of a fiber arrow followed by a pushforward arrow. Noting that in $\Delta^{op} \ltimes \mathcal{I}_\bullet$ this composite can be rewritten as $([n], i_n) \to ([m], \phi^*(i_n)) \xrightarrow{\phi^*(I_n)} ([m], \phi^*(i'_n))$ this amounts to checking that

$$\begin{tikzcd}
{[0, i_n]} \arrow[r] \arrow[d, "I_n"'] & {[\phi(0), i_n)]} \arrow[r, equal] \arrow[d, "I_n"'] & {[0, \phi^*(i_n)]} \arrow[d, "\phi^*(I_n)"] \\
{[0, i'_n]} \arrow[r] & {[\phi(0), i'_n]} \arrow[r, equal] & {[0, \phi^*(i_n)]}
\end{tikzcd} \tag{5.45}$$

commutes in $|\mathcal{I}_\bullet|$, which is the case since the left square is encoded by a square in $[n] \times \mathcal{I}_n$ and the right square is encoded by an arrow in $[m] \times \mathcal{I}_n$.

We now turn to showing that s is final.

Fix $j \in \mathcal{I}_0$. We will show that $[0, j] \downarrow \Delta^{op} \ltimes \mathcal{I}_\bullet$ is indeed connected. By Lemma 5.37 any object in this undercategory has a description (not necessarily unique) as a pair

$$\Big(([n], i_n) \, , [0, j] \xrightarrow{f_1} \cdots \xrightarrow{f_r} s([n], i_n) \Big)$$

where each f_i is a generating arrow of $|\mathcal{I}_\bullet|$ induced by either an arrow I_0 of $\mathcal{I}_0$ or object $i_1 \in \mathcal{I}_1$.

We will connect this object to the canonical object $(([0], h), [0, h] = [0, h])$, arguing by induction on r. If $n \neq 0$, the map $d_{\hat{0}}\colon ([n], i_n) \to ([0], d^*_{\hat{0}}(i_n))$ and the fact that $s\big(d^*_{\hat{0}}\big) = id_{[0, d^*_{\hat{0}}(i_n)]}$ provides an arrow to an object with $n = 0$ without changing r. If $n = 0$, one can apply the induction hypothesis by lifting f_r to $\Delta^{op} \ltimes \mathcal{I}_\bullet$ according to one of two cases: (i) if f_r is induced by an arrow I_0 of $\mathcal{I}_0$, the lift of f_r is simply $([0], i'_0) \xrightarrow{I_0} ([0], i_0)$; (ii) if f_r is induced by $i_1 \in \mathcal{I}_1$ the lift is provided by the map $([1], i_1) \to ([0], d_0(i_1))$. □

In practice, we will need to know that s satisfies the following stronger finality condition with respect to left Kan extensions.

URCELANFINAL COR

Corollary 5.46. *Consider a map $\mathcal{I}_\bullet \to \mathcal{J}$ between $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ and a constant object $\mathcal{J} = \mathcal{J}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then the source map s*

$$\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "s"] \arrow[dr] & & {|\mathcal{I}_\bullet|} \arrow[dl] \\
& \mathcal{J} &
\end{tikzcd}$$

is Lan-*final over $\mathcal{J}$, i.e. the functors $s \downarrow j\colon (\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \to |\mathcal{I}_\bullet| \downarrow j$ are final for all $j \in \mathcal{J}$.*

Proof. It is clear that $(\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \simeq \Delta^{op} \ltimes (\mathcal{I}_\bullet \downarrow j)$ while Lemma 2.2 guarantees that, since $(-) \downarrow j$ is a left adjoint, $|\mathcal{I}_\bullet| \downarrow j \simeq |\mathcal{I}_\bullet \downarrow j|$. One thus reduces to Proposition 5.41. ☐

We end this section with two basic lemmas that will allows us to apply Corollary 5.46 to the tree categories we will be interested in.

TWISTING LEMMA

Lemma 5.47. *Let $\mathcal{I}^F_\bullet \in \mathsf{Span}(*,\mathcal{C})^{\Delta^{op}}$ be such that the diagrams*

$$
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] \arrow[ur, phantom, "\Nearrow\delta_i"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] \arrow[ur, phantom, "\Nearrow\sigma_j"] & \mathcal{C}
\end{tikzcd}
\tag{5.48}
$$

IDENTSIMPRELSISO EQ

commute up to isomorphism for $0 < i \le n$, $0 \le j \le n$.

Then the functors $\tilde{F}_n : \mathcal{I}_n \to \mathcal{C}$ given by the composites

$$\mathcal{I}_n \xrightarrow{d_{1,\cdots,n}} \mathcal{I}_0 \xrightarrow{F_0} \mathcal{C}$$

assemble to an object $\mathcal{I}^{\tilde{F}}_\bullet \in \mathsf{Span}(*,\mathcal{C})^{\Delta^{op}}$ which is isomorphic to $\mathcal{I}^F_\bullet$ and such that the corresponding diagrams (5.48) for $0 < i \le n$, $0 \le j \le n$ are strictly commutative.

Proof. This follows by a straightforward verification. ☐

SOURCEFACT LEM

Lemma 5.49. *A (necessarily unique) factorization*

$$
\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr] \arrow[dr, "s"'] & & \mathcal{C} \\
& |\mathcal{I}_\bullet| \arrow[ur, dashed] &
\end{tikzcd}
\tag{5.50}
$$

SOURCEFACT EQ

exists iff for the associated object $\mathcal{I}_\bullet \in \mathsf{Span}(*,\mathcal{C})^{\Delta^{op}}$ (cf. Lemma 5.31) all faces d_i for $0 < i \le n$ and degeneracies s_j for $0 \le j \le n$ are strictly commutative, i.e. they are given by diagrams

$$
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_0"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] \arrow[ur, phantom, "\Swarrow\varphi_n"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C}
\end{tikzcd}
\tag{5.51}
$$

IDENTSIMPRELS EQ

Proof. For the "if" direction, it suffices to note that s sends all pushout arrows of $\Delta^{op} \ltimes \mathcal{I}_\bullet$ for faces d_i, $0 < i \le n$ and degeneracies s_j, $0 \le j \le n$ to identities and this yields the commutative diagrams (5.51).

