Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf-Piacenza type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

A surprising feature of topological algebra is that the category of (connected) topological commutative monoids is quite small, consisting only of products of Eilenberg-MacLane spaces (e.g [9, 4K.6]). Instead, the more interesting structures are those monoids which are commutative and associative only up to homotopy, and moreover up to "all higher homotopies". To capture these more subtle algebraic notions, Boardman-Vogt [4] and May [14] developed the theory of *operads*. Informally, an operad $\mathcal{O}$ consists of sets/spaces $\mathcal{O}(n)$ of "n-ary operations" carrying a Σ_n-action recording "reordering the inputs of the operations", and a suitable notion of "composition of operations". The purpose of the theory is then the study of "objects X with operations indexed by $\mathcal{O}$", refereed to as *algebras*, with the notions of monoid, commutative monoid, Lie algebra, algebra with a module, and more, all being recovered as algebras over some fixed operad in an appropriate category. Of special importance are the E_∞-operads, introduced by May in [14], which are "homotopical replacements" for the commutative operad and encode the aforementioned "commutative monoids up to homotopy". In particular, while an E_∞-algebra structure on X does not specify unique maps $X^n \to X$, it nonetheless specifies such maps "uniquely up to homotopy".

E_∞-operads are characterized by the homotopy type of their levels $\mathcal{O}(n)$: $\mathcal{O}$ is E_∞ iff each $\mathcal{O}(n)$ is Σ_n-free and contractible. That is, for each subgroup $\Gamma \leq \Sigma_n$ one has

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \Gamma = \{*\}, \\ \varnothing & \Gamma \neq \{*\}. \end{cases}$$

Notably, when studying the homotopy theory of operads in topological spaces the preferred notion of weak equivalence is usually that of "naive equivalence", with a map of operads $\mathcal{O} \to \mathcal{O}'$ deemed a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a weak equivalence of spaces upon forgetting the Σ_n-actions. In this context, E_∞-operads are then equivalent to the commutative operad Comm and, moreover, any cofibrant replacement of Comm is E_∞. However, naive equivalences differ from the equivalences in "genuine equivariant homotopy theory", where a map of G-spaces $X \to Y$ is deemed a G-equivalence only if the induced fix point maps $X^H \to Y^H$ are weak equivalences for all $H \leq G$. This contrast hints at a number of novel subtleties that appear in the study of equivariant operads, which we now discuss.

Firstly, noting that for a G-operad $\mathcal{O}$ (i.e. an operad $\mathcal{O}$ together with a G-action commuting with all the structure) the n-th level $\mathcal{O}(n)$ has a $G \times \Sigma_n$-action, one might guess that a map of G-operads $\mathcal{O} \to \mathcal{O}'$ should be called a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a G-equivalence after forgetting the Σ_n-actions, i.e. if the maps

$$\mathcal{O}(n)^H \xrightarrow{\sim} \mathcal{O}'(n)^H, \qquad H \leq G \leq G \times \Sigma_n, \tag{1.1}$$

are weak equivalences of spaces. However, the notion of equivalence suggested in (1.1) turns out to not be "genuine enough". To see why, we first consider a homotopical replacement for Comm using this theory: if one simply equips an E_∞-operad $\mathcal{O}$ with a trivial G-action, the resulting G-operad has fixed points for each subgroup $\Gamma \leq G \times \Sigma_n$ determined by

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \leq G, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.2}$$

However, as first noted by Costenoble-Waner in [6] in their study of equivariant infinite loop spaces, the *G-trivial E_∞-operads* of (1.2) do not provide the correct replacement of Comm in the G-equivariant context. Rather, that replacement is provided instead by the *G-E_∞-operads*, characterized by the fixed point conditions

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \cap \Sigma_n = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.3}$$

In contrasting (1.2) and (1.3), we note that the subgroups $\Gamma \leq G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \{*\}$ are readily shown to be precisely the graphs of partial homomorphisms $G \geq H \to \Sigma_n$, and that $\Gamma \leq G$ iff Γ is the graph of such a trivial homomorphism. However, the notion of weak equivalence described in (1.1) fails to distinguish (1.2) and (1.3), and indeed it is possible to build maps $\mathcal{O} \to \mathcal{O}'$ where $\mathcal{O}$ is a G-trivial E_∞-operad (as in (1.2)) and $\mathcal{O}'$ is a G-E_∞-operad (as in (1.3)). Therefore, in order to distinguish such operads, one needs to replace the notion of weak equivalence in (1.1) with the notion of *graph equivalence*, so that $\mathcal{O} \to \mathcal{O}'$ is considered a weak equivalence only if

$$\mathcal{O}(n)^{\Gamma} \xrightarrow{\sim} \mathcal{O}'(n)^{\Gamma}, \qquad \Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}. \tag{1.4}$$

are weak equivalences.

As mentioned above, the original evidence [6] that (1.3), rather than (1.2), provides the best up to homotopy replacement for Comm in the equivariant context comes from the study of equivariant infinite loop spaces. For our purposes, however, we instead focus on the perspective of Blumberg-Hill in [3], which concerns the Hill-Hopkins-Ravenel norm maps featured in the solution of the Kervaire invariant problem [11].

Given a G-spectrum R and finite G-set X with n elements, the corresponding *norm* is a G-spectrum $N^X R$ whose underlying spetrum is $R^{\wedge X} \simeq R^{\wedge n}$ but equipped with a mixed G-action that combines the actions on R and X in the natural way. Moreover, for any Comm-algebra R, i.e. strictly commutative G-ring spectrum, ring multiplication further induces so called *norm maps*

$$N^X R \to R. \tag{1.5}$$

Furthermore, by reducing structure on R the maps (1.5) are also defined when X is only a H-set for some subgroup $H \leq G$, and the maps (1.5) then satisfy a number of natural equivariance and associativity conditions. Crucially, we note that the more interesting of these associativity conditions involve H-sets for varying H (for an example packaged in operadic language, see (1.10) below).

The key observation at the source of the work in [3] is then that, operadically, norm maps are encoded by the graph fixed points as in (1.4). More explicitly, noting that a H-set X with n elements is encoded by a partial homomorphism $G \geq H \to \Sigma_n$, one obtains an associated graph subgroup $\Gamma_X \leq G \times \Sigma_n$, $\Gamma_X \cap \Sigma_n = \{*\}$, well defined up to conjugation. It then follows that for R an $\mathcal{O}$-algebra, maps of the form (1.5) are parametrized by the fixed point space $\mathcal{O}(n)^{\Gamma_X}$. The flaw with the G-trivial E_∞-operad described in (1.2) is then that it lacks all norms maps other than those for H-trivial X, thus lacking some of the data encoded by Comm. Further, from this perspective one may regard the more naive notion of weak equivalence in (1.1), according to which (1.2) and (1.3) are equivalent, as studying "operads without norm maps" (in the sense that equivalences ignore norm maps), while the equivalences (1.4) study "operads with norm maps".

Our first main result, Theorem I, establishes the existence of a model structure on operads with weak equivalences the graph equivalences of (1.4), though our analysis goes significantly further, again guided by Blumberg-Hill's work in [3].

The main novelty of [3] is the definition, for each finite group G, of a finite lattice of new types of equivariant operads, which they dub N_∞-operads. The minimum type of N_∞-operads is that of the G-trivial E_∞-operads in (1.2) while the maximal type is that of the G-E_∞-operads in (1.3). The remaining types, which interpolate between G-trivial E_∞ and

G-E_∞, can hence be thought of as encoding varying degrees of "up to homotopy equivariant commutativity". More concretely, each type of N_∞-operad is determined by a collection $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ where each $\mathcal{F}_n$ is itself a collection of graph subgroups of $G \times \Sigma_n$, with an operad $\mathcal{O}$ being called a $N\mathcal{F}$-operad if it satisfies the fixed point condition

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.6}$$

Such collections $\mathcal{F}$ are, however, far from arbitrary, with much of the work in [3, §3] spent cataloging a number of closure conditions that such $\mathcal{F}$ must satisfy. The simplest of these conditions state that each $\mathcal{F}_n$ is a *family*, i.e. closed under subgroups and conjugation. These first conditions, which are common in equivariant homotopy theory, are a simple consequence of each $\mathcal{O}(n)$ being a space. However, the remaining conditions, all of which involve $\mathcal{F}_n$ for various n simultaneously and are a consequence of operadic multiplication, are both novel and subtle. In loose terms, these conditions, which are more easily described in terms of the H-sets X associated to graph subgroups, concern closure of those under disjoint union, cartesian product, subobjects, and an entirely novel key condition called *self-induction*. The precise conditions are collected in [3, Def. 3.22], which also introduces the term *indexing system* for a $\mathcal{F}$ satisfying all such conditions. The main result of [3, §4] is then that whenever a $N\mathcal{F}$-operad $\mathcal{O}$ as in (1.6) exists, the associated collection $\mathcal{F}$ must be an indexing system. However, the converse statement, that given any indexing system $\mathcal{F}$ such an $\mathcal{O}$ can be produced, was left as a conjecture.

One of the key motivating goals of the present work was to verify this conjecture of Blumberg-Hill, which we obtain in Corollary V, and, moreover, to produce models of $N\mathcal{F}$-operads that are as explicit as possible.

To motivate our approach, we first recall the solution of a closely related but simpler problem: that of building universal spaces for families of subgroups. Given a family $\mathcal{F}$ of subgroups of G (i.e. a collection closed under conjugation and subgroups), a *universal space* X for $\mathcal{F}$, also called a $E\mathcal{F}$-space, is a space with fixed points X^H characterized just as in (1.6). In particular, whenever $\mathcal{O}$ is a $N\mathcal{F}$-operad, each $\mathcal{O}(n)$ is necessarily a $E\mathcal{F}_n$-space. The existence of $E\mathcal{F}$-spaces for any choice of the family $\mathcal{F}$ is best understood in light of Piacenza's classical result from [18] (with the key insight dating back to Elmendorf in [7]) stating that there is a Quillen equivalence (where O_G is the *orbit* category, formed by the G-sets G/H)

$$\begin{tikzcd} \mathsf{Top}^{\mathsf{O}_G^{op}} \arrow[r, shift left, "\iota^*"] & \mathsf{Top}^G \arrow[l, shift left, "\iota_*"] \\ (G/H \mapsto Y(G/H)) \arrow[r, mapsto] & Y(G) \\ (G/H \mapsto X^H) & X \arrow[l, mapsto] \end{tikzcd} \tag{1.7}$$

where the weak equivalences (and fibrations) on Top^G are detected on all fixed points and the weak equivalences (and fibrations) on the category $\mathsf{Top}^{\mathsf{O}_G^{op}}$ of *coefficient systems* are detected at each presheaf level. Noting that the fixed point characterization of $E\mathcal{F}$-spaces define an obvious object $\delta_\mathcal{F} \in \mathsf{Top}^{\mathsf{O}_G^{op}}$ by $\delta_\mathcal{F}(G/H) = *$ if $H \in \mathcal{F}$ and $\delta_\mathcal{F}(G/H) = \varnothing$ otherwise, $E\mathcal{F}$-spaces can then be built as $\iota^*(C\delta_\mathcal{F}) = C\delta_\mathcal{F}(G)$, where C denotes cofibrant replacement in $\mathsf{Top}^{\mathsf{O}_G^{op}}$. Moreover, we note that, as in [7], these cofibrant replacements can be built via explicit simplicial realizations.

The overarching goal of this paper is then that of proving the analogue of Elmendorf-Piacenza's Theorem (1.7) in the context of operads with norm maps (i.e. with equivalences as in (1.4)), which we state as our main result, Theorem III. However, in trying to formulate such a result one immediately runs into a fundamental issue: it is unclear which category should take the role of the coefficient systems $\mathsf{Top}^{\mathsf{O}_G^{op}}$ in that context. This last remark likely requires justification. Indeed, it may at first seem tempting to simply employ one of the known formal generalizations of Elmendorf-Piacenza's result (see, e.g. [22, Thm.

3.17]) which simply replace Top on either side of (1.7) with a more general model category $\mathcal{V}$. However, if one applies such a result when $\mathcal{V} = \mathsf{Op}$ to establish a Quillen equivalence $\mathsf{Op}^{\mathsf{O}_G^{op}} \rightleftarrows \mathsf{Op}^G$, the fact that the levels of each $\mathcal{P} \in \mathsf{Op}^{\mathsf{O}_G^{op}}$ correspond only to those fixed-point spaces appearing in (1.1) would require working in the context of operads *without* norm maps, and thereby forgo the ability to distinguish the many types of $N\mathcal{F}$-operads.

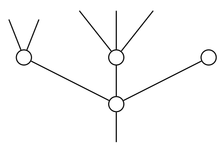
In order to work in the context of operads with norm maps we will need to replace $\mathsf{Top}^{\mathsf{O}_G^{op}}$ with a category Op_G of new algebraic objects we dub *genuine equivariant operads* (as opposed to (regular) equivariant operads Op^G). Each genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will consist of a list of spaces indexed in the same way as in (1.4) along with obvious restriction maps and, more importantly, suitable *composition maps*. Precisely identifying the required composition maps is one of the main challenges of this theory, and again we turn to [3] for motivation.

When analyzing the proofs of the results in [3, §4] concerning the closure properties for indexing systems $\mathcal{F}$ a common motif emerges: when performing an operadic composition

$$\begin{aligned} \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) &\longrightarrow \mathcal{O}(m_1 + \cdots + m_n) \\ (f, g_1, \cdots, g_n) &\longmapsto f(g_1, \cdots, g_n) \end{aligned} \tag{1.8}$$

careful choices of fixed point conditions on the operations $f, g_1, \cdots, g_n$ yield a fixed point condition on the composite operation $f(g_1, \cdots, g_n)$. The desired multiplication maps for a genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will then abstract such interactions between multiplication and fixed points for a equivariant operad $\mathcal{O} \in \mathsf{Op}^G$. However, these interactions can be challenging to write down explicitly and, indeed, the arguments in [3, §4] do not quite provide the sort of unified conceptual approach to these needed for our purposes. The cornerstone of the current work was then the joint discovery by the authors of such a conceptual framework: equivariant trees.

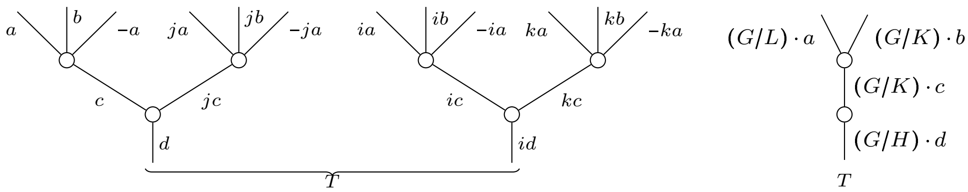
Non-equivariantly, it has long been known that the combinatorics of operadic composition is best visualized by means of tree diagrams. For instance, the tree



encodes the operadic composition

$$\mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0) \to \mathcal{O}(5)$$

where the inputs $\mathcal{O}(3), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(0)$ correspond to the nodes (i.e. circles) in the tree, with arity given by number of incoming edges (i.e. edges immediately above) and the output $\mathcal{O}(5)$ has arity given by counting leaves (i.e. edges at the top, not capped by a node). Similarly, the role of equivariant trees is, in the context of equivariant operads, to encode such operadic compositions together with fixed point compatibilities. A detailed introduction to equivariant trees can be found in [17, §4], where the second author develops the theory of equivariant dendroidal sets (which is a parallel approach to equivariant operads), though here we include a single representative example. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternionic units and $G \geq H \geq K \geq L$ denote the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$. There is then a G-tree T with *expanded representation* given by the two trees on the left below and *orbital representation* given by the (single) tree on the right.



(1.9) D6SMALLER EQ

We note that G acts on the expanded representation of T as indicated by the edge labels (so that the edges a, b, c, d have stabilizers L, K, K, H respectively), and the orbital representation is obtained by collapsing the edge orbits of the expanded representation. As explained in [17, Example 4.9], T then encodes the fact that for any equivariant operad $\mathcal{O} \in \mathsf{Op}^G$ the composition $\mathcal{O}(2) \times \mathcal{O}(3)^{\times 2} \to \mathcal{O}(6)$ restricts to a fixed point composition

$$\mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \to \mathcal{O}(H/L \amalg H/K)^H \tag{1.10}$$

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where $\mathcal{O}(X)$ for an H-set (resp. K-set) X denotes $\mathcal{O}(|X|)$ together with a suitably mixed H-action (K-action). We note that the inputs $\mathcal{O}(H/K)^H$, $\mathcal{O}(K/L \amalg K/K)^K$ in (1.10) correspond to the nodes of the orbital representation in (1.9), though in contrast to the non-equivariant case arity is now determined by both incoming and outgoing edge *orbits*, while the output $\mathcal{O}(H/L \amalg H/K)^H$ is similarly determined by both the leaf and root edge *orbits*. The existence of maps of the form (1.10) is essentially tantamount to the subtlest closure property for indexing systems $\mathcal{F}$, self-induction (cf. [3, Def. 3.20]), and similar tree descriptions exist for all other closure properties, as detailed by the second author in [17, §9].

We can now at last give a full informal description of the category Op_G featured in our main result, Theorem III. A genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ has levels $\mathcal{P}(X)$ for each H-set X, $H \le G$, that mimic the role of the fixed points $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$ for $\mathcal{O} \in \mathsf{Op}^G$. More explicitly, there are restriction maps $\mathcal{P}(X) \to \mathcal{P}(X|_K)$ for $K \le L$, isomorphisms $\mathcal{P}(X) \simeq \mathcal{P}(gX)$ where gX denotes the conjugate gHg^{-1}-set, and composition maps given by

$$\mathcal{P}(H/K) \times \mathcal{P}(K/L \amalg K/K) \to \mathcal{P}(H/L \amalg H/K)$$

in the case of the abstraction of (1.10), and more generally by

$$\begin{gathered}\mathcal{P}(H/K_1 \amalg \cdots \amalg H/K_n) \times \mathcal{P}(K_1/L_{11} \amalg \cdots \amalg K_1/L_{1m_1}) \times \cdots \times \mathcal{P}(K_n/L_{n1} \amalg \cdots \amalg K_n/L_{nm_n}) \\ \big\downarrow \\ \mathcal{P}(H/L_{11} \amalg \cdots \amalg H/L_{1m_1} \amalg \cdots \amalg H/L_{n1} \amalg \cdots \amalg H/L_{nm_n}).\end{gathered}$$

Lastly, these composition maps must satisfy associativity, unitality, compatibility with restriction maps, and equivariance conditions, as encoded by the theory of G-trees. Rather than making such compatibilities explicit, it will be more convenient and effective for our purposes to simply define genuine equivariant operads intrinsically in terms of G-trees.

1.1 Main results

For much of the discussion, our base model category will need to been sufficiently nice (see Definition 6.8). In particular, the strongest statements all hold when $\mathcal{V} = \mathsf{sSet}$.

Our first result, proved in §5.5, shows that G-operads can be endowed with a (semi) model structure encoding the theory of operads with norm maps. In fact, it shows more, in particular that the partially-genuine (semi) model structures also exist.

MAINEXIST1 THM

Theorem I. *Let $\mathcal{V}$ be a strongly cellular model category.*

Then there exists a semi model structure on $\mathsf{Op}^G(\mathcal{V})$ where weak equivalences and fibrations are determined by graph subgroup fixed points as in 1.4; that is, by the forgetful functor

$$\mathsf{Op}^G(\mathcal{V}) \xrightarrow{\text{fgt}} \mathsf{Sym}^G(\mathcal{V}) \cong \prod_n \mathcal{V}^{G\times\Sigma_n}_{\Gamma_n}.$$

More generally, for any collection $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ of sets $\mathcal{F}_n$ of arbitrary subgroups of $G \times \Sigma_n$ closed under conjugation, there exists an $\mathcal{F}$ semi model structure on $\mathsf{Op}^G(\mathcal{V})$, where weak equivalences and fibrations determined as in 1.6; that is, by the forgetful functor

$$\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\text{fgt}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \cong \prod_n \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}.$$

In the cases $\mathcal{V} = \mathsf{sSet}$ or sSet_*, these are actual model structures.

One important remark to make after this existence result is that these model structures need not be "well-behaved", specifically with respect to the forgetful functors preserving cofibrancy, unless the collections $\mathcal{F}$ are highly structured. The necessary condition, found in §6.4 that $\mathbb{F}$ is a *weak* indexing system, is also precisely the necessary condition from §4.4 to build the new algebraic structures of genuine equivariant operads.

In §5.6, we show these categories of algebras have a projective model structure.

MAINEXIST2 THM

Theorem II. *Let $\mathcal{V}$ be a strongly cellular model category with diagonals. Then the projective genuine semi model structure on $\mathsf{Op}_G(\mathcal{V})$ exists, and is determined by the forgetful functor*

$$\mathsf{Op}_G(\mathcal{V}) \xrightarrow{\text{fgt}} \mathsf{Sym}_G(\mathcal{V}).$$

More generally, for any weak indexing system $\Sigma_{\mathcal{F}} \subseteq \Sigma_G$, the projective genuine semi model structure on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ exists, and is determined by the forgetful functor

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\text{fgt}} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}).$$

In the cases $\mathcal{V} = \mathsf{sSet}$ or sSet_*, these are both actual model structures.

Our main result, proved in §6.4, says both notions of equivariant operads are Quillen equivalent.

QUILLENEQUIV THM

Theorem III. *Let $\mathcal{V}$ be strongly cellular with diagonals.*

Then the adjunction

$$\mathsf{Op}_G(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G(\mathcal{V}). \tag{1.11}$$

is a Quillen equivalence of semi model categories.

More generally, for $\mathcal{F}$ a weak indexing system, the analogue adjunction

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}). \tag{1.12}$$

are also Quillen equivalences.

If $\mathcal{V} = \mathsf{sSet}$ or sSet_*, then the above are equivalences of Quillen model categories.

We lastly return to the conjecture of [3], which we resolve in the affirmative using two different methods in §6.5. In fact, we show the existence of a larger class of operads, realizing any weak indexing system.

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Corollary IV. *For $\mathcal{V} = \mathsf{sSet}$ or Top and $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ any weak indexing system, $N\mathcal{F}$-operads exist. That is, there exist explicit operads $\mathcal{O}$ such that*

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.13}$$

In particular, $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathbb{I}$ is an equivalence of categories.

1.2 Outline

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There are multiple distinct steps along the path to proving our main result. Specifically, we need better understanding and control of the interplay between the subtle equivariant structures and the combinatorics which underly operads. To that end, we rebuild operads in a way which allows us to exploit the equivariant generalization Ω_G of the dendroidal category Ω discussed in [17].

We begin by observing that the free operad $\mathbb{F}X$ generated by a symmetric sequence $X \in \mathsf{Sym}(\mathcal{V})$, discussed in [21, 1, 19] etc., can be repackaged as a left Kan extension out of Ω.

$$\begin{tikzcd} \Omega^{op} \arrow[r, "N_X"] \arrow[d, "\mathsf{lr}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma^{op} \arrow[ur, "\mathbb{F}X"'] & \end{tikzcd} \qquad \begin{tikzcd} \Omega_G^{op} \arrow[r, "N_Y"] \arrow[d, "\mathsf{lr}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma_G^{op} \arrow[ur, "\mathbb{F}_G Y"'] & \end{tikzcd}$$

Here, lr is the "leaf-root" or "valence" functor, and N_X sends T to $\prod_{v \in V(T)} X(v)$.

This can be equivariantly generalized, replacing Ω and Σ with Ω_G and Σ_G, as seen on the right-hand-side. This yields our "genuine G-operad" monad $\mathbb{F}_G$, acting on the new category of G-sequences $Y \in \mathsf{Sym}_G(\mathcal{V}) = \mathcal{V}^{\Sigma_G^{op}}$. Now, having defined our notion of "coefficient systems", we can begin further analysis. As is often the case, we desire a clear understanding of free $\mathbb{F}_G$-extensions (5.1).

First, we produce multiple categorical manipulations of this functor. We modify both Ω_G and $\mathbb{F}_G$ to construct a description of coproducts of $\mathbb{F}_G$-algebras as a similar looking left Kan extension. Then, we"add in relations" found in our free $\mathbb{F}_G$-extension by inserting new maps into this modified Ω_G, via a realization of a simplicial category of strings of such maps. Finally, using this description, we can build a filtration of such free extensions.

Secondly, we consider the homotopical characteristics of $\mathbb{F}_G$, in particular a detailed study of the interactions of cofibrancy with composition. We also compare $\mathbb{F}$ and $\mathbb{F}_G$, categorically and homotopically, in order to display the Quillen equivalence.

We now give a brief overview of the remaining sections.

Section 2: Preliminaries. Here, we identify the major constructions we will be using throughout the paper, as well as establish notion and background material.

Section 3: Planar and tall maps. Our machery requires our trees to record an additional structure, namely a planarization. In this section, we define this notion for G-trees, and explore the notion of "substitution", a second operad-like structure on the class of trees, which many of our constructions employ.

Section 4: The genuine equivariant operad monad. Defining $\mathbb{F}_G$ as an endofunctor is fairly straightforward, as is defining composition and unit maps. However, showing that this data forms a monad requires a more subtle analysis. We define a more general monad N on the category of weak spans $\mathsf{WSpan}(\Sigma_G, \mathcal{V})$, and simplify the discussion through the flexibility of this more general category. It is also here that the concept of "planar strings" of maps are introduced.

Section 5: Free extensions and the existence of model structures. In this section, we develop the technology to produce free T-extensions for a particular class of monads T, utilizing the realizations of simplicial categories of planar strings. After providing a filtration of such constructions, we prove the existence of variuos model structures on our categories of algebras.

Section 6: Cofibrancy and Quillen equivalences. This section contains the bulk of the homotopical analysis. It is here that we show that weak indexing systems $\mathcal{F}$ provide sufficient structure so that their associated $\mathcal{F}$-model structures are well-behaved, in particular proving the desired Quillen equivalences. We end this section, and this paper, by providing two proofs of the N_∞ realization conjecture.

Much of the machinery in Sections 4 and 5 are built in large categorical generality, designed such that the intuitive descriptions of, for example, the free operad monad and operations on forests, are verified with strict categorical rigor; as such, they may be of broader interest.

1.3 Future Work

come back here

While this paper focuses soley on *single coloured* operads, we expect the theory to extend immediately to the coloured setting. One key difference is that coloured genuine G-operads will have a *coefficient system* of colours: evaluation at a G-corolla will include the additional data of a colour for each G-orbit of edges, and the isotropy of the colours must match that of its asscoatied edge. This feature is inspired by those seen in other models for G-operads, on which we now elaborate.

As stated earlier, the main goal of the authors' current project is to detail the full story of the homotopy theory of equivariant operads. This paper and [17] form the basis of a generalization of the work of Cisinski-Moerdijk-Weiss, describing the homotopy theory of non-equivariant operads, into the G-equivariant context. Importantly, the notion of "multicoloured genuine G-operads" will provide the natural (and most highly structured) target for the strictification, or "homotopy operad" construction, on the G-∞-operads of [17].

Moreover, sequels will extend the definitions of "dendroidal complete Segal spaces" and "Segal pre-operads", as well as the comparison Diagram $(\star)$ of Quillen equivalences from [5], reproduced here below, to the G-equivariant context.

$$\begin{tikzcd}
\mathsf{PreOp} \arrow[d] & \mathsf{sOp} \arrow[l] \arrow[d, "hcN_d"] \\
\mathsf{sdSet} & \mathsf{dSet} \arrow[l]
\end{tikzcd} \tag{$\star$}$$

CM_EQ

We note that, as in [17], there are both multiple possible *categorical* generalizations (i.e. dSet^G verses dSet_G, or Op^G verses Op_G) and multiple possible *model categorical* generalizations (i.e $\mathsf{dSet}^G_{\mathcal{F}}$ for varying weak indexing systems $\mathcal{F}$) for each corner of ($\star$). Analogous to the main result of this paper, we expect all compatible notions at each corner to be Quillen equivalent (i.e. $\mathsf{dSet}^G_{\mathcal{F}} \simeq_Q \mathsf{dSet}^{\mathcal{F}}_G \simeq_Q \mathsf{dSet}^{\mathcal{F}}_{\mathcal{F}}$). In future papers, we will study this precise question, as well as comparisons between the different models.

We end by interpreting these different models of G-operads, focusing on the right half of $\star$. We recall that, intuitively, ∞-operads can be thought of as operads where composition is "weakly defined", and similiarly G-coefficient systems as spaces with a "relaxed" fixed-point condition. In this fashion, genuine G-operads can be thought of as G-operads where composition is still rigidly defined, but with relaxed fixed-point conditions. Comparitively, the G-∞-operads of [17] have rigid fixed-point conditions but weak composition. The remaining missing link is then the suitable notion of G-∞-operad (that is, a suitable model structure) in the true pre-sheaf category dSet_G, in which both composition and fixed-point conditions are weak. This would yield an expanded equivariant right half of $\star$, given by the following.

$$\begin{tikzcd}
\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \arrow[r, "i_*"] \arrow[d, "hcN^G"] & \mathsf{Op}^{\mathcal{F}}_G(\mathcal{V}) \arrow[d, "hcN_G"'] \\
\mathsf{dSet}^G_{\mathcal{F}} \arrow[r, "i_*"'] \arrow[u, dashed, bend left, "Ho^G"] \arrow[ur, dashed, "Ho_G"] & \mathsf{dSet}^{\mathcal{F}}_G \arrow[u, dashed, bend right, "\mathrm{Ho}_G"']
\end{tikzcd}$$

We expect that all solid arrows are Quillen equivalences. However, analogous to the work of Cisinski-Moerdijk, direct vertical comparisons between G-operads and G-dendroidal sets will be challenging, inspiring futher need to explore other notions of G-∞-operads in the previous paragraph.

We also record our finding of an unexpected complication, that G-∞-operads in dSet^G themselves satisfy a rigid fixed point condition, but their *genuine* homotopy operad does *not*, in particular that

$$\mathrm{Ho}_G(X) \ncong \mathrm{Ho}_G i_*(X) \ncong i_* \mathrm{Ho}^G(X)$$

for G-∞-operads X.

A full exploration of this and the other above topics will be discussed in sequels.

Mention colored operad versions, equivalence with dendroidal sets

PRELIM_SECTION

2 Preliminaries

This section lists some elementary concepts and results that will be used throughout the paper, but may not be entirely standard.

2.1 Grothendieck fibrations

Recall that a functor $\pi: \mathcal{E} \to \mathcal{B}$ is called a *Grothendieck fibration* if for every arrow $f: b' \to b$ in $\mathcal{B}$ and $e \in \mathcal{E}$ such that $\pi(e) = b$, there exists a cartesian arrow $f^*e \to e$ lifting f, meaning that for any choice of solid arrows

$$\begin{tikzcd} e'' \arrow[rr] \arrow[dr, dashed, "\exists!"'] & & e \\ & f^*e \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} b'' \arrow[rr] \arrow[dr] & & b \\ & b' \arrow[ur, "f"'] & \end{tikzcd}$$

such that the rightmost diagram commutes and $e'' \to e$ lifts $b'' \to b$ there exists a unique dashed arrow $e'' \to f^*e$ lifting $b'' \to b'$ and making the leftmost diagram commute.

In most contexts the cartesian arrows $f^*e \to e$ are assumed to be defined only up to unique isomorphism, but in all examples considered in this paper we will in fact be able so identify preferred choices of cartesian arrows to which we will refer to as *pullbacks*. Moreover, pullbacks will be compatible with composition and units in the obvious way, i.e. $g^* f^* e = (fg)^* e$ and $id_b^* e = e$. As a terminological note, one sometimes encounters the term *split fibration* to refer to a Grothendieck fibration together with such a choice of pullbacks, though we will rarely have need to make the distinction outside of the present discussion.

A map of Grothendieck fibrations (resp. split fibrations) is then a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\delta"] \arrow[dr, "\pi"'] & & \bar{\mathcal{E}} \arrow[dl, "\bar{\pi}"] \\ & \mathcal{B} & \end{tikzcd} \tag{2.1}$$ GROTHFIBMAP EQ

such that δ preserves cartesian arrows (pullbacks).

There is a well known equivalence between Grothendieck fibrations over $\mathcal{B}$ and contravariant pseudo-functors $\mathcal{B}^{op} \to \mathsf{Cat}$ with split fibrations corresponding to (regular) contravariant functors. We now recall how this works in the (covariant) split case.

GROTHCONS DEF

Definition 2.2. Given a diagram category $\mathcal{B}$ and functor $\mathcal{E}_\bullet$

$$\begin{aligned} \mathcal{B} &\xrightarrow{\mathcal{E}_\bullet} \mathsf{Cat} \\ b &\longmapsto \mathcal{E}_b \end{aligned} \tag{2.3}$$

the (covariant) Grothendieck construction $\mathcal{B} \ltimes \mathcal{E}_\bullet$ has objects pairs (b, e) with $b \in \mathcal{B}$, $e \in \mathcal{E}_d$ and arrows $(b, e) \to (b', e')$ given by pairs

$$(f: b \to b', g: g_*(e) \to e'),$$

where $f_*\colon \mathcal{E}_b \to \mathcal{E}_{b'}$ is a shorthand for the functor $\mathcal{E}_\bullet(f)$.

Note that the chosen pushforward of (b,e) along $:b \to b'$ is then $(b', f_* e)$.

One useful property of Grothendieck constructions is that right Kan extensions can be computed using fibers, i.e., given a functor $F\colon \mathcal{E} \to \mathcal{V}$ into a complete category $\mathcal{V}$ one has

$$\mathsf{Ran}_\pi F(b) \simeq \mathsf{lim} F|_{b\downarrow\mathcal{E}} \simeq \mathsf{lim} F|_{\mathcal{E}_b} \tag{2.4}$$ FIBERKAN EQ

where the first identification is the usual pointwise formula for Kan extensions (cf. [13, X.3.1]) and the second identification follows by noting that due to the existence of cartesian arrows the fibers $\mathcal{E}_b$ are initial (in the sense of [13, IX.3]) in the undercategories $b \downarrow \mathcal{E}$. In fact, a little more is true: a choice of cartesian arrows yields a right adjoint to the inclusion $\mathcal{E}_b \hookrightarrow b \downarrow \mathcal{E}$, so that $\mathcal{E}_b$ is a coreflexive subcategory of $b \downarrow \mathcal{E}$, a well known sufficient condition for initiality. In practice, we will also need a generalization of the Kan extension formula (2.4) for maps of Grothendieck fibrations as in (2.1). Keeping the notation therein, given an $\bar{e} \in \bar{\mathcal{E}}$ we will write $\bar{e} \downarrow_\pi \mathcal{E} \hookrightarrow \bar{e} \downarrow \mathcal{E}$ for the full subcategory of those pairs $(e, f\colon \bar{e} \to \delta(e))$ such that $\bar{\pi}(f) = \bar{\pi}(\bar{e})$.

FIBERKANMAP PROP **Proposition 2.5.** *Given a map of Grothendieck fibrations each subcategory $\bar{e} \downarrow_\pi \mathcal{E}$ is an initial subcategory of $\bar{e} \downarrow \mathcal{E}$ so that for each functor $\mathcal{E} \to \mathcal{V}$ with $\mathcal{V}$ complete one has*

$$\mathsf{Ran}_\delta F(\bar{e}) \simeq \mathsf{lim} F|_{\bar{e}\downarrow\mathcal{E}} \simeq \mathsf{lim} F|_{\bar{e}\downarrow_\pi\mathcal{E}}. \tag{2.6}$$ FIBERKANMAP EQ

Proof. One readily checks that the assignment $(e, f\colon \bar{e} \to \delta(e)) \mapsto ((\pi(f)^* e, \bar{e} \to \delta\pi(f)^*(e)))$ (where $\delta\pi(f)^* = \bar{\pi}^*(f)\delta$) is right adjoint to the inclusion $\bar{e} \downarrow_\pi \mathcal{E} \hookrightarrow \bar{e} \downarrow \mathcal{E}$, so that the claim follows by coreflexivity (note that if not in the split case pullbacks may be chosen arbitrarily). ☐

We also record the following, the proof of which is straightforward.

GROTHSTAB PROP **Proposition 2.7.** *Suppose that $\mathcal{E} \to \mathcal{B}$ is a (split) Grothendieck fibration. Then so is the map of functor categories $\mathcal{E}^{\mathcal{C}} \to \mathcal{B}^{\mathcal{C}}$ for any category $\mathcal{C}$ as well as the map $\bar{\mathcal{E}} \to \bar{\mathcal{B}}$ in any pullback of categories*

$$\begin{tikzcd} \bar{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \bar{\mathcal{B}} \arrow[r] & \mathcal{B}. \end{tikzcd}$$

2.2 Wreath product over finite sets

Throughout we will let F denote the usual skeleton of the category of (ordered) finite sets and all set maps. Explicitly, its objects are the finite sets $\{1, 2, \cdots, n\}$ for $n \geq 0$.

Definition 2.8. For a category $\mathcal{C}$, we write $\mathsf{F} \wr \mathcal{C} = (\mathsf{F}^{op} \ltimes \mathcal{C}^{\times\bullet})^{op}$ for the opposite of the Grothendieck construction (cf. Definition 2.2) of the functor

$$\begin{aligned} \mathsf{F}^{op} &\longrightarrow \mathsf{Cat} \\ I &\longmapsto \mathcal{C}^{\times I} \end{aligned}$$

Explicitly, the objects of $\mathsf{F} \wr \mathcal{C}$ are tuples $(c_i)_{i\in I}$ and a map $(c_i)_{i\in I} \to (d_j)_{j\in J}$ consists of a pair

$$(\phi\colon I \to J, (f_i\colon c_i \to d_{\phi(i)})_{i\in I}),$$

henceforth abbreviated as $(\phi, (f_i))$.

Notation 2.9. Using the coproduct functor $\mathsf{F}^{\wr 2} = \mathsf{F}^{\wr\{0,1\}} = \mathsf{F} \wr \mathsf{F} \xrightarrow{\amalg} \mathsf{F}$ (where $\coprod_{i\in I} J_i$ is ordered lexicographically) and the simpleton $\{1\} \in \mathsf{F}$ one can regard the collection of categories $\mathsf{F}^{\wr n+1} \wr \mathcal{C} = \mathsf{F}^{\wr\{0,\cdots,n\}} \wr \mathcal{C}$ for $n \geq -1$ as a coaugmented cosimplicial object in Cat. As such, we will denote by

$$\delta^i\colon \mathsf{F}^{\wr n} \wr \mathcal{C} \to \mathsf{F}^{n+1} \wr \mathcal{C}, \qquad 0 \leq i \leq n$$

the cofaces obtained by inserting simpletons $\{1\} \in \mathsf{F}$ and by

$$\sigma^i \colon \mathsf{F}^{n+2} \wr \mathcal{C} \to \mathsf{F}^{n+1} \wr \mathcal{C}, \qquad 0 \le i \le n$$

the codegeneracies obtained by applying the coproduct $\mathsf{F}^{\wr 2} \xrightarrow{\amalg} \mathsf{F}$ to adjacent F coordinates.

Further, note that there are identifications $\mathsf{F} \wr \delta^i = \delta^{i+1}$, $\mathsf{F} \wr \sigma^i = \sigma^{i+1}$.

Remark 2.10. If $\mathcal{V}$ has all finite coproducts then injections and fold maps assemble into a functor as on the left below. Dually, if $\mathcal{V}$ has all finite products then projections and diagonals assemble into a functor as on the right.

$$\begin{array}{ccc} \mathsf{F} \wr \mathcal{V} & \xrightarrow{\quad\amalg\quad} & \mathcal{V} \\ (v_i)_{i \in I} & \longmapsto & \coprod_{i \in I} v_i \end{array} \qquad\qquad \begin{array}{ccc} (\mathsf{F} \wr \mathcal{V}^{op})^{op} & \xrightarrow{\quad\Pi\quad} & \mathcal{V} \\ (v_i)_{i \in I} & \longmapsto & \prod_{i \in I} v_i \end{array} \tag{2.11}$$

WREATHPROD EQ

Moreover, these functors satisfy a number of additional coherence conditions. Firstly, there is a natural isomorphism α as on the left below

$$\begin{tikzcd} \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F} \wr \mathcal{V} \arrow[rr, "\amalg"'] & & \mathcal{V} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{V} \arrow[d, "\delta^0"'] \arrow[dr, equal] & \\ \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"'] & \mathcal{V} \end{tikzcd} \tag{2.12}$$

COHER EQ

that encodes both changes in parenthesizing of coproducts and removal of initial objects (note that the empty tuple $()_{i \in \varnothing} \in \mathsf{F} \wr \mathcal{V}$ maps under $\coprod$ to an initial object of $\mathcal{V}$). Additionally, we are free to assume that the triangle on the right of (2.12) strictly commutes, i.e. that "unary coproducts" of simpletons (v) are given simply by v itself. α is then associative in the sense that the composite natural isomorphisms between the two functors $\mathsf{F}^{\wr 3} \wr \mathcal{V} \to \mathcal{V}$ in the diagrams below coincide.

$$\begin{tikzcd} \mathsf{F}^{\wr 3} \wr \mathcal{V} \arrow[r, "\mathsf{F}^{\wr 2}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^1"'] & \mathsf{F} \wr \mathcal{V} \arrow[rr, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & & \mathcal{V} \arrow[d, equal] \\ \mathsf{F} \wr \mathcal{V} \arrow[rrr, "\amalg"'] & & & \mathcal{V} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{F}^{\wr 3} \wr \mathcal{V} \arrow[r, "\mathsf{F}^{\wr 2}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, phantom, "\Leftarrow\mathsf{F}\wr\alpha"] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[rr, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^0"'] & & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F} \wr \mathcal{V} \arrow[rrr, "\amalg"'] & & & \mathcal{V} \end{tikzcd} \tag{2.13}$$

COHER2 EQ

Similarly, α is also unital in the sense that both of the following diagrams strictly commute or, more precisely, if the composite natural transformation in either diagram is the identify for the functor $\coprod \colon \mathsf{F} \wr \mathcal{V} \to \mathcal{V}$.

$$\begin{tikzcd} \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, "\delta^0"'] & \mathcal{V} \arrow[r, equal] \arrow[d, "\delta^0"'] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F} \wr \mathcal{V} \arrow[rr, "\amalg"'] & & \mathcal{V} \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathsf{F} \wr \mathcal{V} \arrow[r, equal] \arrow[d, "\delta^1"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] \arrow[d, phantom, "\Leftarrow\alpha"] & \mathcal{V} \arrow[d, equal] \\ \mathsf{F} \wr \mathcal{V} \arrow[rr, "\amalg"'] & & \mathcal{V} \end{tikzcd} \tag{2.14}$$

COHER3 EQ

SIGMA_WR_REM

Remark 2.15. More generally, if $\mathcal{V}$ is an arbitrary symmetric monoidal category, one instead has a functor $\Sigma \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ (where as usual $\Sigma \hookrightarrow \mathsf{F}$ denotes finite sets and isomorphisms) satisfying the obvious analogues of (2.12), (2.13), (2.14), as is readily shown using the standard coherence results for symmetric monoidal categories (moreover, we note that α itself encodes all associativity, unital and symmetry isomorphisms, with the right side of (2.12) and (2.14) being mere common sense desiderata for "unary products").

It is likely no surprise that the converse is also true, i.e. that a functor $\Sigma \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ satisfying the analogues of (2.12), (2.13), (2.14) endows $\mathcal{V}$ with a symmetric monoidal structure. We will however have no direct need to use this fact, and as such include only a few pointers concerning the associativity pentagon axiom (the hardest condition to check) that the interested reader may find useful. Firstly, it becomes convenient to write expressions such as $(A \otimes B) \otimes C$ instead as $(A \otimes B) \otimes (C)$, so as to encode notationally the fact that this is the image of $((A,B),(C)) \in \mathsf{F}^{\wr 2} \wr \mathcal{V}$ under the top map in (2.12). The associativity isomorphisms are hence given by the composites $(A \otimes B) \otimes (C) \xrightarrow{\simeq} A \otimes B \otimes C \xleftarrow{\simeq} (A) \otimes (B \otimes C)$ obtained by combining $\alpha_{((A,B),(C))}$ and $\alpha_{((A),(B,C))}$. The pentagon axiom is then checked by combining *six* instances of each of the squares in (2.13) (i.e. twelve squares total), most of which are obvious except for the fact that the $(A \otimes B) \otimes (C \otimes D)$ vertex of the pentagon contributes two pairs of squares rather than just one, with each pair corresponding to the two alternate expressions $((A \otimes B)) \otimes ((C) \otimes (D))$ and $((A) \otimes (B)) \otimes ((C \otimes D))$.

