Graph equivalences for spaces

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1 Main results

The following are our key results concerning operads and genuine G-spectra.

modmodelexist thm

Theorem 1.1. *Let $\mathcal{O}$ be any operad in $(\mathsf{Sp}^{\Sigma})^G$, and let $(\mathsf{Sp}^{\Sigma})^G$, Sym^G be equipped with the respective positive S G-graph stable model structures*[1].

Then the respective induced projective model structures on $\mathsf{Alg}_{\mathcal{O}}$, $\mathsf{Mod}^L_{\mathcal{O}}$ exist, and these are simplicial model structures.

Further, if $\mathcal{O} \to \bar{\mathcal{O}}$ is a G-graph stable equivalence of operads then the induce-forget adjunctions

$$\bar{\mathcal{O}} \circ_{\mathcal{O}} \cdot : \mathsf{Alg}_{\mathcal{O}} \rightleftarrows \mathsf{Alg}_{\bar{\mathcal{O}}} : fgt, \qquad \bar{\mathcal{O}} \circ_{\mathcal{O}} \cdot : \mathsf{Mod}^L_{\mathcal{O}} \rightleftarrows \mathsf{Mod}^L_{\bar{\mathcal{O}}} : fgt$$

are Quillen equivalences.

We will refer to the model structures in Theorem 1.1 as the projective positive S G-graph stable model structures.

gsymseq thm

Theorem 1.2. *Let $f : A \to B$ be a map in $(\mathsf{Sp}^{\Sigma})^{G \times \Sigma_n}$.*

Then the maps

$$A \wedge_{\Sigma_n} (\cdot)^{\wedge n} : (\mathsf{Sp}^{\Sigma})^G \to (\mathsf{Sp}^{\Sigma})^G \quad , \quad B \wedge_{\Sigma_n} (\cdot)^{\wedge n} : (\mathsf{Sp}^{\Sigma})^G \to (\mathsf{Sp}^{\Sigma})^G$$

are left derivable and the induced natural transformation

$$f \wedge^L_{\Sigma_n} (\cdot)^{\wedge n} : A \wedge^L_{\Sigma_n} (\cdot)^{\wedge n} \to B \wedge^L_{\Sigma_n} (\cdot)^{\wedge n}$$

of left derived functors is a levelwise G-stable equivalence if and only if f is a G-graph stable equivalence.

circo pos thm

Theorem 1.3. *Let $\mathcal{O}$ be an operad in $(\mathsf{Sp}^{\Sigma})^G$ and consider the relative composition product*

$$\mathsf{Mod}^R_{\mathcal{O}} \times \mathsf{Mod}^L_{\mathcal{O}} \xrightarrow{\cdot \circ_{\mathcal{O}} \cdot} \mathsf{Sym}^G.$$

Regard $\mathsf{Mod}^L_{\mathcal{O}}$ as equipped with the projective positive S G-graph stable model structure and Sym^G as equipped with the S G-graph stable model structure[2].

Suppose f_2 is a cofibration between cofibrant objects in $\mathsf{Mod}^L_{\mathcal{O}}$. Then if the map f_1 in $\mathsf{Mod}^R_{\mathcal{O}}$ is an underlying cofibration (respectively monomorphism) in Sym^G, then so is the pushout product

$$f_1 \square^{\circ_{\mathcal{O}}} f_2$$

with respect to $\cdot \circ_{\mathcal{O}} \cdot$. Further, this map is also a w.e. if either f_1 or f_2 are.

Proofs of Theorems 1.1, 1.2 and 1.3 can be found in subsection 4.

[1] Note that for $(\mathsf{Sp}^{\Sigma})^G$ this is just the G genuine model structure.

[2] Note that we do not equip $\mathsf{Mod}^R_{\mathcal{O}}$ with a model structure in this statement.

2 G-graph model structures

2.1 Definitions

HERE

We first set some notation. Throughout the paper we let S denote the category of simplicial sets.

In this paper we will be interested in spaces that are acted on simultaneously by two groups G and H with differing roles. This is reflected in the notion of G-graph equivalence, which we now define.

GGRAPHSTABLE DEF

Definition 2.1. Let $\mathsf{S}^{G\times H}$ be the category of $G\times H$-simplicial sets. A *G-graph stable equivalence* is a map $X\to Y$ such that $X^W\to Y^W$ is a w.e. for any graph subgroup $W\subset G\times H$ associated to a homomorphism $\varphi\colon\bar{G}\to H$ with domain a subgroup $\bar{G}\subset G$.

EXISTMODELSTR THM

Theorem 2.2. *There exist a cofibrantly generated simplicial model category on $\mathsf{S}^{G\times H}$ with w.e.s the G-graph equivalences and cofibrations the monomorphisms.*

Proof. This is proven just as in [6]. □

2.2 Key properties of the G-graph model structures

We know list the key results concerning the G-graph model structures we will need. These are modeled after the results in [6, Section 4].

PROPINDUCE

Proposition 2.3. *Let $\bar{H}\subset H$ be finite groups, and suppose each category is equipped with the respective G-graph stable model structure. Then both adjunctions*

$$\mathsf{fgt}\colon\mathsf{S}^{G\times H}\rightleftarrows\mathsf{S}^{G\times\bar{H}}\colon((-)^{S\otimes H_+})^{\bar{H}},\qquad H\times_{\bar{H}}(-)\colon\mathsf{S}^{G\times\bar{H}}\rightleftarrows\mathsf{S}^{G\times H}\colon\mathsf{fgt}$$

are Quillen adjunctions.

Proof. In both cases it is clear that the left adjoints preserve cofibrations. Further, fgt preserves all G-graph equivalences since the graph subgroups of $G\times\bar{H}$ are a subset of those in $G\times H$.

To deal with $H\times_{\bar{H}}(-)$, one fixes a graph subgroup W associated to a homomorphism $\varphi\colon\bar{G}\to H$ and notes that for any map $f\colon A\to B$ one has a decomposition of the W action as

$$H\times_{\bar{H}}f\cong\bigvee_{H/\bar{H}}f\cong\bigvee_{i\in W\backslash H/\bar{H}}W\times_{W_i}\varphi_{h_i}^{*}f \tag{2.4}$$

decomp eq

where W_i, the intersection of W with the isotropy $G\times h_i\bar{H}h_i^{-1}$ of the $h_i\bar{H}$ summand in the intermediate decomposition, is a graph subgroup associated to a homomorphism $\varphi_i\colon\bar{\bar{G}}_i\to h_i\bar{H}h_i^{-1}$ (and where $\varphi_{h_i}^{*}$ denotes the pullback of the action along the conjugation isomorphism $\varphi_{h_i}\colon h_i\bar{H}h_i^{-1}\to H$). It is now clear that the W-fixed points are weak equivalences whenever f is a graph equivalence. □

biquillen thm

Theorem 2.5. *Consider the functor*

$$\mathsf{S}^{G\times H\times T}\times\mathsf{S}^{G\times\hat{H}\times T}\xrightarrow{\times_T}\mathsf{S}^{G\times H\times\hat{H}},$$

where the second category $\mathsf{S}^{G\times\bar{H}\times T}$ *is regarded as equipped with the* $G\times T$*-graph model structure. Then* $\times_T$ *is a left Quillen bifunctor if both* $\mathsf{S}^{G\times H\times T}$ *and the target* $\mathsf{S}^{G\times H\times\hat{H}}$ *are equipped with the respective* G*-graph model structures.*

Proof. First we note that by fixing the graph subgroup $W \subset G\times H\times\bar{H}$ to consider one immediately reduces to the case $H = \bar{H} = *$, i.e., one reduces to proving the result for the functor

$$\mathsf{S}^{G\times T}\times\mathsf{S}^{G\times T}\xrightarrow{\times_T}\mathsf{S}^G$$

when the second $\mathsf{S}^{G\times T}$ has the G-graph model structure and the remaining categories their genuine model structures. To see this, one first proves the analogue result when all model structures are genuine. Choosing generating cofibrations (resp. a generating cofibration and a generating trivial cofibration)

$$f = (G\times T)/H\cdot\bar{f}, \qquad g = (G\times T)/\bar{H}\cdot\bar{g}$$

where f, f' are generating cofibrations in S (resp. a generating cofibration and a generating trivial cofibration) one has

$$f\square^{\times_T}g = \big((G\times T\times G\times T)/(H\times\bar{H})\cdot\bar{f}\,\square^{\times}\,\bar{g}\big)_T = T\backslash(G{\times}T{\times}G{\times}T)/(H{\times}\bar{H}){\cdot}\bar{f}\square^{\times}\bar{g},$$

which is clearly a genuine G-cofibration (resp. genuine trivial G-cofibration).

To obtain the desired result, we now apply the localizing criterion provided by Pe14 [6, ~~blabla~~]. I.e., we need to show that for any spectrum X, the map

$$(G\times T\times_W EW\times X)_T\to((G\times T)/W\times X)_T$$

is a genuine equivalence. But since H is a graph subgroup one has that T acts freely on wedge summands, so that the previous map is identified with

$$G\times_{\bar{G}}\varphi^*(EW\times X)\to G\times_{\bar{G}}\varphi^*(X)$$

where φ^* denotes the action pulled back along the isomorphism $\varphi\colon\bar{G}\to W$, and the result is now clear. □

SIGMANPUSHPROD THM **Theorem 2.6.** *For* $f\colon A\to B$ *a trivial cofibration in* S^H *in the genuine model structure, the maps*

$$f^{\wedge n}A^{\wedge n}\to B^{\wedge n}, \qquad f^{\square n}\colon Q^n_{n-1}(f)\to B^n$$

are trivial cofibrations in $\mathsf{S}^{\Sigma_n\wr H}$ *for the genuine model structure.*

Proof. Let $W\subset\mathsf{S}^{\Sigma_n\wr H}$ be any subgroup. Considering the induced action of W on $\underline{n}$ via the projection $\Sigma_n\wr H\to\Sigma_n$, one has a decomposition

$$X^{\wedge n}\cong\prod_{i\in W\backslash\underline{n}}N^W_{W_i}(X) \tag{2.7}$$ THISISAN EQ

where W_i is the isotropy of the i-th factor and $N^W_{W_i}$ represents the norm construction. It now follows that $(X^{\wedge n})^W\simeq\prod_{i\in W\backslash\underline{n}}X^{W_i}$, and the first part of the result is now clear. The second part follows by an easy filtration argument. □

SIGMANPUSHPROD THM2

Theorem 2.8. *For $f\colon A \to B$ a trivial cofibration in $\mathsf{S}^{G\times H}$ in the G-graph model structure, the maps*

$$f^{\wedge n} A^{\wedge n} \to B^{\wedge n}, \qquad f^{\square n}\colon Q^n_{n-1}(f) \to B^n$$

are trivial cofibrations in $\mathsf{S}^{G\times(\Sigma_n \wr H)}$ for the G-graph model structure.

HERE

Proof. This follows exactly as in the proof of Theorem 2.6 by fixing a graph subgroup W corresponding to $\phi\colon G \to \Sigma_r \wr H$, and noting that in (2.7) the appearing W_i are now graph subgroups. □

3 Cofibrancy of operadic constructions

aplications sec

opdefinitions sec

3.1 Terminology: operads, modules and algebras

We now recall some notation and terminology concerning operads. We refer to [6] for the full definitions.

Definition 3.1. Let $(\mathcal{C}, \otimes, \mathbb{1})$ denote a closed symmetric monoidal category.

Then the category $\mathsf{Sym}(\mathcal{C})$ of *symmetric sequences in* $\mathcal{C}$ is the category of functors $\Sigma \to \mathcal{C}$.

$\mathsf{Sym}(\mathcal{C})$ can be equipped with two usual monoidal structures.

twomon def

Definition 3.2. Given $X, Y \in \mathsf{Sym}(\mathcal{C})$ we define their *tensor product* to be

$$(X \check{\otimes} Y)(r) = \bigvee_{0 \le \bar{r} \le r} \Sigma_r \times_{\Sigma_{\bar{r}} \times \Sigma_{r-\bar{r}}} X(\bar{r}) \otimes Y(r - \bar{r})$$

and their *composition product* to be

$$(X \circ Y)(r) = \bigvee_{\bar{r} \ge 0} X(\bar{r}) \otimes_{\Sigma_{\bar{r}}} (Y^{\check{\otimes}\bar{r}})(r). \tag{3.3}$$

circ def

Definition 3.4. An *operad* $\mathcal{O}$ in $\mathcal{C}$ is a monoid object in $\mathsf{Sym}(\mathcal{C})$ with respect to $\circ$, i.e., a symmetric sequence $\mathcal{O}$ together with multiplication and unit maps

$$\mathcal{O} \circ \mathcal{O} \to \mathcal{O}, \qquad I \to \mathcal{O}$$

satisfying the usual associativity and unit conditions.

opermod def

Definition 3.5. Let $\mathcal{O}$ be an operad in $\mathcal{C}$. A *left module* L (resp. *right module* R) over $\mathcal{O}$ is an object in $\mathsf{Sym}(\mathcal{C})$ together with a map

$$\mathcal{O} \circ L \to L \qquad (\text{resp. } R \circ \mathcal{O} \to R)$$

which satisfies the usual associativity and unit conditions. The category of left modules (resp right modules) over $\mathcal{O}$ is denoted $\mathsf{Mod}^L_{\mathcal{O}}$ (resp. $\mathsf{Mod}^R_{\mathcal{O}}$).

Further, left modules X over $\mathcal{O}$ concentrated[3] in degree 0 are called *algebras* over $\mathcal{O}$. The category of algebras over $\mathcal{O}$ is denoted $\mathsf{Alg}_{\mathcal{O}}$.

Definition 3.6. The category $\mathsf{BSym}(\mathcal{C})$ of *bi-symmetric sequences in* $\mathcal{C}$ is the category $\mathsf{Sym}(\mathsf{Sym}(\mathcal{C}))$ of symmetric sequences of symmetric sequences in $\mathcal{C}$.

Following Definition 3.2 one can hence build two monoidal structures on $\mathsf{BSym}(\mathcal{C})$ which we denote, respectively, by $\check{\check{\otimes}}$ and $\check{\circ}^{\mathsf{s}}$, where $\check{\circ}^{\mathsf{s}}$ is built using s (the "second" index) as the "operadic" index. Note that while $\check{\check{\otimes}}$ behaves symmetrically with respect to the two indexes r and s, $\check{\circ}^{\mathsf{s}}$ does not.

[3] I.e. such that $X(r) = \varnothing$ for $r \ge 1$.

3.2 Model structures on Sym^G

possym sec

Throughout the remainder of the paper we shall abbreviate $\mathsf{Sym}((\mathsf{Sp}^\Sigma)^G)$ simply as Sym^G.

We now introduce the key model structures on Sym^G that we will need.

monsym def

Definition 3.7. The *monomorphism G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is obtained by combining the monomorphism G-graph stable model structures in $(\mathsf{Sp}^\Sigma)^{G\times H\times\Sigma_r}$ in all degrees.

ssym def

Definition 3.8. The *S G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is obtained by combining the S G-graph stable model structures in $(\mathsf{Sp}^\Sigma)^{G\times H\times\Sigma_r}$ in all degrees.

possym def

Definition 3.9. The *positive S G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is the model structure obtained by combining the positive S G-graph stable model structure in $(\mathsf{Sp}^\Sigma)^{G\times H}$ on degree $r=0$ with the S G-graph stable model structures in $(\mathsf{Sp}^\Sigma)^{G\times H\times\Sigma_r}$ in degrees $r\geq 1$.

subtlecof rmk sym

Remark 3.10. By analogy to Remark ??, we will also use briefly S $\Sigma\times\Sigma\times G\times H$-inj T-proj cofibrations in $\mathsf{Sym}^{G\times H\times T}$, which are those maps which are underlying S $\Sigma\times\Sigma$-inj T-proj cofibrations in the sense of [6, Sec. 5.3] after forgetting the $G\times H$ action.

These cofibrations are necessary for the formulation of Proposition 3.15 bellow.

Propositions ?? and Theorems 2.5 and ?? directly imply analogous results in the context of symmetric sequences, which we now state.

propinduce sym

Proposition 3.11. *Let $\bar{H}\subset H$ be finite groups, and suppose each category is equipped with the respective S G-graph stable model structure. Then both adjuntions*

$$res^{G\times H}_{G\times\bar{H}}\colon \mathsf{Sym}^{G\times H} \rightleftarrows \mathsf{Sym}^{G\times\bar{H}}\colon ((\cdot)^{S\otimes H_+})^{\bar{H}}$$

$$H\times_{\bar{H}}\cdot\colon \mathsf{Sym}^{G\times\bar{H}} \rightleftarrows \mathsf{Sym}^{G\times H}\colon res^{G\times H}_{G\times\bar{H}}$$

are Quillen adjunctions.

Proof. This follows by applying Proposition ?? at each level. ☐

sigmanpushprod thm sym

Proposition 3.12. *For $f\colon A\to B$ a trivial cofibration in $\mathsf{Sym}^{G\times H}$ for the S G-graph stable model structure the map*

$$f^{\square\check{\wedge}n}\colon Q^n_{n-1}(f)\to B^n$$

is a trivial cofibration in $\mathsf{Sym}^{G\times(\Sigma_n\wr H)}$ for the S G-graph stable model structure.

Proof. Computing $X_1\check{\wedge}\cdots\check{\wedge}X_n$ iteratively and regrouping terms we get

$$(X_1\check{\wedge}\cdots\check{\wedge}X_n)(r) = \bigvee_{\phi:\underline{r}\to\underline{n}} X_1(\phi^{-1}(1))\wedge\cdots\wedge X_n(\phi^{-1}(n)).$$

The $\Sigma_n\times\Sigma_r$ action[4] interchanges these wedge summands via post- and precomposition with ϕ. It is hence sufficient to prove the result for the subfunctor

[4]Note that since we are not yet assuming all X_i are the same the Σ_n action actually corresponds to the structure symmetric monoidal structure isomorphisms of $\check{\wedge}$.

formed by the wedge summands corresponding to the $\Sigma_n \times \Sigma_r$ orbit of a single ϕ, and w.l.o.g. we can decompose $\underline{n} = \underline{n_0} \amalg \underline{n_1} \amalg \cdots \amalg \underline{n_k}$ where $\#\phi^{-1}(*) = i$ for any element $* \in \underline{n_i}$. It now follows that the $\Sigma_n \times \Sigma_r$ isotropy of ϕ is the subgroup $\Sigma_{n_0} \wr \Sigma_0 \times \cdots \times \Sigma_{n_k} \wr \Sigma_k$.

Now letting $f\colon A \to B$ be a trivial cofibration in $\mathsf{Sym}^{G\times H}$, we conclude that the summand of $f^{\square n}$ corresponding to the ϕ subfunctor is isomorphic to

$$\Sigma_n \times \Sigma_r \times_{\Sigma_{n_0}\wr\Sigma_0\times\cdots\times\Sigma_{n_k}\wr\Sigma_k} f(0)^{\square n_0} \square f(1)^{\square n_1} \square \cdots \square f(k)^{\square n_k}. \qquad (3.13)$$ sigmankind eq

By Theorem ?? each $f(i)^{\square n_i}$ is a S G-graph stable trivial cofibration, and by Theorem 2.5 (when $T = *$) so is $f(0)^{\square n_0} \square \cdots \square f(k)^{\square n_k}$. The proof is now complete by applying Proposition ?? (while noting that the induction from $\Sigma_{n_0} \wr \Sigma_0 \times \cdots \times \Sigma_{n_k} \wr \Sigma_k$ to $\Sigma_n \times \Sigma_r$ in (3.13) is in fact an induction from $G \times \Sigma_{n_0} \wr (\Sigma_0 \times H) \times \cdots \times \Sigma_{n_k} \wr (\Sigma_k \times H)$ to $G \times (\Sigma_n \wr H) \times \Sigma_r$).

□

combine rmk sym **Remark 3.14.** Just as in Remark ??, Proposition 3.12 needs to be combined with [6, Prop. 5.35] so as to obtain the "lax Σ_n-projective cofibrancy" conditions needed for one to apply Proposition 3.15 bellow.

biquillen thm sym **Proposition 3.15.** *Consider the functor*

$$\mathsf{Sym}^{G\times H\times T} \times \mathsf{Sym}^{G\times \bar{H}\times T} \xrightarrow{\cdot \check{\wedge}_T \cdot} \mathsf{Sym}^{G\times H\times \bar{H}},$$

where the first category $\mathsf{Sym}^{G\times H\times T}$ *is regarded as equipped with the* S $\Sigma \times \Sigma \times G \times H$*-inj* T*-proj* G*-graph stable model structure (see Remark 3.10).*

Then $\cdot\check{\wedge}_T\cdot$ *is a left Quillen bifunctor if either:*

(a) Both $\mathsf{Sym}^{G\times \bar{H}\times T}$ *and the target* $\mathsf{Sym}^{G\times H\times \bar{H}}$ *are equipped with the respective monomorphism* G*-graph stable model structures;*

(b) Both $\mathsf{Sym}^{G\times \bar{H}\times T}$ *and the target* $\mathsf{Sym}^{G\times H\times \bar{H}}$ *are equipped with the respective* S G*-graph stable model structures.*

Proof. First recall that

$$(X \check{\otimes} Y)(r) = \bigvee_{0\leq \bar{r}\leq r} \Sigma_r \times_{\Sigma_{\bar{r}}\times\Sigma_{r-\bar{r}}} X(\bar{r}) \wedge Y(r-\bar{r}).$$

Applying the second part of Proposition ?? we reduce to showing that pushout products for the bifunctors $(X, Y) \mapsto X(\bar{r}) \wedge_T Y(r - \bar{r})$ are left biQuillen, and that follows by applying Theorem 2.5 in the case

$$(\mathsf{Sp}^\Sigma)^{G\times H\times \Sigma_{\bar{r}}\times T} \times (\mathsf{Sp}^\Sigma)^{G\times \bar{H}\times\Sigma_{r-\bar{r}}\times T} \xrightarrow{\cdot\wedge_T\cdot} (\mathsf{Sp}^\Sigma)^{G\times H\times\bar{H}\times\Sigma_r\times\Sigma_{\bar{r}}}.$$

□

bsym rmk **Remark 3.16.** All of this subsection immediately generalizes to the category $\mathsf{BSym}^{G\times H} = \mathsf{Sym}(\mathsf{Sym}^{G\times H})$.

Indeed, one can define monomorphism G-graph stable, S G-graph stable and positive S G-graph stable model structures in $\mathsf{BSym}^{G\times H} = \mathsf{Sym}(\mathsf{Sym}^{G\times H})$ and S $\Sigma\times\Sigma\times\Sigma\times G\times H$-inj T-proj G-graph stable model structures in $\mathsf{BSym}^{G\times H\times T}$ by simply repeating Definitions 3.7, 3.8, 3.9 and Remark 3.10 except replacing the initial model structures in $(\mathsf{Sp}^\Sigma)^{G\times H}$ by their analogues in $\mathsf{Sym}^{G\times H}$.

Analyzing the proofs of 3.11, 3.15 and 3.12 it is then clear that those results themselves imply their $\mathsf{BSym}^{G\times H}$ versions as well.

4 Proofs of the main theorems

Proof of Theorem 1.1. To prove the first part asserting the existence of the model structures it suffices, by [?, Lem. 2.3], to show that, for J a set of generating trivial cofibrations in $\mathsf{Sp}^{\Sigma}/\mathsf{Sym}$, then any transfinite composition of pushouts of maps in $\mathcal{O} \circ J$ is a w.e.. Since the proof of Theorem 1.3 follows precisely by using such a decomposition of f_2, the result then follows from that proof by setting $f_1 = * \to \mathcal{O}$ and noting that $* = *_A \to \mathcal{O}_A$ is automatically a monomorphism even if not assuming a cofibrancy condition on A.

For the second part concerning Quillen equivalences we see, after unpacking definitions, that it suffices to show that the unit of the adjunctions

$$A \to \bar{\mathcal{O}} \circ_{\mathcal{O}} A$$

are w.e.s whenever A is cofibrant. Applying Theorem 1.3 we see that any functor of the form $\cdot \circ A$ preserves all monomorphisms which are w.e.s (note that this uses as f_2 the map $\mathcal{O}(0) \to A$, so one must use the fact that $\mathcal{O}(0) \simeq \mathcal{O} \circ_{\mathcal{O}} \mathcal{O}(0) \to \bar{\mathcal{O}} \circ_{\mathcal{O}} \mathcal{O}(0) \simeq \bar{\mathcal{O}}(0)$ is a w.e.), and hence by Ken Brown's lemma combined with Theorem A.6 it in fact preserves all w.e.s, finishing the proof.

□

Proof of Theorem 1.2. The first claim, that the functors $A \wedge_{\Sigma_n} (\cdot)^{\wedge n}$ and $B \wedge_{\Sigma_n} (\cdot)^{\wedge n}$ are left derivable, follows by Ken Brown's Lemma and Theorem 1.3 applied to the unit operad I, since it then follows that those functors send trivial cofibrations between cofibrant objects in the positive S G-stable model structure to G-stable equivalences.

For the second claim, the "if" part again follows by Ken Brown's Lemma combined with Theorem 1.3 applied to the unit operad I, since it then follows that, for positive S cofibrant X, the functor $\cdot \wedge_{\Sigma_n} X^{\wedge n}$ sends trivial cofibrations between cofibrant objects in the monomorphism G-graph stable model structure to G-stable equivalences.

For the "only if" claim, fix a G-graph subgroup W associated to $\varphi\colon \bar{G} \to \Sigma_n$. Now let $\bar{S}$ denote a positive cofibrant replacement for the sphere S and consider the positive S cofibrant spectrum

$$X = G \times_{\bar{G}} \underline{n} \times \bar{S},$$

where $\bar{G}$ acts on $\underline{n}$ via φ. Computing $X^{\wedge n}$ one obtains a natural decomposition

$$X^{\wedge n} = \bigvee_{G^n \times_{\bar{G}^n} \underline{n}^n} \bar{S}^{\wedge n}$$

such that the $\Sigma_n \wr G$ action, and hence also the diagonal $\Sigma_n \times G$ action, interchanges wedge summands.

