Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

No content yet.

2 Basic definitions

In this section we recall some definitions that will be used throughout.

2.1 Grothendieck constructions

Recall that for a diagram category $\mathcal{D}$ and functor $\mathcal{I}_\bullet$

$$\begin{aligned} \mathcal{D} &\xrightarrow{\mathcal{I}_\bullet} \mathsf{Cat} \\ d &\longmapsto \mathcal{I}_d \end{aligned} \tag{2.1}$$

the (covariant) Grothendieck construction $\mathcal{D} \ltimes \mathcal{I}_\bullet$ has objects pairs (d,i) with $d \in \mathcal{D}$, $i \in \mathcal{I}_d$ and arrows $(d,i) \to (d',i')$ given by pairs

$$(f\colon d \to d', g\colon f_*(i) \to i'),$$

where $f_*\colon \mathcal{I}_d \to \mathcal{I}_{d'}$ is a shorthand for the functor $\mathcal{I}_\bullet(f)$.

We now discuss a basic property of over and under categories that will be used in Section TRANSFSIMP SEC 5.4.

Given $\mathcal{J}, \mathcal{C} \in \mathsf{Cat}$ and $j \in \mathcal{J}$ we will let $\mathcal{C}^{\downarrow j}$ denote the Grothendieck construction for the functor

$$\begin{aligned} \mathcal{J} &\longrightarrow \mathsf{Cat} \\ i &\longmapsto \mathcal{C}^{\mathcal{J}(i,j)} \end{aligned}$$

Explicitly, an object of $\mathcal{C}^{\downarrow j}$ is a pair $\left(i, \mathcal{J}(i,j) \xrightarrow{\varphi} \mathcal{C}\right)$ and an arrow $(i,\varphi) \to (i',\varphi')$ is a pair $(I\colon i \to i', \gamma\colon \varphi \circ I^* \to \varphi')$.

UNDERLEFTADJ LEM **Lemma 2.2.** *Let $\mathcal{J} \in \mathsf{Cat}$ be a small category and $j \in \mathcal{J}$. One then has adjunctions*

$$(- \downarrow j)\colon \mathsf{Cat}_{/\mathcal{J}} \rightleftarrows \mathsf{Cat}\colon (-)^{\downarrow j}, \qquad (j \downarrow -)\colon \mathsf{Cat}_{/\mathcal{J}} \rightleftarrows \mathsf{Cat}\colon (-)^{j\downarrow}.$$

Proof. Since $j \downarrow \mathcal{I} = (\mathcal{I}^{op} \downarrow j)^{op}$ by defining $(\mathcal{C}^{j\downarrow}) = ((\mathcal{C}^{op})^{\downarrow j})^{op}$ one reduces to the leftmost adjuntion.

Given $\mathcal{I} \xrightarrow{\pi} \mathcal{J}$ and $\mathcal{C}$ we will show that functors $\mathcal{I} \downarrow j \xrightarrow{F} \mathcal{C}$ correspond to functors $\mathcal{I} \xrightarrow{G} \mathcal{C}^{\downarrow j}$ over $\mathcal{J}$.

On objects, F associates to each pair $(i, J{:}\,\pi(i) \to j)$ an object $F(i,J) \in \mathcal{C}$. One thus sets $G(i) = (\pi(i), F(i,-))$ and these are clearly inverse processes.

On arrows F associates to $(i, J' \circ \pi(I)) \xrightarrow{I} (i', J')$ an arrow $F(i, J' \circ \pi(I)) \xrightarrow{F(I)} F(i', J')$. One thus defines

$$G(I) = \left(\pi(i) \xrightarrow{\pi(I)} \pi(i'), F\left(i, (-) \circ \pi(i)\right) \xrightarrow{F(I)} F\left(i', -\right)\right)$$

and again it is clear that these are inverse processes. Finally, the fact that the associativity and unit conditions for F, G coincide is likewise clear. □

2.2 Monads

We will make multiple uses of the following straightforward results.

MONADADJ1 PROP

Proposition 2.3. *Let $L{:}\,\mathcal{C} \rightleftarrows \mathcal{D}{:}\,R$ be an adjunction and T a monad on $\mathcal{D}$. Then*

(i) RTL is a monad and R induces a functor $R{:}\,\mathsf{Alg}_T(\mathcal{D}) \to \mathsf{Alg}_{RTL}(\mathcal{C})$;

(ii) if $LRTL \xrightarrow{\epsilon} TL$ is an isomorphism one further has an induced adjunction

$$L{:}\,\mathsf{Alg}_{RTL}(\mathcal{C}) \rightleftarrows \mathsf{Alg}_T(\mathcal{D}){:}\,R.$$

MONADADJ PROP

Proposition 2.4. *Let $L{:}\,\mathcal{C} \rightleftarrows \mathcal{D}{:}\,R$ be an adjunction, T a monad on $\mathcal{C}$, and suppose further that*

$$LR \xrightarrow{\epsilon} id_{\mathcal{D}}, \qquad LT \xrightarrow{\eta} LTRL$$

are natural isomorphisms (so that in particular $\mathcal{D}$ is a reflexive subcategory of $\mathcal{C}$).

Then

(i) LTR is a monad, with multiplication and unit given by

$$LTRLTR \xrightarrow{\eta^{-1}} LTTR \to LTR, \qquad id_{\mathcal{D}} \xrightarrow{\epsilon^{-1}} LR \to LTR;$$

(ii) $d \in \mathcal{D}$ is a LTR-algebra iff Rd is a T-algebra;

(iii) there is an induced adjunction

$$L{:}\,\mathsf{Alg}_T(\mathcal{C}) \rightleftarrows \mathsf{Alg}_{LTR}(\mathcal{D}){:}\,R.$$

3 Planar and tall maps

3.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

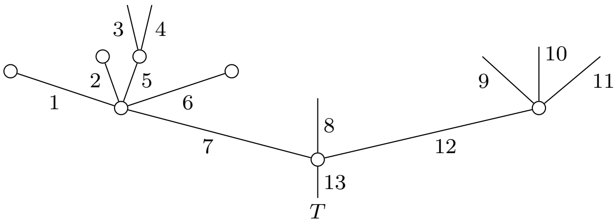
Our preferred model for trees will be that of broad posets first introduced by Weiss in We12 [8] and further worked out by the second author in Pe17 [6]. We now define planar structures in this context.