FINSURJ REM **Remark 2.16.** In lieu of the two previous remarks, and writing $\mathsf{F}_s \hookrightarrow \mathsf{F}$ for the subcategory of surjections, we define a *symmetric monoidal category with fold maps* as a category $\mathcal{V}$ together with a functor $\mathsf{F}_s \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ satisfying the analogues of (2.12), (2.13), (2.14). Further, the dual of such $\mathcal{V}$ is called a *symmetric monoidal category with diagonals.*

Maybe mention that if both folds and inclusions then cocartesian

Remark 2.17. Replacing F_s in the previous remark with the subcategory $\mathsf{F}_i \hookrightarrow \mathsf{F}$ of injections yields the notion of a *symmetric monoidal category with injection maps* or, dually, *symmetric monoidal category with diagonals.*

maybe mention that "cat with diagonals" is the same as semi-cartesian

We end this section by collecting some straightforward lemmas that will be used in §4.

FWRGROTH LEM **Lemma 2.18.** *If $\mathcal{E} \to \mathcal{B}$ a (split) Grothendieck fibration then so is $\mathsf{F} \wr \mathcal{E} \to \mathsf{F} \wr \mathcal{B}$.*

Moreover, if $\mathcal{E} \to \bar{\mathcal{E}}$ is a map of (split) Grothendieck fibrations over $\mathcal{B}$ then $\mathsf{F} \wr \mathcal{E} \to \mathsf{F} \wr \bar{\mathcal{E}}$ is a map of (split) Grothendieck fibrations over $\mathsf{F} \wr \mathcal{B}$.

Proof. Given a map $(\phi, (f_i)) \colon (b'_i)_{i \in I} \to (b_j)_{j \in J}$ in $\mathsf{F} \wr \mathcal{B}$ and object $(e_j)_{j \in J}$ one readily checks that its pullback can be defined by $(f^*_{\phi(i)} e_{\phi(i)})_{i \in I}$. ☐

WREATPRODLIM LEM **Lemma 2.19.** *Suppose that $\mathcal{V}$ is a bicomplete category such that coproducts commute with limits in each variable. If the leftmost diagram*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "k"'] & \mathcal{V} \\
\mathcal{D} \arrow[ur, "H"'] \arrow[u, phantom, shift left=3, "\Uparrow\eta"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathsf{F} \wr \mathcal{C} \arrow[r, "\mathsf{F}\wr G"] \arrow[d, "\mathsf{F}\wr k"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] & \mathcal{V} \\
\mathsf{F} \wr \mathcal{D} \arrow[ur, "\mathsf{F}\wr H"'] \arrow[urr, bend right, "\amalg \circ \mathsf{F}\wr H"'] \arrow[u, phantom, shift left=3, "\Uparrow \mathsf{F}\wr\eta"]
\end{tikzcd}
\tag{2.20}$$

WRRAN EQ

is a right Kan extension diagram then so is the composite of the rightmost diagram.

Dually, if in $\mathcal{E}$ products commute with colimits in each variable, and the leftmost diagram

$$\begin{tikzcd}
\mathcal{C}^{op} \arrow[r, "G"] \arrow[d, "k^{op}"'] & \mathcal{V} \\
\mathcal{D}^{op} \arrow[ur, "H"'] \arrow[u, phantom, shift left=3, "\Swarrow\epsilon"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
(\mathsf{F} \wr \mathcal{C})^{op} \arrow[r, "(\mathsf{F}\wr G^{op})^{op}"] \arrow[d, "(\mathsf{F}\wr k)^{op}"'] & (\mathsf{F} \wr \mathcal{V}^{op})^{op} \arrow[r, "\Pi"] & \mathcal{V} \\
(\mathsf{F} \wr \mathcal{D})^{op} \arrow[ur, "(\mathsf{F}\wr H^{op})^{op}"'] \arrow[urr, bend right, "\Pi \circ (\mathsf{F}\wr H^{op})^{op}"'] \arrow[u, phantom, shift left=3, "\Swarrow"]
\end{tikzcd}
\tag{2.21}$$

WRLAN EQ

is a left Kan extension diagram then so is the composite of the rightmost diagram.

Proof. Unpacking definitions using the pointwise formula for Kan extensions ([13, X.3.1]), the claim concerning (2.20) amounts to showing that for each $(d_i) \in \mathsf{F} \wr \mathcal{D}$ one has natural isomorphisms

$$\lim_{((d_i) \to (kc_j)) \in ((d_i) \downarrow \mathsf{F}\wr\mathcal{C})} \left(\coprod_j G(c_j) \right) \simeq \coprod_i \lim_{(d_i \to kc_i) \in d_i \downarrow \mathcal{C}} (G(c_i)) \,. \tag{2.22}$$

POINTKAN EQ

Proposition FIBERKANMAP PROP 2.5 now applies to the map $\mathsf{F} \wr \mathcal{C} \to \mathsf{F} \wr \mathcal{D}$ of Grothendieck fibrations over F and one readily checks that $(d_i) \downarrow_\pi \mathsf{F} \wr \mathcal{C} \simeq \prod_i (d_i \downarrow \mathcal{C})$ so that

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow\mathsf{F}\wr\mathcal{C})} \left(\coprod_j G(c_j) \right) \simeq \lim_{((d_i)\to(kc_i))\in\prod_i(d_i\downarrow\mathcal{D})} \left(\coprod_i G(c_i) \right)$$

and the isomorphisms (POINTKAN EQ 2.22) now follow from the assumption that coproducts commute with limits in each variable. $\square$

2.3 Monads and adjunctions

In §4 we will make use of the following straightforward results concerning the transfer of monads along adjunctions (note that L (resp. R) denotes the left (right) adjoint).

MONADADJ1 PROP

Proposition 2.23. *Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be an adjunction and T a monad on $\mathcal{D}$. Then*

(i) RTL is a monad and R induces a functor $R: \mathsf{Alg}_T(\mathcal{D}) \to \mathsf{Alg}_{RTL}(\mathcal{C})$;

(ii) if $LRTL \xrightarrow{\epsilon} TL$ is an isomorphism one further has an induced adjunction

$$L: \mathsf{Alg}_{RTL}(\mathcal{C}) \rightleftarrows \mathsf{Alg}_T(\mathcal{D}): R.$$

MONADADJ PROP

Proposition 2.24. *Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be an adjunction, T a monad on $\mathcal{C}$, and suppose further that*

$$LR \xrightarrow{\epsilon} id_{\mathcal{D}}, \qquad LT \xrightarrow{\eta} LTRL$$

are natural isomorphisms (so that in particular $\mathcal{D}$ is a reflexive subcategory of $\mathcal{C}$).

Then

(i) LTR is a monad, with multiplication and unit given by

$$LTRLTR \xrightarrow{\eta^{-1}} LTTR \to LTR, \qquad id_{\mathcal{D}} \xrightarrow{\epsilon^{-1}} LR \to LTR;$$

(ii) $d \in \mathcal{D}$ is a LTR-algebra iff Rd is a T-algebra;

(iii) there is an induced adjunction

$$L: \mathsf{Alg}_T(\mathcal{C}) \rightleftarrows \mathsf{Alg}_{LTR}(\mathcal{D}): R.$$

Any monad T on $\mathcal{C}$ induces obvious monads $T^{\times l}$ on $\mathcal{C}^{\times l}$. More generally, and letting I denote the identity monad, a partition $\{1, \cdots, l\} = \lambda_a \amalg \lambda_i$, which we denote by λ, determines a monad $T^{\times \lambda} = T^{\times \lambda_a} \times I^{\times \lambda_i}$ on $\mathcal{C}$. Here "a" stands for "active" and "i" for "inert".

Such monads satisfy a number of compatibility conditions. Firstly, if $\lambda'_a \subset \lambda_a$ there is a monad map $T^{\times \lambda'} \Rightarrow T^{\times \lambda}$, and we write $\lambda' \le \lambda$. Moreover, writing $\alpha^*: \mathcal{C}^{\times m} \to \mathcal{C}^{\times l}$ for the forgetful functor induced by a map $\alpha: \{1, \cdots, l\} \to \{1, \cdots, m\}$, one has an equality $T^{\times \alpha^* \lambda} \alpha^* = \alpha^* T^{\times \lambda}$, where $\alpha^* \lambda$ is the pullback partition. The following is straightforward.

MONADICFUN PROP

Proposition 2.25. *Suppose $\mathcal{C}$ has finite coproducts and write $\alpha_!: \mathcal{C}^{\times l} \to \mathcal{C}^{\times m}$ for the left adjoint of α^*. Then the map*

$$T^{\times \alpha^* \lambda} \Rightarrow \alpha^* T^{\times \lambda} \alpha_! \tag{2.26}$$

MONADFUNCTORALPHA EQ

adjoint to the identity $T^{\times \alpha^* \lambda} \alpha^* = \alpha^* T^{\times \lambda}$ is a map of monads on $\mathcal{C}^{\times l}$.

Hence, since $T^{\times \lambda} \alpha_!$ is a right $\alpha^* T^{\times \lambda} \alpha_!$-module, it is also a right $T^{\times \lambda'}$ whenever $\lambda' \le \alpha^* \lambda$. Finally, the natural map

$$\alpha_! T^{\times \alpha^* \lambda} \Rightarrow T^{\times \lambda} \alpha_! \tag{2.27}$$

MONADFUNCTORALPHADOU

is a map of right $T^{\times \alpha^* \lambda}$-modules, and thus also a map of right $T^{\times \lambda'}$-modules whenever $\lambda' \le \alpha^* \lambda$.

Remark 2.28. We unpack the content of (2.27) when $\alpha\colon\{1,\cdots,l\}\to *$ is the unique map to the simpleton $*$, in which case we write $\alpha_! = \coprod$. We thus have commutative diagrams

$$\begin{tikzcd}
\coprod_{j\in\lambda_a} TTA_j \amalg \coprod_{j\in\lambda_i} A_j \arrow[r] \arrow[d] & T\left(\coprod_{j\in\lambda_a} TA_j \amalg \coprod_{j\in\lambda_i} A_j\right) \arrow[d] \\
\coprod_{j\in\lambda_a} TA_j \amalg \coprod_{j\in\lambda_i} A_j \arrow[r] & T\left(\coprod_{j\in\lambda_a} A_j \amalg \coprod_{j\in\lambda_i} A_j\right)
\end{tikzcd} \tag{2.29}$$

for each collection $(A_j)_{j\in\underline{l}}$ in $\mathcal{C}$, where the vertical maps come from the right $T^{\times\lambda}$-module structure. Writing $\check{\amalg}$ for the coproduct of T-algebras and recalling the canonical identifications $\check{\coprod}_{k\in K}(TA_k) \simeq T\left(\coprod_{k\in K} A_k\right)$, (2.29) shows that the right $T^{\times\lambda}$-module structure on $T\circ\coprod$ codifies the multiplication maps

$$\check{\coprod}_{j\in\lambda_a} TTA_j \,\check{\amalg}\, \check{\coprod}_{j\in\lambda_i} TA_j \to \check{\coprod}_{j\in\lambda_a} TA_j \,\check{\amalg}\, \check{\coprod}_{j\in\lambda_i} TA_j.$$

3 Planar and tall maps

Throughout we will assume that the reader is familiar with the category Ω of trees. A good introduction to Ω is given by [15, §3], where arrows are described both via the "colored operad generated by a tree" and by identifying explicit generating arrows, called faces and degeneracies. Alternatively, Ω can also be described using the algebraic model of *broad posets* introduced by Weiss in [23] and further worked out by the second author in [17, §5]. This latter will be our "official" model, though a detailed understanding of broad posets is needed only to follow our formal discussion of planar structures in §3.1, and the reader willing to accept the results therein should be able to read the remainder of the paper.

Given a finite group G, there is also a category Ω_G of G-trees, jointly discovered by the authors and first discussed by the second author in [17, §4.3,§5.3], which we now recall. Firstly, we let Φ denote the category of forests, i.e. "formal coproducts of trees". A broad poset description of Φ is found in [17, §5.2], but here we prefer the alternative definition $\Phi = \mathsf{F} \wr \Omega$. The category of G-forests is then Φ^G, i.e. the category of G-objects in Φ. Identifying the G-orbit category as the subcategory $\mathsf{O}_G \hookrightarrow \mathsf{F}^G$ of those sets with transitive actions, Ω_G can then be described as given by the pullback of categories

$$\begin{tikzcd}
\Omega_G \arrow[r] \arrow[d] & \Phi^G \arrow[d] \\
\mathsf{O}_G \arrow[r] & \mathsf{F}^G,
\end{tikzcd} \tag{3.1}$$

which is a repackaging of [17, Prop. 5.46]. Explicitly, a G-tree T is then a tuple $T = (T_x)_{x\in X}$ with $X \in \mathsf{O}_G$ together with isomorphisms $T_x \to T_{gx}$ that are suitably associative and unital.

3.1 Planar structures

The specific model for the orbit category O_G used in (3.1) has extra structure not found in the usual model (i.e. that of the G-sets G/H for $H \leq G$), namely the fact that each $X \in \mathsf{O}_G$ comes with a canonical total order (the underlying set of X being one of the sets $\{1,\cdots,n\}$).

We will find it convenient to use a model of Ω with similar extra structure, given by planar structures on trees. Intuitively, a planar structure on a tree is the data of a planar representation of the tree, and definitions of *planar trees* along those lines are found throughout the literature. However, to allow for precise proofs of some key results concerning the interaction of planar structures with the maps in Ω (namely Propositions 3.28 and 3.47) we will instead use a combinatorial definition of planar structure in the context of broad posets.

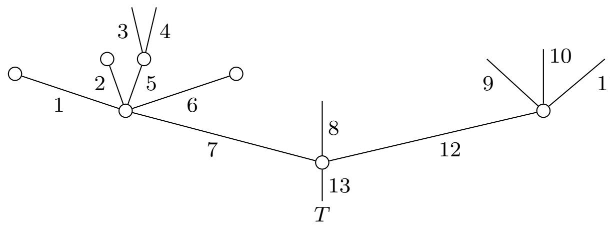
In what follows a tree will be a *dendroidally ordered broad poset* as in [23], [17, Def. 5.9].

PLANARIZE DEF

Definition 3.2. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\le_d$ to a total order $\le_p$ such that:

- *Planar*: if $e \le_p f$ and $e \nleq_d f$ then $g \le_d f$ implies $e \le_p g$.

Example 3.3. An example of a planar structure on a tree T follows, with $\le_p$ encoded by the number labels.



(3.4) PLANAREX EQ

Intuitively, given a planar depiction of a tree T, $e \le_d f$ holds when the downward path from e passes through f and $e \le_p f$ holds if either $e \le_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

It is visually clear that a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 3.2 is equivalent to such choices of total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT

Notation 3.5. Let $T \in \Omega$ be a tree and $e \in T$ an edge. We will denote

$$I(e) = \{f \in T : e \le_d f\}$$

and refer to this poset as the *input path of* e.

We will repeatedly use the following, which is a consequence of [17, Cor. 5.26].

INCOMPNOTOP

Lemma 3.6. *If $e \le_d f$, $e \le_d f'$, then f, f' are $\le_d$-comparable.*

INPUTPATHS PROP

Proposition 3.7. *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \le_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 3.6. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee_i e_i$. ☐

Notation 3.8. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^\uparrow \in T$ the predecessor of b in $I(e)$.

ECESSORPROP PROP

Proposition 3.9. *Suppose e, f are $\le_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^\uparrow \neq b_f^\uparrow$;

(b) $b_e^\uparrow, b_f^\uparrow \in b^\uparrow$. In fact $\{b_e^\uparrow\} = I(e) \cap b^\uparrow$, $\{b_f^\uparrow\} = I(f) \cap b^\uparrow$;

(c) if $e' \le_d e$, $f' \le_d f$ then $b = e' \vee f'$ and $b_{e'}^\uparrow = b_e^\uparrow$, $b_{f'}^\uparrow = b_f^\uparrow$.

Proof. (a) is immediate: the condition $e = b$ (resp. $f = b$) would imply $f \le_d e$ (resp. $e \le_d f$) while the condition $b_e^\uparrow = b_f^\uparrow$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \le_d b_a^* <_d b$ for some unique $b_a^* \in b^\uparrow$, where uniqueness follows from Lemma 3.6. Choosing $a = e$ implies $I(e) \cap b^\uparrow = \{b_e^*\}$ and letting a range over edges such that $e \le_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \le_d a <_d f$ and $e \le_d a$, and this would imply $e \le_d f$, contradicting our hypothesis. ☐

TERNARYJOIN PROP

Proposition 3.10. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$.*

Therefore, all ternary joins in $(T, \le_d)$ are binary, i.e.

$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{3.11}$$

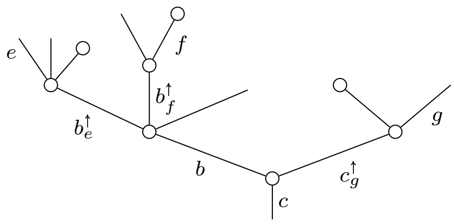
TERNJOIN EQ

for some $1 \le i < j \le 3$, and (3.11) fails for at most one choice of $1 \le i < j \le 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$ then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 3.9(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. $\square$

Example 3.12. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c^{\uparrow}_e = c^{\uparrow}_f = b$.



Notation 3.13. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will freely abuse notation by regarding its objects as pairs $(S, \le)$ where $\le$ is a total order in S.

IZATIONCHAR PROP

Proposition 3.14. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \le_p)\} & \xrightarrow{\simeq} & \prod_{(a^{\uparrow} \le a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \le_p & \longmapsto & (\le_p |_{a^{\uparrow}}) \end{array} \tag{3.15}$$

PLANAR EQ

Proof. We will keep the notation of Proposition 3.9 throughout: e, f are $\le_d$-incomparable edges and we write $b = e \vee f$.

We first show that (3.15) is injective, i.e. that the restrictions $\le_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_e <_p b^{\uparrow}_f$, the relations $e \le_d b^{\uparrow}_e <_p b^{\uparrow}_f \ge_d f$ and Definition 3.2 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_f <_p b^{\uparrow}_e$ then $f <_p e$. Thus $b^{\uparrow}_e <_p b^{\uparrow}_f \Leftrightarrow e <_p f$ and hence (3.15) is indeed injective.

To check that (3.15) is surjective, it suffices (recall that e, f are assumed $\le_d$-incomparable) to check that defining $e \le_p f$ to hold iff $b^{\uparrow}_e < b^{\uparrow}_f$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\le_p$ in the case $e' \le_d e <_p f$ and the planar condition, which is the case $e <_p f \ge_d f'$, follow from Proposition 3.9(c). Transitivity of $\le_p$ in the case $e <_p f \le_d f'$ follows since either $e \le_d f'$ or else e, f' are $\le_d$-incomparable, in which case one can apply 3.9(c) with the roles of f, f' reversed.

It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\le_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 3.10, either (i) $e \vee f <_d c$, in which case Proposition 3.10 implies $c = e \vee g$, $c^{\uparrow}_e = c^{\uparrow}_f$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_e < c^{\uparrow}_f < c^{\uparrow}_g$ in $c^{\uparrow}$ so that $c^{\uparrow}_e \neq c^{\uparrow}_g$ and by Proposition 3.10 it is also $c = e \vee g$ and transitivity follows. $\square$

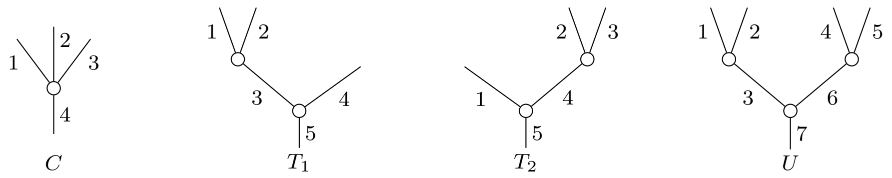
CLOSURE REM

Remark 3.16. Proposition 3.14 states in particular that $\le_p$ is the closure of the relations in $\le_d$ and on the vertices $a^{\uparrow}$ under the planar condition in Definition 3.2.

The discussion of the substitution procedure in §3.2 will be significantly simplified by working with a model for the category Ω possessing exactly one representative of each possible planar structure on each tree or, more precisely, if the only isomorphisms preserving the planar structure are the identities. On the other hand, exclusively using such a model for Ω throughout would, among other issues, make the discussion of faces in §3.2 rather awkward. We now outline our conventions to address such issues.

Let Ω^p, the category of *planarized trees*, denote the category with objects pairs $T_{\leq_p} = (T, \leq_p)$ of trees together with a planar structure and morphisms the *underlying* maps of trees (so that the planar structures are ignored). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq_p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ and for which $\leq_p$ coincides with the canonical order.

STANDMODEL EX **Example 3.17.** Some examples of standard models, i.e. objects of Ω^s, follow (further, (3.4) can also be interpreted as such an example).

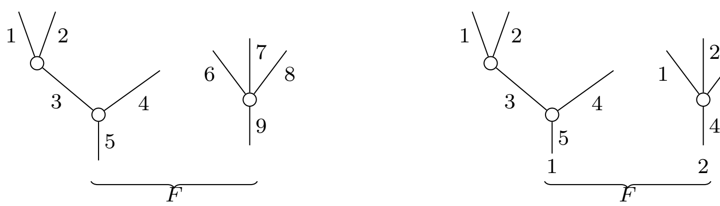


(3.18) PLANAROMEGAEX1 EQ

Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq_p} \in \Omega^p$ there is an obvious standard model $T^s_{\leq_p} \in \Omega^s$ given by replacing each edge by its order following $\leq_p$. Indeed, this defines a retraction $(-)^s \colon \Omega^p \to \Omega^s$ and a natural transformation $\sigma \colon id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is unique characterized by this property).

FORESTPLAN REM **Remark 3.19.** Definition 3.2 readily extends to the broad poset definition of forests $F \in \Phi$ in [17, Def. 5.27], with the analogue of Proposition 3.14 then stating that a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. There are thus two competing notions of standard forests: the [17, Def. 5.27] model Φ^s whose objects are planar forest structures on one of the standard sets $\{1, \cdots, n\}$ and (following the discussion at the start of §3) the model $\mathsf{F} \wr \Omega^s$, whose objects are tuples, indexed by a standard set, of planar tree structures on standard sets. An illustration follows.



(3.20) TWOPLAFORCONV EQ

It is however clear that there is a *canonical* isomorphism $\Phi^s \simeq \mathsf{F} \wr \Omega^s$ (with the two side of (3.20) representing the same planar forest). Moreover, while the similarly defined categories Φ^p and $\mathsf{F} \wr \Omega^p$ are only equivalent (rather than isomorphic), their retractions onto $\Phi^s \simeq \mathsf{F} \wr \Omega^s$ are compatible, and we will thus henceforth not distinguish between Φ^s and $\mathsf{F} \wr \Omega^s$.

PLANARCONV CON **Convention 3.21.** From now on we write simply Ω, Ω_G to denote the categories Ω^s, Ω^s_G of standard models (where planar structures are defined in the underlying forest as in Remark 3.19). Therefore, whenever a construction produces an object/diagram in Ω^p, Ω^p_G (of trees, G-trees) we always implicitly reinterpret it by using the standardization functor $(-)^s$.

Similarly, any finite set or orbital finite G-set together with a total order is implicitly reinterpreted as an object in F, O^p_G.

Example 3.22. To illustrate our convention, consider the trees in Example 3.17.

There are subtrees $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures.

Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

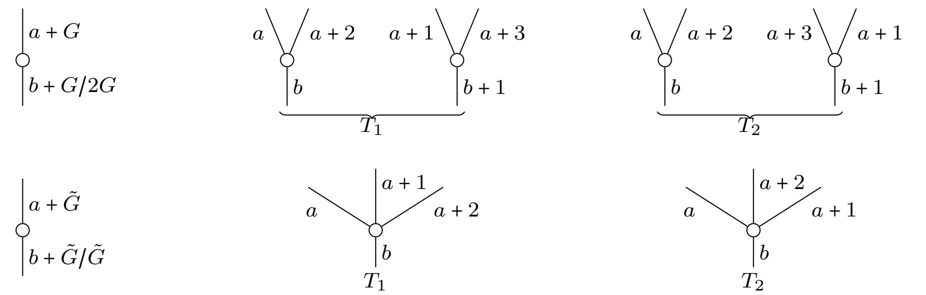
$$\begin{tikzcd} F_1 \arrow[rr, hook] \arrow[dr, hook] & & U \\ & F_2 \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & U \\ & T_1 \arrow[ur] & \end{tikzcd}$$

Similarly, let $\leq_{(12)}$ and $\leq_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs $1, 2$ and $4, 5$, so that one has objects $U_{\leq_{(12)}}$, $U_{\leq_{(45)}}$ in Ω^p. Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.

$$\begin{tikzcd} U \arrow[rr, "id"] \arrow[dr, "id"'] & & U_{\leq_{(45)}} \\ & U_{\leq_{(12)}} \arrow[ur, "id"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "(45)"] \arrow[dr, "(12)"'] & & U \\ & U \arrow[ur, "(12)(45)"'] & \end{tikzcd}$$

Example 3.23. An additional reason to leave the use of $(-)^s$ implicit as detailed in Convention 3.21 is that when depicting G-trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, when $G = \mathbb{Z}_{/4}$, $\tilde{G} = \mathbb{Z}_{/3}$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting only of the two trees $T_1, T_2 \in \Omega_G$ on the right.



In general, isomorphism classes are of course far bigger. The interested reader may show that there are $3 \cdot 3! \cdot 2 \cdot 3! \cdot 3!$ trees in the isomorphism class of the tree depicted in (1.9).

The attentive reader may have noted that it follows from Proposition 2.7 that both vertical maps in (3.1) are split Grothendieck fibrations. We now introduce some terminology.

ROOTPULL DEF **Definition 3.24.** The map $\mathsf{r}: \Omega_G \to \mathsf{O}_G$ in (3.1) is called the *root functor*.

Further, fiber maps (i.e. maps inducing identities, i.e. ordered bijections, on $\mathsf{r}(-)$) are called *rooted maps* and pullbacks with respect to r are called *root pullbacks*.

To motivate the terminology, note first that unpacking definitions shows that $\mathsf{r}(T)$ is the ordered set of tree components of $T \in \Omega_G$, which coincides with the ordered set of roots. The exact choice of name is further meant to accentuate the connection with another key functor which we call the *leaf-root functor*, described in §3.3.

Further, unpacking the construction in (3.1), one sees that the pullback of the G-tree $T = (T_x)_{x \in X}$ with structure maps $T_x \to T_{gx}$ along the map $\varphi: Y \to X$ is simply the G-tree $(T_{\varphi(y)})_{y \in Y}$ with structure maps $T_{\varphi(y)} \to T_{g\varphi(y)} = T_{\varphi(gy)}$.

ROOTPULL EX **Example 3.25.** Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$, $H = \langle j \rangle$ and $K = \langle -1 \rangle$. Figure 1 illustrates the pullbacks of two G-trees T and S along the twist map $\tau: G/H \to G/H$ and the unique map $\pi: G/H \to G/G$ (or, more precisely, noting that in our model the underlying set of G/H is actually $\{1, 2\}$, τ is the permutation (12)). We note that the stabilizers of a, b, c are $\{1\}, K, H$ for T and K, H, G for S. The top depictions of $\tau^* T$, $\pi^*(S)$ then use the edge orbit generators suggested by T, S while the bottom depictions choose generators that are minimal with regard to the planar structure, so that in $\tau^* T$ it is $d = ic$, $e = ib$, $f = ia$ and in $\pi^* S$ it is $e = ib'$, $d = ia'$.

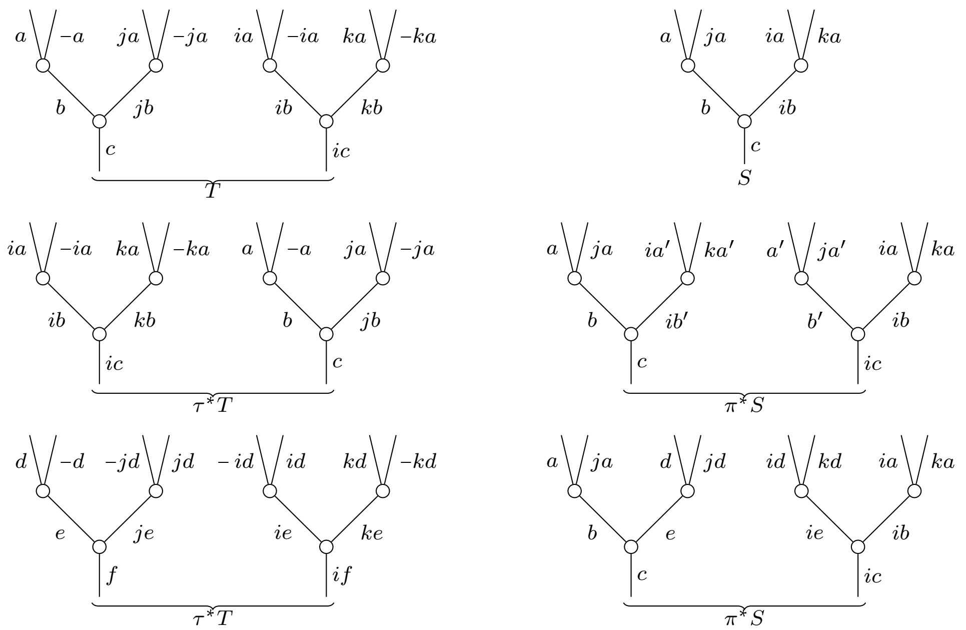


Figure 1: Root pullbacks

FIGURE

Definition 3.26. A map $S \xrightarrow{\varphi} T$ in Ω preserving the planar structure $\leq_p$ is called a *planar map*.

More generally, a map $F \to G$ in one of the categories Φ, Φ^G, Ω^G of forests, G-forests, G-trees is called a *planar map* if it is an independent map (cf. [17, Def. 5.28]) compatible with the planar structures $\leq_p$.

Remark 3.27. The need for the independence condition is justified by [17, Lemma 5.33] and its converse, since non independent maps do not reflect $\leq_d$-comparability.

However, we note that in the case of Ω_G independence admits simpler characterizations: φ is independent iff φ is injective on each edge orbit iff φ is injective on the root orbit.

PLANARPULL PROP

Proposition 3.28. *Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization*

$$F \xrightarrow{\simeq} \bar{F} \to G$$

such that $F \xrightarrow{\simeq} \bar{F}$ is an isomorphism and $\bar{F} \to G$ is planar.

Proof. We need to show that there is a unique planar structure $\leq_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of the broad poset G ensures that for any vertex $e^\uparrow \leq e$ of F the edges in $\varphi(e^\uparrow)$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(\underline{r}_F)$ are distinct. By (the forest version of) Proposition 3.14 the only possible planar structure $\leq_p^{\bar{F}}$ is the one which orders each set $e^\uparrow$ and the root tuple $\underline{r}_F$ according to their images. The claim that φ is then planar follows from Remark 3.16 together with the fact ([17, Lemma 5.33]) that φ reflects $\leq_d$-comparability. $\square$

PULLPLANAR REM

Remark 3.29. Proposition 3.28 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As an example, in the notation of (3.18), consider the map $C \to T_1$ defined by $1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 5$.

OUTSUBS SEC
3.2 Outer faces, tall maps, and substitution

One of the key ideas needed for our description of operads is the notion of substitution of tree nodes, a process that we will prefer to repackage in terms of maps of trees.

In preparation for that discussion, we first recall some basic definitions and results concerning outer subtrees and tree grafting, as in [17, §5].

OUTFACE DEF

Definition 3.30. Let $T \in \Omega$ be a tree and $e_1 \cdots e_n = \underline{e} \leq e$ a broad relation in T.

We define the *planar outer face* $T_{\underline{e} \leq e}$ to be the subtree with underlying set those edges $f \in T$ such that

$$f \leq_d e, \quad \forall_i f \nleq_d e_i, \tag{3.31}$$ OUTERFACE EQ

generating broad relations the relations $f^{\uparrow} \leq f$ for those $f \in T$ satisfying (3.31) but $\forall_i f \neq e_i$, and planar structure pulled back from T (in the sense of Remark 3.29).

Remark 3.32. If one forgoes the requirement that $T_{\underline{e} \leq e}$ be equipped with the pullback planar structure, the inclusion $T_{\underline{e} \leq e} \hookrightarrow T$ is usually called simply an *outer face*.

We now recap some basic results.

Proposition 3.33. *Let $T \in \Omega$ be a tree.*

(a) $T_{\underline{e} \leq e}$ is a tree with root e and edge tuple $\underline{e}$;

(b) there is a bijection

$$\{\textit{planar outer faces of } T\} \leftrightarrow \{\textit{broad relations of } T\};$$

(c) if $R \to S$ and $S \to T$ are outer face maps then so is $R \to T$;

(d) any pair of broad relations $\underline{g} \leq v$, $\underline{f}v \leq e$ induces a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "v"] \arrow[d, "v"'] & T_{\underline{g} \leq v} \arrow[d] \\ T_{\underline{f}v \leq e} \arrow[r] & T_{\underline{f}\underline{g} \leq e}. \end{tikzcd} \tag{3.34}$$ GRAPTPUSH EQ

Further, $T_{\underline{f}\underline{g} \leq e}$ is the unique choice of pushout that makes the maps in (3.34) planar.

Proof. We first show (a). That $T_{\underline{e} \leq e}$ is indeed a tree is the content of [17, Prop. 5.20]: more precisely, $T_{\underline{e} \leq e} = (T^{\leq e})_{< \underline{e}}$ in the notation therein. That the root of $T_{\underline{e} \leq e}$ is e is clear and that the root tuple is $\underline{e}$ follows from [17, Remark 5.23].

(b) follows from (a), which shows that $\underline{e} \leq e$ can be recovered from $T_{\underline{e} \leq e}$.

(c) follows from the definition of outer face together with [17, Lemma 5.33], which states that the $\leq_d$ relations on S, T coincide.

Since by (b) and (c) both $T_{\underline{g} \leq v}$ and $T_{\underline{f}v \leq e}$ are outer faces of $T_{\underline{f}\underline{g} \leq e}$, the first part of (d) is a restatement of [17, Prop. 5.15], while the additional planarity claim follows by Proposition 3.14 together with the vertex identification $V(T_{\underline{f}\underline{g} \leq e}) = V(T_{\underline{f}v \leq e}) \amalg V(T_{\underline{g} \leq v})$. $\square$

Definition 3.35. A map $S \xrightarrow{\varphi} T$ in Ω is called a *tall map* if

$$\varphi(\underline{l}_S) = \underline{l}_T, \qquad \varphi(r_S) = r_T,$$

where $l_{(-)}$ denotes the (unordered) leaf tuple and $r_{(-)}$ the root.

The following is a restatement of [17, Cor. 5.24]

TALLOUTERDEC PROP

Proposition 3.36. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphism,*

$$S \xrightarrow{\varphi^t} U \xrightarrow{\varphi^u} T$$

as a tall map followed by an outer face (in fact, $U = T_{\varphi(\underline{l}_S) \leq r_S}$).

We recall that a face $F \to T$ is called inner if is obtained by iteratively removing inner edges, i.e. edges other than the root or the leaves. In particular, it follows that a face is inner iff it is tall. The usual face-degeneracy decomposition thus combines with Proposition 3.36 to give the following.

Corollary 3.37. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^u} T \tag{3.38}$$ TRIPLEFACT EQ

as a degeneracy followed by an inner face followed by an outer face.

Proof. The factorization (3.38) can be built by first performing the degeneracy-face decomposition and then performing the tall-outer decomposition on the face map. □

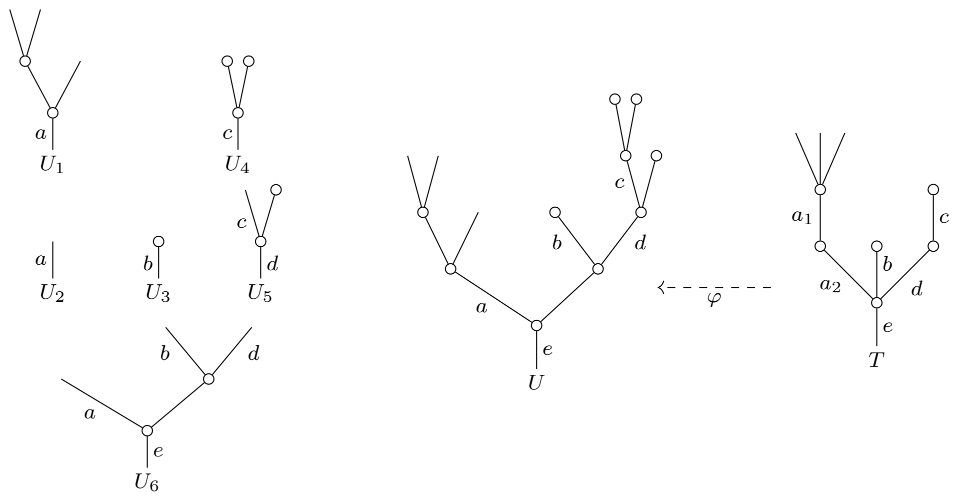
We will find it convenient throughout to regard the groupoid Σ of finite sets as the subcategory $\Sigma \hookrightarrow \Omega$ consisting of *corollas* (i.e. trees with a single vertex) and isomorphisms.

UNIQCOR NOT **Notation 3.39.** Given a tree $T \in \Omega$ there is a unique corolla $\mathsf{lr}(T) \in \Sigma$ and planar tall map $\mathsf{lr}(T) \to T$, which we call the *leaf-root* of T (this name is motivated by the equivariant analogue, discussed in §3.3). Explicitly, the number of leaves of $\mathsf{lr}(T)$ matches that of T, together with the inherited order.

We now turn to discussing the substitution operation. We start with an example, focused on the closely related notion of iterated graftings of trees (as described in (3.34)).

Example 3.40. The trees $U_1, U_2, \cdots, U_6$ on the left below can be grafted to obtain the tree U in the middle. More precisely (among other possible grafting orders), one has

$$U = (((((U_6 \amalg_a U_2)) \amalg_a U_1) \amalg_b U_3) \amalg_d U_5) \amalg_c U_4 \tag{3.41}$$ UFORMULA EQ

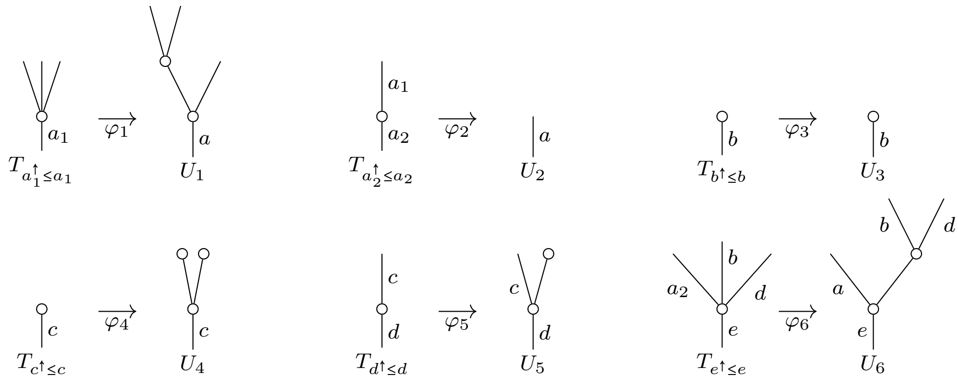


(3.42) SUBSDATUMTREES EQ

We now consider the tree T, which is built by converting each U_i into the corolla $\mathsf{lr}(U_i)$, and then performing the same grafting operations as in (3.41). T can then be regarded as encoding the combinatorics of the iterated grafting in (3.41), with alternative ways to reorder operations in (3.41) in bijection with ways to assemble T out of its nodes.

One can now therefore think of the iterated grafting (3.41) as being instead encoded by the tree T together with the (unique) planar tall maps φ_i below.

UFORMULA EQ



(3.43) SUBSDATUMTREES2 EQ

From this perspective, U can then be thought of as obtained from T by *substituting* each of its nodes with the corresponding U_i. Moreover, the φ_i assemble to a planar tall map $\varphi: T \to U$ (such that $a_i \mapsto a, b \mapsto b, \cdots, e \mapsto e$), which likewise encodes the same information.

One of the fundamental ideas that shape our perspective on operads is then that data for substitution of nodes as in (3.43) can equivalently be repackaged using planar tall maps.

SUBSDATUMTREES2 EQ

UBSTITUTIONDATUM **Definition 3.44.** Let $T \in \Omega$ be a tree.

A *T-substitution datum* is a tuple $(U_{e^\uparrow \leq e})_{(e^\uparrow \leq e) \in V(T)}$ together with tall maps $T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}$. Further, a map of T-substitution data $(U_{e^\uparrow \leq e}) \to (V_{e^\uparrow \leq e})$ is a tuple of tall maps $(U_{e^\uparrow \leq e} \to V_{e^\uparrow \leq e})$ compatible with the substitution maps.

Lastly, a substitution datum is called a *planar T-substitution datum* if the chosen maps are planar (so that $\mathsf{lr}(U_{e^\uparrow \leq e}) = T_{e^\uparrow \leq e}$) and a morphism of planar data is called a planar morphism if it consists of a tuple of planar maps.

We denote the category of (resp. planar) T-substitution data by $\mathsf{Sub}(T)$ (resp. $\mathsf{Sub}_{\mathsf{p}}(T)$).

Definition 3.45. Let $T \in \Omega$ be a tree. The *Segal core poset* $\mathsf{Sc}(T)$ is the poset with objects the single edge subtrees η_e and vertex subtrees $T_{e^\uparrow \leq e}$, ordered by inclusion.

Remark 3.46. Note that the only maps in $\mathsf{Sc}(T)$ are inclusions of the form $\eta_a \subset T_{e^\uparrow \leq e}$. In particular, there are no pairs of composable non-identity relations in $\mathsf{Sc}(T)$.

Given a T-substitution datum $\{U_{\{e^\uparrow \leq e\}}\}$ we abuse notation by writing

$$U_{(-)}: \mathsf{Sc}(T) \to \Omega$$

for the functor $\eta_a \mapsto \eta$, $T_{e^\uparrow \leq e} \mapsto U_{e^\uparrow \leq e}$ and sending the inclusions $\eta_a \subset T_{e^\uparrow \leq e}$ to the composites

$$\eta \xrightarrow{a} T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}.$$

TAUNDERPLAN PROP **Proposition 3.47.** *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\begin{array}{ccc}
\mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & T \downarrow \Omega^{\mathsf{pt}} \\
(U_{e^\uparrow \leq e}) & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \\
\left(U_{\varphi(e^\uparrow) \leq \varphi(e)}\right) & \longleftarrow\!\shortmid & (T \xrightarrow{\varphi} U)
\end{array} \tag{3.48}$$

SUBDATAUNDERPLAN EQ

where $T \downarrow \Omega^{\mathsf{pt}}$ denotes the category of planar tall maps under T and $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ *is chosen in the unique way that makes the inclusions of the $U_{e^\uparrow \leq e}$ planar.*

Proof. We first show in parallel that: (i) $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, which we denote U_T, exists; (ii) for the datum $(T_{e^{\uparrow}\le e})$, it is $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$; (iii) $V(U_T) = \coprod_{V(T)} V(U_{e^{\uparrow}\le e})$; (iv) the induced map $T \to U_T$ is planar tall.

The argument is by induction on the number of vertices of T, with the base cases of T with 0 or 1 vertices being immediate, since then T is the terminal object of $\mathsf{Sc}(T)$. Otherwise, one can choose a non trivial grafting decomposition so as to write $T = R \amalg_e S$, resulting in identifications $\mathsf{Sc}(R) \subset \mathsf{Sc}(T)$, $\mathsf{Sc}(S) \subset \mathsf{Sc}(T)$ so that $\mathsf{Sc}(R) \cup \mathsf{Sc}(S) = \mathsf{Sc}(T)$ and $\mathsf{Sc}(R) \cap \mathsf{Sc}(S) = \{\eta_e\}$. The existence of $U_T = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is thus equivalent to the existence of the pushout below (where the rightmost diagram merely simplifies notation).

$$\begin{tikzcd}
\eta \arrow[r, "e"] \arrow[d, "e"'] & \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)} \arrow[d, dashed] \\
\operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)} \arrow[r, dashed] & \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\eta \arrow[r, "e"] \arrow[d, "e"'] & U_R \arrow[d, dashed] \\
U_S \arrow[r, dashed] & U_T
\end{tikzcd} \tag{3.49}$$

ASSEMBLYGRAFT EQ

By induction, U_R and U_S exist for any $U_{(-)}$, equal R and S in the case $U_{(-)} = T_{(-)}$, $V(U_R) = \coprod_{V(R)} V(U_{e^{\uparrow}\le e})$ and likewise for S (so that there are unique choices of U_R, U_S making the inclusions of $U_{e^{\uparrow}\le e}$ planar), and the maps $R \to \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)}$, $S \to \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)}$ are planar tall. But it now follows that (3.49) is a grafting pushout diagram (cf. (3.34)), so that the pushout indeed exists. The conditions $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$, $V(U_T) = \coprod_{V(T)} V(U_{e^{\uparrow}\le e})$, and that $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall follow.

