Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads and which can be thought of as a hybrid between operads and coefficient systems. We then prove an Elmendorf-Piacenza type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads, with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

A surprising feature of topological algebra is that the category of (connected) topological commutative monoids is quite small, consisting only of products of Eilenberg-MacLane spaces (e.g. [18, 4K.6]). Instead, the more interesting structures are those monoids which are commutative and associative only up to homotopy and, moreover, up to "all higher homotopies". To capture these more subtle algebraic notions, Boardman-Vogt [4] and May [23] developed the theory of *operads*. Informally, an operad $\mathcal{O}$ consists of a sequence of sets/spaces $\mathcal{O}(n)$ of "n-ary operations" carrying a Σ_n-action (recording "reordering of the inputs of the operations"), and a suitable notion of "composition of operations". The purpose of the theory is then the study of "objects X with operations indexed by $\mathcal{O}$", referred to as *algebras*, with the notions of monoid, commutative monoid, Lie algebra, algebra with a module, and more, all being recovered as algebras over some fixed operad in an appropriate category. Of special importance are the E_∞-operads, introduced by May in [23], which are homotopical replacements for the commutative operad and encode the aforementioned "commutative monoids up to homotopy". In particular, while an E_∞-algebra structure on X does not specify unique maps $X^n \to X$, it nonetheless specifies such maps "uniquely up to homotopy".

E_∞-operads are characterized by the homotopy type of their levels $\mathcal{O}(n)$: $\mathcal{O}$ is E_∞ if and only if each $\mathcal{O}(n)$ is Σ_n-free and contractible. That is, for each subgroup $\Gamma \leq \Sigma_n$ one has

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Notably, when studying the homotopy theory of operads in topological spaces the preferred notion of weak equivalence is usually that of "naive equivalence", with a map of operads $\mathcal{O} \to \mathcal{O}'$ deemed a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a weak equivalence of spaces upon forgetting the Σ_n-actions (e.g. BM03 [2, 3.2]). In this context, E_∞-operads are then equivalent to the commutative operad Com and, moreover, any cofibrant replacement of Com is E_∞. These naive equivalences differ from the equivalences in "genuine equivariant homotopy theory", where a map of G-spaces $X \to Y$ is deemed a G-equivalence only if the induced fix point maps $X^H \to Y^H$ are weak equivalences for all $H \leq G$. This contrast hints at a number of novel subtleties that appear in the study of equivariant operads, which we now discuss.

First, note that for a finite group G and G-operad $\mathcal{O}$ (i.e. an operad $\mathcal{O}$ together with a G-action commuting with all the structure), the n-th level $\mathcal{O}(n)$ has a $G \times \Sigma_n$-action. As such, one might guess that a map of G-operads $\mathcal{O} \to \mathcal{O}'$ should be called a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a G-equivalence after forgetting the Σ_n-actions, i.e. if the maps

$$\mathcal{O}(n)^H \xrightarrow{\sim} \mathcal{O}'(n)^H, \qquad H \leq G \leq G \times \Sigma_n, \tag{1.1}$$ NAIVEOPEQ EQ

are weak equivalences of spaces. However, the notion of equivalence suggested in (1.1) turns out to not be "genuine enough". To see why, we consider a homotopical replacement for Com using this theory: if one simply equips an E_∞-operad $\mathcal{O}$ with a trivial G-action, the resulting G-operad has fixed points for each subgroup $\Gamma \leq G \times \Sigma_n$ determined by

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \leq G, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.2}$$ NAIVEGEINFTY EQ

However, as first noted by Costenoble-Waner CW91 [12] in their study of equivariant infinite loop spaces, the *G-trivial E_∞-operads* of (1.2) do not provide the correct replacement of Com in the G-equivariant context. Rather, that replacement is provided instead by the *G-E_∞-operads*, characterized by the fixed point conditions

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \cap \Sigma_n = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.3}$$ GENGEINFTY EQ

In contrasting (1.2) and (1.3), we note first that the subgroups $\Gamma \leq G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \{*\}$ are characterized as being the *graph subgroups*, i.e. the subgroups of the form

$$\Gamma = \{(h, \phi(h)) \in G \times \Sigma_n | h \in H\} \tag{1.4}$$ GRAPHSUBIN EQ

for some subgroup $H \leq G$ and homomorphism $\phi \colon H \to \Sigma_n$. On the other hand, $\Gamma \leq G$ if and only if Γ is the graph subgroup (1.4) for ϕ a trivial homomorphism. As it turns out, the notion of weak equivalence described in (1.1) fails to distinguish (1.2) and (1.3), and indeed it is possible to build maps $\mathcal{O} \to \mathcal{O}'$ where $\mathcal{O}$ is a G-trivial E_∞-operad (as in (1.2)) and $\mathcal{O}'$ is a G-E_∞-operad (as in (1.3)). Therefore, in order to differentiate such operads, one needs to replace the notion of weak equivalence in (1.1) with the finer notion of *graph equivalence*, so that $\mathcal{O} \to \mathcal{O}'$ is considered a weak equivalence only if the maps

$$\mathcal{O}(n)^\Gamma \xrightarrow{\sim} \mathcal{O}'(n)^\Gamma, \qquad \Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}. \tag{1.5}$$ GENEOPEQ EQ

are all weak equivalences.

As mentioned above, the original evidence [12] that (1.3), rather than (1.2), provides the best up-to-homotopy replacement for Com in the equivariant context comes from the study of equivariant infinite loop spaces. For our purposes, however, we instead focus on the perspective of Blumberg-Hill in [3], which concerns the Hill-Hopkins-Ravenel norm maps featured in the solution of the Kervaire Invariant One Problem [19].

Given a G-spectrum R and finite G-set X with n elements, the corresponding *norm* is another G-spectrum $N^X R$, whose underlying spectrum is $R^{\wedge X} \simeq R^{\wedge n}$, but equipped with a "mixed G-action" which both permutes wedge factors via the action on X and acts diagonally on each factor (alternatively, $N^X R$ can be described via graph subgroups; see the next paragraph). Moreover, for any Com-algebra R, i.e. any strictly commutative G-ring spectrum, ring multiplication further induces so called *norm maps*

$$N^X R \to R. \tag{1.6}$$

NORMMAPS EQ

Furthermore, by restricting the structure on R, the maps (1.6) are also defined when X is only an H-set for some subgroup $H \leq G$, and the maps (1.6) then satisfy a number of natural equivariance and associativity conditions. Crucially, we note that the more interesting of these associativity conditions involve H-sets for various H simultaneously (for an example packaged in operadic language, see (1.12) below).

The key observation at the source of the work in [3] is then that, operadically, norm maps are encoded by the graph fixed points appearing in (1.5). More explicitly, noting that, for $H \leq G$, an H-set X with n elements is encoded by a homomorphism $H \to \Sigma_n$, one obtains an associated graph subgroup $\Gamma_X \leq G \times \Sigma_n$, well-defined up to conjugation. Next, using the natural $(G \times \Sigma_n)$-action on $R^{\wedge n}$, the H-action on $N^X R \simeq R^{\wedge n}$ is obtained via the obvious identification $H \simeq \Gamma_X$. It then follows that, for any $\mathcal{O}$-algebra R, maps of the form (1.6) are parametrized by the fixed point space $\mathcal{O}(n)^{\Gamma_X}$. The flaw of the G-trivial E_∞-operads described in (1.2) is then that they lack all norms maps other than those for H-trivial X, thus lacking some of the data encoded by Com. Further, from this perspective one may regard the more naive notion of weak equivalence in (1.1), according to which (1.2) and (1.3) are equivalent, as studying "operads without norm maps" (in the sense that equivalences ignore norm maps), while the equivalences (1.5) study "operads with norm maps".

Our first main result, Theorem I, establishes the existence of a model structure on G-operads with weak equivalences the graph equivalences of (1.5), though our analysis goes significantly further, again guided by Blumberg and Hill's work in [3].

The main novelty of [3] is the definition, for each finite group G, of a finite lattice of new types of equivariant operads, which they dub N_∞-operads. The minimal type of N_∞-operads is that of the G-trivial E_∞-operads in (1.2) while the maximal type is that of the G-E_∞-operads in (1.3). The remaining types, which interpolate between the two, can hence be thought of as encoding varying degrees of "up to homotopy equivariant commutativity". More concretely, each type of N_∞-operad is determined by a collection $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ where each $\mathcal{F}_n$ is itself a collection of graph subgroups of $G \times \Sigma_n$, with an operad $\mathcal{O}$ being called a *$N\mathcal{F}$-operad* if it satisfies the fixed point condition

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.7}$$

NFINFTY EQ

Such collections $\mathcal{F}$ are, however, far from arbitrary, with much of the work in [3, §3] spent cataloging a number of closure conditions that these $\mathcal{F}$ must satisfy. The simplest of these conditions state that each $\mathcal{F}_n$ is a *family*, i.e. closed under subgroups and conjugation. These first two conditions, which are common in equivariant homotopy theory, are a simple consequence of each $\mathcal{O}(n)$ being a space. However, the remaining conditions, all of which involve $\mathcal{F}_n$ for various n simultaneously and are a consequence of operadic multiplication, are both novel and subtle. In loose terms, these conditions, which are more easily described in terms of the H-sets X associated to the graph subgroups, concern closure of those under disjoint union, cartesian product, subobjects, and an entirely new key condition called *self-induction*. The precise conditions are collected in [3, Def. 3.22], which also introduces the term *indexing system* for an $\mathcal{F}$ satisfying all of those conditions. A main result of [3, §4] is then that whenever an $N\mathcal{F}$-operad $\mathcal{O}$ as in (1.7) exists, the associated collection $\mathcal{F}$ must be an indexing system. However, the converse statement, that given any indexing system $\mathcal{F}$ such an $\mathcal{O}$ can be produced, was left as a conjecture.

One of the key motivating goals of the present work was to verify this conjecture of Blumberg-Hill, which we obtain in Corollary IV. We note here that this conjecture has also been concurrently verified by Gutiérrez-White in [17] and by Rubin in [29], with each of their approaches having different advantages: Gutiérrez-White's model for $N\mathcal{F}$ is cofibrant while Rubin's model is explicit. Our model, which emerges from a broader framework, satisfies both of these desiderata.

To motivate our approach, we first recall the solution of a closely related but simpler problem: that of building universal spaces for families of subgroups. Given a family $\mathcal{F}$ of subgroups of G (i.e. a collection closed under conjugation and subgroups), a *universal space* X for $\mathcal{F}$, also called an *$E\mathcal{F}$-space*, is a space with fixed points X^H characterized just as in (1.7). In particular, whenever $\mathcal{O}$ is an $N\mathcal{F}$-operad, each $\mathcal{O}(n)$ is necessarily an $E\mathcal{F}_n$-space. The existence of $E\mathcal{F}$-spaces for any choice of the family $\mathcal{F}$ is best understood in light of Elmendorf's classical result from [13] (modernized by Piacenza in [28]) stating that there is a Quillen equivalence (recall that O_G is the *orbit* category, formed by the G-sets G/H)

$$\begin{array}{ccc} \mathsf{Top}^{\mathsf{O}_G^{op}} & \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} & \mathsf{Top}^G \\ (G/H \mapsto Y(G/H)) & \longmapsto & Y(G) \\ (G/H \mapsto X^H) & \longleftarrow\!\shortmid & X \end{array} \tag{1.8}$$

COFADJINT EQ

where the weak equivalences (and fibrations) on Top^G are detected on all fixed points and the weak equivalences (and fibrations) on the category $\mathsf{Top}^{\mathsf{O}_G^{op}}$ of *coefficient systems* are detected at each presheaf level. Noting that the fixed point characterization of $E\mathcal{F}$-spaces defines a natural object $\delta_{\mathcal{F}} \in \mathsf{Top}^{\mathsf{O}_G^{op}}$ by $\delta_{\mathcal{F}}(G/H) = *$ if $H \in \mathcal{F}$ and $\delta_{\mathcal{F}}(G/H) = \varnothing$ otherwise, $E\mathcal{F}$-spaces can then be built as $\iota^*(C\delta_{\mathcal{F}}) = C\delta_{\mathcal{F}}(G)$, where C denotes cofibrant replacement in $\mathsf{Top}^{\mathsf{O}_G^{op}}$. Moreover, we note that, as in [13, §3], these cofibrant replacements can be built via explicit simplicial realizations.

