Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

No content yet.

2 Planar and tall maps

2.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

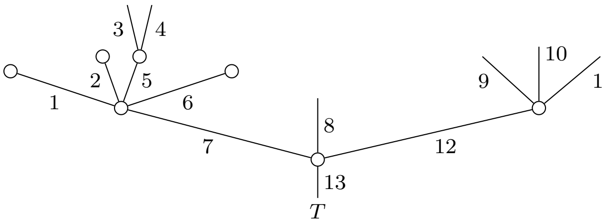
Our preferred model for trees will be that of broad posets first introduced by Weiss in We12 [2] and further worked out by the second author in Pe16b [1]. We now define planar structures in this context.

PLANARIZE DEF

Definition 2.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\leq_d$ to a total order $\leq_p$ such that:

- *Planar*: if $e \leq_p f$ and $e \nleq_d f$ then $g \leq_d f$ implies $e \leq_p g$.

Example 2.2. An example of a planar structure on a tree T follows, with $\leq_r$ encoded by the number labels.



(2.3) PLANAREX EQ

Intuitively, given a planar depiction of a tree T, $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 2.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT **Notation 2.4.** Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path of e*.

We will repeatedly use the following, which is a consequence of [1, Cor. 5.26].

INCOMPNOTOP **Lemma 2.5.** *If $e \leq_d f$, $e \leq_d f'$, then f, f' are $\leq_d$-comparable.*

INPUTPATHS PROP **Proposition 2.6.** *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \leq_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 2.5. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee e_i$. □

Notation 2.7. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^{\uparrow} \in T$ the predecessor of b in $I(e)$.

ECESSORPROP PROP **Proposition 2.8.** *Suppose e, f are $\leq_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^{\uparrow} \neq b_f^{\uparrow}$;

(b) $b_e^{\uparrow}, b_f^{\uparrow} \in b^{\uparrow}$. In fact $\{b_e^{\uparrow}\} = I(e) \cap b^{\uparrow}$, $\{b_f^{\uparrow}\} = I(f) \cap b^{\uparrow}$;

(c) if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^{\uparrow} = b_e^{\uparrow}$, $b_{f'}^{\uparrow} = b_f^{\uparrow}$.

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^{\uparrow} = b_f^{\uparrow}$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^{\uparrow}$, where uniqueness follows from Lemma 2.5. Choosing $a = e$ implies $I(e) \cap b^{\uparrow} = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. □

TERNARYJOIN PROP

Proposition 2.9. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$.*

Therefore, all ternary joins in $(T, \leq_d)$ are binary, i.e.

$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{2.10}$$

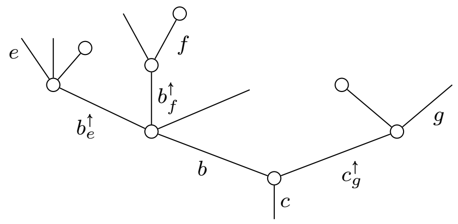
TERNJOIN EQ

for some $1 \leq i < j \leq 3$, and (2.10) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$ then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 2.8(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

Example 2.11. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c^{\uparrow}_e = c^{\uparrow}_f = b$.



Notation 2.12. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will usually abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order in S.

IZATIONCHAR PROP

Proposition 2.13. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \leq_p)\} & \longrightarrow & \prod_{(a^{\uparrow} \leq a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p |_{a^{\uparrow}}) \end{array} \tag{2.14}$$

PLANAR EQ

Proof. We will keep the setup of Proposition 2.8 throughout: e, f are $\leq_d$-incomparable edges and we write $b = e \vee f$.

We first show that (2.14) is injective, i.e. that the restrictions $\leq_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_e <_p b^{\uparrow}_f$, the relations $e \leq_d b^{\uparrow}_e <_p b^{\uparrow}_f \geq_d f$ and Definition 2.1 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_f <_p b^{\uparrow}_e$ then $f <_p e$. Thus $b^{\uparrow}_e <_p b^{\uparrow}_f \Leftrightarrow e <_p f$ and hence (2.14) is indeed injective.

To check that (2.14) is surjective, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $b^{\uparrow}_e < b^{\uparrow}_f$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\leq_p$ in the case $e' \leq_d e <_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 2.8(c). Transitivity of $\leq_p$ in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are $\leq_d$-incomparable, in which case one can apply 2.8(c) with the roles of f, f' reversed.

It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\leq_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 2.9, either (i) $e \vee f <_d c$, in which case Proposition 2.9 implies $c^{\uparrow}_e = c^{\uparrow}_f$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_e < c^{\uparrow}_f < c^{\uparrow}_g$ in $c^{\uparrow}$ so that $c^{\uparrow}_e \neq c^{\uparrow}_g$ and by Proposition 2.9 it is also $e \vee g = c$ and transitivity follows. □

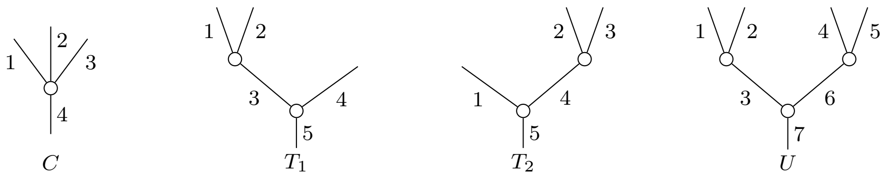
FORESTPLAN REM

Remark 2.15. Definition 2.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 2.13 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [1, Def. 7.44]), which adds a single edge η to F, serving as the (unique) root of $F \star \eta$.

