Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads and which can be thought of as a hybrid between operads and coefficient systems. We then prove an Elmendorf-Piacenza type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads, with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

A surprising feature of topological algebra is that the category of (connected) topological commutative monoids is quite small, consisting only of products of Eilenberg-MacLane spaces (e.g. [16, 4K.6]). Instead, the more interesting structures are those monoids which are commutative and associative only up to homotopy and, moreover, up to "all higher homotopies". To capture these more subtle algebraic notions, Boardman-Vogt [4] and May [21] developed the theory of *operads*. Informally, an operad $\mathcal{O}$ consists of a sequence of sets/spaces $\mathcal{O}(n)$ of "n-ary operations" carrying a Σ_n-action (recording "reordering of the inputs of the operations"), and a suitable notion of "composition of operations". The purpose of the theory is then the study of "objects X with operations indexed by $\mathcal{O}$", referred to as *algebras*, with the notions of monoid, commutative monoid, Lie algebra, algebra with a module, and more, all being recovered as algebras over some fixed operad in an appropriate category. Of special importance are the E_∞-operads, introduced by May in [21], which are homotopical replacements for the commutative operad and encode the aforementioned "commutative monoids up to homotopy". In particular, while an E_∞-algebra structure on X does not specify unique maps $X^n \to X$, it nonetheless specifies such maps "uniquely up to homotopy".

E_∞-operads are characterized by the homotopy type of their levels $\mathcal{O}(n)$: $\mathcal{O}$ is E_∞ if and only if each $\mathcal{O}(n)$ is Σ_n-free and contractible. That is, for each subgroup $\Gamma \leq \Sigma_n$ one has

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma = \{*\},\\ \varnothing & \text{otherwise.} \end{cases}$$

Notably, when studying the homotopy theory of operads in topological spaces the preferred notion of weak equivalence is usually that of "naive equivalence", with a map of operads $\mathcal{O} \to \mathcal{O}'$ deemed a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a weak equivalence of spaces upon forgetting the Σ_n-actions (e.g. [2, 3.2]). In this context, E_∞-operads are then equivalent to the commutative operad Com and, moreover, any cofibrant replacement of Com is E_∞. These naive equivalences differ from the equivalences in "genuine equivariant homotopy theory", where a map of G-spaces $X \to Y$ is deemed a G-equivalence only if the induced fix point maps $X^H \to Y^H$ are weak equivalences for all $H \leq G$. This contrast hints at a number of novel subtleties that appear in the study of equivariant operads, which we now discuss.

First, note that for a finite group G and G-operad $\mathcal{O}$ (i.e. an operad $\mathcal{O}$ together with a G-action commuting with all the structure), the n-th level $\mathcal{O}(n)$ has a $G \times \Sigma_n$-action. As such, one might guess that a map of G-operads $\mathcal{O} \to \mathcal{O}'$ should be called a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a G-equivalence after forgetting the Σ_n-actions, i.e. if the maps

$$\mathcal{O}(n)^H \xrightarrow{\sim} \mathcal{O}'(n)^H, \qquad H \leq G \leq G \times \Sigma_n, \tag{1.1}$$

NAIVEOPEQ EQ

are weak equivalences of spaces. However, the notion of equivalence suggested in (1.1) turns out to not be "genuine enough". To see why, we consider a homotopical replacement for Com using this theory: if one simply equips an E_∞-operad $\mathcal{O}$ with a trivial G-action, the resulting G-operad has fixed points for each subgroup $\Gamma \leq G \times \Sigma_n$ determined by

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \leq G, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.2}$$

NAIVEGEINFTY EQ

However, as first noted by Costenoble-Waner [12] in their study of equivariant infinite loop spaces, the *G-trivial E_∞-operads* of (1.2) do not provide the correct replacement of Com in the G-equivariant context. Rather, that replacement is provided instead by the *G-E_∞-operads*, characterized by the fixed point conditions

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma \cap \Sigma_n = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.3}$$

GENGEINFTY EQ

In contrasting (1.2) and (1.3), we note first that the subgroups $\Gamma \leq G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \{*\}$ are characterized as being the *graph subgroups*, i.e. the subgroups of the form

$$\Gamma = \{(h, \phi(h)) \in G \times \Sigma_n | h \in H\} \tag{1.4}$$

GRAPHSUBIN EQ

for some subgroup $H \leq G$ and homomorphism $\phi\colon H \to \Sigma_n$. On the other hand, $\Gamma \leq G$ if and only if Γ is the graph subgroup (1.4) for ϕ a trivial homomorphism. As it turns out, the notion of weak equivalence described in (1.1) fails to distinguish (1.2) and (1.3), and indeed it is possible to build maps $\mathcal{O} \to \mathcal{O}'$ where $\mathcal{O}$ is a G-trivial E_∞-operad (as in (1.2)) and $\mathcal{O}'$ is a G-E_∞-operad (as in (1.3)). Therefore, in order to differentiate such operads, one needs to replace the notion of weak equivalence in (1.1) with the finer notion of *graph equivalence*, so that $\mathcal{O} \to \mathcal{O}'$ is considered a weak equivalence only if the maps

$$\mathcal{O}(n)^\Gamma \xrightarrow{\sim} \mathcal{O}'(n)^\Gamma, \qquad \Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}. \tag{1.5}$$

GENEOPEQ EQ

are all weak equivalences.

As mentioned above, the original evidence [12] that (1.3), rather than (1.2), provides the best up-to-homotopy replacement for Com in the equivariant context comes from the study of equivariant infinite loop spaces. For our purposes, however, we instead focus on the perspective of Blumberg-Hill in [3], which concerns the Hill-Hopkins-Ravenel norm maps featured in the solution of the Kervaire Invariant One Problem [17].

Given a G-spectrum R and finite G-set X with n elements, the corresponding *norm* is another G-spectrum $N^X R$, whose underlying spectrum is $R^{\wedge X} \simeq R^{\wedge n}$, but equipped with a "mixed G-action" which both permutes wedge factors via the action on X and acts diagonally on each factor (alternatively, $N^X R$ can be described via graph subgroups; see the next paragraph). Moreover, for any Com-algebra R, i.e. any strictly commutative G-ring spectrum, ring multiplication further induces so called *norm maps*

$$N^X R \to R. \tag{1.6}$$

NORMMAPS EQ

Furthermore, by restricting the structure on R, the maps (1.6) are also defined when X is only an H-set for some subgroup $H \leq G$, and the maps (1.6) then satisfy a number of natural equivariance and associativity conditions. Crucially, we note that the more interesting of these associativity conditions involve H-sets for various H simultaneously (for an example packaged in operadic language, see (1.12) below).