Now recall that by hypothesis

$$A \wedge_{\Sigma_n} X^{\wedge n} \to B \wedge_{\Sigma_n} X^{\wedge n}$$

is a G-stable equivalence, and hence so must be each of its summands corresponding to a $\Sigma_n \times G$ orbit of $G^n \times_{\bar{G}^n} \underline{n}^n$. In particular, this must be the case for the (diagonal) orbit $G \times_{\bar{G}} \Sigma_n \subset G^n \times_{\bar{G}^n} \underline{n}^n$, whose summand is then

$$A \wedge_{\Sigma_n} (G \times_{\bar{G}} \times \Sigma_n \times \bar{S}^{\wedge n}) \to B \wedge_{\Sigma_n} (G \times_{\bar{G}} \times \Sigma_n \times \bar{S}^{\wedge n})$$

and using the freeness of the Σ_n action this is in turn identified with

$$G \times_{\bar{G}} \varphi^*(A) \wedge \bar{S}^{\wedge n} \to G \times_{\bar{G}} \varphi^*(B) \wedge \bar{S}^{\wedge n},$$

where $\varphi^*(A), \varphi^*(B)$ denote A, B with the $\bar{G}$ actions obtained by pulling back the W action. Forgetting the full G action to a $\bar{G}$ action and focusing on the identity summand one sees that $\varphi^*(A) \to \varphi^*(B)$ must be a $\bar{G}$-stable equivalence, completing the proof.

□

Proof of Theorem 1.3. The result to be shown is an equivariant analogue of [?, Thm. 1.5] and it follows by repeating the same exact proof as in [?] except now using the G-graph stable results developed in this paper at certain key points. Rather than repeating the full proof, we list only the specific steps where the new results are used.

The case of regular (i.e. non trivial) cofibrations immediately reduces to [?, Thm. 1.5].

For the case where one of the cofibrations is trivial, assume first for simplicity (as in the proof in [?]) that f_2 is a map of algebras.

If f_2 is the trivial cofibration, the argument in [?] reduces to the case where f_2 is the pushout of a generating trivial cofibration $f\colon \mathcal{O} \circ X \to \mathcal{O} \circ Y$ and to checking that the "pushout corner map" in

$$\begin{array}{ccc} M_A(r) \wedge_{\Sigma_r} Q^r_{r-1} & \longrightarrow & M_A(r) \wedge_{\Sigma_r} Y^r \\ \downarrow & & \downarrow \\ N_A(r) \wedge_{\Sigma_r} Q^r_{r-1} & \longrightarrow & N_A(r) \wedge_{\Sigma_r} Y^r \end{array} \tag{4.1}$$

bla

is a G-graph stable equivalence. This follows by Theorems ??, 2.5 and Remark ??.

If instead f_1 is the trivial cofibration then it suffices to show that, in (4.1), $M_A \to N_A$ is a G-graph equivalence in Sym^G, and following along the proof in [?], namely the inductive argument along $\beta \leq \kappa$, it suffices for the "pushout corner map" in the diagram

$$\begin{array}{ccc} M_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Q^s_{s-1,\beta} & \longrightarrow & M_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Y^s_\beta \\ \downarrow & & \downarrow \\ N_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Q^s_{s-1,\beta} & \longrightarrow & N_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Y^s_\beta \end{array}$$

to be a G-graph stable equivalence in Sym^G. Hence by Propositions 3.12, 3.15 and Remark 3.14 it suffices to show that $M_{\mathcal{O} \amalg A_\beta}(r,s) \to N_{\mathcal{O} \amalg A_\beta}(r,s)$ is a G-graph stable equivalence in $(\mathsf{Sp}^\Sigma)^{G \times \Sigma_r \times \Sigma_s}$. But by [?, Prop. 5.24] this map can be identified $M_{A_\beta}(r+s) \to N_{A_\beta}(r+s)$, which is a G-graph stable equivalence in $(\mathsf{Sp}^\Sigma)^{G \times \Sigma_{r+s}}$ by induction on β and the result follows by Proposition 3.11.

In the more general case where f_2 is a general map of left modules one proceeds as described in the last paragraph of the proof in [?], except now using the bisymmetric sequence analogues of Theorems ?? and 2.5 described in Remark 3.16.

□

A Existence of model structures

existence app

proof of Theorem existmodelstr ??. In either the monomorphism or the S cofibration case one starts by building a levelwise G-graph model category structure, i.e. a model structure where a map $X \to Y$ of $G \times H$-spectra is a w.e. if all maps

$$X_n^W \to Y_n^W$$

are w.e.s of simplicial sets for all n and graph subgroup W associated to a morphism $\varphi: \bar{G} \to H \times \Sigma_n$. In the S cofibration case the existence of such a model structure follows either from Hausmann2014 [?, Prop. 2.30] or by repeating the proof of Pereira2014 [?, Thm. 3.5]. For the monomorphism case, one instead repeats the arguments in Pereira2014 [?, App. A]. The only non trivial part of this is to show the analogue of Pereira2014 [?, Lem. A.2], itself just an adaptation of HSS [4, Lem. 5.1.7], and again the the proof of HSS [4] can be adapted directly by making sure that the spectrum FC built in the proof kills off relative homotopy groups of fixed points with respect to all G-graph subgroups W.

To produce the G-graph stable model structure, consider the commuting square of left Bousfield localizations

$$\begin{tikzcd}
((\mathsf{Sp}^\Sigma)^{G\times H})_{G\times H-\mathrm{lv}} \arrow[r] \arrow[d] & ((\mathsf{Sp}^\Sigma)^{G\times H})_{G\times H-\mathrm{st}} \arrow[d] \\
((\mathsf{Sp}^\Sigma)^{G\times H})_{G\text{-gr-lv}} \arrow[r] & ((\mathsf{Sp}^\Sigma)^{G\times H})_{G\text{-gr-st}}
\end{tikzcd} \tag{A.1}$$

leftbousfield diag

where the horizontal arrows localize the maps

$$G \times H \ltimes_W (F_{U \amalg V} S^V \to F_U S^0) \tag{A.2}$$

Gstable loc

for U, V representations of $W \subset H \times W$ and the vertical maps localize the maps[5]

$$F_n(E\mathcal{F}_n)_+ \to F_n S^0 \tag{A.3}$$

Ggraphfamily loc

for $\mathcal{F}_n$ the family of G-graph subgroups of $G \times H \times \Sigma_n$.

We claim that the w.e.s in the bottom right corner are precisely the G-graph stable equivalences of Definition ggraphstable def ??. The latter contains the former since $res_W^{G\times H}$ is both a left and right Quillen functor for the genuine model structures Hausmann2014 [[?], 6.2] and the maps in (Ggraphfamily loc A.3) restrict to w.e.s when W is a graph subgroup. For the converse, it now suffices to check that the two classes of equivalences coincide when restricted to fibrant objects in the bottom right corner model structure. These are precisely those fibrant Ω-spectra X such that $X_n^W \simeq *$ for all n and W any non G graph subgroup of $G \times H \times \Sigma_n$. The first notion of w.e. then says that a map of Ω-spectra $X \to Y$ is a w.e. if

$$X_n^W \to Y_n^W$$

is a w.e. of simplicial sets for all subgroups W, while the second demands that condition only for W a G-graph subgroup, but these conditions are clearly the same.

□

[5] Notational note: $F_n(E\mathcal{F}_n)_+$ is actually a semi-free spectrum.

reducedlocal rmk

Remark A.4. In the proof of Theorem 2.5 below we will use the fact that in the lower horizontal localization in diagram (A.1) it in fact suffices to localize those maps of the form

$$\lambda_{U,V}^{(W)}: G \times H \ltimes_W (F_{U \amalg V} S^V \to F_U S^0) \tag{A.5}$$

blabla loc

for U, V representations of a G**-graph subgroup** $W \subset G \times H$.

To see this is the case it suffices to show that localizing with respect to this smaller set produces the same local objects. Now note that any G-graph level fibrant object that is local with respect to the maps in (A.5) becomes W-genuine fibrant after applying $res_W^{G\times H}$, and hence level equivalences between such objects are the same as G-graph stable equivalences. The desired conclusion follows.

oinjmodel thm

Theorem A.6. *Let $\mathcal{O}$ be any operad in $(\mathsf{Sp}^\Sigma)^{G\times H}$.*

There exists a cofibrantly generated model structure on $\mathsf{Mod}_{\mathcal{O}}^R$, which we call the **monomorphism G-graph stable model structure***, such that*

- *cofibrations are the maps $X \to Y$ such that $X_n \to Y_n$ is a monomorphism of pointed simplicial sets for each $n \geq 0$.*
- *weak equivalences are the maps are the maps $X \to Y$ which are underlying G-graph stable equivalences of spectra.*

Further, this is a left proper cellular simplicial model category.

Proof. This result is an analogue of [?, Thm. A.4] and essentially the same proof applies. The only differences worth mentioning are that in the levelwise stage of the proof one uses levelwise G-graph equivalences as defined in the proof of Theorem ?? above and that when proving the analogue of [?, Lem. A.2] one needs the spectrum FC to kill relative homotopy groups with respect to all G-graph subgroups W.

□

proof of Theorem 2.5. Since cofibrancy is defined by forgetting structure, the case of regular cofibrations reduces to that of [?, Thm. 1.7], so that we need only worry about the case involving trivial cofibrations. It hence suffices to prove part (a).

Further, by fixing a graph subgroup W associated to a homomorphism $\varphi: \bar{G} \to H \times \hat{H}$ we further reduce to the case $H = \hat{H} = *$.

The remainder of the proof closely follows that of [?, Thm . 1.7], by first proving a level G-graph equivalence version of the result (see the proof of Theorem ??), then showing the left biQuillen functor localizes to stabilizations.

For the level structure result, the fact that all generating (trivial) cofibrations in the $\Sigma \times G$-inj H-proj level model structure have the form $S \otimes f$ (cf. (??)) one reduces to the analogous statement for the bifunctor

$$\mathsf{S}_*^{G\times T\times\Sigma} \times (\mathsf{Sp}^\Sigma)^{G\times T} \xrightarrow{\otimes_T} (\mathsf{Sp}^\Sigma)^G.$$

For $A_m \xrightarrow{i} B_m$ a $\Sigma \times G$-inj H-proj cofibration in $\mathsf{S}_*^{G\times T\times\Sigma}$ and $C \xrightarrow{f} D$ any monomorphism one then has

$$(i \square^{\otimes} f)_{\bar{m}} = \Sigma_{\bar{m}} \times_{\Sigma_m \times \Sigma_{\bar{m}-m}} i \square^{\wedge} f_{\bar{m}-m}.$$

We need to check this is a levelwise G-graph equivalence if either i or all f_m are. Fixing the graph subgroup W associated to $\varphi\colon \bar{G} \to \Sigma_{\bar{m}}$ one sees that the domains and codomains of the fixed point map $((i \square^{\otimes} f)_{\bar{m}})^W$ can be non trivial only if φ factors through $\Sigma_m \times \Sigma_{\bar{m}-m}$ up to conjugation by some $\sigma \in \Sigma_{\bar{m}}$. It then follows that one can forget the $\Sigma_m, \Sigma_{\bar{m}-m}$ actions, so that the claim follows from the Quillen biadjunction $\mathsf{S}_*^{G\times T} \times \mathsf{S}_*^{G\times T} \xrightarrow{\cdot\wedge_T\cdot} \mathsf{S}_*^G$ where one of the $\mathsf{S}_*^{G\times T}$ has the projective G-graph projective model structure and the other categories their respective (projective) G-genuine model structures.

To localize the level structure result to the desired stable version, we repeat the argument in the last paragraph of the proof of [?, Thm. 1.7] to conclude that, by [?, Cor. B.4] and Remark A.4 above, it suffices to check that

$$X \wedge_T (G \times T \ltimes_W \lambda_{U,V}^{(W)})$$

is a G-stable equivalence for U, V representations of any G-graph subgroup W associated to $\varphi\colon \bar{G} \to H$. But since $W \cap T = \{*\}$ the map above can be rewritten G-equivariantly as (abusing notation by writing φ for the isomorphism $\bar{G} \xrightarrow{\cong} W$)

$$G \ltimes_{\bar{G}} \varphi^*(X \wedge \lambda_{U,V}^{(W)})$$

so that the result now follows by [?, Prop. 7.1] together with [?, Sec. 6.2]. □

B Fixed points of powers of G-spaces

HERE

Proposition B.1. *Let $X \in \mathsf{S}_*$. Then the possile Σ_n-isotropies of $X^{\times n}$ are the partition subgroups*

$$\Sigma_{\pi^{-1}(1)} \times \cdots \times \Sigma_{\pi^{-1}(r)}$$

associated to surjections $\underline{n} \twoheadrightarrow \underline{r}$.

Proof. Obvious. □

Proposition B.2. *Let $X \in \mathsf{S}_*^G$. Then the possible $\Sigma_n \wr G$-isotropies of $X^{\times n}$ are the partition described by the following information:*

Proposition B.3. *Given groups $H \leq G$ with $|G/H| = n$, a choice of representatives $g_1, \cdots, g_n$ for G/H induces a factorization*

$$G \xrightarrow{\phi} \Sigma_{|G/H|} \ltimes H^{\times |G/H|} \to \Sigma_{|H||G/H|} = \Sigma_n$$

or, equivalently,

$$G \xrightarrow{\phi} \Sigma_{G/H} \ltimes \Pi_i (g_i H g_i^{-1}) \to \Sigma_G.$$

Proof. Throughout we let $G \xrightarrow{\sigma_{(-)}} \Sigma_{|G/H|} \simeq \Sigma_{G/H}$ denote the left action of G on G/H.

Using the first notation for ϕ, we define

$$\phi(x) = (\sigma_x, (g_i^{-1} x g_{\sigma_x^{-1}(i)})).$$

To check $g_i^{-1}xg_{\sigma_x^{-1}(i)} \in H$ we compute

$$\sigma_{g_i^{-1}xg_{\sigma_x^{-1}(i)}}(1) = \sigma_{g_i^{-1}}\sigma_x\sigma_{g_{\sigma_x^{-1}(i)}}(1) = \sigma_{g_i^{-1}}\sigma_x\sigma_x^{-1}(i) = \sigma_{g_i^{-1}}(i) = 1.$$

To check that this defines a homomorphism we compute

$$\begin{aligned}\phi(x)\phi(y) =&(\sigma_x,(g_i^{-1}xg_{\sigma_x^{-1}(i)}))(\sigma_y,(g_i^{-1}yg_{\sigma_y^{-1}(i)})) = \\ =&(\sigma_x\sigma_y,(g_i^{-1}xg_{\sigma_x^{-1}(i)})\sigma_x(g_i^{-1}yg_{\sigma_y^{-1}(i)})) = \\ =&(\sigma_x\sigma_y,(g_i^{-1}xg_{\sigma_x^{-1}(i)})(g_{\sigma_x^{-1}(i)}^{-1}yg_{\sigma_y^{-1}(\sigma_x^{-1}(i))})) = \\ =&(\sigma_{xy},(g_ixyg_{\sigma_{xy}^{-1}(i)})).\end{aligned}$$

Using the second notation for ϕ, we define the action of $\Sigma_{G/H}$ on $\Pi_i(g_iHg_i^{-1})$ by

$$\sigma(g_ik_ig_i^{-1}) = (g_ik_{\sigma^{-1}(i)}g_i^{-1}) \qquad \text{or} \qquad \sigma(k_i) = (g_ig_{\sigma^{-1}(i)}^{-1}k_{\sigma^{-1}(i)}g_{\sigma^{-1}(i)}g_i^{-1}).$$

We now define

$$\phi(x) = (\sigma_x,(xg_{\sigma_x^{-1}(i)}g_i^{-1})).$$

To check that this defines a homomorphism we compute

$$\begin{aligned}\phi(x)\phi(y) =&(\sigma_x,(xg_{\sigma_x^{-1}(i)}g_i^{-1}))(\sigma_y,(yg_{\sigma_y^{-1}(i)}g_i^{-1})) = \\ =&(\sigma_x\sigma_y,(xg_{\sigma_x^{-1}(i)}g_i^{-1})\sigma_x(yg_{\sigma_y^{-1}(i)}g_i^{-1})) = \\ =&(\sigma_x\sigma_y,(xg_{\sigma_x^{-1}(i)}g_i^{-1})(g_ig_{\sigma_x^{-1}(i)}^{-1}yg_{\sigma_y^{-1}(\sigma_x^{-1}(i))}g_{\sigma_x^{-1}(i)}^{-1}g_{\sigma_x^{-1}(i)}g_i^{-1}) = \\ =&(\sigma_{xy},(xyg_{\sigma_{xy}^{-1}(i)}g_i^{-1})).\end{aligned}$$

□

bla bla bla

When given $\phi: G \to \Sigma_n$, then all points of

$$(G \times \Sigma_n/\Gamma_\phi)^{\times 2} = G^{\times 2} \times \Sigma_n^{\times 2}/\Gamma_\phi^{\times 2}$$

are in the image of $G \times \Sigma_2 \ltimes \Sigma_n^{\times 2}$ and there is hence an isomorphism

$$G \times \Sigma_2 \ltimes \Sigma_n^{\times 2}/\Sigma_2 \ltimes \Gamma_{\phi^{\times 2}} \to \Sigma_2 \ltimes (G^{\times 2} \times \Sigma_n^{\times 2})/\Sigma_2 \ltimes \Gamma_\phi^{\times 2}$$

and since $\Sigma_2 \ltimes \Gamma_{\phi^{\times 2}} = \Sigma_2 \times \Gamma_{\phi^{\times 2}}$ we conclude that this is the isotropy.

Suppose now given $\phi: H \to \Sigma_n$ with $[G:H] = 2$ and let g_1, g_2 be representatives for G/H. It is now only clear that, up to multiplication by $\Sigma_n^{\times 2}$, any point of $G^{\times 2} \times \Sigma_n^{\times 2}/\Gamma_\phi^{\times 2}$ can be reduced to either $(g_1,g_1),(g_1,g_2),(g_2,g_1),(g_2,g_2)$

bla bla bla

C Partitions

Definition C.1. Given a partition λ of $\underline{n}$, we denote its isotropy subgroup by $\Sigma_\lambda \subset \Sigma_n$, the corresponding normalizer $N(\Sigma_\lambda)$ by N_λ and the corresponding Weyl group N_λ/Σ_λ by W_λ.

Further, given partitions $\lambda \le \bar{\lambda}$, we denote the normalizer $N_{\Sigma_{\bar{\lambda}}}(\Sigma_\lambda)$ by $N_\lambda^{\bar{\lambda}}$ and the corresponding Weyl group $N_\lambda^{\bar{\lambda}}/\Sigma_\lambda$ by $W_\lambda^{\bar{\lambda}}$.

Definition C.2. A G-action $G \xrightarrow{\phi} \Sigma_n$ of G on $\underline{n}$ is called λ-transitive if $\underline{n}/G = \underline{n}/\lambda$.

Remark C.3. Note that if ϕ is λ transitive one has a natural factorization $G \to \Sigma_\lambda \le \Sigma_n$, and that the restriction of the G-action to each λ equivalence class is transitive.

Definition C.4. A *uniform partition* of a finite set $\underline{n}$ is a partition λ such that all equivalence classes have the same cardinality $k|n$.

Definition C.5. We say that $\bar{\lambda}$ is a *uniform refinement* of λ if $\bar{\lambda} \le \lambda$ and $\bar{\lambda}$ restricts to a uniform partition in each of the equivalence classes for λ.

Remark C.6. Given any partition λ there is a maximal partition λ^u such that $\lambda \le \lambda^u$ is a uniform partition.

Uniform refinements possess the following very convenient property.

Proposition C.7. *For any uniform refinement $\lambda \le \bar{\lambda}$ one has a natural isomorphism*

$$N_\lambda^{\bar{\lambda}} \simeq \Sigma_{\bar{\lambda}/\lambda}.$$

Proof. (DESHOU NE) $\square$

Definition C.8. An *array partition* of a finite set $\underline{n}$ is a pair of partitions (λ_x, λ_y) such that $\lambda_x \cap \lambda_y = \hat{0}$ and λ_x, λ_y are uniform partitions of $\lambda_x \cup \lambda_y$.

Remark C.9. In the definition above it is enough to demand that λ_x (or λ_y) is a uniform partition of $\lambda_x \cup \lambda_y$. (DESHOU NE)

D Power Isotropies

Proposition D.1. *Let $X \in \mathsf{S}$. Then the possible Σ_n-isotropies of $X^{\times n}$ are the partition subgroups Σ_λ associated to partitions λ of $\underline{n}$.*

Proposition D.2. *Let $X \in \mathsf{S}^G$ with isotropies in a family $\mathcal{F}$ closed under subgroups. Then the possible $G \times \Sigma_n$-isotropies of $X^{\times n}$ are described by the following information:*

- *a uniform refinement $\lambda \le \bar{\lambda}$;*
- *a $\bar{\lambda}/\lambda$ transitive action with $\mathcal{F}$ isotropies $H \xrightarrow{\phi} \Sigma_{\bar{\lambda}/\lambda}$.*

Further, the isotropy subgroup is then given by $\Gamma_\phi \ltimes \Sigma_\lambda$.

Proof. Note first that any tuple $\underline{x} = (x_i)_{i\in\underline{n}} \in X^{\times n}$ induces a "diagonal partition" $\lambda_{\underline{x}}$ corresponding to the entries of $\underline{x}$ that coincide.

Since G acts diagonally it is clear that if (g,σ) is in the isotropy of $\underline{x}$, then the action of g must preserve $\lambda_{\underline{x}}$. Indeed, the action of g on $(x_i)_{\underline{i\in n}/\lambda_{\underline{x}}} \in X^{\times n/\lambda_{\underline{x}}}$ must in fact lie in $\Sigma_{\lambda_{\underline{x}}^u/\lambda_{\underline{x}}}$. Indeed, we define $H = Isot((x_i)_{\underline{i\in n}/\lambda_{\underline{x}}}) \cap \Sigma_{\lambda_{\underline{x}}^u/\lambda_{\underline{x}}}$ and note that the image of H in $\Sigma_{\lambda_{\underline{x}}^u/\lambda_{\underline{x}}}$ must be contained in $\Sigma_{\bar{\lambda}/\lambda_{\underline{x}}}$ for a minimal subuniform refinement $\lambda_{\underline{x}} \le \bar{\lambda} \le \lambda_{\underline{x}}^u$. This matches the description above since minimality of $\bar{\lambda}$ ensures the action of H is $\bar{\lambda}$-transitive and the isotropies of the action $H \to \Sigma_{\bar{\lambda}}$ must be in $\mathcal{F}$ since they are subgroups of the G-isotropy subgroups of the coordinates x_i of $\underline{x}$.

We now show that all such isotropy subgroups can in fact arise. Given $\lambda \leq \bar{\lambda}$ and $H \xrightarrow{\phi} \Sigma_{\bar{\lambda}/\lambda}$, let $H_i \leq H$ be the ϕ-action isotropy for each $i \in \underline{n}/\bar{\lambda}$ and choose elements x_i of X, all lying in distinct G orbits, whose isotropy is precisely H_i. One example of a n-tuple in $X^{\times n}$ whose isotropy is precisely $\Gamma_\phi \ltimes \Sigma_\lambda$ is given by

$$\left(\left((hx_i)_{h\in H/H_i}\right)_{j\in(\underline{n}/\lambda)_i}\right)_{i\in\underline{n}/\bar{\lambda}}.$$

□

Example D.3. It is not enough to assume that the collection $\mathcal{F}$ is closed under conjugation and intersections.

For example, for $G = \Sigma_5$ let $\mathcal{F}$ be generated by $K = \Sigma_{\{1,2,3,4\}}$ under conjugation and intersections, so that $\mathcal{F}$ consists of the conjugates of the subgroups $\Sigma_{\{1,2,3,4\}}$, $\Sigma_{\{1,2,3\}}$, $\Sigma_{\{1,2\}}$, $\Sigma_{\{1\}}$.

Then the $G \times \Sigma_3$ isotropy subgroup of the point $(K, (345)K, (354)K) = (K, (53)K, (54)K)$ in $(\Sigma_5/K)^{\times 3}$ is the graph subgroup of the projection action $\Sigma_{\{1,2\}} \times \Sigma_{\{3,4,5\}} \to \Sigma_{\{3,4,5\}} \simeq \Sigma_3$. However, the isotropy of this action is $\Sigma_{\{12\}} \times \Sigma_{\{34\}}$, which is not in $\mathcal{F}$.

For a second example, still with $G = \Sigma_5$, let $\mathcal{F}$ be generated by $A = \Sigma_{1,2,3}$ and by $B =< (12345) >$. Then the $G \times \Sigma_3$ isotropy subgroup of the point $(A, B, (123)B, (132)B)$ in $(G/A \amalg G/B)^{\times 4}$ is the graph subgroup of the action $\mathbb{Z}_{/3} \subset \Sigma_3 \simeq \Sigma_{\{1\}} \times \Sigma_{\{2,3,4\}}$.

Lemma D.4. *Let $G \xrightarrow{\phi} \Sigma_r \wr T$ be a homomorphism such that $\pi\colon G \to \Sigma_r$ is a transitive action. Then, setting $H = \pi^{-1}(\Sigma_{\{2,\cdots,r\}})$ and up to conjugation by some $\underline{t} \in T^{\times n}$, one has a natural factorization of ϕ as*

$$G \to \Sigma_r \wr H \xrightarrow{\Sigma_t \wr \psi} \Sigma_r \wr T.$$

Proof. Since $\phi|_H$ factors through $(\Sigma_{\{1\}} \times \Sigma_{\{2,\cdots,r\}}) \wr T \simeq T \times \Sigma_{\{2,\cdots,r\}} \wr T$ one defines ψ by restricting ϕ and then projecting to the first T component.

Choose now $e, g_2, g_3, \cdots, g_r$ to be representatives of G/H and set $t_i = \pi_i^T(\phi(g_i))$. Setting $\underline{t} = (t_i)$ and computing

$$\pi_i^T\left(\underline{t}^{-1}\phi(g_i)\underline{t}\right) = \pi_i^T(\underline{t}^{-1})\pi_i^T(\phi(g_i))\pi_i^T(\sigma_{g_i}(\underline{t})) = \pi_i^T(\underline{t}) = e$$

we see that after conjugating by $\underline{t}^{-1}$ we are free to assume that $\pi_i^T(\phi(g_i)) = e$.

