

Genuine equivariant operads

Peter Bonventre, Luís A. Pereira

April 16, 2017

Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞ -operads of Blumberg and Hill.

Contents

1	Introduction	1
2	Planar and tall maps	1
2.1	Planar structures	1
2.2	Outer faces and tall maps	5
2.3	Substitution	7
3	The genuine equivariant operad monad	9
3.1	Wreath product over finite sets	9
3.2	Equivariant leaf-root and vertex functors	10
3.3	Planar strings	12
3.4	A monad on spans	14
3.5	The free genuine operad monad	17
4	Free extensions	20
4.1	Extensions over general monads	21
4.2	Mixed strings	22

1 Introduction

No content yet.

2 Planar and tall maps

2.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

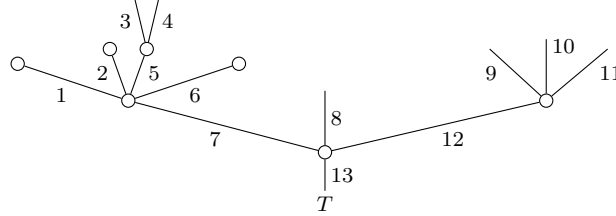
Our preferred model for trees will be that of broad posets first introduced by Weiss in [3] and further worked out by the second author in [2]. We now define planar structures in this context.

PLANARIZE DEF

Definition 2.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendant partial order \leq_d to a total order \leq_p such that:

- *Planar*: if $e \leq_p f$ and $e \not\leq_d f$ then $g \leq_d f$ implies $e \leq_p g$.

Example 2.2. An example of a planar structure on a tree T follows, with \leq_r encoded by the number labels.



(2.3)

PLANAREX EQ

Intuitively, given a planar depiction of a tree T , $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 2.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

Notation 2.4. Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path* of e .

We will repeatedly use the following, which is a consequence of [Pe16b, Cor. 5.26].

Lemma 2.5. If $e \leq_d f$, $e \leq_d f'$, then f, f' are \leq_d -comparable.

Proposition 2.6. Let $T \in \Omega$ be a tree. Then

- for any $e \in T$ the finite poset $I(e)$ is totally ordered;
- the poset (T, \leq_d) has all joins, denoted \vee . In fact, $\vee_i e_i = \min(\cap_i I(e_i))$.

Proof. (a) is immediate from Lemma 2.5. To prove (b) we note that $\min(\cap_i I(e_i))$ exists by (a), and that this is clearly the join $\vee e_i$. \square

Notation 2.7. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^\dagger \in T$ the predecessor of b in $I(e)$.

Proposition 2.8. Suppose e, f are \leq_d -incomparable edges of T and write $b = e \vee f$. Then

- $e <_d b$, $f <_d b$ and $b_e^\dagger \neq b_f^\dagger$;
- $b_e^\dagger, b_f^\dagger \in b^\dagger$. In fact $\{b_e^\dagger\} = I(e) \cap b^\dagger$, $\{b_f^\dagger\} = I(f) \cap b^\dagger$;
- if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^\dagger = b_e^\dagger$, $b_{f'}^\dagger = b_f^\dagger$.

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^\dagger = b_f^\dagger$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^\dagger$, where uniqueness follows from Lemma 2.5. Choosing $a = e$ implies $I(e) \cap b^\dagger = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b .

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. \square

TERNARYJOIN PROP

Proposition 2.9. Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c_{e_i}^\dagger \neq c_{e_j}^\dagger$.
Therefore, all ternary joins in (T, \leq_d) are binary, i.e.

$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \quad (2.10)$$

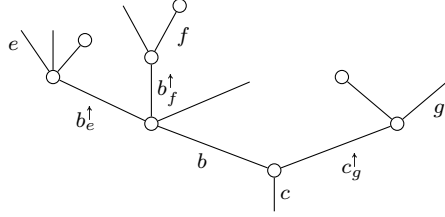
TERNJOIN EQ

for some $1 \leq i < j \leq 3$, and (2.10) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c_{e_i}^\dagger \neq c_{e_j}^\dagger$, then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 2.8(a).

The “therefore” part follows by noting that $c_{e_1}^\dagger, c_{e_2}^\dagger, c_{e_3}^\dagger$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. \square

Example 2.11. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c_e^\dagger = c_f^\dagger = b$.



Notation 2.12. Given a set S of size n we write $\text{Ord}(S) \simeq \text{Iso}(S, \{1, \dots, n\})$. We will usually abuse notation by regarding its objects as pairs (S, \leq) where \leq is a total order in S .

Proposition 2.13. Let $T \in \Omega$ be a tree. There is a bijection

$$\begin{aligned} \{\text{planar structures } (T, \leq_p)\} &\longrightarrow \prod_{(a^\dagger \leq a) \in V(T)} \text{Ord}(a^\dagger) \\ \leq_p &\longmapsto (\leq_p \mid a^\dagger) \end{aligned} \quad (2.14)$$

PLANAR EQ

Proof. We will keep the setup of Proposition 2.8 throughout: e, f are \leq_d -incomparable edges and we write $b = e \vee f$.

We first show that (2.14) is injective, i.e. that the restrictions $\leq_p \mid a^\dagger$ determine if $e <_p f$ holds or not. If $b_e^\dagger <_p b_f^\dagger$, the relations $e \leq_d b_e^\dagger <_p b_f^\dagger \leq_d f$ and Definition 2.1 imply it must be $e <_p f$. Dually, if $b_f^\dagger <_p b_e^\dagger$ then $f <_p e$. Thus $b_e^\dagger <_p b_f^\dagger \Leftrightarrow e <_p f$ and hence (2.14) is indeed injective.

To check that (2.14) is surjective, it suffices (recall that e, f are assumed \leq_d -incomparable) to check that defining $e \leq_p f$ to hold iff $b_e^\dagger < b_f^\dagger$ holds in b^\dagger yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of \leq_p in the case $e' <_p e <_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 2.8(c). Transitivity of \leq_p in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are \leq_d -incomparable, in which case one can apply 2.8(c) with the roles of f, f' reversed.

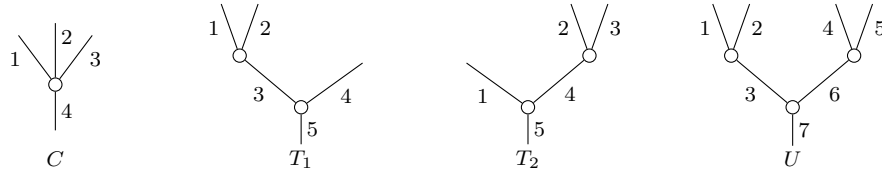
It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with e, f, g pairwise incomparable. We write $c = e \vee f \vee g$. By the “therefore” part of Proposition 2.9, either (i) $e \vee f <_d c$, in which case Proposition 2.9 implies $c_e^\dagger = c_f^\dagger$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c_e^\dagger < c_f^\dagger < c_g^\dagger$ in c^\dagger so that $c_e^\dagger \neq c_g^\dagger$ and by Proposition 2.9 it is also $e \vee g = c$ and transitivity follows. \square

Remark 2.15. Definition 2.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 2.13 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [2, Def. 7.44]), which adds a single edge η to F , serving as the (unique) root of $F \star \eta$.

When discussing the substitution procedure in §2.3 we will find it convenient to work with a model for the category Ω that possesses exactly one representative of each possible planar structure on each tree or, more precisely, such that the only isomorphisms preserving the planar structures are the identities. On the other hand, using such a model for Ω throughout would, among other issues, make the discussion of faces in §2.2 rather awkward. We now outline our conventions to address such issues.

Let Ω^p , the category of *planarized trees*, denote the category with objects pairs $T_{\leq p} = (T, \leq_p)$ of trees together with a planar structure and morphisms the *underlying* maps of trees (so that the planar structures are ignored). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \dots, n\}$ and for which \leq_p coincides with the canonical order.

Example 2.16. Some examples of standard models, i.e. objects of Ω^s , follow (further, (2.3) can also be interpreted as such an example).



(2.17)

PLANAROMEGAEX1 EQ

Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq p} \in \Omega^p$ there is an obvious standard model $T_{\leq p}^s \in \Omega^s$ given by replacing each edge by its order following \leq_p . Indeed, this defines a retraction $(-)^s: \Omega^p \rightarrow \Omega^s$ and a natural transformation $\sigma: id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is clearly unique).

Convention 2.18. From now on, we will write simply Ω , Ω_G to denote the categories Ω^s , Ω_G^s of standard models (where planar structures are defined in the underlying forest as in Remark 2.15). Similarly \mathbf{O}_G will denote the model \mathbf{O}_G^s for the orbital category whose objects are the orbital G -sets whose underlying set is one of the sets $\underline{n} = \{1, 2, \dots, n\}$.

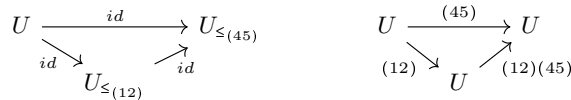
Therefore, whenever one of our constructions produces an object/diagram in Ω^p , Ω_G^p , \mathbf{O}_G^p (of trees, G -trees, orbital G -sets with a planarization/total order) we will hence implicitly reinterpret it by using the standardization functor $(-)^s$.

Example 2.19. To illustrate our convention, we consider the trees in Example 2.16.

One has subfaces $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures. Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

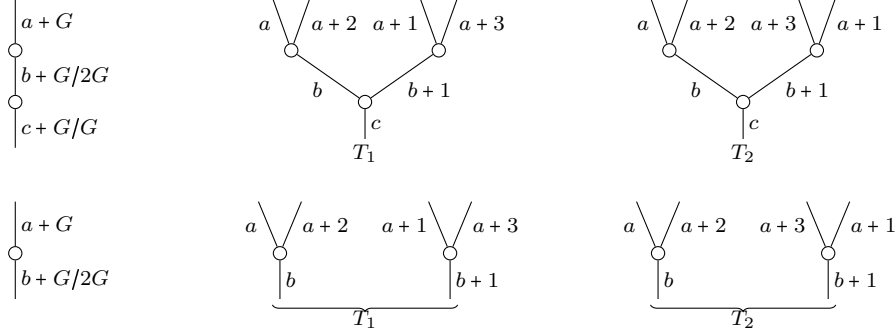


Similarly, let $\leq_{(12)}$ and $\leq_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs 1, 2 and 4, 5, so that one has objects $U_{\leq_{(12)}}$, $U_{\leq_{(45)}}$ in Ω^p . Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.



Example 2.20. An additional reason to leave the use of $(-)^s$ implicit is that when depicting G -trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, when $G = \mathbb{Z}/4$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting of the two trees $T_1, T_2 \in \Omega_G$ on the right.



Definition 2.21. A morphism $S \xrightarrow{\varphi} T$ in Ω that is compatible with the planar structures \leq_p is called a *planar map*.

More generally, a morphism $F \rightarrow G$ in the categories Φ, Φ^G, Ω^G of forests, G -forests, G -trees is called a *planar map* if it is an independent map (cf. [2, Def. 5.28]) compatible with the planar structures \leq_p .

Remark 2.22. The need for the independence condition is justified by [2, Lemma 5.33] and its converse, since non independent maps do not reflect \leq_d inequalities.

We note that in the Ω_G case a map φ is independent iff φ does not factor through a non trivial quotient iff φ is injective on each edge orbit.

Proposition 2.23. Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization

$$F \xrightarrow{\sim} \bar{F} \rightarrow G$$

such that $F \xrightarrow{\sim} \bar{F}$ is an isomorphism and $\bar{F} \rightarrow G$ is planar.

Proof. We need to show that there is a unique planar structure $\leq_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of G ensures that for any vertex $e^\dagger \leq e$ of F the edges in $\varphi(e^\dagger)$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(e^\dagger)$ are distinct. The result now follows from (the forest version of) Proposition 2.13: one simply orders each set e^\dagger and \bar{r}_F according to its image.

not quite complete... maybe that \leq_p is the closure of \leq_d and the vertex relations under transitivity and the planar condition \square

Remark 2.24. Proposition 2.23 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As an example, in the notation of (2.17), consider the map $C \rightarrow T_1$ defined by $1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 2, 4 \mapsto 5$.

Remark 2.25. Given any tree $T \in \Omega$ there is a unique corolla $\text{lr}(T) \in \Sigma$ and planar tall map $\text{lr}(T) \rightarrow T$. Explicitly, the number of leaves of $\text{lr}(T)$ matches that of T , together with the inherited order.