For the "only if" direction, this will follow by building a functor $\mathcal{I}_\bullet \xrightarrow{\bar{F}} \mathcal{C}^{[\bullet]}$ together with the naturality of the source map s (recall that $|\mathcal{C}^{[\bullet]}| \simeq \mathcal{C}$). We define $\bar{F}_n|_{k\to k+1}$ as the map

$$F_{n-k} d_{0,\cdots,k-1} \xrightarrow{\varphi_{n-k} d_{0,\cdots,k-1}} F_{n-k-1} d_{0,\cdots,k}. \tag{5.52}$$

EQUIVALENCEDEF EQ

The claim that $s \circ (\Delta^{op} \ltimes \bar{F})$ recovers the horizontal map in (5.50) is straightforward, hence the real task is to prove that (5.52) indeed defines a map of simplicial objects.

$$\varphi_{n-1} d_i = \varphi_n,\ \ 1 < i \qquad \varphi_{n-1} d_1 = (\varphi_{n-1} d_0) \circ \varphi_n, \qquad \varphi_{n+1} s_i = \varphi_n,\ \ 0 < i, \qquad \varphi_{n+1} s_0 = id_{F_n} \tag{5.53}$$

Next, note that there is no ambiguity in writing simply $\varphi_{n-k} d_{0,\cdots,k-1}$ to denote the map (5.52). We now check that $\bar{F}_{n-1} d_i = d_i \bar{F}_n$, $0 \le i \le n$, which must be verified after restricting to each $k \to k+1$, $0 \le k \le n-2$. There are three cases, depending on i and k:

$(i < k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{0,\cdots,k}$;

$(i = k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_1 d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_0 \circ \varphi_{n-k}) d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_{0,\cdots,k}) \circ (\varphi_{n-k} d_{0,\cdots,k-1})$;

$(i > k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{i-k} d_{0,\cdots,k-1} = \varphi_{n-k} d_{0,\cdots,k-1}$.

The case of degeneracies is similar. $\square$

DUALRESULTS REM

Remark 5.54. One can twist all results by the opposite functor

$$\Delta \xrightarrow{(-)^{op}} \Delta$$

which sends $[n]$ to itself and d_i, s_i to d_{n-i}, s_{n-i}. In doing so, one obtains vertical isomorphisms

$$\begin{tikzcd} \Delta^{op} \ltimes (\mathcal{J}_\bullet \circ (-)^{op}) \arrow[r, "s"] \arrow[d, "\simeq"'] & {|\mathcal{J}_\bullet \circ (-)^{op}|} \arrow[d, "\simeq"] \\ \Delta^{op} \ltimes \mathcal{J}_\bullet \arrow[r, "t"'] & {|\mathcal{J}_\bullet|} \end{tikzcd}$$

which reinterpret the "source" functor as what one might call the "target" functor, with $t([n], i_n) = [n, i_n]$ rather than $s([n], i_n) = [0, i_n]$.

Corollary 5.46 now says that t is Lan-final and Lemmas 5.47, 5.49 generalize in the obvious way by replacing s with t and d_0 with d_n.

5.5 The category of extension trees

In this section we combine the previous sections to obtain a compact description of free extension pushouts

$$\begin{tikzcd} \mathbb{F}_G A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{F}_G B \arrow[r] & Y \end{tikzcd} \tag{5.55}$$

FREEEXT EQ

as a left Kan extension over a convenient category of trees.

For simplicity, we first explain how to obtain a similar description for the simpler case of a coproduct $X \amalg^a Y$. By (5.30), one has a description

$$\begin{aligned} X \amalg^a Y \simeq & \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}_{\Omega^{\underline{2},op}_{G,\bullet} \to \Sigma^{op}_G} N^{(X,Y)}_\bullet \right) \\ \simeq & \mathsf{Lan}_{\Delta^{op} \ltimes \Omega^{\underline{2},op}_{G,\bullet} \to \Sigma^{op}_G} N^{(X,Y)}_\bullet \end{aligned}$$

where the second identification follows from formal properties of Grothendieck constructions.

Combining the fact that (5.27) consists of natural isomorphisms with (the Remark 5.54 dual of) Lemma 5.47, yields an isomorphic twisted functor $\tilde{N}^{(X,Y)}_\bullet$ with strictly commutative s_i and d_i for $i \neq n$. The dual of Lemma 5.49 now says that $\tilde{N}^{(X,Y)}_\bullet$ factors via the target map t though $\Omega^{\underline{2},op}_G \simeq |\Omega^{\underline{2},op}_{G,\bullet}|$ (writing $\tilde{N}^{(X,Y)}$ for the factorization) and thus the dual of Corollary 5.46 finally yields

$$X \amalg^a Y \simeq \mathsf{Lan}_{\Omega^{\underline{2},op}_G \to \Sigma^{op}_G} \tilde{N}^{(X,Y)}. \tag{5.56}$$

We recall that by Example 5.40, $\Omega^{\underline{2}}_G$ is simply the category of $\underline{2}$-labeled trees and tall label maps.

More generally, one has

$$\coprod^a_J X_j \amalg^a \coprod^a_{\underline{l}-J} \mathbb{F}_G X_j \simeq \mathsf{Lan}_{\Omega^{J,op}_G \to \Sigma^{op}_G} \tilde{N}^{(\underline{X})}. \tag{5.57}$$

LANCOPRODDESC

where Ω^J_G is the category of $\underline{l}$-labeled trees and tall $(\underline{l} - J)$-inert label maps.

Remark 5.58. We note that the twisting $\tilde{N}_{\bullet}^{(X,Y)}$ is fairly harmless. For explicitness, we focus on the simplest case of a "unary coprodut", in which case (5.57) is simply recovering the genuine equivariant operad X from its bar resolution. In that case $N_2^X : \Omega_{G,2}^{op} \to \mathcal{V}$ is given by the top map in (4.53) or, equivalently, by the top map in (4.54) (we note that, in the notation therein, it is $A = \Sigma_G$). On the other hand, the twisted map $\tilde{N}_2^X : \Omega_{G,2}^{op} \to \mathcal{V}$ is given by the left bottom composite in either of (4.53), (4.54). Informally, the role of this twisting is therefore simply that of replacing the order on $V_G(T_n)$ induced lexicographically by planar strings $T_0 \to \cdots \to T_n$ with the simpler order induced directly from T_n.

In what follows we will largely be able to ignore this technicality. Indeed, the role of lexicographic orders in building (5.57) is that of guaranteeing that $\tilde{N}_{\bullet}$ satisfies the necessary simplicial identities, which are ensured by appealing to the bar construction for the monad N.

We now turn to the task of building (5.55) as a left Kan extension. One has a colimit description

$$\mathbb{F}B \coprod_{\mathbb{F}A} X \simeq \mathrm{colim}_{\Delta^{op}} \Big(\mathbb{F}B \amalg \mathbb{F}A \amalg X \leftleftarrows \mathbb{F}B \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg X \mathrel{\substack{\leftarrow\\ \leftarrow\\ \leftarrow}} \mathbb{F}B \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg X \quad \cdots \Big) \tag{5.59}$$

FREEEXTUSEFCOL EQ

where all differentials are fold maps of $\mathbb{F}A$ except to the n-th differential d_n, which is induced by the two maps $\mathbb{F}A \to X$, $\mathbb{F}A \to \mathbb{F}B$.