It now remains to verify that, under these assumptions, we have

$$\phi(x) = (\sigma_x, (\psi\left(g_i^{-1} x g_{\sigma_x^{-1}(i)}\right))),$$

or, equivalently,

$$\pi_i^T(\phi(x)) = \psi\left(g_i^{-1} x g_{\sigma_x^{-1}(i)}\right) =$$

But

$$\begin{aligned}\psi\left(g_i^{-1} x g_{\sigma_x^{-1}(i)}\right) &= n\pi_1^T\phi\left(g_i^{-1} x g_{\sigma_x^{-1}(i)}\right) = \pi_i^T\phi\left(x g_{\sigma_x^{-1}(i)}\right) = \pi_i^T\phi(x)\,\pi_i^T\phi\left(\sigma_x g_{\sigma_x^{-1}(i)}\right) = \\ &= \pi_i^T\phi(x)\,\pi_{\sigma_x^{-1}(i)}^T\phi\left(g_{\sigma_x^{-1}(i)}\right) = \pi_i^T\phi(x),\end{aligned}$$

finishing the proof. □

Remark D.5. Note that in the previous proof the $\underline{t}$ with which to conjugate was seen to be chosen in $T^{\{2,\cdots,r\}}$, and hence conjugation by it does not in fact alter ψ.

Example D.6. The need to conjugate by some $\underline{t}$ in the previous proposition can not be done away with. For example, for the homomorphism $\mathbb{Z}_{/6} \to \Sigma_2 \wr \Sigma_3$ given by $1 \mapsto ((12), ((12), (13)))$ it can easily be checked that since ψ factors through A_3 then $\Sigma_2 \wr \psi$ must factor through $\Sigma_2 \wr A_3$ for any choice of the representatives g_1, g_2.

Proposition D.7. *Let $X \in \mathsf{S}^{G \times T}$ with isotropies in a family $\mathcal{F}$ that is T-free and closed under subgroups. Then the possible $G \times \Sigma_n \wr T$-isotropies of $X^{\times n}$ are, up to conjugation by $T^{\times n}$, described by the following information:*

- *a uniform refinement $\lambda \leq \bar{\lambda}$;*
- *a $\bar{\lambda}/\lambda$ transitive action with $\mathcal{F}$ isotropies $H \xrightarrow{\phi} \Sigma_{\bar{\lambda}/\lambda} \wr T$.*

Further, the isotropy subgroup is then given by $\Gamma_\phi \ltimes \Sigma_\lambda$.

Proof. Note first that any tuple $\underline{x} = (x_i)_{i \in \underline{n}} \in X^{\times n}$ induces a "T-diagonal partition" $\lambda_{\underline{x},T}$ corresponding to the entries of $\underline{x}$ that differ by the action of some (unique) element in T. Hence, after conjugating by some $\underline{t} \in T^{\times n}$ (unique up to $\Delta(\lambda_{\underline{x},T})$), we can assume that $\underline{x}$ is a T-diagonal tuple, i.e., that its entries in each equivalence class of $\lambda_{\underline{x},T}$ in fact coincide, or, put another way, that $\lambda_{\underline{x}} = \lambda_{\underline{x},T}$

Now suppose $(g, \sigma, \underline{t})$ is in the isotropy of the T-diagonal tuple $\underline{x}$. The action of g is diagonal, it preserves both $\lambda_{\underline{x}}$ and $\lambda_{\underline{x},T}$; the action of σ conjugates both $\lambda_{\underline{x}}$ and $\lambda_{\underline{x},T}$; finally, the action of $\underline{t}$ tautologically preserves $\lambda_{\underline{x},T}$ and it preserves $\lambda_{\underline{x}}$ only if $\underline{t} \in \Delta(\lambda_{\underline{x},T})$. All together, we conclude that it must be $\underline{t} \in \Delta(\lambda_{\underline{x},T})$ and that $\sigma \in N_{\lambda_{\underline{x},T}}^{\lambda^u_{\underline{x},T}}$.

Since the $G \times \Sigma_{\lambda^u_{\underline{x},T}/\lambda_{\underline{x},T}} \wr T$-isotropy of $(x_i)_{\underline{i \in n}/\lambda_{\underline{x},T}} \in X^{\times n/\lambda_{\underline{x},T}}$ can not intersect $\Sigma_{\lambda^u_{\underline{x},T}/\lambda_{\underline{x},T}} \wr T$ it must be codified by a graph subgroup associated to

$$H \xrightarrow{\phi} \Sigma_{\lambda^u_{\underline{x},T}/\lambda_{\underline{x},T}} \wr T,$$

and by choosing the minimal subuniform refinement $\lambda_{\underline{x},T} \leq \bar{\lambda} \leq \lambda^u_{\underline{x},T}$ with a factorization

$$H \xrightarrow{\phi} \Sigma_{\lambda^u_{\underline{x},T}/\bar{\lambda}} \wr T,$$

HERE

showing that all isotropies have the desired form.

Since G acts diagonally it is clear that if (g, σ) is in the isotropy of $\underline{x}$, then the action of g must preserve $\lambda_{\underline{x}}$. Indeed, the action of g on $(x_i)_{\underline{i \in n}/\lambda_{\underline{x}}} \in X^{\times n/\lambda_{\underline{x}}}$ must in fact lie in $\Sigma_{\lambda^u_{\underline{x}}/\lambda_{\underline{x}}}$. Indeed, we define $H = Isot((x_i)_{\underline{i \in n}/\lambda_{\underline{x}}}) \cap \Sigma_{\lambda^u_{\underline{x}}/\lambda_{\underline{x}}}$ and note that the image of H in $\Sigma_{\lambda^u_{\underline{x}}/\lambda_{\underline{x}}}$ must be contained in $\Sigma_{\bar{\lambda}/\lambda_{\underline{x}}}$ for a minimal subuniform refinement $\lambda_{\underline{x}} \leq \bar{\lambda} \leq \lambda^u_{\underline{x}}$. This matches the description above since minimality of $\bar{\lambda}$ ensures the action of H is $\bar{\lambda}$-transitive and the isotropies of the action $H \to \Sigma_{\bar{\lambda}}$ must be in $\mathcal{F}$ since they are subgroups of the G-isotropy subgroups of the coordinates x_i of $\underline{x}$.

We now show that all such isotropy subgroups can in fact arise. Given $\lambda \leq \bar{\lambda}$ and $H \xrightarrow{\phi} \Sigma_{\bar{\lambda}/\lambda}$, let $H_i \leq H$ be the ϕ-action isotropy for each $i \in \underline{n}/\bar{\lambda}$ and choose elements x_i of X, all lying in distinct G orbits, whose isotropy is precisely H_i. One example of a n-tuple in $X^{\times n}$ whose isotropy is precisely $\Gamma_\phi \ltimes \Sigma_\lambda$ is given by

$$\left(\left((hx_i)_{h \in H/H_i}\right)_{j \in (\underline{n}/\lambda)_i}\right)_{i \in \underline{n}/\bar{\lambda}}.$$

□

E Family model structures

The following is well known.

GSPAC PROP **Proposition E.1.** *For $\mathcal{F}$ any family of subgroups of G closed under conjugation there is a $\mathcal{F}$-projective model structure on S^G, i.e., a model structure such that weak equivalences (resp. fibrations) are the maps $A \to B$ such that $A^H \to B^H$ is a weak equivalence (resp. fibration) in S for any $H \in \mathcal{F}$.*

Further, this is a left proper cellular simplicial model category.

Proof. We apply the usual small object argument in [3, Thm. 2.1.19] with the generating sets I, J built from those in S by inducing along each H. Explicitly

$$I = \bigcup_{H \in \mathcal{F}} \{G/H \cdot (\partial\Delta^k_+ \to \Delta^k_+)\}, \; J = \bigcup_{H \in \mathcal{F}} \{G/H \cdot (\Lambda^k_{l+} \to \Delta^k_+)\}.$$

Only the claim that maps in J-cell are weak equivalences is non obvious. This follows for maps in J by direct calculation, for pushouts of those by Proposition ?? and for transfinite compositions since those commute with $(-)^H$. Left properness, cellularity, and the simplicial model structure axioms are clear. □

Proposition E.2. *For any finite group G there exists a model structure on S^G, which we call the fixed point model structure, such that a map $A \to B$ in S^G is*

- *a cofibration if it is a monomorphism;*
- *a weak equivalence if $A^G \to B^G$ is a weak equivalence in S;*
- *a fibration if $A^H \to B^H$ is a trivial fibration (resp. fibration) in S for each $H < G$ (resp. $H = G$);*
- *a trivial fibration if $A^H \to B^H$ is a trivial fibration in S for each $H \leq G$.*

Proof. Just as in Proposition E.1, this follows by applying [3, Thm. 2.1.19]. We set

$$I = \bigcup_{H \leq G} \{G/H \cdot (\partial\Delta^k_+ \to \Delta^k_+)\}, \; J = \bigcup_{H < G} \{G/H \cdot (\partial\Delta^k_+ \to \Delta^k_+)\} \cup \{G/G \cdot (\Lambda^k_{l+} \to \Delta^k_+)\}.$$

The only non obvious part is to show that the maps in J-cell are weak equivalences, but this follows by repeating the same exact argument as in Proposition E.1. □

NGSPAC PROP **Proposition E.3.** *For $\mathcal{F} \subset \bar{\mathcal{F}}$ any two families of subgroups of G closed under conjugation there is a $\bar{\mathcal{F}}$-projective $\mathcal{F}$-equivalence model structure on S^G model structure such that a map $A \to B$ is*

- *a cofibration if $A^H \to B^H$ is a cofibration in S for each $H \in \bar{\mathcal{F}}$;*
- *a weak equivalence if $A^H \to B^H$ is a weak equivalence in S for each $H \in \mathcal{F}$.*

Further, this is a left proper cellular simplicial model category.

Proof. This follows by left Bousfield localization using [2, Thm. 4.1.1] with respect to a suitably chosen set of maps $\mathcal{S}$. Letting $c_{\mathcal{F}}$ denote a cofibrant replacement functor for the $\mathcal{F}$-projective model structure we set (cf. [7, Prop. 1.3])

$$\mathcal{S} = \left\{ c_{\mathcal{F}}(G/H) \to G/H : H \in \bar{\mathcal{F}} \right\}.$$

It remains to show that the $\mathcal{S}$-equivalences are precisely the $\mathcal{F}$-equivalences.

Since for each $H \in \mathcal{F}$ the maps in $\mathcal{S}$ are H-fixed point equivalences between $\bar{\mathcal{F}}$-cofibrant objects, [2, Prop. 3.3.18(1)] applied to the adjunction

$$\mathsf{fgt} \colon \left(\mathsf{S}^G\right)_{\mathcal{F}} \rightleftarrows \left(\mathsf{S}^H\right)_{H-\text{fix}} : Hom_H(G, -)$$

yields that all $\mathcal{S}$-local equivalences are $\mathcal{F}$-equivalences.

To prove the converse it suffices to show that between $\mathcal{S}$-local objects any $\mathcal{F}$-equivalence is in fact also a $\bar{\mathcal{F}}$-equivalence. Since a fibrant object X is $\mathcal{S}$-local precisely if, for each $H \in \bar{\mathcal{F}}$, one has induced weak equivalences

$$X^H = \mathrm{Map}_G\left(G/H, X\right) \xrightarrow{\sim} \mathrm{Map}_G\left(c_{\mathcal{F}}(G/H), X\right),$$

the result follows due to $\mathrm{Map}_G\left(c_{\mathcal{F}}(G/H), -\right)$ preserving $\mathcal{F}$-equivalences. □

EMPTYFIX LEM

Lemma E.4. *Suppose that $\mathcal{F}$ is closed under taking subgroups. Then for $X \in \mathsf{S}^G$ a $\mathcal{F}$ cofibrant object and $H \notin F$ it is*

$$X^H = \varnothing.$$

More generally, for $X \to Y$ a $\mathcal{F}$-cofibration, then

$$X^H \to Y^H$$

is an isomorphism.

Proof. This is obvious by induction on the generating cofibrations. □

Proposition E.5. *Suppose $\mathcal{F} \subset \bar{\mathcal{F}}$ are both closed under subgroups.*

Then the functor

$$\left(\mathsf{S}^H\right)_{\mathcal{F}-pr} \times \left(\mathsf{S}^H\right)_{\bar{\mathcal{F}}-cof, \mathcal{F}-eq} \to \left(\mathsf{S}^H\right)_{\mathcal{F}-pr}$$

is a left Quillen bifunctor.

Proof. We note first that the analogue statement for

$$\left(\mathsf{S}^H\right)_{\mathcal{F}-\mathrm{pr}} \times \left(\mathsf{S}^H\right)_{\bar{\mathcal{F}}-\mathrm{pr}} \to \left(\mathsf{S}^H\right)_{\mathcal{F}-\mathrm{pr}}$$

is obvious by looking at generating cofibrations/generating trivial cofibrations.

To obtained the desired localized version, one applies the localization criterion of [?]. It suffices to check that for X a $\mathcal{F}$ cofibrant object then

$$X \times \left(c_{\mathcal{F}}(G/H) \to G/H\right)$$

is a weak equivalence on all fixed points. This is obvious for fixed points by subgroups in $\mathcal{F}$ and follows for the remaining subgroups by Lemma E.4.

□

F The colored operad of operads

Definition F.1. Let Σ denote the usual skeleton of the category of finite sets and bijections.

Then, given any category $\mathcal{C}$, define $\Sigma \wr \mathcal{C}$ to be the Grothendieck construction for the functor

$$\Sigma \to \mathsf{Cat}$$
$$\underline{n} \mapsto \mathcal{C}^{\times n}$$

Definition F.2. The groupoid of trees is defined inductively as follows

$$\mathsf{T}_0 = \{|\}, \qquad \mathsf{T}_{n+1} = (\Sigma \wr \mathsf{T}_n) \amalg \{|\}.$$

There are obvious inclusions $i_n : \mathsf{T}_n \subset \mathsf{T}_{n+1}$ described inductively by

$$i_0 = i_{\{|\}}, \qquad i_{n+1} = \Sigma \wr i_n$$

and we hence define

$$\mathsf{T} = \operatorname{colim}_{n \to \infty} \mathsf{T}_i.$$

Remark F.3. These can be regarded as trees.

Proposition F.4. *Let $\underline{x} = (x_i)_{i \in \underline{r}} \in \mathsf{T}_{n+1}$ and let $\lambda_{\underline{x}}$ denote the "connected component" partition on $\underline{r}$ induced by $\underline{x}$.*

Then

$$Iso(\underline{x}) = \Sigma_{\lambda_{\underline{x}}} \ltimes \prod Iso(x_i)$$

Proof. This is similar to the statement for spaces □

Any tree has an associated set of leaves and an associated set of nodes. Further, the groupoid permutations of trees in T should functorially induce permutations of both leaves and nodes.

We now produce inductive descriptions of these functors.

Definition F.5. The **leaf functor**

$$L : \mathsf{T} \to \Sigma$$

is defined by

$$L(|) = \underline{1}, \qquad L((x_i)_{i \in \underline{r}}) = \coprod_{i \in \underline{r}} L(x_i).$$

The **node functor**

$$N : \mathsf{T} \to \Sigma_\Sigma$$

is defined by

$$N(|) = \varnothing, \qquad N((x_i)_{i \in \underline{r}}) = \underline{r} N(x_1) \cdots N(x_r).$$

Here $\Sigma_\Sigma \simeq \Sigma \wr \Sigma$, although we prefer the first notation for the sake of coherence with the $\Sigma_{\mathfrak{C}}$ notation for colored operads.

Proposition F.6. *Let $N \in \Sigma_\Sigma$ and $L \in \Sigma$. Then the colored operad of operads* Op *is given by*

$$\mathsf{Op}(N; L) = \coprod_{[T] \in T(N;L)} \Sigma_N \times \Sigma_L / \Sigma_T$$

or, more compactly,

$$\mathsf{Op} = \Sigma_\Sigma \times \Sigma \otimes_{\mathsf{T}} *.$$

F.1 Colored trees

Definition F.7. The groupoid of trees with $\{\circ,\bullet\}$-colored nodes is defined inductively as follows

$$\mathsf{T}_0^{\circ,\bullet} = \{|\}, \qquad \mathsf{T}_{n+1}^{\circ,\bullet} = (\Sigma^\circ \wr \mathsf{T}_n^{\circ,\bullet}) \amalg (\Sigma^\bullet \wr \mathsf{T}_n^{\circ,\bullet}) \amalg \{|\}.$$

There are obvious inclusions $i_n\colon \mathsf{T}_n^{\circ,\bullet} \subset \mathsf{T}_{n+1}^{\circ,\bullet}$ described inductively by

$$i_0 = i_{\{|\}}, \qquad i_{n+1} = \Sigma^\circ \wr i_n \amalg \Sigma^\bullet \wr i_n$$

and we hence define

$$\mathsf{T}^{\circ,\bullet} = \mathrm{colim}_{n\to\infty} \mathsf{T}_i^{\circ,\bullet}.$$

Definition F.8. The subgroupoid $\bar{\mathsf{T}}^{\circ,\bullet} \subset \mathsf{T}^{\circ,\bullet}$ of trees with non sequential $\bullet$ nodes is defined inductively by

$$\bar{\mathsf{T}}_0^{\circ,\bullet} = \{|\}, \qquad \bar{\mathsf{T}}_{n+1}^{\circ,\bullet} = (\Sigma^\circ \wr \bar{\mathsf{T}}_n^{\circ,\bullet}) \amalg (\Sigma^\bullet \wr \left(\Sigma^\circ \wr \bar{\mathsf{T}}_{n-1}^{\circ,\bullet} \amalg \{|\}\right)) \amalg \{|\}.$$

Definition F.9. The **leaf functor**

$$L\colon \mathsf{T} \to \Sigma$$

is defined by

$$L(|) = \underline{1}, \qquad L((\circ,(x_i)_{i\in\underline{r}})) = \coprod_{i\in\underline{r}} L(x_i), \qquad L((\bullet,(x_i)_{i\in\underline{r}})) = \coprod_{i\in\underline{r}} L(x_i).$$

The $\circ$**-node functor**

$$N^\circ\colon \mathsf{T}^{\circ,\bullet} \to \Sigma_\Sigma$$

is defined by

$$N^\circ(|) = \varnothing, \qquad N^\circ((\circ,(x_i)_{i\in\underline{r}})) = \underline{r}N^\circ(x_1)\cdots N^\circ(x_r), \qquad N^\circ((\bullet,(x_i)_{i\in\underline{r}})) = N^\circ(x_1)\cdots N^\circ(x_r).$$

Here $\Sigma_\Sigma \simeq \Sigma \wr \Sigma$, although we prefer the first notation for the sake of coherence with the $\Sigma_{\mathfrak{C}}$ notation for colored operads.

Proposition F.10. *Let $T \in \bar{\mathsf{T}}^{\circ,\bullet}$ be any tree and set $N^\circ = N^\circ(T)$, $L = L(T)$. Then the map*

$$\Sigma_T \to \Sigma_{N^\circ}^{\wr} \times \Sigma_L$$

is a monomorphism.

Proof. This follows by induction on the height of the trees.

The case of $|$ is obvious.

For the $T = (\circ,(x_i)_{i\in\underline{r}})$ case, the required map is

$$\begin{aligned}\Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma_{x_i} &\to (\Sigma_1 \ltimes \Sigma_{\underline{r}}) \times \Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} (\Sigma^{\wr}_{N^\circ(x_i)} \times \Sigma_{L(x_i)}) \simeq \\ &\simeq (\Sigma_1 \ltimes \Sigma_{\underline{r}}) \times \Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma^{\wr}_{N^\circ(x_i)} \times \Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma_{L(x_i)} \to \Sigma^{\wr}_{N^\circ(T)} \times \Sigma_{L(T)}\end{aligned}$$

which is a monomorphism since the first map is a monomorphism by the induction hypothesis and the second map is a partition inclusion, due to the fact that it is never simultaneously $N^\circ(x_i) = L(x_i) = \varnothing$.

In the $T = (\bullet, (x_i)_{i \in \underline{r}})$ case, we instead have the map

$$\begin{aligned}\Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma_{x_i} \to & \Sigma_{\lambda_{\underline{x}}} \ltimes \Big(\prod_{i\in\underline{r}} \Sigma^{\wr}_{N^{\circ}(x_i)} \times \prod_{i\in\underline{r}} \Sigma_{L(x_i)}\Big) \simeq \\ \simeq & \Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma^{\wr}_{N^{\circ}(x_i)} \times \Sigma_{\lambda_{\underline{x}}} \ltimes \prod_{i\in\underline{r}} \Sigma_{L(x_i)} \to \Sigma^{\wr}_{N^{\circ}(T)} \times \Sigma_{L(T)}\end{aligned}$$

which is again a monomorphism for the same reason.

□

Remark F.11. This does not quite work for a general $T \in \mathsf{T}^{\circ,\bullet}$ due to the possible presence of nullary $\bullet$ nodes attached to other $\bullet$ nodes.

F.2 pushout step

Here we assume that the family $\mathcal{F}$ is closed under generalized self-inductions.

Definition F.12. Let $T = (x_i)_{i\in\underline{n}} \in \mathsf{T}$ be a tree. We say that a map $G \to \Sigma_T \simeq \Sigma_{\underline{x}} \ltimes \prod \Sigma_{x_i}$ is $\mathcal{F}$-admissible if the map $G \to \Sigma_{\underline{x}} \subset \Sigma_n$ is in $\mathcal{F}$ and the induced maps $G_i \to \Sigma_{x_i}$ are also $\mathcal{F}$-admissible.

HERE jkjlkj

Proposition F.13. *Let $A(N;L) \xrightarrow{i} B(N;L)$ be a $\mathcal{F}$-admissible arboreal cofibration in $\mathsf{S}^{\Sigma_N \times \Sigma_L}$-spaces and $X \xrightarrow{f} Y$ be an $\mathcal{F}$-admissible sequence in S^{Σ}. Then*

$$i \,\square^{\times_{\Sigma_N}} f^N$$

is a $\mathcal{F}$-admissible cofibration in S^{Σ_L}.

Proof. Since the condition is linear on i, one is free to assume that i is one of the generating cofibrations, which have the form

$$\Sigma^{\wr}_N \times \Sigma_L / \Sigma_T \cdot g$$

for $T \in \mathsf{T}^{\circ,\bullet}$ and g a generating cofibration in S.

HERE □

G W construction

$$WP = \int^{T\in\Omega_i} P(T) \times I(T)$$

H Classifying spaces

Given a family of subgroups $\mathcal{F}$ of G, one can encode it as a functor

$$\mathsf{O}_G^{op} \xrightarrow{\mathcal{F}} \mathsf{Set}$$

which can only assume the values $\varnothing, *$.

One can then build $E\mathcal{F} \in \mathsf{S}$ as

$$E\mathcal{F} = B_{\bullet}(\mathcal{F}, \mathsf{O}_G, \mathsf{O}_G)(G/*) = B_{\bullet}(\mathcal{F}, \mathsf{O}_G, \mathsf{O}_G(G/*, -))$$

This of course comes with an action of $\mathsf{O}_G(G/*, G/*) = G$ and since for $H \le G$ one has $\mathsf{O}_G(G/*, -)^H = (-)^H = \mathsf{O}_G(G/H, -)$ and hence indeed

$$(E\mathcal{F})^H \simeq B_\bullet(\mathcal{F}, \mathsf{O}_G, \mathsf{O}_G(G/H, -)) \sim \mathcal{F}(G/H).$$

Alternatively, one can define the obvious full subcategory $\mathsf{O}_\mathcal{F} \subset \mathsf{O}_G$ determined by those objects G/H such that $H \in \mathcal{F}$.

Noting that $\mathcal{F}|_{\mathsf{O}_\mathcal{F}} = *$ and unpacking the definition of $B_\bullet(\mathcal{F}, \mathsf{O}_G, \mathsf{O}_G)(G/*)$ one sees that only the objects in $\mathsf{O}_\mathcal{F}$ contribute to it, so that one also has

$$E\mathcal{F} = B_\bullet(*, \mathsf{O}_\mathcal{F}, \mathsf{O}_\mathcal{F})(G/*) = B_\bullet(*, \mathsf{O}_\mathcal{F}, \mathsf{O}_\mathcal{F}(G/*, -)).$$

H.1 Graph families

Let $\mathcal{F}^n_G$ be the family of G-graph subgroups of $G \times \Sigma_n$. Throughout we will denote $\mathsf{O}^n_G = \mathsf{O}_{\mathcal{F}^n_G}$.

Our first goal is to produce an alternate model for O^n_G in terms of G-sets rather than $G \times \Sigma_n$ sets.

Let $\mathcal{G}^n_G$ denote the category whose objects are equivariant maps of G-sets

$$T \xrightarrow{\phi} G/H, \quad H \le G, \forall_{gH \in G/H} |\phi^{-1}(gH)| = n$$

and an arrow from $\phi\colon T \to G/H$ to $\bar{\phi}\colon \bar{T} \to G/\bar{H}$ is a commutative **pullback diagram**

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "\phi"] & \bar{T} \arrow[d, "\bar{\phi}"] \\
G/H \arrow[r] & G/\bar{H}.
\end{tikzcd}$$

Note that there is an obvious forgetful functor

$$\mathcal{G}^n_G \to \mathsf{O}_G.$$

Proposition H.1. *The functor*

$$F = (\underline{n} \to *) \times_{\Sigma_n} (-) \colon \mathsf{O}^n_G \to \mathcal{G}^n_G$$

is an equivalence of categories.

Proof. Consider an auxiliary category $\bar{\mathcal{G}}^n_G$ whose objects are equivariant maps of G-sets $T \xrightarrow{\phi} G/H$ together with a chosen isomorphism $\phi^{-1}(eH) \simeq \{1, \cdots, n\}$ and arrows as in $\bar{\mathcal{G}}^n_G$ (i.e. arrows ignore the chosen identification $\phi^{-1}(eH) \simeq \{1, \cdots, n\}$). It is then clear that the forgetful functor $\bar{\mathcal{G}}^n_G \xrightarrow{\mathsf{fgt}} \mathcal{G}^n_G$ is an equivalence of categories and furthermore that our desired equivalence of categories factors as

$$\mathsf{O}^n_G \xrightarrow{\bar{F}} \bar{\mathcal{G}}^n_G \xrightarrow{\mathsf{fgt}} \mathcal{G}^n_G.$$

We now define an inverse

$$\bar{\mathcal{G}}^n_G \xrightarrow{K} \mathsf{O}^n_G.$$

On objects, we set $K(T \xrightarrow{\phi} G/H) = G \times \Sigma_n/\Gamma_{T_e}$, where Γ_{T_e} is the graph of the map $H \to \Sigma_n \simeq \Sigma_{T_e}$ classifying the H-action on the fiber $T_e = \phi^{-1}(eH)$. On an arrow

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\phi"] & \bar{T} \arrow[d, "\bar{\phi}"] \\ G/H \arrow[r, "eH \mapsto g\bar{H}"'] & G/\bar{H}. \end{tikzcd}$$

we define K as the map

$$G \times \Sigma_n/\Gamma_{T_e} \xrightarrow{(e,e)\Gamma_{T_e} \mapsto (g, f|^{-1}_{T_e} g)\Gamma_{\bar{T}_e}} G \times \Sigma_n/\Gamma_{\bar{T}_e}$$

where $f|^{-1}_{T_e} g$ denotes the composite

$$\{1, \cdots, n\} \simeq \bar{T}_e \xrightarrow[g]{\simeq} \bar{T}_{g\bar{H}} \xrightarrow[f^{-1}_{T_e}]{\simeq} T_e \simeq \{1, \cdots, n\}$$

regarded as an element of Σ_n. It is clear that this map does not depend on the choice of representative $g \in g\bar{H}$.