2.2 Outer faces and tall maps

In preparation for our discussion of the substitution operation in §2.3, we now recall some basic notions and results concerning outer subtrees and tree grafting, as in [2, §5].

Definition 2.26. Let $T \in \Omega$ be a tree and $e_1 \dots e_n = \underline{e} \leq e$ a broad relation in T .

We define the *planar outer face* $T_{\underline{e} \leq e}$ to be the subtree with underlying set those edges $f \in T$ such that

$$f \leq_d e, \quad \forall i e_i \not\leq_d f, \quad (2.27)$$

generating broad relations the relations $f^\dagger \leq f$ for f satisfying (2.27) and $\forall i f \neq e_i$, and planar structure pulled back from T .

Remark 2.28. If one forgoes the requirement that $T_{\underline{e} \leq e}$ be equipped with the pullback planar structure, the inclusion $T_{\underline{e} \leq e} \rightarrow T$ is usually called simply an *outer face*.

We now recap some basic results.

Proposition 2.29. *Let $T \in \Omega$ be a tree.*

- (a) $T_{\underline{e} \leq e}$ is a tree with root e and edge tuple \underline{e} ;
- (b) there is a bijection

$$\{\text{planar outer faces of } T\} \leftrightarrow \{\text{broad relations of } T\};$$

- (c) if $R \rightarrow S$ and $S \rightarrow T$ are outer face maps then so is $R \rightarrow T$;
- (d) any pair of broad relations $\underline{g} \leq v$, $\underline{f}v \leq e$ induces a grafting pushout diagram

$$\begin{array}{ccc} \eta & \xrightarrow{v} & T_{\underline{g} \leq v} \\ v \downarrow & & \downarrow \\ T_{\underline{f}v \leq e} & \longrightarrow & T_{\underline{f}g \leq v} \end{array} \quad (2.30) \quad \boxed{\text{GRATPUSH EQ}}$$

Proof. We first show (a). That $T_{\underline{e} \leq e}$ is indeed a tree is the content of [Pe16b, Prop. 5.20]: more precisely, $T_{\underline{e} \leq e} = (T^{\leq e})_{< \underline{e}}$ in the notation therein. That the root of $T_{\underline{e} \leq e}$ is e is clear and that the root tuple is \underline{e} follows from [Pe16b, Remark 5.23].

(b) follows from (a), which shows that $\underline{e} \leq e$ can be recovered from $T_{\underline{e} \leq e}$.

(c) follows from the definition of outer face together with [Pe16b, Lemma 5.33], which states that the \leq_d relations on S, T coincide.

Since by (c) both $T_{\underline{g} \leq v}$ and $T_{\underline{f}v \leq e}$ are outer faces of $T_{\underline{f}g \leq v}$, (d) is a restatement of [Pe16b, Prop. 5.15]. \square

Definition 2.31. A map $S \xrightarrow{\varphi} T$ in Ω is called a *tall map* if

$$\varphi(l_S) = l_T, \quad \varphi(r_S) = r_T,$$

where $l_{(-)}$ denotes the leaf tuple and $r_{(-)}$ the root.

The following is a restatement of [Pe16b, Cor. 5.24]

Proposition 2.32. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphism,*

$$S \xrightarrow{\varphi^t} U \xrightarrow{\varphi^u} T$$

as a tall map followed by an outer face (in fact, $U = T_{\varphi(l_S) \leq r_S}$).

We recall that a face $F \rightarrow T$ is called inner if it is obtained by iteratively removing inner edges, i.e. edges other than the root or the leaves. In particular, it follows that a face is inner iff it is tall. The usual face-degeneracy decomposition thus combines with Corollary 2.32 to give the following.

Corollary 2.33. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^u} T \quad (2.34) \quad \boxed{\text{TRIPLEFACT EQ}}$$

as a degeneracy followed by an inner face followed by an outer face.

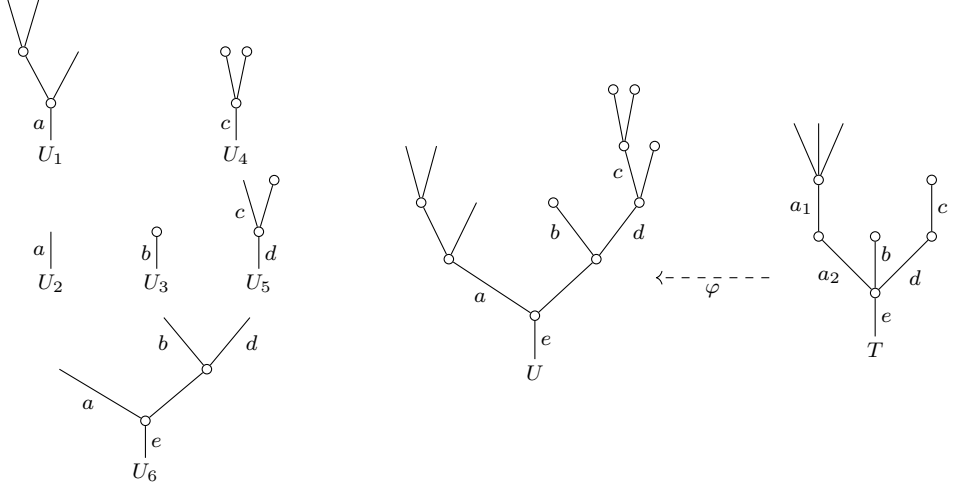
Proof. The factorization (2.34) can be built by first performing the degeneracy-face decomposition and then performing the tall-outer decomposition on the face map. \square

2.3 Substitution

One of the key ideas needed to describe operads is that of substitution of tree nodes, a process that we will prefer to repackage in terms of maps of trees. We start by discussing an example, focusing on the related notion of iterated graftings of trees (as described in (2.30)).

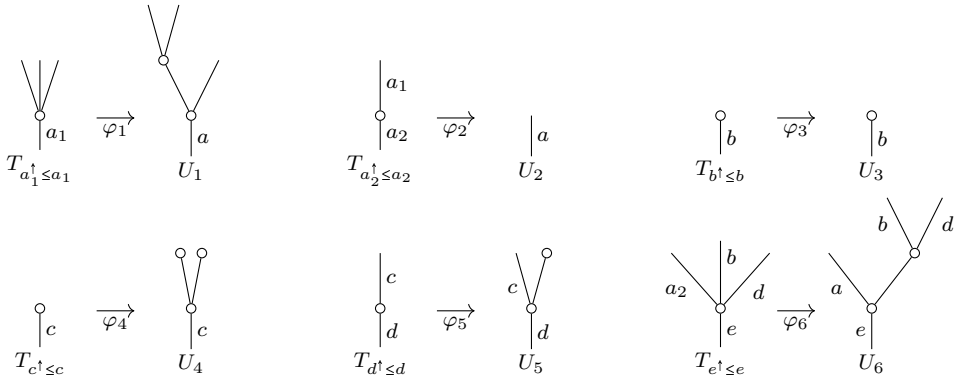
Example 2.35. The trees U_1, U_2, \dots, U_6 on the left below can be grafted into the tree U in the middle. More precisely (among other possible grafting orders), one has

$$U = (((((U_6 \sqcup_a U_2)) \sqcup_a U_1) \sqcup_b U_3) \sqcup_d U_5) \sqcup_c U_4) \quad (2.36) \quad \text{UFORMULA EQ}$$



We now consider the tree T , which is built by converting each U_i into the corolla $\text{Ir}(U_i)$ (cf. Remark 2.25), and then performing the same grafting operations as in (2.36). T can then be regarded as encoding the combinatorics of the iterated grafting in (2.36), with alternative ways to reorder operations in (2.36) in bijection with ways to assemble T out of its nodes.

One can now therefore think of the iterated grafting (2.36) as being instead encoded by the tree T together with the (unique) planar tall maps φ_i below.



From this perspective, U can now be thought as obtained from T by substituting each of its nodes with the corresponding U_i . Moreover, the φ_i assemble to a planar tall map $\varphi: T \rightarrow U$ (such that $a_i \mapsto a, b \mapsto b, \dots, e \mapsto e$), which likewise encodes the same information.

Our perspective will then be that data for substitution of tree nodes such as in (2.38) can equivalently be repackaged using planar tall maps.

UBSTITUTIONDATUM

Definition 2.39. Let $T \in \Omega$ be a tree.

A T -substitution datum is a tuple $\{U_{e^\dagger \leq e}\}_{(e^\dagger \leq e) \in V(T)}$ such that $\text{lr}(U_{e^\dagger \leq e}) = T_{e^\dagger \leq e}$.

Further, a map of T -substitution data $\{U_{e^\dagger \leq e}\} \rightarrow \{V_{e^\dagger \leq e}\}$ is a tuple of planar tall maps $\{U_{e^\dagger \leq e} \rightarrow V_{e^\dagger \leq e}\}$.

Definition 2.40. Let $T \in \Omega$.

The *Segal core poset* $\text{Sc}(T)$ is the poset with objects the edge subtrees η_e and vertex subtrees $T_{e^\dagger \leq e}$. The order relation is given by inclusion.

Remark 2.41. Note that the only maps in $\text{Sc}(T)$ are inclusions of the form $\eta_a \subset T_{e^\dagger \leq e}$. In particular, there are no pairs of composable non-identity relations in $\text{Sc}(T)$.

Given a T -substitution datum $\{U_{\{e^\dagger \leq e\}}\}$ we abuse notation by writing

$$U_{(-)} : \text{Sc}(T) \rightarrow \Omega$$

for the functor $\eta_a \mapsto \eta$, $T_{e^\dagger \leq e} \mapsto U_{e^\dagger \leq e}$ and sending the inclusions $\eta_a \subset T_{e^\dagger \leq e}$ to the composites

$$\eta \xrightarrow{a} T_{e^\dagger \leq e} = \text{lr}(U_{e^\dagger \leq e}) \rightarrow U_{e^\dagger \leq e}.$$

TAUNDERPLAN PROP

Proposition 2.42. Let $T \in \Omega$ be a tree. There is an isomorphism of categories

$$\begin{aligned} \text{Sub}(T) &\xrightleftharpoons{\quad} \Omega_{T|}^{\text{pt}} \\ \{U_{e^\dagger \leq e}\} &\longmapsto (T \rightarrow \text{colim}_{\text{Sc}(T)} U_{(-)}) \\ \{U_{\varphi(e^\dagger) \leq \varphi(e)}\} &\longleftarrow (T \xrightarrow{\varphi} U) \end{aligned} \tag{2.43}$$

SUBDATAUNDERPLAN EQ

Where $\text{Sub}(T)$ denotes the category of T -substitution data and $\Omega_{T|}^{\text{pt}}$ the category of planar tall maps under T .

Proof. We first claim that (i) the $\text{colim}_{\text{Sc}(T)} U_{(-)}$ indeed exists; (ii) for the canonical datum $\{T_{e^\dagger \leq e}\}$, it is $T = \text{colim}_{\text{Sc}(T)} T_{(-)}$; (iii) the induced map $T \rightarrow \text{colim}_{\text{Sc}(T)} U_{(-)}$ is planar tall.

The argument is by induction on the number of vertices of T , with the base cases of T with 0 or 1 vertices being immediate, since then T is the terminal object of $\text{Sc}(T)$. Otherwise, one can choose a non trivial grafting decomposition so as to write $T = R \sqcup_e S$, resulting in identifications $\text{Sc}(R) \subset \text{Sc}(T)$, $\text{Sc}(S) \subset \text{Sc}(T)$ so that $\text{Sc}(R) \cup \text{Sc}(S) = \text{Sc}(T)$ and $\text{Sc}(R) \cap \text{Sc}(S) = \{\eta_e\}$. The existence of $\text{colim}_{\text{Sc}(T)} U_{(-)}$ is thus equivalent to the existence of the pushout below.

$$\begin{array}{ccc} \eta & \longrightarrow & \text{colim}_{\text{Sc}(R)} U_{(-)} \\ \downarrow & & \downarrow \\ \text{colim}_{\text{Sc}(S)} U_{(-)} & \dashrightarrow & \text{colim}_{\text{Sc}(T)} U_{(-)} \end{array} \tag{2.44}$$

ASSEMBLYGRAFT EQ

By induction, the top right and bottom left colimits exist for any $U_{(-)}$, equal R and S in the case $U_{(-)} = T_{(-)}$, and the maps $R \rightarrow \text{colim}_{\text{Sc}(R)} U_{(-)}$, $S \rightarrow \text{colim}_{\text{Sc}(S)} U_{(-)}$ are planar tall. But it now follows that (2.44) is a grafting pushout diagram, so that the pushout indeed exists. The conditions that $T = \text{colim}_{\text{Sc}(T)} T_{(-)}$ and $T \rightarrow \text{colim}_{\text{Sc}(T)} U_{(-)}$ is planar tall follow.