The fact that the two functors in (3.48) are inverse to each other is clear from the same inductive argument. ☐

ATAUNDERPLAN COR

Corollary 3.50. *Let $T \in \Omega$ be a tree. The formulas in (3.48) give an isomorphism of categories*

$$\mathsf{Sub}(T) \rightleftarrows T \downarrow \Omega^{\mathrm{t}} \tag{3.51}$$

SUBDATAUNDERNONPL EQ

where $T \downarrow \Omega^{\mathrm{t}}$ denotes the category of tall maps under T.

Proof. This is a consequence of Proposition 3.28 together with the previous result. Indeed, Proposition 3.14 can be restated as saying that isomorphisms $T \to T'$ are in bijection with substitution data consisting of isomorphisms, and thus bijectiveness of $\mathsf{Sub}(T) \to T \downarrow \Omega^{\mathrm{t}}$ reduces to that in the previous result. ☐

VERTEXDECOMP REM

Remark 3.52. As noted in the proof of Proposition 3.47, writing $U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, one has

$$V(U) = \coprod_{(e^{\uparrow}\le e)\in V(T)} V(U_{e^{\uparrow}\le e}). \tag{3.53}$$

VERTEXDECOMP EQ

Alternatively, (3.53) can be regarded as a map $\varphi^*: V(U) \to V(T)$ induced by the planar tall map $\varphi: T \to U$. Explicitly, $\varphi^*(U_{u^{\uparrow}\le u})$ is the unique $T_{t^{\uparrow}\le t}$ such that there is an inclusion of outer faces $U_{u^{\uparrow}\le u} \hookrightarrow U_{t^{\uparrow}\le t}$, so that φ^* indeed depends contravariantly on the tall map φ.

INPPATH REM

Remark 3.54. Suppose that $e \in T$ has input path $I_T(e) = (e = e_n < e_{n-1} < \cdots < e_0)$. It is intuitively clear that for a tall map $\varphi: T \to U$ the input path of $\varphi(e)$ is built by gluing input paths in the $U_{t^{\uparrow}\le t}$. More precisely (and omitting φ for readability), one has

$$I_U(e_n) \simeq I_{n-1}(e_n) \amalg_{e_{n-1}} I_{n-2}(e_{n-1}) \amalg_{e_{n-2}} \cdots \amalg_{e_1} I_1(e_0).$$

where $I_k(-)$ denotes the input path in $U_{e_k^{\uparrow}\le e_k}$. More formally, this follows from the characterization of predecessors in Proposition 3.9(b).

We end this section with a couple of lemmas that will allow us to reverse the substitution procedure of Proposition 3.47 and will be needed in §5.2.

BUILDABLE PROP

Proposition 3.55. *Let $U \in \Omega$ be a tree. Then:*

(i) given non stick outer subtrees U_i such that $V(U) = \coprod_i V(U_i)$ there is a unique tree T and planar tall map $T \to U$ such that the sets $\{U_i\}$, $\{U_{e^{\uparrow}\le e}\}$ coincide;

(ii) given multiplicities $m_e \geq 1$ for each edge $e \in U$, there is a unique planar degeneracy $\rho\colon T \to U$ such that $\rho^{-1}(e)$ has m_e elements;

(iii) planar tall maps $T \to U$ are in bijection with collections $\{U_i\}$ of outer subtrees such that $V(U) = \coprod_i V(U_i)$ and U_j is not an inner edge of any U_i whenever $U_j \simeq \eta$ is a stick.

Proof. We first show (i) by induction on the number of subtrees U_i. The base case $\{U_i\} = \{U\}$ is immediate, setting $T = \mathsf{lr}(U)$. Otherwise, U must not be a corolla and letting e be an edge that is both an inner edge of U and a root of some U_i, and one can form a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "e"] \arrow[d, "e"'] & U^{\leq e} \arrow[d] \\ U_{\nleq e} \arrow[r] & U \end{tikzcd} \tag{3.56}$$

where $U^{\leq e}$ (resp. $U_{\nleq e}$) are the outer faces consisting of the edges $u \leq_d e$ (resp. $u \nleq_d e$). Since there is a unique U_i containing the vertex $e^{\uparrow} \leq e$, it follows from the definition of outer face that there is a nontrivial partition $\{U_i\} = \{U_i | U_i \hookrightarrow V\} \amalg \{U_i | U_i \hookrightarrow W\}$. Existence of $T \to U$ now follows from the induction hypothesis. For uniqueness, the condition that no U_i is a stick guarantees that T possesses a single inner edge mapping to e, and thus admits a compatible decomposition as in (3.56), so that uniqueness too follows from the induction hypothesis.

For (ii), we argue existence by nested induction on the number of vertices $|V(U)|$ and the sum of the multiplicities m_e. The base case $|V(U)| = 0$, i.e. $U = \eta$ is immediate. Otherwise, writing $m_e = m'_e + 1$, one can form a decomposition (3.56) where either $|V(V)|, |V(W)| < |V(U)|$ or one of V, W is η, so that $T \to U$ can be built via the induction hypothesis. For uniqueness, note first that by [17, Lemma 5.33] each pre-image $\rho^{-1}(e)$ is linearly ordered and by the "further" claim in [17, Cor. 5.39] the remaining broad relations are precisely the pre-image of the non-identity relations in U, showing that underlying broad poset of the tree T is unique up to isomorphism. Strict uniqueness is then Proposition 3.28.

(iii) follows by combining (i) and (ii). Indeed, any planar tall map $T \to U$ has a unique decomposition $T \twoheadrightarrow \bar{T} \hookrightarrow U$ as a planar degeneracy followed by a planar inner face, and each of these maps is classified by the data in (b) and (a). ∎

Lemma 3.57. *Suppose $T_1, T_2 \hookrightarrow T$ are two outer faces with at least one common edge e. Then there exists an unique outer face $T_1 \cup T_2$ such that $V(T_1 \cup T_2) = V(T_1) \cup V(T_2)$.*

Proof. If either T is a corolla or one of T_1, T_2 consists only of the root or a leaf stick subtrees the result is obvious. Otherwise, one can necessarily choose e to be an inner edge of T, in which case all of three of T_1, T_2, T admit compatible decompositions as in (3.56) and the result follows by induction on $|V(T)|$. ∎

3.3 Equivariant leaf-root and vertex functors

This section introduces two functors that are the very center of our definition of the category Op_G of genuine equivariant operads: the leaf-root and vertex functors.

We start by recalling a key class of maps of G-trees.

Definition 3.58. Let $S = (S_y)_{y \in Y}$ and $T = (T_x)_{x \in X}$ be G-trees. A map of G-trees

$$\varphi = (\phi, (\varphi_y))\colon S \to T$$

is called a *quotient* if each of the constituent tree maps

$$\varphi_y\colon S_y \to T_{\phi(y)}$$

is an isomorphism of trees.

The category of G-trees and quotients is denoted Ω^0_G (this notation is justified in §3.4).

Remark 3.59. Quotients can alternatively be described as the cartesian arrows for the Grothendieck fibration $\Omega_G \xrightarrow{\mathsf{r}} \mathsf{O}_G$. We note that this differs from the notion of root pullbacks, which are the *chosen* cartesian arrows, and include only those quotients such that each $\varphi_y \colon S_y \to T_{\phi(y)}$ is a planar isomorphism, i.e., an identity.

Definition 3.60. The *G-symmetric category*, whose objects we call *G-corollas*, is the full subcategory $\Sigma_G \hookrightarrow \Omega_G^0$ of those G-trees $C = (C_x)_{x \in X}$ such that some (and thus all) C_x is a corolla $C_x \in \Sigma \hookrightarrow \Omega$ (cf. Notation 3.39).

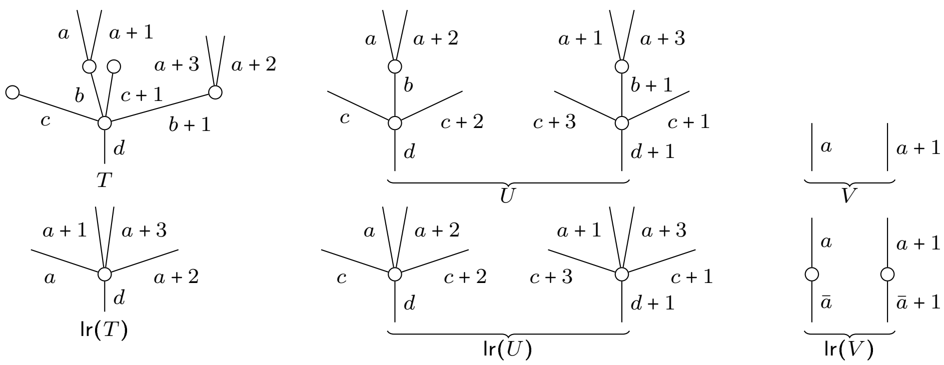
Definition 3.61. The *leaf-root functor* is the functor $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$ defined by

$$\mathsf{lr}\left((T_x)_{x \in X}\right) = \left(\mathsf{lr}(T_x)\right)_{x \in X}.$$

Remark 3.62. The leaf-root functor extends to a functor $\mathsf{lr} \colon \Omega_G^{\mathsf{t}} \to \Sigma_G$, where Ω_G^{t} is the category of tall maps, defined exactly as in Definition 3.58, but not to a functor defined on all arrows in Ω_G. However, we will mostly be concerned with the restriction $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$.

EAFROOTEXAMP REM

Remark 3.63. Generalizing the remark in Notation 3.39, $\mathsf{lr}(T)$ can alternatively be characterized as being the *unique* G-corolla which admits an also unique tall planar map $\mathsf{lr}(T) \to T$. Moreover, $\mathsf{lr}(T)$ can usually be regarded as the "smallest inner face" of T, obtained by removing all the inner edges, although this characterization fails when $T = (\eta_x)_{x \in X}$ is a stick G-tree. Some examples with $G = \mathbb{Z}_{/4}$ follow.



LRROOTMAP REM

Remark 3.64. Since planarizations can not be pushed forward along tree maps (cf. Remark 3.29) the leaf-root functor $\mathsf{lr} \colon \Omega_G^0 \to \Sigma_G$ is not a Grothendieck fibration, but instead only a map of Grothendieck fibrations over O_G (for the obvious root functor $\mathsf{r} \colon \Sigma_G \to \mathsf{O}_G$).

VG DEF

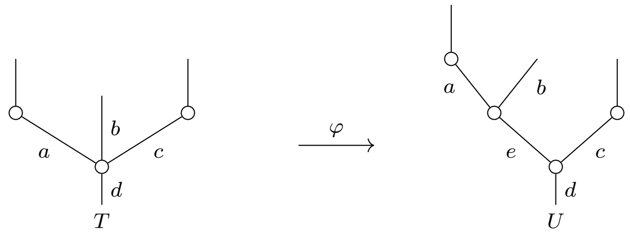
Definition 3.65. Given $T = (T_x)_{x \in X} \in \Omega_G$ we define its set of *vertices* to be $V(T) = \coprod_{x \in X} V(T_x)$ and its set of *G-vertices* to be the orbit set $V(T)/G$.

Furthermore, we will regard $V(T)$ as an object of F by using the induced planar order (with $e^{\uparrow} \le e$ ordered according to e) and likewise $V_G(T)$ will be regarded as an object of F by using the lexicographic order: i.e. vertex equivalence classes $[e^{\uparrow} \le e]$ are ordered according to the planar order $\le_p$ of the smallest representative ge, $g \in G$.

ERTEXDECOMPG REM

Remark 3.66. Following Remark 3.52, a tall map $\varphi \colon T \to U$ of G-trees induces a G-equivariant map $\varphi^* \colon V(U) \to V(T)$ and thus also a map of orbits $\varphi^* \colon V_G(U) \to V_G(T)$. We note, however, that φ^* is not in general compatible with the order on $V_G(-)$ even if φ is planar, as is indeed the case even in the non-equivariant setting.

A minimal example follows.



In $V(T)$ the vertices are ordered as $a < c < d$ while in $V(U)$ they are ordered as $a < e < c < d$ but the map $\varphi^*: V(U) \to V(T)$ is given by $a \mapsto a, c \mapsto c, d \mapsto d, e \mapsto d$.

GVERT NOT **Notation 3.67.** Given $T = (T_x)_{x \in X} \in \Omega_G$ and $(e^\uparrow \le e) \in V(T)$ we write $T_{e^\uparrow \le e}$ as a shorthand for $T_{x, e^\uparrow \le e}$, where $e \in T_x$.

Further, each element $V_G(T)$ corresponds to an unique edge orbit Ge for e not a leaf. We will prefer to write G-vertices as v_{Ge}, and write

$$T_{v_{Ge}} = (T_{f^\uparrow \le f})_{f \in Ge} \tag{3.68}$$ TVGE DEF

where Ge inherits the planar order.

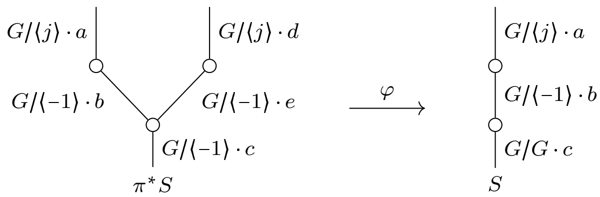
We note that $T_{v_{Ge}}$ is always a G-corolla, leading to the following definition.

Definition 3.69. The *G-vertex functor* is the functor

$$\begin{aligned} \Omega_G^0 &\xrightarrow{V_G} \mathsf{F}_s \wr \Sigma_G \\ T &\longmapsto (T_{v_{Ge}})_{v_{Ge} \in V_G(T)}, \end{aligned} \tag{3.70}$$ VFUNCTOR EQ

where F_s is the category of finite sets and surjections of Remark FINSURJ REM 2.16.

ED_WREATH_REMARK **Remark 3.71.** In the non-equivariant case the vertex functor can be defined to land instead in $\Sigma \wr \Sigma$. The need to introduce the $\mathsf{F} \wr (-)$ construction comes from the fact that in general quotient maps do not preserve the number of G-vertices. For a simple example, let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ and consider the pullback map $\varphi: \pi^* S \to S$ of Example ROOTPULL EX 3.25 determined by the assignments $a \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto ia$, $e \mapsto ib$, and presented below in orbital notation.



We note that $T = \pi^* S$ has three G-vertices v_{Gc}, v_{Gb}, v_{Ge} while S has only two G-vertices v_{Gc} and v_{Gb}. $V_G(\varphi)$ then maps the two G-corollas $T_{v_{Gb}}$ and $T_{v_{Ge}}$ isomorphically onto $S_{v_{Gb}}$ and the G-corolla $T_{v_{Gc}}$ by a non-isomorphism quotient onto $S_{v_{Gc}}$.

The following elementary statement will play an important auxiliary role.

VGPULL LEM **Lemma 3.72.** *The G-vertex functor*

$$\Omega_G^0 \xrightarrow{V_G} \mathsf{F}_s \wr \Sigma_G$$

sends pullbacks over O_G (i.e. root pullbacks) to pullbacks over $\mathsf{F}_s \wr \mathsf{O}_G$ (cf. Lemma FWRGROTH LEM 2.18).

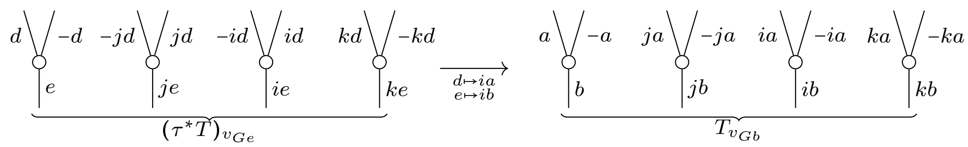
Proof. Note first that an arrow $(\phi,(\varphi_i)):(C_i)_{i\in I}\to(C'_j)_{j\in J}$ is a pullback for the split fibration $\mathsf{F}_s\wr\Sigma_G\to\mathsf{F}_s\wr\mathsf{O}_G$ iff each of the constituent arrows $\varphi_i:C_i\to C'_{\phi(i)}$ are pullbacks for the split fibration $\Sigma_G\to\mathsf{O}_G$.

The pullback $\psi^*T\xrightarrow{\bar\psi}T$ of $T=(T_x)_{x\in X}\in\Omega_{G,0}$ over $\psi:Y\to X$ has the form $(T_{\psi(y)})_{y\in Y}\to(T_x)_{x\in X}$ and it now suffices to check that each of the vertex maps $(\psi^*T)_{v_{Ge}}\to T_{v_{G\bar\psi(e)}}$ is itself a pullback. By (3.68), this is the statement that for $f\in Ge$ the induced map

$$(\psi^*T)_{f^\uparrow\leq f}\to T_{\bar\psi(f^\uparrow)\leq\bar\psi(f)} \tag{3.73}$$ VGPULL EQ

is an identity (i.e. planar isomorphism), and letting y be such that $f\in T_{\psi(y)}$ one sees that (3.73) is the identity $T_{\psi(y),f^\uparrow\leq f}=T_{x,\bar\psi(f)^\uparrow\leq\bar\psi(f)}$, where $x=\psi(y)$, finishing the proof. $\square$

Example 3.74. The following depicts one of the maps (3.73) for the pullback $\tau^*T\to T$ appearing in Example 3.25.



Note that $(\tau^*T)_{v_{Ge}}=\rho^*T_{v_{Gb}}$ for ρ the map $\{e,je,ie,ke\}\to\{b,jb,ib,kb\}$ defined by $e\mapsto ib$ so that, accounting for orders, ρ is the block permutation $\rho=(13)(24)$.

We are now in a position to generalize Definition 3.44.

TUTIONDATUMG DEF **Definition 3.75.** Let $T\in\Omega_G$ be a G-tree.

A *(resp. planar) T-substitution datum* is a tuple $\left(U_{f^\uparrow\leq f}\right)_{V(T)}$ of G-trees together with

(i) associative and unital G-action maps $U_{f^\uparrow\leq f}\to U_{gf^\uparrow\leq gf}$;

(ii) (resp. planar) tall maps $T_{f^\uparrow\leq f}\to U_{f^\uparrow\leq f}$ compatible with the G-action maps.

Further, a map of (resp. planar) T-substitution data $\left(U_{f^\uparrow\leq f}\right)\to\left(V_{f^\uparrow\leq f}\right)$ is a compatible tuple of (resp. planar) tall maps $\left(U_{f^\uparrow\leq f}\to V_{f^\uparrow\leq f}\right)$.

We denote the category of (resp. planar) T-substitution data by $\mathsf{Sub}(T)$ (resp. $\mathsf{Sub}_{\mathsf{p}}(T)$).

Recall that a map of G-trees is called *rooted* if it induces an ordered isomorphism on the root orbit (cf. Definition 3.24).

SUBSGREF DEF **Remark 3.76.** Writing $U^{\mathsf{r}}_{v_{Ge}}=(U_{f^\uparrow\leq f})_{f\in Ge}$ a T-substitution datum can equivalently be encoded by the tuple $\left(U^{\mathsf{r}}_{v_{Ge}}\right)_{V_G(T)}$ together with *rooted* tall maps $T_{v_{Ge}}\to U^{\mathsf{r}}_{v_{Ge}}$. The need to include r (which stands for "rooted") in the notation is explained by Remark 3.81.

Further, the T-substitution datum is planar iff the so are the maps $T_{v_{Ge}}\to U^{\mathsf{r}}_{v_{Ge}}$.

UBSDATUMCONV REM **Remark 3.77.** Writing $T=(T_x)_{x\in X}$ as usual one obtains (non-equivariant) T_x-substitution data $U_{x,(-)}$ for each T_x. We again write $U_{x,(-)}:\mathsf{Sc}(T_x)\to\Omega$ and note that these are compatible with the G-action in the sense that the obvious diagram

$$\begin{tikzcd}
\mathsf{Sc}(T_x) \arrow[rr, "U_{x,(-)}"] \arrow[dr, "g"'] & & \Omega \\
& \mathsf{Sc}(T_{gx}) \arrow[ur, "U_{gx,(-)}"'] &
\end{tikzcd} \tag{3.78}$$ EQUIVSCMAP EQ

commutes. Writing $\mathsf{Sc}(T)=\coprod_x\mathsf{Sc}(T_x)$, (3.78) is then equivalent to a functor $G\ltimes\mathsf{Sc}(T)\to\Omega$, where $G\ltimes\mathsf{Sc}(T)$ is the Grothendieck construction for the G-action (which, explicitly, adds arrows $\eta_a\to\eta_{ga}$, $T_{e^\uparrow\leq e}\to T_{ge^\uparrow\leq ge}$ to $\mathsf{Sc}(T)$ that satisfy obvious compatibilities).

In the following we write $\operatorname{colim}_{\mathsf{Sc}(T)}U_{(-)}$ to mean $(\operatorname{colim}_{\mathsf{Sc}(T_x)}U_{x,(-)})_{x\in X}$ or, in other words, we take the colimit in $\Phi=\mathsf{F}\wr\Omega$ rather than in Ω (as is needed since Ω lacks coproducts).

TAUNDERPLANG COR

Corollary 3.79. *Let $T \in \Omega_G$ be a G-tree. There are isomorphisms of categories*

$$\begin{array}{ccc} \mathsf{Sub_p}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{pt}} \\ \left(U_{f^\uparrow \le f}\right)_{V(T)} & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{rt}} \\ \left(U_{f^\uparrow \le f}\right)_{V(T)} & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \end{array} \tag{3.80}$$

SUBDATAUNDERPLANG EQ

where $T \downarrow \Omega_G^{\mathsf{pt}}$ (resp. $T \downarrow \Omega_G^{\mathsf{rt}}$) is the category of planar tall (resp. rooted tall) maps under T.

Proof. This is a direct consequence of the non-equivariant analogues Proposition 3.47 and Corollary 3.50 applied to each individual T_x together with the equivariance analysis in Remark 3.77. □

WHYR REM

Remark 3.81. Writing $U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, it follows from the non-equivariant results Proposition 3.47 and Corollary 3.50 that each inclusion map $U_{f^\uparrow \le f} \to U$ is planar, so that there is no conflict with Notation 3.67.

However, some care is needed concerning the $U^{\mathsf{r}}_{v_{Ge}}$ appearing in the reformulation of substitution data given in Remark 3.76. Letting $\varphi: T \to U$ be the induced map, one sees that while $U^{\mathsf{r}}_{v_{Ge}}$ and $U_{v_{G\varphi(e)}}$ have the same constituent trees (with the latter defined by Notation 3.67), the roots of $U^{\mathsf{r}}_{v_{Ge}}$ are ordered by Ge while those of $U_{v_{G\varphi(e)}}$ are ordered by $G\varphi(e)$. More succinctly, it is then $U^{\mathsf{r}}_{v_{Ge}} = \varphi^*_{Ge} U_{v_{G\varphi(e)}}$ for $\varphi_{Ge}: Ge \to G\varphi(e)$ the induced map.

Lastly, we note that such distinctions are unnecessary for planar data, since then the φ_{Ge} are ordered isomorphisms (i.e. identities), so that $U^{\mathsf{r}}_{v_{Ge}} = U_{v_{G\varphi(e)}}$.

PULLCOMP REM

Remark 3.82. The isomorphisms in Corollary 3.79 are compatible with root pullbacks of trees. More concretely, as in the proof of Lemma 3.72 each pullback $\bar{\psi}: \psi^* T \to T$ determines pullback maps $\bar{\psi}_{Ge}: (\psi^* T)_{v_{Ge}} \to T_{v_{G\bar{\psi}(e)}}$, which we now note are pullbacks over the maps $\bar{\psi}_{Ge}: Ge \to G\bar{\psi}(e)$ in O_G. The definition of pullback then allows us to uniquely fill any diagram (where we reformulate substitution data as in Remark 3.76)

$$\begin{array}{ccc} (\psi^* T)_{v_{Ge}} & \dashrightarrow & \bar{\psi}^*_{Ge} U^{\mathsf{r}}_{v_{G\bar{\psi}(e)}} \\ \downarrow & & \downarrow \\ T_{v_{G\bar{\psi}(e)}} & \longrightarrow & U^{\mathsf{r}}_{v_{G\bar{\psi}(e)}} \end{array}$$

defining the left vertical functors (with the right functors defined analogously) in the commutative diagrams below.

$$\begin{array}{ccc} \mathsf{Sub_p}(\psi^* T) & \rightleftarrows & \psi^* T \downarrow \Omega_G^{\mathsf{pt}} \\ {\scriptstyle (\bar{\psi}^*_{Ge})}\uparrow & & \uparrow{\scriptstyle \psi^*} \\ \mathsf{Sub_p}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{pt}} \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}(\psi^* T) & \rightleftarrows & \psi^* T \downarrow \Omega_G^{\mathsf{rt}} \\ {\scriptstyle (\bar{\psi}^*_{Ge})}\uparrow & & \uparrow{\scriptstyle \psi^*} \\ \mathsf{Sub}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{rt}} \end{array} \tag{3.83}$$

SUBDATAUNDERPLANG2 EQ

PLANARSTRING SEC

3.4 Planar strings

We now use the leaf-root and vertex functors to repackage our substitution results in a format that will be more convenient for our discussion of operads in §4.

PLANSTR DEF

Definition 3.84. The category Ω_G^n of *planar n-strings* is the category whose objects are strings

$$T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} T_n \tag{3.85}$$

STRINGOBJ EQ

where $T_i \in \Omega_G$ and the φ_i are tall planar maps, while arrows are commutative diagrams

$$\begin{array}{ccccccc} T_0 & \xrightarrow{\varphi_1} & T_1 & \xrightarrow{\varphi_2} & \cdots & \xrightarrow{\varphi_n} & T_n \\ {\scriptstyle \pi_0}\downarrow & & {\scriptstyle \pi_1}\downarrow & & & & \downarrow{\scriptstyle \pi_n} \\ T'_0 & \xrightarrow[\varphi'_1]{} & T'_1 & \xrightarrow[\varphi'_2]{} & \cdots & \xrightarrow[\varphi'_n]{} & T'_n \end{array} \tag{3.86}$$

PTNARROW EQ

where each π_i is a quotient map.

IMPOPERATORS NOT

Notation 3.87. Since compositions of planar tall arrows are planar tall and identity arrows are planar tall it follows that $\Omega_G^{\bullet}$ forms a simplicial object in Cat, with faces given by composing and degeneracies by inserting identities.

Further setting $\Omega_G^{-1} = \Sigma_G$, the leaf-root functor $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$ makes $\Omega_G^{\bullet}$ into an augmented simplicial object and, furthermore, the maps $s_{-1}\colon \Omega_G^n \to \Omega_G^{n+1}$ sending $T_0 \to T_1 \to \cdots \to T_n$ to $\mathsf{lr}(T_0) \to T_0 \to T_1 \to \cdots \to T_n$ equip it with extra degeneracies.

Remark 3.88. The identification $\Omega_G^{-1} = \Sigma_G$ can be understood by noting that a string (3.85) is equivalent to a string

$$T_{-1} \xrightarrow{\varphi_0} T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} T_n \qquad (3.89)$$

STRINGOBJALT EQ

where $T_{-1} = \mathsf{lr}(T_0) = \cdots = \mathsf{lr}(T_n)$.

ALLSPLITMAPS REM

Remark 3.90. Since for any planar n-string it is $\mathsf{r}(T_i) = \mathsf{r}(T_j)$ for any $1 \leq i, j \leq n$, one has a well defined functor $\mathsf{r}\colon \Omega_G^n \to \mathsf{O}_G$, which is readily seen to be a split Grothendieck fibration. Furthermore, generalizing Remark 3.64, all operators d_i, s_j are maps of split Grothendieck fibrations.

VGDEF NOT

Notation 3.91. We extend the vertex functor to a functor $V_G\colon \Omega_G^{n+1} \to \mathsf{F}_s \wr \Omega_G^n$ by

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_Ge} \to \cdots \to T_{n,v_Ge})_{v_Ge \in V_G(T_0)} \qquad (3.92)$$

VGDEF EQ

where we abuse notation by writing T_{i,v_Ge} for $(T_{i,\bar{\varphi}_i(f)^{\uparrow} \leq \bar{\varphi}_i(f)})_{f \in v_Ge}$, where $\bar{\varphi}_i = \varphi_i \circ \cdots \circ \varphi_1$.

Alternatively, regarding $T_0 \to \cdots \to T_n$ as a string of $n-1$ arrows in $T_0 \downarrow \Omega_G^{\mathsf{pt}}$, the object $V_G(T_0 \to \cdots \to T_n)$ can be thought of as the image of the inverse functor in Corollary 3.79, written according to the reformulation in Remark 3.76 (where since we are in the planar case we need not distinguish between $U^{\mathsf{r}}_{(-)}$ and $U_{(-)}$ notation (cf. Remark 3.81)). Note however that from this perspective functoriality needs to be checked separately.

We now obtain a key reinterpretation (and slight strengthening) of Corollary 3.79.

SUBSASPULL PROP

Proposition 3.93. *For any $n \geq 0$ the commutative diagram*

$$\begin{tikzcd} \Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n}"'] & \mathsf{F}_s \wr \Omega_G^{n-1} \arrow[d, "\mathsf{F} \wr d_{0,\cdots,n-1}"] \\ \Omega_G^0 \arrow[r, "V_G"'] & \mathsf{F}_s \wr \Sigma_G \end{tikzcd} \qquad (3.94)$$

PTPULL EQ

is a pullback diagram in Cat.

Proof. Let us write $P = \Omega_G^0 \times_{\mathsf{F}_s \wr \Sigma_G} \mathsf{F}_s \wr \Omega_G^{n-1}$ for the pullback, so that our goal is to show that the canonical map $\Omega_G^n \to P$ is an isomorphism.

That $\Omega_G^n \to P$ is an isomorphism on objects follows by combining the alternative description of V_G in Notation 3.91 with the planar half of Corollary 3.79 (in fact, this yields isomorphisms of the fibers over Ω_G^0, but we will not directly use this fact). We will hence write $T_0 \to \cdots \to T_n$ to denote an object of P as well.

An arrow in P from $T_0 \to \cdots \to T_n$ to $T_0' \to \cdots \to T_n'$ then consists of a quotient $\pi_0\colon T_0 \to T_0'$ together with a $V_G(T_0)$ indexed tuple of quotients of strings (where we write $e' = \pi_0(e)$)

$$\begin{tikzcd} T_{0,v_Ge} \arrow[r] \arrow[d, "\pi_{0,e}"'] & T_{1,v_Ge} \arrow[r] \arrow[d, "\pi_{1,e}"'] & \cdots \arrow[r] & T_{n,v_Ge} \arrow[d, "\pi_{n,e}"] \\ T'_{0,v_Ge'} \arrow[r] & T'_{1,v_Ge'} \arrow[r] & \cdots \arrow[r] & T'_{n,v_Ge'}. \end{tikzcd} \qquad (3.95)$$

PTNARROWLOC EQ

That $\Omega_G^n \to P$ is injective on arrows is then clear.

For surjectivity, note first that by Lemma 3.72 the composite $P \to \Omega_G^0 \to \mathsf{O}_G$ is a split Grothendieck fibration and $P \to \Omega_G^0$ is a map of split Grothendieck fibrations. Indeed, pullbacks in P can be built explicitly as those arrows such that π_0 and all $\pi_{i,e}$ in (3.95) are pullbacks (alternatively, an abstract argument also works). The alternative description of V_G in Notation 3.91 combined with (3.83) then show that $\Omega_G^n \to P$ preserves pullbacks, so that injectivity needs only be checked for maps in the fibers over O_G, i.e. on rooted maps. Tautologically, a map in P is rooted iff $\pi_0 : T_0 \to T_0'$ is. But since a quotient is an isomorphism iff it is so on roots, we further have that a map in P is rooted iff $\pi_0 : T_0 \to T_0'$ is a rooted isomorphism and each $\pi_{i,e}$ in (3.95) is an isomorphism. But now rewriting 3.95 as a tuple of diagrams indexed by $f \in Ge$ one obtains a diagram in $\mathsf{Sub}(T_0)$ of the same shape which, after converted to a diagram in $T_0 \downarrow \Omega_G^{\mathrm{rt}}$ using the rooted half of Corollary 3.79, yields the desired rooted map (3.86) in Ω_G^n lifting the rooted map in P. □

INDVNG NOT **Notation 3.96.** For $0 \le k \le n$ we will let

$$V_G^k : \Omega_G^n \to \mathsf{F}_s \wr \Omega_G^{n-k-1}$$

be inductively defined by $V_G^0 = V_G$ and $V_G^{n+1} = \sigma^0 \circ (\mathsf{F}_s \wr V_G^n) \circ V_G$.

VGN REM **Remark 3.97.** When $n = 2$, V_G^2 is thus the composite

$$\Omega_G^2 \xrightarrow{V_G} \mathsf{F}_s \wr \Omega_G^1 \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \Omega_G^0 \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \mathsf{F}_s \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \mathsf{F}_s \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Sigma_G$$

while for $n = 4$, V_G^1 is the composite

$$\Omega_G^4 \xrightarrow{V_G} \mathsf{F}_s \wr \Omega_G^3 \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \Omega_G^2 \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Omega_G^2.$$

In light of Remarks 3.52 and 3.66, $V_G^n(T_0 \to \cdots \to T_n)$ is identified with the tuple

$$(T_{k,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_k)}, \tag{3.98}$$ VGNISO EQ

where we note that strings are written in prepended notation as in (3.89), so that $T_{i,v_{Ge}}$ is superfluous unless $k = n$. Further, note that this requires changing the order of $V_G(T_k)$. Rather than using the order induced by T_k, one instead equips $V_G(T_k)$ with the order induced lexicographically from the maps $V_G(T_k) \to V_G(T_{k-1}) \to \cdots \to V_G(T_0)$ of Remark 3.52. I.e., for $v, w \in V_G(T_k)$ the condition $v < w$ is determined by the lowest l such that the images of $v, w \in V_G(T_l)$ are distinct.

Therefore, for each d_i with $i < k$ there are natural isomorphisms as on the left below which interchange the lexicographical order on the indexing set $V_G(T_k)$ induced by the string $V_G(T_k) \to V_G(T_{k-1}) \to \cdots \to V_G(T_0)$ with the one induced by the string that omits $V_G(T_i)$. For d_i with $i > k$ one has a commutative diagram as on the right below. Note that no such diagram is defined for d_k.

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[d, equal] \arrow[dl, Leftarrow, "\pi_i"] \\
\Omega_G^{n-1} \arrow[r, "V_G^{k-1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[d, "d_{i-k-1}"] \\
\Omega_G^{n-1} \arrow[r, "V_G^k"'] & \mathsf{F}_s \wr \Omega_G^{n-k-2}
\end{tikzcd} \tag{3.99}$$ PIIDEFDI EQ

Similarly, for s_j with $j < k$ (resp. $j \ge k$) one has commutative diagrams as on the left (resp. right) below. Note that for s_k one uses the extra degeneracy $s_{k-k-1} = s_{-1}$.

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[d, equal] \\
\Omega_G^{n+1} \arrow[r, "V_G^{k+1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[d, "s_{j-k-1}"] \\
\Omega_G^{n+1} \arrow[r, "V_G^k"'] & \mathsf{F}_s \wr \Omega_G^{n-k}
\end{tikzcd} \tag{3.100}$$ PIIDEFDI2 EQ

The functors V_G^k and isomorphisms π_i satisfy a number of useful conditions that we now catalog.

PIIPROP PROP

Proposition 3.101. *(a) The composite*

$$\Omega_G^n \xrightarrow{V_G^k} \mathsf{F}_s \wr \Omega_G^{n-k-1} \xrightarrow{V_G^l} \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-l-2} \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Omega_G^{n-k-l-2}$$

equals the functor V_G^{k+l+1}.

(b) The functors V_G^k send pullbacks for the split Grothendieck fibration $\Omega_G^k \to \mathsf{O}_G$ to pullbacks for $\mathsf{F}_s \wr \Omega_G^{n-k-1} \to \mathsf{F}_s$.

(c) The isomorphisms $\pi_i(T_0 \to \cdots \to T_n)$ are pullbacks for the split Grothendieck fibration $\mathsf{F}_s \wr \Omega_G^{n-k-1} \to \mathsf{F}_s$. Moreover, the projection of $\pi_i(T_0 \to \cdots \to T_n)$ onto F_s depends only on $T_0 \to \cdots \to T_i$.

(d) The rightmost diagrams in both PIIDEFDI EQ *(3.99) and* PIIDEFDI2 EQ *(3.100) are pullbacks diagrams in* Cat.

(e) For $i < k$ the composite natural transformation in the diagram below is π_i.

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[r, "\mathsf{F}_s \wr V_G^l"] \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_i"] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-l-2} \arrow[r, "\sigma^0"] \arrow[d, equal] & \mathsf{F}_s \wr \Omega_G^{n-k-l-2} \arrow[d, equal] \\
\Omega_G^{n-1} \arrow[r, "V_G^{k-1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[r, "\mathsf{F}_s \wr V_G^l"'] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-l-2} \arrow[r, "\sigma^0"'] & \mathsf{F}_s \wr \Omega_G^{n-k-l-2}
\end{tikzcd} \tag{3.102}$$

INDPI1 EQ

For $k < i < k + l + 1$ the composite natural transformation in the diagram below is π_{i+1}.

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1} \arrow[r, "\mathsf{F}_s \wr V_G^l"] \arrow[d, "\mathsf{F}_s \wr d_{i-k-1}"'] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-l-2} \arrow[r, "\sigma^0"] \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \mathsf{F}_s \wr \pi_i"] & \mathsf{F}_s \wr \Omega_G^{n-k-l-2} \arrow[d, equal] \\
\Omega_G^{n-1} \arrow[r, "V_G^k"'] & \mathsf{F}_s \wr \Omega_G^{n-k-2} \arrow[r, "\mathsf{F}_s \wr V_G^{l-1}"'] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-l-2} \arrow[r, "\sigma^0"'] & \mathsf{F}_s \wr \Omega_G^{n-k-l-2}
\end{tikzcd} \tag{3.103}$$

INDPI2 EQ

(f) Restricting to the case $k = n$, the pairs (d_i, π_i) and $(s_j, id_{V_G^n})$ satisfy all possible simplicial identities (i.e. those with $i \neq n$). Explicitly, for $0 \leq i' < i < n$ the composite natural transformations in the diagrams

$$\begin{tikzcd}
\Omega_G^n \arrow[r] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_i"] \\
\Omega_G^{n-1} \arrow[r] \arrow[d, "d_{i'}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i'}"] \\
\Omega_G^{n-2} \arrow[r] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^n \arrow[r] \arrow[d, "d_{i'}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i'}"] \\
\Omega_G^{n-1} \arrow[r] \arrow[d, "d_{i-1}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i-1}"] \\
\Omega_G^{n-2} \arrow[r] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd} \tag{3.104}$$

SIMPPI EQ

coincide, and similarly for the face-degeneracy relations.

Proof. (a) follows by induction on k, with $k = 0$ being the definition. More generally (and writing F for F_s) one has

$$\begin{aligned}
\sigma^0(\mathsf{F} \wr V_G^l)V_G^{k+1} &= \sigma^0(\mathsf{F} \wr V_G^l)\sigma^0(\mathsf{F} \wr V_G^k)V_G = \sigma^0\sigma^0(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G \\
&= \sigma^0\sigma^1(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G = \sigma^0(\mathsf{F} \wr \sigma^0)(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G \\
&= \sigma^0\left(\mathsf{F} \wr \left(\sigma^0(\mathsf{F} \wr V_G^l)V_G^k\right)\right)V_G = \sigma^0\left(\mathsf{F} \wr V_G^{k+l-1}\right)V_G = V_G^{k+l}.
\end{aligned}$$

(b) generalizes Lemma VGPULL LEM 3.72, and follows by induction using that result, Lemma FWRGROTH LEM 2.18, and the obvious claim that $\mathsf{F} \wr \mathsf{F} \wr A \xrightarrow{\sigma^0} \mathsf{F} \wr A$ sends pullbacks over $\mathsf{F} \wr \mathsf{F}$ to pullbacks over F.

(c) is clear. Also, (e) and (f) are easy consequences of (b) and (c): since all natural transformations involved consist of pullbacks, one needs only check each claim after forgetting to the F_s coordinate, which is straightforward.

Lastly, (d) is argued by induction on k and n. The case $k=0$ for the rightmost diagram in (3.99) PIIDEFDI EQ follows by the diagram on the left below, combined with Proposition 3.93 SUBSASPULL PROP applied to the bottom and total squares. The general case then follows from the right diagram, with the left square being in the case $k=0$, the middle square being a pullback by induction (and since $\mathsf{F}\wr(-)$ preserves pullback squares), and the rightmost square by direct verification.

$$
\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_i"'] & \mathsf{F}_s\wr\Omega_G^{n-1} \arrow[d, "d_{i-1}"] \\
\Omega_G^{n-1} \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n}"'] & \mathsf{F}_s\wr\Omega_G^{n-2} \arrow[d, "d_{0,\cdots,n-1}"] \\
\Omega_G^0 \arrow[r, "V_G"] & \mathsf{F}_s\wr\Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_i"'] & \mathsf{F}_s\wr\Omega_G^{n-1} \arrow[r, "V_G^k"] \arrow[d, "\mathsf{F}_s\wr d_{i-1}"'] & \mathsf{F}_s^{\wr 2}\wr\Omega_G^{n-k-2} \arrow[r, "\sigma^0"] \arrow[d, "\mathsf{F}_s^{\wr 2}\wr d_{i-1}"'] & \mathsf{F}_s\wr\Omega_G^{n-k-2} \arrow[d, "\mathsf{F}_s\wr d_{i-1}"'] \\
\Omega_G^{n-1} \arrow[r, "V_G"'] & \mathsf{F}_s\wr\Omega_G^{n-3} \arrow[r, "V_G^k"'] & \mathsf{F}_s^{\wr 2}\wr\Omega_G^{n-k-3} \arrow[r, "\sigma^0"'] & \mathsf{F}_s\wr\Omega_G^{n-k-3}
\end{tikzcd}
\tag{3.105}
$$

The claim for the rightmost diagram in (3.100) PIIDEFDI2 EQ follows by the analogous diagrams with the d_i (but not $d_{1,\cdots,n}$, $d_{0,\cdots,n-1}$) replaced by s_j. □

4 Genuine equivariant operads

OP_MONAD_SECTION

In this section we now build the category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads. We will do so by building a monad $\mathbb{F}_G$ on the category $\mathsf{Sym}_G(\mathcal{V}) = \mathsf{Fun}(\Sigma_G^{op},\mathcal{V})$, that we refer to as the category of G-symmetric sequences on $\mathcal{V}$. The underlying endofunctor of $\mathbb{F}_G$ is easy enough to describe. Given $X\in\mathsf{Sym}_G(\mathcal{V})$, $\mathbb{F}_G X$ is given by the left Kan extension diagram

$$
\begin{tikzcd}
(\Omega_G^0)^{op} \arrow[r, "V_G^{op}"] \arrow[d, "\mathsf{lr}"'] & (\mathsf{F}\wr\Sigma_G)^{op} \arrow[r, "(\mathsf{F}\wr X^{op})^{op}"] \arrow[dl, phantom, "\swArrow"] & (\mathsf{F}\wr\mathcal{V}^{op})^{op} \arrow[r, "\Pi"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[urrr, "\mathbb{F}_G X"'] & & &
\end{tikzcd}
\tag{4.1}
$$
FGXDEF EQ

To describe the monad structure on $\mathbb{F}_G$, however, we will find it preferable to separate the left Kan extension step from the remaining construction. As such, we will first build a monad N on a larger category $\mathsf{WSpan}^l(\Sigma_G^{op},\mathcal{V})$ which we then transfer via the (Lan,ι) adjunction in Remark 4.6. RANLANADJ REM

4.1 A monad on spans

WSPAN DEF

Definition 4.2. We will write $\mathsf{WSpan}^l(\mathcal{C},\mathcal{D})$ (resp. $\mathsf{WSpan}^r(\mathcal{C},\mathcal{D})$), which we call the category of *left weak spans* (resp. *right weak spans*), to denote the category with objects the spans

$$\mathcal{C} \xleftarrow{k} A \xrightarrow{X} \mathcal{D},$$

arrows the diagrams as on the left (resp. right) below

$$
\begin{tikzcd}
 & A_1 \arrow[dl, "k_1"'] \arrow[dr, "X_1"] \arrow[dd, "i"] & \\
\mathcal{C} & \arrow[r, phantom, "\swArrow\varphi"] & \mathcal{D} \\
 & A_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A_1 \arrow[dl, "k_1"'] \arrow[dr, "X_1"] \arrow[dd, "i"] & \\
\mathcal{C} & \arrow[r, phantom, "\neArrow\varphi"] & \mathcal{D} \\
 & A_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] &
\end{tikzcd}
\tag{4.3}
$$
TWISTEDARROWRIGHT EQ

which we write as $(i,\varphi)\colon(k_1,X_1)\to(k_2,X_2)$, and composition given in the obvious way.