By the previous discussion each individual object $X \amalg (\mathbb{F}A)^{\amalg 2n+1} \amalg \mathbb{F}B$ in (5.59) can be described as a left Kan extension over the tree category $\Omega_G^{\{X\}}$ where $\{X\} \subset \{B, A, \cdots, A, X\}$ is a simpleton. The maps in (5.59) can themselves be encoded as span maps between the $\Omega_G^{\{X\}}$. To see this, we make (5.59) more precise. Firstly, we write $\langle n \rangle$ for the poset

$$-\infty \le -n \le -n+1 \le \cdots \le -1 \le 0 \le 1 \le \cdots \le n-1 \le n \le +\infty.$$

The posets $\langle n \rangle$ together with antisymmetric (i.e. such that $f(-x) = -f(x)$) poset maps preserving all three of $-\infty, 0, +\infty$ then form a simplicial object[1] $\langle - \rangle : \Delta^{op} \to \mathsf{F}$.

(5.55) thus induces a simplicial object $(B, A, X)_{\langle n \rangle} \in \mathsf{F} \wr \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$.

Each level of $(\iota B, \iota A, \iota X)_{\langle n \rangle}$ is then a $N^{\times \{+\infty\}}$-algebra on $\big(\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})\big)^{\times \langle n \rangle}$, compatibly with the simplicial maps. One thus obtains a *bisimplicial* object

$$\Sigma_G^{op} \leftarrow \Omega_{G,\bullet}^{\{+\infty\}\langle\bullet\rangle, op} \xrightarrow{N_{\bullet}^{(B,A,X)_{\langle\bullet\rangle}}} \mathcal{V}$$

on $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$ whose realization along the string direction yields the spans

$$\Sigma_G^{op} \leftarrow \Omega_{G}^{\{+\infty\}\langle\bullet\rangle, op} \xrightarrow{N^{(B,A,X)_{\langle\bullet\rangle}}} \mathcal{V} \tag{5.60}$$

PARTREALSPAN EQ

discussed above, except now assembled into a simplicial object in $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$.

All degeneracies s_i and differentials d_i of (5.60) other than the top differential d_n are induced by maps ρ^* described in §5.1 and thus given by strictly commutative diagrams, so that Lemma 5.49 and Corollary 5.46 can be applied (this time with no need to appeal to Lemma 5.47) so as to allow (5.59) to be repackaged as

$$\mathbb{F}B \coprod_{\mathbb{F}A} X \simeq \mathsf{Lan}_{\Omega_G^{e,op} \to \Sigma_G^{op}} N^{(B,A,X)} \tag{5.61}$$

FREEEXTUSEFCOLNEW EQ

where we write Ω_G^e for $|\Omega_G^{\{+\infty\}\langle\bullet\rangle}|$. We now turn to the task of describing Ω_G^e, starting with by defining it directly.

[1] Indeed, we recall that the opposite simplex category Δ^{op} can equivalently described as the category of *intervals*, i.e. finite ordered posets with distinct top and bottom, along with order maps preserving both top and bottom. $\langle n \rangle$ can then be regarded as obtained by gluing the interval $0 \le 1 \le \cdots \le n \le +\infty$ with its opposite.

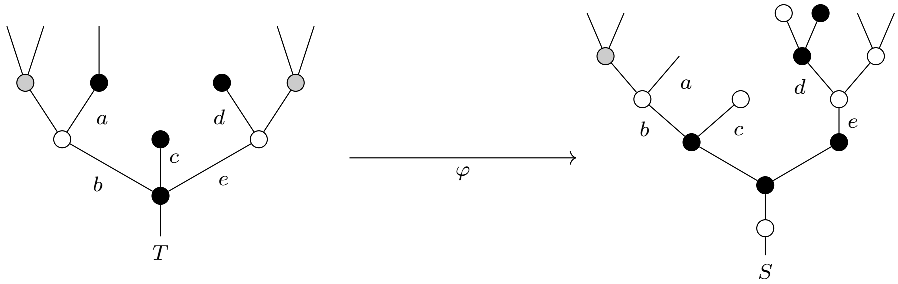
EXTTREECAT DEF **Definition 5.62.** The *extension tree category* Ω_G^e is the category whose objects are $\{B, A, X\}$-labeled trees and whose maps $\varphi: T \to S$ are tall maps of trees such that

(i) if $T_{v_{Ge}}$ has an A-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has an A-label;

(ii) if $T_{v_{Ge}}$ has a B-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has either an A-label or a B-label;

(iii) if $T_{v_{Ge}}$ has a X-label, then $S_{v_{Ge}}$ has only A and X-labels.

Example 5.63. The following is an example of a planar map in Ω_G^e, where black nodes represent X-labeled nodes, grey nodes represent B-labeled nodes and white nodes represent A-labeled nodes.



(5.64) REGALTERNMAP EQ

Proposition 5.65. *One has an identification*

$$\Omega_G^e \simeq |\Omega_G^{\{+\infty\}\langle\bullet\rangle}|.$$

Proof. We note first that Ω_G^e contains all label maps that are $\{A, B\}$-inert. In fact, any map of Ω_G^e clearly has a unique factorization as such a label map followed by an underlying planar isomorphism of trees that replaces some of the X and B labels with A labels. We will refer to the former as label maps and to the latter as relabel maps.

We recall that $\Omega_G^{\{+\infty\}\langle n\rangle}$ consists of trees with $2n+3$ types of labels: X-labels, B-labels and $2n+1$ distinct types of A-labels. One can equivalently encode such a tree as a string $T_0 \to \cdots \to T_n$ of relabel maps. Indeed, the A-label nodes of T_n in such a string are partitioned into $2n+1$ types according to that node's labels one the T_i (which are either all A's, some X's and then A's or some B's and then A's). Moreover, a diagram

$$\begin{array}{ccccccc}
T_0 & \longrightarrow & T_1 & \longrightarrow & \cdots & \longrightarrow & T_n \\
{\scriptstyle f_0}\big\downarrow & & {\scriptstyle f_1}\big\downarrow & & & & {\scriptstyle f_n}\big\downarrow \\
T_0' & \longrightarrow & T_1' & \longrightarrow & \cdots & \longrightarrow & T_n'
\end{array}$$

with f_i label maps of Ω_G^e is then equivalent to a label map $f_n: T_n \to T_n'$ respecting all $2n+3$ labels in $\Omega_G^{\{+\infty\}\langle n\rangle}$. Since the string description above is also compatible with the simplicial structure maps in the obvious way, the result is now clear. $\square$

Our next task will be that of identifying a convenient Lan-final subcategory $\bar{\Omega}_G^e \hookrightarrow \Omega_G^e$. We first introduce the auxiliary notion of alternating trees. We recall the notion of input path (Notation INPUTPATH NOT 3.4) $I(e) = \{f \in T : e \leq_d f\}$ for an edge $e \in T$, which naturally extends to T in any of $\in \Omega, \Phi, \Omega_G, \Phi_G$.