The key observation at the source of the work in [3] is then that, operadically, norm maps are encoded by the graph fixed points appearing in (1.5). More explicitly, noting that, for $H \leq G$, an H-set X with n elements is encoded by a homomorphism $H \to \Sigma_n$, one obtains an associated graph subgroup $\Gamma_X \leq G \times \Sigma_n$, well-defined up to conjugation. Next, using the natural $(G \times \Sigma_n)$-action on $R^{\wedge n}$, the H-action on $N^X R \simeq R^{\wedge n}$ is obtained via the obvious identification $H \simeq \Gamma_X$. It then follows that, for any $\mathcal{O}$-algebra R, maps of the form (1.6) are parametrized by the fixed point space $\mathcal{O}(n)^{\Gamma_X}$. The flaw of the G-trivial E_∞-operads described in (1.2) is then that they lack all norms maps other than those for H-trivial X, thus lacking some of the data encoded by Com. Further, from this perspective one may regard the more naive notion of weak equivalence in (1.1), according to which (1.2) and (1.3) are equivalent, as studying "operads without norm maps" (in the sense that equivalences ignore norm maps), while the equivalences (1.5) study "operads with norm maps".

Our first main result, Theorem I, establishes the existence of a model structure on G-operads with weak equivalences the graph equivalences of (1.5), though our analysis goes significantly further, again guided by Blumberg and Hill's work in [3].

The main novelty of [3] is the definition, for each finite group G, of a finite lattice of new types of equivariant operads, which they dub N_∞-operads. The minimal type of N_∞-operads is that of the G-trivial E_∞-operads in (1.2) while the maximal type is that of the G-E_∞-operads in (1.3). The remaining types, which interpolate between the two, can hence be thought of as encoding varying degrees of "up to homotopy equivariant commutativity". More concretely, each type of N_∞-operad is determined by a collection $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ where each $\mathcal{F}_n$ is itself a collection of graph subgroups of $G \times \Sigma_n$, with an operad $\mathcal{O}$ being called a *$N\mathcal{F}$-operad* if it satisfies the fixed point condition

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.7}$$

NFINFTY EQ

Such collections $\mathcal{F}$ are, however, far from arbitrary, with much of the work in [3, §3] spent cataloging a number of closure conditions that these $\mathcal{F}$ must satisfy. The simplest of these conditions state that each $\mathcal{F}_n$ is a *family*, i.e. closed under subgroups and conjugation. These first two conditions, which are common in equivariant homotopy theory, are a simple consequence of each $\mathcal{O}(n)$ being a space. However, the remaining conditions, all of which involve $\mathcal{F}_n$ for various n simultaneously and are a consequence of operadic multiplication, are both novel and subtle. In loose terms, these conditions, which are more easily described in terms of the H-sets X associated to the graph subgroups, concern closure of those under disjoint union, cartesian product, subobjects, and an entirely new key condition called *self-induction*. The precise conditions are collected in [3, Def. 3.22], which also introduces the term *indexing system* for an $\mathcal{F}$ satisfying all of those conditions. A main result of [3, §4] is then that whenever an $N\mathcal{F}$-operad $\mathcal{O}$ as in (1.7) exists, the associated collection $\mathcal{F}$ must be an indexing system. However, the converse statement, that given any indexing system $\mathcal{F}$ such an $\mathcal{O}$ can be produced, was left as a conjecture.

One of the key motivating goals of the present work was to verify this conjecture of Blumberg-Hill, which we obtain in Corollary IV. We note here that this conjecture has also been concurrently verified by Gutiérrez-White in [15] and by Rubin in [27], with each of their approaches having different advantages: Gutiérrez-White's model for $N\mathcal{F}$ is cofibrant while Rubin's model is explicit. Our model, which emerges from a broader framework, satisfies both of these desiderata.

To motivate our approach, we first recall the solution of a closely related but simpler problem: that of building universal spaces for families of subgroups. Given a family $\mathcal{F}$ of subgroups of G (i.e. a collection closed under conjugation and subgroups), a *universal space* X for $\mathcal{F}$, also called an $E\mathcal{F}$*-space*, is a space with fixed points X^H characterized just as in (1.7). In particular, whenever $\mathcal{O}$ is an $N\mathcal{F}$-operad, each $\mathcal{O}(n)$ is necessarily an $E\mathcal{F}_n$-space. The existence of $E\mathcal{F}$-spaces for any choice of the family $\mathcal{F}$ is best understood in light of Elmendorf's classical result from [13] (modernized by Piacenza in [26]) stating that there is a Quillen equivalence (recall that O_G is the *orbit* category, formed by the G-sets G/H)

$$\begin{array}{ccc} \mathsf{Top}^{\mathsf{O}_G^{op}} & \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} & \mathsf{Top}^G \\ (G/H \mapsto Y(G/H)) & \longmapsto & Y(G) \\ (G/H \mapsto X^H) & \longleftarrow & X \end{array} \tag{1.8}$$

where the weak equivalences (and fibrations) on Top^G are detected on all fixed points and the weak equivalences (and fibrations) on the category $\mathsf{Top}^{\mathsf{O}_G^{op}}$ of *coefficient systems* are detected at each presheaf level. Noting that the fixed point characterization of $E\mathcal{F}$-spaces defines a natural object $\delta_{\mathcal{F}} \in \mathsf{Top}^{\mathsf{O}_G^{op}}$ by $\delta_{\mathcal{F}}(G/H) = *$ if $H \in \mathcal{F}$ and $\delta_{\mathcal{F}}(G/H) = \varnothing$ otherwise, $E\mathcal{F}$-spaces can then be built as $\iota^*(C\delta_{\mathcal{F}}) = C\delta_{\mathcal{F}}(G)$, where C denotes cofibrant replacement in $\mathsf{Top}^{\mathsf{O}_G^{op}}$. Moreover, we note that, as in [13, §3], these cofibrant replacements can be built via explicit simplicial realizations.

The overarching goal of this paper is then that of proving the analogue of Elmendorf-Piacenza's Theorem (1.8) in the context of operads with norm maps (i.e. with equivalences as in (1.5)), which we state as our main result, Theorem III. However, in trying to formulate such a result one immediately runs into a fundamental issue: it is unclear which category should take the role of the coefficient systems $\mathsf{Top}^{\mathsf{O}_G^{op}}$ in this context. This last remark likely requires justification. Indeed, it may at first seem tempting to simply employ one of the known formal generalizations of Elmendorf-Piacenza's result (see, e.g. [29, Thm. 3.17]) which simply replace Top on either side of (1.8) with a more general model category $\mathcal{V}$. However, if one applies such a result when $\mathcal{V} = \mathsf{Op}$ to establish a Quillen equivalence $\mathsf{Op}^{\mathsf{O}_G^{op}} \rightleftarrows \mathsf{Op}^G$ (the existence of this equivalence is due to upcoming work of Bergner-Gutiérrez), the fact that the levels of each $\mathcal{P} \in \mathsf{Op}^{\mathsf{O}_G^{op}}$ correspond only to those fixed-point spaces appearing in (1.1) would require working in the context of operads *without* norm maps, and thereby forgo the ability to distinguish the many types of $N\mathcal{F}$-operads.