PLANARIZE DEF

Definition 3.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\leq_d$ to a total order $\leq_p$ such that:

- *Planar*: if $e \leq_p f$ and $e \nleq_d f$ then $g \leq_d f$ implies $e \leq_p g$.

Example 3.2. An example of a planar structure on a tree T follows, with $\leq_r$ encoded by the number labels.



(3.3) PLANAREX EQ

Intuitively, given a planar depiction of a tree T, $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 3.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

INPUTPATH NOT

Notation 3.4. Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path of* e.

We will repeatedly use the following, which is a consequence of [6, Cor. 5.26].

INCOMPNOTOP

Lemma 3.5. *If $e \leq_d f$, $e \leq_d f'$, then f, f' are $\leq_d$-comparable.*

INPUTPATHS PROP

Proposition 3.6. *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \leq_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 3.5. To prove (b) we note that $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee e_i$. ☐

Notation 3.7. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^{\uparrow} \in T$ the predecessor of b in $I(e)$.

ECESSORPROP PROP

Proposition 3.8. *Suppose e, f are $\leq_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^{\uparrow} \neq b_f^{\uparrow}$;

(b) $b_e^{\uparrow}, b_f^{\uparrow} \in b^{\uparrow}$. In fact $\{b_e^{\uparrow}\} = I(e) \cap b^{\uparrow}$, $\{b_f^{\uparrow}\} = I(f) \cap b^{\uparrow}$;

(c) if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^{\uparrow} = b_e^{\uparrow}$, $b_{f'}^{\uparrow} = b_f^{\uparrow}$.

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^{\uparrow} = b_f^{\uparrow}$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^{\uparrow}$, where uniqueness follows from Lemma 3.5. Choosing $a = e$ implies $I(e) \cap b^{\uparrow} = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. ☐

Proposition 3.9. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$.*

Therefore, all ternary joins in $(T, \leq_d)$ are binary, i.e.

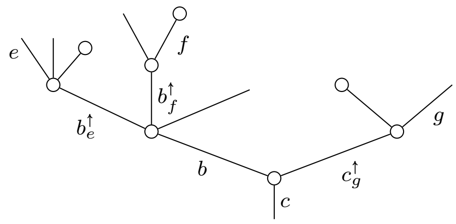
$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{3.10}$$

for some $1 \leq i < j \leq 3$, and (3.10) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$ then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 3.8(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

Example 3.11. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c^{\uparrow}_e = c^{\uparrow}_f = b$.



Notation 3.12. Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will usually abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order in S.

Proposition 3.13. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \leq_p)\} & \longrightarrow & \prod_{(a^{\uparrow} \leq a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p |_{a^{\uparrow}}) \end{array} \tag{3.14}$$

Proof. We will keep the setup of Proposition 3.8 throughout: e, f are $\leq_d$-incomparable edges and we write $b = e \vee f$.

We first show that (3.14) is injective, i.e. that the restrictions $\leq_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_e <_p b^{\uparrow}_f$, the relations $e \leq_d b^{\uparrow}_e <_p b^{\uparrow}_f \geq_d f$ and Definition 3.1 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_f <_p b^{\uparrow}_e$ then $f <_p e$. Thus $b^{\uparrow}_e <_p b^{\uparrow}_f \Leftrightarrow e <_p f$ and hence (3.14) is indeed injective.

To check that (3.14) is surjective, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $b^{\uparrow}_e < b^{\uparrow}_f$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\leq_p$ in the case $e' \leq_d e <_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 3.8(c). Transitivity of $\leq_p$ in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are $\leq_d$-incomparable, in which case one can apply 3.8(c) with the roles of f, f' reversed.

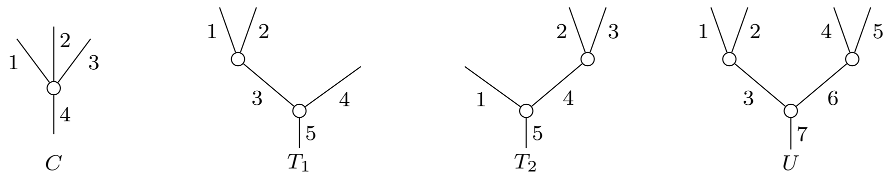
It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\leq_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 3.9, either (i) $e \vee f <_d c$, in which case Proposition 3.9 implies $c^{\uparrow}_e = c^{\uparrow}_f$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_e < c^{\uparrow}_f < c^{\uparrow}_g$ in $c^{\uparrow}$ so that $c^{\uparrow}_e \neq c^{\uparrow}_g$ and by Proposition 3.9 it is also $e \vee g = c$ and transitivity follows. □

Remark 3.15. Definition 3.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 3.13 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [6, Def. 7.44]), which adds a single edge η to F, serving as the (unique) root of $F \star \eta$.

When discussing the substitution procedure in Section SUBS SEC 3.3 we will find it convenient to work with a model for the category Ω that possesses exactly one representative of each possible planar structure on each tree or, more precisely, such that the only isomorphisms preserving the planar structures are the identities. On the other hand, using such a model for Ω throughout would, among other issues, make the discussion of faces in Section OUTTALL SEC 3.2 rather awkward. We now outline our conventions to address such issues.

Let Ω^p, the category of *planarized trees*, denote the category with objects pairs $T_{\leq_p} = (T, \leq_p)$ of trees together with a planar structure and morphisms the *underlying* maps of trees (so that the planar structures are ignored). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq_p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ and for which $\leq_p$ coincides with the canonical order.

STANDMODEL EX **Example 3.16.** Some examples of standard models, i.e. objects of Ω^s, follow (further, (PLANAREX EQ 3.3) can also be interpreted as such an example).



(3.17) PLANAROMEGAEX1 EQ

Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq_p} \in \Omega^p$ there is an obvious standard model $T^s_{\leq_p} \in \Omega^s$ given by replacing each edge by its order following $\leq_p$. Indeed, this defines a retraction $(-)^s : \Omega^p \to \Omega^s$ and a natural transformation $\sigma : id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is clearly unique).

PLANARCONV CON **Convention 3.18.** From now on, we will write simply Ω, Ω_G to denote the categories Ω^s, Ω^s_G of standard models (where planar structures are defined in the underlying forest as in Remark FORESTPLAN REM 3.15). Similarly O_G will denote the model O^s_G for the orbital category whose objects are the orbital G-sets whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$.