The overarching goal of this paper is then that of proving the analogue of Elmendorf-Piacenza's Theorem (1.8) in the context of operads with norm maps (i.e. with equivalences as in (1.5)), which we state as our main result, Theorem III. However, in trying to formulate such a result one immediately runs into a fundamental issue: it is unclear which category should take the role of the coefficient systems $\mathsf{Top}^{\mathsf{O}_G^{op}}$ in this context. This last remark likely requires justification. Indeed, it may at first seem tempting to simply employ one of the known formal generalizations of Elmendorf-Piacenza's result (see, e.g. [31, Thm. 3.17]) which simply replace Top on either side of (1.8) with a more general model category $\mathcal{V}$. However, if one applies such a result when $\mathcal{V} = \mathsf{Op}$ to establish a Quillen equivalence $\mathsf{Op}^{\mathsf{O}_G^{op}} \rightleftarrows \mathsf{Op}^G$ (the existence of this equivalence is due to upcoming work of Bergner-Gutiérrez), the fact that the levels of each $\mathcal{P} \in \mathsf{Op}^{\mathsf{O}_G^{op}}$ correspond only to those fixed-point spaces appearing in (1.1) would require working in the context of operads *without* norm maps, and thereby forgo the ability to distinguish the many types of $N\mathcal{F}$-operads.

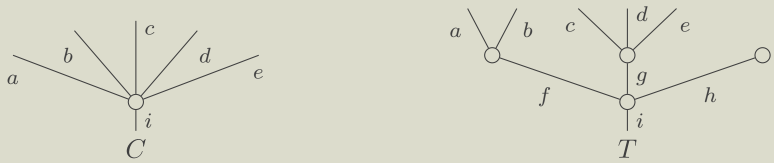
As such, to obtain an Elemendorf-Piacenza Theorem in the context of operads with norm maps, we will need to replace $\mathsf{Top}^{\mathsf{O}_G^{op}}$ with a category Op_G of new algebraic objects we dub *genuine equivariant operads* (as opposed to (regular) equivariant operads Op^G). Each genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will consist of a list of spaces, indexed in the same way as in (1.5), along with obvious restriction maps and, more importantly, suitable *composition maps*. Precisely identifying the required composition maps is one of the main challenges of this theory, and again we turn to [3] for motivation.

Analyzing the proofs of the results in BH15 [3, §4] concerning the closure properties for indexing systems $\mathcal{F}$, a common motif emerges: when performing an operadic composition

$$\begin{aligned} \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) &\longrightarrow \mathcal{O}(m_1 + \cdots + m_n), \\ (f, g_1, \cdots, g_n) &\longmapsto f(g_1, \cdots, g_n) \end{aligned} \tag{1.9}$$ OPMULT EQ

careful choices of fixed point conditions on the operations $f, g_1, \cdots, g_n$ yield a fixed point condition on the composite operation $f(g_1, \cdots, g_n)$. The desired multiplication maps for a genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will then abstract such interactions between multiplication and fixed points for an equivariant operad $\mathcal{O} \in \mathsf{Op}^G$. However, these interactions can be challenging to write down explicitly and indeed, the arguments in BH15 [3, §4] do not quite provide the sort of unified conceptual approach to these interactions needed for our purposes. The cornerstone of the current work was then the joint discovery by the authors of such a conceptual framework: equivariant trees.

Non-equivariantly, it has long been known that the combinatorics of operadic composition is best visualized by means of tree diagrams. For instance, the tree T on the right below



encodes the operadic composition

$$\mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0) \to \mathcal{O}(5) \tag{1.10}$$ COMPEX EQ

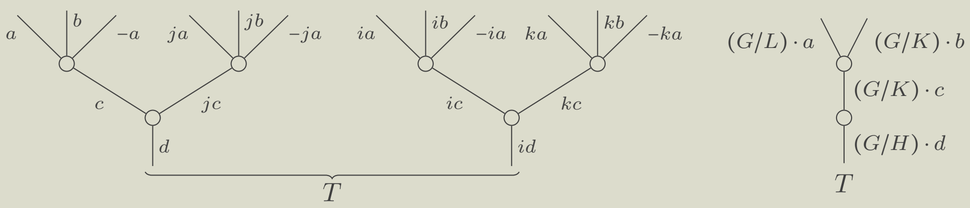
where the inputs $\mathcal{O}(3), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(0)$ correspond to the nodes/vertices (i.e. circles) in the tree T, with arity given by number of incoming edges (i.e. edges immediately above) and the output $\mathcal{O}(5)$ has arity given by counting leaves (i.e. edges at the top, not capped by a node). Before recalling equivariant trees, it is worth making the connection between T and (1.10) more precise. Recall MW07 [24, §3] that T gives rise to a colored operad[1] $\Omega(T)$, as follows. The colors/objects of $\Omega(T)$ are the edges $a, b, c, \cdots, i$ while the generating operations, determined by the nodes, are $(a,b) \to f$, $(c,d,e) \to g$, $() \to h$, $(f,g,h) \to i$ (i.e., for each node, incoming edges are viewed as inputs and the outgoing edge as an output). Let C be the corolla (i.e. tree with a single node) above, which is formed by the leaves and root of T. There is then a natural map of colored operads $\Omega(C) \to \Omega(T)$ so that, writing $\mathsf{Op}_\bullet$ for the category of colored operads, (1.10) is the induced map of mapping sets $\mathsf{Op}_\bullet(\Omega(T), \mathcal{O}) \to \mathsf{Op}_\bullet(\Omega(C), \mathcal{O})$. Indeed, $\mathsf{Op}_\bullet(\Omega(T), \mathcal{O}) \simeq \mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0)$ and $\mathsf{Op}_\bullet(\Omega(C), \mathcal{O}) \simeq \mathcal{O}(5)$ since maps $\Omega(T) \to \mathcal{O}$ and $\Omega(C) \to \mathcal{O}$ are determined by the image of the generating operations.

Analogously, the role of equivariant trees is, in the context of equivariant operads, to encode operadic compositions as in (1.10) together with fixed point compatibilities. Briefly, a G-tree Pe16b [26, Def. 5.44] is a forest diagram (i.e. a collection of trees) together with a G-action that is transitive on tree components. A detailed introduction to (and motivation for) equivariant trees can be found in Pe17 [27, §4], where the second author develops the theory of equivariant dendroidal sets (a parallel approach to equivariant operads), though here we include only a single representative example.

Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternionic units and $G \geq H \geq K \geq L$ be the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$. One has a G-tree T with *expanded representation*

[1]Recall that colored operads, also known as multicategories, are a generalization of the notion of category where each arrow/operation has multiple inputs but a single output.

given by the two leftmost trees below and *orbital representation* given by the rightmost tree.



(1.11)

Here, the expanded representation of T is just a forest with edges labels that indicate the G-action. Note that all edges are conjugate to one of the edges a, b, c, d which have, respectively, stabilizers L, K, K, H. For example, the labels of T imply that $\pm id = \pm kd$ and $\pm jd = \pm d = d$. Given the expanded representation, the orbital representation is obtained by collapsing each edge orbit into a single edge, which is labeled by the corresponding orbit set of edges in the expanded representation (one may also reverse this process, though we will not need to do so). We note that orbital representations always "look like a tree".

As explained in [27, Example 4.9], the G-tree T encodes the fact that, for $\mathcal{O} \in \mathsf{Op}^G$ a G-operad, the composition $\mathcal{O}(2) \times \mathcal{O}(3)^{\times 2} \to \mathcal{O}(6)$ restricts to a fixed point composition

$$\mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \to \mathcal{O}(H/L \amalg H/K)^H \tag{1.12}$$

(we discuss how (1.12) is obtained in the next paragraph) where $\mathcal{O}(X)$ for X an H-set denotes $\mathcal{O}(|X|)$ with the H-action given by the identification $H \simeq \Gamma_X$ (the graph subgroup Γ_X is as discussed after (1.6)), and likewise for K-sets. In particular, $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$.

We recall the precise connection between T and (1.12). Let $\mathsf{Op}^G_\bullet$ be the category of G-objects in colored operads. As in the non-equivariant case, one builds $\Omega(T)$ in $\mathsf{Op}^G_\bullet$ and a map $\Omega(C) \to \Omega(T)$ in $\mathsf{Op}^G_\bullet$, where C is the G-corolla (i.e. G-tree composed of corollas) formed by the leaves and roots of T. The composition (1.12) is then the induced map $\mathsf{Op}^G_\bullet(\Omega(T), \mathcal{O}) \to \mathsf{Op}^G_\bullet(\Omega(C), \mathcal{O})$. The implicit claim $\mathsf{Op}^G_\bullet(\Omega(T), \mathcal{O}) \simeq \mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K$ follows since: by equivariance, a G-map $\phi\colon \Omega(T) \to \mathcal{O}$ is determined by the images of the operations $(a, b, -a) \to c$ and $(c, jc) \to d$; the operation $\phi((a, b, -a) \to c)$ must be in $\mathcal{O}(K/L \amalg K/K)^K$, since K is the isotropy of c and $\{a, b, -a\} \simeq K/L \amalg K/K$ as K-sets; likewise $\phi((c, jc) \to d)$ must be in $\mathcal{O}(H/K)^H$. The claim $\mathsf{Op}^G_\bullet(\Omega(C), \mathcal{O}) \simeq \mathcal{O}(H/L \amalg H/K)^H$ is similar.

We note that the two inputs $\mathcal{O}(H/K)^H$, $\mathcal{O}(K/L \amalg K/K)^K$ in (1.12) correspond to the two nodes of the orbital representation in (1.11). Notice that now the arity (i.e. the associated "type of input") of such a node does not just count incoming edge orbits, but depends on the labels of both incoming and outgoing edge orbits (in particular, the fixed point condition depends on the latter). Similarly, the output $\mathcal{O}(H/L \amalg H/K)^H$ is determined by both the leaf and root edge orbits. The existence of maps of the form (1.12) is essentially tantamount to the subtlest closure property for indexing systems $\mathcal{F}$, self-induction (cf. [3, Def. 3.20]), and similar tree descriptions exist for all other closure properties, as detailed in [27, §9].