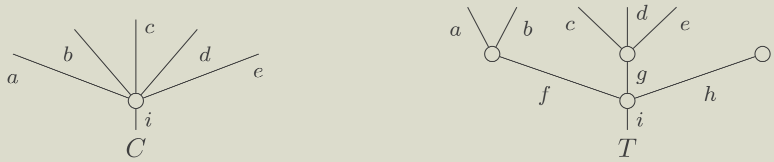
As such, to obtain an Elemendorf-Piacenza Theorem in the context of operads with norm maps, we will need to replace $\mathsf{Top}^{\mathsf{O}_G^{op}}$ with a category Op_G of new algebraic objects we dub *genuine equivariant operads* (as opposed to (regular) equivariant operads Op^G). Each genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will consist of a list of spaces, indexed in the same way as in (1.5), along with obvious restriction maps and, more importantly, suitable *composition maps*. Precisely identifying the required composition maps is one of the main challenges of this theory, and again we turn to [3] for motivation.

Analyzing the proofs of the results in [3, §4] concerning the closure properties for indexing systems $\mathcal{F}$, a common motif emerges: when performing an operadic composition

$$\begin{array}{ccc} \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) & \longrightarrow & \mathcal{O}(m_1 + \cdots + m_n), \\ (f, g_1, \cdots, g_n) & \longmapsto & f(g_1, \cdots, g_n) \end{array} \tag{1.9}$$

careful choices of fixed point conditions on the operations $f, g_1, \cdots, g_n$ yield a fixed point condition on the composite operation $f(g_1, \cdots, g_n)$. The desired multiplication maps for a genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will then abstract such interactions between multiplication and fixed points for an equivariant operad $\mathcal{O} \in \mathsf{Op}^G$. However, these interactions can be challenging to write down explicitly and indeed, the arguments in [3, §4] do not quite provide the sort of unified conceptual approach to these interactions needed for our purposes. The cornerstone of the current work was then the joint discovery by the authors of such a conceptual framework: equivariant trees.

Non-equivariantly, it has long been known that the combinatorics of operadic composition is best visualized by means of tree diagrams. For instance, the tree T on the right below



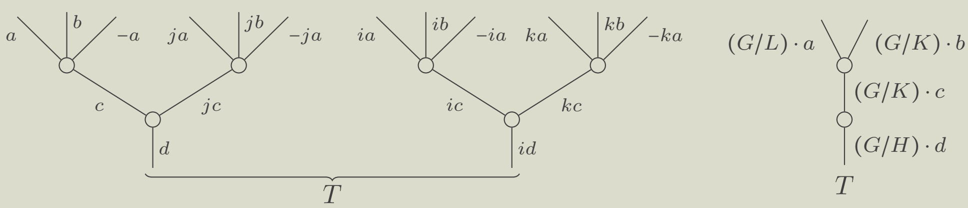
encodes the operadic composition

$$\mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0) \to \mathcal{O}(5) \tag{1.10}$$ COMPEX EQ

where the inputs $\mathcal{O}(3), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(0)$ correspond to the nodes/vertices (i.e. circles) in the tree T, with arity given by number of incoming edges (i.e. edges immediately above) and the output $\mathcal{O}(5)$ has arity given by counting leaves (i.e. edges at the top, not capped by a node). Before recalling equivariant trees, it is worth making the connection between T and (1.10) more precise. Recall [22, §3] that T gives rise to a colored operad[1] $\Omega(T)$, as follows. The colors/objects of $\Omega(T)$ are the edges $a, b, c, \cdots, i$ while the generating operations, determined by the nodes, are $(a, b) \to f$, $(c, d, e) \to g$, $() \to h$, $(f, g, h) \to i$ (i.e., for each node, incoming edges are viewed as inputs and the outgoing edge as an output). Let C be the corolla (i.e. tree with a single node) above, which is formed by the leaves and root of T. There is then a natural map of colored operads $\Omega(C) \to \Omega(T)$ so that, writing $\mathsf{Op}_\bullet$ for the category of colored operads, (1.10) is the induced map of mapping sets $\mathsf{Op}_\bullet(\Omega(T), \mathcal{O}) \to \mathsf{Op}_\bullet(\Omega(C), \mathcal{O})$. Indeed, $\mathsf{Op}_\bullet(\Omega(T), \mathcal{O}) \simeq \mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0)$ and $\mathsf{Op}_\bullet(\Omega(C), \mathcal{O}) \simeq \mathcal{O}(5)$ since maps $\Omega(T) \to \mathcal{O}$ and $\Omega(C) \to \mathcal{O}$ are determined by the image of the generating operations.

Analogously, the role of equivariant trees is, in the context of equivariant operads, to encode operadic compositions as in (1.10) together with fixed point compatibilities. Briefly, a G-tree [24, Def. 5.44] is a forest diagram (i.e. a collection of trees) together with a G-action that is transitive on tree components. A detailed introduction to (and motivation for) equivariant trees can be found in [25, §4], where the second author develops the theory of equivariant dendroidal sets (a parallel approach to equivariant operads), though here we include only a single representative example.

Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ be the group of quaternionic units and $G \geq H \geq K \geq L$ be the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$. One has a G-tree T with *expanded representation* given by the two leftmost trees below and *orbital representation* given by the rightmost tree.



(1.11) D6SMALLER EQ

[1]Recall that colored operads, also known as multicategories, are a generalization of the notion of category where each arrow/operation has multiple inputs but a single output.

Here, the expanded representation of T is just a forest with edges labels that indicate the G-action. Note that all edges are conjugate to one of the edges a, b, c, d which have, respectively, stabilizers L, K, K, H. For example, the labels of T imply that $\pm id = \pm kd$ and $\pm jd = \pm d = d$. Given the expanded representation, the orbital representation is obtained by collapsing each edge orbit into a single edge, which is labeled by the corresponding orbit set of edges in the expanded representation (one may also reverse this process, though we will not need to do so). We note that orbital representations always "look like a tree".

As explained in [25, Example 4.9], the G-tree T encodes the fact that, for $\mathcal{O} \in \mathsf{Op}^G$ a G-operad, the composition $\mathcal{O}(2) \times \mathcal{O}(3)^{\times 2} \to \mathcal{O}(6)$ restricts to a fixed point composition

$$\mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \to \mathcal{O}(H/L \amalg H/K)^H \tag{1.12}$$

INTFIXPTCOMP EQ

(we discuss how (1.12) is obtained in the next paragraph) where $\mathcal{O}(X)$ for X an H-set denotes $\mathcal{O}(|X|)$ with the H-action given by the identification $H \simeq \Gamma_X$ (the graph subgroup Γ_X is as discussed after (1.6)), and likewise for K-sets. In particular, $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$.