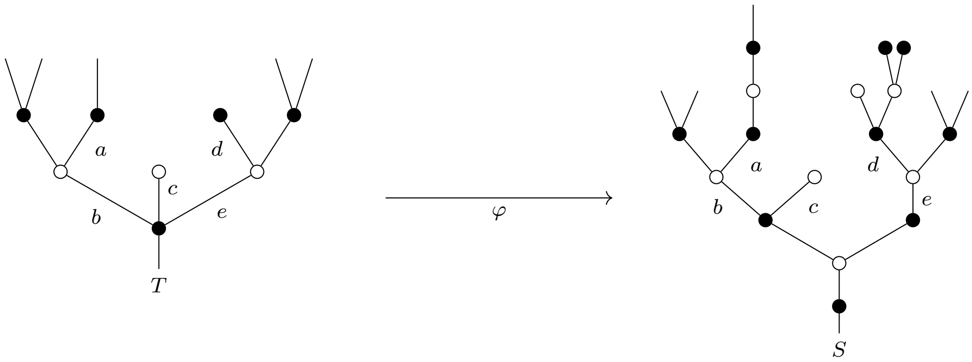
Definition 5.66. A G-tree $T \in \Omega_G$ is called *alternating* if, for all leafs $l \in T$ one has that the input path $I(l)$ has an even number of elements.

Further, a vertex $e^\uparrow \leq e$ is called *active* if $|I(e)|$ is odd and *inert* otherwise.

Finally, a tall map $T \xrightarrow{\varphi} T'$ between alternating G-trees is called a *tall alternating map* if for any inert vertex $e^\uparrow \leq e$ of T one has that $T'_{e^\uparrow \leq e)}$ is an inert vertex of T'.

We will denote the category of alternating G-trees and tall alternating maps by Ω_G^a.

Example 5.67. Two alternating trees (for $G = *$ the trivial group) and a planar tall alternating map between them follow, with active nodes in black ($\bullet$) and white nodes in white ($\circ$).



$$(5.68)$$

The term "alternating" comes from the fact that no adjacent nodes have the same color. We note, however, that there is additional restriction: the "outer" vertices, i.e. those immediately below a leaf or the one immediately above the root, are necessarily black/active (not, however, that this does *not* apply to stumps).

Remark 5.69. One can extend Definition 3.39 to the alternating context by defining a substitution datum to be alternating if it is given by isomorphisms for inert nodes and by alternating maps for active nodes. It is then straightforward to check that Proposition 3.42 and its equivariant analogue Proposition 4.19 extend to give alternating analogues.

Definition 5.70. $\bar{\Omega}^e_G \hookrightarrow \Omega^e_G$ is the full subcategory of (B, A, X)-labeled trees whose underlying trees is alternating, active nodes are labeled by X, and passive nodes are labeled by A or B.

We note that conditions (i) and (ii) in Definition 5.62 imply that maps in $\bar{\Omega}^e_G$ are underlying alternating maps.

The following establishes the required finality of $\bar{\Omega}^e_G$ in Ω^e_G.

Proposition 5.71. *For each $U \in \Omega^e_G$ there exists a unique $\mathsf{lr}_X(U) \in \bar{\Omega}^e_G$ together with a unique planar label map of Ω^e_G*

$$\mathsf{lr}_X(U) \to U.$$

Furthermore, lr_X extends to a right retraction $\mathsf{lr}_X : \Omega^e_G \to \bar{\Omega}^e_G$.

Proof. Given U, we form a collection of outer faces $\{U^A_i\} \amalg \{U^B_j\} \amalg \{U^X_k\}$ where the U^A_i, U^b_j are simply the A, B-labeled nodes and the $\{U^X_k\}$ are the maximal outer subtrees whose nodes have only X-labels (we note that these may possible be sticks). Lemma 3.51 then guarantees that the $V_G(U^X_k)$ are disjoint, so that one can apply (the equivariant version of Proposition 3.49) to build

$$T = \mathsf{lr}(U) \to U \tag{5.72}$$

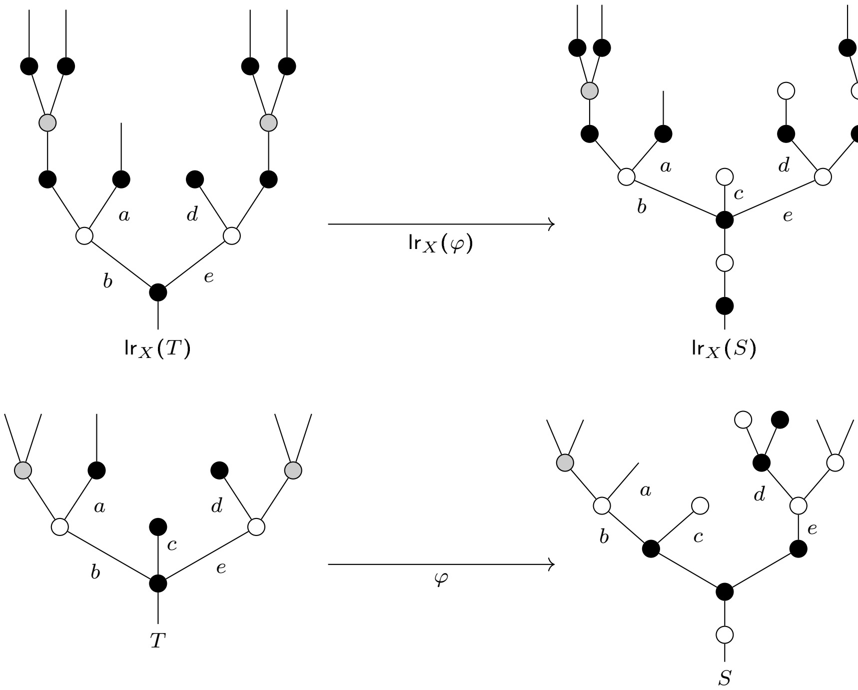
such that $\{U_{v_{Ge}}\} = \{U^A_i\} \amalg \{U^B_j\} \amalg \{U^X_k\}$. T has an obvious (B, A, X)-labeling making (5.72) into a label map, but we must still check $T \in \bar{\Omega}^e_G$, i.e. that T is alternating with the X-labeled vertices being precisely the X-labeled vertices. Let us now write any input path of T as $I(e) = (e = e_n \le e_{n-1} \le \cdots \le e_1 \le e_0)$. By Lemma 3.51 and maximality of the U^X_k, no pair of consecutive vertices v_{Ge_i} and $v_{Ge_{i+1}}$ can be both X-labeled. On the other hand, again by Lemma 3.51 any edge of U belongs to some U^X_k and therefore: (i) at least one of in each pair of consecutive vertices v_{Ge_i} and $v_{Ge_{i+1}}$ is X-labeled; (ii) if $r \in T$ is a root, v_{Gr} is X-labeled; (iii) if $l \in T$ is a leaf $v_{Gl_{n-1}}$ is X-labeled. This suffices to conclude $T \in \bar{\Omega}^e_G$, and uniqueness of T is immediate from the uniqueness in Lemma 3.51.