We can now at last give a full informal description of the category Op_G featured in our main result, Theorem III. A genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ has levels $\mathcal{P}(X)$ for each H-set X, $H \leq G$, that mimic the role of the fixed points $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$ for $\mathcal{O} \in \mathsf{Op}^G$. More explicitly, there are restriction maps $\mathcal{P}(X) \to \mathcal{P}(X|_K)$ for $K \leq H$, isomorphisms $\mathcal{P}(X) \simeq \mathcal{P}(gX)$ where gX denotes the conjugate gHg^{-1}-set, and composition maps given by

$$\mathcal{P}(H/K) \times \mathcal{P}(K/L \amalg K/K) \to \mathcal{P}(H/L \amalg H/K)$$

in the case of the abstraction of (1.12), and more generally by

$$\begin{gathered}\mathcal{P}(H/K_1 \amalg \cdots \amalg H/K_n) \times \mathcal{P}(K_1/L_{11} \amalg \cdots \amalg K_1/L_{1m_1}) \times \cdots \times \mathcal{P}(K_n/L_{n1} \amalg \cdots \amalg K_n/L_{nm_n}) \\ \downarrow \\ \mathcal{P}(H/L_{11} \amalg \cdots \amalg H/L_{1m_1} \amalg \cdots \amalg H/L_{n1} \amalg \cdots \amalg H/L_{nm_n}).\end{gathered} \tag{1.13}$$

GENGENMULT EQ

Lastly, these composition maps must satisfy associativity, unitality, compatibility with restriction maps, and equivariance conditions, as encoded by the theory of G-trees. Rather than making such compatibilities explicit, however, we will find it preferable for our purposes to simply define genuine equivariant operads intrinsically in terms of G-trees.

We end this introduction with an alternative perspective on the role of genuine equivariant operads. The Elmendorf-Piacenza theorem in (1.8) is ultimately a strengthening of the basic observation that the homotopy groups $\pi_n(X)$ of a G-space X are coefficient systems rather than just G-objects. Similarly, the generalized Elmendorf-Piacenza result [31, Thm. 3.17] applied to the category $\mathcal{V} = \mathsf{sCat}$ of simplicial categories strengthens the observation that for a G-simplicial category $\mathcal{C}$ the associated homotopy category $\mathrm{ho}(\mathcal{C})$ is a coefficient system of categories rather than just a G-category. Likewise, Theorem III strengthens the (not so basic) observation that for a G-simplicial operad $\mathcal{O}$ the associated homotopy operad $\mathrm{ho}(\mathcal{O})$ is neither just a G-operad nor just a coefficient system of operads but rather the richer algebraic structure that we refer to as a "genuine equivariant operad".

1.1 Main results

We now discuss our main results.

Fixing a finite group G, we recall that $\mathsf{Op}^G(\mathcal{V}) = (\mathsf{Op}(\mathcal{V}))^G$ denotes G-objects in $\mathsf{Op}(\mathcal{V})$.

MAINEXIST1 THM

Theorem I. *Let* $(\mathcal{V}, \otimes)$ *denote either* $(\mathsf{sSet}, \times)$ *or* $(\mathsf{sSet}_*, \wedge)$.

Then there exists a model category structure on $\mathsf{Op}^G(\mathcal{V})$ *such that* $\mathcal{O} \to \mathcal{O}'$ *is a weak equivalence (resp. fibration) if all the maps*

$$\mathcal{O}(n)^\Gamma \to \mathcal{O}'(n)^\Gamma \tag{1.14}$$

GENEOPEQMT EQ

for $\Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}$, *are weak equivalences (fibrations) in* $\mathcal{V}$.

More generally, for $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ *with* $\mathcal{F}_n$ *an arbitrary collection of subgroups of* $G \times \Sigma_n$ *there exists a model category structure on* $\mathsf{Op}^G(\mathcal{V})$, *which we denote* $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$, *with weak equivalences (resp. fibrations) determined by* (1.14) *for* $\Gamma \in \mathcal{F}_n$.

Lastly, analogous semi-model category structures $\mathsf{Op}^G(\mathcal{V})$, $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ *exist provided that* $(\mathcal{V}, \otimes)$*: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers.*

We note that a similar result has also been proven by Gutiérrez-White in [17].

Theorem I is proven in §**??**. Condition (i) can be found in [21, Def. 2.1.17] while (ii) can be found in [21, Def. 4.2.6]. The additional conditions (iii) and (iv), which are less standard, are discussed in §**??** and §**??**, respectively. Further, by *semi-model category* we mean the notion in [32, Def. 2.2.1][2], which relaxes the definition of model structure by requiring that some of the axioms need only apply if the domains of certain cofibrations are cofibrant.

Our next result concerns the model structure on the new category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads introduced in this paper. Before stating the result, we must first outline how $\mathsf{Op}_G(\mathcal{V})$ itself is built. Firstly, the levels of each $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$, i.e. the H-sets in (1.13), are encoded by a category Σ_G of G-*corollas*, introduced in §**??**, which generalizes the usual

[2]We note that the role of $\mathcal{M}$ in [32, Def. 2.2.1] is auxiliary, as one is always free to replace $\mathcal{M}$ with the terminal category $*$, recovering the notion of a J-*semi-model structure (over* $*$*)* from [30, Def. 1]. In practice, the purpose of choosing $\mathcal{M} \neq *$ in [32] is that the existence of the semi-model structure on $\mathcal{D}$ therein is typically established via transfer from $\mathcal{M}$. We also caution that, when $\mathcal{M} \neq *$, the notion in [30] is more demanding than that in [32].

category Σ of finite sets and isomorphisms. We then define G-symmetric sequences by $\mathsf{Sym}_G(\mathcal{V}) = \mathcal{V}^{\Sigma_G^{op}}$ and, whenever $\mathcal{V}$ is a closed symmetric monoidal category with diagonals (cf. Remark **??**), we define in §**??** a *free genuine equivariant operad* monad $\mathbb{F}_G$ on $\mathsf{Sym}_G(\mathcal{V})$ whose algebras form the desired category $\mathsf{Op}_G(\mathcal{V})$.

Moreover, inspired by the analogues $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}} \rightleftarrows \mathsf{Top}^G_{\mathcal{F}}$ of the Elmendorf-Piacenza equivalence where $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}}$ are partial coefficient systems determined by a family $\mathcal{F}$, we show in §**??** that (a slight generalization of) Blumberg-Hill's indexing systems $\mathcal{F}$ give rise to sieves $\Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ and partial G-symmetric sequences $\mathsf{Sym}_{\mathcal{F}}(\mathcal{F}) = \mathcal{V}^{\Sigma_{\mathcal{F}}^{op}}$ which are suitably compatible with the monad $\mathbb{F}_G$, thus giving rise to categories $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ of *partial genuine equivariant operads.*

MAINEXIST2 THM

Theorem II. *Let $(\mathcal{V},\otimes)$ denote either $(\mathsf{sSet},\times)$ or $(\mathsf{sSet}_*,\wedge)$. Then the projective model structure on $\mathsf{Op}_G(\mathcal{V})$ exists. Explicitly, a map $\mathcal{P} \to \mathcal{P}'$ is a weak equivalence (resp. fibration) if all maps*

$$\mathcal{P}(C) \to \mathcal{P}'(C) \tag{1.15}$$

GENEQTHM EQ

are weak equivalences (fibrations) in $\mathcal{V}$ for each $C \in \Sigma_G$.

More generally, for $\mathcal{F}$ a weak indexing system, the projective model structure on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ exists. Explicitly, weak equivalences (resp. fibrations) are determined by (1.15) for $C \in \Sigma_{\mathcal{F}}$.

Lastly, analogous semi-model structures on $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ exist provided that $(\mathcal{V},\otimes)$: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals.

Theorem II is proven in §**??** in parallel with Theorem I. We note that the condition (v) that $(\mathcal{V},\otimes)$ has diagonals (cf. Remark **??**), which is not needed in Theorem I, is required to build the monad $\mathbb{F}_G$, and hence the categories $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$.

The following is our main result. The explicit formulas for the functors ι^*, ι_* are found in (**??**) (also, see Corollaries **??** and **??**).

QUILLENEQUIV THM

Theorem III. *Let $(\mathcal{V},\otimes)$ denote either $(\mathsf{sSet},\times)$ or $(\mathsf{sSet}_*,\wedge)$.*

Then the adjunctions, where in the more general rightmost case $\mathcal{F}$ is a weak indexing system,

$$\mathsf{Op}_G(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G(\mathcal{V}), \qquad \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}). \tag{1.16}$$

are Quillen equivalences.

Morover, analogous Quillen equivalences of semi-model structures[3] *$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \simeq \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ exist provided that $(\mathcal{V},\otimes)$: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals; (vi) has cartesian fixed points.*

Theorem III is proven in §**??**. Condition (vi), which is not needed in either of Theorems I,II is discussed in §**??**.

Lastly, our techniques also verify the main conjecture of BH15 [3], which we discuss in §**??**. Moreover, we note that our models for $N\mathcal{F}$-operads are given by explicit bar constructions.

TY_REAL_COR_MAIN

Corollary IV. *For $\mathcal{V} = \mathsf{sSet}$ or Top and $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ any weak indexing system, $N\mathcal{F}$-operads exist. That is, there exist explicit operads $\mathcal{O}$ such that*

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.17}$$

NFINFTY2 EQ

In particular, the map $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathcal{I}$ in BH15 [3, Cor. 5.6] is an equivalence of categories.

Moreover, if $\mathcal{O}'$ has fixed points as in (1.17) for some collection of graph subgroups $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$, then $\mathcal{F}$ must be a weak indexing system.

[3]See Fre09 [14, §12.1.8] for a precise definition.

1.2 Context, applications and future work

Models for equivariant operads with norm maps

This article is closely linked to the authors project in [27, 6, 8, 7], which culminates in the existence of a Quillen equivalence [7, Thm. I]

$$\mathsf{dSet}^G \rightleftarrows \mathsf{sOp}^G_\bullet. \qquad (1.18)$$

DSETSOP_EQ

Here $\mathsf{sOp}^G_\bullet$ is the model category of G-equivariant colored operads with norm maps (in sSet) given by [8, Thm. III], which is the colored extension of Theorem I, while $\mathsf{dSet}^G = \mathsf{Set}^{\Omega^{op}\times G}$ (where Ω is the category of trees) is the model category of G-∞-operads [27, Thm. 2.1], whose model structure is defined using the category Ω_G of G-trees.

The equivalence (1.18) generalizes the equivalence $\mathsf{dSet} \rightleftarrows \mathsf{sOp}$, [11, Thm. 8.15], which culminates the project of Cisinski, Moerdijk, Weiss in [24, 25, 9, 10, 11]. Crucially, we note that, while the underlying categories in (1.18) are obtained from those in [11, Thm. 8.15] by taking G-objects, the model structures in (1.18) are more subtle, needing the use of G-trees.

As a result, when generalizing the arguments and constructions in [24, 25, 9, 10, 11] one must often think in terms of genuine operads. For example, in [27, §8.2], to understand the homotopy theory of G-∞-operads in dSet^G, which are "G-operads with norm maps up to homotopy", one considers objects [27, Not. 8.11] in $\mathsf{dSet}_G = \mathsf{Set}^{\Omega_G^{op}}$ that are "genuine operads up to homotopy". Similarly, in [6, §5] one must consider the "homotopy genuine operad of a Segal space" [6, Def. 5.8] (notably, [6] uses an alternative description of genuine operads [6, Def. 3.35]; see Theorem B.1). Heuristically, this need for genuine operads comes from the observation that taking fixed points does not commute with taking homotopy/isomorphism classes link to below. As such, while the categories in (1.18) are "described in terms of $\Omega^{op} \times G$", any construction in those categories involving homotopy classes needs to be "described in terms of Ω_G^{op}" so as to account for norm maps.

Algebras over genuine operads

Just like usual operads, genuine operads admit a notion of algebra. A full formal definition of such algebras is forthcoming, but the following example illustrates the main idea.