We recall the precise connection between T and (1.12). Let $\mathsf{Op}^G_\bullet$ be the category of G-objects in colored operads. As in the non-equivariant case, one builds $\Omega(T)$ in $\mathsf{Op}^G_\bullet$ and a map $\Omega(C) \to \Omega(T)$ in $\mathsf{Op}^G_\bullet$, where C is the G-corolla (i.e. G-tree composed of corollas) formed by the leaves and roots of T. The composition (1.12) is then the induced map $\mathsf{Op}^G_\bullet(\Omega(T), \mathcal{O}) \to \mathsf{Op}^G_\bullet(\Omega(C), \mathcal{O})$. The implicit claim $\mathsf{Op}^G_\bullet(\Omega(T), \mathcal{O}) \simeq \mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K$ follows since: by equivariance, a G-map $\phi \colon \Omega(T) \to \mathcal{O}$ is determined by the images of the operations $(a, b, -a) \to c$ and $(c, jc) \to d$; the operation $\phi((a, b, -a) \to c)$ must be in $\mathcal{O}(K/L \amalg K/K)^K$, since K is the isotropy of c and $\{a, b, -a\} \simeq K/L \amalg K/K$ as K-sets; likewise $\phi((c, jc) \to d)$ must be in $\mathcal{O}(H/K)^H$. The claim $\mathsf{Op}^G_\bullet(\Omega(C), \mathcal{O}) \simeq \mathcal{O}(H/L \amalg H/K)^H$ is similar.

We note that the two inputs $\mathcal{O}(H/K)^H$, $\mathcal{O}(K/L \amalg K/K)^K$ in (1.12) correspond to the two nodes of the orbital representation in (1.11). Notice that now the arity (i.e. the associated "type of input") of such a node does not just count incoming edge orbits, but depends on the labels of both incoming and outgoing edge orbits (in particular, the fixed point condition depends on the latter). Similarly, the output $\mathcal{O}(H/L \amalg H/K)^H$ is determined by both the leaf and root edge orbits. The existence of maps of the form (1.12) is essentially tantamount to the subtlest closure property for indexing systems $\mathcal{F}$, self-induction (cf. [3, Def. 3.20]), and similar tree descriptions exist for all other closure properties, as detailed in [25, §9].

We can now at last give a full informal description of the category Op_G featured in our main result, Theorem III. A genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ has levels $\mathcal{P}(X)$ for each H-set X, $H \leq G$, that mimic the role of the fixed points $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$ for $\mathcal{O} \in \mathsf{Op}^G$. More explicitly, there are restriction maps $\mathcal{P}(X) \to \mathcal{P}(X|_K)$ for $K \leq H$, isomorphisms $\mathcal{P}(X) \simeq \mathcal{P}(gX)$ where gX denotes the conjugate gHg^{-1}-set, and composition maps given by

$$\mathcal{P}(H/K) \times \mathcal{P}(K/L \amalg K/K) \to \mathcal{P}(H/L \amalg H/K)$$

in the case of the abstraction of (1.12), and more generally by

$$\begin{array}{c}\mathcal{P}(H/K_1 \amalg \cdots \amalg H/K_n) \times \mathcal{P}(K_1/L_{11} \amalg \cdots \amalg K_1/L_{1m_1}) \times \cdots \times \mathcal{P}(K_n/L_{n1} \amalg \cdots \amalg K_n/L_{nm_n}) \\ \downarrow \\ \mathcal{P}(H/L_{11} \amalg \cdots \amalg H/L_{1m_1} \amalg \cdots \amalg H/L_{n1} \amalg \cdots \amalg H/L_{nm_n}).\end{array} \tag{1.13}$$

GENGENMULT EQ

Lastly, these composition maps must satisfy associativity, unitality, compatibility with restriction maps, and equivariance conditions, as encoded by the theory of G-trees. Rather than making such compatibilities explicit, however, we will find it preferable for our purposes to simply define genuine equivariant operads intrinsically in terms of G-trees.

We end this introduction with an alternative perspective on the role of genuine equivariant operads. The Elmendorf-Piacenza theorem in (1.8) is ultimately a strengthening of the basic observation that the homotopy groups $\pi_n(X)$ of a G-space X are coefficient systems rather than just G-objects. Similarly, the generalized Elmendorf-Piacenza result [29, Thm. 3.17] applied to the category $\mathcal{V} = \mathsf{sCat}$ of simplicial categories strengthens the observation that for a G-simplicial category $\mathcal{C}$ the associated homotopy category $\mathrm{ho}(\mathcal{C})$ is a coefficient system of categories rather than just a G-category. Likewise, Theorem III strengthens the (not so basic) observation that for a G-simplicial operad $\mathcal{O}$ the associated homotopy operad $\mathrm{ho}(\mathcal{O})$ is neither just a G-operad nor just a coefficient system of operads but rather the richer algebraic structure that we refer to as a "genuine equivariant operad".

1.1 Main results

We now discuss our main results.

Fixing a finite group G, we recall that $\mathsf{Op}^G(\mathcal{V}) = (\mathsf{Op}(\mathcal{V}))^G$ denotes G-objects in $\mathsf{Op}(\mathcal{V})$.

MAINEXIST1 THM **Theorem I.** *Let* $(\mathcal{V}, \otimes)$ *denote either* $(\mathsf{sSet}, \times)$ *or* $(\mathsf{sSet}_*, \wedge)$.

Then there exists a model category structure on $\mathsf{Op}^G(\mathcal{V})$ *such that* $\mathcal{O} \to \mathcal{O}'$ *is a weak equivalence (resp. fibration) if all the maps*

$$\mathcal{O}(n)^{\Gamma} \to \mathcal{O}'(n)^{\Gamma} \tag{1.14}$$

GENEOPEQMT EQ

for $\Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}$, *are weak equivalences (fibrations) in* $\mathcal{V}$.

More generally, for $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ *with* $\mathcal{F}_n$ *an arbitrary collection of subgroups of* $G \times \Sigma_n$ *there exists a model category structure on* $\mathsf{Op}^G(\mathcal{V})$, *which we denote* $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$, *with weak equivalences (resp. fibrations) determined by* (1.14) *for* $\Gamma \in \mathcal{F}_n$.

Lastly, analogous semi-model category structures $\mathsf{Op}^G(\mathcal{V})$, $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ *exist provided that* $(\mathcal{V}, \otimes)$*: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers.*

We note that a similar result has also been proven by Gutiérrez-White in [15].

Theorem I is proven in §**??**. Condition (i) can be found in [19, Def. 2.1.17] while (ii) can be found in [19, Def. 4.2.6]. The additional conditions (iii) and (iv), which are less standard, are discussed in §**??** and §**??**, respectively. Further, by *semi-model category* we mean the notion in [31, Def. 2.2.1][2], which relaxes the definition of model structure by requiring that some of the axioms need only apply if the domains of certain cofibrations are cofibrant.