Remark 4.4. There are canonical natural isomorphisms

$$\mathsf{WSpan}^r(\mathcal{C},\mathcal{D}) \simeq \mathsf{WSpan}^l(\mathcal{C}^{op},\mathcal{D}^{op}). \tag{4.5}$$ LRSPANISO EQ

RANLANADJ REM **Remark 4.6.** The terms *left/right* are motivated by the existence of adjunctions (which are seen to be equivalent by using (4.5))

$$\mathsf{Lan}\colon \mathsf{WSpan}^l(\mathcal{C},\mathcal{D}) \rightleftarrows \mathsf{Fun}(\mathcal{C},\mathcal{D})\colon \iota$$

$$\iota\colon \mathsf{Fun}(\mathcal{C},\mathcal{D}) \rightleftarrows \mathsf{WSpan}^r(\mathcal{C},\mathcal{D})^{op}\colon \mathsf{Ran}$$

where the functors ι denote the obvious inclusions (note the need for the $(-)^{op}$ in the second adjunction) and $\mathsf{Lan}/\mathsf{Ran}$ denote the left/right Kan extension functors.

We will mainly be interested in the span categories $\mathsf{WSpan}^l(\Sigma_G^{op},\mathcal{V}) \simeq \mathsf{WSpan}^r(\Sigma_G,\mathcal{V}^{op})$.

OMEGAGNA NOT **Notation 4.7.** Given a functor $\rho\colon A \to \Sigma_G$, $n \geq 0$, we let $\Omega_G^n \wr A$ denote the pullback in Cat

$$\begin{tikzcd} \Omega_G^n \wr A \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F}_s \wr A \arrow[d] \\ \Omega_G^n \arrow[r, "V_G^n"'] & \mathsf{F}_s \wr \Sigma_G \end{tikzcd} \tag{4.8}$$ OMGGNA

We will write the top V_G^n functor as $V_G^n \wr A$ whenever we need to distinguish such functors.

Explicitly, by Remark 3.97 the objects of $\Omega_G^n \wr A$ are pairs

$$(T_0 \to \cdots \to T_n, (a_{v_{Ge}})_{v_{Ge} \in V_G(T_n)}) \tag{4.9}$$ OMEGAGNA EQ

such that $\rho(a_{v_{Ge}}) = T_{n,v_{Ge}}$, and where $V_G(T_n)$ is ordered lexicographically according to the string $T_0 \to \cdots \to T_n$.

Remark 4.10. Generalizing the notation $\Omega_G^{-1} = \Sigma_G$, we will also write $\Omega_G^{-1} \wr A = A$, in which case $V_G^{-1} \wr A\colon \Omega_G^{-1} \wr A \to \mathsf{F}_s \wr A$ is the obvious "simpleton map" $\delta^0\colon A \to \mathsf{F}_s \wr A$.

Remark 4.11. An alternative, and arguably more suggestive, notation for $\Omega_G^n \wr A$ would be $\Omega_G^n \wr_{\Sigma_G} A$, since we are really defining a "relative" analogue of the wreath product (so that in particular $\Omega_G^n \wr_{\Sigma_G} \Sigma_G \simeq \Omega_G^n$). However, we will prefer $\Omega_G^n \wr A$ due to space concerns.

Remark 4.12. The definition of $\Omega_G^n \wr A$ in (4.8) is unchanged by replacing F_s with F. As such, to avoid cluttering the diagrams in this section we will from now on abuse notation by writing simply F instead of F_s.

Expand on if necessary

Our primary interest here will be in the $\Omega_G^0 \wr (-)$ construction, which can be iterated thanks to the existence of the composite maps $\Omega_G^0 \wr A \to \Omega_G^0 \to \Sigma_G$. The role of the higher strings $\Omega_G^n \wr A1$ will then be to provide more convenient models for iterated $\Omega_G^0 \wr (-)$ constructions. Indeed, Proposition 3.93 can be reinterpreted as providing a canonical identification $\Omega_G^0 \wr \Omega_G^n \simeq \Omega_G^{n+1}$ with the functor $V_G^0 \wr \Omega_G^n$ identified with the functor V_G as defined in Notation 3.91. Moreover, arguing by induction on n, the fact that the rightmost squares in (3.99) are pullbacks (Proposition 3.101) provides further identifications $\Omega_G^k \wr \Omega_G^n \simeq \Omega_G^{n+k+1}$ with $V_G^k \wr \Omega_G^n$ identified with V_G^k as defined by Notation 3.96.

Our first task is now to produce analogous identifications between $\Omega_G^k \wr \Omega_G^n \wr A = \Omega_G^k \wr (\Omega_G^n \wr A)$ and $\Omega_G^{n+k+1} \wr A$ (note that iterated wreath expressions should always be read as bracketed on the right, i.e. we do *not* define the expression $(\Omega_G^k \wr \Omega_G^n) \wr A$). We start by generalizing the key functors from §3.4.

PIIPROPA PROP **Proposition 4.13.** *There are functors*

$$\Omega_G^n \wr A \xrightarrow{V_G^k} \mathsf{F}_s \wr \Omega_G^{n-k-1} \wr A \qquad \Omega_G^n \wr A \xrightarrow{d_i} \Omega_G^{n-1} \wr A \qquad \Omega_G^n \wr A \xrightarrow{s_j} \Omega_G^{n+1} \wr A$$

where $i < n$, and natural isomorphisms

$$\pi_i : V_G^k \Rightarrow V_G^{k-1} \circ d_i$$

for $i < k$. Further, all of these are natural in A and they satisfy all the analogues of the properties listed in Proposition 3.101.

Proof. While not hard to explicitly write formulas for V_G^k, d_i, s_j, π_i (which we list in Remark 4.16), and then verify the desired properties, we here instead argue that the desired properties themselves can be used to uniquely, and coherently, define those functors.

Firstly, the functors V_G are defined from the following diagram

$$\begin{tikzcd}
\Omega_G^{n+1} \wr A \arrow[r, dashed, "V_G"] \arrow[d] & \mathsf{F} \wr \Omega_G^n \wr A \arrow[r, "\mathsf{F} \wr V_G^n"] \arrow[d] & \mathsf{F}^{\wr 2} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_G^{n+1} \arrow[r, "V_G"] & \mathsf{F} \wr \Omega_G^n \arrow[r, "\mathsf{F} \wr V_G^n"] & \mathsf{F}^{\wr 2} \wr \Sigma_G \arrow[r, "\sigma^0"] & \mathsf{F} \wr \Sigma_G
\end{tikzcd} \tag{4.14}$$

ALLSQUARES2 EQ

by noting that the center and right squares are pullbacks, and choosing V_G to be the unique functor such that the top composite is V_G^{n+1}. The higher functors V_G^k are defined exactly as in (3.92), and the analogue of Proposition 3.101(a) follows by the same proof.

The analogue of Proposition 3.101(b) is tautological, as pullback arrows for $\Omega_G^n \wr A \to \mathsf{O}_G$ are defined as compatible pairs of pullbacks in Ω_G^n and $\mathsf{F} \wr A$.

To define d_i we consider the diagram below (for some $i < k$).

$$\begin{tikzcd}
\Omega_G^n \wr A \arrow[rrr, "V_G^k"] \arrow[dr, dashed, "d_i"'] \arrow[dd] & & & \mathsf{F} \wr \Omega_G^{n-k-1} \wr A \arrow[dr, equal] \arrow[dd] & \\
& \Omega_G^{n-1} \wr A \arrow[rrr, "V_G^{k-1}"'] \arrow[dd] \arrow[urr, phantom, "\Leftarrow \pi_i"] & & & \mathsf{F} \wr \Omega_G^{n-k-1} \arrow[dd] \\
\Omega_G^n \arrow[rrr] \arrow[dr, "d_i"'] & & & \mathsf{F} \wr \Omega_G^{n-k-1} \arrow[dr, equal] & \\
& \Omega_G^{n-1} \arrow[rrr, "V_G^{k-1}"'] \arrow[urr, phantom, "\Leftarrow"] & & & \mathsf{F} \wr \Omega_G^{n-k-1}
\end{tikzcd} \tag{4.15}$$

PICUBOIDAB EQ

The desiderata that the top π_i consist of pullback arrows lifting the lower π_i implies that it is uniquely defined by the top V_G^k functor, and hence so is the top composite $V_G^{k-1} d_i$. But since the front face is a pullback square (by arguing by induction on k), there is a unique choice for d_i. The fact that this definition of $d_i \wr A$ is not dependent on k is ensured by natural transformation in (3.102) is π_i. Similarly, that the analogues of the left diagrams in (3.100) hold follows by an identical argument from the fact that the composites of (3.103) are π_{i+1}.

The definitions of the s_j are similar, except easier since there are no π_i to contend with.

The analogue of Proposition 3.101(c),(e),(f) are then tautological, and the analogue of Proposition 3.101(d) follows by an identical argument. $\square$

VGDEFA REM **Remark 4.16.** Explicitly, $V_G^k : \Omega_G^n \wr A \to \mathsf{F} \wr \Omega_G^{n-k-1} \wr A$ is defined by sending (4.9) to

$$\left(\left(T_{k, v_{Gf}} \to \cdots \to T_{n, v_{Gf}}, \left(a_{v_{Ge}}\right)_{v_{Ge} \in V_G\left(T_{n, v_{Gf}}\right)} \right) \right)_{v_{Gf} \in V_G(T_k)} \tag{4.17}$$

VGDEFA EQ

where both $V_G(T_k)$ and $T_{n, v_{Gf}}$ are ordered lexicographically accordind to the obvious strings.

Similarly, functors $d_i : \Omega_G^n \wr A \to \Omega_G^{n-1} \wr A$ for $0 \le i < n$ and $s_j : \Omega_G^n \wr A \to \Omega_G^{n+1} \wr A$ for $-1 \le j \le n$ are defined on (4.9) by performing the corresponding operation on the $T_0 \to \cdots \to T_n$ coordinate and, in the d_i case suitably reordering $V_G(T_n)$.

Remark 4.18. One upshot of Proposition 4.13 is that formally applying the symbol $(-) \wr A$ to the diagrams in Proposition 3.101 yields sensible statements. As such, we will simply refer to the corresponding part of Proposition 3.101 when using one of the generalized claims.

IDEN COR

Corollary 4.19. *One has identifications $\Omega^k_G \wr \Omega^n_G \wr A \simeq \Omega^{n+k+1}_G \wr A$ which identify $V^k_G \wr \Omega^n_G \wr A$ with $V^k_G \wr A$. Further, these are associative in the sense that the identifications*

$$\Omega^k_G \wr \Omega^l_G \wr \Omega^n_G \wr A \simeq \Omega^{k+l+1}_G \wr \Omega^n_G \wr A \simeq \Omega^{k+l+n+2}_G \wr A$$

$$\Omega^k_G \wr \Omega^l_G \wr \Omega^n_G \wr A \simeq \Omega^k_G \wr \Omega^{l+n+1}_G \wr A \simeq \Omega^{k+l+n+2}_G \wr A$$

coincide. Lastly, one obtains identifications

$$d_i \wr \Omega^n_G \simeq d_i \qquad \pi_i \wr \Omega^n_G \simeq \pi_i \qquad s_j \wr \Omega^n_G \simeq s_j \qquad \Omega^k_G \wr d_i \simeq d_{i+k+1} \qquad \Omega^k_G \wr \pi_i \simeq \pi_{i+k+1} \qquad \Omega^k_G \wr s_j \simeq s_{j+k+1}$$

PIIPROP PROP

Proof. The identification $\Omega^k_G \wr \Omega^n_G \wr A \simeq \Omega^{n+k+1}_G \wr A$ follows since by Proposition 3.101(a) both expressions compute the limit of the solid part of the diagram below.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F}^{\wr 2} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega^{n+k+1}_G \arrow[r, "V^k_G"'] \arrow[d] & \mathsf{F} \wr \Omega^n_G \arrow[r, "\mathsf{F}\wr V^n_G"'] \arrow[d] & \mathsf{F}^{\wr 2} \wr \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \Sigma_G \\
\Omega^k_G \arrow[r, "V^k_G"'] & \mathsf{F} \wr \Sigma_G & &
\end{tikzcd}$$

Associativity follows similarly. The remaining identifications are obvious. ☐

We now have all the necessary ingredients to define our monad on spans.

MONAD_DEFINITION

Definition 4.20. Suppose $\mathcal{V}$ has finite products or, more generally, that it is a symmetric monoidal category with diagonals in the sense of Remark 2.16. FINSURJ REM

We define an endofunctor N of $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$ by letting $N(\Sigma_G \leftarrow A \to \mathcal{V}^{op})$ be the span $\Sigma_G \leftarrow \Omega^0_G \wr A \to \mathcal{V}^{op}$ given composition along the diagram

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{op}"] & \mathcal{V}^{op} \\
\Omega^0_G \arrow[r, "V_G"'] \arrow[d] & \mathsf{F} \wr \Sigma_G & & \\
\Sigma_G & & &
\end{tikzcd}$$

and defined on maps of spans in the obvious way.

One has a multiplication $\mu\colon N \circ N \Rightarrow N$ given by the natural isomorphism

$$\begin{tikzcd}
\Sigma_G \arrow[d, equal] & \Omega^1_G \wr A \arrow[l] \arrow[r, "V_G"] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \wr A \arrow[r, "\mathsf{F}\wr V_G"] \arrow[dl, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{op}"] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{op}"] \arrow[dl, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma_G & \Omega^0_G \wr A \arrow[l] \arrow[rr, "V_G"'] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\Pi^{op}"'] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.21}$$

MULTDEFSPAN EQ

where we note that the top right composite in the π_0 square is indeed V^1_G using the inductive description in (the $(-) \wr A$ analogue of) Notation 3.96. INDVNG NOT

Lastly, there is a unit $\eta\colon id \Rightarrow N$ given by the strictly commutative diagrams (where to see that the second square commutes we recall that $V^{-1}_G = \delta^0$)

$$\begin{tikzcd}
\Sigma_G \arrow[d, equal] & A \arrow[l] \arrow[r, equal] \arrow[d, "s_{-1}"'] & A \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma_G & \Omega^0_G \wr A \arrow[l] \arrow[r, "V_G"'] & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{op}"'] & \mathcal{V}^{op}.
\end{tikzcd} \tag{4.22}$$

UNITSPAN EQ

MONSPAN PROP

Proposition 4.23. *(N, μ, η) form a monad on* $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$.

Proof. The natural transformation component of $\mu \circ (N\mu)$ is given by the composite diagram

$$\begin{tikzcd}
\Omega_G^2 \wr A \arrow[r] \arrow[d, "d_1"'] & \mathsf{F} \wr \Omega_G^1 \wr A \arrow[r] \arrow[d, "\mathsf{F}\wr d_0"'] & \mathsf{F}^{\wr 2} \wr \Omega_G^0 \wr A \arrow[r] \arrow[dl, phantom, "\mathsf{F}\wr\pi_0"] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\mathsf{F}\wr\alpha"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^1 \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega_G^0 \wr A \arrow[rr] \arrow[dl, phantom, "\pi_0"] & & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[rr] \arrow[d, "\sigma^0"] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^0 \wr A \arrow[rrr] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.24}$$

ASSOCSPAN1 EQ

whereas the natural transformation component of $\mu \circ (\mu N)$ is given by

$$\begin{tikzcd}
\Omega_G^2 \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega_G^1 \wr A \arrow[r] \arrow[dl, phantom, "\pi_0"] & \mathsf{F}^{\wr 2} \wr \Omega_G^0 \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d] & \mathsf{F}^{\wr} \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^1 \wr A \arrow[rr] \arrow[d, "d_0"'] & & \mathsf{F} \wr \Omega_G^0 \wr A \arrow[r] \arrow[dll, phantom, "\pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] \arrow[dl, phantom, "\alpha"] & & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^0 \wr A \arrow[rrr] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.25}$$

ASSOCSPAN2 EQ

That the rightmost sections of (4.24) and (4.25) coincide follows from the associativity of the isomorphisms α in (2.13). On the other hand, the leftmost sections coincide since they are instances of the "simplicial relation" diagrams in (3.104), as is seen by using (3.102) and (3.103) to reinterpret the top left sections.

As for unit conditions, $\mu \circ (N\eta)$ is represented by

$$\begin{tikzcd}
\Omega_G^0 \wr A \arrow[r] \arrow[d, "s_0"'] & \mathsf{F} \wr A \arrow[r, equal] \arrow[d, "s_{-1}"'] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^1 \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega_G^0 \wr A \arrow[r] \arrow[dl, phantom, "\pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^0 \wr A \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.26}$$

UNITSPAN1 EQ

while $\mu \circ (\eta N)$ is represented by

$$\begin{tikzcd}
\Omega_G^0 \wr A \arrow[r, equal] \arrow[d, "s_{-1}"'] & \Omega_G^0 \wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^1 \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega_G^0 \wr A \arrow[r] \arrow[dl, phantom, "\pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_G^0 \wr A \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.27}$$

UNITSPAN2 EQ

That (4.26) and (4.27) coincide follows analogously by the unital condition for α and the face degeneracy relations in Proposition 3.101(f). $\square$

FGMON SEC

4.2 The genuine equivariant operad monad

Since $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op}) \simeq \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$, Proposition 4.23 and Remark 4.6 give an adjunction

$$\mathsf{Lan} \colon \mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V}) \colon \iota \tag{4.28}$$

LANIOTAADJ EQ

together with a monad N in the leftmost category $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$.

We will now show that under reasonable conditions on $\mathcal{V}$ this monad can be transferred by using Proposition MONADADJ PROP 2.24, i.e. we will show that the natural transformations $\mathsf{Lan} \circ N \Rightarrow \mathsf{Lan} \circ N \circ \iota \circ \mathsf{Lan}$ and $\mathsf{Lan} \circ \iota \Rightarrow id$ are isomorphisms.

This will require us to introduce a slight modification of the category of spans. For motivation, note that iterations $N^{\circ n+1} \circ \iota$ produce spans of the form $\Sigma_G \leftarrow \Omega_G^n \rightarrow \mathcal{V}^{op}$ (where we use the identification $\Omega_G^n \wr \Sigma_G \simeq \Omega_G^n$). As noted in Remark ALLSPLITMAPS REM 3.90, the maps $\Omega_G^n \to \Sigma_G$ are maps of split fibrations over O_G, as are all other simplicial operators d_i, s_j.

Definition 4.29. The category $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$ of *rooted (left) spans* has as objects spans $\Sigma_G^{op} \leftarrow A^{op} \rightarrow \mathcal{V}$ together with a split Grothendieck fibration $\mathsf{r}\colon A \to \mathsf{O}_G$ such that $A \to \Sigma_G$ is a map of split fibrations.

Similarly, arrows are maps of spans that induce maps of split fibrations.

We refer split fibrations $A \to \mathsf{O}_G$ as *root fibrations* and to maps between them as *root fibration maps.*

Remark 4.30. The condition that $A \to \mathsf{O}_G$ be a root fibration requires additional *choices* of root pullbacks. Therefore, the forgetful functor $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V}) \to \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$ is not quite injective on objects.

The relevance of rooted spans is given by the following couple of lemmas.

ROOTFIBPULL LEM **Lemma 4.31.** *If $A \to \Sigma_G$ is a root fibration map then so is $\Omega_G^0 \wr A \to \Omega_G^0$, naturally in A.*

Proof. The hypothesis that $A \to \Sigma_G$ is root fibration map implies that the rightmost map below in is a map of split fibrations over $\mathsf{F} \wr \mathsf{O}_G$.

$$\begin{tikzcd} \Omega_G^0 \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\ \Omega_G^0 \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd} \tag{4.32}$$ ROOTIMPLIESROOT EQ

Since by Lemma VGPULL LEM 3.72 the map V_G sends pullback arrows in Ω_G^0 (over O_G) to pullback arrows of $\mathsf{F} \wr \Sigma_G$ (over $\mathsf{F} \wr \mathsf{O}_G$), the root pullback arrows in $\Omega_G^0 \wr A$ can be defined as compatible pairs of pullback arrows in Ω_G^0 and $\mathsf{F} \wr A$, and the result follows. $\square$

Remark 4.33. Explicitly, if $\psi\colon Y \to X$ is a map in O_G, and $\tilde{T} = (T, (A_{v_{Ge}})_{V_G(T)}) \in \Omega_G^0 \wr A$, the pullback $\psi^* \tilde{T}$ is given by

$$\left(\psi^* T, (\bar{\psi}^*_{Ge} A_{v_{Ge}})_{V_G(\psi^* T)}\right)$$

where $\bar{\psi}$ is the map $\bar{\psi}\colon \psi^* T \to T$ and $\bar{\psi}_{Ge}$ denote the restrictions $\bar{\psi}\colon Ge \to G\bar{\psi}(e)$, as in Remark PULLCOMP REM 3.82.

LANPULLCOMA LEM **Lemma 4.34.** *Suppose that $\mathcal{V}$ is complete and that $\rho\colon A \to \Sigma_G$ is a root fibration map. If the rightmost triangle in*

$$\begin{tikzcd} \Omega_G^0 \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathcal{V}^{op} \\ \Omega_G^0 \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \arrow[ur] \arrow[u, phantom, shift left=3, "\Uparrow"] & \end{tikzcd} \tag{4.35}$$

is a right Kan extension diagram then so is the composite diagram.

Proof. Unpacking definitions using the pointwise formula for right Kan extensions ([McL 13, X.3.1]), it suffices to check that for each $T \in \Omega_G^0$ the induced functor

$$T \downarrow \Omega_G^0 \wr A \xrightarrow{V_G} V_G(T) \downarrow \mathsf{F} \wr A \tag{4.36}$$ LANPULLCOMA EQ

is initial. We will slightly abuse notation by writing $(T \to U, (A_{v_{Gf}})_{V_G(U)})$ for the objects of $T \downarrow \Omega^0_G$, as well as $\big((T_{v_{Ge}} \to U_{\phi(v_{Ge})})_{v_{Ge} \in V_G(T)}, (A_v)_{v \in V}\big)$ for the objects of $V_G(T) \downarrow \mathsf{F} \wr A$, with the map $\phi : V_G(T) \to V$ and the condition $\rho(A_v) = U_v$ left implicit.

By Proposition 2.5, $T \downarrow \Omega^0_G \wr A$ has an initial subcategory $T \downarrow_{\mathsf{r}} \Omega^0_G \wr A$ of those objects such that $T \to U$ is the identity on roots. Similarly, again by Proposition 2.5, $V_G(T) \downarrow \mathsf{F} \wr A$ has an initial subcategory

$$\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r}} A \tag{4.37}$$ INITCAT EQ

of those objects inducing an identity on $\mathsf{F} \wr \mathsf{O}_G$. Moreover, (4.37) comes together with a right retraction r, i.e. a right adjoint to the inclusion i into $V_G(T) \downarrow \mathsf{F} \wr A$, which is built using pullbacks. We now compute the following composite (where we abbreviate expressions $T_{v_{Ge}}$ as $T_{v_{G}e}$ and implicitly assume that tuples with index Ge (resp. Gf) run over $V_G(T)$ (resp. $V_G(U)$)).

$$\begin{array}{ccccc}
T \downarrow_{\mathsf{r}} \Omega^0_G \wr A & \xrightarrow{\quad V_G \quad} & V_G(T) \downarrow \mathsf{F} \wr A & \xrightarrow{\qquad r \qquad} & \displaystyle\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r}} A \\[2ex]
(T \xrightarrow{\psi} U, (A_{Gf})) & \longmapsto & \big((T_{Ge} \to U_{G\psi(e)}), (A_{Gf})\big) & \longmapsto & \big((T_{Ge} \to \psi^*_{Ge} U_{G\psi(e)}), (\psi^*_{Ge} A_{G\psi(e)})\big)
\end{array}$$

Since rooted quotients are isomorphisms, the ψ and ψ_{Ge} appearing above are isomorphisms, and hence the natural transformation $i \circ r \circ V_G \Rightarrow V_G$ is a natural isomorphism. Therefore, to check that V_G is initial it suffices to verify that $r \circ V_G$ is an isomorphism.

But now note that an arbitrary choice of rooted isomorphisms $T_{v_{Ge}} \to U^{\mathsf{r}}_{v_{Ge}}$ uniquely determines a compatible planar structure on T, and thus a unique isomorphism $\psi : U$. Therefore, arbitrary choices of $\psi^*_{Ge} U_{G\psi(e)}$, $\psi^*_{Ge} A_{G\psi(e)}$ uniquely determine U, A_{Gf}, finishing the proof. ☐

Lemmas 4.31 implies that copying Definition 4.20 yields a monad N_{r} on $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma^{op}_G, \mathcal{V})$ lifting the monad N.

Corollary 4.38. *Suppose that finite products in $\mathcal{V}$ commute with colimits in each variable or, more generally, that $\mathcal{V}$ is a monoidal category with diagonals such that $\otimes$ preserves colimits in each variable. Then the functors*

$$\mathsf{Lan} \circ N_{\mathsf{r}} \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan}, \qquad \mathsf{Lan} \circ \iota \Rightarrow id$$

are natural isomorphisms.

Proof. This follows by combining Lemma 4.34 with Lemma 2.19. ☐

THEMONAD DEF **Definition 4.39.** The *genuine equivariant operad monad* is the monad $\mathbb{F}_G$ on $\mathsf{Fun}(\Sigma^{op}_G, \mathcal{V})$ given by

$$\mathbb{F}_G = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

and with multiplication and unit given by the composites

$$\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ N_{\mathsf{r}} \circ N_{\mathsf{r}} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

$$id \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota.$$

We will write $\mathsf{Op}_G(\mathcal{V})$ for the category $\mathsf{Alg}_{\mathbb{F}_G}(\mathcal{V})$ of *genuine equivariant operads.*

Remark 4.40. The functor $\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$ is isomorphic to $\mathsf{Lan} \circ N \circ \iota$, and this isomorphism is compatible with the multiplication and unit in Definition 4.39, and hence we will henceforth simply write N rather than N_{r}.

From this point of view, the role of root fibrations is to guarantee that $\mathsf{Lan} \circ N \circ \iota$ is indeed a monad, though unnecessary to describe the monad structure itself.

REPACKAGERES REM

Remark 4.41. Since a map

$$\mathbb{F}_G X = \mathsf{Lan} \circ N \circ \iota X \to X$$

is adjoint to a map

$$N \circ \iota X \to \iota X$$

one easily verifies that X is a genuine equivariant operad, i.e. a $\mathbb{F}_G$-algebra, iff ιX is a N-algebra.

Moreover, the bar resolution $\mathbb{F}_G^{\circ n+1} X$ is isomorphic to $\mathsf{Lan}\left(N^{\circ n+1}\iota X\right)$.

_REGULAR_SECTION

4.3 Comparison with (regular) equivariant operads

We start by noting that in the case $G = *$, genuine operads simply recover the usual notion of symmetric operads, i.e. $\mathsf{Sym}_*(\mathcal{V}) \simeq \mathsf{Sym}(\mathcal{V})$ and $\mathsf{Op}_*(\mathcal{V}) \simeq \mathsf{Op}(\mathcal{V})$, and in what follows we will adopt the notations $\mathsf{Sym}^G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$ for the corresponding categories of G-objects. Our goal will be to relate these to the categories $\mathsf{Sym}_G(\mathcal{V})$ and $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant sequences and genuine operads.

We will throughout this section fix a total order of G such that the identity e is the first element, though we note that the exact order is unimportant, as any other such choice would lead to unique isomorphisms between the constructions in this section.

We thus have an inclusion functor

$$\begin{aligned} \iota : G \times \Sigma &\hookrightarrow \Sigma_G \\ C &\longmapsto G \cdot C \end{aligned}$$

where $G \cdot C$ is the constant tuple $(C)_{g \in G}$, which we think of as $|G|$ copies of C, planarized according to C and the order on G. Moreover, letting $\Sigma_G^{\mathrm{fr}} \hookrightarrow \Sigma_G$ denote the full subcategory of G-free corollas, there is an induced retraction $\rho : \Sigma_G^{\mathrm{fr}} \to G \times \Sigma$ defined by $\rho\left((C_i)_{1 \le i \le |G|}\right) = G \cdot C_1$ together with isomorphisms $C \simeq \rho(C)$ uniquely determined by the condition that they are the identity on the first tree component C_1.

We now consider the associated adjunctions.

$$\begin{tikzcd} \mathsf{Sym}_G(\mathcal{V}) \arrow[r, "\iota^*"'] & \mathsf{Sym}^G(\mathcal{V}) \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"] \end{tikzcd} \tag{4.42}$$

TWOADJOINTS EQ

Explicitly, we have the formulas (where we write G-corollas as $(C_i)_I$ for $I \in \mathsf{O}_G$)

$$\iota_! Y\left((C_i)_I\right) = \begin{cases} Y(C_1), & (C_i)_I \in \Sigma_G^{\mathrm{fr}} \\ \varnothing, & (C_i)_I \notin \Sigma_G^{\mathrm{fr}} \end{cases}, \quad \iota^* X(C) = X(G \cdot C), \quad \iota_* Y((C_i)_I) = \left(\prod_I Y(C_i)\right)^G,$$

where in the formula for $\iota_*(-)$ the action of G interchanges factors according to the action on the indexing set I. As a side note, note that the formulas for $\iota_!$ and ι_* are independent of the chosen order of G.

REFLCOREFL REM

Remark 4.43. $\iota_!$ essentially identifies $\mathsf{Sym}^G(\mathcal{V})$ as the coreflexive subcategory of sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C) = \varnothing$ whenever C is not a free corolla.

By contrast, ι_* identifies $\mathsf{Sym}^G(\mathcal{V})$ with a far more interesting reflexive subcategory of sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C)$ for each C not a free corolla must satisfy a fixed point condition. Concretely, letting $\varphi : G \to \mathsf{r}(C)$ denote the unique map preserving the minimal element, one has

$$X(C) \xrightarrow{\simeq} X(\varphi^* C)^\Gamma$$

for $\Gamma \le \mathsf{Aut}(\varphi^* C)$ the subgroup preserving the quotient map $\varphi^* C \to C$ under precomposition (note that $\varphi^* C \in \Sigma_G^{\mathrm{fr}}$).

There is an obvious natural transformation $\beta\colon \iota_! \Rightarrow \iota_*$ which for $(C_i)_I \in \Sigma_G^{\mathrm{fr}}$ sends $Y(C_1)$ to the "G-twisted diagonal" of $\prod_I Y(C_i)$. Moreover, letting $\eta_!, \epsilon_!$ (resp. η_*, ϵ_*) denote the unit and counit of the $(\iota_!, \iota^*)$ adjunction (resp. (ι^*, ι_*) adjunction) it is straightforward to check that the following diagram commutes.

$$\begin{tikzcd}
\iota_!\iota^*\iota_* \arrow[r, Rightarrow, "\epsilon_!"] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota_* \arrow[d, Rightarrow, "\eta_!", "\simeq"'] \\
\iota_! \arrow[ur, Rightarrow, "\beta"'] \arrow[r, Rightarrow, "\eta_*"'] & \iota_*\iota^*\iota_!
\end{tikzcd} \tag{4.44}$$

BETADEFSQUARE EQ

Remark 4.45. An exercise in adjunctions shows that any outer square as in (4.44) commutes provided at least one of the adjunctions in 4.42 is (co)reflexive, so that (4.44) can be regarded as an alternative definition of β.

Proposition 4.46. *One has the following:*

(i) the map $\iota^*\mathbb{F}_G \xrightarrow{\eta_*} \iota^*\mathbb{F}_G\iota_*\iota^*$ is an isomorphism, and thus (cf. Prop. 2.24) $\iota^*\mathbb{F}_G\iota_*$ is a monad;

(ii) the map $\iota^*\mathbb{F}_G\iota_! \xrightarrow{\beta} \iota^*\mathbb{F}_G\iota_*$ is an isomorphism of monads;

(iii) the map $\iota_!\iota^*\mathbb{F}_G\iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G\iota_!$ is an isomorphism;

(iv) there is a natural isomorphism of monads $\alpha\colon \mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$.

Proof. We first show (i), starting with some notation. In analogy with Σ_G^{fr}, we write $\Omega_G^{0,\mathrm{fr}}$ for the subcategory of free trees and note that the leaf-root and vertex functors then restrict to functors $\mathsf{lr}\colon \Omega_G^{0,\mathrm{fr}} \to \Sigma_G^{\mathrm{fr}}$, $V_G\colon \Omega_G^{0,\mathrm{fr}} \to \mathsf{F} \wr \Sigma_G^{\mathrm{fr}}$. Moreover, for each $C \in \Sigma_G^{\mathrm{fr}}$ one has an equality of rooted undercategories between $C \downarrow_{\mathsf{r}} \Omega_G^0$ and $C \downarrow_{\mathsf{r}} \Omega_G^{0,\mathrm{fr}}$, and thus $\iota^*\mathbb{F}_G X$ is computed by the Kan extension of the following diagram.

$$\begin{tikzcd}
\Omega_G^{0,\mathrm{fr}} \arrow[r] \arrow[d] & \mathsf{F} \wr \Sigma_G^{\mathrm{fr}} \arrow[r, "\mathsf{F} \wr X"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma_G^{\mathrm{fr}}
\end{tikzcd} \tag{4.47}$$

IFGI EQ

(i) now follows by noting that $X \to \iota_*\iota^* X$ is an isomorphism when restricted to Σ_G^{fr}.

For (ii), to show that $\iota^*\mathbb{F}_G\iota_! \to \iota^*\mathbb{F}_G\iota_*$ is an isomorphism one just repeats the argument in the previous paragraph by noting that $\iota_! \to \iota_*$ is an isomorphism when restricted to Σ_G^{fr}. To check that this is a map of monads, we recall first that the monad structure on $\iota^*\mathbb{F}_G\iota_*$ is given as described in Proposition 2.24. Unpacking definitions, compatibility with multiplication reduces to showing that the composite $\iota_!\iota^* \xrightarrow{\epsilon_!} id \xrightarrow{\eta_*} \iota_*\iota^*$ coincides with $\beta\iota^*$ while compatibility with units reduces to showing that the composite $id \xrightarrow{\eta_!} \iota^*\iota_! \xrightarrow{\iota^*\beta} \iota^*\iota_* \xrightarrow{\epsilon_*} id$ is the identity. Both of these are a consequence of (4.44), following from the diagrams below (where the top composites are identities).

$$\begin{tikzcd}
\iota_!\iota^* \arrow[r, Rightarrow, "\iota_!\iota^*\eta_*"] \arrow[d, Rightarrow, "\epsilon_!"'] & \iota_!\iota^*\iota_*\iota^* \arrow[r, Rightarrow, "\iota_!\epsilon_*\iota^*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_!\iota_*\iota^*"'] & \iota_!\iota^* \arrow[dl, Rightarrow, "\beta\iota^*"] \\
id \arrow[r, Rightarrow, "\eta_*"'] & \iota_*\iota^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota^*\iota_* \arrow[r, Rightarrow, "\eta_!\iota^*\iota_*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota^*\iota_!\iota^*\iota_* \arrow[r, Rightarrow, "\iota^*\epsilon_!\iota_*"] \arrow[d, Rightarrow, "\iota^*\iota_!\epsilon_*"', "\simeq"] & \iota^*\iota_* \\
id \arrow[r, Rightarrow, "\simeq", "\eta_!"'] & \iota^*\iota_! \arrow[ur, Rightarrow, "\iota^*\beta"'] &
\end{tikzcd} \tag{4.48}$$

(iii) amounts to showing that if $X(C) = \varnothing$ whenever $C \notin \Sigma_G^{\mathrm{fr}}$ then it is also $\mathbb{F}_G X(C) = \varnothing$. But since for such $C \notin \Sigma_G^{\mathrm{fr}}$ the undercategory $C \downarrow \Omega_G^0$ consists of trees with at least one non-free vertex (namely the root vertex), the composite

$$C \downarrow \Omega_G^0 \xrightarrow{V_G} \mathsf{F} \wr \Sigma_G \xrightarrow{\mathsf{F}\wr X} \mathsf{F} \wr \mathcal{V}^{op} \xrightarrow{\Pi} \mathcal{V}^{op}$$

is constant equal to $\varnothing$, and (iii) follows.

Finally, we show (iv). We will slightly abuse notation by writing $G\times\Sigma \hookrightarrow \Sigma_G$ for the image of ι and similarly $G\times\Omega^0 \hookrightarrow \Omega^0_G$ for the image of the obvious analogous functor $\iota\colon G\times\Omega^0 \to \Omega^0_G$. The map $\alpha\colon \mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$ is the adjoint to the map $\tilde{\alpha} : \mathbb{F}\iota^* \to \iota^*\mathbb{F}_G$ encoded on spans by the following diagram.

$$\begin{tikzcd}
G\times\Omega^0 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr(G\times\Sigma) \arrow[rr, "\iota^*X"] \arrow[dr] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] \arrow[dr, equal] & & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega^0_G \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\Sigma_G \arrow[rr, "X"'] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op} \\
G\times\Sigma \arrow[dr] & & & & & & & \\
& \Sigma_G & & & & & &
\end{tikzcd} \tag{4.49}$$

MONADEQUIV DEF

That α is a natural isomorphism follows by the previous identifications $C\downarrow_{\mathsf{r}}\Omega^0_G \simeq C\downarrow_{\mathsf{r}}\Omega^{0,\mathrm{fr}}_G$ for $C\in G\times\Sigma$ together with the fact that the retraction $\rho\colon\Omega^{0,\mathrm{fr}}_G \to G\times\Omega^0$ (built just as the retraction $\rho\colon\Sigma^{\mathrm{fr}}_G \to G\times\Sigma$) retracts $C\downarrow_{\mathsf{r}}\Omega^{0,\mathrm{fr}}_G$ to the undercategory $C\downarrow_{\mathsf{r}} G\times\Omega^0$, which is thus initial (as well as final).

Intuitively, the final claim that α is a map of monads follows from the fact that the composite $\mathbb{F}\mathbb{F} \to \iota^*\mathbb{F}_G\iota_!\iota^*\mathbb{F}_G\iota_! \to \iota^*\mathbb{F}_G\mathbb{F}_G\iota_!$ is encoded by the analogous natural transformation of diagrams for strings $G\times\Omega^1 \hookrightarrow \Omega^{1,\mathrm{fr}}_G$. However, since the presence of left Kan extensions in the definitions of $\mathbb{F}$, $\mathbb{F}_G$ can make a rigorous direct proof of this last claim fairly cumbersome, we sketch here a workaround argument. We first consider the adjunction $\iota_!\colon\mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V}) \rightleftarrows \mathsf{WSpan}^l(\Sigma^{op}_G,\mathcal{V})\colon\iota^*$ where $\iota_!$ is composition with ι and ι^* is the pullback of spans. Writing N, N_G for the monads on the span categories, mimicking (4.49) yields a map $\tilde{\alpha}\colon N \to \iota^*N_G\iota_!$ encoded by the diagram (where the front and back squares are pullbacks).

$$\begin{tikzcd}
(G\times\Omega^0)\wr\iota^*A \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr\iota^*A \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] \arrow[dr, equal] & & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega^0_G\wr A \arrow[rr] \arrow[dd] & & \mathsf{F}\wr A \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op} \\
G\times\Omega^0 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr(G\times\Sigma) \arrow[dr] & & & & & \\
& \Omega^0_G \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\Sigma_G & & & & \\
G\times\Sigma \arrow[dr] & & & & & & & \\
& \Sigma_G & & & & & &
\end{tikzcd}$$

The claim that $\tilde{\alpha}$ is a map of monads is then straightforward. Writing

$$\mathsf{Lan}\colon\mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V}) \rightleftarrows \mathsf{Fun}((G\times\Sigma)^{op},\mathcal{V})\colon j \qquad \mathsf{Lan}_G\colon\mathsf{WSpan}^l(\Omega^{op}_G,\mathcal{V}) \rightleftarrows \mathsf{Fun}(\Omega^{op}_G,\mathcal{V})\colon j_G$$

for the span functor adjunctions, $\alpha\colon\mathbb{F}\to\iota^*\mathbb{F}_G\iota_!$ can then be written as the composite

$$\mathsf{Lan}Nj \to \mathsf{Lan}\iota^*N_G\iota_!j \to \iota^*\mathsf{Lan}_G N_G j_G\iota_!$$

where the first map is the isomorphism of monads induced by $\tilde{\alpha}$ and the second map can be shown directly to be a monad map by unpacking the monad structures in Propositions 2.23 and 2.24. ☐

Combining the previous result with Propositions 2.23 and 2.24 now gives the following.

Corollary 4.50. *The adjunctions (4.42) extends to adjunctions*

$$\begin{tikzcd}
\mathsf{Op}_G(\mathcal{V}) \arrow[r, "\iota^*"'] & \mathsf{Op}^G(\mathcal{V}). \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"]
\end{tikzcd} \tag{4.51}$$

TWOADJOINTSOP EQ

In particular, ι_* identifies $\mathsf{Op}^G(\mathcal{V})$ as a reflexive subcategory of $\mathsf{Op}_G(\mathcal{V})$.

Remark 4.52. Remark 4.43 extends to operads mutatis mutandis.

Moreover, the isomorphism $\iota_!\iota^*\mathbb{F}_G\iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G\iota_!$ then shows that $\mathbb{F}_G$ essentially preserves the image of $\iota_!$, and can thus be identified with $\mathbb{F}$ over it.

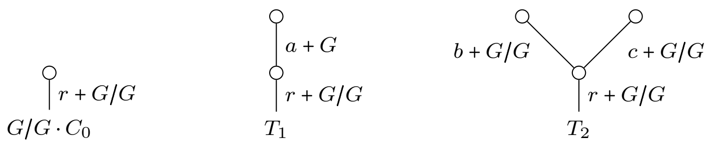
However, the analogous statement fails for ι_*, i.e., one does not always have that

$$\mathbb{F}_G\iota_* \xrightarrow{\eta_*} \iota_*\iota^*\mathbb{F}_G\iota_* \tag{4.53}$$

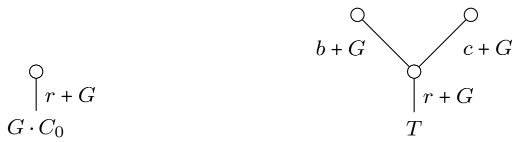
is an isomorphism. In fact, showing that (4.53) *does* become an isomorphism when restricted to suitably cofibrant objects is one of the key technical ingredients for our proof of the Quillen equivalence between $\mathsf{Op}_G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$, and will be the subject of §6.

For now, we end this section with a minimal counterexample to the more general claim. Let $G = \mathbb{Z}_{/2}$ and $Y = * \in \mathsf{Sym}^G(\mathcal{V})$ be the simpleton.

When evaluating $\mathbb{F}_G Y$ at the G-fixed stump corolla $G/G\cdot C_0$, the two G-trees T_1 and T_2 below encode two distinct points (since T_1, T_2 are not isomorphic as objects under $G/G\cdot T_0$).



However, when pulling these points back to the G-free stump corolla $G\cdot C_0$ one obtains the same point, namely that encoded by the G-tree T below.



Moreover, it is not hard to modify the example above to produce similar examples when evaluating $\mathbb{F}_G Y$ at non-empty corollas.

However, such counter-examples all require the use of trees with stumps. Indeed, it can be shown that (4.53) is an isomorphism whenever evaluated at a Y such that $Y(C_0) = \varnothing$.

4.4 Indexing systems and partial genuine operads

In (1.7), universal spaces for families $E\mathcal{F}$ are constructed by cofibrantly replacing the coefficient system $\delta_{\mathcal{F}}$ in $\mathcal{V}^{O_G^{op}}$ (where, to construct EG, $\delta_{\mathcal{F}}$ is in fact the terminal object). Equivalently, we could have cofibrantly replaced the terminal object in the category of *partial* coefficient systems $\mathcal{V}^{O_{\mathcal{F}}^{op}}$, which moreover are the natural target for the Quillen adjunction (equivalence) in [22]

$$\mathcal{V}^{O_{\mathcal{F}}^{op}} \rightleftarrows \mathcal{V}^G_{\mathcal{F}}.$$

In the parallel operadic story, N_∞-operads replace the $E\mathcal{F}$, and the category Op_G just built plays the role of $\mathcal{V}^{O_G^{op}}$. To complete our investigation, we need to characterize those "$\delta_{\mathcal{F}}$" which *can* exist in Op_G, and moreover develop the notion of *partial* genuine operads, to provide a natural setting for the construction of the $N\mathcal{F}$.