Example 1.19. Let $G = \mathbb{Z}_{/2}$, $\mathcal{O} \in \mathsf{sOp}^G$ be a G-operad and $X \in \mathsf{sSet}^G$ be an $\mathcal{O}$-algebra. It is immediate that $\pi_0(X)$ is a $\pi_0(\mathcal{O})$-algebra while $\pi_0(X^G)$ is a $\pi_0(\mathcal{O}^G)$-algebra. However, the sets $\pi_0(X)$, $\pi_0(X^G)$ admit additional structure. Consider the following G-tree

(1.20) ALGTREEEX EQ

where we regard white (resp. black) nodes as corresponding to $\mathcal{O}$ (resp. X). Then, as in (1.12), the tree in (1.20) encodes a "multiplication" (note that $\Gamma_G \leq G \times \Sigma_2$ is the diagonal)

$$\begin{array}{ccc} \pi_0\left(\mathcal{O}(2)^{\Gamma_G}\right) \times \pi_0(X) & \longrightarrow & \pi_0\left(X^G\right) \\ ([p],[x]) & \longmapsto & [p(x, x+1)] \end{array} \qquad (1.21)$$

ALGTREECOM EQ

More generally, analogues of (1.21) are obtained for any G-tree which, as in (1.20), has a single white node topped by black 0-ary nodes.

Writing $\pi_0(\iota_* X) \in \mathsf{Set}^{\mathsf{O}_G^{op}}$ for the coefficient system $H \mapsto \pi_0(X^H)$, the multiplications (1.21) for such G-trees describe the algebra structure of $\pi_0(\iota_* X)$ over the genuine operad $\pi_0(\iota_* \mathcal{O}) \in \mathsf{Op}_G(\mathsf{Set})$, where ι_* is now as in Theorem III.

HERE

Remark 1.22. One way to formalize algebras over genuine operads is to adapt the composition product $\circ$ on symmetric sequences GJ94 [15, Def. 1.4] to a product on G-symmetric sequences $\mathsf{Sym}_G(\mathcal{V}) = \mathcal{V}^{\Sigma_G^{op}}$ (here Σ_G is the category of G-corollas, cf. Definition **??**). Loosely, this G-composition product is encoded by G-trees as in (1.20) but where black nodes need not be 0-ary.

The composition product approach has some advantages over the "free genuine operad monad on $\mathsf{Sym}_G(\mathcal{V})$" approach described in §**??**. Namely, one can both define genuine operads as "algebras over $\circ$" and algebras over genuine operads as "left modules in arity 0". However, it is hard to describe *free* (genuine) operads using the composition product, making such an approach poorly suited for proving Theorems I and II.

conjecture Elmendorf Piacenza for algebras

HERE

In order to simplify our discussion, this paper focuses exclusively on the theory of single colored (genuine) equivariant operads. As mentioned above, BP_HGOP [8, Theorem III] is an extension of Theorem I to the colored setting. Moreover, we conjecture the many-colored variants of Theorems II and III also hold, and intend to show this in upcoming work. We note, however, that an important new subtlety emerges in the equivariant setting: while usual equivariant colored operads have G-sets of objects, genuine equivariant colored operads will instead have *coefficient systems* of objects. Furthermore, one can show that there is a genuine equivariant colored operad encoding single-colored genuine equivariant operads, analogous to the colored operad encoding single-colored operads from e.g. GV12 [16].

The combinatorial structure underlies key steps in the proofs of the main results from each of Pe17,BP_edss,BP_HGOP,BP_TAS [27, 6, 8, 7], for example BP_edss [6, Def. 5.8] and BP_TAS [7, Prop. 4.47].

transition

Let $\mathcal{T}$ denote the colored operad whose algebras are operads, as defined in Gutierrez-Vogt.

In terms of our language, the colors of $\mathcal{T}$ are the arities $C \in \Sigma$, and an operation with signature $(C_1, \cdots, C_n; C_0)$ consists of a tree T, a permutation $\sigma \in \Sigma_n$ such that $V(T) = (C_{\sigma(i)})$ and a tall map $C_0 \to T$.

Their construction can be modified to construct a G-equivariant colored operad $\mathcal{T}_G^{fr}$, as follows. Operations are now the G-free corollas $C \in \Sigma_G^{fr}$ and operations with signature $(C_1, \cdots, C_n; C_0)$ are encoded by G-free trees T, a permutation $\sigma \in \Sigma_n$ such that such that $V_G(T) = (C_{\sigma(i)})$ and a tall rooted map $C_0 \to T$. Moreover, the G-action on $\mathcal{T}_G^{fr}$ is described on objects by $gC = g(C_i)_{i\in I} = (C_{gi})_{i\in I}$ and similar by $gT = g(T_i)_{i\in I} = (T_{gi})_{i\in I}$ on operations (some caution is needed concerning the permutation σ, since while the vertices of gT are conjugate to the vertices of T, they do not share the same order).

Note: The operad $\mathcal{T}_G^{fr}$ does *not* include "G-action operations". Instead, both $\mathcal{T}_G^{fr}$ and its algebras come with prescribed G-actions, which then impose equivariance conditions on the algebra structure maps.

One then has a map of G-colored operads (where G acts trivially on $\mathcal{T}$)

$$\mathcal{T} \to \mathcal{T}_G^{fr}$$

Moreover, upon forgetting the G-action, this map is fully faithful and essentially surjective. However, this map is not G-essentially surjective. More precisely, for any $* \neq H \leq G$ one has that the induced map on H-fixed points is not essentially surjective.

separation

$\mathcal{H}_n(X) = \mathcal{C}(X^{\otimes n}, X)$

For coefficient system consisting of fixed points, an algebra structure is a map $\mathcal{P} \to \iota_* \mathcal{H}_\bullet(X)$.

transition

Second, much of the machinery in this paper, presenting the free operad monad and its equivariant variants, can be applied in different contexts to define new equivariant algebraic notions. It is used in [8] to construct a many-color variant of Theorem I. Moreover, as previewed in [5], there is a subcategory $\mathsf{Sym}_G \to \mathsf{Op}_G$ of "genuine symmetric monoidal categories", constructed by blending the description of $\mathbb{F}_G$ used here reference with the monad $\Sigma \wr (-)$ encoding symmetric monoidal categories (cf. Remark ??). Our formal investigation of Sym_G is in progress. Furthermore, we conjecture that this bookkeeping can be applied to produce genuine equivariant analogues of properads and other algebraic theories.

transition

Finally, the comparison between simplicial G-operads sOp^G and the parametrized G-∞-operads of [1] factors most naturally through the category of genuine G-operads sOp_G. Non-equivariantly, this comparison is given by the operadic nerve functor $N^{\otimes}\colon \mathsf{sOp} \to \mathsf{Op}_\infty$ [22, Def. 2.1.1.3]. This construction first converts a simplicial operad $\mathcal{O}$ into a simplicial category $\mathcal{O}^{\otimes} \to \mathsf{F}_*$ equipped with a functor to the category of pointed finite sets, which acts like a fibration over a certain wide subcategory, and then takes the homotopy coherent nerve $hcN(\mathcal{O}^{\otimes}) = N^{\otimes}(\mathcal{O})$. This process motivates Lurie's definition of an ∞-operad in sSet.

In [5], the first-named author generalizes this process, by first building from a genuine equivariant operad $\mathcal{P}$ an simplicial category $\mathcal{P}^{\otimes} \to \mathsf{F}_{*,G}$ equipped with a partial fibration to the coefficient system of pointed finite G-sets (e.g. [5, Def. 3.3]), and then showing that the homotopy coherent nerve $hcN(\mathcal{P}^{\otimes}) = N^{\otimes \mathcal{P}}$ yields a G-∞-operad in the sense of [1]. Moreover, this transformation induces a functor on the categories of algebras $\mathsf{Alg}(\mathcal{O}) \to \mathsf{Alg}(N^{\otimes}\mathcal{O})$. This has been applied by [20] in the case where $\mathcal{O} = \mathcal{D}_V$ is the equivariant little disks operad over a G-representation V. Specifically, Horev shows [20, §3.9] that $N^{\otimes}(\mathcal{D}_V)$ is equivalent to the G-∞-operad of V-framed representations, which allows for $\mathcal{D}_V$-algebras to be used as input into his genuine equivariant factorization homology machinery, in particular producing new notions of equivariant topological Hochschild homology.

come back

A Transferring Kan extensions

TRANSKAN AP

The purpose of this appendix is to provide the somewhat long proof of Proposition ??, which is needed when repackaging free extensions of genuine equivariant operads in (??).

We start with a more detailed discussion of the realization functor $|-|$ defined by the adjunction

$$|-|\colon \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat} \colon (-)^{[\bullet]}$$

in Definition ??. More explicitly, one has

$$|\mathcal{I}_\bullet| = coeq\left(\coprod_{[n]\to[m]} [n] \times \mathcal{I}_m \rightrightarrows \coprod_{[n]} [n] \times \mathcal{I}_n \right). \tag{A.1}$$

REALDEF EQ

Example A.2. Any $\mathcal{I} \in \mathsf{Cat}$ induces objects $\mathcal{I}, \mathcal{I}_\bullet, \mathcal{I}^{[\bullet]} \in \mathsf{Cat}^{\Delta^{op}}$ where $\mathcal{I}$ is the constant simplicial object and $\mathcal{I}_\bullet$ is the nerve $N\mathcal{I}$ with each level regarded as a discrete category. It is straightforward to check that $|\mathcal{I}| \simeq |\mathcal{I}_\bullet| \simeq |\mathcal{I}^{[\bullet]}| \simeq \mathcal{I}$.

OBJGENREL LEMMA

Lemma A.3. *Given $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ one has an identification $Ob(|\mathcal{I}_\bullet|) \simeq Ob(\mathcal{I}_0)$. Furthermore, the arrows of $|\mathcal{I}_\bullet|$ are generated by the image of the arrows in $\mathcal{I}_0 \simeq \mathcal{I}_0 \times [0]$ and the image of the arrows in $[1] \times Ob(\mathcal{I}_1)$.*

For each $i_1 \in \mathcal{I}_1$, we will denote the arrow of $|\mathcal{I}_\bullet|$ induced by the arrow in $[1] \times \{i_1\}$ by

$$d_1(i_1) \xrightarrow{i_1} d_0(i_1).$$

Proof. We write $d_{\hat{k}}$, $d_{\hat{k},\hat{l}}$ for the simplicial operators induced by the maps $[0] \xrightarrow{0\mapsto k} [n]$, $[1] \xrightarrow{0\mapsto k, 1\mapsto l} [n]$ which can informally be thought of as the "composite of all faces other than d_k, d_l". Using (A.1) one has equivalence relations between the objects $(k, i_n) \in [n] \times \mathcal{I}_n$ and $(0, d_{\hat{k}}(i_n)) \in [0] \times \mathcal{I}_0$ and since for any generating relation $(k, i_n) \sim (l, i'_m)$ it is $d_{\hat{k}}(i_n) = d_{\hat{l}}(i'_m)$ the identification $\mathrm{Ob}(|\mathcal{I}_\bullet|) \simeq \mathrm{Ob}(\mathcal{I}_0)$ follows.

To verify the claim about generating arrows, note that any arrow of $[n] \times \mathcal{I}_n$ factors as

$$(k, i_n) \to (l, i_n) \xrightarrow{I_n} (l, i'_n) \tag{A.4}$$

FACTORIZATIONREAL EQ

for $I_n : i_n \to i'_n$ an arrow of $\mathcal{I}_n$. The $d_{\hat{l}}$ relation identifies the right arrow in (A.4) with $(0, d_{\hat{l}}(i_n)) \xrightarrow{d_{\hat{l}}(I_n)} (0, d_{\hat{l}}(i'_n))$ in $[0] \times \mathcal{I}_0$ while (if $k < l$) the $d_{\hat{k},\hat{l}}$ relation identifies the left arrow with $(0, d_{\hat{k},\hat{l}}(i_n)) \to (1, d_{\hat{k},\hat{l}}(i_n))$ in $[1] \times \mathcal{I}_1$. The result follows. □

Remark A.5. Given $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$, $\mathcal{C} \in \mathsf{Cat}$, the isomorphisms

$$\mathsf{Hom}_{\mathsf{Cat}}(|\mathcal{I}_\bullet|, \mathcal{C}) \simeq \mathsf{Hom}_{\mathsf{Cat}^{\Delta^{op}}}\left(\mathcal{I}_\bullet, \mathcal{C}^{[\bullet]}\right)$$

together with the fact that $\mathcal{C}^{[\bullet]}$ is 2-coskeletal show that $|\mathcal{I}_\bullet|$ is determined by the categories $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ and maps between them, i.e. by the truncation of formula (A.1) for $n, m \leq 2$.