The fact that the two functors in (2.43) are inverse to each other is clear by the same inductive argument. \square

VERTEXDECOMP REM

Remark 2.45. It follows from the previous proof that, writing $U = \text{colim}_{\text{Sc}(T)} U_{(-)}$, one has

$$V(U) = \coprod_{(e^\dagger \leq e) \in V(T)} V(U_{e^\dagger \leq e}). \tag{2.46}$$

VERTEXDECOMP EQ

Alternatively, (2.46) can be regarded as a map $f^* : V(U) \rightarrow V(T)$ induced by the planar tall map $f : T \rightarrow U$. Explicitly, $f^*(U_{u^\dagger \leq u})$ is the unique $T_{t^\dagger \leq t}$ such that $U_{u^\dagger \leq u} \subset U_{t^\dagger \leq t}$. We note that f^* is indeed contravariant in the tall planar map f .

3 The genuine equivariant operad monad

We now turn to the task of building the monad encoding genuine equivariant operads.

3.1 Wreath product over finite sets

In what follows we will let \mathbf{F} denote the usual skeleton of the category of finite sets and all set maps. Explicitly, its objects are the finite sets $\{1, 2, \dots, n\}$ for $n \geq 0$. However, much as in the discussion in Convention 2.18 we will often find it more convenient to regard the elements of \mathbf{F} as equivalence classes of finite sets equipped with total orders.

Definition 3.1. For a category \mathcal{C} , we let $\mathbf{F} \wr \mathcal{C}$ denote the opposite of the Grothendieck construction for the functor

$$\begin{aligned} \mathbf{F}^{op} &\longrightarrow \mathbf{Cat} \\ I &\longmapsto \mathcal{C}^I \end{aligned}$$

Explicitly, the objects of $\mathbf{F} \wr \mathcal{C}$ are tuples $(c_i)_{i \in I}$ and a map $(c_i)_{i \in I} \rightarrow (d_j)_{j \in J}$ consists of a pair

$$(\phi: I \rightarrow J, (f_i: c_i \rightarrow d_{\phi(i)})_{i \in I}),$$

henceforth abbreviated as $(\phi, (f_i))$.

The following is immediate.

Proposition 3.2. Suppose \mathcal{C} has all finite coproducts. One then has a functor as on the left below. Dually, if \mathcal{C} has all finite products, one has a functor as on the right below.

$$\begin{aligned} \mathbf{F} \wr \mathcal{C} &\xrightarrow{\Pi} \mathcal{C} & (\mathbf{F} \wr \mathcal{C}^{op})^{op} &\xrightarrow{\Pi} \mathcal{C} \\ (c_i)_{i \in I} &\longmapsto \coprod_{i \in I} c_i & (c_i)_{i \in I} &\longmapsto \prod_{i \in I} c_i \end{aligned}$$

Lemma 3.3. Suppose that \mathcal{E} is a bicomplete category such that coproducts commute with limits in each variable. If the leftmost diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{E} \\ k \downarrow & \nearrow \eta & \uparrow G \\ \mathcal{D} & & \end{array} \quad \begin{array}{ccccc} \mathbf{F} \wr \mathcal{C} & \xrightarrow{\mathbf{F} \wr F} & \mathbf{F} \wr \mathcal{E} & \xrightarrow{\Pi} & \mathcal{E} \\ \mathbf{F} \wr k \downarrow & \nearrow \mathbf{F} \wr \eta & \nearrow \mathbf{F} \wr G & & \\ \mathbf{F} \wr \mathcal{D} & \xrightarrow{\Pi \circ \mathbf{F} \wr G} & & & \end{array} \quad (3.4) \quad \boxed{\text{WRRAN EQ}}$$

is a right Kan extension diagram then so is the composite of the rightmost diagram.

Dually, if in \mathcal{E} products commute with colimits in each variable, and the leftmost diagram

$$\begin{array}{ccc} \mathcal{C}^{op} & \xrightarrow{F} & \mathcal{E} \\ k \downarrow & \nearrow \epsilon & \uparrow G \\ \mathcal{D}^{op} & & \end{array} \quad \begin{array}{ccccc} (\mathbf{F} \wr \mathcal{C})^{op} & \xrightarrow{(\mathbf{F} \wr F)^{op}} & (\mathbf{F} \wr \mathcal{E})^{op} & \xrightarrow{\Pi} & \mathcal{E} \\ (\mathbf{F} \wr k)^{op} \downarrow & \nearrow & \nearrow (\mathbf{F} \wr G)^{op} & & \\ (\mathbf{F} \wr \mathcal{D})^{op} & \xrightarrow{\Pi \circ (\mathbf{F} \wr G)^{op}} & & & \end{array} \quad (3.5) \quad \boxed{\text{WRLAN EQ}}$$

is a left Kan extension diagram then so is the composite of the rightmost diagram.

Proof. Unpacking definitions using the pointwise formula for Kan extensions ($\boxed{\text{McL X.3.1}}$), the claim concerning (3.4) amounts to showing that for each $(d_i) \in \mathbf{F} \wr \mathcal{D}$ one has natural isomorphisms

$$\lim_{((d_i) \rightarrow (kc_j)) \in ((d_i) \downarrow \mathbf{F} \wr \mathcal{C})} \left(\coprod_j F(c_j) \right) \simeq \coprod_i \lim_{(d_i \rightarrow kc_i) \in d_i \downarrow \mathcal{C}} (F(c_i)). \quad (3.6) \quad \boxed{\text{POINTKAN EQ}}$$

Noting that the canonical factorizations of each $(\varphi, (f_i)): (d_i)_{i \in I} \rightarrow (kc_j)_{j \in J}$ as

$$(d_i)_{i \in I} \rightarrow (c_{\phi(i)})_{i \in I} \rightarrow (kc_j)_{j \in J}$$

exhibit $\prod_i (d_i \downarrow \mathcal{C})$ as a coreflexive subcategory of $(d_i) \downarrow \mathbf{F} \wr \mathcal{C}$, we see that it is an initial subcategory. Therefore

$$\lim_{((d_i) \rightarrow (kc_j)) \in ((d_i) \downarrow \mathbf{F} \wr \mathcal{C})} \left(\prod_j F(c_j) \right) \simeq \lim_{((d_i) \rightarrow (kc_i)) \in \prod_i (d_i \downarrow \mathcal{D})} \left(\prod_i F(c_i) \right)$$

and hence ^{POINTKAN EQ} (3.6) now follows from the assumption that coproducts commute with limits in each variable. \square

Notation 3.7. Using the coproduct functor $\mathbf{F}^{\wr} = \mathbf{F}^{\wr\{0,1\}} = \mathbf{F} \wr \mathbf{F} \xrightarrow{\mathbf{u}} \mathbf{F}$ (where $\prod_{i \in I} J_i$ is ordered lexicographically) and the simpleton $\{1\} \in \mathbf{F}$ one can regard the collection of categories $\mathbf{F}^{\wr\{0,\dots,n\}} \wr \mathcal{C} = \mathbf{F}^{\wr n} \wr \mathcal{C}$ as a coaugmented cosimplicial object in \mathbf{Cat} . As such, we will denote by

$$\delta^i: \mathbf{F}^{\wr n-1} \wr \mathcal{C} \rightarrow \mathbf{F}^{\wr n} \wr \mathcal{C}, \quad 0 \leq i \leq n$$

the cofaces obtained by inserting simpletons $\{1\} \in \mathbf{F}$ and by

$$\sigma^i: \mathbf{F}^{\wr n+1} \wr \mathcal{C} \rightarrow \mathbf{F}^{\wr n} \wr \mathcal{C}, \quad 0 \leq i \leq n$$

the codegeneracies obtained by applying the coproduct $\mathbf{F}^{\wr} \xrightarrow{\mathbf{u}} \mathbf{F}$ to adjacent \mathbf{F} coordinates.

3.2 Equivariant leaf-root and vertex functors

Definition 3.8. A morphism $T \xrightarrow{\varphi} S$ in Ω_G is called a *quotient* if the underlying morphism of forests

$$\coprod_{[g] \in G/H} T_{[g]} \rightarrow \coprod_{[h] \in G/K} S_{[h]}$$

maps each tree component (or, equivalently, some tree component) isomorphically onto its image component.

We denote the subcategory of G -trees and quotients by Ω_G^q .

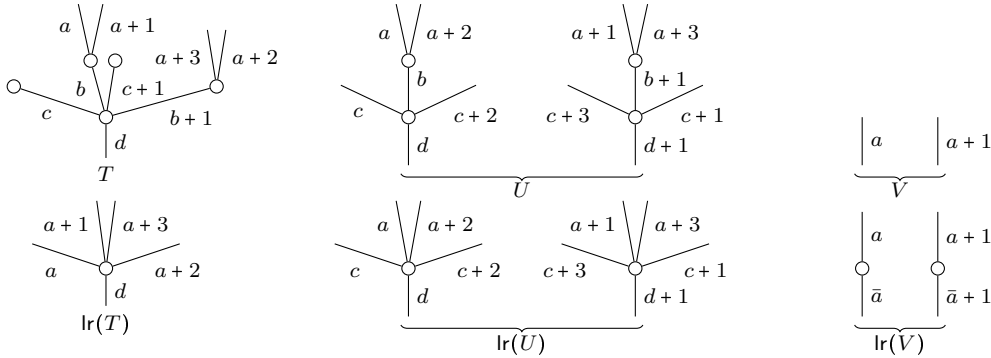
Definition 3.9. The *G -symmetric category*, which we will also call the *category of G -corollas*, is the full subcategory $\Sigma_G \subset \Omega_G^q$ of those G -trees that are corollas, i.e. G -trees such that each edge is either a root or a leaf (but not both).

Definition 3.10. The *leaf-root functor* is the functor $\Omega_G^q \xrightarrow{\text{lr}} \Sigma_G$ defined by

$$\text{lr}(T) = \{\text{leaves of } T\} \sqcup \{\text{roots of } T\}$$

with a broad relation $l_1 \cdots l_n \leq r$ holding in $\text{lr}(T)$ iff its image holds in T and similarly for the planar structure \leq_p .

Remark 3.11. Generalizing Remark ^{UNIQCOR REM} 2.25, $\text{lr}(T)$ can alternatively be characterized as being the *unique G -corolla* which admits an also unique (tree-wise) tall planar map $\text{lr}(T) \rightarrow T$. Moreover, $\text{lr}(T)$ can usually be regarded as the “smallest inner face” of T , obtained by removing all the inner edges, although this characterization fails when $T = G \cdot_H \eta$ is a stick G -tree. Some examples with $G = \mathbb{Z}/4$ follow.



Remark 3.12. One consequence of the fact that planarizations can not be pushed forward along tree maps (cf. Remark 2.24) is that $\mathbb{F}\Omega_G^q \rightarrow \Sigma_G$ is not a categorical fibration. **maybe add to this.**

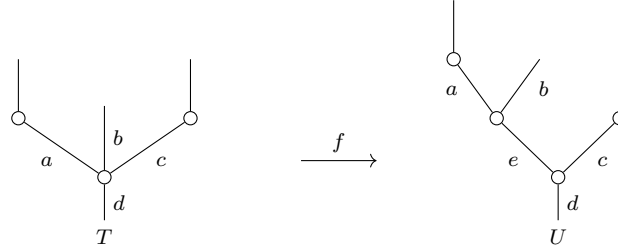
VG DEF

Definition 3.13. Given $T \in \Omega_G$ we define the set $V_G(T)$ of G -vertices of T to be the orbit set $V(T)/G$, i.e. the quotient of the vertex set $V(T)$ by its G -action.

Furthermore, we will regard $V_G(T)$ as an object in \mathbb{F} by equipping it with its lexicographic order: i.e. vertex equivalence classes $[e^\dagger \leq e]$ are ordered according to the planar order \leq_p of the smallest representative ge , $g \in G$.