We begin by identifying certain useful subcategories of Σ_G.

Definition 4.54. For any category $\mathcal{D}$, a subcategory $\widehat{\mathcal{D}}$ is called a *sieve* if for all maps $f : d \to \widehat{d}$, knowing $\widehat{d} \in \widehat{\mathcal{D}}$ implies both d and the map f are also in $\widehat{\mathcal{D}}$. In particular, sieve subcategories are full.

In §6.1, we will identify sieve subcategories of the orbit category O_G with *families* of subgroups of G. Following this nomenclature, we make the following definition.

Definition 4.55. A *family* of G-corollas is a sieve subcategory $\Sigma_{\mathcal{F}}$ of Σ_G.

Remark 4.56. In fact, families of corollas $\Sigma_{\mathcal{F}}$ are particular examples of (coproducts of) families of subgroups; see Lemma 6.53) for more details.

Due to this, we will abuse notation and abbreviate either set of data simply as $\mathcal{F}$.

Definition 4.57. Let $\mathcal{F}$ be a family of G-corollas.

We say that a G-tree T is a *$\mathcal{F}$-tree* if all of its G-vertices T_v, $v \in V_G(T)$ are in $\Sigma_{\mathcal{F}}$, and we denote the full subcategory spanned by the $\mathcal{F}$-trees $\Omega_{\mathcal{F}} \hookrightarrow \Omega_G$.

Remark 4.58. By vacuousness the stick G-trees $G \cdot_H \eta \simeq G/H \cdot \eta$ are always $\mathcal{F}$-trees.

Now, given a family $\mathcal{F}$ of G-corollas and writing $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) = \mathcal{V}^{\Sigma_{\mathcal{F}}^{op}}$ for the partial G-symmetric sequences determined by $\mathcal{F}$, one may then ask under which conditions the construction of the monad $\mathbb{F}_G$ on $\mathsf{Sym}_G(\mathcal{V})$ of §4.2 can be adapted to build a monad $\mathbb{F}_{\mathcal{F}}$ on $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$.

Writing $\Omega^0_{\mathcal{F}} \hookrightarrow \Omega^0_G$ for the full subcategory of $\mathcal{F}$-trees and quotients, this amounts to asking whether the leaf-root and vertex functors of §3.3 restrict to the $\mathcal{F}$ context. That the vertex functor restricts to a functor $V_G \colon \Omega^0_{\mathcal{F}} \to \mathsf{F} \wr \Sigma_{\mathcal{F}}$ is in fact tautological: indeed, $\Omega_{\mathcal{F}}$ can be defined to be the pre-image $(V_G)^{-1}(\mathsf{F} \wr \Sigma_{\mathcal{F}})$. Compatibility with the leaf-root functor, however, requires an additional closure condition on $\mathcal{F}$, which we now formally introduce.

Definition 4.59. A family $\mathcal{F}$ of G-corollas is called a *weak indexing system* if for any $\mathcal{F}$-tree $T \in \Omega^0_{\mathcal{F}}$ it is $\mathsf{lr}(T) \in \Sigma_{\mathcal{F}}$, i.e. if the leaf-root functor restricts to a functor $\mathsf{lr} \colon \Omega^0_{\mathcal{F}} \to \Sigma_{\mathcal{F}}$.

Additionally, $\mathcal{F}$ is called simply an *indexing system* if all trivial corollas $(G/H) \cdot C_n$ are in $\Sigma_{\mathcal{F}}$.

Remark 4.60. In light of Remark 4.58 any weak coefficient system must contain the 1-corollas $(G/H) \cdot C_1$.

Remark 4.61. The notion and terminology of indexing system was first introduced in [3, Def. 3.22], though packaged quite differently. Moreover, an alternate third definition of indexing systems, also in terms of G-trees but differing slightly from Definition 4.59, was presented by the second author in [17, §9]. The equivalence between the definitions in [3] and [17] was addressed in [17, Rmk. 9.7], hence here we address only the easier equivalence between Definition 4.59 and the description in [17, §9].

In [17, Def. 9.5], $\mathcal{F}$ was defined to be an indexing system if $\mathcal{F}$-trees form a sieve $\Omega_{\mathcal{F}} \hookrightarrow \Omega_G$ (cf. 4.55). The existence of canonical maps $\mathsf{lr}(T) \to T$ is sufficient to show that the condition in [17, Def. 9.5] implies that in Definition 4.59. Conversely, as discussed immediately preceding [17, Def. 9.5], the sieve condition needs only be checked for inner faces and degeneracies, i.e. tall maps, and thus follows from Definition 4.59 since planar tall strings $\Omega^1_{\mathcal{F}}$ between $\mathcal{F}$-trees can be defined as the pullback of $\Omega^0_{\mathcal{F}} \to \mathsf{F} \wr \Sigma_{\mathcal{F}} \leftarrow \mathsf{F} \wr \Omega^0_{\mathcal{F}}$.

Now, for any family of corollas $\mathcal{F}$, let $\gamma : \Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ denote the inclusion. We then have a pair of adjunctions

$$\begin{tikzcd} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \arrow[r, bend left, "\gamma_!"] \arrow[r, bend right, "\gamma_*"'] & \mathsf{Sym}_G(\mathcal{V}) \arrow[l, "\gamma^*"'] \end{tikzcd} \tag{4.62}$$

Remark 4.63. The forgetful map γ^* is always easy to describe. However, only when $\Sigma_{\mathcal{F}}$ is a sieve subcategory of Σ_G (which holds in particular if $\mathcal{F}$ is a weak indexing system) do we have a handle on $\gamma_!$. In this case for $Y \in \mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$, we have

$$\gamma_! Y(C) = \begin{cases} Y(C) & C \in \Sigma_{\mathcal{F}} \\ \varnothing & C \notin \Sigma_{\mathcal{F}}. \end{cases}$$

However, γ_* remains hard to unpack in general[1].

[1] In fact, this formula holds for any $\gamma_! : \mathcal{V}^{\widehat{\mathcal{D}}^{op}} \to \mathcal{V}^{\mathcal{D}^{op}}$ when $\widehat{\mathcal{D}}$ is a sieve of $\mathcal{D}$. One case where γ_* is easy to describe is for $\gamma : G \hookrightarrow O_G$, where it is the fixed-point system functor.

Definition 4.64. Let $\mathbb{F}_{\mathcal{F}}$ denote the endofunctor $\gamma^*\mathbb{F}_G\gamma_!$ on $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$.

It is immediate from Proposition 2.23 that $\mathbb{F}_{\mathcal{F}}$ is a monad for any subcategory $\Sigma_{\mathcal{F}} \subseteq \Sigma_G$.

Definition 4.65. Let $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ denote the category $\mathsf{Alg}_{\mathbb{F}_{\mathcal{F}}}(\mathcal{V})$ of *$\mathcal{F}$-genuine G-operads.*

If $\mathcal{F}$ is not a weak indexing system, our discussion ends here. However, we have the following adjustments of Proposition 4.46 and (4.51) for highly-structured $\mathcal{F}$.

_COMPARISON_PROP **Lemma 4.66.** *Let $\mathcal{F}$ be a weak indexing system.*

(i) the map $\gamma^*\mathbb{F}_G \xrightarrow{\eta_*} \gamma^*\mathbb{F}_G\gamma_*\gamma^*$ is an isomorphism, and thus (cf. Prop. 2.24) $\gamma^*\mathbb{F}_G\gamma_*$ is a monad.

(ii) the map $\gamma^*\mathbb{F}_G\gamma_! \xrightarrow{\beta} \gamma^*\mathbb{F}_G\gamma_*$ is an isomorphism of monads;

(iii) the map $\gamma_!\gamma^*\mathbb{F}_G\gamma_! \xrightarrow{\epsilon_!} \mathbb{F}_G\gamma_!$ is an isomorphism.

Proof. This follows identically as in the proof of Proposition 4.46, by replacing the use of $\mathsf{lr} : \Omega_G^{0,\mathrm{fr}} \to \Sigma_G^{\mathrm{fr}}$ with $\mathsf{lr} : \Omega^0_{\mathcal{F}} \to \Sigma_{\mathcal{F}}$. For example, using the description of $\Omega_{\mathcal{F}}$ as a sieve of Ω_G, it is immediate that for each $C \in \Sigma_{\mathcal{F}}$ one has an equality of rooted undercategories between $C \downarrow_{\mathsf{r}} \Omega^0_G$ and $C \downarrow_{\mathsf{r}} \Omega^0_{\mathcal{F}}$. Thus for $X \in \mathsf{Sym}_G(\mathcal{V})$, $\gamma^*\mathbb{F}_G X$ is computed by the Kan extension of the following diagram,

$$\begin{tikzcd}
\Omega^0_{\mathcal{F}} \arrow[r] \arrow[d] & \mathsf{F} \wr \Sigma_{\mathcal{F}} \arrow[r, "\mathsf{F}\wr X"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma_{\mathcal{F}}
\end{tikzcd} \tag{4.67}$$ F_F_LAN_EQ

and the result follows since $X \to \gamma_*\gamma^* X$ is a isomorphism when restricted to $\Sigma_{\mathcal{F}}$.

The remaining results follow from similarly analogous proofs. ☐

Corollary 4.68. *The adjunction (4.62) extends to an adjunction on algebras*

$$\begin{tikzcd}
\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \arrow[r, bend left, "\gamma_!"] \arrow[r, bend right, "\gamma_*"'] & \mathsf{Op}_G(\mathcal{V}) \arrow[l, "\gamma^*"']
\end{tikzcd}$$

Remark 4.69. Part (iii) above can be interpreted as saying that the monad $\mathbb{F}_G$ restricts to $\mathsf{Sym}_{\mathcal{F}}$, where we view $\mathsf{Sym}_{\mathcal{F}} \xrightarrow{\gamma_!} \mathsf{Sym}_G$ as the inclusion of a full subcategory, when $\mathcal{F}$ is a weak indexing system.

This mimics one possible interpretation of the main results from [3, §4], where indexing systems provide the closure conditions necessary to ensure that composite operations $f(g_1, \ldots, g_n)$ have $\mathcal{F}$-admissible isotropy whenever the x and y_i do; that is, the operadic structure "restricts" to indexing systems.

This view of Part (iii) also indicates a reason why we chose to define $\mathbb{F}_{\mathcal{F}}$ in this way, as opposed to building an entirely new operad given by left Kan extensions as in (4.67) and repeating the arguments from §4.2 mutatis mutandis.

Remark 4.70. We observe that regular G-operads are a type of partially genuine G-operads. Indeed, the family of free G-corollas $\Sigma_G^{\mathrm{fr}} = \Sigma_{\mathcal{F}_{\mathrm{fr}}}$ is an indexing system, and the proof of Proposition 4.46(i) yields that $\mathsf{Op}^G \simeq \mathsf{Op}_{\mathcal{F}_{\mathrm{fr}}}$.

Comparison between different classes of partially-genuine operads can be quite difficult: as hinted at in Remark 4.63, the adjunction maps become less tractible. However, the solid arrows in the combination of (4.42) and (4.62) below

$$\begin{tikzcd}
\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \arrow[r, bend left, "\gamma_!"] \arrow[r, bend right, dashed, "\gamma_*"'] & \mathsf{Op}_G(\mathcal{V}) \arrow[l, "\gamma^*"'] \arrow[r, "\iota^*"'] & \mathsf{Op}^G(\mathcal{V}) \arrow[l, bend right, dashed, "\iota_!"'] \arrow[l, bend left, "\iota_*"]
\end{tikzcd}$$

yields an adjunction comparing all partially-genuine operads with regular G-operads in a managable fashion.

Remark 4.71. A minor warning: if $\Sigma_{\mathcal{F}}$ does not contain the free G-tree $G \cdot C_n$ for some $n \geq 0$ (which is possible even for indexing systems), then the composite $\gamma^* \iota_*$ is no longer injective on objects out of $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$. and hence this adjunction is no longer reflective.

Instead, we will see that it is reflective only on *cofibrant* objects in $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$.

I_STAR_ABUSE **Notation 4.72.** In all statements of results, including the main theorems, we will abuse notation, and denote the composite adjunctions above as simply

$$\iota^* : \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \rightleftarrows \mathsf{Op}^G(\mathcal{V}) : \iota_* \qquad \iota^* : \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \rightleftarrows \mathsf{Sym}^G(\mathcal{V}) : \iota_*.$$

5 Free extensions and the existence of model structures

TENSIONS_SECTION

In order to prove all of our main theorems we will need to homotopically analyze free extensions of genuine equivariant operads, i.e. pushouts of the form

$$\begin{tikzcd} \mathbb{F}_G X \arrow[r] \arrow[d, "\mathbb{F}_G u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_G Y \arrow[r] & \mathcal{P}[u] \end{tikzcd} \tag{5.1}$$ FREE_FG_EXT_EQ

in the category Op_G. As is common in the literature (e.g. [20, 21, 1, 24, 16]), the key technical ingredient will be the identification of a suitable filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \mathcal{P}_\infty = \mathcal{P}[u] \tag{5.2}$$ FILTR EQ

of the map $\mathcal{P} \to \mathcal{P}[u]$ in the underlying category Sym_G. To explain how this filtration is obtained, and abbreviating $\mathbb{F}_G$ as $\mathbb{F}$, note first that $\mathcal{P}[u]$ is given by a coequalizer

$$\mathcal{P} \check{\amalg} \mathbb{F}X \check{\amalg} \mathbb{F}Y \mathrel{\substack{\longrightarrow \\ \leftarrow\!-\!- \\ \longrightarrow}} \mathcal{P} \check{\amalg} \mathbb{F}Y \tag{5.3}$$ REFLCOEQ EQ

where $\check{\amalg}$ denotes the algebraic coproduct, i.e. the coproduct in Op_G, and, a priori, the coequalizer is also calculated in Op_G. However, (5.3) is a so called *reflexive coequalizer*, meaning that the maps being coequalized have a common section, and standard arguments show that it is hence also an underlying coequalizer in Sym_G.

In practice, we will need to enlarge (5.3) somewhat. Firstly, note that (5.3) corresponds to the two bottom levels of the bar construction $B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}Y) = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}Y$, whose colimit (over Δ^{op}) is again $\mathcal{P}[u]$. For technical reasons, we prefer the double bar construction

$$B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y) = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}X \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}Y = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} 2l+1} \check{\amalg} \mathbb{F}Y. \tag{5.4}$$ DOUBAR EQ

To actually describe the individual levels of (5.4) one further resolves $\mathcal{P}$ so as to obtain the bisimplicial object

$$B_l(\mathbb{F}^{n+1}\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y) = \mathbb{F}^{n+1}\mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} 2l+1} \check{\amalg} \mathbb{F}Y \simeq \mathbb{F}\left(\mathbb{F}^n \mathcal{P} \amalg X^{\amalg 2l+1} \amalg Y\right), \tag{5.5}$$ FURRES EQ

where $\amalg$ denotes the coproduct in Sym_G. As in Remark 4.41, each level of (5.5) can then be described as

$$\mathsf{Lan} N(N^n \iota \mathcal{P} \amalg \iota X^{\amalg 2l+1} \amalg \iota Y), \tag{5.6}$$ LANLEVELFOR EQ

for N the span monad (cf. Definition 4.20) and $\amalg$ now the coproduct of spans. In particular, each level of (5.5) is thus a left Kan extension over some category Ω_G^{n,λ_l}, which we explicitly identify in §5.1, giving the first identification below.

$$\mathcal{P}\check{\coprod}_{\mathbb{F}X}\mathbb{F}Y \simeq \mathrm{colim}_{(\Delta\times\Delta)^{op}}\left(\mathsf{Lan}_{\left(\Omega_G^{n,\lambda_l}\to\Sigma_G\right)^{op}} N_{n,l}^{(\mathcal{P},X,Y)}\right) \simeq \mathsf{Lan}_{\left(\Omega_G^{e}\to\Sigma_G\right)^{op}} \tilde{N}^{(\mathcal{P},X,Y)} \tag{5.7}$$

The second identification, which reduces the calculation to a single left Kan extension, is an instance of Proposition 5.40, a result whose proof is straightforward but lengthy, and thus postponed to the appendix. The category Ω_G^e of *extension trees* appearing on the right side is obtained as a categorical realization $\Omega_G^e = |\Omega_G^{n,\lambda_l}|$, which we explicitly describe and analyze in §5.2. In particular, we identify a smaller and more convenient subcategory $\widehat{\Omega}_G^e \hookrightarrow \Omega_G^e$ that is suitably initial, so that Ω_G^e can be replaced with $\widehat{\Omega}_G^e$ in (5.7).

The desired filtration (5.2) then follows from a filtration of the category $\widehat{\Omega}_G^e$ itself, and this is the subject of §5.3.

5.1 Labeled planar strings

In this section we explicitly identify the categories underlying the left Kan extensions in (5.6).

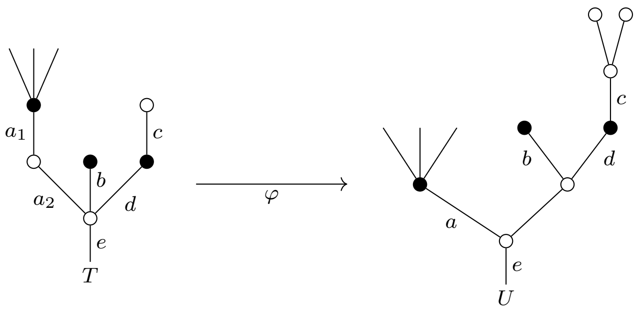
In the notation of Remark 2.28, letting $\langle\langle l\rangle\rangle = \{-\infty,-l,\cdots,-1,0,1,\cdots,l,\infty\}$ and writing λ_l for the partition $\lambda_{l,a} = \{-\infty\}$, $\lambda_{l,i} = \langle\langle l\rangle\rangle - \{-\infty\}$, (5.6) can be repackaged as an instance of the functor $\mathsf{Lan}\circ N\circ\coprod\circ(N^{\times\lambda_l})^{\circ n}\circ\iota^{\times\langle l\rangle}$. Our goal is thus to understand the underlying categories of the spans in the image of the functor $N\circ\coprod\circ(N^{\times\lambda_l})^{\circ n}$, though we will find it preferable and no harder to tackle the more general case of the functors $N^{s+1}\circ\coprod\circ(N^{\times\lambda})^{\circ n-s}$.

Definition 5.8. A *l-node labeled G-tree* (or just *l-labeled G-tree*) is a pair $(T, V_G(T)\to\{1,\cdots,l\})$ with $T\in\Omega_G$, which we think of as a G-tree together with G-vertices labels in $1,\cdots,l$.

Further, a tall map $\varphi\colon T\to S$ between l-labeled trees is called a *label map* if for each G-vertex v_{Ge} of T with label j, the vertices of the subtree $S_{v_{Ge}}$ are all labeled by j.

Lastly, given a subset $J\subset \underline{l}$, a planar label map $\varphi\colon T\to S$ is said to be J-inert if for every G-vertex v_{Ge} of T with label $j\in J$ it is $S_{v_{Ge}} = T_{v_{Ge}}$.

Example 5.9. Consider the 2-labeled trees below (for $G=*$ the trivial group), with black nodes ($\bullet$) denoting labels by the number 1 and white nodes ($\circ$) labels by the number 2. The planar map φ (sending $a_i\mapsto a$, $b\mapsto b$, $c\mapsto c$, $d\mapsto d$, $e\mapsto e$) is a label map which is $\{1\}$-inert.



(5.10)

Definition 5.11. Let $-1\le s\le n$ and $\lambda=\lambda_a\amalg\lambda_i$ a partition of $\{1,2,\cdots,l\}$.

We define $\Omega_G^{n,s,\lambda}$ to have as objects n-planar strings (where $T_{-1}=\mathsf{lr}(T_0)$ as in (3.89))

$$T_{-1}\xrightarrow{\varphi_0}T_0\xrightarrow{\varphi_1}T_1\xrightarrow{\varphi_2}\cdots\xrightarrow{\varphi_s}T_s\xrightarrow{\varphi_{s+1}}T_{s+1}\xrightarrow{\varphi_{s+2}}\cdots\xrightarrow{\varphi_n}T_n \tag{5.12}$$

together with l-labelings of $T_s, T_{s+1},\cdots,T_n$ such that the $\varphi_r, r>s$ are λ_i-inert label maps.

Arrows in $\Omega_G^{n,s,\lambda}$ are quotients of strings $(\pi_r\colon T_r \to T'_r)$ such that $\pi_r, r \geq s$ are label maps.

Further, for any $s < 0$ or $n < s'$ we write

$$\Omega_G^{n,s,\lambda} = \Omega_G^{n,-1,\lambda}, \qquad \Omega_G^{n,s',\lambda} = \Omega_G^{n}. \tag{5.13}$$

EXTRACASES EQ

Intuitively, $\Omega_G^{n,s,\lambda}$ consists of strings that are labeled in the range $s \leq r \leq n$, with the extra cases (5.13) interpreted by infinitely prepending and postpending copies of T_{-1} and T_n to (5.12).

The main case of interest is that of $s = 0$, which we abbreviate as $\Omega_G^{n,\lambda} = \Omega_G^{n,0,\lambda}$, with the remaining $\Omega_G^{n,s,\lambda}$ playing an auxiliary role. The $s = -1$ case also deserves special attention.

Remark 5.14. For $s < 0$ there are identifications

$$\Omega_G^{n,s,\lambda} = \Omega_G^{n,-1,\lambda} \simeq \coprod_{\lambda_a} \Omega_G^n \amalg \coprod_{\lambda_i} \Sigma_G. \tag{5.15}$$

OMEGANMINUSONE EQ

Indeed, since T_{-1} is a G-corolla, the label of its unique G-vertex determines all other labels.

Notation 5.16. We will write $(\Omega_G^n)^{\times\lambda}$ to denote the l-tuple with $(\Omega_G^n)_j^{\times\lambda} = \Omega_G^n$ if $j \in \lambda_a$ and $(\Omega_G^n)_j^{\times\lambda} = \Sigma_G$ if $j \in \lambda_i$. As such, (5.15) is abbreviated as $\Omega_G^{n,-1,\lambda} = \coprod(\Omega_G^n)^{\times\lambda}$.

The $\Omega_G^{n,s,\lambda}$ categories are related by a number of obvious functors, which we now catalog.

Firstly, if $s \leq s'$ there are forgetful functors

$$\Omega_G^{n,s,\lambda} \to \Omega_G^{n,s',\lambda} \tag{5.17}$$

NKNFGT EQ

and the simplicial operators in Notation 3.87 generalize to operators (for $0 \leq i \leq n$, $-1 \leq j \leq n$)

$$\begin{array}{llll} d_i\colon \Omega_G^{n,s,\lambda} \to \Omega_G^{n-1,s-1,\lambda} & i < s & s_j\colon \Omega_G^{n,s,\lambda} \to \Omega_G^{n+1,s+1,\lambda} & j < s \\ d_i\colon \Omega_G^{n,s,\lambda} \longrightarrow \Omega_G^{n-1,s,\lambda} & s \leq i & s_j\colon \Omega_G^{n,s,\lambda} \longrightarrow \Omega_G^{n+1,s,\lambda} & s \leq j \end{array} \tag{5.18}$$

LABSTSIM EQ

which are compatible with the forgetful functors in the obvious way.

We will prefer to reorganize (5.17) and (5.18) somewhat. Defining functions $d_i\colon \mathbb{Z} \to \mathbb{Z}$ and $s_j\colon \mathbb{Z} \to \mathbb{Z}$ by

$$d_i(s) = \begin{cases} s-1, & i < s \\ s, & s \leq i \end{cases} \qquad s_j(s) = \begin{cases} s+1, & j < s \\ s, & s \leq j \end{cases} \tag{5.19}$$

INTERMAPDEF EQ

(5.18) is rewritten as maps $d_i\colon \Omega_G^{n,s,\lambda} \to \Omega_G^{n-1,d_i(s),\lambda}$ and $s_j\colon \Omega_G^{n,s,\lambda} \to \Omega_G^{n+1,s_j(s),\lambda}$. Therefore, we henceforth write simply $\Omega_G^{n,\bullet,\lambda}$ to denote the string of categories $\Omega_G^{n,s,\lambda}$ and forgetful functors, and abbreviate (5.18) as

$$d_i\colon \Omega_G^{n,\bullet,\lambda} \to \Omega_G^{n-1,\bullet,\lambda} \qquad\qquad s_j\colon \Omega_G^{n,\bullet,\lambda} \to \Omega_G^{n+1,\bullet,\lambda} \tag{5.20}$$

LABSTSIM2 EQ

ORDLABEL REM **Remark 5.21.** Considering the ordered sets $\langle n \rangle = \{0 < 1 < \cdots < n < +\infty\}$, the formulas (5.19) define functions $d_i\colon \langle n \rangle \to \langle n-1 \rangle$, $s_j\colon \langle n \rangle \to \langle n+1 \rangle$ which preserve 0 and $+\infty$, except for s_{-1} which preserves only $+\infty$. This recovers the description of Δ^{op} as the category of intervals (i.e. ordered finite sets with a minimum and maximum and maps preserving them).

Next, the vertex functors V_G^k of (3.98) generalize to functors $V_G^k\colon \Omega_G^{n,s,\lambda} \to \mathsf{F}_s \wr \Omega_G^{n-k-1,s-k-1,\lambda}$ given by the same formula

$$(T_{k,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_k)}, \tag{5.22}$$

as in (3.98), except now with the T_m, v_{Ge} inheriting the node labels from T_m (if any).

The diagrams in (3.99) for $i < k$ and $i > k$ now generalize to diagrams

$$
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, equal] \arrow[dl, Rightarrow, "\pi_i"] \\
\Omega_G^{n-1,\bullet,\lambda} \arrow[r, "V_G^{k-1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, "d_{i-k-1}"] \\
\Omega_G^{n-1,\bullet,\lambda} \arrow[r, "V_G^{k}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-2,\bullet,\lambda}
\end{tikzcd}
\tag{5.23}
$$

PIIDEFDILAB EQ

while the diagrams in (3.100) for $j < k$ and $j > k$ generalize to diagrams

$$
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, equal] \\
\Omega_G^{n+1,\bullet,\lambda} \arrow[r, "V_G^{k+1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, "s_{j-k-1}"] \\
\Omega_G^{n+1,\bullet,\lambda} \arrow[r, "V_G^{k}"'] & \mathsf{F}_s \wr \Omega_G^{n-k,\bullet,\lambda}
\end{tikzcd}
\tag{5.24}
$$

PIIDEFDI2LAB EQ

where we note that in all cases the s-index $\bullet$ varies according to (5.18).

Lastly, the $\Omega_G^{n,s,\lambda}$ are also functorial in λ. Explicitly, given $\alpha\colon\{1,\cdots,l\} \to \{1,\cdots,m\}$ and partitions such that $\lambda' \leq \alpha^*\lambda$ one has forgetful functors

$$\Omega_G^{n,s,\lambda'} \to \Omega_G^{n,s,\lambda} \tag{5.25}$$

LAMBINC EQ

compatible with the forgeful functors (5.17), the simplicial operators d_i, s_j and the isomorphisms π_i.

Remark 5.26. When α is the identity and $\lambda' \leq \lambda$ the forgetful functors (5.25) are fully faithful inclusions. However, this is not the case for the (5.17) forgetful functors. Indeed, regarding the map $T \to U$ in (5.10) as an object in $\Omega_G^{1,0,\lambda}$ for $\lambda = \lambda_a \amalg \lambda_i = \{2\} \amalg \{1\} = \{\bullet\} \amalg \{\circ\}$, changing the label of $a_1 \leq a_2$ to a $\bullet$-label produces a non isomorphic object $\bar{T} \to U$ of $\Omega_G^{1,0,\lambda}$ that forgets to the same object of $\Omega_G^{1,1,\lambda}$.

We now extend Notation 4.7.

Notation 5.27. Let $(A_j) = (A_j \to \Sigma_G)_{1\leq j\leq l}$ be a l-tuple of maps over Σ_G. We define $\Omega_G^{n,s,\lambda} \wr (A_j)$ as the pullback

$$
\begin{tikzcd}
\Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[dd] & \mathsf{F} \wr \coprod A_j \arrow[d] \\
 & \mathsf{F} \wr \coprod_l \Sigma_G \arrow[d] \\
\Omega_G^{n,s,\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \Omega_G^{-1,s-n-1,\lambda}
\end{tikzcd}
\tag{5.28}
$$

OMEGAWRTUP EQ

Remark 5.29. To unpack 5.28, note first that by (5.13) $\Omega_G^{-1,r,\lambda}$ is simply either $\Sigma_G^{\amalg l}$ if $r < 0$ or Σ_G for $r \geq 0$. We can thus break down (5.28) into the three cases $s < 0$, $0 \leq s \leq n$ and $n < s$, depicted below.

$$
\begin{tikzcd}
\Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\
\coprod (\Omega_G^n)^{\times\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \coprod_l \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\
\Omega_G^{n,s,\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \coprod_l \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\
\Omega_G^{n} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}
\tag{5.30}
$$

Therefore, for $s > n$ (5.28) coincides with $\Omega_G^n \wr (\coprod_j A_j)$ as defined in Notation 4.7. Moreover, for $s < 0$ both squares in the diagram below are pullbacks and the bottom composites is V_G^n,

$$
\begin{tikzcd}
\coprod(\Omega^n_G)^{\times\lambda} \wr (A_j) \arrow[r, "\coprod(V^n_G)^{\times\lambda}"] \arrow[d] & \coprod \mathsf{F} \wr A_j \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\
\coprod(\Omega^n_G)^{\times\lambda} \arrow[r, "\coprod(V^n_G)^{\times\lambda}"'] & \coprod_l \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F} \wr \coprod_l \Sigma_G
\end{tikzcd}
\tag{5.31}
$$

BOTTOM EQ

so that there is an identification $\Omega^{n,s,\lambda}_G \wr (A_j) \simeq \coprod(\Omega^n_G)^{\times\lambda} \wr (A_j)$, where in the right side $(-) \wr (-)$ is computed entry-wise.

ORDLABEL REM

Remark 5.32. The naturality of the $\Omega^{n,s,\lambda}_G \wr (A_j)$ constructions with regards to λ interacts with the tuple (A_j) in the obvious way, i.e., given $\alpha\colon \{1,\cdots,l\} \to \{1,\cdots,m\}$, $\lambda' \leq \alpha^*\lambda$ and a map $(B_k) \to \alpha^*(A_j)$ one obtains a natural map

$$\Omega^{n,s,\lambda'}_G \wr (B_k) \to \Omega^{n,s,\lambda}_G \wr (A_j).$$

Proposition 5.33. *The analogue statements of Proposition 3.101 hold for the $\Omega^{n,s,\lambda}_G$ and the $\Omega^{n,s,\lambda}_G \wr (A_j)$ constructions, where in the latter case we exclude the statements involving d_n.*

Additionally, the natural squares (for $n \geq -1$)

$$
\begin{tikzcd}
\Omega^{n,n,\lambda}_G \arrow[r, "V^n_G"] \arrow[d] & \mathsf{F} \wr \coprod_l \Sigma_G \arrow[d] \\
\Omega^n_G \arrow[r, "V^n_G"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}
\tag{5.34}
$$

ADDSQUARE EQ

are also pullback squares.

Proof. Firstly, we note that the $\Omega^{n,s,\lambda}_G$ analogues, as well as the claim for (5.34), all follow by keeping track of the labels on the strings, with the only part worthy of note being the analogue of (d), stating that the right squares in (5.23) and (5.24) are pullbacks. Since in these diagrams the s-coordinate $\bullet$ is determined by the top left corner, a direct analysis shows that compatible choices of labels for strings on the top right and bottom left corners do assemble to the correct labels on the top left corner, so that the claim follows by the unlabeled one.

For the more general $\Omega^{n,s,\lambda}_G \wr (A_j)$ constructions, one can either build the general V^k_G, d_i, s_j, π_i explicitly, or simply mimic the argument in Proposition 4.13, thereby reducing to the $\Omega^{n,s,\lambda}_G$ case. □

LABIDEN COR

Corollary 5.35. *For $-1 \leq s \leq n$ there are natural identifications*

$$\Omega^k_G \wr \Omega^{n,s,\lambda}_G \wr (A_j) \simeq \Omega^{n+k+1,s+k+1,\lambda}_G \wr (A_j) \qquad \Omega^{n,s,\lambda}_G \wr (\Omega^k_G)^{\times\lambda} \wr (A_j) \simeq \Omega^{n+k+1,s,\lambda}_G \wr (A_j)$$

which identify $V^k_G \wr \Omega^{n,s,\lambda}_G \wr (A_j)$ with $V^k_G \wr (A_j)$ and $V^n_G \wr (\Omega^k_G)^{\times\lambda} \wr (A_j)$ with $V^n_G \wr (A_j)$.

Further, these identifications are compatible with each other and associative in the obvious ways, and they induce identifications

$$
\begin{array}{lll}
d_i \wr (\Omega^n_G)^{\times\lambda} \simeq d_i & \pi_i \wr (\Omega^n_G)^{\times\lambda} \simeq \pi_i & s_j \wr (\Omega^n_G)^{\times\lambda} \simeq s_j \\
\Omega^k_G \wr d_i \simeq d_{i+k+1} & \Omega^k_G \wr \pi_i \simeq \pi_{i+k+1} & \Omega^k_G \wr s_j \simeq s_{j+k+1}
\end{array}
$$

as well as obvious identifications of forgeful functors.

Proof. This is analogous to Corollary 4.19. For the first identification, the case $s \geq 0$ follows from the diagram below, where we note that the bottom arrow is $V_G^k : \Omega_G^k \to \mathsf{F} \wr \Sigma_G$.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F}^{\wr 2} \wr \coprod (A_j) \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr \coprod (A_j) \arrow[d] \\
\Omega_G^{n+k+1,s+k+1,\lambda} \arrow[r, "V_G^k"'] \arrow[d] & \mathsf{F} \wr \Omega_G^{n,s,\lambda} \arrow[r, "\mathsf{F} \wr V_G^n"'] \arrow[d] & \mathsf{F}^{\wr 2} \wr \coprod_l \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \coprod_l \Sigma_G \\
\Omega_G^{k,k+1,\lambda} \arrow[r, "V_G^k"'] & \mathsf{F} \wr \Omega_G^{-1,0,\lambda} & &
\end{tikzcd}$$

In the $s = -1$ case, the bottom arrow is instead $V_G^k : \Omega_G^{k,k,\lambda} \to \mathsf{F} \wr \Omega_G^{-1,-1,\lambda} = \mathsf{F} \wr \coprod_l \Sigma_G$, in which case one further attaches (5.34) to the diagram.

The second identification is analogous, using the pullback diagram below, with the composite of the central horizontal arrows reinterpreted using (5.31).

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F} \wr \coprod \mathsf{F} \wr A_j \arrow[r] \arrow[d] & \mathsf{F}^{\wr 2} \wr \coprod A_j \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr \coprod A_j \arrow[d] \\
\Omega_G^{n+k+1,s,\lambda} \arrow[r, "V_G^n"'] \arrow[d] & \mathsf{F} \wr \coprod (\Omega_G^k)^{\times\lambda} \arrow[r, "\mathsf{F} \wr \coprod (V_G^n)^{\times\lambda}"'] \arrow[d] & \mathsf{F} \wr \coprod_l \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F}^{\wr 2} \wr \coprod \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \coprod_l \Sigma_G \\
\Omega_G^{n,s,\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \coprod_l \Sigma_G & & &
\end{tikzcd}$$

The additional claims are straightforward. ☐

Remark 5.36. The identifications in Corollary 5.35 allow for the case $n = -1$, which is non-trivial due to the existence of $\Omega_G^{-1,-1,\lambda} = \coprod_l \Sigma_G$, in which case $\Omega_G^{-1,-1,\lambda} \wr (A_j) \simeq \coprod A_j$. For $-1 \leq s \leq n$ the identifications

$$\Omega_G^{n,s,\lambda} = \Omega_G^s \wr \Omega_G^{-1,-1} \wr (\Omega_G^{n-s-1})^{\times\lambda}$$

then show that $\Omega_G^{n,s,\lambda} \wr (-)$ encodes (the underlying category of) the functor $N^{\circ s+1} \coprod (N^{\times\lambda})^{\circ n-s}$. Furthermore, the left commutative square below, where vertical arrows are forgetful functors (and the right diagram merely unpacks notation)

$$\begin{tikzcd}
\Omega_G^{0,-1,\lambda} \arrow[r, "\coprod (V_G^0)^{\times\lambda}"] \arrow[d] & \coprod \mathsf{F} \wr (\Omega_G^{-1})^{\times\lambda} \arrow[r] & \mathsf{F} \wr \Omega_G^{-1,-2,\lambda} \arrow[d, equal] \\
\Omega_G^{0,0,\lambda} \arrow[rr, "V_G^0"] \arrow[d] & & \mathsf{F} \wr \Omega_G^{-1,-1,\lambda} \arrow[d] \\
\Omega_G^{0,1,\lambda} \arrow[rr, "V_G^0"] & & \mathsf{F} \wr \Omega_G^{-1,0,\lambda}
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod (\Omega_G^0)^{\times\lambda} \arrow[r] \arrow[d] & \coprod \mathsf{F} \wr \Sigma_G \arrow[d] \\
\Omega_G^{0,0,\lambda} \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \Sigma_G \arrow[d] \\
\Omega_G^0 \arrow[r] & \mathsf{F} \wr \Sigma_G
\end{tikzcd} \tag{5.37}$$

NATCOP EQ

shows that the forgetful functor $\Omega_G^{0,-1,\lambda} \wr (A_j) \to \Omega_G^{0,0,\lambda} \wr (A_j)$ encodes the natural map $\coprod \circ N \Rightarrow N \circ \coprod$.

EXTTREE SEC

5.2 The category of extension trees

The purpose of this section is to make (5.7) explicit. We start by discussing realizations of simplicial objects in Cat.

Recalling the standard cosimplicial object $[\bullet] \in \mathsf{Cat}^{\Delta}$ given by $[n] = (0 \to 1 \to \cdots \to n)$ yields the following definition.

REAL DEF

Definition 5.38. The left adjoint below is called the *realization* functor.

$$|-| : \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat} : (-)^{[\bullet]}$$

REALEX REM

Remark 5.39. Suppose that $\mathcal{C} \in \mathsf{Cat}$ contains subcategories $\mathcal{C}_h$, $\mathcal{C}^v$ whose arrows span those of $\mathcal{C}$. Defining $\mathcal{C}^v_{h,\bullet} \in \mathsf{Cat}^{\Delta^{op}}$ so that the objects of $\mathcal{C}^v_{h,n}$ are n-strings in $\mathcal{C}_h$ and the arrows are compatible n-tuples of arrows in $\mathcal{C}^v$, it is straightforward to show that it is $|\mathcal{C}^v_{h,\bullet}| = \mathcal{C}$.

An immediate example is given by the planar strings in Definition 3.84. Writing $\mathcal{C} = \Omega^{\mathsf{t}}_G$ the category of tall maps, $\mathcal{C}_h = \Omega^{\mathsf{pt}}_G$ the category of planar tall maps and $\mathcal{C}^v = \Omega^0_G$ the category of quotients, one has $\mathcal{C}^v_{h,\bullet} = \Omega^n_G$ and thus $|\Omega^n_G| = \Omega^{\mathsf{t}}_G$.

Similarly, noting that the $\Omega^{n,\lambda}_G = \Omega^{n,0,\lambda}_G$ form a simplicial object, we have that the $|\Omega^{n,\lambda}_G| = \Omega^{\mathsf{t},\lambda}_G$ is the category of tall maps between l-labeled trees that induce quotients on nodes with λ-inert labels.

In the following statement, we note that it is shown in Lemma A.3 that $ob(|A_\bullet|) \simeq ob(A_0)$ and that arrows in $|A_\bullet|$ are generated by the arrows in A_0 together with arrows $d_1(a) \to d_0(a)$ for each $a \in A_1$.

RANTRANS PROP

Proposition 5.40. *Given a simplicial object $\Sigma_G \leftarrow A_\bullet \xrightarrow{N_\bullet} \mathcal{V}^{op}$ in $\mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$ such that the natural transformation components of the differential operators d_i, $0 \le i < n$ and s_j, $0 \le j \le n$ are isomorphisms, there is an identification*

$$\lim_{\Delta} \left(\mathsf{Ran}_{A_n \to \Sigma_G} N_n\right) \simeq \mathsf{Ran}_{|A_\bullet| \to \Sigma_G} \tilde{N}$$

where $\tilde{N}\colon |A_\bullet| \to \mathcal{V}^{op}$ is given by N_0 on objects and generating arrows in A_0, and on generating arrows $d_1(a) \to d_0(a)$ for $a \in A_1$ as the composite

$$\begin{tikzcd}
A_0 \arrow[dr] & A_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d] & A_0 \arrow[dl] \\
& \mathcal{V}^{op} &
\end{tikzcd}$$

Proposition 5.40 applies to both directions of the bisimplicial object $N(N^{\circ n}\iota\mathcal{P} \amalg \iota X^{\amalg 2l+1}) \amalg Y)$ in (5.6). Indeed, in the n direction all d_i with $0 < i < n$ are induced by the multiplication $NN \to N$ defined in (4.21) while d_0 is induced by the composite $N \circ \coprod \circ N \to NN \circ \coprod \to N \circ \coprod$, with the second map again given by composition and the first induced by the natural map $\coprod \circ N \to N \circ \coprod$, which is encoded by a strictly commutative diagram of spans, as seen using the top part of (5.37) (or, more abstractly, it also suffices to note that N preserves arrows in $\mathsf{WSpan}^l(\Sigma^{op}_G, \mathcal{V})$ given by strictly commutative diagrams). Degeneracies are similar.

As for the l direction, we note that our convention on the double bar construction $B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y)$, is symmetric, with d_l encoding both of the maps $\mathbb{F}X \to \mathbb{F}Y$ and $\mathbb{F}X \to \mathbb{F}\mathcal{P}$ and the remaining differentials given by fold maps. Or, more precisely, the action of the differential operators on the sets of labels $\langle\langle l \rangle\rangle = \{-\infty, -l, \cdots -1, 0, 1, \cdots, l, +\infty\}$ is given by extending the functions in Remark 5.32 anti-symmetrically. But then the differential operators d_i, s_j for $0 \le i < l$ and $0 \le j \le l$ correspond to instances of the naturality in Remark 5.32 when $(B_k) = \alpha^*(A_j)$, and are hence given by strictly commutative maps of spans.

Our next task is thus that of identifying the category of extension trees Ω^{e}_G appearing in (5.7), i.e. to produce an explicit model for the double realization $|\Omega^{n,\lambda_l}_G|$. By Remark 5.39 we can first perform the realization in the n direction, so as to obtain $|\Omega^{n,\lambda_l}_G| = |\Omega^{\mathsf{t},\lambda_l}_G|$, where we recall that $\Omega^{\mathsf{t},\lambda_l}_G$ consists of $\langle l \rangle$-labelled trees together with tall maps that induce quotients on all nodes not labeled by $-\infty$.

We now identify Ω^{e}_G directly.

EXTTREECAT DEF

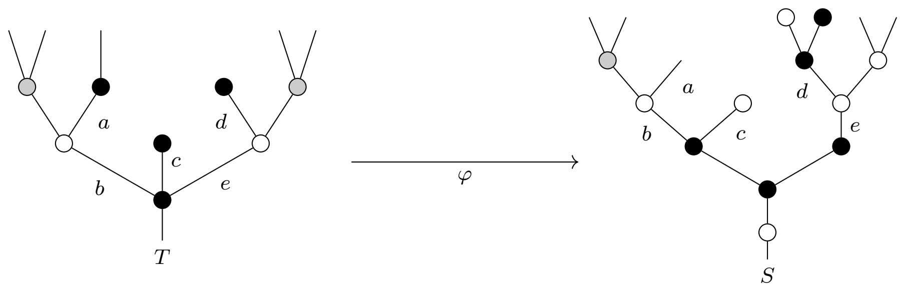
Definition 5.41. The *extension tree category* Ω^{e}_G is the category whose objects are $\{\mathcal{P}, X, Y\}$-labeled trees and whose maps $\varphi\colon T \to S$ are tall maps of trees such that

(i) if $T_{v_{Ge}}$ has a X-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has a X-label;

(ii) if $T_{v_{Ge}}$ has a Y-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has either an X-label or a Y-label;

(iii) if $T_{v_{Ge}}$ has a $\mathcal{P}$-label, then $S_{v_{Ge}}$ has only X and $\mathcal{P}$-labels.