It remains to check that lr_X in fact defines a functor. We consider the following diagram.

$$\begin{tikzcd} \mathsf{lr}_X(U) \arrow[r] \arrow[d, dashed, "\mathsf{lr}_X(f)"'] & U \arrow[d, "f"] \\ \mathsf{lr}_X(V) \arrow[r] & V \end{tikzcd}$$

When f is a root pullback map, we define $\mathsf{lr}_X(f)$ to likewise be a root pullback map. When f is a rooted tall map, writing $T = \mathsf{lr}_X(U)$ one has a map of rooted T-substitution data $\{\mathsf{lr}_X(V_{v_{G_e}})\} \to \{V_{v_{G_e}}\}$, which after converted to a tree map yields the desired map $\mathsf{lr}_X(f)$. To check that lr_X respects composition of maps, the only non immediate case is that of a root pullback followed by a rooted map, in which case this follows from Remark PULLCOMP_REM 4.22. $\square$

Example 5.73. The following illustrates the lr_X construction when applied to the map φ in REGALTERNMAP_EQ (5.68).



(5.74)

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6 Filtration of Cellular Extensions

As we saw above, we have the the free extension $\mathcal{P}[u]$ given by the pushout

$$\begin{tikzcd} \mathbb{F}_G X \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_G Y \arrow[r] & \mathcal{P}[u] \end{tikzcd}$$

can be built via Kan extensions over $\Omega_{G,e}$. Thus, in order to filter the map $\mathcal{P} \to \mathcal{P}[u]$, it will suffice, via naturality of Kan extensions, to filter the category $\Omega_{G,e}$.

We begin by analyzing the objects of this category. These are $(\mathcal{P}; X, Y)$-alternating G-trees T; that is, an odd tree T with each odd vertex labeled by $\mathcal{P}$, and the even vertices labeled by either X or Y.

Adjusting the general notation of the previous section to this setting, given $T \in \Omega_{G,e}$ we let $V_{\mathcal{P}}(T)$, $V_X(T)$, and $V_Y(T)$ denote the G-sets of vertices labeled by $\mathcal{P}$, X, or Y, respectiely, and $V_{G,\mathcal{P}}(T)$ (and similarly) the set $V_{\mathcal{P}}(T)/G$ of orbits. Further, let $V_{G,in}(T) = V_{G,X}(T) \amalg V_{G,Y}(T)$ denote the set of *inert* or *passive* nodes. Moreover, we let $\mathbb{V}_{G,in}(T) \in \mathsf{F} \wr \mathcal{V}$ denote the map $V_{G,in}(T) \to \mathcal{V}$ which sends T_v to $X(T_v)$ or $Y(T_v)$, depending on the labeling of the vertex v.

Further, we define the *degree* of T, denoted $|T|$, to be the sum $|T|_X + |T|_Y$, where $|T|_X$ is defined by

$$|T|_X = \frac{|V_X(T)|}{|G.r|} = \sum_{G.v \in V_{G,X}(T)} \frac{|G.v|}{|G.r|}$$

where $G.r$ is the root G-set of T, and similarly for $|T|_Y$. Intuitively, $|T|_X$ is the number of X-labeled vertices in any (every) single tree component of T.

6.1 Filtration Pieces

We begin our filtration of $\Omega_{G,e}$.

IECES_DEFINITION **Definition 6.1.** We define subcategories of Ω^e_G.

1. Let $\Omega^e_G[\le k]$ (respectively $\Omega^e_G[k]$) be the full subcategory of Ω^e_G spanned by trees T with $|T| \le k$ (respectively, $|T| = k$).
2. Let $\Omega^e_G[\le k, -]$ (respectively $\Omega^e_G[k, -]$) be the full subcategory of $\Omega^e_G[\le k]$ (respectively $\Omega^e_G[k]$) spanned by trees T with $|T|_Y \ne k$.
3. Let $\Omega^e_G[\le k, 0]$ (respectively $\Omega^e_G[k, 0]$) be the full subcategory of $\Omega^e_G[\le k]$ (respectively $\Omega^e_G[k]$) spanned by trees T with $|T|_X = 0$ (equivalently, $|T|_Y = k$).
4. If Ξ is any of the above categories, and $C \in \Sigma_G$, let $\Xi(C)$ denote the full subcategory of Ξ spanned by those trees T with $val(T) \simeq C$.

Remark 6.2. The categories $\Omega^e_G[k]$ and $\Omega^e_G[k, -]$ have only very limited morphisms, as there cannot be any "active substitutions". Thus, any map $S \to T$ in these categories just changes some Y-labelings into X-labelings, while the underlying $(\mathcal{P}; Z)$-alternating tree remains fixed (where here the one passive colour Z encompasses both Y and X).

_LAN_FINAL_LEMMA **Lemma 6.3.** *$\Omega^e_G[\le k-1]^{op}$ is* Lan*-final in $\Omega^e_G[\le k, -]^{op}$ over Σ^{op}_G.*

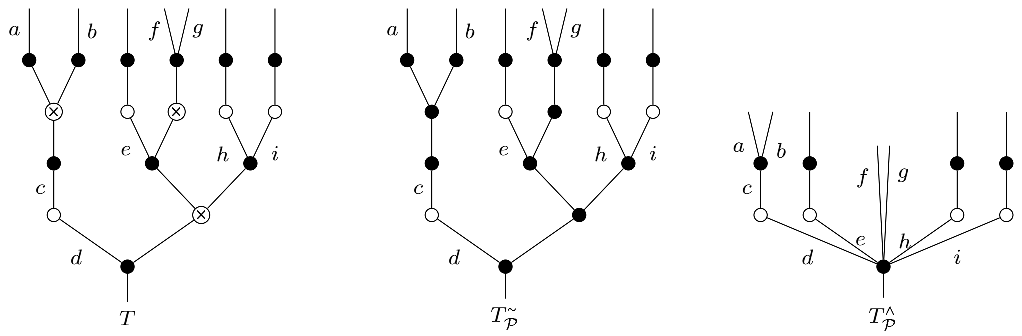
In order to prove this, we will first need a particular construction $T \mapsto T^{\wedge}_{\mathcal{P}}$ on $\Omega_{G,e}$.

Definition 6.4. Given a $(\mathcal{P}; X, Y)$-alternating G-tree T, let $T^{\wedge}_{\mathcal{P}}$ denote the $(\mathcal{P}; Y)$-alternating tree created from T by

1. relabelling all X-nodes by $\mathcal{P}$ (yielding a $(\mathcal{P}, Y)$-labeled tree); then
2. collapsing all connected components of $\mathcal{P}$-labeled nodes.