Therefore, whenever one of our constructions produces an object/diagram in Ω^p, Ω^p_G, O^p_G (of trees, G-trees, orbital G-sets with a planarization/total order) we will hence implicitly reinterpret it by using the standardization functor $(-)^s$.

Example 3.19. To illustrate our convention, we consider the trees in Example STANDMODEL EX 3.16.

One has subfaces $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures. Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

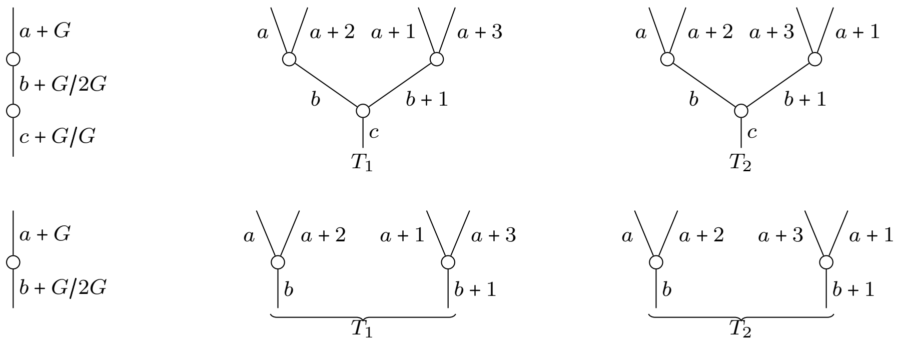
$$\begin{tikzcd} F_1 \arrow[rr, hook] \arrow[dr, hook] & & U \\ & F_2 \arrow[ur, hook] & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr] \arrow[dr] & & U \\ & T_1 \arrow[ur] & \end{tikzcd}$$

Similarly, let $\leq_{(12)}$ and $\leq_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs $1, 2$ and $4, 5$, so that one has objects $U_{\leq_{(12)}}$, $U_{\leq_{(45)}}$ in Ω^p. Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.

$$\begin{tikzcd} U \arrow[rr, "id"] \arrow[dr, "id"'] & & U_{\leq_{(45)}} \\ & U_{\leq_{(12)}} \arrow[ur, "id"'] & \end{tikzcd} \qquad \begin{tikzcd} U \arrow[rr, "(45)"] \arrow[dr, "(12)"'] & & U \\ & U \arrow[ur, "(12)(45)"'] & \end{tikzcd}$$

Example 3.20. An additional reason to leave the use of $(-)^s$ implicit is that when depicting G-trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, when $G = \mathbb{Z}_{/4}$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting of the two trees $T_1, T_2 \in \Omega_G$ on the right.



Definition 3.21. A morphism $S \xrightarrow{\varphi} T$ in Ω that is compatible with the planar structures $\le_p$ is called a *planar map.*

More generally, a morphism $F \to G$ in the categories Φ, Φ^G, Ω^G of forests, G-forests, G-trees is called a *planar map* if it is an independent map (cf. [6, Def. 5.28]) compatible with the planar structures $\le_p$.

Remark 3.22. The need for the independence condition is justified by [6, Lemma 5.33] and its converse, since non independent maps do not reflect $\le_d$ inequalities.

We note that in the Ω_G case a map φ is independent iff φ does not factor through a non trivial quotient iff φ is injective on each edge orbit.

PLANARPULL EQ **Proposition 3.23.** *Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization*

$$F \xrightarrow{\simeq} \bar{F} \to G$$

such that $F \xrightarrow{\simeq} \bar{F}$ is an isomorphism and $\bar{F} \to G$ is planar.

Proof. We need to show that there is a unique planar structure $\le_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of G ensures that for any vertex $e^\uparrow \le e$ of F the edges in $\varphi(e^\uparrow)$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(\underline{r}_F)$ are distinct. The result now follows from (the forest version of) Proposition 3.13: one simply orders each set $e^\uparrow$ and $\underline{r}_F$ according to its image.

not quite complete... maybe that $\le_p$ is the closure of $\le_d$ and the vertex relations under transitivity and the planar condition □

PULLPLANAR REM **Remark 3.24.** Proposition 3.23 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As an example, in the notation of (3.17), consider the map $C \to T_1$ defined by $1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 5$.

UNIQCOR REM **Remark 3.25.** Given any tree $T \in \Omega$ there is a unique corolla $\mathsf{lr}(T) \in \Sigma$ and planar tall map $\mathsf{lr}(T) \to T$. Explicitly, the number of leaves of $\mathsf{lr}(T)$ matches that of T, together with the inherited order.

3.2 Outer faces and tall maps

OUTTALL SEC

In preparation for our discussion of the substitution operation in Section 3.3, we now recall some basic notions and results concerning outer subtrees and tree grafting, as in [6, Section 5].

Definition 3.26. Let $T \in \Omega$ be a tree and $e_1 \cdots e_n = \underline{e} \le e$ a broad relation in T.

We define the *planar outer face* $T_{\underline{e} \le e}$ to be the subtree with underlying set those edges $f \in T$ such that

$$f \le_d e, \quad \forall_i e_i \nleq_d f, \tag{3.27}$$

OUTERFACE EQ

generating broad relations the relations $f^{\uparrow} \leq f$ for f satisfying (3.27) and $\forall_i f \neq e_i$, and planar structure pulled back from T.

Remark 3.28. If one forgoes the requirement that $T_{\underline{e} \leq e}$ be equipped with the pullback planar structure, the inclusion $T_{\underline{e} \leq e} \to T$ is usually called simply an *outer face.*

We now recap some basic results.