Remark 3.14. Following Remark 2.45, a planar tall map $f: T \rightarrow U$ of G -trees induces a G -equivariant map $f^*: V(U) \rightarrow V(T)$ and thus also a map of orbits $f^*: V_G(U) \rightarrow V_G(T)$. We note, however, that f^* is not in general compatible with the order on V_G , as is indeed the case even in the non-equivariant case.

A minimal example follows.



In $V(T)$ the vertices are ordered as $a < c < d$ while in $V(U)$ they are ordered as $a < e < c < d$ but the map $f^*: V(U) \rightarrow V(T)$ is given by $a \mapsto a, c \mapsto c, d \mapsto d, e \mapsto d$.

Note that each element of $V_G(T)$ corresponds to an unique edge orbit Ge for e not a leaf. As such, we will represent the corresponding G -vertex by $v_{Ge} = (Ge)^\dagger \leq Ge$ (which we interpret as the concatenation of the relations $f^\dagger \leq f$ for $f \in Ge$) and write

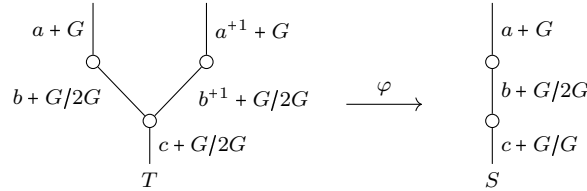
$$T_{v_{Ge}} = T_{(Ge)^\dagger \leq Ge} = \coprod_{f \in Ge} T_{f^\dagger \leq f}.$$

We note that $T_{v_{Ge}}$ is always a G -corolla. Indeed, noting that a quotient map $\varphi: T \rightarrow S$ induces quotient maps $T_{v_{ge}} \rightarrow S_{v_{\varphi(e)}}$ one obtains a functor

$$\begin{aligned} \Omega_G^q &\xrightarrow{V_G} \mathbb{F} \wr \Sigma_G \\ T &\longmapsto (T_{v_{Ge}})_{v_{Ge} \in V_G(T)}. \end{aligned} \tag{3.15}$$

VFUNCTOR EQ

Remark 3.16. The need to introduce the $\mathbb{F} \wr \mathcal{C}$ categories comes from the fact that general quotient maps do not preserve the number of G -vertices. For a simple example, let $G = \mathbb{Z}/4$ and consider the quotient map



sending edges labeled a, b, c to the edges with the same name and the edges a^{+1}, b^{+1} to the edges $a+1, b+1$. We note that T has three G -vertices v_{Ge}, v_{Gb}, v_{Gb+1} while S has only two G -vertices v_{Gc} and v_{Gb} . $V(\phi)$ then maps the two corollas $T_{v_{Gb}}$ and $T_{v_{Gb+1}}$ isomorphically onto $T_{S_{Gb}}$ and the corolla $T_{v_{Gc}}$ non-isomorphically onto $S_{v_{Gc}}$.

Definition [SUBSTITUTIONDATUM 2.39](#) now immediately generalizes.

Definition 3.17. Let $T \in \Omega_G$ be a G -tree.

A T -substitution datum is a tuple $\{U_{v_{Ge}}\}_{v_{Ge} \in V_G(T)}$ such that $\text{lr}(U_{v_{Ge}}) = T_{v_{Ge}}$.

Further, a map of T -substitution data $\{U_{v_{Ge}}\} \rightarrow \{V_{v_{Ge}}\}$ is a tuple of planar tall maps $\{U_{v_{Ge}} \rightarrow V_{v_{Ge}}\}$.

Remark 3.18. To establish the equivariant analogue of Proposition [SUBDATAUNDERPLAN PROP 2.42](#) we will prefer to repackage equivariant substitution data in terms of non-equivariant terms.

Noting that there are decompositions $U_{v_{Ge}} = \coprod_{ge \in Ge} U_{ge \uparrow \leq ge}$ and letting $G \ltimes V(T)$ denote the Grothendieck construction for the action of G on the non-equivariant vertices $V(T)$ (often called the action groupoid), it is immediate that an equivariant T -substitution datum is the same as a functor $G \ltimes V(T) \rightarrow \Omega$ whose restriction to $V(T) \subset G \ltimes V(T)$ is a (non-equivariant) substitution datum.

Proposition 3.19. Let $T \in \Omega_G$ be a G -tree. There is an isomorphism of categories

$$\begin{aligned} \text{Sub}(T) &\xrightarrow{\sim} \Omega_{G,T}^{\text{pt}} \\ \{U_{v_{Ge}}\} &\longmapsto (T \rightarrow \text{colim}_{\text{Sc}(T)} U_{(-)}) \end{aligned} \tag{3.20}$$

Proof. This is a minor adaptation of the non-equivariant analogue Proposition [SUBDATAUNDERPLAN PROP 3.19](#). Since $\text{Sc}(T)$ inherits a G -action, one can form the Grothendieck construction $G \ltimes \text{Sc}(T)$ and by Remark [SUBDATUMCONV REM 3.18](#) equivariant substitution data $\{U_{v_{Ge}}\}$ therefore induce functors $U_{(-)}: G \ltimes \text{Sc}(T) \rightarrow \Omega$. It is then immediate that $\text{colim}_{\text{Sc}(T)} U_{(-)}$ inherits a G -action, provided it exists. The key observation is then that, since $\text{Sc}(T)$ is now a disconnected poset, this colimit is to be interpreted as taken in the category Φ of forests rather than in Ω . \square

Remark 3.21. We will need to know that each of the maps

$$U_{v_{Ge}} \rightarrow U = \text{colim}_{\text{Sc}(T)} U_{(-)}$$

induced by the previous proof is a planar map of G -trees. This requires two observations: (i) the restrictions to each of the constituent non-equivariant trees $U_{ge \uparrow \leq ge}$ is planar by Proposition [SUBDATAUNDERPLAN PROP 3.19](#); (ii) the restriction to the roots of $U_{v_{Ge}}$ is injective and order preserving since it matches the inclusion of the roots of $T_{v_{Ge}}$, and the map $T \rightarrow U$ is a planar map of G -trees.

3.3 Planar strings

The leaf-root and vertex functors will allow us to reinterpret our results concerning substitution.

Definition 3.22. The category $\Omega_{G,n}$ of *substitution n -strings* is the category whose objects are strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} T_n$$

where $T_i \in \Omega_G$ and the f_i are tall planar maps, and arrows are commutative diagrams

$$\begin{array}{ccccccc} T_0 & \xrightarrow{f_1} & T_1 & \xrightarrow{f_2} & \dots & \xrightarrow{f_n} & T_n \\ q_0 \downarrow & & q_1 \downarrow & & & & q_n \downarrow \\ T'_0 & \xrightarrow{f'_1} & T'_1 & \xrightarrow{f'_2} & \dots & \xrightarrow{f'_n} & T'_n \end{array} \tag{3.23}$$

where each q_i is a quotient map.

IMPOPERATORS NOT

Notation 3.24. Since compositions of planar tall arrows are planar tall and identity arrows are planar tall it follows that $\Omega_{G,\bullet}$ forms a simplicial object in \mathbf{Cat} , with faces given by composing and degeneracies by inserting identities.

Noting that $\Omega_{G,0} = \Omega_G^q$ and setting $\Omega_{G,-1} = \Sigma_G$, the leaf-root functor $\Omega_G^q \xrightarrow{\text{lr}} \Sigma_G$ makes $\Omega_{G,\bullet}^q$ into an augmented simplicial object and, furthermore, the maps $s_{-1}: \Omega_{G,n}^q \rightarrow \Omega_{G,n+1}^q$ sending $T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_n$ to $\text{lr}(T_0) \rightarrow T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_n$ equip it with extra degeneracies.

Notation 3.25. We extend the vertex functor to a functor $V_G: \Omega_{G,n+1} \rightarrow \mathbf{F} \wr \Omega_{G,n}$ by

$$V_G(T_0 \rightarrow T_1 \rightarrow \dots \rightarrow T_n) = (T_{1,v_{Ge}} \rightarrow \dots \rightarrow T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)} \quad (3.26)$$

VG DEF

where we abuse notation by writing $T_{i,v_{Ge}}$ for $T_{i,(f_i \circ \dots \circ f_1)(v_{Ge})}$.

The following is a reinterpretation of Proposition 3.19.

SUBDATAUNDERPLANG PROP

SUBSASPULL PROP

Proposition 3.27. *The diagram*

$$\begin{array}{ccc} \Omega_{G,n+1} & \xrightarrow{V_G} & \mathbf{F} \wr \Omega_{G,n} \\ d_{1,\dots,n+1} \downarrow & & \downarrow \text{Fld}_{0,\dots,n} \\ \Omega_{G,0} & \xrightarrow{V_G} & \mathbf{F} \wr \Sigma_G \end{array} \quad (3.28)$$

PTPULL EQ

is a pullback diagram in \mathbf{Cat} .

Proof. An object in the pullback (3.28) over $T \in \Omega_{G,0} = \Omega_G^q$ is precisely the same as a n -string in $\mathbf{Sub}(T)$, and thus by Proposition 3.19 equivalent to a $n+1$ planar tall string starting at T .

The case of arrows is slightly more subtle. A quotient map $\pi: T \rightarrow T'$ induces a G -equivariant poset map $\pi_*: \mathbf{Sc}(T) \rightarrow \mathbf{Sc}(T')$ (or equivalently, a map of Grothendieck constructions $G \times \mathbf{Sc}(T) \rightarrow G \times \mathbf{Sc}(T')$) and diagrams as on the left below (where v_{Ge} ranges over $V_G(T)$ and $e' = \varphi(e)$) induce diagrams (of functors $\mathbf{Sc}(T) \rightarrow \Omega$) as on the right below.

$$\begin{array}{ccc} T_{v_{Ge}} \rightarrow T_{1,v_{Ge}} \rightarrow \dots \rightarrow T_{n,v_{Ge}} & T_{(-)} \longrightarrow T_{1,(-)} \longrightarrow \dots \longrightarrow T_{n,(-)} \\ \downarrow & \downarrow & \downarrow \\ T'_{v_{Ge'}} \rightarrow T'_{1,v_{Ge'}} \rightarrow \dots \rightarrow T'_{n,v_{Ge'}} & T'_{(-)} \circ \pi_* \rightarrow T'_{1,(-)} \circ \pi_* \rightarrow \dots \rightarrow T'_{n,(-)} \circ \pi_* \end{array} \quad (3.29)$$

PTNARROWLOC EQ

Passing to colimits then gives the desired commutative diagram (3.23). Moreover, diagrams of the form (3.23) clearly induce diagrams as in (3.28) and it is straightforward to check that these are inverse processes. \square

DSCOM REM

Remark 3.30. The diagrams (with back and lower slanted faces instances of (3.27))

$$\begin{array}{ccc} \Omega_{G,n+2} & \longrightarrow & \mathbf{F} \wr \Omega_{G,n+1} \\ \downarrow & \searrow d_{i+1} & \downarrow \text{Fld}_i \\ & \Omega_{G,n+1} & \longrightarrow \mathbf{F} \wr \Omega_{G,n} \\ \downarrow & \swarrow & \downarrow \\ \Omega_{G,0} & \longrightarrow & \mathbf{F} \wr \Sigma_G \end{array} \quad \begin{array}{ccc} \Omega_{G,n+1} & \longrightarrow & \mathbf{F} \wr \Omega_{G,n} \\ \downarrow & \searrow s_i & \downarrow \text{Fld}_{s_{i-1}} \\ & \Omega_{G,n+2} & \longrightarrow \mathbf{F} \wr \Omega_{G,n+1} \\ \downarrow & \swarrow & \downarrow \\ \Omega_{G,0} & \longrightarrow & \mathbf{F} \wr \Sigma_G \end{array}$$

commute whenever defined (i.e. $0 \leq i \leq n+1$).

INDVNG NOT

Notation 3.31. We will let

$$V_{G,n}: \Omega_{G,n} \rightarrow \mathbf{F} \wr \Sigma_G$$

be inductively defined by $V_{G,n} = \sigma_0 \circ V_{G,n-1} \circ V_G$.