To see that this gives a well defined map one must further check that $\Gamma_{T_e}(g, f|^{-1}_{T_e} g) \subset (g, f|^{-1}_{T_e} g)\Gamma_{\bar{T}_e}$. This follows by the following calculation

$$(h,h)(g, f|^{-1}_{T_e} g) = (hg, hf|^{-1}_{T_{eH}} g) = (hg, f|^{-1}_{T_{eH}} hg) = (gk, f|^{-1}_{T_{eH}} gk) = (g, f|^{-1}_{T_{eH}} g)(k,k).$$

It is straightforward to check that K is indeed a quasi-inverse to $\bar{F}$. $\square$

I Equivariant trees

Let F_G denote the category of finite G-sets.

Definition I.1. We define $\mathsf{T}_{n,G}$ inductively as

- $$\begin{tikzcd} & \mathsf{T}_{0,G} \arrow[dl, "="'] \arrow[dr, "="] & \\ \mathsf{F}_G^{op} & & \mathsf{F}_G^{op} \end{tikzcd}$$

- $$\begin{tikzcd} & \mathsf{F}_G^{\to} & & \mathsf{T}_{n,G} & \\ \mathsf{F}_G^{op} & & \mathsf{F}_G^{op} & & \mathsf{F}_G^{op} \end{tikzcd}$$

J Equivariant trees

For $H < G$ let $\underline{G/H}$ denote the category with object set G/H and arrow set $G \times (G/H)$ where (g, kH) is an arrow $kH \to gkH$ in the obvious way.

Note that this construction is functorial, i.e. one obtains a functor $\underline{(-)} \colon \mathsf{O}_G \to Cat$

Definition J.1. The category Ω_G of G-trees is the Grothendieck construction for the functor

$$\mathsf{O}_G^{op} \xrightarrow{\Omega^{(-)}} Cat \qquad \text{(J.2)}$$
$$G/H \mapsto \Omega^{\underline{G/H}}.$$

treefunctor eq

We will call the canonical functor $\pi_0 \colon \Omega_G \to \mathsf{O}_G$ the *connected component functor.*

fiberReedy lemma

Lemma J.3. *Let $\mathbb{R}$ be a generalized Reedy category as defined in [1, Defn. 1.1].*

Then for any group G one has that $\mathbb{R}^G$ is again a generalized Reedy category with the degree function defined by forgetting the action and $(\mathbb{R}^G)^{\pm} = (\mathbb{R}^{\pm})^G$.

Proof. Conditions (i), (ii) and (iv) of [1, Defn. 1.1] are immediate.

It remains to verify condition (iii), i.e. that any G-map $X \xrightarrow{f} Y$ between G-objects in $\mathbb{R}^G$ has a suitable factorization by G-maps. By forgetting the G-action one can write $f = f^+ \circ f^-$ for $f^{\pm} \in \mathbb{R}^{\pm}$. For each $g \in G$ consider now the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f^-"] \arrow[d, "g_X"] & Z \arrow[r, "f^+"] \arrow[d, dashed, "g_Z"] & Y \arrow[d, "g_Y"] \\
X \arrow[r, "f^-"] & Z \arrow[r, "f^+"] & Y
\end{tikzcd}$$

The goal is to find the dashed arrows g_Z equipping Z with a compatible G-action. Since the two factorizations $f = f^+ \circ f^-$, $f = (g_Y^{-1} \circ f^+) \circ (f^- \circ g_X)$ must differ by a unique automorphism of Z (as explained in [1, Rmk. 1.2]), we set that automorphism to be g_Z. It is straightforward to check that the uniqueness condition implies that the g_Z assemble to a G-action. ☐

Corollary J.4. *Ω_G is a generalized Reedy model structure where Ω_G^- consists of fiberwise composites of degeneracy maps and Ω_G^+ consists of composites of face maps followed by a pullback maps.*

Proof. This is an application of [1, Cor. 1.10] by giving O_G the $(\mathsf{O}_G)^+ = \mathsf{O}_G$ generalized Reedy structure of [1, Examples 1.8(e)] together with the Reedy structures on fibers induced by Lemma J.3. ☐

Remark J.5. The Reedy structure thus obtained on Ω_G is **not** dualizable in the sense of [1, Defn. 1.1]. This is a direct consequence of the fact that corresponding Reedy structure on O_G is also not dualizable.

Remark J.6. There is an additional Reedy structure on O_G such that $(\mathsf{O}_G)^- = \mathsf{O}_G$. However, the pullback maps $\alpha^* \colon \Omega^{\underline{G/K}} \to \Omega^{\underline{G/H}}$ in (J.2) fail to have left adjoints $\alpha_* \colon \Omega^{\underline{G/H}} \to \Omega^{\underline{G/K}}$. Indeed, while α_{**} possesses left adjoints after extending to either the categories of forests or to the category of trees and poset morphisms (with $\alpha_*(\cdot \xrightarrow{G/H_2} \cdot \xrightarrow{G/H_1} \cdot \xrightarrow{G/H} \cdot) = (\cdot \xrightarrow{G/H_2} \cdot \xrightarrow{G/H_1} \cdot \xrightarrow{G/K} \cdot)$), neither of these adjunctions restricts to a tree adjunction (although, interestingly, the latter provides a functor between categories of trees, but the tree category obtained by the corresponding Grothendieck construction does not match the tree category that is naturally induced by equivariant operads).

K Remarks on generalized Reedy categories

One can run the skeletal/coskeletal part of the story without using axiom (iv).

Indeed, one can define $L_r X = colim_{(c\to r)\in\mathbb{R}_{<n}\downarrow r} X_c$ instead of $L_r X = colim_{(c\to r)\in\mathbb{R}^+_{<n}\downarrow r} X_c$ since

Proposition K.1. *For any R, $\mathbb{R}^+_{<n}\downarrow r$ is final in $\mathbb{R}_{<n}\downarrow r$.*

Proof. This is straightforward using axiom (iii) of generalized Reedy categories. □

Remark K.2. It then automatically follows that lifts in the left diagram

$$\begin{tikzcd} A_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d] \\ B_\bullet \arrow[r] \arrow[ur, dashed] & Y_\bullet \end{tikzcd} \qquad \begin{tikzcd} A_n \coprod_{L_n A} L_n B \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed] & Y_n \underset{M_n Y}{\times} M_n X \end{tikzcd}$$

can be built via a sequence of compatible **equivariant** lifts in the the right diagram.

Remark K.3. The main point of axiom (iv) in the definition of generalized Reedy is that $x\downarrow\mathbb{R}^-$ becomes (equivalent) to a downward closed strict Reedy, so that the generalized Reedy structures on $\mathcal{M}^{x\downarrow\mathbb{R}^-}$, $\mathcal{M}^{\mathbb{R}^+\downarrow x}$ are in fact the injective and projective model structures.

L Broad posets

In this section we use the broad poset framework of Weiss. Recall that $\le$ denotes the broad poset relation while $\le_d$ denotes the corresponding induced "descendacy" poset relation.

EDGESTREE LEMMA

Lemma L.1. *Let $e_1, e_2, \cdots, e_n, s$ be edges in a tree. The following are equivalent.*

(i) the e_i are distinct and there exists an E such that $E \le s$ and all $e_i \in E$;

(ii) the e_i are non comparable by $\le_d$ and all $e_i \le_d s$.

Proof. The proof is by upward nested induction on both s and n. The base cases are those of s a leaf and any $n \ge 1$ or of any s and $n = 1$, all of which are obvious.

Otherwise, when $n \ge 2$ it is immediate that $(i) \Rightarrow (ii)$, since if it was $e_i \le_d e_j$, $i \ne j$, replacing e_j in E by a tuple containing e_i would yield an $E' \le E \le s$ with repeated elements. To see that $(ii) \Rightarrow (i)$, note that since $n \ge 2$ one must have $e_i \ne s$ for all i, or else the e_i would be comparable. Hence, there are distinct $f_{\pi(i)} \in s^\uparrow$ such that $e_i \le_d f_{\pi(i)}$. If all $f_{\pi(i)}$ coincide (i.e. if $\pi(1) = \cdots = \pi(n)$) the result follows by the induction hypothesis with $s = f_{\pi(i)}$. Otherwise, the induction hypothesis on n allows us to find $E_{\pi(i)} \le f_{\pi(i)}$ such that $e_i \in E_{\pi(i)}$ and replacing the $f_{\pi(i)}$ by $E_{\pi(i)}$ in $s^\uparrow$ produces the desired E. □

COMPAR COR

Corollary L.2. *For any map $f\colon T \to T'$ the edges $e, e' \in T$ are comparable by $\le_d$ iff $f(e), f(e')$ are.*

FIBERS LEMMA

Lemma L.3. *If $\pi{:}T \to T'$ is a (edge) surjection of trees (regarded as broad posets). Then (this needs to be restated as pullbacks in broad posets)*

(i) $\pi^{-1}(T'_{\leq_d e})$ is a tree for any edge $e \in T'$;

(ii) $\pi^{-1}(e)$ is a linearly ordered tree for any edge $e \in T'$.

Proof. We prove this by induction on the father of each edge.

For the root edge r (i) is obvious. As for (ii), we first verify that $\pi^{-1}(r)$ is a tree. That $\leq$ is simple is automatic. Further, $\pi^{-1}(r)$ certainly contains a minimum, namely the root of T. Finally, if $a \in \pi^{-1}(r)$ is a leaf in T, it certainly is also a leaf in $\pi^{-1}(r)$. Finally if a has children either $\pi(a^{\uparrow}) < r$, in which case none of the children are in $\pi^{-1}(r)$, so that a has no children in $\pi^{-1}(r)$ (note that this DOES cover the case $a^{\uparrow} = \epsilon$), or $\pi(a^{\uparrow}) = r$, in which case a has children in $\pi^{-1}(r)$ as well.

We now show (i) for a child e of the root. That $\leq$ remains simple and that the child condition still hold are immediate, and it remains hence to show that $\pi^{-1}(T'_{\leq_d e})$ has a minimum. Suppose there are distinct minimal elements f, f'. WLOG, surjectivity allows us to assume $\pi(f) = e$. But then $\pi(f), \pi(f')$ must be comparable, contradicting the Corollary L.2.

The remainder of the proof follows by induction on the tree $\pi^{-1}(T'_{\leq_d e})$. ☐

EMPTYTREE LEMMA

Lemma L.4. *If $A \leq s$, $AB \leq s$ then $\epsilon \leq B$.*

Proof. We proceed by upward induction on s and the length of B. The base cases of s a leaf and $B = \epsilon$ are obvious, as is the case $B = s$.

Otherwise, it suffices to show $\epsilon \leq b_i$ for each $b_i \in B$. Letting $s^{\uparrow} = s_1 \cdots s_k$ one can hence find decompositions A_i, $\bar{A}_i B_i$ of A, AB such that $A_i \leq s_i$, $\bar{A}_i B_i \leq s_i$. Noting that one must in fact have $A_i = \bar{A}_i$ (since no element of A can be comparable to more than one s_i), the result now follows by the induction hypothesis. ☐

SPLITEPI COR

Corollary L.5. *A (edge) surjection $\pi{:}T \to T'$ in Ω is split iff $\epsilon \leq e \Leftrightarrow \epsilon \leq \pi(e)$ for all edges e in T.*

Proof. The "only if part" is obvious. Otherwise, we claim that any section of the underlying map of sets is in fact a map of broad posets as well. It hence suffices to show that if $\pi(E) \leq \pi(s)$ then $E \leq s$ for any choice of E, s in T. By L.2 the elements in such an E must be non comparable and comparable with s, hence by Lemma L.1 there exists an F such that $EF \leq s$ and hence also $\pi(E)\pi(F) \leq s$. Since we are assuming $\pi(E) \leq s$, Lemma L.4 and the $\epsilon \leq e \Leftrightarrow \epsilon \leq \pi(e)$ hypothesis finish the proof. ☐

Corollary L.6. *Surjections in Ω^H that are underlyingly split in Ω are in fact split in Ω^H.*

Proof. By Lemma L.3 (ii) the action of H on each fiber of a surjection $\pi{:}T \to T'$ must be trivial. It is hence possible to choose an equivariant section of the underlying map of sets, and hence by the proof of Corollary L.5 this section is indeed a map of trees. ☐

M EZ, boundaries and normal maps

It will be useful for us to know that pairs of split epimorphisms in Ω^H with common domain have absolute pushouts. To see, denote such a diagram as D and note that since the adjunction $Set^{\Omega^{op}} \rightleftarrows Set^{\Omega^{op}\times H^{op}}$ is monadic, the right adjoint creates absolute pushouts, so that D induces an absolute pushout in $Set^{\Omega^{op}\times H^{op}}$, hence the result follows from the commutative diagram

$$\begin{tikzcd}
\Omega^{(H^{op})} \arrow[r, "\simeq", "((-)^{-1})^*"'] \arrow[d, "(Y_\Omega)^{(H^{op})}"'] & \Omega^H \arrow[dr, "Y_{\Omega^{H^{op}}}"] & \\
Set^{\Omega^{op}\times H^{op}} \arrow[r, "(\epsilon,\pi_{H^{op}})^*"'] & Set^{(\Omega^H)^{op}\times H^{op}} \arrow[r, "(-)^{H^{op}}"'] & Set^{(\Omega^H)^{op}}
\end{tikzcd}$$

where $Y_{(-)}$ denote Yoneda embeddings and $\epsilon : \Omega^H \times H \to \Omega$ is the counit.

As in [1], we can call maps in $(\Omega_G)^-$ degeneracy operators, and for $X \in Set^{\Omega_G^{op}}$ we hence call an element $x : \Omega_G[T] \to X$ *degenerate* if it factors through a non invertible degeneracy operator and *non-denegerate* otherwise. One then has the following result, which is lifted from [1, Prop. 6.9].

UNIQUEFACT LEMMA

Proposition M.1. *Let $X \in Set^{\Omega_G^{op}}$. Then any element $x : \Omega_G[T] \to X$ has a factorization, unique up to unique isomorphism,*

$$\Omega_G[T] \xrightarrow{\rho_x} \Omega_G[S] \xrightarrow{\bar{x}} X$$

as a degeneracy operator ρ_x followed by a non degenerate element $\bar{x}$.

Proof. The existence of such a factorization follows since non invertible degeneracy operators lower degree. We now show uniqueness. Considering a second factorization $\Omega_G[T] \xrightarrow{\rho'_x} \Omega_G[S'] \xrightarrow{\bar{x}'} X$ one can build the absolute pushout

$$\begin{tikzcd}
\Omega_G[T] \arrow[r, two heads, "\rho_x"] \arrow[d, two heads, "\rho'_x"'] & \Omega_G[S] \arrow[d, two heads, "\tau'_x"] \\
\Omega_G[S'] \arrow[r, two heads, "\tau_x"'] & \Omega_G[R].
\end{tikzcd}$$

Since there is an induced map $\Omega_G[R] \to X$ one has that τ_x, τ'_x must be isomorphisms (or else $\bar{x}, \bar{x}'$ would be degenerate), showing uniqueness up to isomorphism. That the isomorphism is itself unique follows from the fact that degeneracies are epic. $\square$

SKELL COR

Corollary M.2. *The counit $sk_n X \to X$ for $X \in Set^{\Omega_G}$ is a monomorphism whose image consists of those elements of X that factor through some $\Omega_G[S] \to X$ for $|S| \leq n$.*

Proof. One has

$$(sk_n X)_T \simeq \underset{S \to T \in (\Omega_G)^{op}_{\leq n} \downarrow T}{\operatorname{colim}} X_S \simeq \underset{S \to T \in ((\Omega_G)^{op})^+_{\leq n} \downarrow T}{\operatorname{colim}} X_S \simeq \underset{T \to S \in T \downarrow (\Omega_G)^-_{\leq n}}{\operatorname{colim}} X_S$$

and it is hence clear that the image of $sk_n X \to X$ is as described. To see that this is an injective, consider the diagram, where maps marked $-$ denote degeneracies.

$$\begin{tikzcd}
 & & \Omega[T] \arrow[dll, "-"'] \arrow[dl, dashed, "-"] \arrow[dr, dashed, "-"'] \arrow[drr, "-"] & & \\
\Omega[S] \arrow[r, dashed, "-"] \arrow[drr] & \Omega[R] \arrow[rr, dashed, leftrightarrow, "\simeq"] \arrow[dr, dashed] & & \Omega[R'] \arrow[dl, dashed] & \Omega[S'] \arrow[l, dashed, "-"'] \arrow[dll] \\
 & & X. & &
\end{tikzcd}$$

The left and right solid paths represent two possible points of $sk_n X$ that are identified in X. $\Omega[R], \Omega[R']$ are then built from the decompositions from Lemma M.1 of the two lower solid diagonal maps. But since the maps marked $-$ are degeneracy operators this shows that $\Omega[R], \Omega[R']$ also form the decompositions of the full vertical composite, so that the horizontal isomorphism must exist, showing that the two points of $sk_n X$ already coincided in $sk_n X$. $\square$

The *formal boundary* $\partial\Omega_G[T]$ is defined to be the subobject of $\Omega_G[T]$ formed by those maps that factor through a non invertible map in Ω_G^+, i.e. through a non invertible face and or pushout. Note that by the Reedy axioms this is equivalent to saying that the map factors through some S with $|S| < |T|$ and by Corollary M.2 one in fact has $\partial\Omega_G[T] = sk_{|T|-1}\Omega_G[T]$.

Remark M.3. By the adjunction $sk_n \dashv cosk_n$,

$$Hom(\partial\Omega[T], X) \simeq Hom(\Omega[T], cosk_{|T|-1} X) \simeq M_T X.$$

Our next goal is generalize the notion of "normal monomorphism" of [1] (add more refs) to our context. In the non-equivariant case, these can be defined as those monomorphisms obtained by sequentially attaching dendrices along their formal boundary. The direct attempt to generalize this definition to Ω_G runs into a problem: the boundaries $\partial\Omega_G[T]$ often can not be built cellularly out of boundary inclusions. The source of the problem is the fact some maps in Ω^+ are not monomorphisms (more specifically, the pushouts). The solution is to regard certain quotients of the boundary inclusions as normal.

Note that for trees $T, T' \in \Omega^{\underline{G/H}} \subset \Omega_G$ there is an induced map

$$Map(T, T') \xrightarrow{(\pi_0)_*} Aut(G/H)$$

satisfying the obvious associativity condition.

A map $f: T \to T'$ will be called *arboreal* if $\pi_0(f) = e$.

Definition M.4. For $T \in \Omega_G$, the *arboreal isotropy subgroup* is the normal subgroup

$$Iso_a(T) \subset Iso(T)$$

of those isomorphisms which are also arboreal maps.

Further, a subgroup $N \subset Iso(T)$ is called *non-arboreal* if $N \cap Iso_a(T) = *$.

Definition M.5. A *non-arboreal dendrex quotient* is any quotient $\Omega_G[T]/N$ for $N \subset Iso(T)$ a non-arboreal subgroup. Further, any such inclusion

$$\partial\Omega_G[T]/N \to \Omega_G[T]/N$$

will be called a *non arboreal boundary inclusion.*

The following generalizes BeMo08 [1, Proposition 7.2].

NORMCHAR PROP

Proposition M.6. *Let $\phi: X \to Y$ be a map in $Set^{\Omega_G^{op}}$. Then the following are equivalent.*

(i) for each tree $T \in \Omega_G$, the relative latching map $l_T(\phi): X_T \coprod_{L_T X} L_T Y \to Y_T$ is an $Iso_a(T)$-free extension;

(ii) ϕ is monic and, for each $T \in \Omega_G$ and non degenerate $y \in Y_T - \phi(X)_T$, the isotropy group $\{g \in Iso(T) | gy = y\}$ is non-arboreal;

(iii) for each $n \geq 0$, the relative n-skeleton $sk_n(\phi) = X \coprod_{sk_n X} sk_n Y$ is obtained from the relative $(n-1)$-skeleton by attaching a coproduct of non arboreal boundary inclusions.

We shall refer to maps satisfying any of these equivalent properties as *G-normal monomorphisms*. Further, a presheaf X such that $\varnothing \to X$ is a G-normal monomorphism is called *G-normal*.

Remark M.7. Note that when $G = **$ all isomorphisms are in fact arboreal, hence our definition does coincide with that of BeMo08 [1].

Proof. $(i) \Rightarrow (iii)$ Noting that $sk_n(\phi)$ is obtained from $sk_n X \coprod_{sk_{n-1}X} sk_{n-1} Y$ by attaching along the inclusion $sk_n X \to X$, one can in fact assume that X, Y are n-skeletal and hence so are $sk_{n-1}(\phi), sk_n(\phi)$. At degrees $< n$ the map $sk_{n-1}(\phi) \to sk_n(\phi)$ is an isomorphism and at each T of degree n it evaluates to the relative latching map $X_T \coprod_{L_T X} L_T Y \to Y_T$. Since this is an $Iso_a(T)$-free extension, the dendrices not in the image can be obtained by a attaching copies of non-arboreal dendrex quotients. Note that the resulting presheaf must match $sk_n(\phi)$ since both are n-skeletal.

$(iii) \Rightarrow (ii)$ Since the condition in (ii) is weakly saturated (i.e. closed under pushouts, transfinite composites and retracts), it suffices to verify that non arboreal boundary inclusions satisfy (ii). This is clear.

$(ii) \Rightarrow (i)$ Note first that since ϕ is assumed monic, all of X, $sk_i X$, $sk_i Y$ can be regarded as subobjects of Y. The relative matching map $X_T \coprod_{L_T X} L_T Y \to Y_T$ for $|T| = n$ is obtained by evaluating $X \coprod_{sk_{n-1}X} sk_{n-1} Y \to Y$ at T. It is now clear that the dendrices not in the image of $l_T(\phi)$ are precisely the non-degenerate dendrices in Y_T not in the image of ϕ. □

GNORMDEND COR

Corollary M.8. *The non arboreal dendroidal quotients $\Omega_G[T]/N$ and their boundaries $\partial\Omega_G[T]/N$ are G-normal dendroidal sets.*

Proof. We apply the characterization in Proposition NORMCHAR PROP M.6(ii). Unwinding definitions, we see that it suffices to check that if $f\theta = f$ for $f: T \to T'$ in Ω_G^+ and $\theta \in Iso_a(T)$ an arboreal automorphism, then $\theta = id_T$. Now note that f canonically factorizes as $T \xrightarrow{\bar{f}} (\pi_0)^*(T') \to T'$ and it suffices to show that if $\bar{f}\theta = \bar{f}$ then $\theta = id_T$. But $\bar{f}$ is an arboreal morphism by construction, hence the result now follows from the fact that Ω^H is dualizable Reedy. ☐

Remark M.9. Maybe explain why this doesn't apply to other maps.

N Reedy-ish model structure

NONARBMODEL PROP **Proposition N.1.** *Let $T \in \Omega_G$ be a G-tree. Then there is a model structure on $\mathsf{SSet}^{Iso(T)^{op}}$ such that*

- *cofibrations are the monomorphisms $A \to B$ which attach only $Iso_a(T)$-free simplices;*
- *weak equivalences (resp. fibrations) are the maps $A \to B$ such that $A^H \to B^H$ is a Quillen weak equivalence (resp. Kan fibration) for each non-arboreal subgroup $N \subset Iso(T)$.*

Proof. This is the standard model projective model structure with regards to the family $\{N \leq G | N \cap Iso_a(T) = *\}$ of non-arboreal subgroups. ☐

Let I be a category with a G action and let

$$I \xrightarrow{i} G \ltimes I \xrightarrow{p} G$$

denote the projection for the Grothendieck construction of the obvious functor $G \xrightarrow{I} Cat$. The projection formula (as described in, for example, BeMo08 [1, Section 3]) then implies that one has an adjunction

$$\mathrm{colim}_I : \mathcal{C}^{G \ltimes I} \rightleftarrows \mathcal{C}^G : p^*.$$

Further, if I is generalized Reedy with $I = I^+$ it is clear that so is $G \ltimes I$ with $(G \ltimes I)^+ = G \ltimes I$ and the projection formula (applied to over categories) then shows that for $X : G \ltimes I \to \mathcal{C}$ on has, after forgetting the G action, $L_i X \simeq L_i X|_I$.

GINJ LEM **Lemma N.2.** *Let I be generalized Reedy with $I = I^+$. Then if a map $A \to B$ in $\mathsf{SSet}^{G \ltimes I}$ is such that*

$$A_i \coprod_{L_i A} L_i B \to B_i \tag{N.3}$$ RELLATCH EQ

is a $Aut_{G \ltimes I}(i)$-genuine trivial cofibration for all i, then

$$\mathrm{colim}_I A \to \mathrm{colim}_I B$$

is a genuine G-trivial cofibration.