There is a unique planar-tall map $\partial_{\mathcal{P}} : T^{\wedge}_{\mathcal{P}} \to T$, and in fact this map factors through all maps of the form $\partial_{\mathcal{P}} : S \to T$.

Example 6.5. We observe that this construction is symmetric across all tree components, and hence, to give an example, it suffices to show want happens on a single component (i.e. when $G = \{e\}$). Consider the $(\mathcal{P}; X, Y)$-alternating tree T below, where black nodes are $\mathcal{P}$-labeled, white nodes filled with × are X-labeled, and empty white nodes are Y-labeled. After Step (1), we produce the tree $T^{\sim}_{\mathcal{P}}$ in the middle, and collapsing connected $\mathcal{P}$-components yields the tree $T^{\wedge}_{\mathcal{P}}$ on the right.



MINUS_LAN_FINAL_LEMMA

Proof of Lemma 6.3. Fix an arbitrary $C \in \Sigma_G$, and consider an element $q_S : val(S) \leftarrow C$ in $\Omega^e_G[\leq k, -]^{op} \downarrow C$ (so in particular $S \in \Omega^e_G[\leq k, -]$). We must show that the overcategory

$$(\Omega^e_G[\leq k-1]^{op} \downarrow C) \downarrow (val(S) \leftarrow C)$$

is non-empty and connected. If in fact $S \in \Omega^e_G[\leq k-1]$, the result is immediate. Otherwise, consider the map

$$S^{\wedge}_{\mathcal{P}} \xrightarrow{\partial_{\mathcal{P}}} S.$$

Since $|S|_Y \neq k$, $|S^{\wedge}_{\mathcal{P}}| \leq k-1$, and hence we have a diagram

$$\begin{tikzcd} val(S) & & val(S^{\wedge}_{\mathcal{P}}) \arrow[ll, "\partial_{\mathcal{P}}"'] \\ & C \arrow[ul, "q_S"] \arrow[ur, "q_S"'] & \end{tikzcd} \tag{6.6}$$

K-1_LAN_FINAL_EQ1

showing that the desired overcategory is inhabited. Further, given any other element

$$\begin{tikzcd} val(S) & & val(T) \arrow[ll, "f"'] \\ & C \arrow[ul, "q_T"] \arrow[ur, "q_S"'] & \end{tikzcd} \tag{6.7}$$

K-1_LAN_FINAL_EQ2

K-1_LAN_FINAL_EQ1

in the overcategory, consider the following zig-zag of maps connecting the objects (6.6) and

K-1_LAN_FINAL_EQ2
(6.7):

$$
\begin{tikzcd}
 & & S & & \\
 & & \tilde{q}_T^*(S) \arrow[u, "\tilde{q}_T"'] & & \\
S^{\wedge}_{\mathcal{P}} \arrow[uurr, "\partial_{\mathcal{P}}"] & \tilde{q}_T^*(S^{\wedge}_{\mathcal{P}}) \arrow[l, "\tilde{q}_T"'] \arrow[ur, "\partial_{\mathcal{P}}"] \arrow[rr, "\partial_{\mathcal{P}}"] & & T' \arrow[ul, "\partial_{\mathcal{P}}"'] & T \arrow[l, "\partial_Y"'] \arrow[uull, "f"'] \\
 & & C \arrow[ull, "q_S"] \arrow[ul, "q_T"'] \arrow[ur, "q_T"] \arrow[urr, "q_T"'] & &
\end{tikzcd}
\tag{6.8}
$$

K-1_LAN_FINAL_DIAGRAM

Here, we have omitted the notation "*val*" from the top three rows. To understand this diagram, we first record that we have a factorization:

$$q_S = \tilde{q}_T q_T,$$

Then, if we let $C_S = val(S) = val(S^{\wedge}_{\mathcal{P}})$ and $C_T = val(T)$, we have

$$C \xrightarrow{q_T} C_T \xrightarrow{\tilde{q}_T} C_S$$

and hence, by the unique factorization of maps in $\Omega_{G,e}$, a factorization

$$
\begin{tikzcd}
C_T \arrow[r, dashed] \arrow[d, "\tilde{q}_T"'] & \tilde{q}_T^*(S^{\wedge}_{\mathcal{P}}) \arrow[d, dashed, "\tilde{q}_T"] \\
C_S \arrow[r] & S^{\wedge}_{\mathcal{P}}
\end{tikzcd}
$$

(where we are recording $C \to val(S)$ as a planar-tall map $C \to S$). A similar analysis shows that the top left trapezoid commutes.

The other regions also commute by a straightforward analysis. Indeed, the top right trapezoid commutes by unique factorization, and finally the middle triangle of $\partial_{\mathcal{P}}$ maps commutes since $(\tilde{q}_T^* S)^{\wedge}_{\mathcal{P}} = \tilde{q}_T^*(S^{\wedge}_{\mathcal{P}})$.

Lastly, we must check that the middle two maps are in fact elements of the appropriate overcategory. This follows from the fact that $S^{\wedge}_{\mathcal{P}}$ and T have $|-|_Y < k$. Thus, the overcategory in question is connected, as desired. $\square$

come back: define $S^{\wedge}_{Y}$.

N_FINALITY_LEMMA

Lemma 6.9. *$\Omega^e_G[k,0]^{op}$ is* Lan-*final in $\Omega^e_G[k]$ over Σ^{op}_G.*

Similarly, we need a construction $T \mapsto T^{\wedge}_Y$ in order to prove this lemma. However, in this case, the analogous notion is much simpler, as $T^{\wedge}_Y$ has the same underlying $(\mathcal{P}; Z)$-alternating G-tree, but we just relabel all X-vertices as being Y-labeled.

Proof of Lemma ZERO_LAN_FINALITY_LEMMA 6.9. This follows analogously to Lemma MINUS_LAN_FINAL_LEMMA 6.3, by replacing Diagram K-1_LAN_FINAL_DIAGRAM 6.8 with the diagram below:

$$\begin{tikzcd}
 & S & \\
 & \tilde{q}_T^*(S) \arrow[u, "\tilde{q}_T"] & \\
S_Y^\wedge \arrow[uur, "\partial_Y"] & \tilde{q}_T^*(S_Y^\wedge) \arrow[l, "\tilde{q}_T"'] \arrow[u, "\partial_Y"] \arrow[r, "\partial_Y"] & T \arrow[uul, "f"'] \arrow[ul, "\partial_Y"'] \\
 & C \arrow[ul, "q_S"] \arrow[u, "q_T"] \arrow[ur, "q_T"'] &
\end{tikzcd}$$

□

Finally, we show that each layer $\Omega_G^e[\leq k]$ can be built from $\Omega_{G,e}[\leq k-1]$ via a pushout which attaches trees with precise degree k. While dealing with general pushouts of categories requires solving a "word problem" on morphisms, we will only work in cases where the problem collapses. We recall that, given a square of categories

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\
\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

if the nerve of this square is a pushout in sSet, then this is a pushout of categories (since the nerve is the inclusion of a reflective subcategory).