Indeed, one can show that a sufficient set of generating relations for $|\mathcal{I}_\bullet|$ is given by: (i) the relations in $\mathcal{I}_0$ (including relations stating that identities of $\mathcal{I}_0$ are identities of $|\mathcal{I}_\bullet|$); (ii) relations stating that for each $i_0 \in \mathcal{I}_0$ the arrow $i_0 = d_1(s_0(i_0)) \xrightarrow{s_0(i_0)} d_1(s_0(i_0)) = i_0$ is an identity; (iii) for each arrow $I_1 : i_1 \to i'_1$ in $\mathcal{I}_1$ the relation that the square below commutes

$$\begin{tikzcd}
d_1(i_1) \arrow[r, "i_1"] \arrow[d, "d_1(I_1)"'] & d_0(i_1) \arrow[d, "d_0(I_1)"] \\
d_1(i'_1) \arrow[r, "i'_1"] & d_0(i'_1)
\end{tikzcd}$$

and; (iv) for each object $i_2 \in \mathcal{I}_2$ the relation that the following triangle commutes.

$$\begin{tikzcd}
d_{1,2}(i_2) \arrow[rr, "d_1(i_2)"] \arrow[dr, "d_2(i_2)"'] & & d_{0,1}(i_2) \\
& d_{0,2}(i_2) \arrow[ur, "d_0(i_2)"'] &
\end{tikzcd}$$

We now relate diagrams in the span categories of §?? with the Grothendieck constructions of Definition ??.

MPSPANREIN LEMMA

Lemma A.6. *Functors $F : \mathcal{D} \ltimes \mathcal{I}_\bullet \to \mathcal{C}$ are in bijection with lifts*

$$\begin{tikzcd}
& \mathsf{WSpan}^l(*, \mathcal{C}) \arrow[d, "\mathsf{fgt}"] \\
\mathcal{D} \arrow[ur, dashed, "\mathcal{I}^F_\bullet"] \arrow[r, "\mathcal{I}_\bullet"'] & \mathsf{Cat}.
\end{tikzcd}$$

where fgt *is the functor forgetting the maps to $*$ and $\mathcal{C}$.*

Proof. This is a matter of unpacking notation. The restrictions $F|_{\mathcal{I}_d}$ to the fibers $\mathcal{I}_d \hookrightarrow \mathcal{D} \ltimes \mathcal{I}_\bullet$ are precisely the functors $\mathcal{I}^F_d : \mathcal{I}_d \to \mathcal{C}$ describing $\mathcal{I}^F_\bullet(d)$.

Furthermore, the images $F((d, i) \to (d', f_*(i)))$ of the pushout arrows over a fixed arrow $f : d \to d'$ of $\mathcal{D}$ assemble to a natural transformation

$$\begin{tikzcd}
\mathcal{I}_d \arrow[dr, "I^F_d"] \arrow[dd, "f_*"'] & \\
& \mathcal{C} \arrow[dl, phantom, "\Swarrow"] \\
\mathcal{I}_{d'} \arrow[ur, "I^F_{d'}"'] &
\end{tikzcd}$$

which describes $\mathcal{I}^F_\bullet(f)$. One readily checks that the associativity and unitality conditions coincide. □

In the cases of interest we have $\mathcal{D} = \Delta^{op}$. The following is the key result in this section.

SOURCEFINAL PROP

Proposition A.7. *Let $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then there is a natural functor*

$$\Delta^{op} \ltimes \mathcal{I}_\bullet \xrightarrow{\;s\;} |\mathcal{I}_\bullet|\,.$$

Further, s is final.

Remark A.8. The s in the result above stands for *source*. This is because, for $\mathcal{I} \in \mathsf{Cat}$, the map $\Delta^{op} \ltimes \mathcal{I}^{[\bullet]} \to \left|\mathcal{I}^{[\bullet]}\right| \simeq \mathcal{I}$ is given by $s(i_0 \to \cdots \to i_n) = i_0$.

Proof. Recall that $|\mathcal{I}_\bullet|$ is the coequalizer (A.1). Given $(k, g_m) \in [n] \times \mathcal{I}_m$, we write $[k, g_m]$ for the corresponding object in $|\mathcal{I}_\bullet|$. To simplify notation, we write objects of $\mathcal{I}_n$ as i_n and implicitly assume that $[k, i_n]$ refers to the class of the object $(k, i_n) \in [n] \times \mathcal{I}_n$.

We define s on objects by $s([n], i_n) = [0, i_n]$ and on an arrow $(\phi, I_m) \colon (n, i_n) \to (m, i'_m)$ as the composite (note that $\phi \colon [m] \to [n]$ and $I_m \colon \phi^* i_n \to i_m$)

$$[0, i_n] \to [\phi(0), i_n] = [0, \phi^* i_n] \xrightarrow{I_m} [0, i'_m]. \tag{A.9}$$

TARGETDEFINITON EQ

To check compatibility with composition, the cases of a pair of either two fiber arrows (i.e. arrows where ϕ is the identity) or two pushforward arrows (i.e. arrows where I_m is the identity) are immediate from (A.9), hence we are left with the case $([n], i_n) \xrightarrow{I_n} ([n], i'_n) \to ([m], \phi^* i'_n)$ of a fiber arrow followed by a pushforward arrow. Noting that in $\Delta^{op} \ltimes \mathcal{I}_\bullet$ this composite can be rewritten as $([n], i_n) \to ([m], \phi^* i_n) \xrightarrow{\phi^* I_n} ([m], \phi^* i'_n)$ this amounts to checking that

$$\begin{tikzcd}
{[0, i_n]} \arrow[r] \arrow[d, "I_n"'] & {[\phi(0), i_n)]} \arrow[r, equal] \arrow[d, "I_n"'] & {[0, \phi^* i_n]} \arrow[d, "\phi^* I_n"] \\
{[0, i'_n]} \arrow[r] & {[\phi(0), i'_n]} \arrow[r, equal] & {[0, \phi^* i_n]}
\end{tikzcd}$$

commutes in $|\mathcal{I}_\bullet|$, which is the case since the left square is encoded by a square in $[n] \times \mathcal{I}_n$ and the right square is encoded by an arrow in $[m] \times \mathcal{I}_n$.

We now show that s is final. Fix $h \in \mathcal{I}_0$. We must check that $[0, h] \downarrow \Delta^{op} \ltimes \mathcal{I}_\bullet$ is connected. By Lemma A.3 any object in this undercategory has a description (not necessarily unique) as a pair

$$\left(([n], i_n)\,, [0, h] \xrightarrow{f_1} \cdots \xrightarrow{f_r} s([n], i_n) \right) \tag{A.10}$$

UNDERCATOB EQ

where each f_i is a generating arrow of $|\mathcal{I}_\bullet|$ induced by either an arrow I_0 of $\mathcal{I}_0$ or object $i_1 \in \mathcal{I}_1$. We will connect (A.10) to the canonical object $(([0], h), [0, h] = [0, h])$, arguing by induction on r. If $n \neq 0$, the map $d_{\hat{0}} \colon ([n], i_n) \to ([0], d^*_{\hat{0}}(i_n))$ and the fact that $s\left(d^*_{\hat{0}}\right) = id_{[0, d^*_{\hat{0}}(i_n)]}$ provides an arrow to an object with $n = 0$ without changing r. If $n = 0$, one can apply the induction hypothesis by lifting f_r to $\Delta^{op} \ltimes \mathcal{I}_\bullet$ according to one of two cases: (i) if f_r is induced by an arrow I_0 of $\mathcal{I}_0$, the lift of f_r is simply $([0], i'_0) \xrightarrow{I_0} ([0], i_0)$; (ii) if f_r is induced by $i_1 \in \mathcal{I}_1$ the lift is provided by the map $([1], i_1) \to ([0], d_0(i_1))$. □

DUALRESULTS REM

Remark A.11. The involution

$$\Delta \xrightarrow{\tau} \Delta$$

which sends $[n]$ to itself and d_i, s_i to d_{n-i}, s_{n-i} induces vertical isomorphisms

$$\begin{tikzcd}
\Delta^{op} \ltimes (\mathcal{I}_\bullet \circ \tau) \arrow[r, "s"] \arrow[d, "\simeq"'] & {|\mathcal{I}_\bullet \circ \tau|} \arrow[d, "\simeq"] \\
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[r, "t"'] & {|\mathcal{I}^{op}_\bullet|^{op}}
\end{tikzcd}$$

which reinterpret the "source" functor as what one might call the "target" functor, with $t([n], i_n) = [n, i_n]$ rather than $s([n], i_n) = [0, i_n]$. The target functor is thus also final.

Moreover, the source/target formulations of all the results that follow are equivalent.

In practice, we will need to know that the source s and target t satisfy a stronger finality condition with respect to left Kan extensions.

UNDERLEFTADJ LEM

Lemma A.12. *Let $\mathcal{J} \in \mathsf{Cat}$ be a small category and $j \in \mathcal{J}$. Then the under and over category functors*

$$\mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{(-)\downarrow j} \mathsf{Cat}, \qquad \mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{j\downarrow(-)} \mathsf{Cat}$$

are left adjoints, and hence preserve colimits.

Proof. The right adjoint to $(-) \downarrow j$, which we denote $(-)^{\downarrow j} \colon \mathsf{Cat} \to \mathsf{Cat} \downarrow \mathcal{J}$, is given on a category $\mathcal{C} \in \mathsf{Cat}$ by the Grothendieck construction $\mathcal{C}^{\downarrow j} = \mathcal{J} \ltimes \mathcal{C}^{\times \mathcal{J}(-,j)}$ for the functor

$$\begin{aligned} \mathcal{J} &\longrightarrow \mathsf{Cat} \\ k &\longmapsto \mathcal{C}^{\times \mathcal{J}(k,j)}. \end{aligned}$$

Given $(\mathcal{I} \xrightarrow{\pi} \mathcal{J}) \in (\mathsf{Cat} \downarrow \mathcal{J})$ and $\mathcal{C} \in \mathsf{Cat}$ we will show that functors $F \colon (\mathcal{I} \downarrow j) \to \mathcal{C}$ are in bijection with functors $\hat{F} \colon \mathcal{I} \to \mathcal{C}^{\downarrow j}$ over $\mathcal{J}$. Given F, we now describe the corresponding $\hat{F}$.

First, F associates to each object $(i, J \colon \pi(i) \to j)$ of $\mathcal{I} \downarrow j$ an object $F(i,J) \in \mathcal{C}$. Write $F_i \in \mathcal{C}^{\times \mathcal{J}(\pi(i),j)}$ for the assignment $J \mapsto F(i,J)$, i.e. $F_i(J) = F(i,J)$. On objects $i \in \mathcal{I}$ one then sets $\hat{F}(i) = (\pi(i), F_i)$.