Our next result concerns the model structure on the new category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads introduced in this paper. Before stating the result, we must first outline how $\mathsf{Op}_G(\mathcal{V})$ itself is built. Firstly, the levels of each $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$, i.e. the H-sets in (1.13), are encoded by a category Σ_G of G-*corollas*, introduced in §**??**, which generalizes the usual category Σ of finite sets and isomorphisms. We then define G-symmetric sequences by $\mathsf{Sym}_G(\mathcal{V}) = \mathcal{V}^{\Sigma_G^{op}}$ and, whenever $\mathcal{V}$ is a closed symmetric monoidal category with diagonals (cf. Remark **??**), we define in §**??** a *free genuine equivariant operad* monad $\mathbb{F}_G$ on $\mathsf{Sym}_G(\mathcal{V})$ whose algebras form the desired category $\mathsf{Op}_G(\mathcal{V})$.

Moreover, inspired by the analogues $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}} \rightleftarrows \mathsf{Top}^G_{\mathcal{F}}$ of the Elmendorf-Piacenza equivalence where $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}}$ are partial coefficient systems determined by a family $\mathcal{F}$, we show in §**??** that (a slight generalization of) Blumberg-Hill's indexing systems $\mathcal{F}$ give rise to sieves $\Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ and partial G-symmetric sequences $\mathsf{Sym}_{\mathcal{F}}(\mathcal{F}) = \mathcal{V}^{\Sigma_{\mathcal{F}}^{op}}$ which are suitably compatible with the monad $\mathbb{F}_G$, thus giving rise to categories $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ of *partial genuine equivariant operads*.

MAINEXIST2 THM **Theorem II.** *Let* $(\mathcal{V}, \otimes)$ *denote either* $(\mathsf{sSet}, \times)$ *or* $(\mathsf{sSet}_*, \wedge)$. *Then the projective model structure on* $\mathsf{Op}_G(\mathcal{V})$ *exists. Explicitly, a map* $\mathcal{P} \to \mathcal{P}'$ *is a weak equivalence (resp. fibration) if all maps*

$$\mathcal{P}(C) \to \mathcal{P}'(C) \tag{1.15}$$

GENEQTHM EQ

[2] We note that the role of $\mathcal{M}$ in [31, Def. 2.2.1] is auxiliary, as one is always free to replace $\mathcal{M}$ with the terminal category $*$, recovering the notion of a *J-semi-model structure (over $*$)* from [28, Def. 1]. In practice, the purpose of choosing $\mathcal{M} \neq *$ in [31] is that the existence of the semi-model structure on $\mathcal{D}$ therein is typically established via transfer from $\mathcal{M}$. We also caution that, when $\mathcal{M} \neq *$, the notion in [28] is more demanding than that in [31].

are weak equivalences (fibrations) in $\mathcal{V}$ for each $C \in \Sigma_G$.

More generally, for $\mathcal{F}$ a weak indexing system, the projective model structure on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ exists. Explicitly, weak equivalences (resp. fibrations) are determined by (1.15) for $C \in \Sigma_{\mathcal{F}}$.

Lastly, analogous semi-model structures on $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ exist provided that $(\mathcal{V}, \otimes)$: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals.

Theorem II is proven in §?? in parallel with Theorem I. We note that the condition (v) that $(\mathcal{V}, \otimes)$ has diagonals (cf. Remark ??), which is not needed in Theorem I, is required to build the monad $\mathbb{F}_G$, and hence the categories $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$.

The following is our main result. The explicit formulas for the functors ι^*, ι_* are found in (??) (also, see Corollaries ?? and ??).

QUILLENEQUIV THM **Theorem III.** *Let $(\mathcal{V}, \otimes)$ denote either $(\mathsf{sSet}, \times)$ or $(\mathsf{sSet}_*, \wedge)$.*

Then the adjunctions, where in the more general rightmost case $\mathcal{F}$ is a weak indexing system,

$$\mathsf{Op}_G(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G(\mathcal{V}), \qquad \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}). \tag{1.16}$$

are Quillen equivalences.

Morover, analogous Quillen equivalences of semi-model structures[3] $\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \simeq \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ exist provided that $(\mathcal{V}, \otimes)$: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals; (vi) has cartesian fixed points.

Theorem III is proven in §??. Condition (vi), which is not needed in either of Theorems I,II is discussed in §??.

Lastly, our techniques also verify the main conjecture of [3], which we discuss in §??. Moreover, we note that our models for $N\mathcal{F}$-operads are given by explicit bar constructions.

TY_REAL_COR_MAIN **Corollary IV.** *For $\mathcal{V} = \mathsf{sSet}$ or Top and $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ any weak indexing system, $N\mathcal{F}$-operads exist. That is, there exist explicit operads $\mathcal{O}$ such that*

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.17}$$ NFINFTY2 EQ

In particular, the map $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathcal{I}$ in [3, Cor. 5.6] is an equivalence of categories.

Moreover, if $\mathcal{O}'$ has fixed points as in (1.17) for some collection of graph subgroups $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$, then $\mathcal{F}$ must be a weak indexing system.

1.2 Context, applications and future work

This article is part of a series of papers by the authors [25, 6, 8, 7] producing and analyzing different models for the homotopy theory of equivariant operads with norm maps. These papers generalize the non-equivariant program of Cisinski, Moerdijk, and Weiss [22, 23, 9, 10, 11]. A major result is the existence of a Quillen equivalence

$$\mathsf{dSet}^G \rightleftarrows \mathsf{sOp}^G, \tag{1.18}$$ DSETSOP_EQ

where dSet^G is the model category of equivariant dendroidal sets encoding G-∞-operads from [25, Theorem 2.1], and sOp^G is the category of equivariant many-colored simplicial operads equipped with the many-colored variant of the model structure in Theorem I from [8, Theorem III]. In particular, this is a generalization of the equivalence between the homotopy

[3]See [14, §12.1.8] for a precise definition.

theories of simplicial categories and quasicategories. More details can be found in the introduction to [8].

In order to simplify our discussion, this paper focuses exclusively on the theory of single colored (genuine) equivariant operads. As mentioned above, [8, Theorem III] is an extension of Theorem I to the colored setting. Moreover, we conjecture the many-colored variants of Theorems II and III also hold, and intend to show this in upcoming work. We note, however, that an important new subtlety emerges in the equivariant setting: while usual equivariant colored operads have G-sets of objects, genuine equivariant colored operads will instead have *coefficient systems* of objects.