Example 5.42. The following is an example of a planar map in Ω^e_G for $G = *$, where black nodes represent $\mathcal{P}$-labeled nodes, grey nodes represent Y-labeled nodes and white nodes represent X-labeled nodes.



(5.43) REGALTERNMAP EQ

Remark 5.44. By changing any X-labels in $S_{v_{Ge}}$ into Y-labels (resp. $\mathcal{P}$-labels) whenever T_{v_G} has a Y-label (resp. $\mathcal{P}$-label), one obtains a factorization

$$T \to \bar{S} \to S \tag{5.45}$$

LABRE EQ

such that $T \to \bar{S}$ is a label map (cf. Definition 5.8) and $\bar{S} \to S$ is an underlying identity of trees that merely changes some if the Y and $\mathcal{P}$ labels into X labels. We refer to the latter kind of map as a *relabel map*. It is clear that the label-relabel factorization (5.45) is unique.

Proposition 5.46. *There is an identification*

$$\Omega^e_G \simeq |\Omega^{\mathrm{t},\lambda_l}_G|.$$

Proof. We will show that Remark 5.39 applies to $\mathcal{C} = \Omega^e_G$, with $\mathcal{C}_h$ and $\mathcal{C}^v$ the categories of relabel and label maps. More precisely, we claim that there is an isomorphism $\mathcal{C}^v_{h,l} \simeq \Omega^{\mathrm{t},\lambda_l}_G$ of objects in $\mathsf{Cat}^{\Delta^{op}}$. Unpacking notation, one must first show that strings

$$T_0 \to T_1 \to \cdots \to T_l \tag{5.47}$$

RELABSTR EQ

of relabel arrows in Ω^e_G are in bijection with objects of $\Omega^{\mathrm{t},\lambda_l}_G$, i.e. with trees labeled by $\langle l \rangle = \{-\infty, -l, \cdots, -1, 0, 1, \cdots, l, +\infty\}$. Noting that the maps in (5.47) are simply underlying identities on some fixed tree T that convert some of the $\mathcal{P}$, Y labels into X labels, we label a vertex $T_{v_{Ge}}$ by (i) $0 < j \leq +\infty$ if the last j labels of $T_{v_{Ge}}$ in (5.47) are Y labels (where $+\infty = l+1$); (ii) $-\infty \leq -j < 0$ if the last j labels of $T_{v_{Ge}}$ in (5.47) are $\mathcal{P}$ labels; (iii) $j = 0$ if all labels in (5.47) are X labels. This process clearly established the desired bijection on objects.

The compatibilities with arrows and with the simplicial structure are straightforward. ☐

Our next task will be that of identifying a convenient initial subcategory $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$. We first introduce the auxiliary notion of alternating trees. Recall the notion of input path (Notation 3.5) $I(e) = \{f \in T : e \leq_d f\}$ for an edge $e \in T$, which naturally extends to T in any of $\in \Omega, \Phi, \Omega_G, \Phi_G$.

OMEGAA DEF **Definition 5.48.** A G-tree $T \in \Omega_G$ is called *alternating* if, for all leafs $l \in T$ one has that the input path $I(l)$ has an even number of elements.

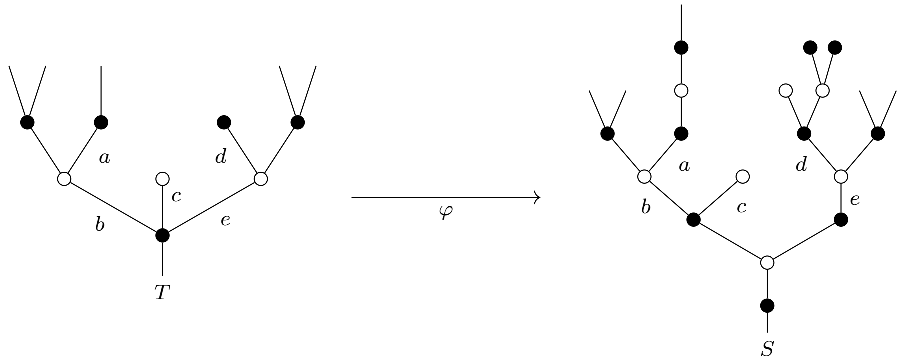
Further, a vertex $e^{\uparrow} \leq e$ is called *active* if $|I(e)|$ is odd and *inert* otherwise.

Finally, a tall map $T \xrightarrow{\varphi} S$ between alternating G-trees is called a *tall alternating map* if for any inert vertex $e^{\uparrow} \leq e$ of T one has that $S_{e^{\uparrow} \leq e}$ is an inert vertex of S.

We will denote the category of alternating G-trees and tall alternating maps by Ω^a_G.

Remark 5.49. A $G-tree$ (resp. map of G-trees) is alternating (resp. an alternating map) iff each component is.

Example 5.50. Two alternating trees (for $G = *$ the trivial group) and a planar tall alternating map between them follow, with active nodes in black ($\bullet$) and white nodes in white ($\circ$).



(5.51) REGALTERNMAPLR EQ

The term "alternating" reflects the fact that adjacent nodes have different colors, though there is an additional restriction: the "outer vertices", i.e. those immediately below a leaf or above the root, are necessarily black/active (this does not, however, apply to stumps).

ALTSUB REM **Remark 5.52.** If $T \in \Omega$ is alternating, it follows from Remark 3.54 that a tall map $\varphi: T \to U$ is an alternating map iff the corresponding substitution datum under Proposition 3.47 is given by the identity $U_{e^\uparrow \le e} = T_{e^\uparrow \le e}$ when $e^\uparrow \le e$ is inert and by an alternating tree $U_{e^\uparrow \le e}$ when $e^\uparrow \le e$ is active.

HATOMEGAE DEF **Definition 5.53.** $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$ is the full subcategory of $(\mathcal{P}, X, Y)$-labeled trees whose underlying tree is alternating, active nodes are labeled by $\mathcal{P}$, and passive nodes are labeled by X or Y.

Note that conditions (i) and (ii) in Definition 5.41 imply that for any map in $\widehat{\Omega}^e_G$ the underlying map is an alternating map.

The following is the key to establishing the desired initiality of $\widehat{\Omega}^e_G$ in Ω^e_G.

LXP PROP **Proposition 5.54.** *For each $U \in \Omega^e_G$ there exists a unique $\mathsf{lr}_{\mathcal{P}}(U) \in \widehat{\Omega}^e_G$ together with a unique planar label map in Ω^e_G*

$$\mathsf{lr}_{\mathcal{P}}(U) \to U.$$

Furthermore, $\mathsf{lr}_{\mathcal{P}}$ extends to a right retraction $\mathsf{lr}_{\mathcal{P}}: \Omega^e_G \to \widehat{\Omega}^e_G$.

Proof. We first address the non-equivariant case $U \in \Omega^e$.

To build $\mathsf{lr}_{\mathcal{P}}(U)$, consider the collection of outer faces $\{U_i^X\} \amalg \{U_j^Y\} \amalg \{U_k^{\mathcal{P}}\}$ where the U_i^X, U_j^Y are simply the X,Y-labeled nodes and the $\{U_k^{\mathcal{P}}\}$ are the maximal outer subtrees whose nodes have only $\mathcal{P}$-labels (these may possibly be sticks). Lemma 3.57 guarantees that each edge and each $\mathcal{P}$-labeled node belong to exactly one of the $V_G(U_k^X)$, and applying Proposition 3.55(iii) yields a planar tall map

$$T = \mathsf{lr}_{\mathcal{P}}(U) \to U \qquad (5.55)$$ LRXDEF EQ

such that $\{U_{e^\uparrow \le e}\}_{(e^\uparrow \le e) \in V(T)} = \{U_i^X\} \amalg \{U_j^Y\} \amalg \{U_k^{\mathcal{P}}\}$. T has an obvious $(\mathcal{P}, X, Y)$-labeling making (5.55) into a label map, but we must still check $T \in \widehat{\Omega}^e_G$, i.e. that T is alternating with active vertices precisely those labeled by $\mathcal{P}$. But since the image of each $e \in T$ belongs to precisely one $U_k^{\mathcal{P}}$, e belongs to precisely one of the $\mathcal{P}$-labeled nodes of T, so that any leaf input path $I(l) = (l = e_n \le e_{n-1} \le \cdots \le e_1 \le e_0)$ must start with, end with, and alternate between $\mathcal{P}$-nodes, and thus have even length.

To check uniqueness, note that for any other planar label map $S \to U$ with S alternating and $e^\uparrow \le e$ a $\mathcal{P}$ vertex of S the outer face $U_{e^\uparrow \le e}$ must be a maximal $\mathcal{P}$-labeled outer face since the vertices adjacent to its root and leaves are labeled by either X or Y. The condition

$V(U) = \coprod_{V(S)} V(U_{e^\uparrow \le e})$ thus guarantees that the collection of outer faces determined by S matches that determined by T except perhaps in the number of stick faces, so that the degeneracy-face factorizations $S \to F \to U$, $T \to F \to U$ factor through the same planar inner face F, with the unique labeling that makes the inclusion a label map. S, T are thus both trees in $\widehat{\Omega}^e_G$ obtained from F by adding degenerate $\mathcal{P}$ vertices, and since this can be done in at most one way, we conclude $S = T$.

To check functoriality, consider the diagram below, where $T \to U$ is the map defined above and $\varphi\colon U \to V$ any map in Ω^e_G.

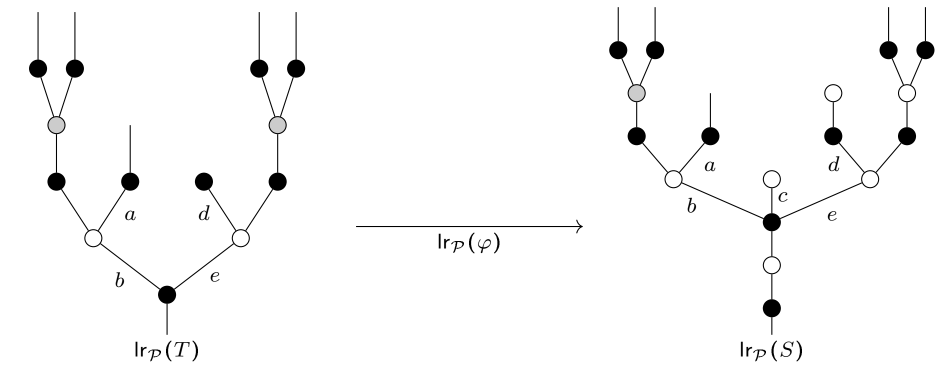
$$\begin{tikzcd} T \arrow[r] \arrow[d, dashed] & U \arrow[d, "\varphi"] \\ S \arrow[r, dashed] & V \end{tikzcd} \tag{5.56}$$

LRPFUN EQ

The composite $T \to V$ is encoded by a substitution datum $\{T_{e^\uparrow \le e} \to V_{e^\uparrow \le e}\}$ which is given by an isomorphism if $e^\uparrow \le e$ has label X or Y (possibly changing a Y label to a X label), and by some $X, \mathcal{P}$-labeled tree $V_{e^\uparrow \le e}$ if $e^\uparrow \le e$ has a $\mathcal{P}$-label. We now consider the factorization problem in (5.56), where we want $S \in \widehat{\Omega}^e_G$ and for the map $S \to V$ to the a planar label map. Combining Remark 5.52 with the uniqueness of the $\mathsf{lr}_{\mathcal{P}}(V_{e^\uparrow \le e})$, the only possibility is for S to be defined using the T substitution datum that replaces $T_{e^\uparrow \le e} \to V_{e^\uparrow \le e}$ with $T_{e^\uparrow \le e} \to \mathsf{lr}_{\mathcal{P}}(V_{e^\uparrow \le e})$ whenever $e^\uparrow \le e$ has a $\mathcal{P}$-label. Uniqueness of $\mathsf{lr}_{\mathcal{P}}(V)$ then implies $S = \mathsf{lr}_{\mathcal{P}}(V)$, and one sets $\mathsf{lr}_{\mathcal{P}}(\varphi)$ to be the map $T \to S$. Associativity and unitality are automatic from the uniqueness of the factorization of (5.56).

In the case of $T = (T_x)_{x \in X}$ in Ω^e_G for a general group G, one sets $\mathsf{lr}_{\mathcal{P}}(T) = (\mathsf{lr}_{\mathcal{P}}(T_x))_{x \in X}$. ☐

Example 5.57. The following illustrates the $\mathsf{lr}_{\mathcal{P}}$ construction when applied to the map φ in (5.43). Intuitively, the functor $\mathsf{lr}_{\mathcal{P}}$ replaces each of the maximal *mathcalP*-labeled subtrees $T^{\mathcal{P}}_k$, $S^{\mathcal{P}}_k$ with the corresponding leaf-root $\mathsf{lr}(T^{\mathcal{P}}_k)$, $\mathsf{lr}(S^{\mathcal{P}}_k)$, which is then $\mathcal{P}$-labeled.



(5.58)

KANRED COR

Corollary 5.59. *The inclusion $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$ is* Ran*-initial over Σ_G. I.e., for $\mathcal{C}$ any a complete category and functor $N\colon \Omega^e_G \to \mathcal{C}$ it is*

$$\mathsf{Ran}_{\Omega^e_G \to \Sigma_G} N \simeq \mathsf{Ran}_{\widehat{\Omega}^e_G \to \Sigma_G} N.$$

Proof. Since $\mathsf{lr}_{\mathcal{P}}$ is a right retraction over Σ_G, the undercategories $C \downarrow \widehat{\Omega}^e_G$ are right retractions of $C \downarrow \Omega^e_G$ for any $C \in \Sigma_G$. ☐

LTRATION_SECTION

5.3 Filtrations of free extensions

Summarizing the previous section, the discussion following Proposition 5.40 establishes (5.7), and hence Corollary 5.59 gives the alternate formula

$$\mathcal{P}[u] \simeq \mathcal{P} \check{\coprod}_{\mathbb{F}X} \mathbb{F}Y \simeq \mathsf{Lan}_{\left(\widehat{\Omega}^e_G \to \Sigma_G\right)^{op}} \tilde{N}^{(\mathcal{P},X,Y)}, \tag{5.60}$$

ALTFOR EQ

which we will now use to filter the map $\mathcal{P} \to \mathcal{P}[u]$ in the underlying category $\mathsf{Sym}_G(\mathcal{V})$.

First, given $T = (T_i)_{i\in I} \in \Omega^e_G$, we write $V^X(T_i)$ (resp. $V^Y(T_i)$) to denote the set of X-labeled (Y-labeled) vertices of T_i. We define *degrees* of T by

$$|T|_X = |V^X(T_i)|, \qquad |T|_Y = |V^Y(T_i)|, \qquad |T| = |T|_X + |T|_Y,$$

which we note do not depend on the choice of $i \in I$.

Similarly, for $T = (T_i)_{i\in I} \in \Omega^a_G$ we write $V^{in}(T_i)$ for the inert vertices and $|T| = |V^{in}(T_i)|$.

Remark 5.61. One key property of the degrees $|T|$, $|T|_X$, $|T|_Y$ is that they are invariant under root pullbacks, which are defined by generalizing Definition 3.24 in the obvious way.

IECES_DEFINITION

Definition 5.62. We specify some rooted (i.e. closed under root pullbacks) full subcategories of $\widehat{\Omega}^e_G$:

(i) $\widehat{\Omega}^e_G[\leq k]$ (resp. $\widehat{\Omega}^e_G[k]$) is the subcategory of T with $|T| \leq k$ ($|T| = k$);

(ii) $\widehat{\Omega}^e_G[\leq k \backslash Y]$ (resp. $\widehat{\Omega}^e_G[k \backslash Y]$) is the subcategory of $\widehat{\Omega}^e_G[\leq k]$ ($\widehat{\Omega}^e_G[k]$) of T with $|T|_Y \neq k$.

Similarly, we define subcategories $\Omega^a_G[\leq k]$, $\Omega^a_G[k]$ of Ω^a_G by the conditions $|T| \leq k$, $|T| = k$.

LIMMOR REM

Remark 5.63. The categories $\widehat{\Omega}^e_G[k]$, $\widehat{\Omega}^e_G[k \backslash Y]$ and $\Omega^a_G[k]$ have rather limited morphisms.

Indeed, it is clear from Definitions 5.41 and 5.48 that maps never lower degree, and Remark 5.52 further ensures that degree is preserved iff $\mathcal{P}$-vertices are substituted by $\mathcal{P}$-vertices (rather than larger trees which would necessarily have inert vertices, thus increasing degree). Therefore, all maps in $\Omega^a_G[k]$ are quotients while maps in $\widehat{\Omega}^e_G[k]$, $\widehat{\Omega}^e_G[k \setminus Y]$ are underlying quotients of G-trees that relabel some Y-vertices to X-vertices. Moreover, this can be repackaged as saying that the diagonal forgetful functors in

$$\begin{tikzcd}
\widehat{\Omega}^e_G[k \setminus Y] \arrow[rr, hook] \arrow[dr] & & \widehat{\Omega}^e_G[k] \arrow[dl] \\
& \Omega^a_G[k] &
\end{tikzcd}$$

are Grothendieck fibrations whose fibers over $T \in \Omega^a_G[k]$ are the punctured cube and cube categories

$$(Y \to X)^{\times V^{in}_G(T)} - Y^{\times V^{in}_G(T)}, \qquad (Y \to X)^{\times V^{in}_G(T)}$$

for $V^{in}_G(T)$ the set of inert G-vertices.

Note that though $|V^{in}(T_i)| = k$ for each T_i composing $T = (T_i)_{i\in I}$, one can only guarantee $|V^{in}_G(T)| \leq k$.

_LAN_FINAL_LEMMA

Lemma 5.64. *$\widehat{\Omega}^e_G[\leq k-1]$ is Ran-initial in $\widehat{\Omega}^e_G[\leq k \setminus Y]$ over Σ_G.*

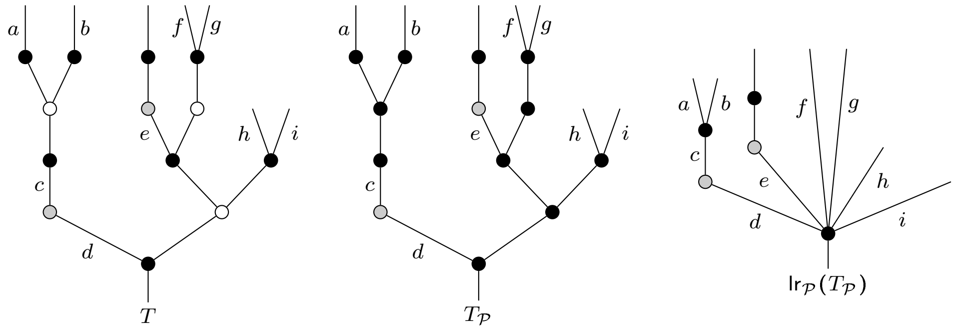
The proof will make use of an additional construction on Ω^e_G: given $T \in \Omega^e_G$ let $T_{\mathcal{P}}$ denote the result of replacing all X-labeled nodes of T with $\mathcal{P}$-labeled nodes.

YINERT REM

Remark 5.65. In contrast to the functor $\mathsf{lr}_{\mathcal{P}}\colon \Omega^e_G \to \widehat{\Omega}^e_G$ of Proposition 5.54, the $(-)_{\mathcal{P}}$ construction does not define a full functor $\Omega^e_G \to \Omega^e_G$, instead being functorial, and the obvious maps $T_{\mathcal{P}} \to T$ natural, only with respect to the Y-inert maps of Ω^e_G.

Example 5.66. Combining the $(-)_{\mathcal{P}}$ and $\mathsf{lr}_{\mathcal{P}}$ constructions one obtains a construction sending trees in $\widehat{\Omega}^e_G$ to trees in $\widehat{\Omega}^e_G$. We illustrate this for the tree $T \in \widehat{\Omega}^e$ below (so that $G = *$), where black nodes are $\mathcal{P}$-labeled, white nodes are X-labeled, and grey nodes are

Y-labeled.



Proof of Lemma 5.64. By Proposition 2.5 it suffices to show that for each $C \in \Sigma_G$ the map of rooted undercategories

$$C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\le k-1] \to C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\le k \setminus Y]$$

is initial, i.e. (cf. ([13, X.3.1])) that for each $(S, \pi : C \to \mathsf{lr}(S))$ in $C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\le k \setminus Y]$ the overcategory

$$(C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\le k-1]) \downarrow (S, \pi) \tag{5.67}$$ UNDERCATPR EQ

is non-empty and connected. By definition of rooted undercategory, π is the identity on roots and thus an isomorphism on Σ_G, so that objects of (5.67) correspond to maps $T \to S$ that induce a rooted isomorphism on lr, i.e. rooted tall maps.

The case $S \in \widehat{\Omega}^e_G[\le k-1]$ is immediate, since then the identity $S = S$ is terminal in (5.67). Otherwise, since $|S|_Y \neq k$ we have $|\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}})| < k$ and the map $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$, which is a rooted tall, shows that (5.67) is indeed non-empty.

Now, consider a rooted tall map $T \to S$ with $T \in \widehat{\Omega}^e_G[\le k-1]$. One can form a diagram

$$\begin{tikzcd}
 & S & \mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \arrow[l] \\
T \arrow[ur] \arrow[r] & T' \arrow[u, "Y\text{-inert}"'] & \mathsf{lr}_{\mathcal{P}}(T'_{\mathcal{P}}) \arrow[l] \arrow[u]
\end{tikzcd} \tag{5.68}$$ K-1LANFINAL EQ

where $T \to T' \to S$ is the natural factorization such that $T' \to S$ is Y-inert, i.e., T' is obtained from T by simply relabeling to X those Y-labeled vertices of T that become X-vertices in S. Note that by Remark 5.65 the existence of the right square relies on $T' \to S$ being Y-inert. Since all maps in (5.68) are rooted tall, this produces the necessary zigzag connecting the objects $T \to S$ and $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$ in the category (5.67), finishing the proof. □

In what follows we write $\tilde{N} : \widehat{\Omega}^{e,op}_G \to \mathcal{V}$ for the functor in (5.60) and any of its restrictions.

We are now in a position to produce the filtration (5.2) of the map $\mathcal{P} \to \mathcal{P}[u]$ in (5.1).

PK_DEFN

Definition 5.69. $\mathcal{P}_k$ is the left Kan extension

$$\begin{tikzcd}
\widehat{\Omega}^e_G[\le k]^{op} \arrow[r, "\tilde{N}"] \arrow[d, "\mathsf{lr}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\Sigma^{op}_G \arrow[ur, "\mathcal{P}_k"'] &
\end{tikzcd}$$

Noting that $\widehat{\Omega}^e_G[\le 0] \simeq \Sigma_G$ (since $|T| = 0$ only if T is a G-corolla with $\mathcal{P}$-labeled vertex) and that $\widehat{\Omega}^e_G$ is the union of (the nerves of) the $\widehat{\Omega}^e_G[\le k]$, we obtain the desired filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \operatorname{colim}_k \mathcal{P}_k = \mathcal{P}[u]. \tag{5.70}$$ FILT EQ

To analyze (5.70) homotopically we will further need a pushout description of each map $\mathcal{P}_{k-1} \to \mathcal{P}_k$. To do so, note that the diagram of inclusions

$$\begin{tikzcd} \widehat{\Omega}^e_G[k \setminus Y] \arrow[r] \arrow[d] & \widehat{\Omega}^e_G[\leq k \setminus Y] \arrow[d] \\ \widehat{\Omega}^e_G[k] \arrow[r] & \widehat{\Omega}^e_G[\leq k] \end{tikzcd} \tag{5.71}$$

INCDIAG EQ

is a pushout of at the level of nerves. Indeed, this follows since

$$\widehat{\Omega}^e_G[k] \cap \widehat{\Omega}^e_G[\leq k \setminus Y] = \widehat{\Omega}^e_G[k \setminus Y], \qquad \widehat{\Omega}^e_G[k] \cup \widehat{\Omega}^e_G[\leq k \setminus Y] = \widehat{\Omega}^e_G[\leq k],$$

and since a map $T \to S$ in $\widehat{\Omega}^e_G[\leq k]$ is in one of subcategories in (5.71) iff T is.

Since Lemma 5.64 provides an identification $\mathsf{Lan}_{\widehat{\Omega}^e_G[\leq k \setminus Y]^{op}} \tilde{N} \simeq \mathsf{Lan}_{\widehat{\Omega}^e_G[\leq k-1]^{op}} \tilde{N} = \mathcal{P}_{k-1}$, applying left Kan extensions to (5.71) yields the pushout diagram below.

$$\begin{tikzcd} \mathsf{Lan}_{\widehat{\Omega}^e_G[k \setminus Y]^{op}} \tilde{N} \arrow[r] \arrow[d] & \mathcal{P}_{k-1} \arrow[d] \\ \mathsf{Lan}_{\widehat{\Omega}^e_G[k]^{op}} \tilde{N} \arrow[r] & \mathcal{P}_k \end{tikzcd} \tag{5.72}$$

FILTRATION_LAN_SQUARE_

We will also make use of an explicit levelwise description of (5.72).

Proposition 5.73. *For each G-corolla $C \in \Sigma_G$, (5.72) is given by the following pushout in $\mathcal{V}^{\mathsf{Aut}(C)}$*

$$\begin{tikzcd} \displaystyle\coprod_{[T] \in \mathsf{Iso}\left(C \downarrow_{\mathsf{r}} \Omega^a_G[k]\right)} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes Q^{in}_T[u] \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C) \arrow[r] \arrow[d] & \mathcal{P}_{k-1}(C) \arrow[d] \\ \displaystyle\coprod_{[T] \in \mathsf{Iso}\left(C \downarrow_{\mathsf{r}} \Omega^a_G[k]\right)} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \bigotimes_{v \in V^{in}_G(T)} Y(T_v) \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C) \arrow[r] & \mathcal{P}_k(C) \end{tikzcd} \tag{5.74}$$

FILTRATION_LAN_LEVEL

where $V^{ac}_G(T)$, $V^{in}_G(T)$ denote the active and inert vertices of $T \in \Omega^a_G[k]$, and $Q^{in}_T[u]$ is the domain of the iterated pushout product

$$\mathop{\square}_{v \in V^{in}_G(T)} u(T_v) \colon Q^{in}_T[u] \to \bigotimes_{v \in V^{in}_G(T)} Y(T_v).$$

Proof. This is a consequence of Remark 5.63. Iteratively computing left Kan extensions by first left Kan extending to $\Omega^a_G[k]$, we can rewrite the leftmost map in (5.72) as

$$\mathsf{Lan}_{\Omega^a_G[k]^{op} \to \Sigma^{op}_G} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v \in V^{in}_G(T)} u(T_v) \right). \tag{5.75}$$

FILTINTALT EQ

The desired description of the leftmost map given in (5.74) now follows by noting that the rooted undercategories $C \downarrow_{\mathsf{r}} \Omega^a_G[k]$ are groupoids. $\square$

HERE

We can use the previous result to build an analogous filtration for partially genuine G-operads.

LEVEL_FILTRATION **Corollary 5.76.** *Let $\mathcal{F}$ be a weak indexing system. For any $\mathbb{F}_{\mathcal{F}}$-free extension*

$$\begin{tikzcd} \mathbb{F}_{\mathcal{F}}X \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_{\mathcal{F}}Y \arrow[r] & \mathcal{P}[u] \end{tikzcd} \tag{5.77}$$ FF_FREE_EXTENSION

in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$, the map $\mathcal{P} \to \mathcal{P}[u]$ has a filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \operatorname{colim}_k \mathcal{P}_k = \mathcal{P}[u].$$

Moreover, for each $\mathcal{F}$-corolla $C \in \Sigma_{\mathcal{F}}$, the map $\mathcal{P}_{k-1}(C) \to \mathcal{P}_k(C)$ is given by a pushout in $\mathcal{V}^{\mathsf{Aut}(C)}$ as in (5.74), except replacing both instances of Ω^a_G (used in indexing the coproduct) with $\Omega^a_{\mathcal{F}}$, where $\Omega^a_{\mathcal{F}}[k]$ denotes alternating $\mathcal{F}$-trees with exactly k passive vertices.

Proof. Any span defining the pushout in (5.77) is equivalent to the data of the solid span of the following pushout diagram in $\mathsf{Op}_G(\mathcal{V})$.

$$\begin{tikzcd} \mathbb{F}_G\gamma_!X \arrow[r] \arrow[d, "u"'] & \gamma_*\mathcal{P} \arrow[d, dashed] \\ \mathbb{F}_G\gamma_!Y \arrow[r, dashed] & (\gamma_*\mathcal{P})[u] \end{tikzcd} \tag{5.78}$$ F_FG_FREE_EXTENSION

Further, since γ^* is a left adjoint into $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ and $\gamma^*\gamma_*$ is the identity, (5.77) is in fact the image of (5.78) under γ^*, and hence the filtration and pushout description for (5.78) from (5.74) induce a filtration and pushout description on the map

$$\mathcal{P} \to \mathcal{P}[u] = \gamma^*((\gamma_*\mathcal{P})[u]).$$

Thus, we see that for any $C \in \Sigma_{\mathcal{F}}$, the map $\mathcal{P}_{k-1}(C) \to \mathcal{P}_k(C)$ is given by the pushout in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ over the map (cf. (5.75))

$$\gamma^* \operatorname{Lan}_{\Omega^a_G[k]^{op}} \left(\bigotimes_{v \in V^{ac}_G(T)} \gamma_*\mathcal{P}(T_v) \otimes \mathop{\square}_{v \in V^{in}_G(T)} \gamma_! u(T_v) \right) \tag{5.79}$$ F_FILTINTALT_EQ

However, since $\Omega_{\mathcal{F}}$ is a seive of Ω_G, for any $C \in \Sigma_{\mathcal{F}}$ we again have an equality of undercategories between $C \downarrow \Omega^a_G[k]$ and $C \downarrow \Omega^a_{\mathcal{F}}[k]$, and thus

$$\gamma^* \operatorname{Lan}_{\Omega^a_G[k]^{op}} \simeq \operatorname{Lan}_{\Omega^a_{\mathcal{F}}[k]^{op}},$$

yielding the desired filtration (as $\gamma^*\gamma_* = \gamma^*\gamma_! = id$). □

XISTENCE_SECTION

5.4 Projective model structures

We begin our homotopical analysis by describing general notions of projective model structures on diagram categories, starting with a categorically simple case.

F_MODEL_DEFN **Definition 5.80** ([22]). Let Π be a finite group, and $\mathcal{F}$ a set of subgroups of Π. A map $f \in \mathcal{V}^{\Pi}$ is called an *$\mathcal{F}$-weak equivalence* (resp. *$\mathcal{F}$-fibration*) if f^H is so in $\mathcal{V}$ for all $H \in \mathcal{F}$.

Definition 5.81. The *$\mathcal{F}$-model structure* on $\mathcal{V}^{\Pi}$, if it exists, is the unique model structure with $\mathcal{F}$-weak equivalences and $\mathcal{F}$-fibrations.

F_MODEL_REM **Remark 5.82.** (i) If $\mathcal{F}$ is the set of all subgroups of Π, this is called the *genuine* model structure, and denoted $\mathcal{V}^{\Pi}_{\text{gen}}$.

(ii) If $\mathcal{F}$ is *empty*, then all maps are both fibrations and weak equivalences in $\mathcal{V}^{\Pi}_{\mathcal{F}}$, while cofibrations are just the isomorphisms.

Definition 5.83. We say $\mathcal{V}$ is *admissible for finite groups* if, for any finite group Π and any collection of subgroups $\mathcal{F}$, the $\mathcal{F}$ model structure $\mathcal{V}^{\Pi}_{\mathcal{F}}$ exists. In this case, the generating (trivial) cofibrations for $\mathcal{V}^{\Pi}_{\mathcal{F}}$ are given by $\Pi/H \cdot i$ for $H \in \mathcal{F}$ and i a generating (trivial) cofibration of $\mathcal{V}$.

In particular, if $\mathcal{V}$ has *cellular fixed points* (see Definition 6.4), then [22, 2.6] says preicsely that $\mathcal{V}$ is admissible for finite groups.

_FGTRIGHT_REMARK **Remark 5.84.** We record the following standard facts (generalized in 6.9, 6.11, and 6.23):

(i) If $\phi : \Pi \to \widehat{\Pi}$ is a group homomorphism, then both the induction map $\phi_! = \widehat{\Pi} \cdot_{\Pi} (-)$ and the forgetful functor ϕ^* in the adjunction

$$\phi_! : \mathcal{V}^{\Pi}_{gen} \leftrightarrows \mathcal{V}^{\widehat{\Pi}}_{gen} : \phi^*$$

are left Quillen for the genuine model structures (where ϕ^* is left Quillen against the coinduction map ϕ_*).

(ii) The symmetric monoidal product on $\mathcal{V}$ extends to a left Quillen functor

$$\mathcal{V}^{\Pi}_{gen} \times \mathcal{V}^{\widehat{\Pi}}_{gen} \to \mathcal{V}^{\Pi \times \widehat{\Pi}}_{gen}.$$

Definition 5.85. Now, let $\mathcal{D}$ be any small category, and let $\mathcal{F} = \{\mathcal{F}_d\}_{d \in \mathcal{D}}$ be a collection of sets $\mathcal{F}_d$ of subgroups of $\mathsf{Aut}(d)$. The *projective $\mathcal{F}$ model structure* on $\mathcal{V}^{\mathcal{D}}$, if it exists, is the model structure denoted $\mathcal{V}^{\mathcal{D}}_{\mathcal{F}}$, where a map f is a weak equivalence (resp. fibration) iff each level $f(d)$ is an $\mathcal{F}_d$-weak equivalence (resp. $\mathcal{F}_d$-fibration).

Equivalently, this is the model structure transferred, via the technology of Kan [12, 11.3.2] or Schwede-Shipley [20, 2.3] along the adjunction

$$\mathcal{V}^{\mathcal{D}}_{\mathcal{F}} \underset{\longrightarrow}{\longleftarrow} \mathcal{V}^{\mathrm{Ob}(\mathcal{D})}_{\mathcal{F}} = \prod_{d \in \mathcal{D}} \mathcal{V}^{\mathsf{Aut}(d)}_{\mathcal{F}_d}.$$

We will let $\mathcal{V}^{\mathcal{D}}$ denote the usual projective model structure (where each $\mathcal{F}_d$ just contains the trivial subgroup), and $\mathcal{V}^{\mathcal{D}}_{\mathrm{gen}}$ denote the *genuine* projective model structure (where each $\mathcal{F}_d$ is the complete set of subgroups of $\mathsf{Aut}(d)$).

VDF_EXISTS_REM **Remark 5.86.** If $\mathcal{V}$ is admissible for finite groups, then $\mathcal{V}^{\mathcal{D}}_{\mathcal{F}}$ exists, and generating (trivial) cofibrations of $\mathcal{V}^{\mathcal{D}}_{\mathcal{F}}$ are of the form

$$\mathcal{D}(d,-)/H \cdot i, \qquad d \in \mathcal{D},\ H \in \mathcal{F}_d,\ i \text{ a generating (trivial) cofibration of } \mathcal{V}.$$

VEL_COFIB_REMARK **Remark 5.87.** We record that for any cofibration f in $\mathcal{V}^{\mathcal{D}}_{\mathrm{gen}}$ (and hence any in $\mathcal{V}^{\mathcal{D}}_{\mathcal{F}}$ for any $\mathcal{F}$), $f(d)$ is a cofibration in $\mathcal{V}^{\mathrm{Aut}(d)}_{\mathrm{gen}}$ for all $d \in \mathcal{D}$. Indeed, pushouts are levelwise, and $D(d', d)/H' \cdot i$, as a map in $\mathcal{V}^{\mathsf{Aut}(d)}_{\mathrm{gen}}$, is a coproduct of generating cofibrations.

In what follows, we will need our base model category $\mathcal{V}$ to be sufficiently well-behaved such that all of the above projective model structures exist. In fact, we will need stronger assumptions than just admissible for finite groups or even having cellular fixed points: we will require $\mathcal{V}$ to be *strongly cellular* (from Definition 6.8)

and possible other things as well

5.5 Model structures on G-operads

P_EXISTS_SECTION

We begin by analyzing the homotopy theory of regular G-operads.

For the remainder of this subsection, fix a collection $\mathcal{F} = \{\mathcal{F}_n\}$ of sets $\mathcal{F}_n$ of subgroups, closed under conjugation, of $G \times \Sigma_n$.

Definition 5.88. The $\mathcal{F}$*-model structure* on $\mathsf{Sym}^G(\mathcal{V})$, if it exists, is the model structure

$$\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \coloneqq \prod_n \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}.$$

Lemma 5.89. *If $\mathcal{V}$ is admissible for finite groups, the model category $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$ exists.* □

We identify the following maps in $\mathsf{Op}^G(\mathcal{V})$ (cf. Definition 5.80).

F_MAPS_DEFN **Definition 5.90.** We say a map $f : \mathcal{O} \to \mathcal{P}$ in $\mathsf{Op}^G(\mathcal{V})$ is a

(i) $\mathcal{F}$*-weak equivalence* (resp. $\mathcal{F}$*-fibration*) if $f(n) : \mathcal{O}(n) \to \mathcal{P}(n)$ is one in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ for all $n \geq 0$.

(ii) *level* $\mathcal{F}$*-cofibration* if $f(n)$ is a cofibration in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ for all $n \geq 0$.

Definition 5.91. The $\mathcal{F}$ *(semi) model structure* is the unique (semi) model structure on $\mathsf{Op}^G(\mathcal{V})$, if it exists, with $\mathcal{F}$-weak equivalences and $\mathcal{F}$-fibrations, and will be denoted $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$.

Equivalently, the $\mathcal{F}$-model structure is the (semi) model structure transferred across the adjunction

$$\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \underset{}{\overset{\mathbb{F}_G}{\leftrightarrows}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$$

If each $\mathcal{F}_n$ is the complete lattice of subgroups of $G \times \Sigma_n$, we call the above the *complete* model structure.

Remark 5.92. A *semi* model structure is a mildly weaker homotopical structure, first discussed in [?] and [21]. Intuitively, they have all of the same properties as Quillen model structures, with the exception that trivial cofibrations only have the usual lifting property if their domain is cofibrant. These structures can arise when attempting to use the machinery of Kan [12, 11.3.2] (or equivalently Schwede-Shipley [20, 2.3]) to transfer model structures. As is often the case (e.g. [?, 21, 1, ?, 24, 25]), Condition (2) can only be proven for relative FJ-cell complexes with cofibrant domains. In this situation, while a full model structure cannot be transferred, Kan's proof implies that semi model structures can. Hence, abusing citations somewhat, the remaining references to [12, 11.3.2] may refer to either that lemma or the analogous semi model structure result.

See Section 2.2 of [25] for a complete definition and further analysis of semi model structures.

The key technical step in the proof of Theorem I follows as an easy particular case, which we now state, of the stronger result Proposition 6.66.

COFPUSH_GEN_PROP **Proposition 5.93.** *Suppose $\mathcal{V}$ is strongly cellular with diagonals, and we are given $T \in \Omega^a$ along with level genuine cofibrations $f_{ac} : Q \to \mathcal{P}$ and $f_{in} : X \to Y$ in $\mathsf{Sym}^G(\mathcal{V})$. Then the iterated box product*

$$f^{\square V(T)} = \mathop{\square}_{v\in V_{ac}(T)} f_{ac}(v) \,\square \mathop{\square}_{v\in V_{in}(T)} f_{in}(v)$$

is a cofibration in $\mathcal{V}^{G\times\Sigma_T}_{gen}$.

EVEL_COFIB_PROP2 **Corollary 5.94.** *For $\mathcal{V}$ strongly cellular, and for any free $\mathbb{F}$-extension*

$$\begin{tikzcd} \mathbb{F}X \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}Y \arrow[r] & \mathcal{P}[u] \end{tikzcd}$$

where $u : X \to Y$ is a level genuine cofibration and $\mathcal{P}$ is level genuine cofibrant in $\mathsf{Sym}^G(\mathcal{V})$, the map $\mathcal{P} \to \mathcal{P}[u]$ is a level cofibration, trivial if u is so.

Proof. As $\mathsf{Op}(\mathcal{V}^G) = \mathsf{Op}^G(V)$, we consider the filtration in (5.74) with $G = *$ and $\mathcal{V} = \mathcal{V}^G$. Then the result follows from Corollary 5.94 and Remark 5.84(i) applied to $\mathcal{V}^{G\times\Sigma_T}_{\text{gen}} \to \mathcal{V}^{G\times\Sigma_n}_{\text{gen}}$, as any map $f \otimes \mathcal{P}(T_v)$ is equivalently given by $f \square (\varnothing \to \mathcal{P}(T_v))$. □

Remark 5.95. Note that the "with diagonals" condition was not necessary for the above proof, even though the cited filtration normally requires it. Indeed, when restricting the machinery of §4 to $G = *$ and $\mathcal{V} = \mathcal{V}^{\mathcal{C}}$, Remark 3.7 no longer applies, and hence we may reformulate the monad on spans from Definition 4.20 by replacing F_s with Σ and using the natural map $\Sigma \wr \mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op}$ which exists for *any* symmetric monoidal product $\otimes$ on $\mathcal{V}$ (see Remark 2.15).

Proof of Theorem I. Using [12, 11.3.2], the fact that weak equivalences and transfinite compositions are levelwise implies that Proposition 5.95 and Remark 5.84(i) are sufficient, as for any such $\mathcal{F}$, $\mathcal{F}$-(trivial) cofibrations are, in particular, *level* (trivial) genuine cofibrations.

Finally, in the case $\mathcal{V} = \mathsf{sSet}$ or sSet_*, for any group Π every object in $\mathcal{V}^{\Pi}_{\text{gen}}$ is cofibrant, and thus the technical hypotheses on $\mathcal{P}$ needed for Corollary 5.104 are always satisfied. $\square$

5.6 Model structures on genuine G-operads

We now turn our consideration to the homotopy theory of genuine G-operads from §4.

Notation 5.96. For this subsection, we fix a weak indexing system $\Sigma_{\mathcal{F}} \subseteq \Sigma_G$ (see Definition 4.59), and let $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ denote the model category $\mathcal{V}^{\Sigma^{op}_{\mathcal{F}}}_{\text{gen}}$, if it exists.

Remark 5.86 immediately implies the following.

Corollary 5.97. *If $\mathcal{V}$ is admissible for finite groups, then $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ exists.* $\square$

We identify the following maps in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ (cf. Definitions 5.80, 5.90).

Definition 5.98. We say a map $f : \mathcal{O} \to \mathcal{P}$ in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ is an

(i) *projective genuine weak equivalence* (resp. *projective genuine fibration*) if $f(C) : \mathcal{O}(C) \to \mathcal{P}(C)$ is a weak equivalence (fibration) in $\mathcal{V}^{\text{Aut}(C)}_{gen}$ for all $\mathcal{F}$-corollas $C \in \Sigma_{\mathcal{F}}$.

(iv) *level genuine cofibration* if $f(C)$ is a cofibration in $\mathcal{V}^{\text{Aut}(C)}_{gen}$ for all $C \in \Sigma_{\mathcal{F}}$.

Definition 5.99. The *projective genuine (semi) model structure* on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$, if it exists, is the unique (semi) model structure with projective genuine weak equivalences and fibrations. Equivalently, it is the (semi) model structure transferred along the adjunction.

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\longrightarrow}{\overset{\mathbb{F}_G}{\longleftarrow}} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$$

The key technical step in the proof of Theorem II, written out below, is a corollary of the simplest case of a result from §6.

Corollary 5.100. *Suppose $\mathcal{V}$ is strongly cellular with diagonals, and we are given $T \in \Omega^a_{\mathcal{F}}$ along with level genuine cofibrations $f_{ac} : \mathcal{Q} \to \mathcal{P}$ and $f_{in} : X \to Y$ in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$. Then the iterated box product*

$$f^{\square V_G(T)} = \mathop{\square}_{[v] \in V^{ac}_G(T)} f_{ac}([v]) \,\square \mathop{\square}_{[v] \in V^{in}_G(T)} f_{in}([v])$$

is a cofibration in $\mathcal{V}^{\text{Aut}(T)}_{gen}$.

Proof. Given $[v] \in V_G(T)$, let $[\![v]\!]$ denote the orbit in $V_G(T)/\mathsf{Aut}(\mathsf{T})$. Then we have a natural group homomorphism

$$\beta : \mathsf{Aut}(T) \to \prod_{[\![v]\!]} \Sigma_{|[\![v]\!]|} \wr \mathsf{Aut}(T_{[v]}),$$

and the forgetful functor

$$\beta^* : V^{\Pi \Sigma_{[\![v]\!]} \wr \mathsf{Aut}(T_{[v]})}_{\text{gen}} \to \mathcal{V}^{\mathsf{Aut}(T)}_{\text{gen}}$$

preserves genuine cofibrations by Remark 5.84(i). Thus, it suffices to show that $f^{\square V_G(T)}$ is cofibrant when considered as a map in the category on the left hand side. This follows immediately by Proposition 6.34 and Remark 5.84(ii).