Proof. It suffices to verify that the maps described have the LLP with respect to $p^*(X \to Y)$ for $X \to Y$ any genuine fibration in $SSet^G$, and this amounts to inductively verifying the LLP of the maps in (RELLATCH EQ N.4) against the maps

$$p^*(X)_i \to (p^*Y)_i \underset{M_i(p^*Y)}{\times} M_i(p^*X). \tag{N.4}$$ RELLATCH EQ

But since $(G \ltimes I)^+ = G \ltimes I$, matching objects are trivial, so that this latter map is simply $X \to Y$ with the $Aut_{G\ltimes I}(i)$-action pulled back along the projection $Aut_{G\ltimes I}(i) \to G$. It follows that the maps in (N.4) are genuine $Aut_{G\ltimes I}(i)$-fibrations, finishing the proof. $\square$

Theorem N.5. *For each $T \in \Omega_G$, equip $\mathsf{SSet}^{Iso(T)^{op}}$ with the model structure from Proposition N.1.*

There is a cofibrantly generated model structure on $\mathsf{SSet}^{\Omega_G^{op}}$, which we call the ***Reedy model structure****, such that a map $A \to B$ is*

- *a cofibration if $A_T \coprod_{L_T A} L_T B \to B_T$ is a cofibration in $\mathsf{SSet}^{Iso(T)^{op}}$, $T \in \Omega_G$;*
- *a weak equivalence if $A_T \to B_T$ is a weak equivalence in $\mathsf{SSet}^{Iso(T)^{op}}$, $T \in \Omega_G$;*
- *a fibration if $A_T \to B_T \underset{M_T B}{\times} M_T A$ is a fibration in $\mathsf{SSet}^{Iso(T)^{op}}$, $T \in \Omega_G$.*

Proof. We will apply [3, Thm.2.1.19]. Let I, J denote the sets of standard generating cofibrations and trivial cofibrations in SSet and

$$B = \{\partial\Omega_G[T]/N \to \Omega_G[T]/N | T \in \Omega_G, N \cap Iso_a(T) = *\}$$

denote the set of non-arboreal boundary inclusions in $\mathsf{Set}^{\Omega_G^{op}}$. We claim that sets of generating cofibrations and trivial cofibrations for the desired model structure are given by

$$I_{\Omega_G} = I \square B, \qquad J_{\Omega_G} = J \square B$$

where $\square$ denotes the pushout product.

Only conditions 4,5,6 of [3, Thm.2.1.19] require explanation. 4 follows since $J \subset I - cof$ and the maps in $J \square B$ are pointwise $Iso(T)$-genuine trivial cofibrations. 5 follows since, by Corollary M.8, $(I \square B)$-cof contains the maps $\{\Omega_G[T] \times I | T \in \Omega_G\}$, showing that maps in $(I \square B)$-inj are weak equivalences. For 6, we will verify that $W \cap (I \square B)$-cof $\subset$ $(J \square B)$-cof. Standard lifting properties arguments show that $(J \square B)$-cof consists of those maps such that $A_T \coprod_{L_T A} L_T B \to B_T$ is a trivial cofibration in $\mathsf{SSet}^{Iso(T)^{op}}$, $T \in \Omega_G$. We will verify this for each map in $W \cap (I \square B)$-cof by induction on T. By the commutative diagram

$$\begin{tikzcd}
L_T A \arrow[r] \arrow[d] & A_T \arrow[d] \arrow[ddr, "\sim"] & \\
L_T B \arrow[r] \arrow[drr] & L_T B \amalg_{L_T A} A_T \arrow[dr] & \\
& & B_T
\end{tikzcd}$$

it suffices to check that $L_T A \to L_T B$ are genuine trivial cofibrations.

Recall that $L_T A = \operatorname{colim}_{T \xrightarrow{f} T' \in (T\downarrow \Omega_G^-)^{op}} A_{T'}$. We will be done if we can apply Lemma N.2 with $I = (T \downarrow \Omega_G^-)^{op}$, $G = Iso(T)$. Note that then $G \ltimes I = (Iso(T) \downarrow \Omega_G^-)^{op}$, and $L_{f:T\to T'} A \circ t = L_{T'} A$ (where t denotes the target/codomain functor; this follows from [1, Lemma 4.4(i)]) with the $Aut_{G\ltimes I}(f)$-action simply being induced by the forgetful homomorphism $Aut_{G\ltimes I}(f) \to Aut_{\Omega_G}(T')$. It is now clear that the induction hypothesis implies the conditions of Lemma N.2, proving that the given generating cofibrations and trivial cofibrations indeed define a model structure.

It remains only to note that the model structure just built has fibrations/cofibrations as advertised. Both cases are clear by simply playing around with the lifting properties versus the generating cofibrations/trivial cofibrations. □

Remark N.6. There most likely is a general abstract proof of this result using exclusively techniques a la [1] to replace the roles of cofibrant generation and Corollary M.8.

O Segal conditions

bla bla bla

P Fully faithfulness of Yoneda

Note that there is a fully faithful embedding $\Omega_G \to Op^G$.

Proposition P.1. *The nerve functor*

$$Op^G \xrightarrow{Hom(\Omega(T),-)} Set^{\Omega_G^{op}}$$

is fully faithful.

Q Equivariant ∞-operads

Desiderata:

Model structure on $dSet^G$ such that:

- cofibrations/ normal maps are created by the forgetful functor to $dSet$;

- trivial cofs are generated by non arboreal inner horn inclusion quotients.

jkljklj

R G-∞-operads

Definition R.1. A G-multicolored operad $\mathcal{O}$ is a G-object in the category of multi-colored operads.

Definition R.2. The *equivariant symmetric groupoid* on the G-set of colors $\mathfrak{C}$ is the Grothendieck category $G \ltimes \Sigma_{\mathfrak{C}}$.

Further, given a word $w \in \Sigma_{\mathfrak{C}}$ we will denote by Σ_w its isotropy subgroup in $\Sigma_{\mathfrak{C}}$ and by $\Sigma_w^{\ltimes G}$ its isotropy subgroup in $G \ltimes \Sigma_{\mathfrak{C}}$. Note that $\Sigma_w \leq \Sigma_w^{\ltimes G}$.

Definition R.3. A G-multicolored operad $\mathcal{O}$ is *locally fibrant* if for each subgroup $H \leq G$ and H-fixed color $b \in \mathfrak{C}^H$ one has that

$$\mathcal{O}(a_1, a_2, \cdots, a_n; c)^K$$

is a Kan complex for each subgroup $K \leq \Sigma_{a_1 a_2 \cdots a_n}^{\ltimes H}$ such that $K \cap \Sigma_{a_1 a_2 \cdots a_n} = *$.

Proposition R.4. *If $\mathcal{O}$ is a locally fibrant operad then $N_{hc}\mathcal{O}$ is a G-∞-operad.*

Proof. It suffices to show that for any equivariant tree $T = (G \cdot |T|)/\Gamma_\phi$, $\phi \colon H \to Aut(|T|)$ one has lifts

$$\begin{tikzcd} W_!\Lambda^e(T) \arrow[r, "f"] \arrow[d] & \mathcal{O} \\ W_!\Omega(T) \arrow[ur, dashed] & \end{tikzcd} \tag{R.5}$$

LOCFIBLIFT EQ

for e any equivariant inner edge of T. Letting $a_1, \cdots, a_n$, c denote the leaves and root of T, one sees that, since e is inner, the vertical map in (LOCFIBLIFT EQ R.5) is an isomorphism at all mapping spaces except that from $a_1, \cdots, a_n$ to c. I.e., it suffices to find a suitable lift in the mapping space diagram

$$\begin{tikzcd} W_!\Lambda^e(T)(a_1, a_2, \cdots, a_n; c) \arrow[r] \arrow[d] & \mathcal{O}\left(f(a_1), f(a_2), \cdots, f(a_n); f(c)\right) \\ W_!\Omega(T)(a_1, a_2, \cdots, a_n; c). \arrow[ur, dashed] & \end{tikzcd} \tag{R.6}$$

LOCFIBLIFT2 EQ

Since the operadic composition maps in $W_!\Omega(T)$ all factor through $W_!\Lambda^e(T)$, one needs only worry about the equivariance of the lift in (LOCFIBLIFT2 EQ R.6). Letting H denote the isotropy of c, it hence suffices to produce a $\Gamma_\phi \simeq \Sigma^{\ltimes H}_{a_1 a_2 \cdots a_n}$-equivariant lift. Noting that $\Sigma_{a_1 a_2 \cdots a_n} = **$ (since the leaves are of course all distinct), the hypothesis that $\mathcal{O}$ is locally fibrant then implies that $\mathcal{O}\left(f(a_1), f(a_2), \cdots, f(a_n); f(c)\right)$ is genuinely $\Sigma^{\ltimes H}_{a_1 a_2 \cdots a_n}$-fibrant, and one hence reduces to checking that

$$W_!\Lambda^e(T)(a_1, a_2, \cdots, a_n; c) \to W_!\Omega(T)(a_1, a_2, \cdots, a_n; c) \tag{R.7}$$

LOCFIBLIFT3 EQ

is a genuine $\Sigma^{\ltimes H}_{a_1 a_2 \cdots a_n}$-trivial cofibration. That this map is both a monomorphism and an underlying weak equivalence follows from the fact that non equivariantly this map is identified with the generalized inner horn of the underlying tree $|T|$ that removes all the faces that collapse a subset of the orbit of e. But now note that each subgroup of Γ_ϕ has the form $\Gamma_{\phi|_{\bar{H}}}$, i.e. the graph of the restriction of ϕ to a subgroup $\bar{H} \leq H$, so that the $\Gamma_{\phi|_{\bar{H}}}$-fixed points of the map (LOCFIBLIFT3 EQ R.7) are identified with a generalized inner horn for the quotient tree $\bar{H}\backslash T$.

Alternatively, one can also compute the mapping space fixed points directly by using the formula

$$\left(\Lambda^A(T) \to \Omega(T)\right)(a_1, \cdots, a_n; c) = (\Lambda^1_1 \to \Delta^1)^{\Box A} \Box (\partial\Delta^1 \to \Delta^1)^{\Box(\{\text{inner edges}\}-A)}$$

$\square$

S Tensor product of open trees

S.1 Broad posets and trees

Throughout we will use the notion of broad poset introduced by Weiss in [We12 8]. We start by briefly recalling the definition and some required notation.

Given a set T we denote by T^+ the free abelian monoid generated by T. Elements of T^+ will be written in tuple notation, such as $\underline{e} = e_1 e_3 e_1 e_2 = e_1 e_1 e_2 e_3 \in$

T^+ for $e_1, e_2, e_3 \in T$ and we will write $e_i \in \underline{e}$ whenever e_i is one of the “letters” appearing in $\underline{e}$. The “empty tuple” of T^+ will be denoted by ϵ.

A (commutative) broad poset structure ([8, Definition 3.2]) on T is a relation $\le$ on (T^+, T) such that

(reflexivity) $e \le e$ (for $e \in T$);

(antisymmetry) if $e \le f$ and $f \le e$ then $e = f$ (for $e, f \in T$);

(broad transitivity) if $f_1 f_2 \cdots f_n = \underline{f} \le e$ and $\underline{g}_i \le f_i$, then $\underline{g}_1 \cdots \underline{g}_n \le e$ (for $e, f_i \in T$, $\underline{f}, \underline{g}_i \in T^+$).

Since the examples of broad posets of interest to us are induced by constructions involving trees, we will refer to the elements of a broad poset as its *edges*.

Definition S.1. A broad poset P is called

- *open* if no broad relations of the form $\epsilon \le e$ hold for any $e \in T$;
- *simple* if for any broad relation $e_1 \cdots e_n \le e$ one has $e_i = e_j$ only if $i = j$.

Notation S.2. A broad poset structure $\le$ on T naturally induces the following preorder relations on T and T^+:

- for $f, e \in T$ we say that f is a *descendent* of e, written $f \le_d e$ if there exists a broad relation $\underline{f} \le e$ such that $f \in \underline{f}$;
- for $\underline{f}, \underline{e} \in T^+$, we write $\underline{f} \le \underline{e}$ if it is possible to write $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$, $\underline{e} = e_1 \cdots e_k$ such that $\underline{f}_i \le e_i$ for $i = 1, \cdots, k$.

Remark S.3. Generally, the preorders just described can be fairly counter-intuitive. For example, it is quite possible to have $ab \le a$, or $aa \le a$ together with $aa \le a$. The case of simple broad posets, however, is much simpler.

SIMPLEBROAD PROP **Proposition S.4.** *Let T be a simple broad poset. Then $\le_d$ (resp. $\le$) is an order relation on T (resp. on T^+). Further:*

(i) if $f_1 \cdots f_n \le e$, then the f_i are $\le_d$ incomparable (in particular, $e\underline{f} \le e$ only if $\underline{f} = \epsilon$);

(ii) if $\underline{f} \le \underline{e} = e_1 \cdots e_n$ with distinct e_i, then the decomposition $\underline{f} = \underline{f}_1 \cdots \underline{f}_n$ with $\underline{f}_i \le e_i$ is unique. In fact, for $g \in \underline{f}$ one has $g \in \underline{f}_i$ if and only if $g \le_d e_i$.

Definition S.5. A leaf $e \in T$ is called

- a *leaf* if there are no $\underline{f} \in T^+$ such that $\underline{f} < e$ (i.e. $\underline{f} \le e$ and $\underline{f} \ne e$);
- a *node* if there is a maximum $\underline{f}$ as above. This maximum is denoted $e^{\uparrow}$.

The following definition is the key purpose of [8].

Definition S.6. A dendroidally ordered set is a finite simple broad poset T such that

- each edge $e \in T$ is either a leaf or a node;
- there is a maximum $r_T \in T$ for $\le_d$, called the *root* of T.

Weiss proves in We12 [8] that the category of dendroidally ordered sets (together with the obvious notion of monotonous function) is equivalent to the category Ω of trees. As such, we will henceforth refer to dendroidally ordered sets simply as *trees*.

INEQMAXIFF LEM

Lemma S.7. *Let T be a tree. For any $e \in T$ there exists a minimum e^l such that $e^l \leq e$. In fact, $e^l = l_1 \cdots l_k$ consists of those leaves l_i such that $l_i \leq_d e$. Further, a broad relation $s_1 \cdots s_n \leq e$ holds if and only if $s_i \leq_d e$ and $s_1^l \cdots s_n^l = e^l$.*

The "further claim" requires T to be open

Proof. Both claims are proven simultaneously by upward $\leq_d$ induction on e. The leaf case is obvious. Otherwise, let $\underline{s} \leq e^{\uparrow} < e$ be any non identity relation. Write $e^{\uparrow} = e_1 \cdots e_r$ and $\underline{s} = \underline{s}_1 \cdots \underline{s}_k$ so that $\underline{s}_i \leq e_i$. By induction, $e_i^l \leq \underline{s}_i \leq e_i$ where e_i^l consists of the leaves above e_i and one can hence set $e^l = e_1^l \cdots e_n^l$. □

S.2 Tensor products

Given $\underline{e} \in S^+$, $\underline{f} \in T^+$, we will let $\underline{e} \times \underline{f} \in (S \times T)^+$ denote the obvious tuple whose elements $(a, b) \in \underline{e} \times \underline{f}$ are those pairs with $a \in \underline{e}$, $b \in \underline{f}$.

Definition S.8. Given broad posets S, T, their *tensor product* $S \otimes T$ is the broad poset whose underlying set is $S \times T$ and whose relations are generated by relations of the form $\underline{s} \times t \leq (s, t)$ (resp. $s \times \underline{t} \leq (s, t)$) for $s \in S$, $t \in T$ and $\underline{s} \leq s$ (resp. $\underline{t} \leq t$) a broad relation in S (resp. T).

TENSORCHILDREN PROP

Proposition S.9. *Suppose S, T are open trees. Then:*

(i) $S \otimes T$ is open and simple. Further, if $(s_1, t_1) \cdots (s_n, t_n) \leq (s, t)$ then the pairs s_i, s_j and t_i, t_j are both $\leq_d$ comparable only if $i = j$;

(ii) an edge $(s, t) \in S \otimes T$ has one of three types:

- *(leaf) it is leaf if both $s \in S$, $t \in T$ are leaves;*
- *(node) it is a node if $s \in S$ is a leaf and $t \in T$ is a node (resp. if $t \in T$ is a leaf and $s \in S$ is a node). In fact, $(s, t)^{\uparrow} = s \times t^{\uparrow}$ (resp. $(s, t)^{\uparrow} = s^{\uparrow} \times t$);*
- *(fork) if $s \in S$, $t \in T$ are both nodes then there are exactly two maximal $\underline{f}$ such that $\underline{f} < (s, t)$, namely $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$. We call such a (s, t) a fork.*

Proof. We first show (i). The open condition is obvious and the "further" condition suffices for simplicity. Proposition SIMPLEBROAD PROP V.4 combined with the fact that in a tree $e \leq_d e', e \leq_d e''$ only if e', e'' are $\leq_d$ comparable (due to the root) implies that the "further" condition is preserved by the generating broad relations (this requires a little induction).

We now show (ii). Only the fork case requires proof. In fact, it is obvious that only $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$ can possibly be maximal, hence one needs only verify that neither $s \times t^{\uparrow} \leq s^{\uparrow} \times t$ nor $s^{\uparrow} \times t \leq s \times t^{\uparrow}$. This follows since the S coordinates of the pairs in the tuple $s^{\uparrow} \times t$ are $<_d$ than those in $s \times t^{\uparrow}$ and vice versa. □

Remark S.10. Note that in the non open case, it is possible to have $e^{\uparrow} = \epsilon$, in which case the final argument in the previous proof fails.

In order to simplify notation, for a fork $e = (s,t) \in S \otimes T$ we will write $e^{\uparrow S} = s^{\uparrow} \times t$, $e^{\uparrow T} = s \times t^{\uparrow}$ and $e^{\uparrow S,T} = s^{\uparrow} \times t^{\uparrow}$.

Proposition S.11. *In the context of the previous result, suppose that*

$$\underline{e} = (s_1,t_1)(s_2,t_2)\cdots(s_n,t_n) \leq (s,t) = e.$$

Then, assuming $s \in S$ or $t \in T$ are nodes as appropriate,

(i) $\underline{e} \leq e^{\uparrow S}$ (resp. $\underline{e} \leq e^{\uparrow T}$)$s^{\uparrow} \times t$ if and only if $s_i \neq s, \forall i$ (resp. $t_i \neq t, \forall i$);

(ii) $a \leq s^{\uparrow} \times t^{\uparrow}$ if and only if both $b_i \neq s$ and $c_i \neq t, \forall i$

Proof. Only the "if" directions need proof, and the proof follows by upward $\leq_d$ induction on s,t. The base cases of either s or t a leaf are obvious.

TENSORCHILDREN PROP

Otherwise, let $\underline{e}$ satisfy the "if" condition in (i). Since (Proposition S.9(ii)) it is necessarily $\underline{e} \leq e^{\uparrow S}$ or $\underline{e} \leq e^{\uparrow T}$ we can assume it is the latter case. One can hence write $t^{\uparrow} = u_1 \cdots u_k$, $\underline{e} = \underline{e}_1 \cdots \underline{e}_k$ so that $\underline{e}_i \leq (s,u_i)$. The induction hypothesis now yields that $\underline{e}_i \leq (s,u_i)^{\uparrow S} = s^{\uparrow} \times u_i$, hence

$$\underline{e} = \underline{e}_1 \cdots \underline{e}_l \leq (s^{\uparrow} \times u_1)\cdots(s^{\uparrow} \times u_l) = s^{\uparrow} \times t^{\uparrow} \leq s^{\uparrow} \times t = e^{\uparrow S}. \tag{S.12}$$

INEQUALS EQ

The proof of (iii) is identical except by disregarding the last inequality in (V.30). INEQUALS EQ $\square$

Corollary S.13. *$\underline{e} \leq e^{\uparrow S,T}$ if and only if both $\underline{e} \leq e^{\uparrow S}$ and $\underline{e} \leq e^{\uparrow T}$.*

OPENREL LEM

Lemma S.14. *Let S,T be open trees. It is impossible for there to simultaneously exist broad relations in $S \otimes T$ of the form $a \leq e$ and $ab \leq e$ with $b \neq \epsilon$.*

Proof. Recall that $e = (s,t)$. The proof will follow by downward induction of (s,t), with the case of two leaves being obvious. W.l.o.g. we can assume $ab \leq e^{\uparrow} \leq e$, and the result now follows by induction by simply breaking ab apart accordingly to $e^{\uparrow}$. $\square$

Proposition S.15. *Let R,S,T be open trees. If $R \xrightarrow{f} S \otimes T$ is an underlying injection of sets, then $R \simeq f(R)$, i.e., the broad relations between elements in $f(R)$ are exactly the ones induced from R.*

Proof. To simplify notation we will represent an edge $r \in R$ and its image $f(r) \in S \otimes T$ by the same letter r, and instead decorate the broad relations $\leq^R$, $\leq^{S \otimes T}$.

Let $r_1 \cdots r_n \leq^{S \otimes T} r$ be a broad relation in $f(R)$. There is a maximal $s \in R$ such that $r_i, r \leq^R_d s$ and we will prove that it must be $r_1 \cdots r_n \leq^R r$ by upward $\leq^R_d$ induction on s, with the leaf case being obvious by openness and simplicity.

Otherwise, either: (i) there exists an i such that $r_i \nleq^R_d r$, in which case one easily builds a nonsimple broad relation in $S \otimes T$ for the root r_R (this needs extra argument); (ii) by Lemma S.7 INEQMAXIFF LEM $r_1 \cdots r_n \leq^R r$ holds if and only if $r^l_1 \cdots r^l_n = r^l$. But since $r^l_1 \cdots r^l_n$ can not have repetitions ($S \otimes T$ is simple) the latter must be the case by Lemma S.14. OPENREL LEM $\square$

Remark S.16. The previous result shows that there is a simple inductive procedure to build sub-broad posets of $S \otimes T$ that are trees: one first picks a "root edge" e and any broad relation $e_1 \cdots e_n < e$, then one picks any broad relations $\underline{e}_i < e_i$, and so on iteratively.

It is hence clear that any subtree of $S \otimes T$ is always a subface of (at least) one maximal subtree where necessarily all the generating broad relations have the form $e^{\uparrow S} \leq e$ or $e^{\uparrow T} \leq e$.

Definition S.17. A subtree R of $S \otimes T$ is called *elementary* if its generating broad relations are all of the form $e^{\uparrow S} \leq e$ or $e^{\uparrow T} \leq e$. We call such relations S-vertices, T-vertices of R.

Further, an elementary subtree is also called *initial* if R contains the double root (r_S, r_T).

We will make use of an order relation on initial elementary subtrees of $S \otimes T$. Write

$$R \leq_{lex} R'$$

whenever R' is obtained from R by replacing the intermediate edges in a string of broad relations $e^{\uparrow S,T} \leq e^{\uparrow S} \leq e$ occurring in R by the intermediate edges in $e^{\uparrow S,T} \leq e^{\uparrow T} \leq e$ occurring in R'.

Proposition S.18. *$\leq_{lex}$ induces a partial order on the set of initial elementary subtrees of $R \otimes S$.*

Further, $\leq_{lex}$ together with the inclusion $\subset$ induce a partial order as well, and we denote this latter order simply $\leq$.

Proof. One needs only check antisymmetry. Let $f(R)$ count the number of pairs (v_S, v_T) of an S-vertex and T-vertex in R such that v_S occurs before v_T (following $\leq$). Since the generating relations of $\leq_{lex}$ strictly decrease f, it defines a partial order. Similarly, if $g(R)$ counts the sum of the distance (i.e. number of generating relations) between each leaf and the root, $\leq_{lex}$ preserves g and $\subset$ decreases it, hence $\leq$ is a partial order. $\square$

Definition S.19. Let e denote a fixed inner edge of T. An initial elementary subtree R of $S \otimes T$ is called *e-internal* if it contains an edge of the form (s, e), henceforth abbreviated as e_s, and the T-vertex $e_s^{\uparrow T} \leq e_s$.

We will denote the subposet of such trees by $(\mathcal{ME}_e(S \otimes T), \leq)$.

INTERCHANGE LEM

Lemma S.20. *Let R be an initial elementary subtree of $S \otimes T$ with root vertex the T-vertex $e_s^{\uparrow T} \leq e_s$ and suppose that $(e_s)_R^l \leq e_s^{\uparrow S}$.*

Then there exists a subtree R' such that $R' \leq_{lex} R$ and R' contains the relations $e_s^{\uparrow S,T} \leq e_s^{\uparrow S} \leq e_s$.

Proof. We argue by induction on the total of the distances from the leaves to the root. The case of e_s a leaf is obvious. Otherwise, for each $e_i \in e^{\uparrow}$ either $(s, e_i)^{\uparrow S} \leq (s, e_i)$ or $(s, e_i)^{\uparrow T} \leq (s, e_i)$ and in the latter case the induction hypothesis allows us find $R'' \leq R$ such that R'' contains all relations $(s, e_i)^{\uparrow S,T} \leq (s, e_i)^{\uparrow S} \leq (s, e_i)$. But now R' contains the relations $e_s^{\uparrow S,T} \leq e_s^{\uparrow T} \leq e_s$ hence a final replacement yields the desired $R' \leq R'' \leq R$. $\square$

COMMONFACE LEM

Lemma S.21. *If F is a common face (resp. inner face) of two initial elementary subtrees R, R', then F is also a face of a subtree R'' such that $R'' \leq R$, $R'' \leq R$ (resp. $R'' \leq_{lex} R$, $R'' \leq_{lex} R$). In fact, in the inner face case the $\leq_{lex}$ inequalites factor through generating $\leq_{lex}$ inequalites involving only trees having F as an inner face.*

Proof. One is clearly free to remove any edges below (with regard to $\leq_d$) the leaves of F, so that F, R, R' now have the same leaves, and to then replace F by the intersection of the sets of edges in R and R' (which is still a tree by Lemma S.7).

Since the root $r = (r_S, r_T)$ is now in all of F, R, R', the case where the root vertices of R, R' match follows by induction. Otherwise, we can assume R contains the vertex $r^{\uparrow S} \leq r$ and R' the vertex $r^{\uparrow T} \leq r$. Let $\underline{e} \leq r$ denote the root vertex of F and set $R'_{\geq \underline{e}}$ be the smallest subtree of R' containing that root vertex. Lemma V.36 then applies to $R'_{\geq \underline{e}}$, and we can hence find $R'' \leq_{lex} R'$ with a strictly larger intersection with R, finishing the argument. □

Recall that a subset B of a poset $\mathcal{P}$ is called *convex* if $a \leq b$ and $b \in B$ implies $a \in B$.

Proposition S.22. *Let S, T be open trees and $e \in T$ an inner edge. Set $A = \Lambda^e S \otimes T \coprod_{\Lambda^e S \otimes \partial T} S \otimes \partial T$. Then for any convex subsets $B \subset B'$ of the poset $\mathcal{ME}_e(S \otimes T)$ one has that*

$$A \cup \bigcup_{Q \in B} Q \to A \cup \bigcup_{Q \in B'} Q$$

is an inner anodyne extension.