Proposition 3.29. *Let $T \in \Omega$ be a tree.*

(a) $T_{\underline{e} \leq e}$ is a tree with root e and edge tuple $\underline{e}$;

(b) there is a bijection

$$\{\textit{planar outer faces of } T\} \leftrightarrow \{\textit{broad relations of } T\};$$

(c) if $R \to S$ and $S \to T$ are outer face maps then so is $R \to T$;

(d) any pair of broad relations $\underline{g} \leq v$, $\underline{f} v \leq e$ induces a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "v"] \arrow[d, "v"'] & T_{\underline{g} \leq v} \arrow[d] \\ T_{\underline{f} v \leq e} \arrow[r] & T_{\underline{f}\underline{g} \leq v} \end{tikzcd} \tag{3.30}$$

Proof. We first show (a). That $T_{\underline{e} \leq e}$ is indeed a tree is the content of [6, Prop. 5.20]: more precisely, $T_{\underline{e} \leq e} = (T^{\leq e})_{< \underline{e}}$ in the notation therein. That the root of $T_{\underline{e} \leq e}$ is e is clear and that the root tuple is $\underline{e}$ follows from [6, Remark 5.23].

(b) follows from (a), which shows that $\underline{e} \leq e$ can be recovered from $T_{\underline{e} \leq e}$.

(c) follows from the definition of outer face together with [6, Lemma 5.33], which states that the $\leq_d$ relations on S, T coincide.

Since by (c) both $T_{\underline{g} \leq v}$ and $T_{\underline{f} v \leq e}$ are outer faces of $T_{\underline{f}\underline{g} \leq v}$, (d) is a restatement of [6, Prop. 5.15]. □

Definition 3.31. A map $S \xrightarrow{\varphi} T$ in Ω is called a *tall map* if

$$\varphi(\underline{l}_S) = \underline{l}_T, \qquad \varphi(r_S) = r_T,$$

where $l_{(-)}$ denotes the leaf tuple and $r_{(-)}$ the root.

The following is a restatement of [6, Cor. 5.24]

Proposition 3.32. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphism,*

$$S \xrightarrow{\varphi^t} U \xrightarrow{\varphi^u} T$$

as a tall map followed by an outer face (in fact, $U = T_{\varphi(\underline{l}_S) \leq r_S}$).

We recall that a face $F \to T$ is called inner if is obtained by iteratively removing inner edges, i.e. edges other than the root or the leaves. In particular, it follows that a face is inner iff it is tall. The usual face-degeneracy decomposition thus combines with Corollary 3.32 to give the following.

Corollary 3.33. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^u} T \tag{3.34}$$

as a degeneracy followed by an inner face followed by an outer face.

Proof. The factorization (3.34) can be built by first performing the degeneracy-face decomposition and then performing the tall-outer decomposition on the face map. □

3.3 Substitution

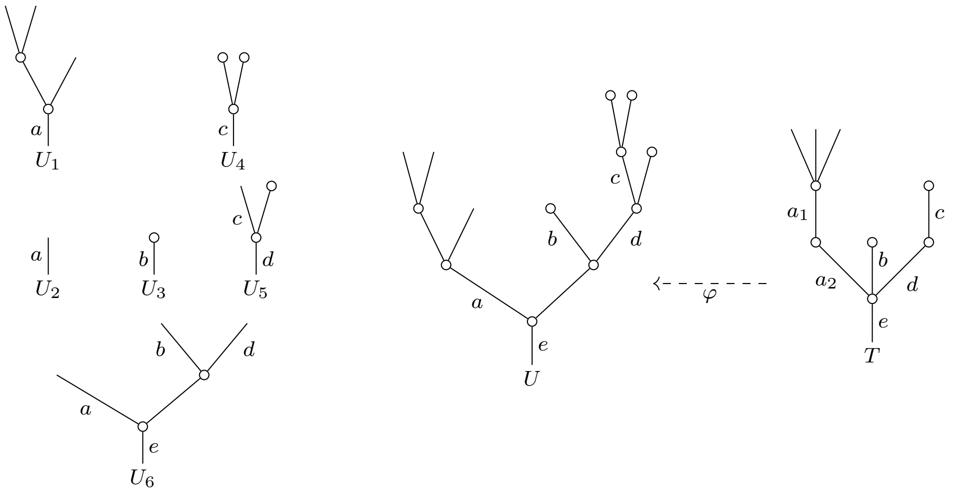
SUBS SEC

One of the key ideas needed to describe operads is that of substitution of tree nodes, a process that we will prefer to repackage in terms of maps of trees. We start by discussing an example, focusing on the related notion of iterated graftings of trees (as described in (3.30)).

Example 3.35. The trees $U_1, U_2, \cdots, U_6$ on the left below can be grafted into the tree U in the middle. More precisely (among other possible grafting orders), one has

$$U = (((((U_6 \amalg_a U_2)) \amalg_a U_1) \amalg_b U_3) \amalg_d U_5) \amalg_c U_4 \tag{3.36}$$

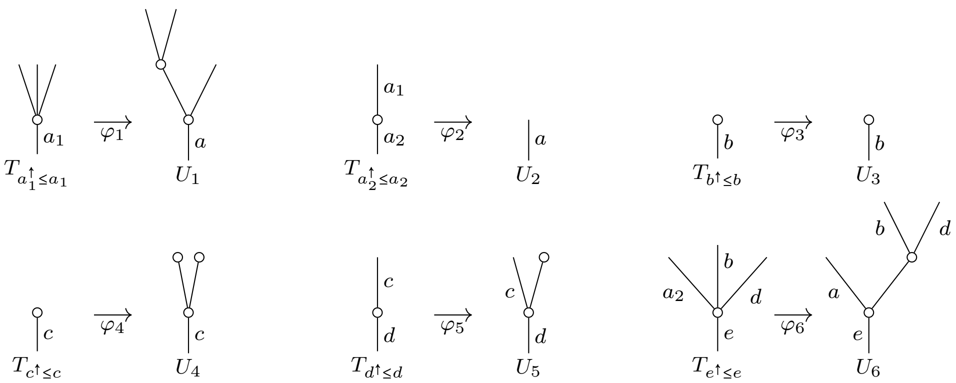
UFORMULA EQ



(3.37) SUBSDATUMTREES EQ

We now consider the tree T, which is built by converting each U_i into the corolla $\mathrm{lr}(U_i)$ (cf. Remark 3.25), and then performing the same grafting operations as in (3.36). T can then be regarded as encoding the combinatorics of the iterated grafting in (3.36), with alternative ways to reorder operations in (3.36) in bijection with ways to assemble T out of its nodes.

One can now therefore think of the iterated grafting (3.36) as being instead encoded by the tree T together with the (unique) planar tall maps φ_i below.