$\square$

Corollary 5.101. *For $\mathcal{V}$ strongly cellular with diagonals, and for any free $\mathbb{F}_{\mathcal{F}}$-extension*

$$\begin{array}{ccc} \mathbb{F}_{\mathcal{F}}X & \longrightarrow & \mathcal{P} \\ {\scriptstyle u}\downarrow & & \downarrow \\ \mathbb{F}_{\mathcal{F}}Y & \longrightarrow & \mathcal{P}[u] \end{array}$$

where $u : X \to Y$ is a level genuine cofibration and $\mathcal{P}$ is level genuine cofibrant in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$, the map $\mathcal{P} \to \mathcal{P}[u]$ is a level genuine cofibration, trivial if u is so.

Proof. By using the levelwise filtration from (5.77), this follows immediately from Corollary 5.103 and Remark 5.84(i). □

Proof of Theorem II. All three parts follow precisely as in the proof of Theorem I, replacing the use of Corollary 5.95 with Corollary 5.104. □

6 Cofibrancy and Quillen equivalences

The key ingredient required to prove our desired Quillen equivalences

$$\iota^* : \mathsf{Op}_{\mathcal{F}} \rightleftarrows \mathsf{Op}^G_{\mathcal{F}} : \iota_*$$

will be an analysis of cofibrant objects in $\mathsf{Op}_{\mathcal{F}}$. To do so will require an intricate discussion of the interplay between different classes of cofibrations, box products, and G-trees, leading to the characterization provided by Lemma 6.76.

6.1 Families of subgroups

This section establishes some useful properties of the model structures associated to families of subgroups. Throughout all groups will be assumed finite.

Definition 6.1. A *family* $\mathcal{F}$ of subgroups of G is a collection of subgroups $H \le G$ such that

- if $H \in \mathcal{F}$ then $H^g = gHg^{-1} \in \mathcal{F}$ for all $g \in G$;
- if $K \le H$ and $H \in \mathcal{F}$ then $K \in \mathcal{F}$.

Remark 6.2. Any collection of subgroups $\mathcal{F}$ determines a full subcategory $\mathsf{O}_{\mathcal{F}} \hookrightarrow \mathsf{O}_G$ of those orbital G-sets G/H for $H \in \mathcal{F}$. Moreover, it is easy to check that $\mathcal{F}$ is a family iff $\mathsf{O}_{\mathcal{F}}$ is a sieve.

Remark 6.3. For fixed G families form a lattice, ordered by inclusion, with meet and join given by intersection and union.

We now recall the following basic notion and result concerning model structures induced by families (cf. [22, Prop. 2.6]).

Definition 6.4. We say a model category $\mathcal{V}$ has *cellular fixed points* if for all finite groups G and subgroups $H, K \le G$ one has that:

(i) fixed points $(-)^H : \mathcal{V}^G \to \mathcal{V}$ preserve direct colimits;

(ii) fixed points $(-)^H$ preserve pushouts where one leg is $(G/K) \cdot f$, for f a cofibration;

(iii) for each object $X \in \mathcal{V}$, the natural map $(G/K)^H \cdot X \to ((G/K) \cdot X)^H$ is an isomorphism.

Proposition 6.5. *If $\mathcal{V}$ is a cofibrantly generated model category with cellular fixed points, then for any finite group G and family $\mathcal{F}$, there is a model structure $\mathcal{V}^G_{\mathcal{F}}$ on the category $\mathcal{V}^G$, called the* $\mathcal{F}$-model structure *such that both weak equivalences and fibrations are determined by the fixed points $(-)^H$ for $H \in \mathcal{F}$.*

When $\mathcal{F}$ is the family of all subgroups the model structure in Proposition 6.5 is called simply the *G-genuine model structure*.

We will find it convenient to strengthen the cellularity conditions in Definition 6.4.

STRONGCELL PROP **Proposition 6.6.** *Suppose that $\mathcal{V}$ is a cofibrantly generated model category with cellular fixed points. Then:*

(i) $(-)^H : \mathcal{V}^G \to \mathcal{V}$ *preserves cofibrations and pushouts where one leg is a genuine cofibration;*

(ii) if X is G-genuine cofibrant the map $(G/K)^H \cdot X^H \to (G \cdot_K X)^H$ is an isomorphism.

Suppose additionally that $\mathcal{V}$ is a closed monoidal model category as well as strongly cofibrantly generated (i.e. that the domains of the generating (trivial) cofibrations are cofibrant). Then:

(iii) for f, g genuine cofibrations between genuine cofibrant objects the natural map

$$f^H \square g^H \xrightarrow{\simeq} (f \square g)^H$$

is an isomorphism. In particular, $X^H \otimes Y^H \xrightarrow{\simeq} (X \otimes Y)^H$ is an isomorphism when X, Y are genuine cofibrant.

Proof. Since all conditions are compatible with retracts, we are free to assume each cofibration $f : X \to Y$ (or, for Y cofibrant, the map $\varnothing \to Y$) is a transfinite composition

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \to Y = X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha \qquad (6.7)$$ TRANSFCOMP EQ

where each $f_\alpha : X_\alpha \to X_{\alpha+1}$ is the pushout of a generating cofibration $(G/H) \cdot i_\alpha$. Both (i) and (ii) now follow by transfinite induction on α in the partial composite map $X_0 \to X_\alpha$, with the successor ordinal case following by Def. 6.4 (ii), (iii) and the limit ordinal case by Def. 6.4 (i). We note that (ii) also includes an obvious base case $X_0 = \varnothing$. To prove (iii), we consider first the case of g a generating cofibration. The exact same induction argument now applies to any f as in (6.7), contingently on a base case $f = (\varnothing \to X)$. But this base case now follows by the exact same argument, now contingent on the base case $f = (\varnothing \to \varnothing)$, which is obvious. The general case now follows by repeating the same argument, now using the analogous filtration (6.7) for g. $\square$

As property (iii) above (and the related Proposition 6.34) will be of increasing importance, we give a name to such categories where it is true.

TRONGLY_CELLULAR **Definition 6.8.** We say a category $\mathcal{V}$ is *strongly cellular* if the following hold:

(i) $\mathcal{V}$ is a strongly cofibrantly-generated closed monoidal model category,

(ii) $\mathcal{V}$ has cellular fixed points,

(iii) $\mathcal{V}$ has cofibrant symmetric pushout powers (see Definition 6.24).

We end this section by cataloging some straightforward interactions of $\mathcal{F}$-model structures with regards to change of group and pushout products.

FGTRIGHT PROP **Proposition 6.9.** *Let $\phi : G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ be cofibrantly generated with cellular fixed points. Then the adjunction*

$$\phi_! = \bar{G} \cdot_G (-) : \mathcal{V}^G_{\mathcal{F}} \rightleftarrows \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} : \mathsf{fgt} = \phi^* \qquad (6.10)$$

is a Quillen adjunction if for any $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$.

Proof. Since one has a canonical isomorphism of fixed points $(\mathsf{fgt}(X))^H \simeq X^{\phi(H)}$, it is immediate that the right adjoint preserves fibrations and trivial fibrations. $\square$

FGTLEFT PROP

Proposition 6.11. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ be cofibrantly generated with cellular fixed points. Then the adjunction*

$$\phi^* = \mathsf{fgt}: \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \longrightarrow \mathcal{V}^{G}_{\mathcal{F}}: \mathsf{Hom}_G(\bar{G}, -) = \phi_* \tag{6.12}$$

is a Quillen adjunction if for any $H \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$.

Proof. Since the double coset formula yields that that

$$\mathsf{fgt}\left(\bar{G}/H \cdot f\right) \simeq \mathsf{fgt}\left(\bar{G}/H\right) \cdot f \simeq \left(\coprod_{[a] \in \phi(G)\backslash \bar{G}/H} G/\phi^{-1}(H^a) \right) \cdot f$$

it is immediate that the left adjoint fgt preserves cofibrations and trivial cofibrations. $\square$

Propositions 6.9 and 6.11 motivate the following definition.

Definition 6.13. Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{F}$ and $\bar{\mathcal{F}}$ families in G and $\bar{G}$. We define

$$\phi^*(\bar{\mathcal{F}}) = \{H \leq G : \phi(H) \in \bar{\mathcal{F}}\} \tag{6.14}$$ PHISTARDEF EQ

$$\phi_!(\mathcal{F}) = \{\phi(H)^{\bar{g}} \leq \bar{G} : \bar{g} \in \bar{G}, H \in \mathcal{F}\} \tag{6.15}$$

$$\phi_*(\mathcal{F}) = \{\bar{H} \leq \bar{G} : \forall_{\bar{g} \in \bar{G}} \left(\phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F}\right)\} \tag{6.16}$$ PHISTARDEF3 EQ

Lemma 6.17. *The $\phi^*(\bar{\mathcal{F}})$, $\phi_!(\mathcal{F})$, $\phi_*(\mathcal{F})$ just defined are themselves families. Furthermore*

(i) The "if" condition in Proposition 6.9 holds iff $\mathcal{F} \subset \phi^*(\bar{\mathcal{F}})$ iff $\phi_!(\mathcal{F}) \subset \bar{\mathcal{F}}$.

(ii) The "if" condition in Proposition 6.11 holds iff $\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F}$ iff $\bar{\mathcal{F}} \subset \phi_*(\mathcal{F})$.

Proof. Since the result is elementary, we include only the proof of the second iff in (ii), which is the hardest step and illustrates the necessary arguments. This follows by the following equivalences.

$$\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F} \Leftrightarrow \left(\forall_{\substack{H \leq G \\ \phi(H) \in \bar{\mathcal{F}}}} H \in \mathcal{F} \right) \Leftrightarrow \left(\forall_{\bar{H} \in \bar{\mathcal{F}}} \phi^{-1}(\bar{H}) \in \mathcal{F} \right) \Leftrightarrow \left(\forall_{\substack{\bar{H} \in \bar{\mathcal{F}} \\ \bar{g} \in \bar{G}}} \phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F} \right) \Leftrightarrow \bar{\mathcal{F}} \subset \phi_*(\mathcal{F})$$

Note that the second equivalence follows since $H \leq \phi^{-1}(\phi(H))$ and $\mathcal{F}$ is closed under subgroups while the third equivalence follows since $\bar{\mathcal{F}}$ is closed under conjugation. $\square$

BIQUILLENG PROP

Proposition 6.18. *Suppose that $\mathcal{V}$ is cofibrantly generated, has cellular fixed points, and is also a closed monoidal model category. Then the bifunctor*

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^G_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^G_{\mathcal{F} \cap \bar{\mathcal{F}}} \tag{6.19}$$ BIQUILLENG EQ

is a left Quillen bifunctor.

Proof. The double coset formula now yields

$$(G/H \cdot f) \,\square\, \left(G/\bar{H} \cdot g\right) \simeq \left(G/H \times G/\bar{H}\right) \cdot (f \,\square\, g) \simeq \left(\coprod_{[a] \in H\backslash G/\bar{H}} G/H \cap \bar{H}^a \cdot (f \,\square\, g) \right) \tag{6.20}$$

and hence the result follows since families are closed under conjugation and subgroups. $\square$

EXTERINT DEF

Definition 6.21. Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be families in G and G, respectively.

We define their *external intersection* to be the family of $G \times \bar{G}$ given by

$$\mathcal{F} \sqcap \bar{\mathcal{F}} = (\pi_G)^*(\mathcal{F}) \cap (\pi_{\bar{G}})^*(\bar{\mathcal{F}})$$

for $\pi_G: G \times \bar{G} \to G$, $\pi_{\bar{G}}: G \times \bar{G} \to \bar{G}$ the projections.

Remark 6.22. Combining Proposition 6.11 with Propositon 6.18 yields that the following composite is a left Quillen bifunctor.

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{G \times \bar{G}}_{(\pi_G)^*(\mathcal{F})} \times \mathcal{V}^{G \times \bar{G}}_{(\pi_{\bar{G}})^*(\bar{\mathcal{F}})} \xrightarrow{\otimes} \mathcal{V}^{G \times \bar{G}}_{\mathcal{F} \sqcap \bar{\mathcal{F}}} \tag{6.23}$$ EXTERINTADJ EQ

6.2 Pushout powers

That (6.23) is a left Quillen bifunctor (and its obvious higher order analogues) is one of the key properties of pushout products of $\mathcal{F}$ cofibrations when those cofibrations (and the group) are allowed to change. However, when those cofibrations (and hence G) coincide there is an additional symmetric group action that we will need to consider.

To handle such actions we introduce the following notion.

Definition 6.24. We say a monoidal model category $\mathcal{V}$ has *cofibrant symmetric pushout powers* if for each cofibration (resp. trivial cofibration) f the pushout product $f^{\square n}$ is a genuine Σ_n-cofibration (resp. trivial cofibration).

Remark 6.25. When $\mathcal{V}$ is cofibrantly generated the condition in Definition 6.24 needs only be checked for generating cofibrations. However, the argument needed is somewhat harder than usual due to $(-)^{\square n}$ not preserving composition of maps: one instead follows the argument in the proof of Proposition 6.34 below when $G = *$.

We now turn to describing the symmetric power analogue of Definition 6.21.

We start with some notation. Letting λ be a partition $E = \lambda_1 \amalg \cdots \amalg \lambda_k$ of a finite set E, we write $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k} \leq \Sigma_E$ for the subgroup of permutations preserving λ. In addition, given any $e \in E$ we write λ_e for the partition $E = \{e\} \amalg (E - e)$, so that Σ_{λ_e} is then the isotropy of e.

Definition 6.26. Let $\mathcal{F}$ be a family of G, E a finite set and $e \in E$ any fixed element.

We define the *n-th semidirect power of* $\mathcal{F}$ to be the family of $\Sigma_E \wr G = \Sigma_E \ltimes G^{\times E}$ given by

$$\mathcal{F}^{\ltimes E} = \left(\iota_{\Sigma_{\lambda_e} \wr G}\right)_* \left(\left(\pi_G\right)^* (\mathcal{F})\right), \tag{6.27}$$

where ι is the inclusion $\Sigma_{\lambda_e} \wr G \to \Sigma_E \wr G$ and π the projection $\Sigma_{\lambda_e} \wr G = \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to G$.

More explicitly, since in (6.16) one needs only consider conjugates by coset representatives of $\bar{G}/\phi(G)$, when computing $\left(\iota_{\Sigma_{\lambda_e} \wr G}\right)_*$ one needs only conjugate by coset representatives of $\Sigma_E \wr G / \Sigma_{\lambda_e} \wr G \simeq \Sigma_E / \Sigma_{\lambda_e}$, so that

$$K \in \mathcal{F}^{\ltimes E} \text{ iff } \underset{e \in E}{\forall}\, \pi_G \left(K \cap (\Sigma_{\lambda_e} \wr G)\right) \in \mathcal{F}, \tag{6.28}$$

showing that in particular (6.27) is independent of the choice of $e \in E$.

Remark 6.29. The previous definition is likely to seem mysterious at first sight. Ultimately, the origin of this definition is best understood by working through this section backwards: the study of the interactions between equivariant trees and graph families, namely Lemma 6.61, requires the study of the families $\mathcal{F}^{\ltimes_G n}$ in Notation 6.45, which are variants of the $\mathcal{F}^{\ltimes n}$ construction for graph families. It then suffices, and is notationally far more convenient, to establish the required results first for the $\mathcal{F}^{\ltimes n}$ families and then translate them to the $\mathcal{F}^{\ltimes_G n}$ families.

Proposition 6.30. *Writing $\iota\colon \Sigma_E \times \Sigma_{\bar{E}} \to \Sigma_{E \amalg \bar{E}}$ for the inclusion, one has*

$$\mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}} \subset \iota^* \left(\mathcal{F}^{\ltimes E \amalg \bar{E}}\right). \tag{6.31}$$

Hence, the following is a left Quillen bifunctor.

$$\Sigma_{E \amalg \bar{E}} \underset{\Sigma_E \times \Sigma_{\bar{E}}}{\cdot} (- \otimes -)\colon \mathcal{V}^{\Sigma_E \wr G} \times \mathcal{V}^{\Sigma_{\bar{E}} \wr G} \to \mathcal{V}^{\Sigma_{E \amalg \bar{E}} \wr G} \tag{6.32}$$

Proof. Let $K \in \mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}}$ and $e \in E$. We write λ_e for the partition of $E \amalg \bar{E}$ and λ_e^E for the partition of E. One then has

$$\pi_G \left(K \cap (\Sigma_{\lambda_e} \wr G)\right) = \pi_G \left(\pi_{\Sigma_E \wr G}(K) \cap \left(\Sigma_{\lambda_e^E} \wr G\right)\right), \tag{6.33}$$

where on the right we write $\pi_{\Sigma_E \wr G}\colon \Sigma_E \wr G \times \Sigma_{\bar{E}} \wr G \to \Sigma_E \wr G$ and $\pi_G\colon \Sigma_{\lambda_e^E} \wr G = \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to G$. Therefore K satisfies (6.28) for $\mathcal{F}^{\ltimes E \amalg \bar{E}}$ since $\pi_{\Sigma_E \wr G}(K)$ does so for $\mathcal{F}^{\ltimes E}$. The case of $e \in \bar{E}$ is identical.

(6.32) simply combines the left Quillen bifunctor (6.23) with Proposition 6.9. $\square$

POWERF PROP **Proposition 6.34.** *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Then, for all n and cofibration (resp. trivial cofibration) f of $\mathcal{V}^G_{\mathcal{F}}$ one has that $f^{\square n}$ is a cofibration (resp. trivial cofibration) of $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$.

Our proof of Proposition 6.34 will essentially repeat the main argument in the proof of [16, Thm. 1.2]. However, both for the sake of completeness and to stress that the argument is independent of the (fairly technical) model structures in [16], we include an abridged version of the proof below, the key ingredient of which is that (6.32) is a left Quillen bifunctor.

Proof. We first note that in the case of $i = (G/H)\cdot \bar{\imath}$, $H \in \mathcal{F}$, a generating (trivial) cofibration it is

$$i^{\square n} = (G/H)^{\times n} \cdot \bar{\imath}^{\square n} \simeq \Sigma_n \wr G \underset{\Sigma_n \wr H}{\cdot} \bar{\imath}^{\square n}.$$

$\bar{\imath}^{\square n}$ is thus a $\Sigma_n \wr - H$-genuine (trivial) cofibration by the cofibrant symmetric pushout powers hypothesis combined with Proposition 6.11 and hence $i^{\square n}$ is a $\mathcal{F}^{\ltimes n}$ (trivial) cofibration by Proposition 6.9 since $\Sigma_n \wr H \in \mathcal{F}^{\ltimes n}$.

For the general case, we start by making the key observation that for composable arrows $\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ the n-fold pushout product $(hg)^{\square n}$ has a factorization

$$\bullet \xrightarrow{k_0} \bullet \xrightarrow{k_1} \cdots \xrightarrow{k_n} \bullet \tag{6.35}$$ COMPNFOLDFACT EQ

where each k_i, $0 \le i \le n$, fits into a pushout product

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\Sigma_n \underset{\Sigma_{n-i}\times\Sigma_i}{\cdot}\left(g^{\square n-i}\square h^{\square i}\right)"'] \arrow[dr, phantom, "\ulcorner"] & \bullet \arrow[d, "k_i"] \\ \bullet \arrow[r] & \bullet. \end{tikzcd} \tag{6.36}$$ COMPNFOLDFACTPUSH EQ

Briefly, (6.35) follows from suitable Σ_n-symmetric convex subposets $P_0 \subset P_1 \subset \cdots \subset P_n$ of the poset $P_n = (0 \to 1 \to 2)^{\times n}$ where P_0 consists of "tuples with at least one 0-coordinate" and P_i is obtained from P_{i-1} by adding the "tuples with $n-i$ 1-coordinates and i 2-coordinates". Additional details concerning this filtration appear in the proof of [16, Lemma 4.8].

The general proof now follows by writing f as a retract of a transfinite composition of pushouts of generating (trivial) cofibrations as in (6.7). As usual, retracts can be ignored, and we can hence assume that there is an ordinal κ and $X_\bullet \colon \kappa \to \mathcal{V}^G$ such that (i) $f_\beta \colon X_\beta \to X_{\beta+1}$ is the pushout of a (trivial) cofibration i_β; (ii) $\operatorname{colim}_{\alpha<\beta} X_\alpha \xrightarrow{\simeq} X_\beta$ for limit ordinals $\beta < \kappa$; (iii) setting $X_\kappa = \operatorname{colim}_{\beta<\kappa} X_\beta$, f equals the transfinite composite $X_0 \to X_\kappa$.

We argue by transfinite induction on κ. Writing $\bar{f}_\beta \colon X_0 \to X_\beta$ for the partial composites, it suffices to check that the natural transformation of κ-diagrams (rightmost map not included)

$$\begin{tikzcd} Q^n(\bar{f}_1) \arrow[r] \arrow[d, "\bar{f}_1^{\square n}"'] & Q^n(\bar{f}_2) \arrow[r] \arrow[d, "\bar{f}_2^{\square n}"'] & Q^n(\bar{f}_3) \arrow[r] \arrow[d, "\bar{f}_3^{\square n}"'] & Q^n(\bar{f}_4) \arrow[r] \arrow[d, "\bar{f}_4^{\square n}"'] & \cdots \arrow[r] & Q^n(\bar{f}_\kappa) \arrow[d, "\bar{f}_\kappa^{\square n} = \operatorname{colim}_{\beta<\kappa} \bar{f}_\beta^{\square n}"] \\ X_1^{\otimes n} \arrow[r] & X_2^{\otimes n} \arrow[r] & X_3^{\otimes n} \arrow[r] & X_4^{\otimes n} \arrow[r] & \cdots \arrow[r] & X_\kappa^{\otimes n}, \end{tikzcd}$$

is κ-cofibrant, i.e. that the maps $Q^n(\bar{f}_\beta) \amalg_{\operatorname{colim}_{\alpha<\beta} Q^n(\bar{f}_\alpha)} \operatorname{colim}_{\alpha<\beta} X_\alpha^{\otimes n} \to X_\beta^{\otimes n}$ are cofibrations in $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$. Condition (ii) above implies that this map is an isomorphism for β a limit ordinal while for $\beta+1$ a successor ordinal it is the map $Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n} \to X_{\beta+1}^{\otimes n}$. But since $Q^n(\bar{f}_{\beta+1}) \to Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n}$ is precisely the map k_0 of (6.35) for $g = \bar{f}_\beta$, $h = f_\beta$, this last map is the composite $k_n k_{n-1} \cdots k_1$ so that the result now follows from (6.36) combined with (6.32), the induction hypothesis applied to $\bar{f}_\beta$, the fact that $f_\beta^{\square k}$ is a pushout of $i_\beta^{\square k}$ (cf. [16, Lemma 4.11]) and the cofibrancy of $i_\beta^{\square k}$ proven at the beginning. □

We will also need to understand the fixed points of $f^{\square n}$ for general subgroups $K \leq \Sigma_n \wr G$. To do so recall first that $f^{\square n}$ can be built from the composite

$$f^{\otimes n} \colon (0 \to 1)^{\times n} \xrightarrow{f^{\times n}} \mathcal{V}^{\times n} \xrightarrow{\otimes} \mathcal{V}$$

as the map

$$\operatorname{colim}_{(0\to 1)^{\times n} - (1,\cdots,1)} f^{\otimes n} \to Y^{\otimes n},$$

where Y is the target of f. Any $K \leq \Sigma_n \wr G$ acts on the poset $(0 \to 1)^{\times n}$ itself (via $K \to \Sigma_n \wr G \to \Sigma_n$). Moreover, the fixed subposet $\left((0 \to 1)^{\times n}\right)^K$ then consists of those tuples in $\{0,1\}^{\times n}$ whose coordinates coincide if their indexes are in the same coset of n/K, i.e. there is an identification $\left((0 \to 1)^{\times n}\right)^K \simeq (0 \to 1)^{\times n/K}$.

Example 6.37. When $n = 3$ and $n/K = \{\{1,2\},\{3\}\}$ the fixed subposet $(0 \to 1)^{n/K}$ is displayed on the right below.

$$\begin{tikzcd}
& 000 \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & 010 \arrow[dl] \arrow[dd] \\
100 \arrow[rr] \arrow[dd] & & 110 \arrow[dd] & \\
& 001 \arrow[rr, dashed] \arrow[dl, dashed] & & 011 \arrow[dl] \\
101 \arrow[rr] & & 111 &
\end{tikzcd}
\qquad
\begin{tikzcd}
000 \arrow[dr] \arrow[dd] & \\
& 110 \arrow[dd] \\
001 \arrow[dr] & \\
& 111
\end{tikzcd}$$

It will be key for our purposes to know that fixed points $\left(f^{\square n}\right)^K$ can be computed by first restricting to the smaller cube $(0 \to 1)^{\times n/K}$, resulting in a cube of objects with K-actions, and then computing a pushout over that smaller cube. The formal result follows.

FIXEDPUSH PROP

Proposition 6.38. *Suppose that $\mathcal{V}$ is as in Proposition 6.34 and that it is also strongly cofibrantly generated — that is, $\mathcal{V}$ is strongly cellular (see Definition 6.8). Let $K \leq \Sigma_n \wr G$ be a subgroup, $f \colon X \to Y$ a map in $\mathcal{V}^G$ and consider the natural maps (in the arrow category)*

$$\mathop{\square}_{[i] \in n/K} \left(f^{\otimes[i]}\right)^K \to \left(f^{\square n}\right)^K. \tag{6.39}$$

FIXEDPUSH EQ

If f is a cofibration between cofibrant objects then (6.39) is an isomorphism.

Proof. The result will follow by induction on n. The base case $n = 1$ is obvious.

Moreover, it is obvious that (6.39), which is a map of arrows, is an isomorphism on the target objects, hence the real claim is that this map is also an isomorphism on sources.

We now note that by considering (6.35) for $g = \varnothing \to X$, $h = f$ and removing the last map k_n one obtains a filtration of the source of $f^{\square n}$. Applying $(-)^K$ to the leftmost map in (6.36) one has isomorphisms

$$\begin{aligned}
\left(\Sigma_n \underset{\Sigma_{n-i}\times\Sigma_i}{\cdot} X^{\otimes n-i} \otimes f^{\square i}\right)^K &\simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A} \otimes f^{\square B}\right)^K \simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A}\right)^K \otimes \left(f^{\square B}\right)^K \\
&\simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(\bigotimes_{[j]\in A/K} \left(X^{\otimes[j]}\right)^K \right) \otimes \left(\mathop{\square}_{[k]\in B/K} \left(f^{\otimes[k]}\right)^K \right)
\end{aligned}$$

where the first step is an instance of Prop. 6.6(ii), the second step follows from Prop. 6.6(iii) (with the required cofibrancy of the objects following from Propositions 6.18 and 6.34), and the last step follows by Prop. 6.6(iii) together with the induction hypothesis (which applies since $|B| \leq i < n$).

We have thus shown that the leftmost map in the pushouts (6.36) for $\left(f^{\square n}\right)^K$ is isomorphic to the leftmost map in the pushouts for the corresponding filtration of $\mathop{\square}_{[i]\in n/K} \left(f^{\otimes[i]}\right)^K$, and since $(-)^K$ preserves such pushouts (cf. Prop. 6.6(i)), the result now follows. $\square$

FIXEDPUSH COR

Corollary 6.40. *Let $\mathcal{V}$ be strongly ceullar. Given a partition λ given by $\{1,2,\cdots,n\} = \lambda_1 \amalg \cdots \amalg \lambda_k$, cofibrations between cofibrant objects f_i in $\mathcal{V}^{G_i}$, $1 \le i \le k$ and a subgroup $K \le \Sigma_{\lambda_1} \wr G_1 \times \cdots \times \Sigma_{\lambda_k} \wr G_k$, the natural map*

$$\mathop{\square}_{1\le i\le k} \mathop{\square}_{[j]\in\lambda_i/K} \left(f_i^{\otimes[j]}\right)^K \to \left(\mathop{\square}_{1\le i\le k} f_i^{\square\lambda_i}\right)^K. \tag{6.41}$$

is an isomorphism.

Proof. This simply combines Propositions 6.6(iii) and 6.38. $\square$

G_GRAPH_SECTION

6.3 G-graph families and G-trees

We will now convert the results in the previous sections to the context of the type of families we are mainly interested in: graph families. We note that in this section we use Σ to denote a general group (usually meant to be some type of permutation group).

Definition 6.42. A subgroup $\Gamma \le G \times \Sigma$ is called a *G-graph subgroup* if $\Gamma \cap \Sigma = *$.

Further, a family $\mathcal{F}$ of $G\times\Sigma$ is called a *G-graph family* if it consists of G-graph subgroups.

GRAPH REM

Remark 6.43. Γ is a G-graph subgroup iff it can be written as

$$\Gamma = \{(k, \varphi(k)) : k \in K \le G\}$$

for some partial homomorphism $G \ge K \xrightarrow{\varphi} \Sigma$, thus motivating the terminology.

Remark 6.44. The collection of all G-graph subgroups is itself a family. Indeed, it coincides with $(\iota_\Sigma)_*(\{*\})$ for the inclusion homomorphism $\iota_\Sigma \colon \Sigma \to G \times \Sigma$.

SEMIDIRG NOT

Notation 6.45. Letting $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G \times \Sigma$ and $G \times \bar{\Sigma}$ we will write

$$\mathcal{F} \sqcap_G \bar{\mathcal{F}} = \Delta^*(\mathcal{F} \sqcap \bar{\mathcal{F}}) \qquad\qquad \mathcal{F}^{\ltimes_G n} = \Delta^*(\mathcal{F}^{\ltimes n})$$

where Δ denotes either of the diagonal inclusions $\Delta\colon G\times\Sigma\times\bar{\Sigma} \to G\times\Sigma\times G\times\bar{\Sigma}$ or $\Delta\colon G\times\Sigma_n\wr\Sigma \to \Sigma_n \wr (G\times\Sigma)$.

PACKINGSQCAP REM

Remark 6.46. Unpacking Definition 6.21 one has that $\Gamma \in \mathcal{F} \sqcap_G \bar{\mathcal{F}}$ iff $\pi_{G\times\Sigma}(\Gamma) \in \mathcal{F}$, $\pi_{G\times\bar{\Sigma}}(\Gamma) \in \bar{\mathcal{F}}$.

ACKINGLTIMES REM

Remark 6.47. Unpacking (6.28) and noting that

$$(G \times \Sigma_E \wr \Sigma) \cap (\Sigma_{\lambda_e} \wr (G \times \Sigma)) = G \times \Sigma_{\lambda_e} \wr \Sigma$$

one has

$$K \in \mathcal{F}^{\ltimes_G E} \text{ iff } \underset{e\in E}{\forall}\, \pi_{G\times\Sigma}\left(K \cap (G \times \Sigma_{\lambda_e} \wr \Sigma)\right) \in \mathcal{F}. \tag{6.48}$$ FLTIMESN2G EQ

Combining either the left Quillen bifunctor (6.23) or Proposition 6.34 with Proposition 6.11 yields the following results.

XTERINTADJG PROP

Proposition 6.49. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points. Let $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G \times \Sigma$ and $G \times \bar{\Sigma}$. Then the following (with diagonal G-action on the images) is a left Quillen bifunctor.*

$$\mathcal{V}^{G\times\Sigma}_{\mathcal{F}} \times \mathcal{V}^{G\times\bar{\Sigma}}_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Sigma\times\bar{\Sigma}}_{\mathcal{F}\sqcap_G\bar{\mathcal{F}}} \tag{6.50}$$ EXTERINTADJG EQ

POWERFG PROP

Proposition 6.51. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Let $\mathcal{F}$ be a G-graph family of $G \times \Sigma$. If f is a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma}_{\mathcal{F}}$ then so is $f^{\square n}$ a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma_n\wr\Sigma}_{\mathcal{F}^{\ltimes_G n}}$.

Remark 6.52. It is straightforward to check that $\mathcal{F} \sqcap_G \bar{\mathcal{F}}$ is in fact also a G-graph family of $G \times \Sigma \times \bar{\Sigma}$. However, $\mathcal{F}^{\ltimes_G n}$ is *not* a G-graph family of $G \times \Sigma_n \wr \Sigma$, due to the need to consider the power Σ_n-action.

The G-graph families we will be interested in encode certain families of G-trees. We start with the case of families of corollas (see Definition FAMILY_COROLLAS_DEF 4.55).

ILY_COROLLAS_LEM **Lemma 6.53.** *The set of families of corollas $\Sigma_{\mathcal{F}}$ is in bijection with the set of collections $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ of G-graph families $\mathcal{F}_n$ of $G \times \Sigma_n$. Moreover, we have an equivalence of categories $\coprod O_{\mathcal{F}_n} \xrightarrow{\simeq} \Sigma_{\mathcal{F}}$.*

Proof. We define $\mathcal{F}_n$ so that any G-corolla C with n leaves per component is in $\Sigma_{\mathcal{F}}$ iff

$$C \simeq (C_\Gamma)_{G/H}$$

for some $C_\Gamma \in \Sigma^H$ encoded by a partial homomophism $G \leq H \to \Sigma_n$ whose graph Γ is in $\mathcal{F}_n$; conversely, any $\mathcal{F}$ dually defines a full subcategory $\Sigma_{\mathcal{F}}$ of Σ_G, and these operations are clearly inverse.

Further, Γ_1 is subconjugate to Γ_2 iff we have a quotient map

$$(C_{\Gamma_1})_{G/H_1} \to (C_{\Gamma_2})_{G/H_2}$$

where $H_i = \pi_G(\Gamma_i)$, and hence $\mathcal{F}$ is a collection of *families* iff $\Sigma_{\mathcal{F}}$ is a sieve of Σ_G.

Finally, the equivalence of categories just sends $\Gamma \leq H \times \Sigma_n$ to $(C_\Gamma)_{G/H}$. □

As in §INDEXING_SECTION 4.4, we will abuse notation and use $\mathcal{F}$ and $\Sigma_{\mathcal{F}}$ interchagably[2].

Proposition 6.54. *Let $\mathcal{F}$ be a family of G-corollas and $T \in \Omega$ a tree with automorphism group Σ_T. Write $\mathcal{F}_T$ for the collection of G-graph subgroups of $G \times \Sigma_T$ encoded by partial homomorphisms $G \geq H \to \Sigma_T$ such that the associated G-tree $G \cdot_H T$ is a $\mathcal{F}$-tree.*

Then $\mathcal{F}_T$ is a G-graph family.

Proof. Closure under conjugation follows since conjugate graph subgroups produce isomorphic G-trees. As for subgroups, they correspond to restrictions $K \leq H \to \Sigma_T$, which induce quotient maps $G \cdot_K T \to G \cdot_H T$. $\mathcal{F}_T$ is thus closed under subgroups since the G-vertices of $G \cdot_H T$ are quotients of those of $G \cdot_K T$. □

Remark 6.55. The closure condition required of weak indexing systems from Definition INDEXSYS_DEF 4.59 can be translated in terms of families as saying that for any tree $T \in \Omega$ and letting $\phi\colon \Sigma_T \to \Sigma_{\mathsf{lr}(T)}$ be the natural homomorphism, one has $(G \times \phi)(\Gamma) \in \mathcal{F}_{\mathsf{lr}(T)}$ for any $\Gamma \in \mathcal{F}_T$. Proposition FGTRIGHT_PROP 6.9 then says that

$$\phi_!\colon \mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T} \to \mathcal{V}^{G\times\mathsf{lr}(T)}_{\mathcal{F}_{\mathsf{lr}(T)}} \tag{6.56}$$ LRLEFTQUILLEN EQ

is a left Quillen functor.

UNPACKFTYPE REM **Remark 6.57.** Unpacking definitions, a partial homomorphism $G \geq H \to \Sigma_T$ encodes a subgroup in $\mathcal{F}_T$ iff, for each vertex $v = e^\uparrow \leq e$ of T with $H_e \leq H$ the H-isotropy of the edge e, the induced homomorphism

$$H_e \to \Sigma_{T_v} \simeq \Sigma_{|v|} \tag{6.58}$$ PARTIALHOMEDGE EQ

encodes a subgroup in $\mathcal{F}_{|v|}$, where $|v| = |e^\uparrow|$.

REEINDUCDESC REM **Remark 6.59.** Recall that any tree $T \in \Omega$ other than the stick η has an essentially unique grafting decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$ where C_n is the root corolla and the leaves of C_n are identified with the roots of the T_i. We now let λ be the partition $\{1, \cdots, n\} = \lambda_1 \amalg \cdots \amalg \lambda_k$ such that $1 \leq i_1, i_2 \leq n$ are in the same class iff $T_{i_1}, T_{i_2} \in \Omega$ are isomorphic.

Writing $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$ and picking representatives $i_j \in \lambda_j$ one then has isomorphisms

$$\Sigma_T \simeq \Sigma_\lambda \wr \prod_i \Sigma_{T_i} \simeq \Sigma_{|\lambda_1|} \wr \Sigma_{T_{i_1}} \times \cdots \times \Sigma_{|\lambda_k|} \wr \Sigma_{T_{i_k}} \tag{6.60}$$ TREEISOT EQ

where the second isomorphism, while not canonical (it depends on choices of isomorphisms $T_{i_j} \simeq T_l$ for each $i_j \neq l \in \lambda_j$) is nonetheless well defined up to conjugation.

[2] Alternatively, the reader can think of $\mathcal{F}$ as a "family in the groupoid Σ of finite sets".

The following, which is the core result in this section, is a reinterpretation of Remark 6.57 in light of the inductive description of trees in Remarks 6.59.

KEYLEMMAGECO LEM

Lemma 6.61. *Let $\Sigma_{\mathcal{F}}$ be a family of G-corollas and $T \in \Omega$ a tree other than η. Then*

$$\mathcal{F}_T = \left(\pi_{G\times\Sigma_n}\right)^* \left(\mathcal{F}_n\right) \cap \left(\mathcal{F}_{T_{i_1}}^{\ltimes_G|\lambda_1|} \sqcap_G \cdots \sqcap_G \mathcal{F}_{T_{i_k}}^{\ltimes_G|\lambda_k|}\right), \tag{6.62}$$

KEYLEMMAGECO EQ

where $\pi_{G\times\Sigma_n}$ denotes the composite $G\times\Sigma_T \to G\times\Sigma_\lambda \to G\times\Sigma_n$.

Proof. The argument is by induction on the decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$ with the base case, that of a corolla, being immediate.

Consider now a partial homomorphism $G \geq H \to \Sigma_T$ encoding a G-graph subgroup $\Gamma \leq G\times\Sigma_T$. The condition that $\Gamma \in \left(\pi_{G\times\Sigma_n}\right)^*(\mathcal{F}_n)$ states that the composite $H \to \Sigma_T \to \Sigma_n$ is in $\mathcal{F}_n$, and this is precisely the condition (6.58) in Remark 6.57 for $e = r$ the root of T.

As for the condition $\Gamma \in \left(\mathcal{F}_{T_{i_1}}^{\ltimes_G|\lambda_1|} \sqcap_G \cdots \sqcap_G \mathcal{F}_{T_{i_k}}^{\ltimes_G|\lambda_k|}\right)$, by unpacking it by combining Remarks 6.46 and 6.47, this translates to the condition that, for each $i \in \{1,\cdots,n\}$, one has

$$\pi_{G\times\Sigma_{T_i}}\left(\Gamma \cap \left(G\times\Sigma_{\{i\}}\times\Sigma_{T_i}\times\Sigma_{\lambda-\{i\}} \wr \prod_{j\neq i}\Sigma_{T_j}\right)\right) \in \mathcal{F}_{T_i} \tag{6.63}$$

KEYLEMMAGECOR EQ

where $\lambda - \{i\}$ denotes the induced partition of $\{1,\cdots,n\} - \{i\}$. Noting that the intersection subgroup inside $\pi_{G\times\Sigma_{T_i}}$ in (6.63) can be rewritten as $\Gamma \cap \pi_{\Sigma_n}^{-1}\left(\Sigma_{\{i\}}\times\Sigma_{\{1,\cdots,n\}-\{i\}}\right)$, we see that this is the graph subgroup encoded by the restriction $H_i \leq H \to \Sigma_T$, where H_i is the isotropy subgroup of the root r_i of T_i (equivalently, this is also the subgroup sending T_i to itself). But since for any edge $e \in T_i$ its isotropy H_e (cf. 6.58) is a subgroup of H_i, the induction hypothesis implies that (6.63) is equivalent to condition (6.58) across all vertices other than the root vertex.

The previous paragraphs show that (6.62) indeed holds when restricted to G-graph subgroups. However, it still remains to show that any group Γ in the right family in (6.62) is indeed a G-graph subgroup, i.e. $\Gamma \cap \Sigma_T = *$ or, in other words, that any element $\gamma \in \Gamma \cap \Sigma_T = G\times\Sigma_\lambda \wr \prod_i \Sigma_{T_i}$ with G-coordinate $\gamma_G = e$ is indeed the identity. But the condition $\pi_{G\times\Sigma_n}(\Gamma) \in \mathcal{F}_n$ now implies that for such γ the Σ_λ-coordinate is $\gamma_{\Sigma_\lambda} = e$ and thus (6.63) in turn implies that the Σ_{T_i}-coordinates are $\gamma_{\Sigma_{T_i}} = e$, finishing the proof. $\square$

The results just developed will allow us to analyze cofibrancy properties of the left maps in the key pushouts (5.74). The first part of the analysis concerns the maps

$$\bigotimes_{v\in V_G^{ac}(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v\in V_G^{in}(T)} u(T_v) \tag{6.64}$$

COFIBMAPSTREE EQ

that constitute the inner part of (5.75), where we recall that $T \in \Omega_G^a$ is an alternating tree. We will in turn subdivide the cofibrancy analysis of (6.64) itself into two parts: (i) showing a $\mathcal{F}_{T_e}$-cofibrancy claim when $T = G\cdot T_e$ is free and; (ii) showing a fixed point claim for non free trees, as in Remark 4.43.

It will find it convenine to slightly reinterpret (6.64): writing $p(T_v)\colon \varnothing \to \mathcal{P}(T_v)$ for the unique map, we can rewrite (6.64) as

$$\mathop{\square}_{v\in V_G^{ac}(T)} p(T_v) \,\square \mathop{\square}_{v\in V_G^{in}(T)} u(T_v).$$

For both the sake of generality and to simply notation in the proof, we extend the context of the following results to the l-labeled trees Ω_G^l of §5.1.

Remark 6.65. l-labeled $\mathcal{F}$-trees can be defined exactly as in Definition 4.57. Moreover, it is then clear that a l-labeled G-tree T is an $\mathcal{F}$-tree iff the underlying G-tree is.

Remarks 6.57, 6.59 and Lemma 6.61 then extend to the l-labeled context, by now writing Σ_T for the group of label isomorphisms (a subgroup of the isomorphisms of the underlying tree) and defining the partition λ in Remark 6.59 by using label isomorphism classes.

Proposition 6.66. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Let $\mathcal{F}$ be a family of corollas and suppose that $f_i \colon A \to B$, $1 \le i \le l$ are level $\mathcal{F}$-cofibrations (resp. trivial cofibrations) in $\mathsf{Sym}^G(\mathcal{V})$, i.e. that $f_i(r) \colon A(r) \to B(r)$ are cofibrations (resp. trivial cofibrations) in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$. Then for any l-labeled tree $T \in \Omega^{\underline{l}}$ the map

$$f^{\square V(T)} = \mathop{\square}_{1\le i\le l, v\in V_i(T)} f_i(v)$$

(where $V_i(T)$ denotes vertices with label i) is a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T}$.