US_PUSHOUTS_DEFN

Definition 6.10. We call such squares *nervous pushouts* of categories.

If we further assume that the span of functors is built out of fully-faithful inclusions, these pushouts behave as nicely as possible with left Kan extensions.

AN_PUSHOUT_LEMMA

Lemma 6.11. *Given any diagram in categories of the form*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "f"] \arrow[d] & \mathcal{C} \arrow[d, "i"] & \\
\mathcal{B} \arrow[r, "g"] & \mathcal{D} \arrow[r, "Y"] \arrow[d, "j"] & \mathcal{V} \\
 & \mathcal{D} &
\end{tikzcd}$$

such that the square is a nervous pushout of fully-faithful functors, then $\operatorname{Lan}_j Y$ *is the pushout of the induced span*

$$\begin{tikzcd}
\operatorname{Lan}_{jif}(Yif) \arrow[r] \arrow[d] & \operatorname{Lan}_{ji}(Yi) \\
\operatorname{Lan}_{jg}(Yg).
\end{tikzcd}$$

Proof. By the universal property of left Kan extensions, it suffices to show that, for any functor $Z : \mathcal{V} \to \mathcal{D}$, the natural map

$$\mathcal{V}^{\mathcal{D}}(Y, Zj) \longrightarrow \mathcal{V}^{\mathcal{B}}(Yg, Zjg) \prod_{\mathcal{V}^{\mathcal{A}}(Yif, Zjif)} \mathcal{V}^{\mathcal{C}}(Yi, Zji)$$

is a bijection. These two sets give the same data: a collection of maps $\Phi_b : Y(b) \to Z(b)$ and $\Phi_c : Y(c) \to Z(c)$ for all $b \in \mathcal{B}$ and $c \in \mathcal{C}$, such that $\Phi_b = \Phi_c$ whenever $b = c \in \mathcal{A}$. In general, the compatibilites required on the right are less demanding. However, with the above assumptions, a map $d \to d'$ in $\mathcal{D}$ is *uniquely* a map in $\mathcal{A}$, $\mathcal{B} \smallsetminus \mathcal{A}$, or $\mathcal{C} \smallsetminus \mathcal{A}$, and thus all the necessary compatibilities are covered by (at least) one of the $\{\Phi_b\}$ or $\{\Phi_c\}$. □

We can now build our category $\Omega_{G,e}[\leq k]$ inductively.

ATS_DECOMP_LEMMA **Lemma 6.12.** *$\Omega^e_G[\leq k]$ is the isomorphic to the pushout below.*

$$\begin{tikzcd}
\Omega^e_G[k,-] \arrow[r] \arrow[d] & \Omega^e_G[\leq k,-] \arrow[d] \\
\Omega^e_G[k] \arrow[r] & \Omega^e_G[\leq k]
\end{tikzcd}$$

In fact, it is a nervous pushout of fully-faithful functors.

Proof. Since maps in Ω^e_G can only increase $|-|$ by adding $|-|_X$, if T is a tree with $|T| = |T|_Y = k$, then any other tree $S \in \Omega^e_G$ connected to T via a zig-zag of maps must have $|S| = k$; that is, if $T \in \Omega^e_G[\leq k] \setminus \Omega^e_G[\leq k,-]$, then the connected component of T is entirely contained in $\Omega^e_G[k]$. Conversely, if $T \in \Omega^e_G[\leq k] \setminus \Omega^e_G[k]$, the connected component of T is entirely contained in $\Omega^e_G[\leq k,-]$. Since the natural induced map

$$\Omega^e_G[k] \amalg \Omega^e_G[\leq k,-] \to \Omega^e_G[\leq k]$$

is clearly full and surjective on objects, the result follows from the above discussion and the obvious fully-faithfulness ofthe span. $\square$

Abusing notation, we will denote by N^e the restriction of that functor to any of the subcategories of Ω^e_G in the above pushout square.

We can now define the sequencers which will make up our filtration of $\mathcal{P}[u]$:

Definition 6.13. Let $\mathcal{P}_k$ denote the left Kan extension

$$\begin{tikzcd}
\Omega^e_G[\leq k]^{op} \arrow[r, "N^e"] \arrow[d, "val"'] \arrow[dr, phantom, "\Rrightarrow"] & \mathcal{V} \\
\Sigma^{op}_G \arrow[ur, "\mathcal{P}_k"'] &
\end{tikzcd}$$

Note that by naturality of Lan, we have maps $\mathcal{P}_{k-1} \to \mathcal{P}_k$.

6.2 Notation

Luis: should this be stated earlier when defining the categorical wreath products?

In order to state our filtration result, we will need to identify another categorical construction. This filtration will be built out of "pushout products over trees of maps of sequences". This subsection is dedicated to making that statement precise.

Definition 6.14. Given a map $u : Y_0 \to Y_1$ of G-symmetric sequences $\mathcal{V}^{\Sigma^{op}_G}$, and $(A, D) \in \mathsf{F} \wr \Sigma_G$, we borrow notation from BM03 [1] and define the functor

$$[u]^D : (0 \to 1)^A \to \mathcal{V}$$

as the composite

$$(0 \to 1)^A \to \mathsf{F} \wr \mathcal{V} \xrightarrow{\times} \mathcal{V}$$

where the first map is defined on $\xi : A \to \{0, 1\}$ by

$$(a \mapsto \xi(a)) \mapsto (A, (a \mapsto Y_{\xi(a)}(D(a))))$$

We recall that, in a general category $\mathcal{C}$, a subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called *convex* if whenever $c' \in \mathcal{C}'$ and $c \to c'$ is an arrow in $\mathcal{C}$¡ then both c' and the map are in $\mathcal{C}'$.

Q_DEFINITION

Definition 6.15. Let $\mathcal{C}$ be a convex subcategory of $(0 \to 1)^A$. We define

$$Q_{\mathcal{C}}[u]^D := \operatorname{colim}_{\mathcal{C}}[u]^D.$$

Moreover, given nested convex subcategories $\mathcal{C}' \subseteq \mathcal{C}$, we let

$$[u]^D \square^{\mathcal{C}}_{\mathcal{C}'} : Q_{\mathcal{C}'}[u]^D \to Q_{\mathcal{C}}[u]^D$$

denote the unique natural map.

In particular, if $\mathcal{C}$ is the full "punctured cube" subcategory $(0 \to 1)^A \smallsetminus \{(1)_a\}$, we simplify the notation as follows:

$$Q[u]^D := Q_{\mathcal{C}}[u]^D$$
$$[u]^{\square D} := [u]^D \square^{(0\to 1)^A}_{\mathcal{C}} : Q[u]^D \to \bigotimes_{a\in A} Y_1(D(a)).$$

6.3 Filtration Result

We can now state our filtration of the cellular extension $\mathcal{P} \to \mathcal{P}[u]$.