Next, recall that arrows in $\mathcal{I} \downarrow j$ have the form $(i', J \circ \pi(I)) \to (i, J)$ for some arrow $I \colon i' \to i$ in $\mathcal{I}$. To each such arrow, F associates an arrow $F_{i'}(J \circ \pi(I)) \to F_i(J)$. Fixing I and allowing $J \in \mathcal{J}(\pi(i), j)$ to vary these arrows form a natural transformation $F_I \colon F_{i'} \circ \pi(I)^* \Rightarrow F_i$, where $\pi(I)^* \colon \mathcal{J}(\pi(i), j) \to \mathcal{J}(\pi(i'), j)$ denotes precomposition with $\pi(I)$. On arrows $I \colon i' \to i$ one now sets $\hat{F}(I) \colon (\pi(i'), F_{i'}) \to (\pi(i), F_i)$ to be $(\pi(I) \colon \pi(i') \to \pi(i), F_I \colon F_{i'} \circ \pi(I)^* \Rightarrow F_i)$.

It is clear that the procedures above relating the values of $F, \hat{F}$ on objects and arrows are invertible. One can readily check that the functoriality requirements on $F, \hat{F}$ match.

Noting that $j \downarrow (-)$ is the composite $\mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{(-)^{op}} \mathsf{Cat} \downarrow \mathcal{J}^{op} \xrightarrow{(-)\downarrow j} \mathsf{Cat} \xrightarrow{(-)^{op}} \mathsf{Cat}$ yields that its right adjoint is the composite $\mathsf{Cat} \xrightarrow{(-)^{op}} \mathsf{Cat} \xrightarrow{(-)^{\downarrow j}} \mathsf{Cat} \downarrow \mathcal{J}^{op} \xrightarrow{(-)^{op}} \mathsf{Cat} \downarrow \mathcal{J}$. ☐

URCELANFINAL COR

Corollary A.13. *Consider a map $\mathcal{I}_\bullet \to \mathcal{J}$ between $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ and a constant object $\mathcal{J} = \mathcal{J}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then the source and target maps*

$$\begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "s"] \arrow[dr] & & {|\mathcal{I}_\bullet|} \arrow[dl] \\ & \mathcal{J} & \end{tikzcd} \qquad \begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "t"] \arrow[dr] & & {|\mathcal{I}_\bullet^{op}|^{op}} \arrow[dl] \\ & \mathcal{J} & \end{tikzcd}$$

are Lan*-final over $\mathcal{J}$, i.e. the functors $s \downarrow j \colon (\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \to |\mathcal{I}_\bullet| \downarrow j$ are final for all $j \in \mathcal{J}$, and similarly for t.*

Proof. It is clear that $(\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \simeq \Delta^{op} \ltimes (\mathcal{I}_\bullet \downarrow j)$ while Lemma A.12 guarantees that, since $(-) \downarrow j$ is a left adjoint, $|\mathcal{I}_\bullet| \downarrow j \simeq |\mathcal{I}_\bullet \downarrow j|$. One thus reduces to Proposition A.7. ☐

We will require two additional straightforward lemmas.

TWISTING LEMMA

Lemma A.14. *Let $\mathcal{I}_\bullet^F \in \mathsf{WSpan}^l(*, \mathcal{C})^{\Delta^{op}}$ be such that the diagrams*

$$\begin{tikzcd} \mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \delta_i"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] \arrow[dr, phantom, shift right=3ex, "\Nearrow \sigma_j"] & \\ \mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C} \end{tikzcd} \tag{A.15}$$

IDENTSIMPRELSISO EQ

are given by natural isomorphisms for $0 < i \le n$*,* $0 \le j \le n$*. Then the functors* $\tilde{F}_n\colon \mathcal{I}_n \to \mathcal{C}$ *given by the composites*

$$\mathcal{I}_n \xrightarrow{d_{1,\cdots,n}} \mathcal{I}_0 \xrightarrow{F_0} \mathcal{C}$$

assemble to an object $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{WSpan}^l(*,\mathcal{C})^{\Delta^{op}}$ *which is isomorphic to* $\mathcal{I}_\bullet^F$ *and such that: (i)* $\mathcal{I}_\bullet^{\tilde{F}}$ *has the same operators* d_i, s_j*; (ii) in* $\mathcal{I}_\bullet^{\tilde{F}}$ *the diagrams (A.15) for* $0 < i \le n$*,* $0 \le j \le n$ *are strictly commutative; in* $\mathcal{I}_\bullet^{\tilde{F}}$ *the natural transformation associated to* d_0 *is the composite*

$$\begin{tikzcd}
\mathcal{I}_n \arrow[r, "d_{2,\cdots,n}"] \arrow[d, "d_0"'] & \mathcal{I}_1 \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, "F_1"] \arrow[dl, phantom, "\Leftarrow \delta_0"] & \mathcal{I}_0 \arrow[d, "F_0"] \arrow[dl, phantom, "\Leftarrow \delta_1"] \\
\mathcal{I}_{n-1} \arrow[r, "d_{1,\cdots,n-1}"] & \mathcal{I}_0 \arrow[r, "F_0"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{A.16}$$

Dually, if (A.15) are natural isomorphisms for $0 \le i < n$ *and* $0 \le j \le n$ *one can form* $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{WSpan}^l(*,\mathcal{C})^{\Delta^{op}}$ *such that the corresponding diagrams are strictly commutative.*

Proof. This follows by a straightforward verification. □

SOURCEFACT LEM **Lemma A.17.** *A (necessarily unique) factorization*

$$\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "F_\bullet"] \arrow[dr, "s"'] & & \mathcal{C} \\
& {|\mathcal{I}_\bullet|} \arrow[ur, dashed, "F"'] &
\end{tikzcd} \tag{A.18}$$ SOURCEFACT EQ

exists iff for the associated object $\mathcal{I}_\bullet \in \mathsf{WSpan}^l(*,\mathcal{C})^{\Delta^{op}}$ *(cf. Lemma A.6) all faces* d_i *for* $0 < i \le n$ *and degeneracies* s_j *for* $0 \le j \le n$ *are strictly commutative, i.e. they are given by diagrams*

$$\begin{tikzcd}
\mathcal{I}_n \arrow[dr, "F_n"] \arrow[d, "d_0"'] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] \arrow[u, phantom, shift left=4, "\Leftarrow \varphi_n"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[dr, "F_n"] \arrow[d, "d_i"'] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[dr, "F_n"] \arrow[d, "s_j"'] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C}
\end{tikzcd} \tag{A.19}$$ IDENTSIMPRELS EQ

Dually, a factorization through the target $t\colon \Delta^{op} \ltimes \mathcal{I}_\bullet \to |\mathcal{I}_\bullet^{op}|^{op}$ *exists iff the faces* d_i *and degeneracies* s_j *are strictly commutative for* $0 \le i < n$*,* $0 \le j \le n$*.*

Proof. For the "only if" direction, it suffices to note that s sends all pushout arrows of $\Delta^{op} \ltimes \mathcal{I}_\bullet$ for faces d_i, $0 < i \le n$ and degeneracies s_j, $0 \le j \le n$ to identities, yielding the required commutative diagrams in (A.19).

For the "if" direction, this will follow by building a functor $\mathcal{I}_\bullet \xrightarrow{\bar{F}_\bullet} \mathcal{C}^{[\bullet]}$ together with the naturality of the source map s (recall that $|\mathcal{C}^{[\bullet]}| \simeq \mathcal{C}$). We define $\bar{F}_n|_{k \to k+1}$ as the map

$$F_{n-k} d_{0,\cdots,k-1} \xrightarrow{\varphi_{n-k} d_{0,\cdots,k-1}} F_{n-k-1} d_{0,\cdots,k}. \tag{A.20}$$ EQUIVALENCEDEF EQ

The claim that $s \circ (\Delta^{op} \ltimes \bar{F})$ recovers the horizontal map in (A.18) is straightforward, hence the real task is to prove that (A.20) defines a map of simplicial objects. First, functoriality of the original $F_\bullet$ yields identities

$$\varphi_{n-1} d_i = \varphi_n, \ \ 1 < i \qquad \varphi_{n-1} d_1 = (\varphi_{n-1} d_0) \circ \varphi_n, \qquad \varphi_{n+1} s_i = \varphi_n, \ \ 0 < i, \qquad \varphi_{n+1} s_0 = id_{F_n}$$

Next, note that there is no ambiguity in writing simply $\varphi_{n-k} d_{0,\cdots,k-1}$ to denote the map (A.20). We now check that $\bar{F}_{n-1} d_i = d_i \bar{F}_n$, $0 \le i \le n$, which must be verified after restricting to each $k \to k+1$, $0 \le k \le n-2$. There are three cases, depending on i and k:

$(i < k+1)$ $\ \varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{0,\cdots,k}$;

$(i = k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_1 d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_0 \circ \varphi_{n-k}) d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_{0,\cdots,k}) \circ (\varphi_{n-k} d_{0,\cdots,k-1})$;

$(i > k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{i-k} d_{0,\cdots,k-1} = \varphi_{n-k} d_{0,\cdots,k-1}$.

The case of degeneracies is similar. □

proof of Proposition **??**. The result follows from the following string of identifications.

$$\begin{aligned}\lim_{\Delta}\left(\mathsf{Ran}_{A_n \to \Sigma_G} N_n\right) \simeq & \mathsf{Ran}_{\Delta \times \Sigma_G \to \Sigma_G}\left(\mathsf{Ran}_{A_n \to \Sigma_G} N_n\right) \simeq \\ \simeq & \mathsf{Ran}_{\Delta \times \Sigma_G \to \Sigma_G}\left(\mathsf{Ran}_{(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Delta \times \Sigma_G} N_\bullet\right) \simeq \\ \simeq & \mathsf{Ran}_{(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Sigma_G} N_\bullet \simeq \mathsf{Ran}_{(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Sigma_G} \tilde{N}_\bullet \simeq \mathsf{Ran}_{|A_\bullet| \to \Sigma_G} \tilde{N}\end{aligned}$$

The first step simply rewrites $\lim_\Delta$. The second step follows from Proposition **??** applied to the map $(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Delta \times \Sigma_G$ of Grothendieck fibrations over Δ, since for each $(n, C) \in \Delta \times \Sigma_G$ one has a natural identification between $(n, C) \downarrow_\Delta (\Delta^{op} \ltimes A_\bullet^{op})^{op}$ and $C \downarrow A_n$. The third step follows since iterated Kan extensions are again Kan extensions. The fourth step twists $N_\bullet$ as in Lemma A.14 to obtain $\tilde{N}_\bullet$ such that the d_i, s_j are given by strictly commutative diagrams for $0 \le i < n$, $0 \le j \le n$. Lastly, the final step uses Lemma A.17 to conclude that $\tilde{N}_\bullet$ factors through the target functor t, obtaining $\tilde{N}$, and then uses Corollary A.13 to conclude that the Kan extensions indeed coincide. □

B The nerve theorem

Our goal in this appendix is to prove the nerve theorem below, adapting MW08 [25, Prop. 5.3, Thm. 6.1]. Throughout we assume that the monoidal structure on $\mathcal{V}$ is the cartesian product.

NERVE THM

Theorem B.1. *There is a fully faithful nerve functor $\mathcal{N} \colon \mathsf{Op}_G(\mathcal{V}) \to \mathcal{V}^{\Omega_G^{op}}$ whose essential image consists of the pointed strict Segal objects, i.e. those $X \in \mathcal{V}^{\Omega_G^{op}}$ such that the natural maps*

$$X(T) \xrightarrow{\simeq} \prod_{v \in V_G(T)} X(T_v) \tag{B.2}$$

SEGCOND EQ

are isomorphisms for all $T \in \Omega_G$.

We will prove Theorem B.1 by building $\mathcal{N}$ in (B.12), describing its partial inverse in (B.18), then finishing the argument at the end of the appendix.