While genuine equivariant (colored) operads as defined in this paper do not explicitly appear in [25, 6, 8, 7], G-trees and the algebra and combinatorics of genuine equivariant operads play a prominent role. In every model category encoding the homotopy theory of equivariant operads with norms, (co)fibrations and weak equivalences are described using G-trees. Moreover, recalling that a category (resp. operad) is equivalent, under a nerve functor, to a strict inner Kan complex in sSet [20, Prop. 1.1.2.2] (resp. dSet [23, Prop. 5.3, Thm. 6.1]), a genuine equivariant operad is equivalent, under a nerve functor $\mathsf{Op}_G \to \mathsf{dSet}_G = \mathsf{Set}^{\Omega_G^{op}}$, to a strict inner Kan complex (cf. [6, Remark 3.42]); the proof of this fact will also be delegated to a sequel. This combinatorial notion of a genuine equivariant operad appears and underlies key steps in the proofs of the main results from each of [25, 6, 8, 7], for example [6, Def. 5.8] and [7, Prop. 4.47]. Additionally, we expect there to be a "genuine" analogue

$$\mathsf{dSet}_G \rightleftarrows \mathsf{sOp}_G,$$

of the Quillen equivalence from (1.18).

transition

Let $\mathcal{T}$ denote the colored operad whose algebras are operads, as defined in Gutierrez-Vogt.

In terms of our language, the colors of $\mathcal{T}$ are the arities $C \in \Sigma$, and an operation with signature $(C_1, \cdots, C_n; C_0)$ consists of a tree T, a permutation $\sigma \in \Sigma_n$ such that $V(T) = (C_{\sigma(i)})$ and a tall map $C_0 \to T$.

Their construction can be modified to construct a G-equivariant colored operad $\mathcal{T}_G^{fr}$, as follows. Operations are now the G-free corollas $C \in \Sigma_G^{fr}$ and operations with signature $(C_1, \cdots, C_n; C_0)$ are encoded by G-free trees T, a permutation $\sigma \in \Sigma_n$ such that such that $V_G(T) = (C_{\sigma(i)})$ and a tall rooted map $C_0 \to T$. Moreover, the G-action on $\mathcal{T}_G^{fr}$ is described on objects by $gC = g(C_i)_{i \in I} = (C_{gi})_{i \in I}$ and similar by $gT = g(T_i)_{i \in I} = (T_{gi})_{i \in I}$ on operations (some caution is needed concerning the permutation σ, since while the vertices of gT are conjugate to the vertices of T, they do not share the same order).

Note: The operad $\mathcal{T}_G^{fr}$ does *not* include "G-action operations". Instead, both $\mathcal{T}_G^{fr}$ and its algebras come with prescribed G-actions, which then impose equivariance conditions on the algebra structure maps.

One then has a map of G-colored operads (where G acts trivially on $\mathcal{T}$)

$$\mathcal{T} \to \mathcal{T}_G^{fr}$$

Moreover, upon forgetting the G-action, this map is fully faithful and essentially surjective. However, this map is not G-essentially surjective. More precisely, for any $* \neq H \leq G$ one has that the induced map on H-fixed points is not essentially surjective.

separation

Example 1.19. Let $G = \mathbb{Z}_{/2}$, $\mathcal{O} \in \mathsf{sOp}^G$ and $X \in \mathsf{sSet}^G$ an $\mathcal{O}$-algebra.

$$\begin{aligned} \pi_0\left(\mathcal{O}(2)^{\Gamma_G}\right) \times \pi_0(X) &\longrightarrow \pi_0\left(X^G\right) \\ ([p],[x]) &\longmapsto [p(x, x+1)] \end{aligned}$$

separation

$\mathcal{H}_n(X) = \mathcal{C}(X^{\otimes n}, X)$

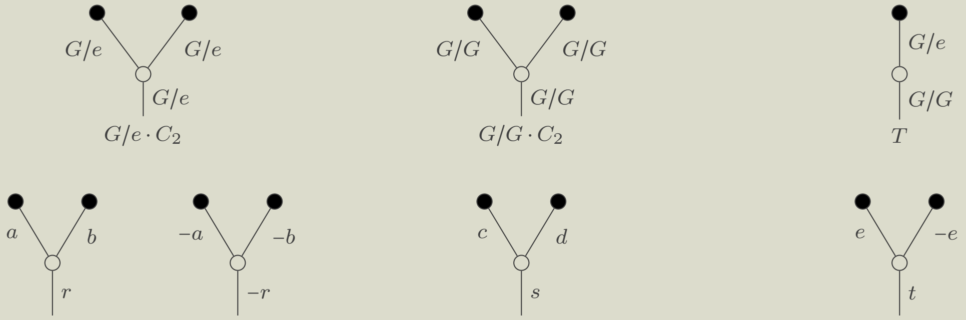
For coefficient system consisting of fixed points, an algebra structure is a map $\mathcal{P} \to \iota_* \mathcal{H}_\bullet(X)$.

transition

The techniques and machinery developed in this paper have applications and extensions in a number of topics outside the homotopy theory of equivariant operads.

First, this work identifies additional structure on the homotopy groups of an $\mathcal{O}$-algebra X for any G-operad $\mathcal{O}$ and G-space X: While it is clear that X^H is an algebra over $\mathcal{O}^H$, and hence $\pi_*(X^H)$ is an algebra over $\pi_*(\mathcal{O}^H)$, the genuine equivariant operad paradigm tells us that in fact there are "twisted structure maps", as the coefficient system $\pi_*(\iota_* X)$ is an algebra over the genuine equivariant operad $\pi_*(\iota_* \mathcal{O})$.

Example 1.20. Let $G = \mathbb{Z}/2 = \{-1, 1\}$ be the cyclic group of order 2, and consider the three G-trees below; the orbital representations are displayed above the extended representations.



(1.21)

For any G-operad $\mathcal{O}$ and $\mathcal{O}$-algebra X, these G-tree diagrams encode structure maps, by placing elements of $\mathcal{O}(2)$ (resp. X) of appropriate isotropy in the root vertices (resp. stumps). The first two trees encode the algebra structure maps

$$\mathcal{O}(2) \times X^{\times 2} \to X, \qquad \mathcal{O}(2)^{\Gamma_{G/G \amalg G/G}} \times (X^G)^{\times 2} \to X^G$$

(where we note that $\Gamma_{G/G \amalg G/G} = G \leq G \times \mathbb{Z}/2$, and hence the right equation is the G-fixed points of the left). This last G-tree encodes the twisted structure map

$$\mathcal{O}(2)^{\Gamma_{G/e}} \times X \longrightarrow X^G,$$

given by the composite (cf. (1.12))

$$\mathcal{O}(2)^{\Gamma_{G/e}} \times X \xrightarrow{\mathrm{id} \times \Delta_{\mathbb{Z}/2}} \mathcal{O}(2)^{\Gamma_{G/2}} \times (X \times X)^{\Gamma_{G/2}} \to X^G$$

where $\Delta_{\mathbb{Z}/2}$ sends x to $(x, -x)$. This induces a twisted action map on homotopy groups, for example

$$\pi_0 \mathcal{O}(2)^{\Gamma_{G/2}} \times \pi_0 X \to \pi_0(X^G).$$

A full discussion of algebras over genuine equivariant operads is forthcoming.