(3.38) SUBSDATUMTREES2 EQ

From this perspective, U can now be thought as obtained from T by substituting each of its nodes with the corresponding U_i. Moreover, the φ_i assemble to a planar tall map $\varphi: T \to U$ (such that $a_i \mapsto a, b \mapsto b, \cdots, e \mapsto e$), which likewise encodes the same information.

Our perspective will then be that data for substitution of tree nodes such as in (3.38) can equivalently be repackaged using planar tall maps.

UBSTITUTIONDATUM

Definition 3.39. Let $T \in \Omega$ be a tree.

A *T-substitution datum* is a tuple $\{U_{e^\uparrow \leq e}\}_{(e^\uparrow \leq e) \in V(T)}$ together with tall maps $T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}$.

Further, a map of planar T-substitution data $\{U_{e^\uparrow \leq e}\} \to \{V_{e^\uparrow \leq e}\}$ is a tuple of tall maps $\{U_{e^\uparrow \leq e} \to V_{e^\uparrow \leq e}\}$ compatible with the chosen maps.

Lastly, a substitution datum is called a *planar T-substitution datum* if the chosen maps are planar (so that $\mathsf{lr}(U_{e^\uparrow \leq e}) = T_{e^\uparrow \leq e}$) and a morphism of planar data is called a planar morphism if it consists of a tuple of planar maps.

Definition 3.40. Let $T \in \Omega$.

The *Segal core poset* $\mathsf{Sc}(T)$ is the poset with objects the edge subtrees η_e and vertex substrees $T_{e^\uparrow \leq e}$. The order relation is given by inclusion.

Remark 3.41. Note that the only maps in $\mathsf{Sc}(T)$ are inclusions of the form $\eta_a \subset T_{e^\uparrow \leq e}$. In particular, there are no pairs of composable non-identity relations in $\mathsf{Sc}(T)$.

Given a T-substitution datum $\{U_{\{e^\uparrow \leq e\}}\}$ we abuse notation by writing

$$U_{(-)} \colon \mathsf{Sc}(T) \to \Omega$$

for the functor $\eta_a \mapsto \eta$, $T_{e^\uparrow \leq e} \mapsto U_{e^\uparrow \leq e}$ and sending the inclusions $\eta_a \subset T_{e^\uparrow \leq e}$ to the composites

$$\eta \xrightarrow{a} T_{e^\uparrow \leq e} \to U_{e^\uparrow \leq e}.$$

TAUNDERPLAN PROP

Proposition 3.42. *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\begin{array}{ccc} \mathsf{Sub}_p(T) & \rightleftarrows & \Omega^{\mathsf{pt}}_{T/} \\ \{U_{e^\uparrow \leq e}\} & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \\ \{U_{\varphi(e^\uparrow) \leq \varphi(e)}\} & \longleftarrow\!\shortmid & (T \xrightarrow{\varphi} U) \end{array} \tag{3.43}$$

SUBDATAUNDERPLAN EQ

where $\mathsf{Sub}_p(T)$ denotes the category of planar T-substitution data and $\Omega^{\mathsf{pt}}_{T/}$ the category of planar tall maps under T.

Proof. We first claim that (i) the $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ indeed exists; (ii) for the canonical datum $\{T_{e^\uparrow \leq e}\}$, it is $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$; (iii) the induced map $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall.

The argument is by induction on the number of vertices of T, with the base cases of T with 0 or 1 vertices being immediate, since then T is the terminal object of $\mathsf{Sc}(T)$. Otherwise, one can choose a non trivial grafting decomposition so as to write $T = R \amalg_e S$, resulting in identifications $\mathsf{Sc}(R) \subset \mathsf{Sc}(T)$, $\mathsf{Sc}(S) \subset \mathsf{Sc}(T)$ so that $\mathsf{Sc}(R) \cup \mathsf{Sc}(S) = \mathsf{Sc}(T)$ and $\mathsf{Sc}(R) \cap \mathsf{Sc}(S) = \{\eta_e\}$. The existence of $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is thus equivalent to the existence of the pushout below.

$$\begin{tikzcd} \eta \arrow[r] \arrow[d] & \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)} \arrow[d, dashed] \\ \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)} \arrow[r, dashed] & \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)} \end{tikzcd} \tag{3.44}$$

ASSEMBLYGRAFT EQ

By induction, the top right and bottom left colimits exist for any $U_{(-)}$, equal R and S in the case $U_{(-)} = T_{(-)}$, and the maps $R \to \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)}$, $S \to \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)}$ are planar tall. But is now follows that (3.44) is a grafting pushout diagram, so that the pushout indeed exists. The conditions that $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$ and $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall follow.

The fact that the two functors in (3.43) are inverse to each other is clear by the same inductive argument. $\square$

ATAUNDERPLAN COR

Corollary 3.45. *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\mathsf{Sub}(T) \rightleftarrows \Omega^{\mathsf{t}}_{T/} \tag{3.46}$$

SUBDATAUNDERNONPL EQ

where $\mathsf{Sub}(T)$ denotes the category of T-substitution data and $\Omega^{\mathsf{t}}_{T/}$ the category of tall maps under T.

Proof. This is a consequence of Proposition 3.23 together with the previous result, with the functor $\mathsf{Sub}(T) \to \Omega^{\mathsf{t}}_{T/}$ given by the same formula. Indeed, Proposition 3.13 can be restated as saying that isomorphisms $T \to T'$ are in bijection with substitution data consisting of isomorphisms, and thus bijectiveness reduces to that in the previous result. □

VERTEXDECOMP REM **Remark 3.47.** It follows from the previous proof that, writing $U = \mathrm{colim}_{\mathsf{Sc}(T)} U_{(-)}$, one has

$$V(U) = \coprod_{(e^{\uparrow} \leq e) \in V(T)} V(U_{e^{\uparrow} \leq e}). \tag{3.48}$$ VERTEXDECOMP EQ

Alternatively, (3.48) can be regarded as a map $f^*: V(U) \to V(T)$ induced by the planar tall map $f: T \to U$. Explicitly, $f^*(U_{u^{\uparrow} \leq u})$ is the unique $T_{t^{\uparrow} \leq t}$ such that $U_{u^{\uparrow} \leq u} \subset U_{t^{\uparrow} \leq t}$. We note that f^* is indeed contravariant in the tall planar map f.

The following is a converse of sorts to Proposition 3.42.