Proof. Without loss of generality we can assume that B' is obtained from B by adding a single e-internal initial elementary tree R with e_s its corresponding edge.

We first note that the outer faces of R are in $A \cup \bigcup_{Q \in B} Q$. For any top outer cluster C, either R/C is still e-internal or else it must have the cluster $C = (e_s)^{\uparrow T} \leq e_s$ and R has no other T-vertex over an edge with T-coordinate e. Hence R/C misses any edge of T below (in $\leq_d$) e, and is hence in A. In the rare case of a root cluster, T/C misses either $r_S \in S$ or $r_T \in T$, and in hence in A as well.

We now show that e_s is a characteristic edge (in the sense of cite), i.e. that, for each inner subface F of R containing the edge e_s, then F is in $A \cup \bigcup_{Q \in B} Q$ if and only if F/e_s is.

Suppose first that F/e_s is in A but F is not in A, in which case the edge missed by F/e_s but not by F is necessarily $s \in S$ (since missing e is irrelevant as far as A is concerned). Letting $R^e_{\leq_d e_s}$ denote the smallest elementary subtree of R with $e_s = (s, e)$ as its root and any edges $(s, \bar{e}) <_d (s, e)$ as its inner edges (note that if any such edge was a leaf, F/e_s could not miss the edge $s \in S$). Clearly F must collapse all of the inner edges of $R^e_{\leq_d e_s}$, hence applying Lemma V.36 to $R^e_{\leq_d e_s}$ we see that F is a subface of some $Q <_{lex} R$, hence already contained in $A \cup \bigcup_{Q \in B} Q$.

Suppose now that F/e_s is an inner subface of some $Q \in B$, $R \nleq Q$. Lemma V.38 then that ensures there is a $Q' < B$ containing F/e_s and, in fact, since F/e_s is an inner face it must be $Q' <_{lex} B$, and by the "in fact" part of Lemma V.38 one can assume that this is a generating $\leq_{lex}$ relation. But then Q' necessarily contains e_s, since generating relations don't remove edges whose vertex is a T-vertex.

Finally, we now let $I^{\hat{e}_i}$ denote the subset of inner edges of R distinct from

e_i and claim that for any concave[6] subsets $A \subset A' \subset \mathcal{P}(I^{\hat{e}_i})$ the map

$$A \cup \bigcup_{Q \in B} Q \cup \bigcup_{E \in A} R/E \to A \cup \bigcup_{Q \in B} Q \cup \bigcup_{E \in A'} R/E \tag{S.23}$$ ATTACHTREE EQ

is inner anodyine. We argue by induction on A and again we can assume that A' is obtained from A by adding a single $D \in \mathcal{P}(I^{\hat{e}_i})$. But since the concavity of A, A' and the characteristic edge condition imply that the only faces of F/D not in the source of (ATTACHTREE EQ V.41) are precisely F/D and $F/(D \cup e_s)$, we conclude that (ATTACHTREE EQ V.41) is in fact a pushout of $\Lambda^{e_i}(F/D) \to \Omega(F/D)$, finishing the proof. □

Notation S.24. The category Ω_G *of* G*-equivariant trees* is the subcategory of the presheaf category $Set^{\Omega^{op} \times G^{op}}$ formed by the non arboreal quotients of representables.

Hence, each G-tree T comes with a canonical choice of graph subgroup Γ_T for a canonical morphism $G \supset H_T \xrightarrow{\phi_T} \Sigma_{|T|}$ (this should need adjusting).

In the G-equivariant case, given an inner edge, write instead $\mathcal{ME}_{Ge}(S \otimes T)$ for the poset of initial elementary trees containing at least one T-vertex of the form $(ge)_s^{\uparrow T} \le (ge)_s$ (alternately, one has $\mathcal{ME}_{Ge}(S \otimes T) = \bigcup_{g \in G} \mathcal{ME}_{ge}(S \otimes T)$). Note that in this case the group G acts on the poset $\mathcal{ME}_{Ge}(S \otimes T)$ as well.

The following is the equivariant version of the previous result.

Proposition S.25. *Let S, T be open G-trees and $e \in T$ an inner edge. Set*

$$A = \Lambda^{Ge} S \otimes T \coprod_{\Lambda^{Ge} S \otimes \partial T} S \otimes \partial T.$$

Then for any convex G-equivariant subsets $B \subset B'$ of the poset $\mathcal{ME}_{Ge}(S \otimes T)$ one has that

$$A \cup \bigcup_{Q \in B} Q \to A \cup \bigcup_{Q \in B'} Q$$

is an inner anodyne extension.

Proof. Note that we are free to assume $H_S = H_T = G$, since otherwise $S \otimes$ splits as a disjoint union where each G-connected component has isotropy an intersection of conjugates of H_S, and H_T, and one easily checks that the result needs only be verified for the restricted actions.

As before we can now assume B' is obtained from B by adding the G-orbit of a single tree R which we can choose to have characteristic edge e_s, and let $H_R \subset G$ be as in the notation above.

Again top faces R/C are either a tree in $\mathcal{ME}_{Ge}(S \otimes T)$ or else C must have been the cluster $e_s^{\uparrow T} \le e_s$ and R has no other T-vertex over an edge with T-coordinate in Ge, hence R/C is certainly in A.

We now check that $H_R e_s$ is a characteristic edge orbit of R: i.e., that for each inner subface F of R containing the edge e_s, then $F/H_F e_s$ is in $A \cup \bigcup_{Q \in B} Q$ if and only if F is.

First, if $F/H_F e_s$ is in A but F isn't, the relevant color that $F/H_F e_s$ misses but F doesn't must be one of the colors in $H_F s$. Therefore, for some $h \in H_F$, F must have have been obtained from R by collapsing at least all the edges in one of the subtrees on $R^{he}_{\le_d he_{h s}}$ (and, in fact, by the Γ_F equivariance of F, F

[6] This is the notion dual to "convex".

must have been obtained by collapsing all edges in all such trees). Lemma INTERCHANGE LEM IV.36 again shows that F must have already been in $A \cup \bigcup_{Q \in B} Q$.

The case of $F/H_F e_s$ a inner subface of some $Q \in B$, $R \nleq Q$ follows by using Lemma COMMONFACE LEM IV.38 exactly as in the previous proof.

Finally, the final part of the argument follows similarly: set $I^{\widehat{H_R e_s}}$ to be the H_R-set of those inner edges of R not in $H_R e_s$, so that we argue by induction over H_R-equivariant concave subsets $A \subset A' \subset \mathcal{P}(I^{\widehat{H_R e_s}})$ that the map

$$A \cup \bigcup_{Q \in B} Q \cup G \cdot_{H_R} \left(\bigcup_{E \in A} R/E \right) \to A \cup \bigcup_{Q \in B} Q \cup G \cdot_{H_R} \left(\bigcup_{E \in A'} R/E \right) \tag{S.26}$$

ATTACHTREE EQ2

is inner anodyne. As before we can assume that A' is obtained from A by adding the H_R-orbit of a single D, and we hence see by the "characterist edge orbit" argument that the map above is a pushout of

$$G \cdot_{H_{F/D}} \left(\Lambda^{H_{F/D} e_s}(F/D) \to \Omega(F/D) \right),$$

finishing the proof. □

T Open vs non open trees

T.1 Characterizing vertices

NSORCHILDREN UNOPEN PROP

Proposition T.1. *Suppose S be an open tree. Then:*

(i) $S \otimes T$ is simple. Further, if $(s_1, t_1) \cdots (s_n, t_n) \leq (s, t)$ then the pairs s_i, s_j and t_i, t_j are both $\leq_d$ comparable only if $i = j$;

(ii) an edge $(s, t) \in S \otimes T$ has one of five types:

(leaf) it is leaf if both $s \in S$, $t \in T$ are leaves;

(stump) it is a stump of $s \in S$ is a leaf and $t \in T$ a stump;

(node) it is a node if $s \in S$ is a leaf and $t \in T$ is a node (resp. if $t \in T$ is a leaf and $s \in S$ is a node). In fact, $(s,t)^{\uparrow} = s \times t^{\uparrow}$ (resp. $(s,t)^{\uparrow} = s^{\uparrow} \times t$);

(stump node) it is a node such that $\epsilon \leq (s, t)$ if $s \in S$ is a node and $t \in T$ a stump. In fact $(s,t)^{\uparrow} = s^{\uparrow} \times t$;

(fork) if $s \in S$, $t \in T$ are both nodes then there are exactly two maximal $\underline{f}$ such that $\underline{f} < (s, t)$, namely $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$. We call such a (s, t) a fork.

Proof. We first show (i). The "further" condition suffices for simplicity. Proposition SIMPLEBROAD PROP IV.4 combined with the fact that in a tree $e \leq_d e', e \leq_d e''$ only if e', e'' are $\leq_d$ comparable (due to the root) implies that the "further" condition is preserved by the generating broad relations (this requires a little induction).

We now show (ii). Only the fork case requires proof. In fact, it is obvious that only $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$ can possibly be maximal, hence one needs only verify that neither $s \times t^{\uparrow} \leq s^{\uparrow} \times t$ nor $s^{\uparrow} \times t \leq s \times t^{\uparrow}$. This follows since the S coordinates of the pairs in the tuple $s^{\uparrow} \times t$ are $<_d$ than those in $s \times t^{\uparrow}$ and vice versa. □

T.2 Full subtrees

INEQMAXIFF NOPEN LEM

Lemma T.2. *Let T be a tree. For any $e \in T$ there exists a minimum e^l such that $e^l \le e$. In fact, $e^l = l_1 \cdots l_k$ consists of those leaves l_i such that $l_i \le_d e$.*

Further, a broad relation $\underline{s} = s_1 \cdots s_n \le e$ holds if and only if $s_i \le_d e$, the s_i are $\le_d$ incomparable and $s_1^l \cdots s_n^l = e^l$.

Proof. The proof is by upward $\le_d$ induction on e. The leaf case is obvious. Otherwise, let $\underline{s} \le e^{\uparrow} < e$ be any non identity relation. Write $e^{\uparrow} = e_1 \cdots e_k$ and $\underline{s} = \underline{s}_1 \cdots \underline{s}_k$ so that $\underline{s}_i \le e_i$ (note that possibly $k = 0$, in which case necessarily $e^{\uparrow} = \underline{s} = \epsilon$). By induction, $e_i^l \le \underline{s}_i \le e_i$ where e_i^l consists of the leaves above e_i and one can hence set $e^l = e_1^l \cdots e_k^l$.

The "only if" half of the "further" condition is obvious, and the "if" half follows by the same induction argument: incomparability guarantees that $e \notin \underline{s}$ and one can hence uniquely write $\underline{s} = \underline{s}_1 \cdots \underline{s}_k$ such that $r \in \underline{s}_i$ if and only if $r \le_d (e_i)$; the induction hypothesis then yields the result. □

EQMAXIFF NOPENTENSOR COR

Corollary T.3. *Let S, T be trees. For any $e = (s,t) \in S \otimes T$ there exists a minimum e^l such that $e^l \le e$. In fact, $e^l = s^l \times t^l$.*

Proof. The proof is by upward $\le_d$ induction on e. The case of s, t both leaves is obvious. Otherwise, for any non-identity relation either $\underline{s} \le e^{\uparrow S} < e$ or $\underline{s} \le e^{\uparrow T} < e$, and the analysis in the previous proof applies in either case. □

Lemma T.4. *Let T be a tree and suppose that $e_i \in T$, $1 \le 1 \le n$ are $\le_d$ incomparable edges. Then there exists a broad relation $\underline{e} \le r_T$ such that $e_i \in \underline{e}$, $1 \le i \le n$.*

Proof. The proof is by induction on the sum of the distances from the leaves to the root in T. Clearly r_T is one of the e_i if and only if $\{e_i\} = \{r\}$. Otherwise one can write $r^{\uparrow} = r_1 \cdots r_k$ then split the e_i. The induction hypothesis applies. □

BROADEXIST LEM

Proposition T.5. *Let R, S, T be trees.*

Suppose $R \xrightarrow{f} S \otimes T$ is an underlying injection of sets such that, if $l \in R$ is a leaf then it is not $\epsilon \le f(l)$. Then $R \simeq f(R)$, i.e., the broad relations between elements in $f(R)$ are exactly the ones induced from R.

Proof. To simplify notation we will represent an edge $r \in R$ and its image $f(r) \in S \otimes T$ by the same letter r, and instead decorate the broad relations $\le^R$, $\le^{S \otimes T}$.

We need only show that no broad relation $e = e_1 \cdots e_n \le^{S \otimes T} e$ can fail any of the three conditions in Lemma V.9 for R: (i) if $e_i \nleq_d^R e$ for some i, necessarily also $e \nleq_d^R e_i$ (since $\le_d^{S \otimes T}$ is a partial order), and hence Lemma applied to e_i and could be used to produce a non simple broad relation in $S \otimes T$ for r_R; (ii) if e_i and e_j are $\le_d^R$ comparable for $i \neq j$, one obtains a non simple broad relation in $S \otimes T$ for e; (iii) finally, if $e_1^{l_R} \cdots e_n^{l_R} \neq e^{l_R}$, by part (ii)it must be that $e_1^{l_R} \cdots e_n^{l_R}$ misses some of the leaves in e^{l_R}. But our hypothesis is that $l^{l_{S \otimes T}} \neq \epsilon$ for l any leaf of R, and it must hence also be $e_1^{l_{S\otimes T}} \cdots e_n^{l_{S \otimes T}} \neq e^{l_{S \otimes T}}$, a contradiction. □

T.3 Missing stumps

Definition T.6. A *stump* v of a tree T is a broad relation of the form

$$\epsilon = a^{\uparrow} \le a.$$

Definition T.7. We say that a face F of T misses a stump v is F is a subface of $T - v$.

Lemma T.8. *Let S, T be trees with S open and let $v = \epsilon \leq a$ be a stump in T. Then a broad relation*

$$\underline{f} = f_1 f_2 \cdots f_k \leq e$$

in $S \otimes T$ is a broad relation in $S \otimes (T - v)$ if and only if

$$e^{l, S \otimes (T-v)} \leq \underline{f}.$$

Proof. Only the "if" direction needs proof. We argue by upward $\leq_d$ induction on e.

The base case is that of $e = (s,t)$ for s a leaf in S and t a leaf in $T - v$. Both cases of t a leaf of T or $t = a$ are obvious.

Otherwise, either $\underline{f} \leq e^{\uparrow S} \leq e$ or $\underline{f} \leq e^{\uparrow T} \leq e$, where in the later case the assumption guarantees $e \neq (s,a)$. Writing $e^{\uparrow *}$ to denote either $e^{\uparrow S}$ or $e^{\uparrow T}$ as appropriate, the relation $e^{\uparrow *} \leq e$ is in $S \otimes (T - v)$ and writing $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$ and $e^{\uparrow *} = e_1 \cdots e_k$ so that $\underline{f}_i \leq e_i$ the induction hypothesis shows that these relations are also in fact in $S \otimes (T - v)$. □

Corollary T.9. *A collection of broad relations of the form $\underline{g}_i \leq f_i$, $f_1 \cdots f_k \leq e$ are all in $S \otimes (T - v)$ if and only if the composite relation $\underline{g}_1 \cdots \underline{g}_k \leq e$.*

Unlike with other types of faces, detecting when a face F misses a stump v of T can be somewhat subtle. The following result gives a useful criterion.

Corollary T.10. *Let S, T be trees with S open and let $v = \epsilon \leq a$ be a stump in T. Then a face $R \subset S \otimes T$ is contained in $S \otimes (T - v)$ if and only if*

$$r_R^{l, S \otimes (T-v)} \leq r_R^{l,R}.$$

Proof. The "only if" direction is obvious and the "if" direction follows by the previous result since any relation in R is a factor of $r_R^{l,R} \leq r_R$. □

T.4 Pushout-product of normal monomorphisms

Definition T.11. Let S,T be trees. A subtree $R \subset S \otimes T$ is called a *closed subtree* if the broad relations in R coincide with those in its image. Or, in other words, a closed subtree is one which satisfies either of the equivalent conditions in Lemma T.2.

Further, given any subtree $R \to S \otimes T$, we define its closure $\bar{R}$ to be the smallest closed subtree containing R. Note that by Lemma T.2 $\bar{R}$ is obtained by simply attaching stump nodes to any leaves of R that are nullary in $S \otimes T$.

Remark T.12. Note that if R is a closed subtree, then any inner subface or non-empty outer cluster subface F of R is necessarily also a closed subtree.

CLOSEDSUBTREE LEM **Lemma T.13.** *If R is a closed subtree of S, then $R \otimes T$ is a closed broad subposet of $S \otimes T$, i.e., it contains all broad relations on its image.*

Proof. Any broad relation in $R \otimes T$ sits inside some elementary maximal subtree of $R \otimes T$ and Lemma T.2 applies to those (regarded now as subtrees of $S \otimes T$). □

Proposition T.14. *Let S, T be trees. Then*

$$\partial S \otimes T \to S \otimes T, \qquad \partial S \otimes \partial T \to S \otimes \partial T$$

are monomorphisms of dendroidal sets.

Proof. Since $\otimes$ commutes with colimits in each variable,

$$\partial S \otimes T = \mathrm{colim}_{F \in faces(S) - \{S\}} F \otimes T.$$

Since each map $F \otimes T \to S \otimes T$ is a monomorphism of dendroidal sets, it suffices to check that, for each subtree $R \to S \otimes T$, there is a minimal subface F of S such that there is a factorization $R \to F \otimes T \to S \otimes T$.

Letting $\{e_i\}$ be the set of S-edges missed by R, we first set $\bar{F} = S/\{e_i\}$ so that, by Lemma CLOSEDSUBTREE LEM V.20, one has factorization $R \to \bar{F} \otimes T \to S \otimes T$. Now letting $\{v_i\}$ denote the $\bar{F}$ stumps missed by R we set $F = \bar{F} - \{v_i\}$ and we claim that F does the trick. I.e., letting K be such that there is a factorization $R \to K \otimes T \to S \otimes T$, we claim that F is a subface of K. Since Lemma V.20 CLOSEDSUBTREE LEM implies one has factorization $R \to K \otimes T \to \bar{K} \otimes T \to S \otimes T$ with the latter map full, we can assume without loss of generality that $\bar{K} = S$. Letting $\{w_i\}$ denote the set of S nodes missed by R, we see that it suffices to deal with the case $K = S - \{w_i\}$, i.e., we need to show that $\bar{F} - \{v_i\} \subset S - \{w_i\}$. In other words, we need to check that any stump v of $\bar{F} - \{v_i\}$ remains a nullary edge in $\subset S - \{w_i\}$. This amounts to showing that none of the w_i can be below v on S. But by the characterization (add char) it must be the case that for some leaf l of T, the descendent of (a_v, t) in F is not a leaf, and for any stump w below v, the descendent of (a_w, t) matches that of (a_v, t), proving the result.

HERE prove second half. □

U BLABLA RIGHT NOW

Definition U.1. Let S, T be trees and $V = \{v_i = (\epsilon \le a_i)\}$, $W = \{w_j = (\epsilon \le b_j)\}$ subsets of the stumps of S, T, respectively.

We say that a subtree R of $S \otimes T$ *misses V and W* if R factors through $(S - V) \otimes (T - W)$.

Remark U.2. Setting $W = \varnothing$ (resp. $V = \varnothing$) one can then analogously define the notion "R misses V" (resp. "R misses W").

Note, however, that it follows from the following lemma that the notion "R misses V and W" does not generaly coincide with the notion "R misses V and R misses W".

Lemma U.3. *Let S, T be trees and $V = \{v_i = (\epsilon \le a_i)\}$, $W = \{w_j = (\epsilon \le b_j)\}$ subsets of the stumps of S, T, respectively. Then a broad relation*

$$\underline{f} = f_1 f_2 \cdots f_k \le e$$

in $S \otimes T$ is a broad relation in $(S - V) \otimes (T - W)$ if and only if

$$e^{l,(S-V)\otimes(T-W)} \le \underline{f}.$$

Proof. Only the "if" direction needs proof. We argue by upward $\leq_d$ induction on $e = (s,t)$.

The base case, that of s a leaf of $S - V$ and T a leaf of $T - W$, is obvious (we note that while this base case may possibly be vacuous, the remainder of the proof nonetheless follows).

Otherwise, either $\underline{f} \leq e^{\uparrow S} \leq e$ or $\underline{f} \leq e^{\uparrow T} \leq e$ and our assumption ensures, accordingly, that $s \notin \{a_i\}$ or $t \notin \{b_j\}$. Writing $e^{\uparrow *}$ to denote either $e^{\uparrow S}$ or $e^{\uparrow T}$ as appropriate, this later observation guarantees that the relation $e^{\uparrow *} \leq e$ is in $(S - V) \otimes (T - W)$. Hence, writing $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$ and $e^{\uparrow *} = e_1 \cdots e_k$ so that $\underline{f}_i \leq e_i$, the induction hypothesis shows that these relations are also in fact in $(S - V) \otimes (T - W)$. $\square$

Proposition U.4. *Let S, T be trees. Then*

$$\partial S \otimes T \to S \otimes T, \qquad S \otimes \partial T \to S \otimes \partial T$$

are monomorphisms of dendroidal sets.

Further, if either of S or T has at most one stump, then so are the maps

$$\partial S \otimes \partial T \to \partial S \otimes T, \qquad \partial S \otimes \partial T \to S \otimes \partial T.$$

Finally, if either of S or T is open (i.e. has no stumps), then the map

$$\partial S \otimes T \coprod_{\partial S \otimes \partial T} S \otimes \partial T \to S \otimes T$$

is a monomorphism as well.

Proof. Since $\otimes$ commutes with colimits in each variable,

$$\partial S \otimes T = \operatorname{colim}_{F \in faces(S) - \{S\}} F \otimes T.$$

Since each map $F \otimes T \to S \otimes T$ is a monomorphism of dendroidal sets, it suffices to check that, for each subtree $R \to S \otimes T$, the poset $\mathcal{P}_R = \{F | F \neq S, R \to F \otimes T\}$ is connected. Suppose first that R misses the inner edge e of S. Then the map $F \mapsto F/e$ retracts the poset $\mathcal{P}_R$ onto the subposet $\{F | F \to S/e, R \to F \otimes T\}$ which is of course connected since there it has a maximum S/e. Likewise, the same argument works whenever R misses a non-empty outer cluster, and we can hence assume that R misses neither, in which case all possible F must be such that $\bar{F} = S$. But then if one lets $\{v_i\}$ denote the subset of the stumps of S that are missed by R, it is clear that $F - \{v_i\}$ is the minimum of $\mathcal{P}_R$.

To prove the second part of the result it suffices, by the first part, to show that $\partial S \otimes \partial T \to S \otimes T$ is a monomorphism. Just as before, the source of this map is given by a colimit over a poset of pairs F_S F_T of subfaces of S and T, and again we need to show that the pairs such that $R \to F_S \otimes F_T$ form a connected poset. Repeating the argument from the previous paragraph, if R misses some internal edge e of S, then one can collapse the poset (in two steps) to the subposet of pairs of the form $(S/E, F_T)$, proving the result. Likewise for the case of R missing a non empty outer cluster. Otherwise, if must be the case that F misses only the unique stump v of S. But in this latter case the condition to be verified reduces to the first part of the result with S replaced by $S - v$.

Finally, for the "finally" statement, by the previous parts, it suffices to check that if a face F factors through both $\partial S \otimes T$ and $S \otimes \partial T$, then it in fact factors through $\partial S \otimes \partial T$ as well. Since S is open, F factors through $\partial S \otimes T$ precisely if it misses some S colors, let us say $\{s_i\}$. But then, if $F \to S \otimes F_T$, for some face F_T of T, it must necessarily also be the case that $F \to (S - \{s_i\}) \otimes F_T$, finishing the proof. □

Remark U.5. The further condition of the previous result in face fails whenever both trees have at least two stumps. For a representative example, consider two stumpy trees $S = (\epsilon \leq ab \leq c)$ and $T = (\epsilon \leq 12 \leq 3)$. Then the subtree of $S \otimes T$ generated by the relations $a_1 a_2 b_1 b_2 \leq c_3$, $\epsilon \leq a_2$, $\epsilon \leq b_1$ provides a counter-example and in fact, it is straightforward to modify this counter-example to always work.

V Pushout-product axiom of equivariant dendroidal sets

V.1 Broad posets and trees

Throughout this section we will use the notion of broad poset introduced by Weiss in [8]. We start by briefly recalling the definition and some notation.

Given a set T we denote by T^+ the free abelian monoid generated by T. Elements of T^+ will be written in tuple notation, such as $\underline{e} = e_1 e_3 e_1 e_2 = e_1 e_1 e_2 e_3 \in T^+$ for $e_1, e_2, e_3 \in T$ and we will write $e_i \in \underline{e}$ whenever e_i is one of the "letters" appearing in $\underline{e}$. The "empty tuple" of T^+ will be denoted by ϵ.

A (commutative) broad poset structure ([8, Definition 3.2]) on T is a relation $\leq$ on (T^+, T) such that

(reflexivity) $e \leq e$ (for $e \in T$);

(antisymmetry) if $e \leq f$ and $f \leq e$ then $e = f$ (for $e, f \in T$);

(broad transitivity) if $f_1 f_2 \cdots f_n = \underline{f} \leq e$ and $\underline{g}_i \leq f_i$, then $\underline{g}_1 \cdots \underline{g}_n \leq e$ (for $e, f_i \in T$, $\underline{f}, \underline{g}_i \in T^+$).

Since the main examples of broad posets are induced by constructions involving trees, we will refer to the elements of a broad poset as its *edges*.

Definition V.1. A broad poset P is called *simple* if for any broad relation $e_1 \cdots e_n \leq e$ one has $e_i = e_j$ only if $i = j$.

Notation V.2. A broad poset structure $\leq$ on T naturally induces the following preorder relations on T and T^+:

- for $f, e \in T$ we say that f is a *descendent* of e, written $f \leq_d e$ if there exists a broad relation $\underline{f} \leq e$ such that $f \in \underline{f}$;
- for $\underline{f}, \underline{e} \in T^+$, we write $\underline{f} \leq \underline{e}$ if it is possible to write $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$, $\underline{e} = e_1 \cdots e_k$ such that $\underline{f}_i \leq e_i$ for $i = 1, \cdots, k$.

Remark V.3. Generally, the preorders just described can be fairly counter-intuitive. For example, it is quite possible to have $ab \leq a$, or even $aa \leq a$ together with $a \leq aa$. The case of simple broad posets, however, is much simpler.