Remark 3.32. When $n = 2$, $V_{G,2}$ is thus the composite

$$\Omega_{G,2} \xrightarrow{V_G} F \wr \Omega_{G,1} \xrightarrow{V_G} F \wr F \wr \Omega_{G,0} \xrightarrow{V_G} F \wr F \wr F \wr \Sigma_G \xrightarrow{\sigma^0} F \wr F \wr \Sigma_G \xrightarrow{\sigma^0} F \wr \Sigma_G$$

In light of Remarks VERTEXDECOMPTRECOMP REM 2.45 and 3.14, $V_{G,n}(T_0 \rightarrow \dots \rightarrow T_n)$ is identified with the tuple

$$(T_n, v_{Ge})_{v_{Ge} \in V_G(T_n)}, \quad (3.33)$$

VGNISO EQ

though this requires changing the total order in $V_G(T_n)$. Rather than using the order induced by T_n , one instead equips $V_G(T_n)$ with the order induced lexicographically from the maps $V_G(T_n) \rightarrow V_G(T_{n-1}) \rightarrow \dots \rightarrow V_G(T_0)$, i.e., for $v, w \in V_G(T_n)$ the condition $v < w$ is determined by the lowest i such that the images of $v, w \in V_G(T_i)$ are distinct.

3.4 A monad on spans

Definition 3.34. We will write $\mathbf{WSpan}^l(\mathcal{C}, \mathcal{D})$ (resp. $\mathbf{WSpan}^r(\mathcal{C}, \mathcal{D})$), which we call the category of *left weak spans* (resp. *right weak spans*), to denote the category with objects the spans

$$\mathcal{C} \xleftarrow{k} A \xrightarrow{F} \mathcal{D},$$

arrows the diagrams as on the left (resp. right) below

$$\begin{array}{ccc} & A_1 & \\ k_1 \swarrow & & \searrow F_1 \\ \mathcal{C} & & \mathcal{D} \\ k_2 \swarrow & i \downarrow & \nearrow \varphi \\ & A_2 & \\ & \searrow F_2 & \end{array} \quad \begin{array}{ccc} & A_1 & \\ k_1 \swarrow & & \searrow F_1 \\ \mathcal{C} & & \mathcal{D} \\ k_2 \swarrow & i \downarrow & \nearrow \varphi \\ & A_2 & \\ & \searrow F_2 & \end{array} \quad (3.35)$$

TWISTEDARROWRIGHT EQ

which we write as $(i, \varphi): (k_1, F_1) \rightarrow (k_2, F_2)$, and composition given in the obvious way.

Remark 3.36. There are natural isomorphisms

$$\mathbf{WSpan}^r(\mathcal{C}, \mathcal{D}) \simeq \mathbf{WSpan}^l(\mathcal{C}^{op}, \mathcal{D}^{op}). \quad (3.37)$$

LRSPANISO EQ

Remark 3.38. The terms *left/right* are motivated by the existence of adjunctions (which are seen to be equivalent by using LRSPANISO EQ (3.36))

$$\mathbf{Lan}: \mathbf{WSpan}^l(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathbf{Fun}(\mathcal{C}, \mathcal{D}) : \iota$$

$$\iota: \mathbf{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathbf{WSpan}^r(\mathcal{C}, \mathcal{D})^{op} : \mathbf{Ran}$$

where the functors ι denote the obvious inclusions (note the need for the $(-)^{op}$ in the second adjunction) and $\mathbf{Lan}/\mathbf{Ran}$ denote the left/right Kan extension functors.

We will mainly be interested in the span categories $\mathbf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \simeq \mathbf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$.

Notation 3.39. Given a functor $\pi: A \rightarrow \Sigma_G$, we let $\Omega_{G,n}^{(A)}$ denote the pullback (in \mathbf{Cat})

$$\begin{array}{ccc} \Omega_{G,n}^{(A)} & \xrightarrow{V_{G,n}^{(A)}} & F \wr A \\ \downarrow & & \downarrow \\ \Omega_{G,n} & \xrightarrow{V_{G,n}} & F \wr \Sigma_G \end{array}$$

Explicitly, the objects of $\Omega_{G,n}^{(A)}$ are pairs

$$(T_0 \rightarrow \dots \rightarrow T_n, (a_{e^\dagger \leq e})_{(e^\dagger \leq e) \in V_G(T_n)}) \quad (3.40)$$

OMEGAGNA EQ

such that $\pi(a_{e^\dagger \leq e}) = T_{n, e^\dagger \leq e}$.

Remark 3.41. Our primary interest here will be in the $\Omega_{G,0}^{(A)}$ construction. Importantly, the composite maps $\Omega_{G,0}^{(A)} \rightarrow \Omega_{G,0} \rightarrow \Sigma_G$ allow us to iterate the $\Omega_{G,0}^{(-)}$ construction. In practice, the role of higher strings $\Omega_{G,n}^{(A)}$ will then be to provide more convenient models for iterated $\Omega_{G,0}^{(-)}$ constructions.

Indeed, the content of Proposition 3.26 is then that there are compatible identifications $\Omega_{G,0}^{(\Omega_{G,n}^{(A)})} \simeq \Omega_{G,n+1}$ which identify $V_G^{(\Omega_{G,n}^{(A)})}$ with V_G .

Moreover, since all squares in the diagram

$$\begin{array}{ccccccc}
 \Omega_{G,n+1}^{(A)} & \xrightarrow{V_G^{(A)}} & F \wr \Omega_{G,n}^{(A)} & \xrightarrow{F \wr V_G^{(A)}} & F \wr F \wr A & \xrightarrow{\sigma^0} & F \wr A \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Omega_{G,n+1} & \xrightarrow{V_G} & F \wr \Omega_{G,n} & \xrightarrow{F \wr V_G} & F \wr F \wr \Sigma_G & \xrightarrow{\sigma^0} & F \wr \Sigma_G \\
 \downarrow & & \downarrow & & & & \\
 \Omega_{G,0} & \longrightarrow & F \wr \Sigma_G & & & &
 \end{array} \tag{3.42}$$

ALLSQUARES EQ

are pullback squares (the top center square is so by induction, the top right square by direct verification, the total top square by definition of $\Omega_{G,n+1}^{(A)}$ and the bottom left square by

Proposition 3.26), we likewise obtain identifications $\Omega_G^{(\Omega_{G,n}^{(A)})} \simeq \Omega_{G,n+1}^{(A)}$.

Proposition 3.43. For any $A \rightarrow \Sigma_G$ there are functors $d_0^{(A)}: \Omega_{G,1}^{(A)} \rightarrow \Omega_G^{(A)}$ and natural isomorphisms

$$\begin{array}{ccc}
 \Omega_{G,1}^{(A)} & \xrightarrow{V_G} & F \wr \Omega_G^{(A)} \xrightarrow{F \wr V_G} F \wr F \wr A \\
 d_0^{(A)} \downarrow & \nearrow \pi^{(A)} & \downarrow \sigma^0 \\
 \Omega_G^{(A)} & \xrightarrow{V_G} & F \wr A,
 \end{array} \tag{3.44}$$

SHUFFLEPERMA EQ

both natural in $A \rightarrow \Sigma$. Here naturality of $\pi^{(-)}$ means that for a functor $H: A \rightarrow B$ with corresponding diagram

$$\begin{array}{ccccccc}
 \Omega_{G,1}^{(A)} & \xrightarrow{V_G^{(A)}} & F \wr \Omega_{G,0}^{(A)} & \xrightarrow{F \wr V_G^{(A)}} & F \wr F \wr A & \xrightarrow{\sigma^0} & F \wr A \\
 \Omega_{G,1}^{(H)} \downarrow & d_0^{(A)} \searrow & \downarrow & \nearrow \pi^{(A)} & \downarrow & & \downarrow \\
 \Omega_{G,0}^{(A)} & \xrightarrow{V_G^{(A)}} & F \wr \Omega_{G,0}^{(A)} & \xrightarrow{F \wr V_G^{(A)}} & F \wr F \wr A & \xrightarrow{\sigma^0} & F \wr A \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Omega_{G,1}^{(B)} & \xrightarrow{V_G^{(B)}} & F \wr \Omega_{G,0}^{(B)} & \xrightarrow{F \wr V_G^{(B)}} & F \wr F \wr B & \xrightarrow{\sigma^0} & F \wr B \\
 d_0^{(B)} \searrow & & \downarrow & \nearrow \pi^{(B)} & \downarrow & & \downarrow \\
 \Omega_{G,0}^{(B)} & \xrightarrow{V_G^{(B)}} & F \wr \Omega_{G,0}^{(B)} & \xrightarrow{F \wr V_G^{(B)}} & F \wr F \wr B & \xrightarrow{\sigma^0} & F \wr B
 \end{array} \tag{3.45}$$

PICUBOIDAB EQ

one has an equality

$$(F \wr H) \pi^{(A)} = \pi^{(B)} \Omega_{G,1}^{(H)}$$

(i.e. the two natural isomorphisms between the two distinct functors $\Omega_{G,1}^{(A)} \Rightarrow F \wr B$ coincide).

Proof. Informally, using the object description in (3.39), $d_0^{(A)}$ is simply given by the formula

$$d_0^{(A)}(T_0 \rightarrow T_1, (a_{e^\dagger \leq e})_{(e^\dagger \leq e) \in V_G(T_1)}) = (T_1, (a_{e^\dagger \leq e})_{(e^\dagger \leq e) \in V_G(T_1)}), \tag{3.46}$$

though one must note that since in (3.39) the order in $V_G(T_1)$ is induced lexicographically from the string, the two orders for $V_G(T_1)$ in each side of (3.44) do not coincide.

It now follows that the composites $\sigma^0 \circ (F \wr V_G^{(A)}) \circ V_G^{(A)}$ and $V_G^{(A)} \circ d_0^{(A)}$ differ by the natural automorphism $\pi^{(A)}$ given by the tuple permutations interchanging the two orders in $V_G(T_1)$ for each $T_0 \rightarrow T_1$. PICUBOIDAB EQ

The commutativity of (3.43) is clear. □

Definition 3.47. Suppose \mathcal{V} has finite products.

We define an endofunctor N of $\mathbf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$ by letting $N(\Sigma_G \leftarrow A \rightarrow \mathcal{V}^{op})$ be the span $\Sigma_G \leftarrow \Omega_G^{(A)} \rightarrow \mathcal{V}^{op}$ given composition along the diagram

$$\begin{array}{ccccc} \Omega_{G,0}^{(A)} & \longrightarrow & F \wr A & \longrightarrow & F \wr \mathcal{V}^{op} \xrightarrow{\Pi^{op}} \mathcal{V}^{op} \\ \downarrow & & \downarrow & & \\ \Omega_{G,0} & \longrightarrow & F \wr \Sigma_G & & \\ \downarrow & & & & \\ \Sigma_G & & & & \end{array}$$

and defined on maps of spans in the obvious way.

One has a multiplication $\mu: N \circ N \Rightarrow N$ given by the natural isomorphisms

$$\begin{array}{ccccccc} \Sigma \longleftarrow \Omega_{G,1}^{(A)} & \xrightarrow{V_G} & F \wr \Omega_G^{(A)} & \xrightarrow{F \wr V_G} & F \wr F \wr A & \longrightarrow & F \wr F \wr \mathcal{V}^{op} \longrightarrow F \wr \mathcal{V}^{op} \longrightarrow \mathcal{V}^{op} \\ \parallel & & \searrow d_0^{(A)} & \nearrow \pi^{(A)} & \downarrow \sigma^0 & & \downarrow \sigma^0 & \nearrow \alpha & \parallel \\ \Sigma \longleftarrow \Omega_G^{(A)} & \xrightarrow{V_G} & F \wr A & \longrightarrow & F \wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op} \end{array} \quad (3.48)$$

MULTDEFSPAN EQ

where α is an associativity isomorphism for the product Π . We note that naturality of μ follows from the commutativity of (3.43). PICUBOIDAB EQ

Lastly, there is a unit $\eta: id \Rightarrow N$ given by the strictly commutative diagrams

$$\begin{array}{ccccccc} \Sigma \longleftarrow A & \xlongequal{\quad} & A & \longrightarrow & \mathcal{V}^{op} & \xlongequal{\quad} & \mathcal{V}^{op} \\ \parallel & & \searrow s_{-1}^{(A)} & & \downarrow & & \parallel \\ \Sigma \longleftarrow \Omega_G^{(A)} & \xrightarrow{V_G} & F \wr A & \longrightarrow & F \wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op}. \end{array} \quad (3.49)$$

UNITSPAN EQ

MONSPAN PROP

Proposition 3.50. (N, μ, η) form a monad on $\mathbf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$.