BUILDABLE PROP **Proposition 3.49.** *Let $U \in \Omega$ be a tree. Then:*

(i) given non stick outer subtrees U_i such that $V(U) = \coprod_i V(U_i)$ there is a unique tree T and planar tall map $T \to U$ such that $\{U_i\} = \{U_{e^{\uparrow} \leq e}\}$;

(ii) given multiplicities $m_e \geq 1$ for each edge $e \in U$, there is a unique planar degeneracy $\rho: T \to U$ such that $\rho^{-1}(e)$ has m_e elements;

(iii) planar tall maps $T \to U$ are in bijection with collections $\{U_i\}$ of outer subtrees such that $V(U) = \coprod_i V(U_i)$ and U_j is not an inner edge of any U_i whenever $U_j \simeq \eta$ is a stick.

Proof. We first show (i) by induction on the number of subtrees U_i. The base case $\{U_i\} = \{U\}$ is immediate, setting $T = \mathsf{lr}(U)$. Otherwise, letting e be edge that is both an inner edge of U and a root of some U_i, and one can form a pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "e"] \arrow[d, "e"'] & V \arrow[d] \\ W \arrow[r] & U \end{tikzcd} \tag{3.50}$$ DECOMPPROOF EQ

inducing a nontrivial partition $\{U_i\} = \{U_i | U_i \hookrightarrow V\} \amalg \{U_i | U_i \hookrightarrow W\}$. Existence of $T \to U$ now follows from the induction hypothesis. For uniqueness, the condition that no U_i is a stick guarantees that T possesses a single inner edge mapping to e, and thus admits a compatible decomposition as in (3.50), and thus uniqueness too follows by the induction hypothesis.

For (ii), we argue existence by nested induction on the number of vertices $|V(U)|$ and the sum of the multiplicities m_e. The base case $|V(U)| = 0$, i.e. $U = \eta$ is immediate. Otherwise, writing $m_e = m'_e + 1$, one can form a decomposition (3.50) where either $|V(V)|, |V(W)| < |V(U)|$ or one of V, W is η, so that $T \to U$ can be built via the induction hypothesis. For uniqueness, note first that by [6, Lemma 5.33] each pre-image $\rho^{-1}(e)$ is linearly ordered and by the "further" claim in [6, Cor. 5.39] the remaining broad relations are precisely the pre-image of the non-identity relations in U, showing that the tree T is uniquely determined.

(iii) follows by combining (i) and (ii). Indeed, any planar tall map $T \to U$ has a unique decomposition $T \twoheadrightarrow \bar{T} \hookrightarrow U$ as a planar degeneracy followed by a planar inner face, and each of these maps is classified by the data in (b) and (a). □

TERFACEUNION LEM **Lemma 3.51.** *Suppose $T_1, T_2 \hookrightarrow T$ are two outer faces with at least one common edge e. Then there exists an unique outer face $T_1 \cup T_2$ such that $V(T_1 \cup T_2) = V(T_1) \cup V(T_2)$.*

Proof. If either of T_1, T_2 is the root or a leaf the result is obvious. Otherwise, one can necessarily choose e to be an inner edge of T, in which case all of T_1, T_2, T admit compatible decompositions (3.50) and the result follows by induction on $|V(T)|$. □

4 The genuine equivariant operad monad

We now turn to the task of building the monad encoding genuine equivariant operads.

4.1 Wreath product over finite sets

In what follows we will let F denote the usual skeleton of the category of finite sets and all set maps. Explicitly, its objects are the finite sets $\{1,2,\cdots,n\}$ for $n \geq 0$. However, much as in the discussion in Convention 3.18 we will often find it more convenient to regard the elements of F as equivalence classes of finite sets equipped with total orders.

Definition 4.1. For a category $\mathcal{C}$, we let $\mathsf{F} \wr \mathcal{C}$ denote the opposite of the Grothendieck construction for the functor

$$\begin{aligned} \mathsf{F}^{op} &\longrightarrow \mathsf{Cat} \\ I &\longmapsto \mathcal{C}^{I} \end{aligned}$$

Explicitly, the objects of $\mathsf{F} \wr \mathcal{C}$ are tuples $(c_i)_{i\in I}$ and a map $(c_i)_{i\in I} \to (d_j)_{j\in J}$ consists of a pair

$$(\phi \colon I \to J, (f_i \colon c_i \to d_{\phi(i)})_{i\in I}),$$

henceforth abbreviated as $(\phi,(f_i))$.

The following is immediate.

Proposition 4.2. *Suppose $\mathcal{C}$ has all finite coproducts. One then has a functor as on the left below. Dually, if $\mathcal{C}$ has all finite products, one has a functor as on the right below.*

$$\begin{aligned} \mathsf{F} \wr \mathcal{C} &\xrightarrow{\ \amalg\ } \mathcal{C} \\ (c_i)_{i\in I} &\longmapsto \textstyle\coprod_{i\in I} c_i \end{aligned} \qquad\qquad \begin{aligned} (\mathsf{F} \wr \mathcal{C}^{op})^{op} &\xrightarrow{\ \Pi\ } \mathcal{C} \\ (c_i)_{i\in I} &\longmapsto \textstyle\prod_{i\in I} c_i \end{aligned}$$

WREATPRODLIM LEM

Lemma 4.3. *Suppose that $\mathcal{E}$ is a bicomplete category such that coproducts commute with limits in each variable. If the leftmost diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "k"'] & \mathcal{E} \\ \mathcal{D} \arrow[ur, "G"'] \arrow[u, phantom, "\Uparrow\eta"] & \end{tikzcd} \qquad \begin{tikzcd} \mathsf{F}\wr\mathcal{C} \arrow[r, "\mathsf{F}\wr F"] \arrow[d, "\mathsf{F}\wr k"'] & \mathsf{F}\wr\mathcal{E} \arrow[r, "\amalg"] & \mathcal{E} \\ \mathsf{F}\wr\mathcal{D} \arrow[ur, "\mathsf{F}\wr G"'] \arrow[urr, bend right, "\amalg\circ\mathsf{F}\wr G"'] \arrow[u, phantom, "\Uparrow\mathsf{F}\wr\eta"] & & \end{tikzcd} \tag{4.4}$$

WRRAN EQ

is a right Kan extension diagram then so is the composite of the rightmost diagram.