SIMPLEBROAD PROP **Proposition V.4.** *Let T be a simple broad poset. Then $\leq_d$ (resp. $\leq$) is an order relation on T (resp. on T^+). Further:*

(i) if $f_1 \cdots f_n \leq e$ the f_i are $\leq_d$ incomparable (in particular, $e\underline{f} \leq e$ only if $\underline{f} = \epsilon$);

(ii) if $\underline{f} \leq \underline{e} = e_1 \cdots e_n$ with distinct e_i, then the decomposition $\underline{f} = \underline{f}_1 \cdots \underline{f}_n$ with $\underline{f}_i \leq e_i$ is unique. In fact, for $g \in \underline{f}$ one has $g \in \underline{f}_i$ if and only if $g \leq_d e_i$.

Make sure if the statement for $\leq$ being order holds for non simple tuples

Definition V.5. An edge $e \in T$ is called

- a *leaf* if there are no $\underline{f} \in T^+$ such that $\underline{f} < e$ (i.e. $\underline{f} \leq e$ and $\underline{f} \neq e$);
- a *node* if there is a non empty maximum $\underline{f} \neq \epsilon$ such that $\underline{f} < e$;
- a *stump* if $\underline{f} = \epsilon$ is the maximum (in fact, only) $\underline{f}$ such that $\underline{f} < e$.

Further, in either the node or stump case the maximum $\underline{f}$ is denoted $e^{\uparrow}$.

Remark V.6. While it is customary to regard stumps simply as a type of node, we find it convenient, in lieu of PROPOSITIONS, to separate the two cases.

The following definition is the key purpose of We12 [8].

DENRORD DEF **Definition V.7.** A *dendroidally ordered set* is a finite simple broad poset T such that

- each edge $e \in T$ is either a leaf, a node or a stump;
- there is a maximum $r_T \in T$ for $\leq_d$, called the *root* of T.

Weiss proves in We12 [8] that the category of dendroidally ordered sets (together with the obvious notion of monotonous function) is equivalent to the category Ω of trees. As such, we will henceforth refer to dendroidally ordered sets simply as *trees*.

We will make use of the following basic results.

MAPOUTTREE PROP **Proposition V.8.** *Let T be a tree and A any broad poset. A set map $g: T \to A$ is a broad poset map if and only if $g(e^{\uparrow}) \leq g(e)$ for each node/stump $e \in T$.*

The omitted proof easily follows by the induction argument in the next proof.

INEQMAXIFF NOPEN LEM **Lemma V.9.** *Let T be a tree. For any $e \in T$ there exists a minimum $e^{\lambda} \in T^+$ such that $e^{\lambda} \leq e$. In fact, $e^{\lambda} = l_1 \cdots l_k$ consists of those leaves l_i such that $l_i \leq_d e$.*

Further, a broad relation $\underline{f} = f_1 \cdots f_n \leq e$ holds if and only if $f_i \leq_d e$, the f_i are $\leq_d$ incomparable and $f_1^{\lambda} \cdots f_n^{\lambda} = e^{\lambda}$.

Proof. The proof is by upward $\leq_d$ induction on e. The leaf case is obvious. Otherwise, let $\underline{f} \leq e^{\uparrow} < e$ be any non identity relation. Write $e^{\uparrow} = e_1 \cdots e_k$ and $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$ so that $\underline{f}_i \leq e_i$ (note that $k = 0$ is allowed, in which case $e^{\uparrow} = \underline{f} = \epsilon$). By induction, $e_i^{\lambda} \leq \underline{f}_i \leq e_i$ where e_i^{λ} consists of the leaves l such that $l \leq_d e_i$ and hence indeed $e^{\lambda} = e_1^{\lambda} \cdots e_k^{\lambda} \leq \underline{f}$.

Only the "if" half of the "'further" statement needs proof. We use the same induction argument: incomparability yields $e \notin \underline{f}$ for $\underline{f} \neq e$ and, writing $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$ so that $s \in \underline{f}_i$ if and only if $s \leq_d e_i$, the induction hypothesis applies. □

INCOMPEDGES LEM **Lemma V.10.** *Let T be a tree and $\underline{e}$ a tuple of $\le_d$ incomparable edges of T. Then, letting r be the root of T, there exists a broad relation of the form $\underline{e}\underline{f} \le r$.*

Proof. The proof is by induction on the sum of the distances (measured in $\le_d$ inequality chains) from the leaves to the root. Clearly $r \in \underline{r}$ only if $r = \underline{e}$. Otherwise one can write $r^{\uparrow} = r_1 \cdots r_k$, $\underline{e} = \underline{e}_1 \cdots \underline{e}_k$ so that $s \in \underline{e}_i$ if and only if $s \le_d r_i$. The induction hypothesis applies to the subtrees $T_i = \{e \in T | e \le_d r_i\}$. □

V.2 Tensor products

Given $\underline{s} \in S^+$, $\underline{t} \in T^+$, we will let $\underline{s} \times \underline{t} \in (S \times T)^+$ denote the obvious tuple whose elements $(s,t) \in \underline{s} \times \underline{t}$ are those pairs with $s \in \underline{s}$, $t \in \underline{t}$.

Definition V.11. Given broad posets S, T, their *tensor product* $S \otimes T$ is the broad poset whose underlying set is $S \times T$ and whose relations are generated by relations of the form $\underline{s} \times t \le (s,t)$ (resp. $s \times \underline{t} \le (s,t)$) for $s \in S$, $t \in T$ and $\underline{s} \le s$ (resp. $\underline{t} \le t$) a broad relation in S (resp. T).

NSORCHILDREN UNOPEN PROP **Proposition V.12.** *Let S, T be trees. Then:*

(i) $S \otimes T$ is simple. Further, if $(s_1,t_1)\cdots(s_n,t_n) \le (s,t)$ then the pairs s_i, s_j and t_i, t_j are both $\le_d$ comparable only if $i = j$;

(ii) an edge $(s,t) \in S \otimes T$ has one of five types:

- *(leaf) it is leaf if both $s \in S$, $t \in T$ are leaves;*
- *(stump) it is a stump if $s \in S$ is a leaf and $t \in T$ is a stump or vice versa, or if both $s \in S$, $t \in T$ are stumps;*
- *(node) it is a node if $s \in S$ is a leaf and $t \in T$ is a node or vice versa. In fact $(s,t)^{\uparrow} = s^{\uparrow} \times t$ or $(s,t)^{\uparrow} = s \times t^{\uparrow}$, accordingly;*
- *(nullary node) it is a node such that $\epsilon \le (s,t)$ if $s \in S$ is a node and $t \in T$ a stump or vice versa. In fact $(s,t)^{\uparrow} = s^{\uparrow} \times t$ or $(s,t)^{\uparrow} = s \times t^{\uparrow}$, accordingly;*
- *(fork) if $s \in S$, $t \in T$ are both nodes then there are exactly two maximal $\underline{f}$ such that $\underline{f} < (s,t)$, namely $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$. We call such a (s,t) a fork.*

Proof. We first show (i). The "further" condition suffices for simplicity. Noting that in a tree $e \le_d e'$ and $e \le_d e''$ can happen simultaneously only if e', e'' are $\le_d$ comparable (by Lemma INCOMPEDGES LEM V.10) and using Proposition SIMPLEBROAD PROP V.4, one sees that composing broad relations satisfying the "further" condition with generating broad relations of $S \otimes T$ yields broad relations still satisfying that condition.

We now show (ii). Only the fork case requires proof. In fact, it is obvious that only $s \times t^{\uparrow}$ and $s^{\uparrow} \times t$ can possibly be maximal, hence one needs only verify that neither $s \times t^{\uparrow} \le s^{\uparrow} \times t$ nor $s^{\uparrow} \times t \le s \times t^{\uparrow}$. This follows since the S coordinates of the pairs in the tuple $s^{\uparrow} \times t$ are $<_d$ than those in $s \times t^{\uparrow}$ and vice versa. □

In order to simplify notation, we will henceforth write $e^{\uparrow S} = s^{\uparrow} \times t$, $e^{\uparrow T} = s \times t^{\uparrow}$ and $e^{\uparrow S,T} = s^{\uparrow} \times t^{\uparrow}$ whenever appropriate.

EQMAXIFF NOPENTENSOR LEM **Lemma V.13.** *Let S, T be trees. For any $e = (s,t) \in S \otimes T$ there exists a minimum $e^{\lambda} \in (S \times T)^+$ such that $e^{\lambda} \le e$. In fact, $e^{\lambda} = s^{\lambda} \times t^{\lambda}$.*

Proof. The proof is by upward $\leq_d$ induction on e. The case of s, t both leaves is obvious. Otherwise, for any non-identity relation either $\underline{f} \leq e^{\uparrow S} < e$ or $\underline{f} \leq e^{\uparrow T} < e$, and the analysis in the proof of Lemma V.9 applies in either case to show that indeed $s^\lambda \times t^\lambda \leq \underline{f}$. $\square$

CLOSEDSUBTREE PROP **Proposition V.14.** *Let U, S, T be trees.*

Suppose $U \xrightarrow{g} S \otimes T$ is an underlying injection of sets such that, if $l \in U$ is a leaf then it is not $\epsilon \leq g(l)$. Then $U \simeq g(U)$, i.e., the broad relations between elements in $g(U)$ coincide with those induced from U.

Proof. To simplify notation we will represent an edge $u \in U$ and its image $g(u) \in S \otimes T$ by the same letter u, and instead decorate the broad relations as $\leq^U$, $\leq^{S\otimes T}$. Likewise, we will write $u^{\lambda,U}$, $u^{\lambda,S\otimes T}$ following Lemmas V.9 and V.13.

We need only show that no broad relation $\underline{u} = u_1 \cdots u_n \leq^{S\otimes T} u$ can fail any of the three conditions in Lemma V.9 for with respect to U: (i) if $u_i \nleq_d^U u$ for some i, necessarily also $u \nleq_d^U u_i$ (since $\leq_d^{S\otimes T}$ is a partial order), and hence Lemma T.2 applied to u_i could be used to produce a non simple broad relation in $S \otimes T$ for the root of U; (ii) if u_i and u_j are $\leq_d^U$ comparable for $i \neq j$, one obtains a non simple broad relation in $S \otimes T$ for u; (iii) finally, if $u_1^{\lambda,U} \cdots u_n^{\lambda,U} \neq u^{\lambda,U}$, by part (ii) it must be that $u_1^{\lambda,U} \cdots u_n^{\lambda,U}$ lacks some of the leaves in $u^{\lambda,U}$. But our hypothesis is that $l^{\lambda,S\otimes T} \neq \epsilon$ for l any leaf of U, and it must hence also be $u_1^{\lambda,S\otimes T} \cdots u_n^{\lambda,S\otimes T} \neq u^{\lambda,S\otimes T}$, a contradiction. $\square$

V.3 Faces and subtrees

Definition V.15. Let S, T be trees.

A *face* of T (resp. *subtree* of $S \otimes T$) is a tree U together with an underlying monomorphism $U \rightarrowtail T$ (resp. $U \rightarrowtail S \otimes T$).

Further, a face $U \rightarrowtail T$ (resp. subtree $U \rightarrowtail S \otimes T$) is called *closed* if the relations in U coincide with those in its image. In this case, we suggestively write $U \subset T$ (resp. $U \subset S \otimes T$) instead.

Definition V.16. Given a face $U \rightarrowtail T$ (resp. subtree $U \rightarrowtail S \otimes T$) we define its *closure* $\bar{U}$ to be the smallest closed face $\bar{U} \subset T$ (resp. $\bar{U} \subset S \otimes T$) containing U.

Note that by Propositions V.8 and V.14 $\bar{U}$ is obtained from U by simply adding generating stump relations $\epsilon \leq^{\bar{U}} l$ for each leaf l of U such that $\epsilon \leq^T l$ (resp. $\epsilon \leq^{S\otimes T} l$).

Remark V.17. Given a tree T, call an edge $e \in T$ is called *external* if e is either a leaf or the root. Otherwise, an edge is called *internal*.

The maximal proper faces of a tree T can then be divided into the following three types:

(inner faces) if e in an inner edge, the corresponding inner face is $T - e \subset T$, the broad poset obtained by removing e from T;

(closed outer faces) if $v_e = (e^\uparrow \leq e)$ is a generating broad relation of T where all but one of the edges involved are external[7], the corresponding outer face $T - v_e \subset T$ is the broad poset obtained by removing the outer edges appearing in v;

[7] This can in fact happen in one of two ways: either e is internal and $e^\uparrow$ consists of leaves or, in rarer cases, e is the root and $e^\uparrow$ consists of leaves together with a single internal edge.

(stump outer faces) if $v_e = (\epsilon \leq e)$ is the generating broad relation for a stump $e \in T$, the corresponding outer face $T - v_e \rightarrowtail T$ is the broad poset with the same underlying set as T but only those generating relations $f^{\uparrow} \leq f$ for $f \neq e$.

We stress that, as implied by the notation, an outer face $T - v_e$ is a closed face if and only if v_e is not induced by a stump e.

ELEMTENS DEF **Definition V.18.** Let S, T be trees. A subtree $U \rightarrowtail S \otimes T$ is called

- *elementary* if all of its generating broad relations are minimal broad relations of $S \otimes T$ (in other words, those relations of the form $e^{\uparrow S} \leq e$, $e^{\uparrow T} \leq e$ appearing in Proposition TENSORCHILDREN UNOPEN PROP V.12 *except* for the $\epsilon \leq e$ relations for nullary nodes);
- *initial* if U contains the "double root" $(r_S, r_T) \in S \otimes T$.

Further, a *maximal elementary subtree* is a subtree that is as large as possible. Note that such a subtree is necessarily initial.

AXIMALSUBFACESLEAVES REM **Remark V.19.** Given initial subtrees $U \rightarrowtail U' \rightarrowtail S \otimes T$ such that $\bar{U} = \bar{U}'$, one has $(r_S, r_T)^{\lambda, U'} \leq (r_S, r_T)^{\lambda, U}$ with $(r_S, r_T)^{\lambda, U'}$ obtained from $(r_S, r_T)^{\lambda, U}$ by removing those leaves of U which become stumps in U'.

CLOSEDSUBTREE LEM **Lemma V.20.** *If U is a closed face of S, then $U \otimes T \subset S \otimes T$, i.e., $U \otimes T$ contains all broad relations in its image.*

Proof. Any broad relation of $S \otimes T$ sits inside some elementary maximal subtree and Proposition CLOSEDSUBTREE PROP V.14 applies to those. □

Definition V.21. Let S, T be trees and $A = \{a_i\}$, $B = \{b_j\}$ subsets of the sets of stumps of S, T, respectively and let $v_A = \{\epsilon \leq a_i\}$, $v_B = \{\epsilon \leq b_j\}$ denote the corresponding subsets of generating stump relations.

We say that a subtree $U \rightarrowtail S \otimes T$ *misses v_A and v_B* if one has a factorization $U \rightarrowtail (S - v_A) \otimes (T - v_B) \rightarrowtail S \otimes T$.

Further, if $B = \varnothing$ (resp. $A = \varnothing$) we say simply that "U misses v_A" (resp. "U misses v_B").

Remark V.22. In MorWeiss, Moerdijk and Weiss also define similar notions of "U missing an inner edge/non empty outer vertex of S/T". Here we ignore those notions, whose behavior is straightforward, so as to focus on the much subtler notion of "missing stumps".

For instance, note that Corollary ROOTCOND COR V.25 implies that the notion "U misses v_A and v_B" does not coincide with the notion "U misses v_A and U misses v_B" whenever A and B are both nonempty.

Lemma V.23. *Let S, T be trees and $A = \{a_i\}$, $B = \{b_j\}$ subsets of the stumps of S, T, respectively. Then a broad relation*

$$\underline{f} = f_1 f_2 \cdots f_k \leq e$$

in $S \otimes T$ is a broad relation in $(S - v_A) \otimes (T - v_B)$ if and only if

$$e^{\lambda, (S - v_A) \otimes (T - v_B)} \leq \underline{f}.$$

Proof. Only the "if" direction needs proof. We argue by upward $\leq_d$ induction on $e = (s, t)$. The base case, that of s a leaf of $S - v_A$ and T a leaf of $T - v_B$, is obvious (we note that the proof will follow even when this case is vacuous).

Otherwise, either $\underline{f} \leq e^{\uparrow S} \leq e$ or $\underline{f} \leq e^{\uparrow T} \leq e$ and our assumption ensures, accordingly, that $s \notin A$ or $t \notin B$. Writing $e^{\uparrow *}$ to denote either $e^{\uparrow S}$ or $e^{\uparrow T}$ as appropriate, this last observation guarantees that the relation $e^{\uparrow *} \leq e$ is in $(S - v_A) \otimes (T - v_B)$. Further, writing $\underline{f} = \underline{f}_1 \cdots \underline{f}_k$ and $e^{\uparrow *} = e_1 \cdots e_k$ so that $\underline{f}_i \leq e_i$, the induction hypothesis shows that these last relations are also in $(S - v_A) \otimes (T - v_B)$. $\square$

Recalling Proposition V.4 hence yields the following.

FACTORIZATION COR

Corollary V.24. *A collection of broad relations of the form $\underline{g}_i \leq f_i$, $f_1 \cdots f_k \leq e$ are all in $(S - v_A) \otimes (T - v_B)$ if and only if the composite relation $\underline{g}_1 \cdots \underline{g}_k \leq e$ is.*

ROOTCOND COR

Corollary V.25. *A face $U \rightarrowtail S \otimes T$ misses v_A and v_B if and only if, for r the root of U, one has*

$$r^{\lambda, (S - v_A) \otimes (T - v_B)} \leq r^{\lambda, U}.$$

Proof. This follows from Corollary V.24 since any relation in U is a factor of $r^{\lambda, U} \leq r$. $\square$

V.4 Pushout product of monomorphisms

Notation V.26. For a tree T we let $\Omega[T] \colon \Omega^{op} \to Set$ denote the corresponding representable dendroidal set.

Note that $\otimes$ canonically extends to dendroidal sets by setting $\Omega[S] \otimes \Omega[T](-) = \Omega(-, S \otimes T)$ and having $\otimes$ preserve colimits in each variable.

Proposition V.27. *Let S, T be trees. Then the following are normal monomorphisms of dendroidal sets given the indicated conditions:*

(i) $\partial\Omega[S] \otimes \Omega[T] \to \Omega[S] \otimes \Omega[T]$ *and* $\Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \partial\Omega[T]$ *for any S, T;*

(ii) $\partial\Omega[S] \otimes \partial\Omega[T] \to \partial\Omega[S] \otimes \Omega[T]$ *and* $\partial\Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \partial\Omega[T]$ *if either of S or T has at most one stump;*

(iii) $\partial\Omega[S] \otimes \Omega[T] \coprod_{\partial\Omega[S] \otimes \partial\Omega[T]} \Omega[S] \otimes \partial\Omega[T] \to \Omega[S] \otimes \Omega[T]$ *if either of S or T is open (i.e. has no stumps).*

Proof. We first show (i). Since $\otimes$ commutes with colimits in each variable,

$$\partial\Omega[S] \otimes \Omega[T] = \operatorname{colim}_{F \in \mathsf{Faces}(S) - \{S\}} \Omega[F] \otimes \Omega[T].$$

Since each map $\Omega[F] \otimes \Omega[T] \to \Omega[S] \otimes \Omega[T]$ is a monomorphism of dendroidal sets, it suffices to check that, for each subtree $U \rightarrowtail S \otimes T$, the poset $\mathcal{P}_U = \{F | F \neq S, U \rightarrowtail F \otimes T\}$ is connected. Suppose first $U \rightarrowtail (S - e) \otimes T$ for some inner edge e of S. Lemma V.20 then ensures that if $F \in \mathcal{P}_U$ it is also $F - e \in \mathcal{P}_U$, and hence the assignment $F \mapsto F - e$ retracts the poset $\mathcal{P}_U$ onto the subposet $\{F | F \rightarrowtail S - e, R \rightarrowtail F \otimes T\}$, which is connected since it has the maximum $S - e$. Likewise, the same argument works if $U \rightarrowtail (S - v_e) \otimes T$ for $(S - v_e)$ a closed outer face. We are hence left with the case where all $F \in \mathcal{P}_U$ are such that $\bar{F} = S$ and by Lemma V.20 we are also free to assume that U is initial and we will write $r = (r_S, r_T)$. Letting $A = \{a_1, \cdots, a_k\}$ denote the set of those stumps of S such that U misses v_{a_i}, i.e. $U \rightarrowtail (S - v_{a_i}) \otimes T$, applying Remark V.19 to the closures $\bar{U}^{(S-v_{a_i})\otimes T}$ yields that $r^{\lambda,(S-v_A)\otimes T} \leq r^{\lambda,U}$, and thus by Corollary V.25 $\mathcal{P}_U$ has the minimum $S - v_A$.

To prove (ii), note that by (i) it suffices to check that $\partial\Omega S \otimes \partial\Omega T \to \Omega S \otimes \Omega T$ is a monomorphism. Unpacking the colimit defining $\partial\Omega S \otimes \partial\Omega T$ and arguing as before, it suffices to show that, for each $U \rightarrowtail S \otimes T$, the poset $\mathcal{P}_U^{S,T} = \{(F_S, F_T) | F_S \neq S, F_T \neq T, U \rightarrowtail F_S \otimes F_T\}$ is connected. W.l.o.g, assume that it is T that has at most one stump. Then, if there is a factorization $U \rightarrowtail S \otimes (T - e)$ for some inner edge e, the assignment $(F_S, F_T) \mapsto (F_S, F_T - e)$ retracts $\mathcal{P}_U^{S,T}$ precisely onto the poset $\mathcal{P}_U$ of part (i) for the $S \otimes (T - e)$ case. The case $U \rightarrowtail S \otimes (T - v_e)$ for $T - v_e$ a closed outer face is identical. Otherwise, letting a denote the unique stump of T, it must be the case that for any $(F_S, F_T) \in \mathcal{P}_U^{S,T}$ it is necessarily $F_T = T - v_a$, and we hence again reduce to part (i) for the $S \otimes (T - v_a)$ case.

Finally, to show (iii) if suffices, by (i) and (ii), to show that if a subtree $U \rightarrowtail S \otimes T$ factors through both $F_S \otimes T$ and $S \otimes F_T$ for some faces F_S of S and F_T of T, then it factors through $F_S \otimes F_T$ as well. But since any face of an open tree is a closed face (say, by Proposition V.14), this follows by Lemma V.20. □

Remark V.28. The further condition of the previous result in fact fails whenever both trees have at least two stumps. For a representative example, consider two stumpy trees $S = (\epsilon \leq ab \leq c)$ and $T = (\epsilon \leq 12 \leq 3)$. Then the subtree of $S \otimes T$ generated by the relations $a_1 a_2 b_1 b_2 \leq c_3$, $\epsilon \leq a_2$, $\epsilon \leq b_1$ induces two distinct points in $\partial\Omega S \otimes \partial\Omega T$.

V.5 Anodyne extensions

Recall the $e^{\uparrow S}$, $e^{\uparrow T}$, $e^{\uparrow S,T}$ notation discussed after Proposition V.12.

Proposition V.29. *Let S, T be trees and consider the broad relation*

$$\underline{e} = (s_1, t_1)(s_2, t_2) \cdots (s_n, t_n) \leq (s, t) = e.$$

in $S \otimes T$. Then:

(i) $\underline{e} \leq e^{\uparrow S}$ (resp. $\underline{e} \leq e^{\uparrow T}$) if and only if $s_i \neq s, \forall i$ (resp. $t_i \neq t, \forall i$);

(ii) $\underline{e} \leq e^{\uparrow S,T}$ if and only if both $s_i \neq s$ and $t_i \neq t, \forall i$.

Proof. Only the "if" directions need proof, and the proof follows by upward $\leq_d$ induction on s, t. The base cases of either s or t a leaf are obvious.

Otherwise, let $\underline{e}$ satisfy the "if" condition in (i). Since (by Proposition V.12(ii)) it must be $\underline{e} \leq e^{\uparrow S}$ or $\underline{e} \leq e^{\uparrow T}$ we can assume it is the latter case. Writing $t^{\uparrow} = u_1 \cdots u_k$, $\underline{e} = \underline{e}_1 \cdots \underline{e}_k$ so that $\underline{e}_i \leq (s, u_i)$ (note that possibly $k = 0$), the induction hypothesis now yields $\underline{e}_i \leq (s, u_i)^{\uparrow S} = s^{\uparrow} \times u_i$, and hence

$$\underline{e} = \underline{e}_1 \cdots \underline{e}_l \leq (s^{\uparrow} \times u_1) \cdots (s^{\uparrow} \times u_l) = s^{\uparrow} \times t^{\uparrow} \leq s^{\uparrow} \times t = e^{\uparrow S}. \tag{V.30}$$

The proof of (iii) is identical except by disregarding the last inequality in (V.30). □

Corollary V.31. $\underline{e} \leq e^{\uparrow S,T}$ *if and only if both* $\underline{e} \leq e^{\uparrow S}$ *and* $\underline{e} \leq e^{\uparrow T}$.

We will make use of an order relation on initial elementary subtrees (cf. Definition ELEMTENS DEF V.18) of $S \otimes T$. Write

$$U \leq_{lex} U'$$

whenever U' is obtained from U by replacing the intermediate edges in a string of broad relations $e^{\uparrow S,T} \leq e^{\uparrow S} \leq e$ for a fork e (cf. Proposition TENSORCHILDREN UNOPEN PROP V.12) occurring in U by the intermediate edges in $e^{\uparrow S,T} \leq e^{\uparrow T} \leq e$ occurring in U'.

NONNULLVERT REM **Remark V.32.** The fact that any e as above needs to be a fork was in fact built in into the definition of elementary subtree. Indeed, while the definition just given would work for e a nullary node, in that case $\epsilon = e^{\uparrow S} \leq e$ (or $\epsilon = e^{\uparrow T} \leq e$, as appropriate) is not valid as a generating relation for an elementary tree.

In what follows we refer to *non stump* generating relations of the form $e^{\uparrow S} \leq e$ (resp. $e^{\uparrow T} \leq e$) in an elementary tree as S-vertices (resp. T-vertices) (the reason to exclude stumps is that those would belong to both types simultaneously).

PARTIALORDER PROP **Proposition V.33.** *$\leq_{lex}$ induces a partial order on the set of initial elementary subtrees of $S \otimes T$.*

Further, $\leq_{lex}$ together with the inclusion $\rightarrowtail$ induce a partial order as well, and we denote this latter order simply $\leq$.