Proof. The natural transformation component of $\mu \circ (N\mu)$ is given by the composite diagram

$$\begin{array}{ccccccccccc} \Omega_{G,2}^{(A)} & \rightarrow & F \wr \Omega_{G,1}^{(A)} & \rightarrow & F^{i2} \wr \Omega_G^{(A)} & \rightarrow & F^{i3} \wr A & \rightarrow & F^{i3} \wr \mathcal{V}^{op} & \rightarrow & F^{i2} \wr \mathcal{V}^{op} \rightarrow F \wr \mathcal{V}^{op} \rightarrow \mathcal{V}^{op} \\ d_1^{(A)} \downarrow & & \downarrow & & \nearrow F \wr \pi^{(A)} & & \downarrow \sigma^1 & & \downarrow \sigma^1 & & \nearrow F \wr \alpha & & \parallel \\ \Omega_{G,1}^{(A)} & \rightarrow & F \wr \Omega_G^{(A)} & \rightarrow & F^{i2} \wr A & \rightarrow & F^{i2} \wr \mathcal{V}^{op} & \rightarrow & F \wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} \\ d_0^{(A)} \downarrow & & \nearrow \pi^{(A)} & & \downarrow \sigma^0 & & \downarrow \sigma^0 & & \nearrow \alpha & & \parallel \\ \Omega_G^{(A)} & \rightarrow & F \wr A & \rightarrow & F \wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} \end{array} \quad (3.51)$$

ASSOCSPAN1 EQ

whereas the natural transformation component of $\mu \circ (\mu N)$ is given by

$$\begin{array}{ccccccccccc}
\Omega_{G,2}^{(A)} & \rightarrow & F\wr\Omega_{G,1}^{(A)} & \rightarrow & F^{i2}\wr\Omega_G^{(A)} & \rightarrow & F^{i3}\wr A & \rightarrow & F^{i3}\wr \mathcal{V}^{op} & \rightarrow & F^{i2}\wr \mathcal{V}^{op} & \rightarrow & F\wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} \\
d_0^{(A)} \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \nearrow \alpha & \parallel & & & & \\
\Omega_{G,1}^{(A)} & \xrightarrow{\pi(\Omega_G^{(A)})} & F\wr\Omega_G^{(A)} & \rightarrow & F^{i2}\wr A & \rightarrow & F^{i2}\wr \mathcal{V}^{op} & \rightarrow & F\wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} & & & & \\
d_0^{(A)} \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \nearrow \alpha & \parallel & & & & \\
\Omega_G^{(A)} & \xrightarrow{\pi(A)} & F\wr A & \rightarrow & F\wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} & & & & & & & &
\end{array} \tag{3.52}$$

ASSOCSPAN2 EQ

That the rightmost sections of (3.49) and (3.50) coincide follows from compatibility of the associativity isomorphisms for Π^{op} .

For the leftmost sections, note first that, in either diagram, the top right and bottom left paths $\Omega_{G,2}^{(A)} \rightarrow F\wr A$ differ only by the induced order on $V_G(T_2)$ for each string $T_0 \rightarrow T_1 \rightarrow T_2$. More explicitly, the top right paths use the order induced lexicographically from the string $T_0 \rightarrow T_1 \rightarrow T_2$ while the bottom left paths use the order induced exclusively by T_2 . The two left sections then coincide since are both given by the permutation interchanging these orders, the only difference being that the intermediate stage of (3.49) uses the order induced lexicographically from $T_0 \rightarrow T_2$ while (3.50) uses the order induced lexicographically from $T_1 \rightarrow T_2$.

As for unit conditions, $\mu \circ (N\eta)$ is represented by

$$\begin{array}{ccccccccccc}
\Omega_G^{(A)} & \longrightarrow & F\wr A & = & F\wr A & \longrightarrow & F\wr \mathcal{V}^{op} & = & F\wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op} \\
s_0^{(A)} \downarrow & & \downarrow & & \downarrow \delta^1 & & \downarrow \delta^1 & & \parallel & & \parallel \\
\Omega_{G,1}^{(A)} & \rightarrow & F\wr\Omega_G^{(A)} & \rightarrow & F^{i2}\wr A & \rightarrow & F^{i2}\wr \mathcal{V}^{op} & \rightarrow & F\wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} \\
d_0^{(A)} \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \nearrow \alpha & \parallel & & \parallel \\
\Omega_G^{(A)} & \xrightarrow{\pi(A)} & F\wr A & \longrightarrow & F\wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op} & & & &
\end{array} \tag{3.53}$$

UNITSPAN1 EQ

while $\mu \circ (\eta N)$ is represented by

$$\begin{array}{ccccccccccc}
\Omega_G^{(A)} & = & \Omega_G^{(A)} & \longrightarrow & F\wr A & \longrightarrow & F\wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op} & = & \mathcal{V}^{op} \\
s_{-1}^{(A)} \downarrow & & \downarrow & & \downarrow \delta^0 & & \downarrow \delta^0 & & \downarrow & & \parallel \\
\Omega_{G,1}^{(A)} & \rightarrow & F\wr\Omega_G^{(A)} & \rightarrow & F\wr F\wr A & \rightarrow & F\wr F\wr \mathcal{V}^{op} & \rightarrow & F\wr \mathcal{V}^{op} & \rightarrow & \mathcal{V}^{op} \\
d_0^{(A)} \downarrow & \nearrow & \downarrow & \downarrow & \downarrow \sigma^0 & \downarrow \sigma^0 & \downarrow \sigma^0 & \nearrow \alpha & \parallel & & \parallel \\
\Omega_G^{(A)} & \xrightarrow{\pi(A)} & F\wr A & \longrightarrow & F\wr \mathcal{V}^{op} & \longrightarrow & \mathcal{V}^{op} & & & &
\end{array} \tag{3.54}$$

UNITSPAN2 EQ

It is straightforward to check that the composites of the left and right sections of both (3.51) and (3.52) are strictly commutative diagrams, and thus that (3.51) and (3.52) coincide. \square

3.5 The free genuine operad monad

Recalling that $\mathbf{Wspan}^r(\Sigma_G, \mathcal{V}^{op}) \simeq \mathbf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$, Proposition 3.48 and Remark 3.37 give an adjunction

$$\mathbf{Lan}: \mathbf{Wspan}^l(\Sigma_G^{op}, \mathcal{V}) \rightleftarrows \mathbf{Fun}(\Sigma_G^{op}, \mathcal{V}): \iota \tag{3.55}$$

LANIOTAADJ EQ

together with a monad N in the leftmost category $\mathbf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$. We now turn to showing that, under reasonable hypothesis on \mathcal{V} , the composite $\mathbf{Lan} \circ N \circ \iota$ inherits a monad structure from N . The key will be to show that under such conditions the map $\mathbf{Lan} \circ N \Rightarrow \mathbf{Lan} \circ N \circ \iota \circ \mathbf{Lan}$ is a natural isomorphism.

Recall that following Convention PLANARCONV CON 2.18 our model for \mathbf{O}_G consists of totally ordered sets. One therefore has *root functors*

$$\Omega_G^q \xrightarrow{r} \mathbf{O}_G, \quad \Sigma_G \xrightarrow{r} \mathbf{O}_G$$

sending each planar G -tree to its ordered orbital G -set of roots.

Root functors are compatible with the leaf-root functor and the inclusion, i.e. the following commute.

$$\begin{array}{ccc} \Omega_G^q & \xrightarrow{lr} & \Sigma_G \\ & \searrow r & \downarrow r \\ & & \mathbf{O}_G \end{array} \quad \begin{array}{ccc} \Sigma_G & \hookrightarrow & \Omega_G^q \\ & \searrow r & \downarrow r \\ & & \mathbf{O}_G \end{array} \quad (3.56) \quad \boxed{\text{ROOTLEAFTROOTCOM EQ}}$$

Moreover, the diagrams ROOTLEAFTROOTCOM EQ (3.54) possess some extra structure we will need to make use of. Indeed, both functors are split Grothendieck fibrations: given a map $\varphi: A \rightarrow B$ in \mathbf{O}_G and G -tree T such that $r(T) = B$ we can build a cartesian arrow $\varphi^*(T) \rightarrow T$ by letting $\varphi^*(T)$ to be the pullback G -tree together with the planariz structure on roots given by A and on non-equivariant nodes given by their image via $\varphi^*(T) \rightarrow T$.

It now follows that ROOTLEAFTROOTCOM EQ (3.54) are diagrams of split Grothendieck fibrations.

Definition 3.57. A split Grothendieck fibration $A \xrightarrow{r} \mathbf{O}_G$ is called a *root fibration* and a split Grothendieck fibration diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ & \searrow r & \downarrow r \\ & & \mathbf{O}_G \end{array}$$

is called a *root fibration functor*.

The relevance of root fibrations is given by the following couple of lemmas.

Lemma 3.58. *If $A \rightarrow \Sigma_G$ is a root fibration functor then so is $\Omega_G^{(A)} \rightarrow \Omega_G$, naturally in A .*

Proof. We consider the pullback diagram below.

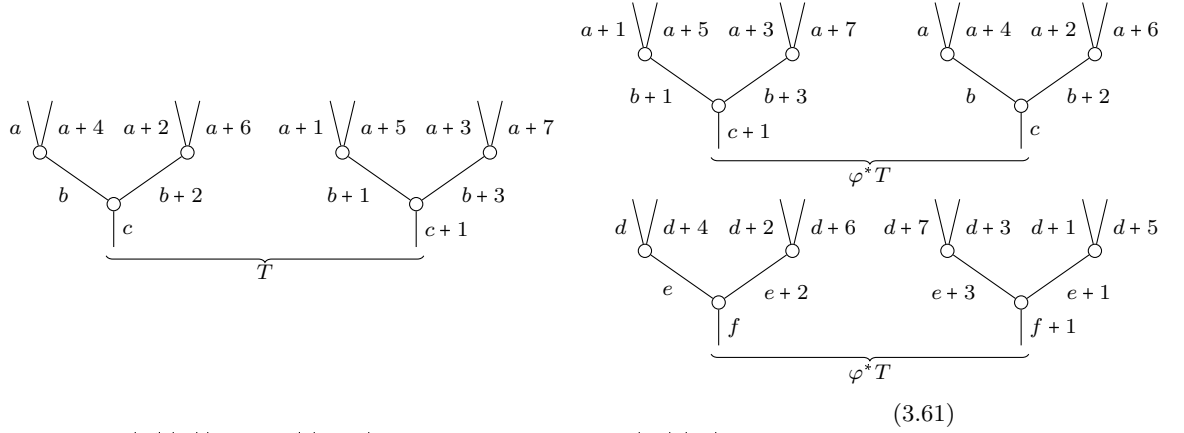
$$\begin{array}{ccc} \Omega_{G,0}^{(A)} & \xrightarrow{V_G^{(A)}} & \mathbf{F} \wr A \\ \downarrow & & \downarrow \\ \Omega_{G,0} & \xrightarrow{V_G} & \mathbf{F} \wr \Sigma_G \end{array} \quad (3.59) \quad \boxed{\text{ROOTIMPLIESROOT EQ}}$$

The hypothesis that $A \rightarrow \Sigma_G$ is root fibration implies that the rightmost map in ROOTIMPLIESROOT EQ (3.57) is a map of split Grothendieck fibrations over $\mathbf{F} \wr \mathbf{O}_G$.

Since the map V_G sends the chosen cartesian arrows in $\Omega_{G,0}$ (over \mathbf{O}_G) to chosen cartesian arrows of $\mathbf{F} \wr \Sigma_G$ (over $\mathbf{F} \wr \mathbf{O}_G$), the result follows. \square

Example 3.60. Let $G = \mathbb{Z}/8$. The following exemplifies a pull back along the twist map $\varphi: G/2G \rightarrow G/2G$ (i.e., accounting for order, φ is the permutation (12)), with the topmost representation of φ^*T maintaining the chosen generators for each edge orbit from T and the bottom representation choosing instead the generators to be minimal with regard to the

planar structure.



We note that $(\varphi^*(T))_{v_{Ge}} = \psi^*(T_{v_{Gb}})$ for ψ the permutation (13)(24) encoded by the composite identifications $\{1, 2, 3, 4\} \simeq \{e, e+2, e+3, e+1\} \simeq \{b+1, b+3, b, b+2\} \simeq \{3, 4, 1, 2\}$.