Let $\Omega^{0,\mathcal{P},Y}_{G,\mathrm{Cl}}$ denote the full subcategory of closed (no leaves) 2-labeled G-trees, with $\lambda_a = \{\mathcal{P}\}$ and $\lambda_i = \{Y\}$, such that all inert vertices are stumps. Then, for $\mathcal{P}$ a genuine G-operad and Y a coefficient system, the *free $\mathcal{P}$-algebra on Y*, $\mathbb{F}_{\mathcal{P}}(Y)$, is given by the opposite of the right Kan extension

$$\begin{tikzcd}
\Omega^{0,\{\mathcal{P},Y\}}_{G,\mathrm{Cl}} \arrow[r, "V_G^{\{\mathcal{P},Y\}}"] \arrow[d, "\mathsf{r}"] & \mathsf{F}_s \wr \Sigma_G \times \mathsf{F}_s \wr \mathsf{O}_G \arrow[r, "{(\mathcal{P},Y)}"] & \mathsf{F}_s \wr \mathcal{V}^{op} \times \mathsf{F}_s \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \\
\mathsf{O}_G \arrow[rrru, "\mathbb{F}_{\mathcal{P}}(Y)^{op} = \mathrm{Ran}"'] & & &
\end{tikzcd}$$

transition

Second, much of the machinery in this paper, presenting the free operad monad and its equivariant variants, can be applied in different contexts to define new equivariant algebraic notions. It is used in [8] to construct a many-color variant of Theorem I. Moreover, as previewed in [5], there is a subcategory $\mathsf{Sym}_G \to \mathsf{Op}_G$ of "genuine symmetric monoidal categories", constructed by blending the description of $\mathbb{F}_G$ used here reference with the monad $\Sigma \wr (-)$ encoding symmetric monoidal categories (cf. Remark ??). Our formal investigation of Sym_G is in progress. Furthermore, we conjecture that this bookkeeping can be applied to produce genuine equivariant analogues of properads and other algebraic theories.

transition

Finally, the comparison between simplicial G-operads sOp^G and the parametrized G-∞-operads of [1] factors most naturally through the category of genuine G-operads sOp_G. Non-equivariantly, this comparison is given by the operadic nerve functor $N^{\otimes}\colon \mathsf{sOp} \to \mathsf{Op}_{\infty}$ [20, Def. 2.1.1.3]. In [5], the first-named author extends this to a functor $N^{\otimes}\colon \mathsf{sOp}^G \to \mathsf{sOp}_G \to \mathsf{Op}_{\infty,G}$, and moreover produces a functor on the categories of algebras $\mathsf{Alg}(\mathcal{O}) \to \mathsf{Alg}(N^{\otimes}\mathcal{O})$. When $\mathcal{O}$ is the genuine equivariant little disks operad, this connection has been used by [18] to give a characterization of equivariant factorization homology.

colored operad of operads, and we have colored genuine operad of genuine operads

- genuine operad for $\underline{\mathfrak{C}} = \iota_* \mathfrak{C}$-colored genuine operad equals ι_* of the operad for $\mathfrak{C}$-colored operads

come back

1.3 The nerve theorem

In this section we assume that the monoidal structure on $\mathcal{V}$ is the cartesian product.

NERVE THM

Theorem 1.22. *There is a fully faithful nerve functor $\mathcal{N}\colon \mathsf{Op}_G(\mathcal{V}) \to \mathcal{V}^{\Omega_G^{op}}$ whose essential image consists of the pointed strict Segal objects, i.e. the $X \in \mathcal{V}^{\Omega_G^{op}}$ such that the natural maps*

$$X(T) \xrightarrow{\simeq} \prod_{v \in V_G(T)} X(T_v) \tag{1.23}$$

SEGCOND EQ

are isomorphisms for all $T \in \Omega_G$.

note that can't do this the usual way since we have no colored genuine operads

Remark 1.24. (1.23) implies that $X(G/H \cdot \eta) = *$ for all $H \leq G$, since $V_G(G/H \cdot \eta) = ()$ is the empty tuple, motivating the term "pointed". If one allows for non-pointed X (i.e. for $X(G/H \cdot \eta) \neq *$), the Segal condition (1.23) needs to be modified.

The proof of Theorem 1.22 will follow by first building N, and then describing a partial inverse (when restricted to pointed Segal objects).

HERE

To build the nerve $\mathcal{N}\mathcal{P}$ for $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$, recall first that $\iota\mathcal{P}$ is a N-algebra and consider the bar construction $B_\bullet = B_\bullet(N, N, \iota\mathcal{P}) = N^{n+1}\iota\mathcal{P}$, whose levels are spans of the form $\Sigma_G \leftarrow \Omega_G^n \xrightarrow{\mathcal{N}_n} \mathcal{V}^{op}$ where

$$\mathcal{N}_n\left(T_0 \to \cdots \to T_n\right) \simeq \prod_{v \in V_G(T_n)} X(T_{n,v})$$

Moreover, the face maps d_i of $B_\bullet$ for $i < n$ are induced by the multiplication $NN \Rightarrow N$, and thus their natural transformation component is an isomorphism. Hence, by Proposition (??), and noting that $|\Omega_G^n| = \Omega_G^{\mathfrak{t}}$ is the category of G-trees and tall maps, one obtains upon realization a functor $\mathcal{N}^{\mathfrak{t}}\colon \Omega_G^{\mathfrak{t}} \to \mathcal{V}^{op}$. This construction almost defines the desired nerve functor in Theorem (1.22), except we have a functor out of the subcategory Ω^{t}_G rather than the entire category Ω_G. To address this, we first enlarge the string categories Ω^n_G to categories $\Omega^{n,o}_G, n \geq 0$ whose objects are still strings but whose maps between strings are the the maps which are component wise outer maps (note that quotients are componentwise isomorphisms). Note that the $\Omega^{n,o}_G$ still form a simplicial object, i.e. one still has simplicial operators $d_i \colon \Omega^{n+1,o}_G \to \Omega^{n,o}_G$ for $0 \leq n, 0 \leq i \leq n+1$ and $s_j \colon \Omega^{n,o}_G \to \Omega^{n+1,o}_G$ for $0 \leq n, 0 \leq j \leq n$, though we note that one no longer has an augmentation to Σ_G nor extra degeneracies. It is clear that one now has $|\Omega^{n,o}_G| = \Omega_G$ so that, to build $\mathcal{N} \colon \Omega_G \to \mathcal{V}^{op}$, it suffices to show that $B_\bullet$ can be extended to a simplicial object $B^o_\bullet$ whose levels are functors $\Omega^{n,o}_G \xrightarrow{\mathcal{N}_n} \mathcal{V}^{op}$. This follows by noting that the vertex functors $V_G \colon \Omega^n_G \to \mathsf{F}_s \wr \Omega^{n-1}_G$ extend to functors $V_G \colon \Omega^{n,o}_G \to \mathsf{F} \wr \Omega^{n-1}_G$. This now follows from the observation that the following diagrams commute (note that all the simplicial operators of $B_\bullet$ are naturally factored so that the leftmost is extended by one of the diagrams below).