Dually, if in $\mathcal{E}$ products commute with colimits in each variable, and the leftmost diagram

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "F"] \arrow[d, "k"'] & \mathcal{E} \\ \mathcal{D}^{op} \arrow[ur, "G"'] \arrow[u, phantom, "\Downarrow\epsilon"] & \end{tikzcd} \qquad \begin{tikzcd} (\mathsf{F}\wr\mathcal{C})^{op} \arrow[r, "(\mathsf{F}\wr F^{op})^{op}"] \arrow[d, "(\mathsf{F}\wr k^{op})^{op}"'] & (\mathsf{F}\wr\mathcal{E}^{op})^{op} \arrow[r, "\Pi"] & \mathcal{E} \\ (\mathsf{F}\wr\mathcal{D})^{op} \arrow[ur, "(\mathsf{F}\wr G^{op})^{op}"'] \arrow[urr, bend right, "\Pi\circ(\mathsf{F}\wr G^{op})^{op}"'] & & \end{tikzcd} \tag{4.5}$$

WRLAN EQ

is a left Kan extension diagram then so is the composite of the rightmost diagram.

Proof. Unpacking definitions using the pointwise formula for Kan extensions ([4, X.3.1]), the claim concerning (4.4) amounts to showing that for each $(d_i) \in \mathsf{F} \wr \mathcal{D}$ one has natural isomorphisms

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow\mathsf{F}\wr\mathcal{C})} \left(\coprod_j F(c_j) \right) \simeq \coprod_i \lim_{(d_i\to kc_i)\in d_i\downarrow\mathcal{C}} (F(c_i)). \tag{4.6}$$

POINTKAN EQ

Noting that the canonical factorizations of each $(\varphi,(f_i)) \colon (d_i)_{i\in I} \to (kc_j)_{j\in J}$ as

$$(d_i)_{i\in I} \to (c_{\phi(i)})_{i\in I} \to (kc_j)_{j\in J}$$

exhibit $\prod_i (d_i \downarrow \mathcal{C})$ as a coreflexive subcategory of $(d_i) \downarrow \mathsf{F} \wr \mathcal{C}$, we see that it is an initial subcategory. Therefore

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow \mathsf{F}\wr\mathcal{C})} \left(\coprod_j F(c_j) \right) \simeq \lim_{((d_i)\to(kc_i))\in\prod_i (d_i\downarrow\mathcal{D})} \left(\coprod_i F(c_i) \right)$$

POINTKAN EQ

and hence (4.6) now follows from the assumption that coproducts commute with limits in each variable. □

Notation 4.7. Using the coproduct functor $\mathsf{F}^{\wr 2} = \mathsf{F}^{\wr\{0,1\}} = \mathsf{F}\wr\mathsf{F} \xrightarrow{\amalg} \mathsf{F}$ (where $\coprod_{i\in I} J_i$ is ordered lexicographically) and the simpleton $\{1\} \in \mathsf{F}$ one can regard the collection of categories $\mathsf{F}^{\wr\{0,\cdots,n\}} \wr \mathcal{C} = \mathsf{F}^{\wr \underline{n}} \wr \mathcal{C}$ as a coaugmented cosimplicial object in Cat. As such, we will denote by

$$\delta^i \colon \mathsf{F}^{\wr \underline{n-1}} \wr \mathcal{C} \to \mathsf{F}^{\wr \underline{n}} \wr \mathcal{C}, \qquad 0 \le i \le n$$

the cofaces obtained by inserting simpletons $\{1\} \in \mathsf{F}$ and by

$$\sigma^i \colon \mathsf{F}^{\wr \underline{n+1}} \wr \mathcal{C} \to \mathsf{F}^{\wr \underline{n}} \wr \mathcal{C}, \qquad 0 \le i \le n$$

the codegeneracies obtained by applying the coproduct $\mathsf{F}^{\wr 2} \xrightarrow{\amalg} \mathsf{F}$ to adjacent F coordinates.

4.2 Equivariant leaf-root and vertex functors

Definition 4.8. A morphism $T \xrightarrow{\varphi} S$ in Ω_G is called a *quotient* if the underlying morphism of forests

$$\coprod_{[g]\in G/H} T_{[g]} \to \coprod_{[h]\in G/K} S_{[h]}$$

maps each tree component (or, equivalently, some tree component) isomorphically onto its image component.

We denote the subcategory of G-trees and quotients by Ω^q_G.

Definition 4.9. The *G-symmetric category*, which we will also call the *category of G-corollas*, is the full subcategory $\Sigma_G \subset \Omega^q_G$ of those G-trees that are corollas, i.e. G-trees such that each edge is either a root or a leaf (but not both).

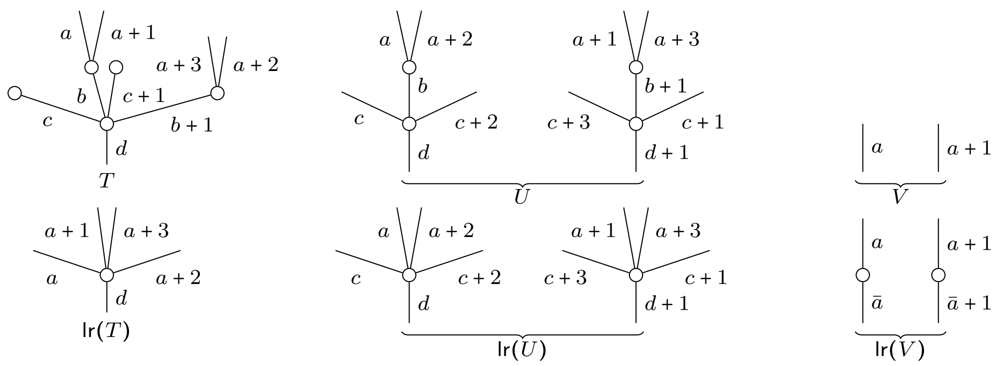
Definition 4.10. The *leaf-root functor* is the functor $\Omega^q_G \xrightarrow{\mathsf{lr}} \Sigma_G$ defined by

$$\mathsf{lr}(T) = \{\text{leaves of } T\} \amalg \{\text{roots of } T\}$$

with a broad relation $l_1 \cdots l_n \le r$ holding in $\mathsf{lr}(T)$ iff its image holds in T and similarly for the planar structure $\le_p$.