Remark B.3. In MW08 [25], which sets $\mathcal{V} = \mathsf{Set}$, $G = *$ and works with *colored* operads $\mathsf{Op}_\bullet$ (of sets), the nerve functor $\mathcal{N} \colon \mathsf{Op}_\bullet \to \mathsf{Set}^{\Omega^{op}}$ is defined by

$$(\mathcal{N}\mathcal{O})(T) = \mathsf{Op}_\bullet(\Omega(T), \mathcal{O}) \tag{B.4}$$

NEREASY EQ

where $\Omega(T)$ for $T \in \Omega$ is the colored operad described in MW07 [24, §3] (or after (1.10)).

However, since this paper does not discuss *colored* genuine operads (due to $\mathsf{Op}_G(\mathcal{V})$ being the single colored case), we can not obtain Theorem B.1 by directly adapting (B.4).

Remark B.5. The term "pointed" in Theorem B.1 is motivated by the fact if X satisfies (B.2) then it is $X(G/H \cdot \eta) = *$ for all $H \le G$, due to $V_G(G/H \cdot \eta) = ()$ being the empty tuple.

This pointedness reflect the fact that $\mathsf{Op}_G(\mathcal{V})$ includes only *single colored* genuine operads. In the multiple color setting, the Segal condition (B.2) needs to be modified BP_edss [6, Def. 3.35].

Our description of $\mathcal{N} \colon \mathsf{Op}_G(\mathcal{V}) \to \mathcal{V}^{\Omega_G^{op}}$ will make use of the monad N on $\mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$ in Definition **??**. Given $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$, so that $\iota\mathcal{P}$ is a N-algebra, consider the bar construction $B_\bullet = B_\bullet(N, N, \iota\mathcal{P}) = N^{n+1}\iota\mathcal{P}$, whose levels we denote as

$$\Sigma_G \leftarrow \Omega_G^n \xrightarrow{N_n^{\mathcal{P}}} \mathcal{V}^{op}. \tag{B.6}$$

BARLEVELS EQ

Ignoring the map to Σ_G, (B.6) determines a simplicial object in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$. Moreover, since the face maps d_i with $i < n$ are given by the multiplication $NN \Rightarrow N$ in (**??**), the opposite of this simplicial object (in $\mathsf{WSpan}^l(*, \mathcal{V})$) satisfies the dual case conditions in Lemma A.14. Thus, Lemma A.14 provides an isomorphic simplicial object $\tilde{N}^{\mathcal{P}}_n : \Omega^n_G \to \mathcal{V}^{op}$ satisfying the dual of the conditions in (A.19). Thus, by Lemma A.17, upon realization this induces a functor

$$|\Omega^n_G| = \Omega^{\mathsf{t}}_G \xrightarrow{\mathcal{N}\mathcal{P}} \mathcal{V}^{op} \tag{B.7}$$ TALLNER EQ

where $\Omega^{\mathsf{t}}_G \subset \Omega_G$ is the subcategory of tall maps. To define the nerve $\mathcal{N}$ in Theorem B.1, we must extend (B.7) to the entire category Ω_G. To do so, we enlarge the string categories Ω^n_G.

PLANSTRO DEF **Definition B.8.** Let $n \geq 0$. The category $\overline{\Omega}^n_G$ has objects the planar tall strings $(T_0 \to \cdots \to T_n) \in \Omega^n_G$ and arrows the diagrams (**??**) where the π_i are outer maps in each tree component.

Remark B.9. In contrasting Definitions **??** and B.8, recall that quotients are the maps which are isomorphisms in each tree component, so that $\Omega^n_G \subseteq \overline{\Omega}^n_G$.

Clearly the $\overline{\Omega}^n_G$ still form a simplicial object, i.e. one has operators $d_i : \overline{\Omega}^n_G \to \overline{\Omega}^{n-1}_G$ for $0 \leq i \leq n+1$ and $s_j : \Omega^n_G \to \overline{\Omega}^{n+1}_G$ for $0 \leq j \leq n$, though we caution that the $\overline{\Omega}^n_G$ have no augmentation to Σ_G nor extra degeneracies s_{-1}. Moreover, it is clear that $|\overline{\Omega}_G| = \Omega_G$.

More importantly, since maps that are outer in each tree component send vertices to vertices, one has that the formula in Notation **??** extends to define a functor

$$\overline{\Omega}^n_G \xrightarrow{V_G} \mathsf{F} \wr \Omega^{n-1}_G \tag{B.10}$$ VGDEF2 EQ

Note that (B.10) requires the full category F of finite sets rather than the subcategory F_s of surjections. By construction of N in Definition **??** one has that the functors in (B.6) extend to functors $N^{\mathcal{P}}_n : \overline{\Omega}^n_G \to \mathcal{V}^{op}$. Moreover, the natural transformations for the associated simplicial object in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$ all factor through one of the diagrams below,

$$\begin{tikzcd}
\overline{\Omega}^n_G \arrow[r, "V_G"] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[r, "\mathsf{F}\wr V_G"] \arrow[dl, Rightarrow, "\pi"] & \mathsf{F}^{\wr 2} \wr \Omega^{n-2}_G \arrow[d, "\sigma^0"] \\
\overline{\Omega}^{n-1}_G \arrow[rr, "V_G"'] & & \mathsf{F} \wr \Omega^{n-2}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Omega}^n_G \arrow[r, "V_G"] \arrow[d, "d_{i+1}"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[d, "\mathsf{F}\wr d_i"] \\
\overline{\Omega}^{n-1}_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Omega^{n-2}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Omega}^n_G \arrow[r, "V_G"] \arrow[d, "s_{j+1}"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[d, "\mathsf{F}\wr s_j"] \\
\overline{\Omega}^{n+1}_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Omega^{n}_G
\end{tikzcd} \tag{B.11}$$

so that $N^{\mathcal{P}}_n : \overline{\Omega}^n_G \to \mathcal{V}^{op}$ extends (B.6) as a simplicial object in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$. Thus, by Lemmas A.14 and A.17 one again obtains an isomorphic simplicial object $\tilde{N}^{\mathcal{P}}_n : \overline{\Omega}^n_G \to \mathcal{V}^{op}$ which, upon realization, extends (B.7) to obtain the desired nerve

$$|\overline{\Omega}^n_G| = \Omega_G \xrightarrow{\mathcal{N}\mathcal{P}} \mathcal{V}^{op}. \tag{B.12}$$ FULLNER EQ

We next describe the partial inverse to $\mathcal{N} : \mathsf{Op}_G(\mathcal{V}) \to \mathcal{V}^{\Omega^{op}_G}$. Choose $X : \Omega_G \to \mathcal{V}^{op}$ whose opposite satisfies the Segal condition (B.2). Letting $\mathcal{P}_X$ be the composite $\Sigma_G \to \Omega_G \to \mathcal{V}^{op}$, we will show that $\mathcal{P}_X$ is a genuine operad or, equivalently, that $\iota \mathcal{P}_X$ is a N-algebra. Throughout, we write $\overline{\Omega}^n_G \to \Omega_G$ for the target functor $(T_0 \to \cdots \to T_n) \mapsto T_n$ and denote by

$$\begin{tikzcd}
\overline{\Omega}^n_G \arrow[rr] \arrow[d, "d_n"'] & {} \arrow[d, phantom, "\Uparrow \varphi_n"] & \Omega_G \\
\overline{\Omega}^{n-1}_G \arrow[urr] & {} &
\end{tikzcd} \tag{B.13}$$ SIMP EQ

the natural transformation induced by $T_{n-1} \to T_n$. We now define spans X_n for $n \geq -1$ by

$$X_n = \left(\Sigma_G \leftarrow \Omega^n_G \to \overline{\Omega}^n_G \to \Omega_G \xrightarrow{X} \mathcal{V}^{op} \right). \tag{B.14}$$ XNSPANS EQ

Note that $X_{-1} = \iota\mathcal{P}_X$. Moreover, the transformations (B.13) make the X_n into a simplicial object in $\mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$. Next, note that one has natural transformations ρ_n

$$\begin{tikzcd}
\overline{\Omega}^n_G \arrow[r] \arrow[d, equal] & \Omega_G \arrow[r, "\delta^0"] & \mathsf{F}\wr\Omega_G \arrow[d, equal] \\
\overline{\Omega}^n_G \arrow[r, "V_G"'] \arrow[urr, phantom, "\Uparrow \rho_n"] & \mathsf{F}\wr\Omega_G^{n-1} \arrow[r] & \mathsf{F}\wr\Omega_G
\end{tikzcd} \tag{B.15}$$

THERHON EQ

which, on $(T_0 \to \cdots \to T_n) \in \Omega^n_G$, are given by the tuple map $(T_{n,v})_{v\in V_G(T_0)} \to (T_n)$ determined by the inclusions $T_{n,v} \to T_n$. Note that, by whiskering with the map $\mathsf{F}\wr\Omega_G \to \mathsf{F}\wr\mathcal{V}^{op} \xrightarrow{\Pi} \mathcal{V}^{op}$, ρ_n determines a map of spans which we likewise denote $\rho_n\colon X_n \to NX_{n-1}$.

SEGALCOND REM

Remark B.16. The Segal condition (B.2) holds iff $\rho_0\colon X_0 \to NX_{-1}$ is an isomorphism and iff the $\rho_n\colon X_n \to NX_{n-1}$ are isomorphisms for all $n \geq 0$.

Proposition B.17. *Suppose the opposite of $X\colon\Omega_G \to \mathcal{V}^{op}$ satisfies the Segal condition (B.2). Then the span X_{-1} in (B.14) is a N-algebra with multiplication*

$$NX_{-1} \xleftarrow[\simeq]{\rho_0} X_0 \xrightarrow{d_0} X_{-1}. \tag{B.18}$$

XMINUSMULT EQ

Proof. The required associativity and unitality conditions for (B.18) say that the outer paths in the diagrams below coincide (μ, η are the multiplication and unit of N, cf. Definition **??**).

$$\begin{tikzcd}
NNX_{-1} \arrow[rr, "\mu"] & & NX_{-1} \\
NX_0 \arrow[u, "N\rho_0", "\simeq"'] \arrow[d, "Nd_0"'] & X_1 \arrow[l, "\rho_1"', "\simeq"] \arrow[r, "d_0"] \arrow[d, "d_1"] & X_0 \arrow[u, "\rho_0"', "\simeq"] \arrow[d, "d_0"] \\
NX_{-1} & X_0 \arrow[l, "\rho_0"', "\simeq"] \arrow[r, "d_0"] & X_{-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_{-1} \arrow[r, "\eta"] \arrow[dr, "s_{-1}"] \arrow[ddr, equal] & NX_{-1} \\
& X_0 \arrow[u, "\rho_0"', "\simeq"] \arrow[d, "d_0"] \\
& X_{-1}
\end{tikzcd} \tag{B.19}$$

ALGCHECK EQ

One readily checks that the unitality diagram commutes. It remains to check that all squares in the associativity diagram commute. The case of the bottom right square is tautological. For the bottom left square note that, up to whiskering with $\mathsf{F}\wr\Omega_G \to \mathcal{V}^{op}$, the composites $X_1 \xrightarrow{d_1} X_0 \xrightarrow{\rho_0} NX_{-1}$ and $X_1 \xrightarrow{\rho_1} NX_0 \xrightarrow{Nd_0} NX_{-1}$ are induced by the diagrams below

$$\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r] \arrow[d, "d_1"'] & \Omega_G \arrow[r, "\delta^0"] \arrow[d, equal] & \mathsf{F}\wr\Omega_G \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[r] \arrow[d, equal] \arrow[ur, phantom, "\Uparrow \varphi_1"] & \Omega_G \arrow[r, "\delta^0"] & \mathsf{F}\wr\Omega_G \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[r, "V_G"'] \arrow[urr, phantom, "\Uparrow \rho_0"] & \mathsf{F}\wr\Sigma_G \arrow[r] & \mathsf{F}\wr\Omega_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Omega}^1_G \arrow[rr] \arrow[d, equal] & & \mathsf{F}\wr\Omega_G \arrow[d, equal] \\
\overline{\Omega}^1_G \arrow[r] \arrow[d, "d_1"'] \arrow[urr, phantom, "\Uparrow \rho_1"] & \mathsf{F}\wr\Omega^0_G \arrow[r] \arrow[d, "d_0"] & \mathsf{F}\wr\Omega_G \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[r, "V_G"'] & \mathsf{F}\wr\Sigma_G \arrow[r] \arrow[ur, phantom, "\Uparrow \varphi_0"] & \mathsf{F}\wr\Omega_G
\end{tikzcd} \tag{B.20}$$

That these composite natural transformations coincide is the observation that, on $(T_0 \to T_1) \in \Omega^1_G$, both compute the map $(T_{0,v})_{v\in V_G(T_0)} \to (T_1)$ given by the maps $T_{0,v} \to T_1$.