$$\begin{tikzcd}
\Omega^{n,o}_G \arrow[r, "V_G"] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[r, "\mathsf{F} \wr V_G"] & \mathsf{F}^{\wr 2} \wr \Omega^{n-2}_G \arrow[d, "\sigma^0"'] \\
\Omega^{n-1,o}_G \arrow[rr, "V_G"'] \arrow[urr, Leftrightarrow, "\pi"'] & & \mathsf{F} \wr \Omega^{n-2}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^{n,o}_G \arrow[r, "V_G"] \arrow[d, "d_{i+1}"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[d, "\mathsf{F} \wr d_i"] \\
\Omega^{n-1,o}_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Omega^{n-2}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^{n,o}_G \arrow[r, "V_G"] \arrow[d, "s_{j+1}"'] & \mathsf{F} \wr \Omega^{n-1}_G \arrow[d, "\mathsf{F} \wr s_j"] \\
\Omega^{n+1,o}_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Omega^{n}_G
\end{tikzcd}
\tag{1.25}$$

Thus, by realizing $\Omega^{n,o}_G \xrightarrow{\mathcal{N}_n} \mathcal{V}^{op}$ we obtain the desired nerve functor $\mathcal{NP} \colon \Omega_G \to \mathcal{V}^{op}$.

HERE

We now turn towards the task of building a partial inverse to $\mathcal{N}$.

1.3.1 Essential surjectivity

$X_1 \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} NX_{-1}$ versus $X_1 \xrightarrow{\epsilon} NX_0 \xrightarrow{Nd_0} NX_{-1}$

$$\begin{tikzcd}
\Omega^1 \arrow[rr] \arrow[d, "d_1"'] & & \mathsf{F} \wr \Omega \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\Omega^0 \arrow[rr] \arrow[d, equal] \arrow[urr, Rightarrow, "m"'] & & \mathsf{F} \wr \Omega \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\Omega^0 \arrow[r] \arrow[urr, Rightarrow, "\epsilon"'] & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega \arrow[r] & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^1 \arrow[rrrr] \arrow[d, equal] & & & & \mathsf{F} \wr \Omega \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\Omega^1 \arrow[r] \arrow[d, "d_1"'] \arrow[urrrr, Rightarrow, "\epsilon"'] & \mathsf{F} \wr \Omega^0 \arrow[r] \arrow[d, "d_0"'] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[r, "\sigma^0"] \arrow[d, equal] & & \mathsf{F} \wr \Omega \arrow[r] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\Omega^0 \arrow[r] & \mathsf{F} \wr \Sigma \arrow[r] \arrow[ur, Rightarrow, "m"'] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[r, "\sigma^0"] & & \mathsf{F} \wr \Omega \arrow[r] & \mathcal{V}
\end{tikzcd}
\tag{1.26}$$

HERE

$NX_0 \xrightarrow{N\epsilon} NNX_{-1} \xrightarrow{\mu} NX_{-1}$

$$\begin{tikzcd}
\Omega^1 \arrow[r] \arrow[d, equal] & \mathsf{F} \wr \Omega^0 \arrow[r] \arrow[d, equal] & \mathsf{F} \wr \Omega \arrow[r, "\delta^1"] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[r] \arrow[d, equal] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\otimes"] \arrow[d, equal] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes"] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^1 \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0 \arrow[r] & \mathsf{F}^{\wr 2} \wr \Sigma \arrow[r] \arrow[d, "\sigma^0"] \arrow[ur, Rightarrow, "\epsilon"'] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\otimes"] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^0 \arrow[rr] \arrow[ur, Leftrightarrow, "\pi_0"'] & & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\otimes"'] \arrow[urr, Leftrightarrow, "\alpha"'] & & \mathcal{V}^{op}
\end{tikzcd}
\tag{1.27}$$

HERE

$$\begin{tikzcd}
\Omega^0 \arrow[r] \arrow[d, equal] & \Omega \arrow[r] & \mathsf{F} \wr \Omega \arrow[d, equal] \\
\Omega^0 \arrow[r] \arrow[urr, Rightarrow, "\epsilon_0"'] & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^1 \arrow[r] \arrow[d, equal] & \Omega \arrow[r] & \mathsf{F} \wr \Omega \arrow[d, equal] \\
\Omega^1 \arrow[r] \arrow[urr, Rightarrow, "\epsilon_1"'] & \mathsf{F} \wr \Omega^0 \arrow[r] & \mathsf{F} \wr \Omega
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^1 \arrow[r] \arrow[d, "d_0"'] & \Omega \arrow[r, equal] & \mathsf{F} \wr \Omega \arrow[d, equal] \\
\Omega^0 \arrow[r] \arrow[urr, Rightarrow, "\beta"'] & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega
\end{tikzcd}
\tag{1.28}$$

HERE
$X_1 \xrightarrow{\epsilon} NX_0 \xrightarrow{N\epsilon} NNX_{-1} \xrightarrow{\mu} NX_{-1}$

$$
\begin{tikzcd}
\Omega^1 \arrow[rr] \arrow[d, equal] & & \mathsf{F} \wr \Omega \arrow[r, "\delta^1"] \arrow[d] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[d, equal] \\
\Omega^1 \arrow[r] \arrow[d, equal] & \mathsf{F} \wr \Omega^0 \arrow[r] \arrow[d, equal] \arrow[ur, Rightarrow, "\epsilon"'] & \mathsf{F} \wr \Omega \arrow[r, "\delta^1"] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[d, equal] \\
\Omega^1 \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0 \arrow[r] & \mathsf{F}^{\wr 2} \wr \Sigma \arrow[r] \arrow[d, "\sigma^0"] \arrow[ur, Rightarrow, "\epsilon"'] \arrow[dll, Rightarrow, "\pi_0"] & \mathsf{F}^{\wr 2} \wr \Omega \arrow[d, "\sigma^0"] \\
\Omega^0 \arrow[rr] & & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega
\end{tikzcd}
\tag{1.29}
$$

versus $X_1 \xrightarrow{d_0} X_0 \xrightarrow{\epsilon} NX_{-1}$

$$
\begin{tikzcd}
\Omega^1 \arrow[rr] \arrow[d, "d_0"'] & & \mathsf{F} \wr \Omega \arrow[d, equal] \\
\Omega^0 \arrow[r] \arrow[d, equal] & \Omega \arrow[r] & \mathsf{F} \wr \Omega \arrow[d, equal] \\
\Omega^0 \arrow[r] \arrow[urr, Rightarrow, "\epsilon_0"'] & \mathsf{F} \wr \Sigma \arrow[r] & \mathsf{F} \wr \Omega
\end{tikzcd}
\tag{1.30}
$$

Glossary of Notation

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