Lemma 3.62. *Suppose that \mathcal{V} is complete and that $A \rightarrow \Sigma_G$ is a root fibration. If the rightmost triangle in*

$$\begin{array}{ccc} \Omega_{G,0}^{(A)} & \xrightarrow{V_G^{(A)}} & F \wr A \xrightarrow{\quad} \mathcal{V} \\ \downarrow & & \downarrow \nearrow \\ \Omega_{G,0} & \xrightarrow{V_G} & F \wr \Sigma_G \end{array} \quad (3.63)$$

is a right Kan extension diagram then so is the composite diagram.

Proof. Unpacking definitions using the pointwise formula for right Kan extensions ([McL, X.3.1]), it suffices to check that for each $T \in \Omega_{G,0}$ the functor

$$T \downarrow \Omega_{G,0}^{(A)} \rightarrow V_G(T) \downarrow F \wr A \quad (3.64)$$

LANPULLCOMA EQ

is initial. In the course of the proof of Lemma 3.3 it was shown that the subcategory

$$\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow A$$

is initial in the $V_G(T) \downarrow F \wr A$.

On the other hand, since $\Omega_G^{(A)} \rightarrow \Omega_G$ is a root fibration functor, $T \downarrow \Omega_G^{(A)}$ has an initial subcategory $T \downarrow_{r,\simeq} \Omega_G^{(A)}$ with objects $(S \in \Omega_G^{(A)}, T \rightarrow u(S))$ such that $T \rightarrow u(S)$ is a quotient map that induces an ordered isomorphism on roots. Note that this can be restated as saying that $T \rightarrow u(S)$ is an isomorphism preserving the order of the roots.

The result now follows from the natural isomorphism

$$T \downarrow_{r,\simeq} \Omega_G^{(A)} \simeq \prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{r,\simeq} A. \quad (3.65)$$

TDOWNISOA EQ

To see this, we focus first on the case $A = \Sigma_G$. In that case, the left hand side of (3.63) encodes replanarizations of T that preserve the root order. On the other hand, the right hand side encodes replanarizations of all the G -vertices that preserve the order of their roots, or, equivalently, replanarizations of the non-equivariant vertices of T . That these are equivalent is the content of Proposition 2.13.

Note that $(T \rightarrow S) \in (T \downarrow_{r,\simeq} \Omega_G)$ is then encoded by a tuple $(T_{v_{Ge}} \rightarrow \varphi_{v_{Ge}}^* S_{v_{Ge}})_{v_{Ge} \in V_G(T)}$ where the pullbacks $\varphi_{v_{Ge}}^*$ are needed to correct the root order.

The case of general A follows likewise, using the corresponding pullbacks $\varphi_{v_{Ge}}^*$.

Note: an addendum is needed to show that (3.63) suffices, since $T \downarrow_{r, \simeq} \Omega_G^{(A)}$ is not sent directly to $\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{r, \simeq} A$ □

Lemma 3.56 can be interpreted as saying that, if one defines a category $\mathbf{Wspan}_r^l(\Sigma_G^{op}, \mathcal{V})$ of *rooted spans*

$$\Sigma_G^{op} \leftarrow A^{op} \rightarrow \mathcal{V}$$

where $A \rightarrow \Sigma_G$ is a root fibration functor, the monad N built in Proposition 3.48 lifts to a monad N_r in $\mathbf{Wspan}_r^l(\Sigma_G^{op}, \mathcal{V})$, and likewise for the adjunction (3.53).

Corollary 3.66. *Suppose that finite products in \mathcal{V} commute with colimits in each variable. The functors*

$$\mathbf{Lan} \circ N_r \Rightarrow \mathbf{Lan} \circ N_r \circ \iota \circ \mathbf{Lan}, \quad \mathbf{Lan} \circ \iota \Rightarrow id$$

are natural isomorphisms.

Proof. This follows by combining Lemma 3.60 with Lemma 3.3. □

Definition 3.67. The *genuine equivariant operad monad* is the monad \mathbb{F}_G on $\mathbf{Fun}(\Sigma_G^{op}, \mathcal{V})$ given by

$$\mathbb{F}_G = \mathbf{Lan} \circ N_r \circ \iota$$

and with multiplication and unit given by the composites

$$\begin{aligned} \mathbf{Lan} \circ N_r \circ \iota \circ \mathbf{Lan} \circ N_r \circ \iota &\stackrel{\sim}{\Leftarrow} \mathbf{Lan} \circ N_r \circ N_r \circ \iota \Rightarrow \mathbf{Lan} \circ N_r \circ \iota \\ id &\stackrel{\sim}{\Leftarrow} \mathbf{Lan} \circ \iota \Rightarrow \mathbf{Lan} \circ N_r \circ \iota. \end{aligned}$$

Remark 3.68. The functor $\mathbf{Lan} \circ N_r \circ \iota$ is isomorphic to $\mathbf{Lan} \circ N \circ \iota$, and this isomorphism is compatible with the multiplication and unit in Definition 3.65, and we will henceforth simply write N rather than N_r .

From this point of view, the role of root fibrations is to guarantee that $\mathbf{Lan} \circ N \circ \iota$ is indeed a monad, but unnecessary to describe the monad structure itself.

Remark 3.69. Since a map

$$\mathbb{F}_G X = \mathbf{Lan} \circ N_r \circ \iota X \rightarrow X$$

is adjoint to a map

$$N_r \circ \iota X \rightarrow \iota X$$

one easily verifies that X is a genuine equivariant operad, i.e. a \mathbb{F}_G -algebra, iff ιX is a N -algebra. Moreover, the bar resolution

$$\mathbb{F}_G^{\bullet+1} X$$

is isomorphic to

$$\mathbf{Lan}(N^{\bullet+1} \iota X).$$

4 Free extensions

Our overall goal in this section will be to produce a description of free genuine operad pushouts, i.e. pushouts of the form

$$\begin{array}{ccc} \mathbb{F}_G A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{F}_G B & \longrightarrow & Y \end{array}$$

in the category \mathbf{Op}_G of genuine equivariant operads.

4.1 Extensions over general monads

Any monad T on \mathcal{C} one obtains induced monads $T^{\times l}$ on $\mathcal{C}^{\times l}$, and we will make use of several standard relations between these. In particular, any map $\alpha: l \rightarrow \underline{m}$ induces a forgetful functor such that for the forgetful functor $\alpha^*: \mathcal{C}^{\times l} \rightarrow \mathcal{C}^{\times n}$ one has $T^{\times l} \alpha^* \simeq \alpha^* T^{\times m}$.

Indeed, we will need to make use of a slightly more general setup. Letting I denote the identity monad on \mathcal{C} , and $K \subset \underline{m}$ be a subset, there is a monad $T^{\times K} \times I^{\times(\underline{m}-K)}$ on $\mathcal{C}^{\times m}$, which we abusively denote simply as $T^{\times K}$. Identities then determine maps of monads $T^J \rightarrow T^{\times K}$ whenever $J \subset K$ and, moreover, there are identifications $T^{\times \alpha^{-1}(K)} \alpha^* \simeq \alpha^* T^{\times K}$. One then has the following.

Proposition 4.1. *The functor*

$$T^{\times \alpha^{-1}(K)} \Rightarrow \alpha^* T^{\times K} \alpha_! \quad (4.2)$$

adjoint to the identification $T^{\times \alpha^{-1}(K)} \alpha^ \simeq \alpha^* T^{\times K}$ is a map of monads on $\mathcal{C}^{\times n}$.*

Proof. We first note that there are identifications of functors $(FG)^{\times K} \simeq F^{\times K} G^{\times K}$ which are compatible with the identifications $F^{\times \alpha^{-1}(K)} \alpha^* \simeq \alpha^* F^{\times K}$ in the sense that the identification $(FG)^{\times \alpha^{-1}(K)} \circ \alpha^* \simeq \alpha^* (FG)^{\times K}$ matches the composite identification $F^{\times \alpha^{-1}(K)} G^{\times \alpha^{-1}(K)} \alpha^* \simeq F^{\times \alpha^{-1}(K)} \alpha^* G^{\times K} \simeq \alpha^* F^{\times K} G^{\times K}$.

Letting η, ϵ denote the unit and counit for the $(\alpha_!, \alpha^*)$ adjunction, (4.2) is then the composite

$$T^{\times \alpha^{-1}(K)} \xrightarrow{\eta} T^{\times \alpha^{-1}(K)} \alpha^* \alpha_! \simeq \alpha^* T^{\times K} \alpha_!.$$

That this is a monad map is the condition that the following multiplication and unit diagrams commute.

$$\begin{array}{ccc} T^{\times \alpha^{-1}(K)} \circ T^{\times \alpha^{-1}(K)} & \longrightarrow & \alpha^* T^{\times K} \alpha_! \circ \alpha^* T^{\times K} \alpha_! \\ \downarrow & & \downarrow \\ T^{\times \alpha^{-1}(K)} & \longrightarrow & \alpha^* T^{\times K} \alpha_! \end{array} \quad \begin{array}{ccc} I^{\times n} & & \\ \downarrow & \searrow & \\ T^{\times \alpha^{-1}(K)} & \longrightarrow & \alpha^* T^{\times K} \alpha_! \end{array}$$

We argue only the case of the leftmost multiplication diagram, with commutativity of the unit diagram following by a similar but simpler argument. Since the precomposition $(-) \circ \alpha^*$ is the left adjoint to the precomposition $(-) \circ \alpha_!$ this follows from the following diagram.

$$\begin{array}{ccccc} T^{\times \alpha^{-1}(K)} T^{\times \alpha^{-1}(K)} \alpha^* & \xrightarrow{\simeq} & T^{\times \alpha^{-1}(K)} \alpha^* T^{\times K} & \xrightarrow{\eta} & T^{\times \alpha^{-1}(K)} \alpha^* \alpha_! \alpha^* T^{\times K} & \xrightarrow{\simeq} & \alpha^* T^{\times K} \alpha_! \alpha^* T^{\times K} \\ \downarrow & & \searrow & & \downarrow \epsilon & & \downarrow \epsilon \\ & & T^{\times \alpha^{-1}(K)} \alpha^* T^{\times K} & \xrightarrow{\simeq} & \alpha^* T^{\times K} T^{\times K} & & \downarrow \\ T^{\times \alpha^{-1}(K)} \alpha^* & \xrightarrow{\simeq} & & & \alpha^* T^{\times K} & & \end{array}$$

□

Remark 4.3. Since $T^{\times K} \alpha_!$ is a right $\alpha^* T^{\times K} \alpha_!$ -module, Proposition 4.1 implies that it is also a right $T^{\times \alpha^{-1}(K)}$ -module or, moreover, a right $T^{\times J}$ -module whenever $\alpha(J) \subset K$.

Remark 4.4. Combining the precomposition and postcomposition adjunctions, the identification $T^{\times \alpha^{-1}(K)} \alpha^* \simeq \alpha^* T^{\times K}$ is then adjoint to a functor $\alpha_! T^{\times \alpha^{-1}(K)} \rightarrow T^{\times K} \alpha_!$ which is readily checked to be a map of right $T^{\times \alpha^{-1}(K)}$ -modules.