Proof. This follows by induction on the decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$, with the base cases of corollas and η being immediate. The description of $\mathcal{F}_T$ in (6.62) combined with the left Quillen functors in Propositions 6.49, 6.18 and 6.11 then yield that

$$\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n} \times \mathcal{V}^{G\times\Sigma_{|\lambda_1|}\wr\Sigma_{T_{i_1}}}_{\mathcal{F}^{\ltimes G|\lambda_1|}_{T_{i_1}}} \times \cdots \times \mathcal{V}^{G\times\Sigma_{|\lambda_k|}\wr\Sigma_{T_{i_k}}}_{\mathcal{F}^{\ltimes G|\lambda_k|}_{T_{i_k}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T}$$

is a left Quillen multifunctor. The result now follows by Proposition 6.51 together with the induction hypothesis. $\square$

Remark 6.67. When $G = *$, Proposition 6.66 coincides with [2, Lemma 5.9]. Moreover, it is not hard to adapt the proof of that non-equivariant result to provide a proof of Proposition 6.66 in the case of the universal family Σ_G of all G-corollas. However, the reader of [2] may note that the proof therein is technically less involved, using no analogue of the rather subtle $\mathcal{F}^{\ltimes_G n}$ families. Indeed, this is reflected on the last paragraph of our proof of Lemma 6.61, which effectively uses the subtle condition (6.63) to deduce the much simpler condition $\Gamma \cap \prod \Sigma_{T_i} = *$ that would suffice for the direct generalization of [2, Lemma 5.9] mentioned above. One may thus wonder if the $\mathcal{F}^{\ltimes_G n}$ families are indeed required to prove Proposition 6.66, or whether a more direct adaptation of the proof of [2, Lemma 5.9] is possible. Reverse engineering our proofs, the most natural "simplification" would be to replace the condition (6.63) with the condition

$$\pi_{G\times\Sigma_{T_i}}(\Gamma \cap \prod \Sigma_{T_i}) \in \mathcal{F}, \tag{6.68}$$

thus replacing the families $\mathcal{F}^{\ltimes n}$ of (6.27) with the families $(\iota_{G^{\times n}})_*(\mathcal{F} \sqcap \cdots \sqcap \mathcal{F})$. However, it is not hard to build indexing systems $\mathcal{F}$ (other than the universal one) for which these simpler families do not satisfy the analogue of Lemma 6.61, and thus for which (6.56) fails.

Proposition 6.69. *Let $\mathcal{V}$ be as in Proposition 6.66, and suppose additionally that $\mathcal{V}$ is strongly cofibrantly generated (so $\mathcal{V}$ is strongly cellular) and that $\mathcal{V}$ is Cartesian monoidal.*

Let $f_i \colon A \to B$, $1 \le i \le l$ be genuine cofibrations between genuine cofibrant objects in $\mathsf{Sym}^G(\mathcal{V})$. For each $T \in \Omega^{\underline{l}}_{G,0}$ define

$$f^{\square V_G(T)} = \mathop{\square}_{1\le i\le l, v\in V_{G,i}(T)} \iota_* f_i(v). \tag{6.70}$$

Then the canonical map

$$f^{\square V_G(-)} \to \iota_*\iota^* f^{\square V_G(-)} \tag{6.71}$$

is an isomorphism in $\mathcal{V}^{\Omega^{0,\underline{l},op}_G}$ (with $G \times \Omega^{\underline{l}}_0 \xrightarrow{\iota} \Omega^{0,\underline{l}}_G$ the inclusion).

Proof. Note first that there is a coproduct decomposition

$$\Omega^{\underline{l}}_{G,0} \simeq \coprod_{U\in\mathsf{Iso}\left(\Omega^{\underline{l}}_0\right)} \Omega^{\underline{l}}_{G,0}[U]$$

where $\Omega^{\underline{l}}_{G,0}[U]$ is the full subcategory formed by the quotients of $G\cdot U$. It suffices to establish (6.71) for each subcategory $\Omega^{\underline{l}}_{G,0}[U]$.

Moreover, writing T as $T \simeq G \cdot_H T_e$ for $T_e \in \Omega^H$, we are free by induction on $|G|$ to assume $H = G$. Indeed, otherwise there are identifications $V_G(T) \simeq V_H(T_e)$ and $f^{\square V_G(T)} \simeq (\mathrm{res}^G_H f)^{\square V_H(T_e)}$ from which the desired isomorphism follows.

We thus reduce to the case where there is a quotient map $G \cdot U \to U_G$ where U_G denotes the underlying tree U together a G-action. Moreover, the automorphisms of $G \cdot U$ compatible with the quotient map $G \cdot U \to U_G$ are the subgroup $K \leq G \times \Sigma_U$ encoding the action $G \to \Sigma_U$ of G on U_G. One then has identifications

$$\left(\mathop{\square}_{[v] \in V_G(G \cdot U)} \iota_* f_\bullet([v]) \right)^K \simeq \left(\mathop{\square}_{v \in V(U)} f_\bullet(v) \right)^G \simeq \mathop{\square}_{[v] \in V(U)/G} \left(\prod_{v \in [v]} f_\bullet(v) \right)^G \simeq \mathop{\square}_{[v] \in V_G(U_G)} \iota_* f_\bullet([v])$$

where the middle step is Corollary 6.40 establishing the desired isomorphim (6.71). ☐

6.4 Cofibrancy and the proof of Theorem III

HM_PROOF_SECTION

Propositions 6.66 and 6.69 from the previous section provide most of the backbone of the main technical step in the proof of our Elmendrof-type result Theorem III. The last remaining step is the following refinement of the key argument in the proof of [22, Thm. 2.10].

COFESSIM PROP

Proposition 6.72. *Let $\mathcal{V}$ be a cofibrantly generated model category with cellular fixed points, $\mathcal{F}$ a non-empty family of subgroups of G, and consider the reflexive adjunction*

$$\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathcal{V}^G_{\mathcal{F}}. \tag{6.73}$$

COFADJ EQ

Then the cofibrant objects of $\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$ are precisely the essential image under ι_* of the cofibrant objects of $\mathcal{V}^G_{\mathcal{F}}$. Moreover, the analogous statement for cofibrations between cofibrant objects also holds.

Proof. Note first that since ι_* identifies $\mathcal{V}^G(\mathcal{F})$ as a reflexive subcategory of $\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$, it is $X \simeq \iota_* Y$ for some $Y \in \mathcal{V}^G(\mathcal{F})$ (i.e. $X \in \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$ is in the essential image of ι_*) iff both $\iota^* X \simeq Y$ and the unit map $X \xrightarrow{\simeq} \iota_* \iota^* X$ is an isomorphism.

Letting $C_{\mathcal{F}}$ (resp. $C^{\mathcal{F}}$) denote the classes of cofibrant objects in $\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$ (resp. $\mathcal{V}^G_{\mathcal{F}}$) we need to show $C_{\mathcal{F}} = \iota_*(C^{\mathcal{F}})$, where we slightly abuse notation by writing $\iota_*(-)$ for the essential image rather than the image. Since $C_{\mathcal{F}}$ is characterized as being the smallest class closed under retracts and transfinite composition of cellular extensions that contains the initial presheaf $\varnothing$, it suffices to show that $\iota_*(C^{\mathcal{F}})$ satisfies this same characterization.

It is immediate that $\iota_*(\varnothing) = \varnothing$. Further, the characterization in the first paragraph yields that $X \in \iota_*(C^{\mathcal{F}})$ iff $\iota^*(X) \in C^{\mathcal{F}}$ and $X \xrightarrow{\simeq} \iota_* \iota^* X$ is an isomorphism, showing that $\iota_*(C^{\mathcal{F}})$ is closed under retracts.

The crux of the proof will be to compare cellular extensions in $C_{\mathcal{F}}$ with the images under ι_* of the cellular extensions in $C^{\mathcal{F}}$. Firstly, note that the generating cofibrations in $\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$ have the form $\mathsf{Hom}(-, G/H) \cdot f$, and that by the cellularity axiom (iii) in Definition 6.4 this map is isomorphic to the map $\iota_*(G/H \cdot f)$. We now claim that the cellular extensions of objects in $\iota_*(C^{\mathcal{F}})$, i.e. pushforward diagrams as on the left below

$$\begin{array}{ccc} \iota_* X & \longrightarrow & \iota_* V \\ {\scriptstyle \iota_* u}\downarrow & & \downarrow \\ \iota_* Y & \dashrightarrow & \tilde{W} \end{array} \qquad\qquad \begin{array}{ccc} X & \longrightarrow & V \\ {\scriptstyle u}\downarrow & & \downarrow \\ Y & \dashrightarrow & W \end{array} \tag{6.74}$$

TWOCELLEXTEAS EQ

are precisely the essential image under ι_* of the cellular extensions of objects in $C^{\mathcal{F}}$, i.e., pushforward diagrams as on the right above. That the solid subdiagrams in either side of (6.74) are indeed in bijection up isomorphism is simply the claim that ι^* is fully faithful, hence the real claim is that $\tilde{W} \simeq \iota_* W$. But this follows since by the cellularity axiom (ii) in Definition CELL DEF 6.4 the map ι_* preserves the rightmost pushforward in (TWOCELLEXTEAS EQ 6.74) (note that $u: X \to Y$ is assumed to be a generating cofibration of $\mathcal{V}^G_{\mathcal{F}}$).

Noting that the cellularity axiom (i) in Definition CELL DEF 6.4 implies that ι_* preserves filtered colimits finishes the proof that $C_{\mathcal{F}} = \iota_*(C^{\mathcal{F}})$.

The additional claim concerning cofibrations between cofibrant objects follows by the same argument. □

FINALCOR COR **Corollary 6.75.** *Let $\mathcal{V}$ be as above, $\phi: G \to \bar{G}$ a homomorphism, and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of G, $\bar{G}$ such that $\phi_! \mathcal{F} \subset \mathcal{F}$. Then the diagram*

$$\begin{tikzcd} \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}} \arrow[d, "\phi_!"'] & \mathcal{V}^G_{\mathcal{F}} \arrow[l, "\iota_*"'] \arrow[d, "\phi_!"] \\ \mathcal{V}^{\mathsf{O}^{op}_{\bar{\mathcal{F}}}} & \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \arrow[l, "\iota_*"] \end{tikzcd}$$

commutes up to isomorphism when restricted to cofibrant objects of $\mathcal{V}^G_{\mathcal{F}}$.

Proof. It is straightforward to check that the left adjoints commute, i.e. that there is a natural isomorphism $\iota^* \phi_! \simeq \phi_! \iota^*$ which by adjunction induces a natural transformation $\phi_! \iota_* \to \iota_* \phi_!$. More explicitly, this natural transformation is the composite

$$\phi_! \iota_* \to \iota_* \iota^* \phi_! \iota_* \xrightarrow{\simeq} \iota_* \phi_! \iota^* \iota_* \xrightarrow{\simeq} \iota_* \phi_!$$

where the last two maps are always isomorphisms. But when restricting to cofibrant objects the previous result guarantees both that $\phi_! \iota_*$ lands in cofibrant objects and that cofibrant objects are in the essential image of ι_*. The result follows. □

We now possess all the technical ingredients needed to prove Theorem MAINQUILLENEQUIV THM III.

MAINLEM LEM **Lemma 6.76.** *Let $\mathcal{V}$ be strongly cellular with diagonals, and let $\mathcal{F}$ be a weak indexing system.*

Then in both of the adjunctions

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \qquad \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \tag{6.77}$$ COFADJ2 EQ

the cofibrant objects in the leftmost category are the essential image under ι_* of the cofibrant objects in the rightmost category.

Moreover, both forgetful functors

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \qquad \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \tag{6.78}$$ FGTFUNC EQ

preserve cofibrant objects.

Remark 6.79. We recall from Notation I_STAR_ABUSE 4.72 that ι_*, ι^* in the statement abusively denote $\gamma^* \iota_*$, $\iota^* \gamma_!$; we will not use this abusive notation in the proof.

Proof. We first observe that the claim concerning the symmetric sequence adjunction of (COFADJ2 EQ 6.77) is not really new. Indeed, as there are equivalences of categories $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \simeq \prod_{n \geq 0} \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}_n}}$, $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \simeq \prod_{n \geq 0} \mathcal{V}^{G \times \Sigma_n}_{\mathcal{F}_n}$, compatible with both the model structures and the (ι^*, ι_*) adjunctions, the symmetric sequence statement merely repackages Proposition COFESSIM PROP 6.72.

come back: fix, and fix notation

For the operad adjunction in (COFADJ2 EQ 6.77), most of the argument in the proof of Proposition COFESSIM PROP 6.72 applies mutatis mutandis except for the claim that $\mathbb{F}_{\mathcal{F}}(\varnothing) \simeq \gamma^* \iota_* \mathbb{F}(\varnothing)$, which is readily checked directly, and the comparison of cellular extensions, which is the key claim.

Explicitly, and borrowing the notation $C_{\mathcal{F}}$ (resp. $C^{\mathcal{F}}$) used in Proposition 6.72 for the classes of cofibrant objects in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ (resp. $\mathsf{Op}^{G}_{\mathcal{F}}(\mathcal{V})$), we need to show that cellular extensions of objects in $\iota_*(C^{\mathcal{F}})$, such as on the left below

$$\begin{tikzcd} (\gamma^*\mathbb{F}_G\gamma_!)\gamma^*\iota_*X \arrow[r] \arrow[d, "\iota_*u"'] & \gamma^*\iota_*\mathcal{O} \arrow[d, dashed] \\ (\gamma^*\mathbb{F}_G\gamma_!)\gamma^*\iota_*Y \arrow[r, dashed] & (\gamma^*\iota_*\mathcal{O})[\iota_*u] \end{tikzcd} \qquad \begin{tikzcd} \mathbb{F}X \arrow[r] \arrow[d, "u"'] & \mathcal{O} \arrow[d, dashed] \\ \mathbb{F}Y \arrow[r, dashed] & \mathcal{O}[u] \end{tikzcd} \tag{6.80}$$

TWOCELLEXT EQ

are precisely the essential image under $\gamma^*\iota_*$ of cellular extensions of objects in $C^{\mathcal{F}}$, as on the right above. In fact, it suffices to show this just when $\mathcal{O} \in C^{\Sigma_{\mathcal{F}}}$, the subcategory of operads in Op^{G} which forget to cofibrant sequences in $\mathsf{Sym}^{G}_{\mathcal{F}}$. Indeed, this would imply that $\iota_*(C^{\Sigma_{\mathcal{F}}})$ is closed under free extensions, all of which are images of free extensions in $C^{\Sigma_{\mathcal{F}}}$, and hence the same is true for $\iota_*(C^{\mathcal{F}})$ (where we can identify this as those objects in $\iota_*(C^{\Sigma_{\mathcal{F}}})$ isomorphic to retracts of those which have a cellular decomposition beginning with $\varnothing$).

To begin, we first note that by Remark 5.82, the unit map $X \to \iota^*\gamma_!\gamma^*\iota_*X$ is an isomorphism of cofibrant objects.

Now, recalling that there are natural isomorphisms

$$\gamma_!\gamma^*\mathbb{F}_G\gamma_! \simeq \mathbb{F}_G\gamma_! \qquad \iota^*\mathbb{F}_G \simeq \iota^*\mathbb{F}_G\iota_*\iota^* \qquad \iota^*\mathbb{F}_G\iota_* \simeq \mathbb{F}$$

we see that the two solid subdiagrams in (6.80) are in fact adjoint up to isomorphism, so that there is a bijection between such data. We claim that it now suffices to check that all four objects in the leftmost diagram of (6.80) are in the essential image of $\gamma^*\iota_*$. Indeed, as we also have a natural isomorphism

$$\gamma^*\iota_*\iota^*\gamma_!\gamma^*\iota_* \xrightarrow{\simeq} \gamma^*\iota_*,$$

if that is the case then

$$\begin{aligned} \gamma^*\mathbb{F}_G\gamma_!\gamma^*\iota_*Z &\simeq \gamma^*\iota_*\iota^*\gamma_!\gamma^*\gamma^*\mathbb{F}_G\gamma_!\gamma^*\iota_*Z \simeq \gamma^*\iota_*\iota^*\mathbb{F}_G\gamma_!\gamma^*\iota_*Z \simeq \gamma^*\iota_*\iota^*\mathbb{F}_G\iota_*\iota^*\gamma_!\gamma^*\iota_*Z \\ &\simeq \gamma^*\iota_*\iota^*\mathbb{F}_G\iota_*Z \simeq \gamma^*\iota_*\mathbb{F}Z \end{aligned}$$

for $Z = X, Y$ and since $\gamma^*\iota_*$ reflects colimits[3] on diagrams of cofibrant objects in $\mathsf{Sym}^{G}_{\mathcal{F}}$,

blue: need that cofibrant $\mathcal{O} \in \mathsf{Op}^{G}_{\mathcal{F}}$ are cofibrant in $\mathsf{Sym}^{G}_{\mathcal{F}}$, as $\gamma^*\iota_*$ does not in general reflect colimits

it must then indeed be that $(\iota_*\mathcal{O})[\iota_*u] \simeq \iota_*(\mathcal{O}[u])$.

To establish the remaining claim that the objects in the leftmost diagram in (6.80) are in the essential image of $\gamma^*\iota_*$, we claim it suffices to show this for the bottom right corner $(\iota_*\mathcal{O})[\iota_*u]$ when $u: X \to Y$ is a general cofibration between cofibrant objects in $\mathsf{Sym}^{G}_{\mathcal{F}}(\mathcal{V})$. Indeed, setting $X = \varnothing$ and $\mathcal{O} = \mathbb{F}(\varnothing)$, one has $(\iota_*\mathcal{O})[\iota_*u] = \gamma^*\mathbb{F}_G\gamma_*\iota_*Y$, and similarly for $\gamma^*\mathbb{F}_G\gamma_*\iota_*X$.

Writing $\mathcal{P} = \iota_*\mathcal{O}$, so that $(\iota_*\mathcal{O})[\iota_*u] = \mathcal{P}[\iota_*u]$, the required condition that $\mathcal{P}[\iota_*u] \to \iota_*\iota^*\mathcal{P}[\iota_*u]$ is an isomorphism can be checked by forgetting to $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$, and we can thus appeal to the filtration (5.70) of the map $\mathcal{P} \to \mathcal{P}[\iota_*u]$ (modified by Corollary 5.76). It thus suffices to verify by induction on k that each $\mathcal{P}_k$ is in the essential image of $\iota_*: \mathsf{Sym}^{G}(\mathcal{V}) \to \mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$. Using the iterative description of the $\mathcal{P}_k$ in (5.74) it suffices by Proposition 6.72 to check that the leftmost map in (5.74) is a cofibration between cofibrant objects in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$. We now recall that that map can also be described (cf. (5.79)) as

$$\mathrm{Lan}_{\Omega^{a}_{\mathcal{F}}[k]^{op} \to \Sigma^{op}_{G}} \left(\bigotimes_{v \in V^{ac}_{G}(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v \in V^{in}_{G}(T)} u(T_v) \right). \tag{6.81}$$

FILTINTALTAG EQ

[3] I.e. any diagram that becomes a colimit upon applying $\gamma^*\iota_*$ must have already been a colimit diagram.

Noting that there is an equivalence $\mathcal{V}^{\Omega^a_{\mathcal{F}}[k]^{op}} \simeq \prod_{T\in \mathsf{Iso}(\Omega^a_{\mathcal{F}}[k])} \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}_T}}$, Propositions 6.66 and 6.69 show that the inner map inside the left Kan extension in (6.81) is in the essential image of the cofibrations between cofibrant objects of $\mathcal{V}_{\mathcal{F}}^{G\times\Omega^a[k]^{op}}$ under the functor $\iota_*\colon \mathcal{V}_{\mathcal{F}}^{G\times\Omega^a[k]^{op}} \to \mathcal{V}^{\Omega^a_{\mathcal{F}}[k]^{op}}$.

blue: the hypotheses of both cited props require we already know $\mathcal{O}$ is (1) level cofibrant in $\mathsf{Sym}^G_{\mathcal{F}}$ and/or (2) genuine cofibrant in Sym^G

Since the Lan in (6.81) can now be identified with an instance of the functor $\phi_!$ in Corollary 6.75 one has (using the further identification $\mathcal{V}_{\mathcal{F}}^{G\times\Omega^a[k]^{op}} \simeq \prod_{T\in\mathsf{Iso}(\Omega^a[k])} \mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T}$) that the main claim now follows by Corollary 6.75 with Propositions 6.72 and 6.9.

As for the additional claim concerning the forgetful functors in (6.78), that $\mathsf{fgt}\colon \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ preserves cofibrant objects is precisely what was argued in the previous paragraph. But since the forgetful functors commute with ι_*, the claim that $\mathsf{fgt}\colon \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$ also preserves cofibrant objects follows from the "essential image characterization" of cofibrant objects in (6.77). $\square$

Remark 6.82. A slightly more careful analysis of the argument in the previous proof shows that we have in fact shown the slightly more general claim that operads (in either $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ or $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$) that forget to cofibrant symmetric sequences (in either $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ or $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$) are closed under cellular extensions of operads.

proof of Theorem III. It suffices to show that both the derived unit and derived counit for the adjunction are given by weak equivalences.

For the counit, it is immediate from Lemma 6.76 that if $X \in \mathsf{Op}^G(\mathcal{V})$ is bifibrant the functor $\iota^*\iota_* X$ is already derived, and hence the derived counit is identified with the counit isomorphism $\iota^*\iota_* X \xrightarrow{\simeq} X$.

For the unit, note first that it is immediate from the definitions that $\iota_*\colon \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ detects fibrations (as well as weak equivalences), and thus by Lemma 6.76 $Y \in \mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ is bifibrant iff $Y \simeq \iota_* X$ for $X \in \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ bifibrant. But then the functor $\iota_*\iota^* Y$ is also already derived (since $\iota^* Y \simeq \iota^*\iota_* X \simeq X$ is fibrant) and the derived unit is thus the isomorphism $Y \xrightarrow{\simeq} \iota_*\iota^* Y$. $\square$

NINFTY_SECTION

6.5 N_∞-operads

HERE

Combining the existence of $\mathcal{F}$-model structures and the above cofibrancy result, we give our first proof of the following conjecture of Blumberg-Hill in [3].

Proof of Corollary V, version one. Recall that $\mathsf{Comm}(n) = *$ for all n. Consider the functorial factorization

$$\varnothing \rightarrowtail N\mathcal{F} \xrightarrow{\sim} \mathsf{Comm}$$

in $\mathsf{Op}^G_{\mathcal{F}}(\mathsf{sSet})$ of the unique map into a cofibration and trivial fibration. Since the initial operad is cofibrant, Theorem 6.76 implies that $\varnothing \to N\mathcal{F}$ is a level $\mathcal{F}$-cofibration, and hence each $N\mathcal{F}(n)$ is cofibrant in $\mathsf{sSet}^{G\times\Sigma_n}_{\mathcal{F}_n}$; thus, for all $\Gamma \notin \mathcal{F}_n$, $N\mathcal{F}(n)^\Gamma = \varnothing$. Further, since $N\mathcal{F}$ is $\mathcal{F}$-equivalent to $*$, $N\mathcal{F}(n)^\Gamma \simeq *$ for all $\Gamma \in \mathcal{F}_n$. Hence, each $N\mathcal{F}(n)$ is a universal space for $\mathcal{F}_n$, as desired. $\square$

come back

N_INFINITY_BAR

Theorem 6.83. *$B(\mathbb{F}_G, \mathbb{F}_G, \delta_{\mathcal{F}})$ is*

- *in the image of i_**
- *weakly equivalent to $\delta_{\mathcal{F}}$.*

Hence, $i^*(B(\mathbb{F}_G, \mathbb{F}_G, \delta_{\mathcal{F}})$ is an $N\mathcal{F}$-operad.

A Transferring Kan extensions

The purpose of this appendix is to provide the somewhat long proof of Proposition 5.40 which is needed when repackaging free extensions of genuine equivariant operads in (5.7).

We start with a more detailed discussion of the realization functor $|-|$ defined by the adjunction

$$|-|\colon \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat}\colon (-)^{[\bullet]}$$

in Definition 5.38. More explicitly, one has

$$|\mathcal{I}_\bullet| = coeq\left(\coprod_{[n]\to[m]} [n]\times\mathcal{I}_m \rightrightarrows \coprod_{[n]} [n]\times\mathcal{I}_n\right). \tag{A.1}$$ REALDEF EQ

Example A.2. Any $\mathcal{I}\in\mathsf{Cat}$ induces objects $\mathcal{I},\mathcal{I}_\bullet,\mathcal{I}^{[\bullet]}\in\mathsf{Cat}^{\Delta^{op}}$ where $\mathcal{I}$ is the constant simplicial object and $\mathcal{I}_\bullet$ is the nerve $N\mathcal{I}$ with each level regarded as a discrete category. It is straightforward to check that $|\mathcal{I}|\simeq|\mathcal{I}_\bullet|\simeq|\mathcal{I}^{[\bullet]}|\simeq\mathcal{I}$.

OBJGENREL LEMMA **Lemma A.3.** *Given $\mathcal{I}_\bullet\in\mathsf{Cat}^{\Delta^{op}}$ one has an identification $ob(|\mathcal{I}_\bullet|)\simeq ob(\mathcal{I}_0)$. Furthermore, the arrows of $|\mathcal{I}_\bullet|$ are generated by the image of the arrows in $\mathcal{I}_0\simeq\mathcal{I}_0\times[0]$ and the image of the arrows in $[1]\times ob(\mathcal{I}_1)$.*

For each $i_1\in\mathcal{I}_1$, we will denote the arrow of $|\mathcal{I}_\bullet|$ induced by the arrow in $[1]\times\{i_1\}$ by

$$d_1(i_1)\xrightarrow{i_1} d_0(i_1).$$

Proof. We write $d_{\hat{k}}$, $d_{\hat{k},\hat{l}}$ for the simplicial operators induced by the maps $[0]\xrightarrow{0\mapsto k}[n]$, $[1]\xrightarrow{0\mapsto k,1\mapsto l}[n]$ which can informally be thought of as the "composite of all faces other than d_k, d_l". Using (A.1) one has equivalence relations of objects

$$[n]\times\mathcal{I}_n\ni(k,i_n)\sim(0,d_{\hat{k}}(i_n))\in[0]\times\mathcal{I}_0$$

and since for any generating relation $(k,i_n)\sim(l,i'_m)$ it is $d_{\hat{k}}(i_n)=d_{\hat{l}}(i'_m)$ the identification $ob(|\mathcal{I}_\bullet|)\simeq ob(\mathcal{I}_0)$ follows.

To verify the claim about generating arrows, note that any arrow of $[n]\times\mathcal{I}_n$ factors as

$$(k,i_n)\to(l,i_n)\xrightarrow{I_n}(l,i'_n) \tag{A.4}$$ FACTORIZATIONREAL EQ

for $I_n\colon i_n\to i'_n$ an arrow of $\mathcal{I}_n$. The $d_{\hat{l}}$ relation identifies the right arrow in (A.4) with $(0,d_{\hat{l}}(i_n))\xrightarrow{d_{\hat{l}}(I_n)}(0,d_{\hat{l}}(i'_n))$ in $[0]\times\mathcal{I}_0$ while (if $k<l$) the $d_{\hat{k},\hat{l}}$ relation identifies the left arrow with $(0,d_{\hat{k},\hat{l}}(i_n))\to(1,d_{\hat{k},\hat{l}}(i_n))$ in $[1]\times\mathcal{I}_1$. The result follows. ☐

Remark A.5. Given $\mathcal{I}_\bullet\in\mathsf{Cat}^{\Delta^{op}}$, $\mathcal{C}\in\mathsf{Cat}$, the isomorphisms

$$\mathrm{Hom}_{\mathsf{Cat}}(|\mathcal{I}_\bullet|,\mathcal{C})\simeq\mathrm{Hom}_{\mathsf{Cat}^{\Delta^{op}}}\left(\mathcal{I}_\bullet,\mathcal{C}^{[\bullet]}\right)$$

together with the fact that $\mathcal{C}^{[\bullet]}$ is always 2-coskeletal show that $|\mathcal{I}_\bullet|$ is determined by the categories $\mathcal{I}_0,\mathcal{I}_1,\mathcal{I}_2$ and maps between them, i.e. by the truncated version of formula (A.1) with $n,m\le 2$.

Indeed, one can show that a sufficient set of generating relations in $|\mathcal{I}_\bullet|$ is given by (i) the relations in $\mathcal{I}_0$ rela(including relations stating that identities of $\mathcal{I}_0$ are identities of $|\mathcal{I}_\bullet|$); (ii) relations stating that for each $i_0\in\mathcal{I}_0$ the arrow $i_0=d_1(s_0(i_0))\xrightarrow{s_0(i_0)}d_1(s_0(i_0))=i_0$ is an identity; (iii) for each arrow $I_1\colon i_1\to i'_1$ in $\mathcal{I}_1$ the relation that the square below commutes

$$\begin{tikzcd}
d_1(i_1) \arrow[r, "i_1"] \arrow[d, "d_1(I_1)"'] & d_0(i_1) \arrow[d, "d_0(I_1)"] \\
d_1(i'_1) \arrow[r, "i'_1"] & d_0(i'_1)
\end{tikzcd}$$

and (iv) for each object $i_2 \in \mathcal{I}_2$ the relation that the following triangle commutes.

$$\begin{tikzcd} d_{1,2}(i_2) \arrow[rr, "d_1(i_2)"] \arrow[dr, "d_2(i_2)"'] & & d_{0,1}(i_2) \\ & d_{0,2}(i_2) \arrow[ur, "d_0(i_2)"'] & \end{tikzcd}$$

We now relate diagrams in the span categories with the Grothendieck constructions in Definition GROTHCONS DEF 2.2.

MPSPANREIN LEMMA **Lemma A.6.** *Functors $F: \mathcal{D} \ltimes \mathcal{I}_\bullet \to \mathcal{C}$ are in bijection with lifts*

$$\begin{tikzcd} & \mathsf{WSpan}^l(*, \mathcal{C}) \arrow[d, "\mathsf{fgt}"] \\ \mathcal{D} \arrow[ur, dashed, "\mathcal{I}_\bullet^F"] \arrow[r, "\mathcal{I}_\bullet"'] & \mathsf{Cat}. \end{tikzcd}$$

where fgt *is the functor forgetting the maps to* $*$ *and* $\mathcal{C}$*.*

Proof. This is a matter of unpacking notation. The restrictions $F|_{\mathcal{I}_d}$ to the fibers $\mathcal{I}_d \subset \mathcal{D} \ltimes \mathcal{I}_\bullet$ are precisely the functors $\mathcal{I}_d^F: \mathcal{I}_d \to \mathcal{C}$ describing $\mathcal{I}_\bullet^F(d)$.

Furthermore, the images $F((d,i) \to (d', f_*(i)))$ of the pushout arrows over a fixed arrow $f: d \to d'$ of $\mathcal{D}$ assemble to a natural transformation

$$\begin{tikzcd} \mathcal{I}_d \arrow[d, "f_*"'] \arrow[dr, "I_d^F"] \arrow[d, phantom, shift left=4ex, "\Swarrow"] & \\ \mathcal{I}_{d'} \arrow[r, "I_{d'}^F"'] & \mathcal{C} \end{tikzcd} \tag{A.7}$$

which describes $\mathcal{I}_\bullet^F(f)$. One readily checks that the associativity and unitality conditions coincide. $\square$

In the cases of interest we have $\mathcal{D} = \Delta^{op}$. The following is the key result in this section.

SOURCEFINAL PROP **Proposition A.8.** *Let $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then there is a natural functor*

$$\Delta^{op} \ltimes \mathcal{I}_\bullet \xrightarrow{\ s\ } |\mathcal{I}_\bullet|. \tag{A.9}$$

Further, s is final.

Remark A.10. The s in the result above stands for *source*. This is because, for $\mathcal{I} \in \mathsf{Cat}$, the map $\Delta^{op} \ltimes \mathcal{I}^{[\bullet]} \to \left|\mathcal{I}^{[\bullet]}\right| \simeq \mathcal{I}$ is given by $s(i_0 \to \cdots \to i_n) = i_0$.

Proof. Recall that $|\mathcal{I}_\bullet|$ is the coequalizer (REALDEF EQ A.1). Given $(k, g_m) \in [n] \times \mathcal{I}_m$, we write $[k, g_m]$ for the corresponding object in $|\mathcal{I}_\bullet|$. To simplify notation, we write objects of $\mathcal{I}_n$ as i_n and implicitly assume that $[k, i_n]$ refers to the class of the object $(k, i_n) \in [n] \times \mathcal{I}_n$.

We define s on objects by $s([n], i_n) = [0, i_n]$ and on an arrow $(\phi, I_m): (n, i_n) \to (m, i'_m)$ as the composite (note that $\phi: [m] \to [n]$ and $I_m: \phi^* i_n \to i_m$)

$$[0, i_n] \to [\phi(0), i_n] = [0, \phi^* i_n] \xrightarrow{I_m} [0, i'_m]. \tag{A.11}$$ TARGETDEFINITON EQ

To check compatibility with composition, the cases of a pair of either two fiber arrows (i.e. arrows where ϕ is the identity) or two pushforward arrows (i.e. arrows where I_m is the identity) are immediate from (TARGETDEFINITON EQ A.11), hence we are left with the case $([n], i_n) \xrightarrow{I_n} ([n], i'_n) \to ([m], \phi^* i'_n)$ of a fiber arrow followed by a pushforward arrow. Noting that in $\Delta^{op} \ltimes \mathcal{I}_\bullet$ this composite can be rewritten as $([n], i_n) \to ([m], \phi^* i_n) \xrightarrow{\phi^* I_n} ([m], \phi^* i'_n)$ this amounts to checking that

$$\begin{tikzcd}
{[0, i_n]} \arrow[r] \arrow[d, "I_n"'] & {[\phi(0), i_n)]} \arrow[r, equal] \arrow[d, "I_n"'] & {[0, \phi^* i_n]} \arrow[d, "\phi^* I_n"] \\
{[0, i'_n]} \arrow[r] & {[\phi(0), i'_n]} \arrow[r, equal] & {[0, \phi^* i_n]}
\end{tikzcd} \tag{A.12}$$

commutes in $|\mathcal{I}_\bullet|$, which is the case since the left square is encoded by a square in $[n] \times \mathcal{I}_n$ and the right square is encoded by an arrow in $[m] \times \mathcal{I}_n$.

We now show that s is final. Fix $h \in \mathcal{I}_0$. We must check that $[0, h] \downarrow \Delta^{op} \ltimes \mathcal{I}_\bullet$ is connected. By Lemma A.3 [OBJGENREL LEMMA] any object in this undercategory has a description (not necessarily unique) as a pair

$$\Big(([n], i_n), [0, h] \xrightarrow{f_1} \cdots \xrightarrow{f_r} s([n], i_n) \Big) \tag{A.13}$$ UNDERCATOB EQ

where each f_i is a generating arrow of $|\mathcal{I}_\bullet|$ induced by either an arrow I_0 of $\mathcal{I}_0$ or object $i_1 \in \mathcal{I}_1$. We will connect (A.13) [UNDERCATOB EQ] to the canonical object $(([0], h), [0, h] = [0, h])$, arguing by induction on r. If $n \neq 0$, the map $d_{\hat{0}} \colon ([n], i_n) \to ([0], d^*_{\hat{0}}(i_n))$ and the fact that $s\left(d^*_{\hat{0}}\right) = id_{[0, d^*_{\hat{0}}(i_n)]}$ provides an arrow to an object with $n = 0$ without changing r. If $n = 0$, one can apply the induction hypothesis by lifting f_r to $\Delta^{op} \ltimes \mathcal{I}_\bullet$ according to one of two cases: (i) if f_r is induced by an arrow I_0 of $\mathcal{I}_0$, the lift of f_r is simply $([0], i'_0) \xrightarrow{I_0} ([0], i_0)$; (ii) if f_r is induced by $i_1 \in \mathcal{I}_1$ the lift is provided by the map $([1], i_1) \to ([0], d_0(i_1))$. □

DUALRESULTS REM **Remark A.14.** The involution

$$\Delta \xrightarrow{\tau} \Delta$$

which sends $[n]$ to itself and d_i, s_i to d_{n-i}, s_{n-i} induces vertical isomorphisms

$$\begin{tikzcd}
\Delta^{op} \ltimes (\mathcal{I}_\bullet \circ \tau) \arrow[r, "s"] \arrow[d, "\simeq"] & {|\mathcal{I}_\bullet \circ \tau|} \arrow[d, "\simeq"] \\
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[r, "t"'] & {|\mathcal{I}_\bullet^{op}|^{op}}
\end{tikzcd}$$

which reinterpret the "source" functor as what one might call the "target" functor, with $t([n], i_n) = [n, i_n]$ rather than $s([n], i_n) = [0, i_n]$. The target functor is thus also final.

Moreover, the source/target formulations of all the results that follow are equivalent.

In practice, we will need to know that the source s and target t satisfy a stronger finality condition with respect to left Kan extensions.

UNDERLEFTADJ LEM **Lemma A.15.** *Let $\mathcal{J} \in \mathsf{Cat}$ be a small category and $j \in \mathcal{J}$. Then the under and over category functors*

$$\mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{(-)\downarrow j} \mathsf{Cat}, \qquad \mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{j \downarrow (-)} \mathsf{Cat} \tag{A.16}$$ UNDEROVER EQ

preserve colimits.

Proof. The result can easily be shown directly, so here we note instead that one can in fact write explicit formulas for the right adjoints of $(-) \downarrow j$, $j \downarrow (-)$. Moreover, since $j \downarrow \mathcal{I} = (\mathcal{I}^{op} \downarrow j)^{op}$ it suffices to do so for $(-) \downarrow j$. The right adjoint $(-)^{\downarrow j} \colon \mathsf{Cat} \to \mathsf{Cat} \downarrow \mathcal{J}$ is then defined on objects by the Grothendieck constructions $\mathcal{C}^{\downarrow j} = \mathcal{J} \ltimes \mathcal{C}^{\mathcal{J}(-, j)}$ for the functors

$$\begin{aligned} \mathcal{J} &\longrightarrow \mathsf{Cat} \\ i &\longmapsto \mathcal{C}^{\mathcal{J}(i, j)}. \end{aligned}$$

□

URCELANFINAL COR

Corollary A.17. *Consider a map $\mathcal{I}_\bullet \to \mathcal{J}$ between $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ and a constant object $\mathcal{J} = \mathcal{J}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then the source and target maps*

$$\begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "s"] \arrow[dr] & & {|\mathcal{I}_\bullet|} \arrow[dl] \\ & \mathcal{J} & \end{tikzcd} \qquad \begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "t"] \arrow[dr] & & {|\mathcal{I}_\bullet^{op}|^{op}} \arrow[dl] \\ & \mathcal{J} & \end{tikzcd}$$

are Lan*-final over $\mathcal{J}$, i.e. the functors $s \downarrow j \colon (\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \to |\mathcal{I}_\bullet| \downarrow j$ are final for all $j \in \mathcal{J}$, and similarly for t.*

Proof. It is clear that $(\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \simeq \Delta^{op} \ltimes (\mathcal{I}_\bullet \downarrow j)$ while Lemma A.15 guarantees that, since $(-) \downarrow j$ is a left adjoint, $|\mathcal{I}_\bullet| \downarrow j \simeq |\mathcal{I}_\bullet \downarrow j|$. One thus reduces to Proposition A.8. □

We will require two additional straightforward lemmas.

TWISTING LEMMA

Lemma A.18. *Let $\mathcal{I}_\bullet^F \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ be such that the diagrams*

$$\begin{tikzcd} \mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] \arrow[d, phantom, shift left=4ex, "\Uparrow \delta_i"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] \arrow[d, phantom, shift left=4ex, "\Uparrow \sigma_j"] & \\ \mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C} \end{tikzcd} \tag{A.19}$$

IDENTSIMPRELSISO EQ

are given by natural isomorphisms for $0 < i \le n$, $0 \le j \le n$. Then the functors $\tilde{F}_n \colon \mathcal{I}_n \to \mathcal{C}$ given by the composites

$$\mathcal{I}_n \xrightarrow{d_{1,\cdots,n}} \mathcal{I}_0 \xrightarrow{F_0} \mathcal{C}$$

assemble to an object $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ which is isomorphic to $\mathcal{I}_\bullet^F$ and such that the corresponding diagrams (A.19) for $0 < i \le n$, $0 \le j \le n$ are strictly commutative.

Dually, if (A.19) are natural isomorphisms for $0 \le i < n$ and $0 \le j \le n$ one can form $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ such that the corresponding diagrams are strictly commutative.

Proof. This follows by a straightforward verification. □

SOURCEFACT LEM

Lemma A.20. *A (necessarily unique) factorization*

$$\begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "F_\bullet"] \arrow[dr, "s"'] & & \mathcal{C} \\ & {|\mathcal{I}_\bullet|} \arrow[ur, dashed, "F"'] & \end{tikzcd} \tag{A.21}$$

SOURCEFACT EQ

exists iff for the associated object $\mathcal{I}_\bullet \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ (cf. Lemma A.6) all faces d_i for $0 < i \le n$ and degeneracies s_j for $0 \le j \le n$ are strictly commutative, i.e. they are given by diagrams

$$\begin{tikzcd} \mathcal{I}_n \arrow[d, "d_0"'] \arrow[dr, "F_n"] \arrow[d, phantom, shift left=4ex, "\Downarrow \varphi_n"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] & \\ \mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C} \end{tikzcd} \tag{A.22}$$

IDENTSIMPRELS EQ

Dually, a factorization through the target $t \colon \Delta^{op} \ltimes \mathcal{I}_\bullet \to |\mathcal{I}_\bullet^{op}|^{op}$ exists iff the faces d_i and degeneracies s_j are strictly commutative for $0 \le i < n$, $0 \le j \le n$.

Proof. For the "only if" direction, it suffices to note that s sends all pushout arrows of $\Delta^{op} \ltimes \mathcal{I}_\bullet$ for faces d_i, $0 < i \le n$ and degeneracies s_j, $0 \le j \le n$ to identities, yielding the required commutative diagrams in (A.22).

For the "if" direction, this will follow by building a functor $\mathcal{I}_\bullet \xrightarrow{\bar{F}_\bullet} \mathcal{C}^{[\bullet]}$ together with the naturality of the source map s (recall that $|\mathcal{C}^{[\bullet]}| \simeq \mathcal{C}$). We define $\bar{F}_n|_{k\to k+1}$ as the map

$$F_{n-k}d_{0,\cdots,k-1} \xrightarrow{\varphi_{n-k}d_{0,\cdots,k-1}} F_{n-k-1}d_{0,\cdots,k}. \tag{A.23}$$

EQUIVALENCEDEF EQ

The claim that $s \circ (\Delta^{op} \ltimes \bar{F})$ recovers the horizontal map in (A.21) is straightforward, hence the real task is to prove that (A.23) defines a map of simplicial objects. First, functoriality of the original $F_\bullet$ yields identities

$$\varphi_{n-1}d_i = \varphi_n, \ \ 1 < i \qquad \varphi_{n-1}d_1 = (\varphi_{n-1}d_0) \circ \varphi_n, \qquad \varphi_{n+1}s_i = \varphi_n, \ \ 0 < i, \qquad \varphi_{n+1}s_0 = id_{F_n} \tag{A.24}$$

Next, note that there is no ambiguity in writing simply $\varphi_{n-k}d_{0,\cdots,k-1}$ to denote the map (A.23). We now check that $\bar{F}_{n-1}d_i = d_i\bar{F}_n$, $0 \le i \le n$, which must be verified after restricting to each $k \to k+1$, $0 \le k \le n-2$. There are three cases, depending on i and k:

$(i < k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_{0,\cdots,k}$;

$(i = k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_1d_{0,\cdots,k-1} = (\varphi_{n-k-1}d_0 \circ \varphi_{n-k})d_{0,\cdots,k-1} = (\varphi_{n-k-1}d_{0,\cdots,k}) \circ (\varphi_{n-k}d_{0,\cdots,k-1})$;

$(i > k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_{i-k}d_{0,\cdots,k-1} = \varphi_{n-k}d_{0,\cdots,k-1}$.

The case of degeneracies is similar. □

proof of Proposition 5.40. The result follows from the following string of identifications.

$$\begin{aligned}\lim_{\Delta}\left(\mathsf{Ran}_{A_n\to\Sigma_G}N_n\right) &\simeq \mathsf{Ran}_{\Delta\times\Sigma_G\to\Sigma_G}\left(\mathsf{Ran}_{A_n\to\Sigma_G}N_n\right) \simeq \\ &\simeq \mathsf{Ran}_{\Delta\times\Sigma_G\to\Sigma_G}\left(\mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Delta\times\Sigma_G}N_\bullet\right) \simeq \\ &\simeq \mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Sigma_G}N_\bullet \simeq \mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Sigma_G}\tilde{N}_\bullet \simeq \mathsf{Ran}_{|A_\bullet|\to\Sigma_G}\tilde{N}\end{aligned}$$

The first step simply rewrites $\lim_\Delta$. The second step follows from Proposition 2.5 applied to the map $(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Delta \times \Sigma_G$ of Grothendieck fibrations over Δ. The third step follows since iterated Kan extensions are again Kan extensions. The fourth step twists $N_\bullet$ as in Lemma A.18 to obtain $\tilde{N}_\bullet$ such that the d_i, s_j are given by strictly commutative diagrams for $0 \le i < n$, $0 \le j \le n$. Lastly, the final step uses Lemma A.20 to conclude that $\tilde{N}_\bullet$ factors through the target functor t, obtaining $\tilde{N}$, and then uses Corollary A.17 to conclude that the Kan extensions indeed coincide. □

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