To show that the top square in (B.19) commutes, we first consider the composite $NX_0 \xrightarrow{N\rho_0} NNX_{-1} \xrightarrow{\mu} NX_{-1}$, which is given by the composite diagram below.

$$\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r] \arrow[d, equal] & \mathsf{F}\wr\Omega^0_G \arrow[r] \arrow[d, equal] & \mathsf{F}\wr\Omega_G \arrow[r, "\delta^1"] & \mathsf{F}^{\wr 2}\wr\Omega_G \arrow[r] \arrow[d, equal] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r, "\Pi"] \arrow[d, equal] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r, "\Pi"] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\overline{\Omega}^1_G \arrow[r] \arrow[d, "d_0"'] & \mathsf{F}\wr\Omega^0_G \arrow[r] \arrow[urr, phantom, "\Uparrow \rho_0"] & \mathsf{F}^{\wr 2}\wr\Sigma_G \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2}\wr\Omega_G \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r, "\Pi"] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r, "\Pi"] & \mathcal{V}^{op} \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[rr] \arrow[urr, phantom, "\Downarrow \pi_0"] & & \mathsf{F}\wr\Sigma_G \arrow[r] & \mathsf{F}\wr\Omega_G \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr, "\Pi"'] \arrow[ur, phantom, "\Uparrow \alpha"] & & \mathcal{V}^{op}
\end{tikzcd} \tag{B.21}$$

INTERM EQ

We now consider the following, where all terms (other than the additional $\mathsf{F} \wr \mathcal{V}^{op}$ term on the top row) retain their relative positions in (B.21).

$$\begin{tikzcd}
\mathsf{F} \wr \Omega_G \arrow[rr] \arrow[dr, "\delta^1"'] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[d, "\delta^1"] & & \\
& \mathsf{F}^{\wr 2} \wr \Omega_G \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] & \mathcal{V}^{op} \arrow[d, equal] \\
& \mathsf{F} \wr \Omega_G \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\Pi"'] \arrow[ur, phantom, "\alpha"] & & \mathcal{V}^{op}
\end{tikzcd} \tag{B.22}$$

By (??), the diagram above is the identity for the functor $\mathsf{F} \wr \Omega_G \to \mathsf{F} \wr \mathcal{V}^{op} \to \mathcal{V}^{op}$. As such, the composite natural transformation in (B.21) can be described by whiskering its 4 leftmost columns (equivalently, the 3 bottom rows of the left diagram in (B.23)) with $\mathsf{F} \wr \Omega_G \to \mathcal{V}^{op}$. It now follows that the composites $X_1 \xrightarrow{\rho_1} NX_0 \xrightarrow{N\rho_0} NNX_{-1} \xrightarrow{\mu} NX_{-1}$ and $X_1 \xrightarrow{d_0} X_0 \xrightarrow{\rho_0} NX_{-1}$ are obtained by whiskering the diagrams below with $\mathsf{F} \wr \Omega_G \to \mathcal{V}^{op}$.

$$\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r] \arrow[d, equal] & \Omega_G \arrow[r, "\delta^0"] & \mathsf{F} \wr \Omega_G \arrow[r, "\delta^1"] \arrow[d] & \mathsf{F}^{\wr 2} \wr \Omega_G \arrow[d, equal] \\
\overline{\Omega}^1_G \arrow[r] \arrow[d, equal] \arrow[ur, phantom, "\rho_1"] & \mathsf{F} \wr \Omega^0_G \arrow[r] \arrow[d, equal] & \mathsf{F} \wr \Omega_G \arrow[r, "\delta^1"] & \mathsf{F}^{\wr 2} \wr \Omega_G \arrow[d, equal] \\
\overline{\Omega}^1_G \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \arrow[r] \arrow[ur, phantom, "\rho_0"] & \mathsf{F}^{\wr 2} \wr \Sigma_G \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \Omega_G \arrow[d, "\sigma^0"] \\
\overline{\Omega}^0_G \arrow[rr] \arrow[ur, phantom, "\pi_0"] & & \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F} \wr \Omega_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r] \arrow[d, "d_0"'] & \Omega_G \arrow[r, "\delta^0"] \arrow[d, equal] & \mathsf{F} \wr \Omega_G \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[r] \arrow[d, equal] & \Omega_G \arrow[r, "\delta^0"] & \mathsf{F} \wr \Omega_G \arrow[d, equal] \\
\overline{\Omega}^0_G \arrow[r] \arrow[ur, phantom, "\rho_0"] & \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F} \wr \Omega_G
\end{tikzcd} \tag{B.23}$$

INTERM2 EQ

That the composites in (B.23) coincide follows since, on $(T_0 \to T_1) \in \Omega^1_G$, both compute the map $(T_{1,v})_{v \in V_G(T_1)} \to (T_1)$ whose components are given by the inclusions $T_{1,v} \to T_1$. $\square$

Proof of Theorem B.1. Let $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$ and $\mathcal{NP}\colon \Omega_G \to \mathcal{V}^{op}$ be as in (B.12). By the construction in Lemmas A.14 and A.17, the composites below coincide, where (d_i, ν_i) denote the simplicial operators of the simplicial object $N^{\mathcal{P}}_n$ in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$ discussed above (B.12).

$$\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r, "d_0"] \arrow[d, "d_1"'] & \overline{\Omega}^0_G \arrow[d] \arrow[dr, "N^{\mathcal{P}}_0"] & \\
\overline{\Omega}^0_G \arrow[r] \arrow[ur, phantom, "\varphi_1"] & \overline{\Omega}_G \arrow[r, "\mathcal{NP}"'] & \mathcal{V}^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\Omega}^1_G \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, "N^{\mathcal{P}}_1"] & \overline{\Omega}^0_G \arrow[d, "N^{\mathcal{P}}_0"] \\
\overline{\Omega}^0_G \arrow[r, "N^{\mathcal{P}}_0"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{B.24}$$

EXPREAL EQ

Setting $X = \mathcal{NP}$, the fact that the left triangle above commutes shows that $X_0 \xrightarrow{\rho_0} NX_{-1}$ is the identity (cf. (B.6) and Definition ??) so that, by Remark B.16, $\mathcal{NP}$ satisfies the Segal condition (B.2). Moreover, $d_0\colon X_0 \to X_{-1}$ is thus computed by the left diagram below, whose composite is the right diagram by the simplicial identities in $N^{\mathcal{P}}_n$.

$$\begin{tikzcd}
\Omega^0_G \arrow[r, "s_{-1}"] \arrow[d, "d_0"'] & \overline{\Omega}^1_G \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, "N^{\mathcal{P}}_1"] & \overline{\Omega}^0_G \arrow[d, "N^{\mathcal{P}}_0"] \\
\Sigma_G \arrow[r, "s_{-1}"'] & \overline{\Omega}^0_G \arrow[r, "N^{\mathcal{P}}_0"'] & \mathcal{V}^{op}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^0_G \arrow[d, "d_0"'] \arrow[dr, "N^{\mathcal{P}}_0"] & \\
\Sigma_G \arrow[r, "N^{\mathcal{P}}_{-1}"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{B.25}$$

BLABLEID EQ

But, by definition, this right diagram is simply the N-algebra multiplication of $\iota\mathcal{P}$, showing that (B.18) indeed inverts $\mathcal{NP}$ by recovering $\mathcal{P}$ with its genuine operad structure.

For the reverse claim characterizing the essential image, suppose the opposite of $X\colon \Omega_G \to \mathcal{V}^{op}$ satisfies the Segal condition (B.2), let X_{-1} be the N-algebra in (B.18) and $\mathcal{P}_X \in \mathsf{Op}_G(\mathcal{V})$ be so that $X_{-1} = \iota\mathcal{P}_X$. It remains to show that $X \simeq \mathcal{N}\mathcal{P}_X$. But now recall that $\mathcal{N}\mathcal{P}_X$ is built by realizing the simplicial object $\tilde{N}_n^{\mathcal{P}_X}\colon \overline{\Omega}_G^n \to \mathcal{V}^{op}$ in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$ built from $N_n^{\mathcal{P}_X}$ via Lemma A.14, so that $\tilde{N}_n^{\mathcal{P}_X}$ is as below, where the right side expands $N_0^{\mathcal{P}_X}$.

$$\overline{\Omega}_G^n \xrightarrow{d_{1,\cdots,n}} \overline{\Omega}_G^0 \xrightarrow{N_0^{\mathcal{P}_X}} \mathcal{V}^{op} \qquad \overline{\Omega}_G^n \xrightarrow{d_{1,\cdots,n}} \overline{\Omega}_G^0 \xrightarrow{V_G} \mathsf{F} \wr \Sigma_G \to \mathsf{F} \wr \Omega_G \xrightarrow{X} \mathsf{F} \wr \mathcal{V}^{op} \xrightarrow{\Pi} \mathcal{V}^{op}$$

Further writing X_n for the simplicial object $\overline{\Omega}_G^n \xrightarrow{d_{1,\cdots,n}} \overline{\Omega}_G^0 \to \Omega_G \xrightarrow{X} \mathcal{V}^{op}$ in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$ (this simply forgets structure in (B.14)), the natural transformations ρ_0 in (B.15) define an isomorphism of simplicial objects $\rho_0\colon X_n \xrightarrow{\simeq} \tilde{N}_n^{\mathcal{P}_X}$. Indeed, the non-trivial claim is that ρ_0 respects the natural transformation components of the differentials $d_1\colon X_1 \to X_0$ and $d_1\colon \tilde{N}_1^{\mathcal{P}_X} \to \tilde{N}_0^{\mathcal{P}_X}$. But the latter is computed by (B.24), and is thus the natural transformation component of the composite $NX_{-1} \xleftarrow{\mu} NNX_{-1} \xleftarrow{N\rho_0} NX_0 \xrightarrow{Nd_0} NX_{-1}$ (as ν_0, ν_1 in (B.24) are induced by μ and (B.18)), which by (B.19) has the same natural transformation component as $NX_{-1} \xleftarrow{\rho_0} X_0 \xleftarrow{d_0} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\rho_0} NX_{-1}$ (note that the natural transformation for d_0 is the identity). Thus $\rho_0\colon X_n \xrightarrow{\simeq} \tilde{N}_n^{\mathcal{P}_X}$ is indeed an isomorphism of simplicial objects in $\mathsf{WSpan}^r(*, \mathcal{V}^{op})$ so that, upon realization, we obtain the desired isomorphism $X \simeq \mathcal{N}\mathcal{P}_X$. $\square$

Glossary of Notation

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