More generally, for $\alpha(J) \subset K$, the composite $T^{\times J} \alpha^* \rightarrow T^{\times \alpha^{-1}(K)} \alpha^* \simeq \alpha^* T^{\times K}$ is thus adjoint to a map of right $T^{\times J}$ -modules

$$\alpha_! T^{\times J} \rightarrow T^{\times K} \alpha_!. \quad (4.5)$$

RIGHTMODULETMAP EQ

We now unpack the content of (4.5) when $\alpha: \underline{l} \rightarrow *$ is the unique map to the singleton $* = \underline{1}$. In this case we can instead write $\alpha_! = \coprod$, $\alpha^* = \Delta$, and we thus have commutative diagrams

$$\begin{array}{ccc} \coprod_J TTA_j \sqcup \coprod_{\underline{l}-J} A_j & \longrightarrow & T(\coprod_J TA_j \sqcup \coprod_{\underline{l}-J} A_j) \\ \downarrow & & \downarrow \\ \coprod_J TA_j \sqcup \coprod_{\underline{l}-J} A_j & \longrightarrow & T(\coprod_J A_j \sqcup \coprod_{\underline{l}-J} A_j) \end{array} \quad (4.6)$$

where the vertical maps come from the right $T^{\times J}$ -module structure. Writing \sqcup^a for the coproduct of T -algebras and recalling the canonical identifications $\coprod_K^a(TA_k) \simeq T(\coprod_K A_k)$, (4.6) in fact shows that the right $T^{\times J}$ -module structure on $T \circ \coprod$ in fact codifies the multiplication maps

$$\coprod_J^a TTA_j \sqcup^a \coprod_{\underline{l}-J}^a TA_j \rightarrow \coprod_J^a TA_j \sqcup^a \coprod_{\underline{l}-J}^a TA_j.$$

4.2 Labeled planar strings

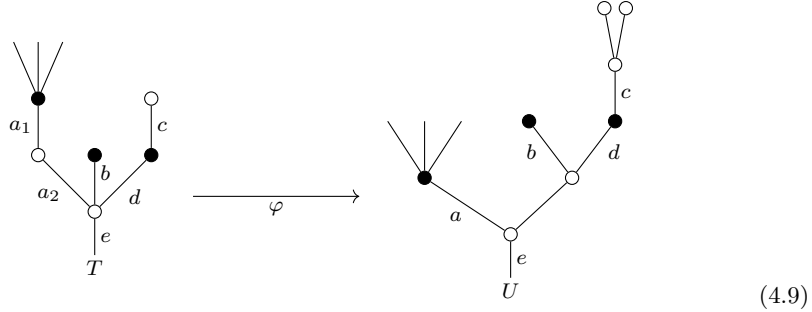
We now translate the results in the previous section to the context of the monad N on $\mathbf{WSpan}^l(\Sigma^{op}, \mathcal{V})$. In analogy to the planar string models $\Omega_{G,n}^{(A)}$ for iterations N^{on+1} of the monad N , we will find it convenient to build similar string models $\Omega_{G,n}^{(A,J)}$ for $N \circ \coprod \circ (N^{\times J})^{on}$.

Definition 4.7. A l -node labeled G -tree (or just l -labeled G -tree) G -tree is a pair $(T, V_G(T) \rightarrow \{1, \dots, l\})$ with $T \in \Omega_G$, which we think of as a G -tree together with G -vertices labels in $1, \dots, l$.

Further, a map $\varphi: T \rightarrow S$ between l -labeled trees is called a *label map* if for each G -vertex v_{Ge} of T with label j , the vertices of the subtree $S_{v_{Ge}}$ are all labeled by j .

Lastly, given a subset $J \subset \underline{l}$, a planar label map $\varphi: T \rightarrow S$ is said to be J -inert if for every G -vertex v_{Ge} of T with label $j \in J$ it is $S_{v_{Ge}} = T_{v_{Ge}}$.

Example 4.8. Consider the 2-labeled trees below (for $G = *$ the trivial group), with black nodes (\bullet) denoting labels by the number 1 and white nodes (\circ) labels by the number 2. The planar map φ (sending $a_i \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto d$, $e \mapsto e$) is a label map which is $\{1\}$ -inert.



Definition 4.10. Let $0 \leq s \leq n$ and $J \subset \underline{l}$ be a subset.

We define $\Omega_{G,n,s}^J$ to have as objects n -planar strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \dots \xrightarrow{f_s} T_s \xrightarrow{f_{s+1}} T_{s+1} \xrightarrow{f_{s+2}} \dots \xrightarrow{f_n} T_n \quad (4.11)$$

together with l -labelings of T_s, T_{s+1}, \dots, T_n such that the $f_r, r > s$ are $(\underline{l} - J)$ -inert label maps.

Arrows in $\Omega_{G,n,s}^J$ are quotients of strings $(q_r: T_r \rightarrow T_r')$ such that $q_r, r \leq s$ are label maps.

Informally, $\Omega_{G,n,s}^J$ consists of n -strings such that trees and maps after T_s are l -labeled.

Remark 4.12. Our main case of interest will that of $s = 0$, in which case we abbreviate $\Omega_{G,n}^J = \Omega_{G,n,0}^J$. Indeed, such strings will suffice to build models for $N \circ \coprod \circ (N^{\times J})^{on}$.

However, to unpack the right $N^{\times J}$ -module structure as in Remark 4.7 one further needs to encode composites $NN \circ \coprod \circ (N^{\times J})^{on-1}$, a role played by strings $\Omega_{G,n,1}^J$.

Notation 4.13. We will further write

$$\Omega_{G,n,-1}^J = \coprod_J \Omega_{G,n} \sqcup \coprod_{\underline{l}=J} \Sigma_G, \quad \Omega_{G,n,n+1}^J = \Omega_{G,n} \quad (4.14)$$

OMEGANMINUSONE EQ

To justify this convention, we note that a string as in (NSTRINGLAB EQ) can be extended by prepending to it the map $\text{lr}(T_0) = T_{-1} \xrightarrow{f_0} T_0$. If one then attempts to define $\Omega_{G,n,-1}^J$ by insisting that T_{-1} also be labeled, it follows that all node labels in each string must coincide, resulting in the coproduct decomposition in (OMEGANMINUSONE EQ).

There are a number of obvious functors relating the $\Omega_{G,n,s}^J$ categories, which we now make explicit. Given $s \leq s'$ or $J \subset J'$ there are forgetful functors

$$\Omega_{G,n,s}^J \rightarrow \Omega_{G,n,s'}^J, \quad \Omega_{G,n,s}^J \rightarrow \Omega_{G,n,s}^{J'} \quad (4.15)$$

NKNFGT EQ

The simplicial operators in Notation SIMPOPERATORS NOT generalize to operators (where $0 \leq i \leq n$, $-1 \leq j \leq n$)

$$\begin{aligned} d_i: \Omega_{G,n,s}^J &\rightarrow \Omega_{G,n-1,s-1}^J & i < s & & s_j: \Omega_{G,n,s}^J &\rightarrow \Omega_{G,n+1,s+1}^J & j < s \\ d_i: \Omega_{G,n,s}^J &\rightarrow \Omega_{G,n-1,s}^J & s \leq i & & s_j: \Omega_{G,n,s}^J &\rightarrow \Omega_{G,n+1,s}^J & s \leq j \end{aligned}$$

which are compatible with the forgetful functors in the obvious way.

Remark 4.16. For $J \subset J'$ the forgetful functor in (NKNFGT EQ) is a fully faithful inclusion. However, and somewhat subtly, this is not the case for the $s \leq s'$ forgetful functors. Indeed, regarding $T \rightarrow U$ in Examples LABELEDTREES EX as an object in $\Omega_{*,n,0}^2$, changing the label of the $a_1 \leq a_2$ vertex of T from a \circ -label to a \bullet -label yields an alternate object $\bar{T} \rightarrow U$ of $\Omega_{*,n,0}^2$ forgetting to the same object of $\Omega_{*,n,1}^2$, yet $T \rightarrow U$ and $\bar{T} \rightarrow U$ are not isomorphic.

We note that this is a consequence of the fact that substitution data can replace unary nodes by stumps, which have no nodes.

Generalizing Notation INDVNG NOT there is a commutative diagram

$$\begin{array}{ccc} \Omega_{G,n,s}^J & \xrightarrow{V_{G,n}} & \text{F} \wr \Sigma_G^{\text{ul}} \\ \downarrow & & \downarrow \\ \Omega_{G,n} & \xrightarrow{V_{G,n}} & \text{F} \wr \Sigma_G \end{array}$$

where for a labeled string it is $V_G(T_0 \rightarrow \dots \rightarrow T_n) = (T_{n,v_{Ge}})_{V_G(T_n)}$, where we regard $T_{n,v_{Ge}} \in \Sigma_G^{\text{ul}} \simeq \Omega_{G,-1,-1}^L$ by using the label in 1. . . .

We now expand Notation OMEGAGNA EQ.

Notation 4.17. Let \underline{A} denote a \underline{l} -tuple $(\pi_j: A_j \rightarrow \Sigma_G)_{\underline{l}}$ of categories over Σ_G . We define $\Omega_{G,n,s}^{(\underline{A}),J}$ by the pullback diagram

$$\begin{array}{ccc} \Omega_{G,n,s}^{(\underline{A}),J} & \xrightarrow{V_{G,n}^{(\underline{A})}} & \text{F} \wr \coprod A_j \\ \downarrow & & \downarrow \\ \Omega_{G,n,s}^J & \xrightarrow{V_{G,n}} & \text{F} \wr \Sigma_G^{\text{ul}} \end{array} \quad (4.18)$$

LTUPLEAPULL EQ

Explicitly, an object of $\Omega_{G,n,s}^{(\underline{A}),J}$ consists of a labeled string $T_0 \rightarrow \dots \rightarrow T_n$ as in (NSTRINGLAB EQ) together with a tuple $(a_{v_{Ge}})_{V_G(T_n)}$ such that $a_{v_{Ge}} \in A_j$ if v_{Ge} has label j and $\pi_j(a_{v_{Ge}}) = T_{n,v_{Ge}}$.

The reader may have noticed a certain asymmetry between our definition of the $V_{G,n}$ functors here versus their analogues in §3.3, where they were defined iteratively in terms of simpler functors V_G . This is because of the possibility that $s = -1$, in which case (PLANARSTRING SEC) applies and some caution is needed in that the following result fails.

Proposition 4.19. Suppose $0 \leq s \leq n$. One has a diagram of pullback squares (generalizing ALLSQUARES EQ (???))

$$\begin{array}{ccccc}
 \Omega_{G,n,s}^{(A),J} & \xrightarrow{V_G^{(A)}} & F \wr \Omega_{G,n-1,s-1}^{(A),J} & \xrightarrow{F \wr V_{G,n}^{(A)}} & F \wr F \wr \coprod A_j & \xrightarrow{\sigma^0} & F \wr \coprod A_j \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \Omega_{G,n,s}^J & \xrightarrow{V_G} & F \wr \Omega_{G,n-1,s-1}^J & \xrightarrow{F \wr V_{G,n}} & F \wr F \wr \Sigma_G^{ul} & \xrightarrow{\sigma^0} & F \wr \Sigma_G^{ul} \\
 \downarrow & & \downarrow & & & & \\
 \Omega_{G,0} & \xrightarrow{V_G} & F \wr \Sigma_G & & & &
 \end{array} \quad (4.20) \quad \boxed{\text{ALLSQUARESJ EQ}}$$

such that the composite of the top squares is LTUPLEAPULL EQ (??).

Proof. The V_G functors are defined just as in VG DEF (3.13) via the formula

$$V_G(T_0 \rightarrow T_1 \rightarrow \cdots \rightarrow T_n) = (T_{1,v_{Ge}} \rightarrow \cdots \rightarrow T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)}$$

with the strings $T_{1,v_{Ge}} \rightarrow \cdots \rightarrow T_{n,v_{Ge}}$ inheriting the extra structure in the obvious way.

Since the top composite square, top center square and top right square are all pullback squares, it remains only to show that the bottom left square is a pullback. This last claim is simply a variation of Proposition 3.26, and follows from the same proof, since both labels and inertness conditions are inherited when assembling substitution data into trees via Proposition SUBDATAUNDERPLAN PRO 2.42. \square

4.3 Bar constructions on spans

kk

HERE

HERE

$$\begin{array}{ccc}
 \Omega_{1,0}^{(A \amalg B, A \amalg B)} & & F \wr (\Omega^{(A \amalg B)} \amalg \Omega^{(A \amalg B)}) \\
 & \searrow & \downarrow \\
 \Omega_{1,1}^{(A \amalg B)} & & F \wr \Omega^{(A \amalg B)} \\
 \Omega_{1,0} \longrightarrow & F(\Omega \amalg \Omega) & \downarrow \\
 \searrow & \searrow & \downarrow \\
 \Omega_{1,1} & \longrightarrow & F \wr \Omega \\
 \Omega_2^{(A,B)} \longrightarrow & F \wr (\Omega_1^{(A)} \amalg \Omega_1^{(B)}) & \\
 \Omega_2^{(A \amalg B, A \amalg B)} \longrightarrow & F \wr (\Omega_1^{(A \amalg B)} \amalg \Omega_1^{(A \amalg B)}) & \\
 \Omega_2^{(A \amalg B)} \longrightarrow & F \wr \Omega_1^{(A \amalg B)} &
 \end{array}$$

References

McL

- [1] S. Mac Lane. Categories for the working mathematician, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 1998.

Pe16b

[2] L. A. Pereira. Equivariant dendroidal sets. Available at: <http://www.faculty.virginia.edu/luisalex/>, 2016.

We12

[3] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: <https://arxiv.org/abs/1201.3987>, 2012.