Proof. One needs only check antisymmetry. Let $g(U)$ count the number of pairs (v_S, v_T) of an S-vertex and T-vertex in U such that v_S occurs before v_T (following $\leq$). Since the generating relations of $\leq_{lex}$ strictly decrease g, it defines a partial order. Similarly, letting $h(U) = |\mathsf{Stumps}(U)| + \sum_{e \in (\mathsf{Leaves}(U) \cup \mathsf{Stumps}(U))} d(e, r)$ (where $d(-, r)$ denotes "distance to the root", measured in generating $\leq_d$ relations), one has that $\leq_{lex}$ preserves h and $\rightarrowtail$ decreases it, hence $\leq$ is a partial order. □

Henceforth we will let η denote a *fixed* inner edge of T.

Definition V.34. An initial elementary subtree $U \rightarrowtail S \otimes T$ is called *η-internal* if it contains an edge of the form (s, η), abbreviated as η_s, and the T-vertex $\eta_s^{\uparrow T} \leq \eta_s$.

We will denote the subposet of such trees by $(\mathcal{IE}_\eta(S \otimes T), \leq)$.

ETSBREAK REM **Remark V.35.** Since η is internal we have that η_s is: (fk) a fork if s, η are both nodes; (nd) a node if s is a leaf and η is a node; (nn) a nullary node if s is a stump and η is a node; (st) a stump if s is a leaf or stump and η is a stump.

INTERCHANGE LEM **Lemma V.36.** *Let $U \rightarrowtail S \otimes T$ be an elementary subtree with root vertex a T-vertex $e^{\uparrow T} \leq e$ and suppose that $e^{\lambda, U} \leq e^{\uparrow S}$.*

Then there exists an elementary subtree U' such that $U' \leq_{lex} U$ and U' contains the relations $e^{\uparrow S,T} \leq e^{\uparrow S} \leq e$.

Proof. We argue by induction on the sum of the distances (cf. proof of Proposition PARTIALORDER PROP V.33) between the leaves and stumps of U and its root e. The base case, that of e a leaf or stump of U (where in the latter case e must be a stump of $S \otimes T$ since U is elementary), is obvious. The case of e a nullary node of $S \otimes T$ is likewise trivial, since then necessarily $\epsilon = e^{\lambda, U} = e^{\uparrow S} \leq e^{\uparrow T}$, so that $U' = U$ works.

Otherwise, writing $e = (s, t)$, for each $t_i \in t^{\uparrow}$ either $(s, t_i)^{\uparrow S} \leq^U (s, t_i)$ or $(s, t_i)^{\uparrow T} \leq^U (s, t_i)$ and iteratively applying the induction hypothesis to each i

in the latter case allows us find $U'' \leq_{lex} U$ such that U'' contains all relations $(s,t_i)^{\uparrow S,T} \leq (s,t_i)^{\uparrow S} \leq (s,t_i)$. But now U'' contains the relations $e^{\uparrow S,T} \leq e^{\uparrow T} \leq e$ and hence a final generating $\leq_{lex}$ relation yields $U' \leq_{lex} U'' \leq_{lex} U$. ☐

INTERTREE LEM

Lemma V.37. *Suppose $U \rightarrowtail S \otimes T$, $U' \rightarrowtail S \otimes T$ are subtrees with common leaves and root. Then $F = U \cap U'$ defines a closed face of both U and U'.*

Proof. As a set, F could alternatively be defined as the underlying set of the composite inner face of U that removes all inner edges of U not in U', or vice versa. Thus, the real claim is that both constructions yield the same broad relations. Noting that the $\leq_d$ order relations on U, U' are induced from $S \otimes T$ (this follows from Proposition CLOSEDSUBTREE PROP V.14), this now follows from Lemma INEQMAXIFF NOPEN LEM V.9. ☐

COMMONFACE LEM

Lemma V.38. *If F is a common face (resp. inner face) of two elementary subtrees U, U', then F is also a face (resp. inner face) of an initial elementary subtree U'' such that $U'' \leq U$, $U'' \leq U'$ (resp. $U'' \leq_{lex} U$, $U'' \leq_{lex} U'$). In fact, in the inner face case the $\leq_{lex}$ inequalites factor through generating $\leq_{lex}$ inequalites involving only trees having F as an inner face.*

Proof. One is free to remove any vertices of U, U' that are below leaves of F (i.e. vertices $e^{\uparrow *} \leq e$ such that $e \leq_d l$ for some leaf l of F) and not below the root of F, so that F, U, U' now have exactly the same leaves and root, with the latter denoted r. Thus, by Lemma INTERTREE LEM V.37 we are free to assume $F = U \cap U'$.

If the root vertices $r^{\uparrow U} \leq r$, $r^{\uparrow U'} \leq r$ coincide, the result follows by induction on $\leq$. Otherwise, we can assume that the root vertex of U is $r^{\uparrow S} \leq r$ and that of U' is $r^{\uparrow T} \leq r$ (note that, cf. Remark NONNULLVERT REM V.32, r must be a fork of $S \otimes T$). Letting $U'_{\geq r^{\uparrow F}}$ be the smallest subtree of U' containing the root vertex $r^{\uparrow F} \leq r$ of F, one can now apply Lemma INTERCHANGE LEM V.36 to $U'_{\geq r^{\uparrow F}}$, and hence build $U'' \leq_{lex} U'$ with a strictly larger intersection with U, finishing the proof. ☐

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Recall that a subset B of a poset $\mathcal{P}$ is called *convex* if $a \leq b$ and $b \in B$ implies $a \in B$.

ANODYNEEXT PROP

Proposition V.39. *Let S, T be trees and $\eta \in T$ an inner edge. Set*

$$A = \Omega[S] \otimes \Lambda^{\eta}\Omega[T] \coprod_{\partial\Omega[S] \otimes \Lambda^{\eta}\Omega[T]} \partial\Omega[S] \otimes \Omega[T]$$

and denote by $i(A)$ the image in $\Omega[S] \otimes \Omega[T]$ of the obvious map.

Then, for any convex subsets $B \subset B'$ of the poset $\mathcal{IE}_{\eta}(S \otimes T)$, one has that

$$i(A) \cup \bigcup_{V \in B} i(\Omega[V]) \to i(A) \cup \bigcup_{V \in B'} i(\Omega[V])$$

is an inner anodyne extension (of subdendroidal sets of $\Omega[S] \otimes \Omega[T]$).

To simplify notation we will throughout the proof suppress i, Ω from the notation, e.g., $i(A) \cup \bigcup_{V \in B} i(\Omega[V])$ will be denoted simply as $A \cup \bigcup_{V \in B} V$.

The key to the proof is the following lemma.

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Lemma V.40. *η_s is a characteristic edge (in the sense of MW09 [5, Lemma 9.7]), i.e., for each inner face F of U containing the edge η_s, then F is in $A \cup \bigcup_{V \in B} V$ if and only if $F - \eta_s$ is.*

Proof. Suppose first that $F - \eta_s$ is in A but that F is not. The required argument depends on which of the scenarios described in Remark ETSBREAK_REM V.35 η_s fails into:

(fk) since s is a node, the only possibility is that $F - \eta_s \rightarrowtail (S - s) \otimes T$ (by definition of A, the scenario $F - \eta_s \rightarrowtail S \otimes (T - \eta)$ is not relevant). Let $U^s_{\le_d \eta_s}$ denote the smallest elementary subtree of U with $\eta_s = (s, \eta)$ as its root and any edges of the form $(s,t) <_d (s,\eta)$ as its inner edges (note that if any such edge was a leaf of U, it could not be $U - \eta_s \rightarrowtail (S - s) \otimes T$). Clearly F could not have contained any of the inner edges of $R^e_{\le_d e_s}$, and thus applying Lemma INTERCHANGE_LEM V.36 to $U^s_{\le_d \eta_s}$ we see that F is a subface of some $V <_{lex} U$, hence already contained in $A \cup \bigcup_{V \in B} V$;

(nd) k;kl

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Clearly F must collapse all of the inner edges of $R^e_{\le_d e_s}$, hence applying Lemma INTERCHANGE_LEM V.36 to $R^e_{\le_d e_s}$ we see that F is a subface of some $Q <_{lex} R$, hence already contained in $A \cup \bigcup_{Q \in B} Q$.

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□

Proof of Proposition ANODYNEEXT_PROP *W.1.* Without loss of generality we can assume that B' is obtained from B by adding a single η-internal initial elementary tree U with η_s its corresponding edge.

We first note that the outer faces of U are in $A \cup \bigcup_{V \in B} V$. For an outer face $U - v_e$ either: (i) $U - v_e$ is still η-internal initial elementary; (ii) $U - v_e$ no longer contains the "double root" $r = (r_S, r_T)$ (this happens precisely if $r = e$), in which case either $U - v_e \rightarrowtail S_{\le r_{S,i}} \otimes T$ or $U - v_e \rightarrowtail S \otimes T_{\le r_{T,j}}$ for $S_{\le r_{S,i}} \rightarrowtail S$, $T_{\le r_{T,j}} \rightarrowtail T$ the subtrees of descendants of some fixed descendant of the corresponding root; (iii) $U - v_e$ is still initial elementary but no longer η-internal, in which case it must have been both that $e = \eta_s$ and that η_s was the only edge qualifying U as η-internal. But in this case $U - v_{\eta_s} \rightarrowtail S \otimes (T - v_\eta)$ (note that this claim is somewhat subtle when η is a stump, which by Remark ETSBREAK_REM V.35 happens only if η_s is itself one. The claim now follows since the only elementary generating relations not in $S \otimes (T - v_\eta)$ have the form $\epsilon \le \eta_l$ for $l \in S$ a leaf).

We now show that η_s is a characteristic edge (in the sense of MW09 [5, Lemma 9.7]), i.e. that, for each inner face F of U containing the edge η_s, then F is in $A \cup \bigcup_{V \in B} V$ if and only if $F - \eta_s$ is.

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Suppose first that $F - \eta_s$ is in A but that F is not. The required argument

HERE note that $S - s$ isn't well defined cause s need not be internal

Suppose first that $F - \eta_s$ is in A but that F is not, in which case the only possibility is that $F - \eta_s \rightarrowtail (S - s) \otimes T$ (by definition of A, the scenario $F - \eta_s \rightarrowtail S \otimes (T - \eta)$ is not relevant).

HERE actually s needs to be internal to be missable at all

Suppose first that $F - \eta_s$ is in A but that F is not, in which case the only possibility is that $F - \eta_s \rightarrowtail (S - s) \otimes T$ (by definition of A, the scenario $F - \eta_s \rightarrowtail S \otimes (T - \eta)$ is not relevant).

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Let $U^s_{\le_d \eta_s}$ denote the smallest elementary subtree of U with $\eta_s = (s, \eta)$ as its root and any edges of the form $(s,t) <_d (s,\eta)$ as its inner edges (note that if any such edge was a leaf of U, it could not be $U - \eta_s \rightarrowtail (S-s)\otimes T$). Note that in $U^s_{\le_d \eta_s}$ all $\le_d$ minimal (s,t) edges correspond to S vertices, and that all other vertices are

, F/e_s could not miss the edge $s \in S$). Clearly F must collapse all of the inner edges of $R^e_{\le_d e_s}$, hence applying Lemma V.36 to $R^e_{\le_d e_s}$ we see that F is a subface of some $Q <_{lex} R$, hence already contained in $A \cup \bigcup_{Q \in B} Q$.

Suppose now that F/e_s is an inner subface of some $Q \in B$, $R \nleq Q$. Lemma V.38 then that ensures there is a $Q' < B$ containing F/e_s and, in fact, since F/e_s is an inner face it must be $Q' <_{lex} B$, and by the "in fact" part of Lemma V.38 one can assume that this is a generating $\le_{lex}$ relation. But then Q' necessarily contains e_s, since generating relations don't remove edges whose vertex is a T-vertex.

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Finally, we now let $I^{\hat{\eta}_s}$ denote the subset of inner edges of U distinct from η_s and claim that for any concave[8] subsets $C \subset C' \subset \mathcal{P}(I^{\hat{\eta}_s})$ the map

$$A \cup \bigcup_{V \in B} V \cup \bigcup_{E \in C} U - E \to A \cup \bigcup_{V \in B} V \cup \bigcup_{E \in C'} U - E \qquad \text{(V.41)}$$

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is inner anodyine.

We argue by induction on C and again we can assume that C' is obtained from C by adding a single $D \in \mathcal{P}(I^{\hat{\eta}_s})$. But since the concavity of C, C' and the characteristic edge condition imply that the only faces of $U - D$ not in the source of (V.41) are precisely $U - D$ and $U - (D \cup \eta_s)$, we conclude that (V.41) is in fact a pushout of $\Lambda^{\eta_s}\Omega[U-D] \to \Omega[U-D]$, finishing the proof. □

W HARD CHARACTERISTIC EDGE EXISTENCE

W.1 Non stump case

ANODYNEEXT PROP

Proposition W.1. *Let S, T be trees and $\eta \in T$ an inner edge. Set*

$$A = \Omega[S] \otimes \Lambda^\eta \Omega[T] \coprod_{\partial\Omega[S]\otimes\Lambda^\eta\Omega[T]} \partial\Omega[S] \otimes \Omega[T]$$

and denote by $i(A)$ the image in $\Omega[S] \otimes \Omega[T]$ of the obvious map.

Then, for any convex subsets $B \subset B'$ of the poset $\mathcal{IE}_\eta(S \otimes T)$, one has that

$$i(A) \cup \bigcup_{V \in B} i(\Omega[V]) \to i(A) \cup \bigcup_{V \in B'} i(\Omega[V])$$

is an inner anodyne extension (of subdendroidal sets of $\Omega[S]\otimes\Omega[T]$).

To simplify notation we will throughout the proof suppress i, Ω from the notation, e.g., $i(A) \cup \bigcup_{V \in B} i(\Omega[V])$ will be denoted simply as $A \cup \bigcup_{V \in B} V$.

The key to the proof is the following lemma.

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[8] This is the notion dual to "convex".

Lemma W.2. *Suppose that η is a node of T and let U be an elementary initial η-internal subtree of $S \otimes T$.*

Then there is an s such that η_s is a characteristic edge (in the sense of MW09 *[5, Lemma 9.7]), i.e., for each inner face F of U containing the edge η_s, then F is in $A \cup \bigcup_{V \in B} V$ if and only if $F - \eta_s$ is.*

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Proof. The proof breaks down according to the possible scenarios for an edge of the form η_a described in Remark ETSBREAK REM V.35.

(fk) Suppose first that U contains the relations $\eta_s^{\uparrow T} \leq \eta_s$ for η_s a fork, i.e., assume that s was a node. We claim that then η_s is indeed characteristic.

Suppose first that $F - \eta_s$ is in A but that F is not. Since s is a node, the only possibility is that $F - \eta_s \rightarrowtail (S - s) \otimes T$ (by definition of A, the scenario $F - \eta_s \rightarrowtail S \otimes (T - \eta)$ is not relevant). Let $U^s_{\leq_d \eta_s}$ denote the smallest elementary subtree of U with $\eta_s = (s, \eta)$ as its root and any edges of the form $(s, t) <_d (s, \eta)$ as its inner edges (note that if any such edge was a leaf of U, it could not be $U - \eta_s \rightarrowtail (S - s) \otimes T$). Clearly F could not have contained any of the inner edges of $R^e_{\leq_d e_s}$, and thus applying Lemma INTERCHANGE LEM V.36 to $U^s_{\leq_d \eta_s}$ we see that F is a subface of some $V <_{lex} U$, hence already contained in $A \cup \bigcup_{V \in B} V$;

Suppose now that $F - \eta_s$ is an inner subface of some $V \in B$, $U \nleq V$. Lemma COMMUNFACE LEM V.38 then that ensures there is a $V' < U$ containing $F - \eta_s$ and, in fact, since $F - \eta_s$ is an inner face it must be $V' <_{lex} U$, and by the "in fact" part of Lemma COMMUNFACE LEM V.38 one can assume that this is a generating $\leq_{lex}$ relation. But then V' necessarily contains η_s, since generating relations don't remove edges whose vertex is a T-vertex.

(nd) Suppose now that U contains no relations $\eta_s^{\uparrow T} \leq \eta_s$ with η_s a fork, but does contain such relations with η_s a node, so that s is a leaf. We claim that such a η_s is in fact a characteristic edge.

Suppose first that $F - \eta_s$ is in A but that F is not. The only possibility is that $F - \eta_s \rightarrowtail (S - v_s) \otimes T$ (by definition of A, the scenario $F - \eta_s \rightarrowtail S \otimes (T - \eta)$ is not relevant). It must then be the face that F was obtained by collapsing all of $U_{\leq \eta_s}$ (which must, in particular, have been a "stumpy tree"). Indeed, it is now clear that we will have found a characteristic edge provided that at least one such η_a edge is not stumpy.

Otherwise, all such trees are stumpy and, denoting them by $\eta_{a_1}, \cdots, \eta_{a_k}$ one setting $U_{\leq i} = U_{\leq \eta_{a_i}}$, one can add $U - U_1 - \cdots - U_i$ in (reverse) order. For $i = k - 1$, one can use η_{a_k} as a characteristic edge. since otherwise one would miss any colors of T above η. Otherwise, in the induction steps one uses each η_{a_i} as a characteristic edge when moving from $U - U_1 - \cdots - U_{i+1}$ to $U - U_1 - \cdots - U_i$ (rephrase this as a uniform induction. The missing color argument belongs on the base case).

The "$F - \eta_s$ is an inner subface of some $V \in B$" case is identical to before.

(nn) The case where all the available edges/T-vertices of the form η_a are nullary nodes, i.e. the case where a is a stump, is identical (in fact, should be included in) to the "stumpy" case above.

□

W.2 The stump case

In this section we assume that S is an open tree and that the characteristic edge η of T is a stump.

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When $\eta \in T$ happens to be a stump, it is necessary to modify both the notion of elementary tree and the $<_{lex}$ order used.

We now call a subtree *elementary* whenever the vertices all have the form $e^{\uparrow S} \le e$ or $e^{\uparrow T} \le e$, even in the case $e = \eta_s$ for s a node, in which case Proposition
TENSORCHILDREN UNOPEN PROP V.12 says that the relation $\epsilon = e^{\uparrow T} \le e$ is not minimal.

We also now reverse the $\le_{lex}$ notion. Namely, we set

$$U \le_{lex} U'$$

whenever U' can be obtained from U by replacing the intermediate edges in the string of relations $e^{\uparrow S,T} \le e^{\uparrow T} \le e$ appearing in U by the intermediate edges in the string of relations $e^{\uparrow S,T} \le e^{\uparrow S} \le e$ appearing in U'.

Remark W.3. An important special case (and, ultimately, the reason why the $\le_{lex}$ order from before needs to be reversed) occurs when $e = \eta_s$ for s a node. In that case, U is in fact the inner face of U' obtained by collapsing all edges in $e^{\uparrow S}$. Therefore, if U is to be allowed as an inner tree it is necessary to reverse the $\le_{lex}$ order if one is to be able to combine $\le_{lex}$ and $\rightarrowtail$ into a coherent order relation.

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Definition W.4. An elementary subtree is called η-internal if both:

- a generating relation of the form $\epsilon \le \eta_s^{\uparrow T} \le \eta_s$ occurs only if η_s is a stump of $S \otimes T$;
- there is at least one generating relation as above.

jkj
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Proposition W.5. *Let S be an open tree and η a stump edge of T.*

Then, when using the ***reverse lex order****, it is always possible to find "intermediate characteristic" edges that work.*

Proof. HERE

Suppose first that T contains at least one leaf. Indeed, let a denote a leaf of T such that the edge where if meets η, say b, is as high as possible.

Pick a stump edge η_s of $U \rightarrowtail S \otimes T$. Next, one finds the highest edge of the form b_x such that $\eta_s \le_d b_x$. Then, is it necessarily the case that the vertex above b_x is a T-vertex the edge of U immediately below a_s is necessarily a leaf of U, which we denote $\bar{a}_l$.

Therefore, one can necessarily find in U an edge of the form $\bar{b}_l$ for $\eta \le_d \bar{b}$. In the case $\eta = \bar{b}$, the edge η_l just found is clearly a characteristic edge for U. Otherwise, we choose $\bar{b}$ as high as possible in T and claim that $\bar{b}_l$ is itself characteristic. Since the vertex above $\bar{b}_l$ is an S-vertex, either (i) there is a leaf of U above $\bar{b}_l$ with a $\bar{b}$ coordinate; (ii) there is not such a leaf, in which case if collapsing $\bar{b}_l$ caused a face to be contained in A, one would have had to collapse all $\bar{b}$ colored edges above it, and hence Lemma INTERCHANGE LEM V.36 would apply, and hence the proof is finished by the induction hypothesis.

More generally, this argument works whenever U contains a leaf of the form a_l such that $\eta \not\leq_d a$ and $s \leq_d l$

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□

W.3 The stump case: for real this time (hopefully)

Throughout S will be an open tree and η will be a stump of T. To deal with this case a more subtle analysis is necessary.

To this effect, we first note that by associating each initial elementary tree U its set of leaves $r_0^{\lambda,U}$, one already obtains a pre-order $\leq_\lambda$ (i.e., "partial order without the anti-symmetry axiom") on the set of initial elementary trees.

Further, noting that whenever $U \leq_{lex} U'$ it is necessarily the case that $r_0^{\lambda,U} = r_0^{\lambda,U}$, one sees that, to refine $\leq_\lambda$ to an actual order relation, one needs only declare that, whenever, $r_0^{\lambda,U} = r_0^{\lambda,U}$, it is $U \leq U'$ whenever either $U \leq U'$ or $U' \leq U$, **provided that the latter choice depends only on $r_0^{\lambda,U} = r_0^{\lambda,U}$.**

With this in mind, we will order initial elementary subtrees according to the following rule: trees U such that $r_0^{\lambda,U}$ contains edges of the form r_i but is not exclusively composed of such edges are ordered according to $\leq_{lex}$; any other trees are ordered according to $\leq_{lex}^{op}$.

Proposition W.6. *With the order just described, then any elementary subtree U always has a characteristic edge (or, possibly, a characteristic edge strategy).*

Proof. We will break the proof into several cases.

(Case I) We first consider the case where U has a leaf of the form r_i, but not all leafs are of that form.

It is then necessarily the case that the root vertex of U is $r_0^{\uparrow,S} \leq r_0$ and, letting t_j denote a leaf of U with $r \neq t$, one can choose a minimal edge of the form r_k such that $t_j \leq_d r_k$. Since the root vertex is a S-vertex, it is necessarily $r_k \neq r_0$ so that, in particular, r_k is an internal edge of U.

We claim that then r_k is characteristic. Indeed, either (i) $k = j$, so that r_k is characteristic since collapsing it can not cause any color of S or of T to be missed; (ii) $j <_d k$, in which case k must be a node of S, and hence r_k is a fork. Hence, if $r_k^{\lambda,U}$ doesn't have any element of the form t_k, it must be the case that in a problematic $F - r_k$ one collapses the necessary edges to use Lemma INTERCHANGE LEM V.36.

(Case II) Now consider the case where U contains no leafs of the form r_i. There are two subcases:

(a) If the root vertex of U is $r_0^{\uparrow T} \leq r_0$, then since we are assuming U is η-internal, at least one $t_0 \in r_0^{\uparrow T}$ sits below a η_s stump. Picking t_0 to be the highest such edge, note that the vertex at t_0 is necessarily $t_0^{\uparrow S} \leq t_0$. We note that t_0 is then internal and claim that it is characteristic, since either (i) it is $t_0 = \eta_0$; (ii) it is $\eta <_d t$, so that t_0 is a fork and, provided $t_0^{\lambda,U}$ does not contain any elements of the form t_i, it must be the case that in a problematic $F - t_0$ one collapses the necessary edges to use Lemma INTERCHANGE LEM IV.36.

(b) It the root vertex of U is $r_0^{\uparrow S} \leq r_0$, let r_i denote a highest possible edge such that the corresponding vertex is $r_i^{\uparrow S} \leq r_i$. In particular, note that the relations $r_i^{\uparrow S,T} \leq r_i^{\uparrow S} \leq r_i$ must be in U. Denote now by $r_{j_1}, \cdots, r_{j_k}$ the elements of $r_i^{\uparrow S}$. If, for some r_{j_i} it is $r_{j_i}^{\lambda,U} \not\leq r_{j_i}^{\uparrow S}$, one of the leaves above r_{j_i} has S leaf j_i and thus r_{j_i} is characteristic.

Otherwise, for each r_{j_i} one can find an edge a_{j_i} together with a vertex $a_{j_i}^{\uparrow S} \leq a_{j_i}$. One can then successively use $a_{j_1}, \cdots, a_{j_k}$ as characteristic for $U - a_{j_l} - a_{j_{l+1}} - \cdots - a_{j_k}$.

(Case III) Finally, in the case where all of the leaves of U have the form r_i, the argument from Case II(b) follows unchanged.

Finally, we note that, to conclude that the scenario where $F - \eta_s$ is a common face of some $V \neq U$, one notes that, since we are dealing with inner faces, we necessarily have that U, V, F all have the same faces, and hence one can apply Lemma INTERCHANGE LEM IV.36. □

HERE

Show the characteristic claim about not meeting guys in the previous part of the order!!!

Add openness/closedness lemma for verifying characteristic conditions

References

BeMo08 [1] Berger and Moerdijk. On an extension of the notion of reedy category.

Hi03 [2] P. S. Hirschhorn. Model categories and their localizations, volume 99 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003.

Hov98 [3] M. Hovey. Model categories, volume 63 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999.

HSS [4] M. Hovey, B. Shipley, and J. Smith. Symmetric spectra. J. Amer. Math. Soc., 13(1):149–208, 2000.

MW09 [5] I. Moerdijk and I. Weiss. On inner Kan complexes in the category of dendroidal sets. Adv. Math., 221(2):343–389, 2009.

Pe14 [6] L. A. Pereira. Cofibrancy of operadic constructions in positive symmetric spectra. Submitted, available at: http://arxiv.org/abs/1410.4816v2, 2014.

Sh04 [7] B. Shipley. A convenient model category for commutative ring spectra. In Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, volume 346 of Contemp. Math., pages 473–483. Amer. Math. Soc., Providence, RI, 2004.

We12 [8] Weiss. Broad posets, trees, and the dendroidal category.