Theorem 6.16. *Let $\mathcal{P}$ be a genuine G-operad, and suppose we are given a map of G-symmetric sequences $u : Y_0 \to Y_1$. Then we have a filtration in G-sequences of the cellular extension*

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \ldots \to \operatorname{colim}(\mathcal{P}_i) = \mathcal{P}[u],$$

where $\mathcal{P}_{k-1} \to \mathcal{P}_k$ is given by the pushout

$$\begin{tikzcd}
\operatorname{Lan}_{\Omega_{G,e}[k,-]^{op}} N^e \arrow[r] \arrow[d] & \mathcal{P}_{k-1} \arrow[d] \\
\operatorname{Lan}_{\Omega_{G,e}[k]^{op}} N^e \arrow[r] & \mathcal{P}_k
\end{tikzcd}$$

Levelwise, for each $C \in \Sigma_G$, in the underlying category $\mathcal{V}^{G\times\Sigma_n}$, we have a filtration on the evaluations at C, where $\mathcal{P}_{k-1}(C) \to \mathcal{P}_k(C)$ is given by the pushout

$$\begin{tikzcd}
\coprod_{[T]\in\Omega_{G,e}[k,0](C)/\simeq} \left(\bigotimes_{v\in V_{G,\mathcal{P}}(T)} \mathcal{P}(T_v) \otimes Q[u]^{\mathbb{V}_{G,in}(T)} \right) \otimes_{\operatorname{Aut}(T)} \operatorname{Aut}(C) \arrow[r] \arrow[d] & \mathcal{P}_{k-1}(C) \arrow[d] \\
\coprod_{[T]\in\Omega_{G,e}[k,0](C)/\simeq} \left(\bigotimes_{v\in V_{G,\mathcal{P}}(T)} \mathcal{P}(T_v) \otimes \bigotimes_{v\in V_{G,in}(T)} Y_1(T_v) \right) \otimes_{\operatorname{Aut}(T)} \operatorname{Aut}(C) \arrow[r] & \mathcal{P}_k(C)
\end{tikzcd}$$

where the left vertical map is the iterated box product

$$\coprod \mathop{\square}_{V_{G,\mathcal{P}}(T)} \iota_{\mathcal{P}(T_v)} \square [u]^{\square \mathbb{V}_{G,in}(T)},$$

$\iota_{\mathcal{P}(T_v)}$ is the canonical map $\varnothing \to \mathcal{P}(T_v)$ out of the initial object, and $\Omega_{G,e}[k,0](C)$ is as in Definition TREE_FILTRATION_PIECES_DEFINITION *6.1.*

Proof. Combining Lemmas LAN_PUSHOUT_LEMMA 6.11 and TREE_LEMMAS_DECOMP_LEMMA 6.12, we have that $\mathcal{P}_k$ can be computed as the pushout

$$\begin{tikzcd}
\operatorname{Lan}_{\Omega_{G,e}[k,-]^{op}} N^e \arrow[r] \arrow[d] & \operatorname{Lan}_{\Omega_{G,e}[\le k,-]^{op}} N^e \arrow[d] \\
\operatorname{Lan}_{\Omega_{G,e}[k]^{op}} N^e \arrow[r] & \operatorname{Lan}_{\Omega_{G,e}[\le k]^{op}} N^e := \mathcal{P}_k
\end{tikzcd} \tag{6.17}$$

FILTRATION_LAN_SQUARE

By Lemma 6.3, the top right corner can be identified with $\mathcal{P}_{k-1}$. Thus, it remains to identify the left hand side.

By Lemma 6.9, we may replace the bottom left corner with $\mathrm{Lan}_{\Omega_{G,e}[k,0]^{op}} N^e$. Now, given $T \in \Omega_{G,e}[k,0]$, let $[T]$ denote the isomorphism class of T in $\Omega_{G,e}[k,0]$. With this notation, the bottom left corner can further be identified with

$$\coprod_{[T]\in\Omega_{G,e}[k,0](C)/\simeq} N^e(T)\otimes_{\mathrm{Aut}(T)}\mathrm{Aut}(C) = \coprod_{[T]}\left(\bigotimes_{v\in V_{G,\mathcal{P}}(T)}\mathcal{P}(T_v)\otimes\bigotimes_{v\in V_{G,Y}(T)}Y_1(T_v)\right)\otimes_{\mathrm{Aut}(T)}\mathrm{Aut}(C).$$

Next, we observe that the non-invertible morphisms of $\Omega_{G,e}[k,-]^{op}\downarrow C$ are just those which change the labeling of some nodes from X to Y. Given S and T in $\Omega_{G,e}[k,-]$, write $S\sim T$ if they are in the same path component, and again note that this implies $|S| = |T|$, and moreover that S and T forget to the same $(\mathcal{P};Z)$-alternating tree. Denote the path component of T by (T).

We note that the set of path components of those trees with $val(T) = C$ is equal to the set of isomorphism classes in $\Omega_{G,e}[k,0](C)$, as both are just determined preicsely by their underlying $(\mathcal{P};Z)$-alternating tree.

To account for the $\mathrm{Aut}(C)$-action on the indexing category, we note that each connected component of $\Omega_{G,e}[k,-]^{op}\downarrow C$ has an action of $\mathrm{Aut}([T])$. Thus, the top left corner of Diagram (6.17) can be identified with the image of the colimit map below:

$$\begin{array}{c}\displaystyle\coprod_{[T]\in\Omega_{G,e}[k](C)/\sim}\left(\coprod_{S\in(T)\smallsetminus\{T\}}N^e(S)\right)\otimes_{\mathrm{Aut}(T)}\mathrm{Aut}(C)\\ \big\downarrow{\scriptstyle \mathrm{colim}}\\ \displaystyle\coprod_{[T]}\left(\bigotimes_{v\in V_{G,\mathcal{P}}(T)}\mathcal{P}(T_v)\otimes Q[u]^{\mathbb{V}_{G,in}(T)}\right)\otimes_{\mathrm{Aut}(T)}\mathrm{Aut}(C)\end{array}$$

where $Q[u]^{\mathbb{V}_{G,in}(T)}$ is the source of the pushout product map defined in Definition 6.15.

Lastly, this left-side map is induced, via the naturality of Kan extesnions, by an inclusion of categories, in particular the product of multiple inclusions of categories, each corresponding the inclusion of a punctured cube into the full cube. Thus, after taking colimits, we have that the left-side map in Diagram (6.17) is in fact (multiple copies of) the pushout-product maps

$$[u]^{\square\mathbb{V}_{G,in}(T)}: Q[u]^{\mathbb{V}_{G,in}(T)}\to\bigotimes_{v\in V_{G,in}(T)}Y_1(T_v)),$$

as desired. □

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