When discussing the substitution procedure in §2.3 we will find it convenient to work with a model for the category Ω that possesses exactly one representative of each possible planar structure on each tree or, more precisely, such that the only isomorphisms preserving the planar structures are the identities. On the other hand, using such a model for Ω throughout would, among other issues, make the discussion of faces in §2.2 rather awkward. We now outline our conventions to address such issues.

Let Ω^p, the category of *planarized trees*, denote the category with objects pairs $T_{\leq_p} = (T, \leq_p)$ of trees together with a planar structure and morphisms the *underlying* maps of trees (so that the planar structures are ignored). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq_p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ and for which $\leq_p$ coincides with the canonical order.

STANDMODEL EX **Example 2.16.** Some examples of standard models, i.e. objects of Ω^s, follow (further, (2.3) can also be interpreted as such an example).



(2.17) PLANAROMEGAEX1 EQ

Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq_p} \in \Omega^p$ there is an obvious standard model $T^s_{\leq_p} \in \Omega^s$ given by replacing each edge by its order following $\leq_p$. Indeed, this defines a retraction $(-)^s : \Omega^p \to \Omega^s$ and a natural transformation $\sigma : id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is clearly unique).

PLANARCONV CON **Convention 2.18.** From now on, we will write simply Ω, Ω_G to denote the categories Ω^s, Ω^s_G of standard models (where planar structures are defined in the underlying forest as in Remark 2.15). Similarly O_G will denote the model O^s_G for the orbital category whose objects are the orbital G-sets whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$.

Therefore, whenever one of our constructions produces an object/diagram in Ω^p, Ω^p_G, O^p_G (of trees, G-trees, orbital G-sets with a planarization/total order) we will hence implicitly reinterpret it by using the standardization functor $(-)^s$.

Example 2.19. To illustrate our convention, we consider the trees in Example 2.16.

One has subfaces $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures. Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

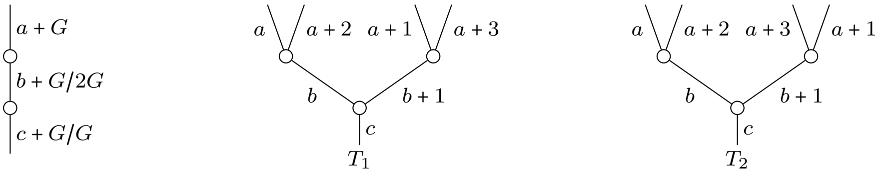
$$\begin{tikzcd} F_1 \arrow[rr, hook] \arrow[dr, hook] & & U \\ & F_2 \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & U \\ & T_1 \arrow[ur] & \end{tikzcd}$$

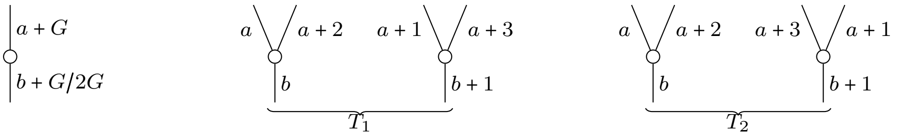
Similarly, let $\leq_{(12)}$ and $\leq_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs $1, 2$ and $4, 5$, so that one has objects $U_{\leq_{(12)}}$, $U_{\leq_{(45)}}$ in Ω^p. Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.

$$\begin{tikzcd} U \arrow[rr, "id"] \arrow[dr, "id"'] & & U_{\leq_{(45)}} \\ & U_{\leq_{(12)}} \arrow[ur, "id"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "(45)"] \arrow[dr, "(12)"'] & & U \\ & U \arrow[ur, "(12)(45)"'] & \end{tikzcd}$$

Example 2.20. An additional reason to leave the use of $(-)^s$ implicit is that when depicting G-trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, when $G = \mathbb{Z}_{/4}$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting of the two trees $T_1, T_2 \in \Omega_G$ on the right.





Definition 2.21. A morphism $S \xrightarrow{\varphi} T$ in Ω that is compatible with the planar structures $\leq_p$ is called a *planar map*.

More generally, a morphism $F \to G$ in the categories Φ, Φ^G, Ω^G of forests, G-forests, G-trees is called a *planar map* if it is an independent map (cf. [1, Def. 5.28]) compatible with the planar structures $\leq_p$.

Remark 2.22. The need for the independence condition is justified by [1, Lemma 5.33] and its converse, since non independent maps do not reflect $\leq_d$ inequalities.

We note that in the Ω_G case a map φ is independent iff φ does not factor through a non trivial quotient iff φ is injective on each edge orbit.

PLANARPULL EQ **Proposition 2.23.** *Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization*

$$F \xrightarrow{\simeq} \bar{F} \to G$$

such that $F \xrightarrow{\simeq} \bar{F}$ is an isomorphism and $\bar{F} \to G$ is planar.

Proof. We need to show that there is a unique planar structure $\leq_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of G ensures that for any vertex $e^{\uparrow} \leq e$ of F the edges in $\varphi(e^{\uparrow})$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(\underline{r}_F)$ are distinct. The result now follows from (the forest version of) Proposition 2.13: one simply orders each set $e^{\uparrow}$ and $\underline{r}_F$ according to its image. $\square$

PULLPLANAR REM **Remark 2.24.** Proposition 2.23 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As an example, in the notation of (2.17), consider the map $C \to T_1$ defined by $1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 5$.

2.2 Outer faces and tall maps

HERE

2.3 Substitution

References

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[2] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: https://arxiv.org/abs/1201.3987, 2012.