UNIQCOR REM

EAFROOTEXAMP REM **Remark 4.11.** Generalizing Remark 3.25, $\mathsf{lr}(T)$ can alternatively be characterized as being the *unique* G-corolla which admits an also unique (tree-wise) tall planar map $\mathsf{lr}(T) \to T$. Moreover, $\mathsf{lr}(T)$ can usually be regarded as the "smallest inner face" of T, obtained by removing all the inner edges, although this characterization fails when $T = G \cdot_H \eta$ is a stick G-tree. Some examples with $G = \mathbb{Z}_{/4}$ follow.



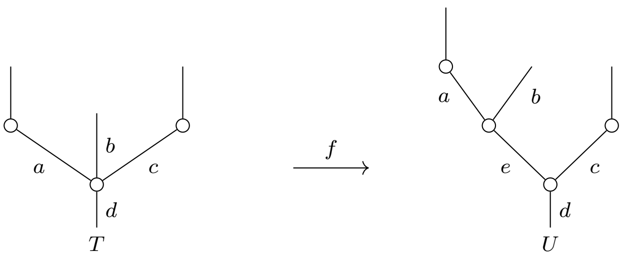
Remark 4.12. One consequence of the fact that planarizations can not be pushed forward along tree maps (cf. Remark PULLPLANAR REM 3.24) is that $\mathsf{lr}\colon \Omega_G^q \to \Sigma_G$ is not a categorical fibration. maybe add to this.

VG DEF **Definition 4.13.** Given $T \in \Omega_G$ we define the set $V_G(T)$ of *G-vertices* of T to be the orbit set $V(T)/G$, i.e. the quotient of the vertex set $V(T)$ by its G-action.

Furthermore, we will regard $V_G(T)$ as an object in F by equipping it with its lexicographic order: i.e. vertex equivalence classes $[e^{\uparrow} \leq e]$ are ordered according to the planar order $\leq_p$ of the smallest representative ge, $g \in G$.

ERTEXDECOMPG REM **Remark 4.14.** Following Remark VERTEXDECOMP REM 3.47, a planar tall map $f\colon T \to U$ of G-trees induces a G-equivariant map $f^*\colon V(U) \to V(T)$ and thus also a map of orbits $f^*\colon V_G(U) \to V_G(T)$. We note, however, that f^* is not in general compatible with the order on V_G, as is indeed the case even in the non-equivariant case.

A minimal example follows.



In $V(T)$ the vertices are ordered as $a < c < d$ while in $V(U)$ they are ordered as $a < e < c < d$ but the map $f^*\colon V(U) \to V(T)$ is given by $a \mapsto a, c \mapsto c, d \mapsto d, e \mapsto d$.

Note that each element of $V_G(T)$ corresponds to an unique edge orbit Ge for e not a leaf. As such, we will represent the corresponding G-vertex by $v_{Ge} = (Ge)^{\uparrow} \leq Ge$ (which we interpret as the concatenation of the relations $f^{\uparrow} \leq f$ for $f \in Ge$) and write

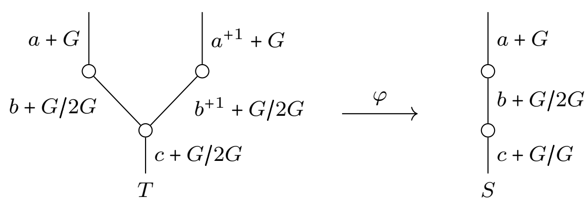
$$T_{v_{Ge}} = T_{(Ge)^{\uparrow} \leq Ge} = \coprod_{f \in Ge} T_{f^{\uparrow} \leq f}.$$

We note that $T_{v_{Ge}}$ is always a G-corolla. Indeed, noting that a quotient map $\varphi\colon T \to S$ induces quotient maps $T_{v_{ge}} \to S_{v_{G\varphi(e)}}$ one obtains a functor

$$\begin{array}{rcl} \Omega_G^q & \xrightarrow{V_G} & \mathsf{F} \wr \Sigma_G \\ T & \longmapsto & (T_{v_{Ge}})_{v_{Ge} \in V_G(T)}. \end{array} \tag{4.15}$$

VFUNCTOR EQ

Remark 4.16. The need to introduce the $\mathsf{F} \wr \mathcal{C}$ categories comes from the fact that general quotient maps do not preserve the number of G-vertices. For a simple example, let $G = \mathbb{Z}_{/4}$ and consider the quotient map



sending edges labeled a, b, c to the edges with the same name and the edges a^{+1}, b^{+1} to the edges $a+1$, $b+1$. We note that T has three G-vertices v_{Gc}, v_{Gb}, $v_{Gb^{+1}}$ while S has only two G-vertices v_{Gc} and v_{Gb}. $V(\phi)$ then maps the two corollas $T_{v_{Gb}}$ and $T_{v_{Gb^{+1}}}$ isomorphically onto $T_{S_{Gb}}$ and the corolla $T_{v_{Gc}}$ non-isomorphically onto $S_{v_{Gc}}$.

SUBSTITUTIONDATUM

Definition 3.39 now immediately generalizes. Here a map is called *rooted* if it induces an ordered isomorphism on the root orbit.

TUTIONDATUMG DEF

Definition 4.17. Let $T \in \Omega_G$ be a G-tree.

A *rooted (resp. planar) T-substitution datum* is a tuple $\{U_{v_{Ge}}\}_{v_{Ge}\in V_G(T)}$ together with rooted (resp. planar) tall maps $T_{v_{Ge}} \to U_{v_{Ge}} = T_{v_{Ge}}$.

Further, a map of rooted (resp. planar) T-substitution data $\{U_{v_{Ge}}\} \to \{V_{v_{Ge}}\}$ is a tuple of rooted (resp. planar) tall maps $\{U_{v_{Ge}} \to V_{v_{Ge}}\}$.

UBSDATUMCONV REM

SUBDATAUNDERPLAN PROP

Remark 4.18. To establish the equivariant analogue of Proposition 3.42 we will prefer to repackage equivariant substitution data in terms of non-equivariant terms.

Noting that there are decompositions $U_{v_{Ge}} = \coprod_{ge\in Ge} U_{ge^{\uparrow}\le ge}$ and letting $G \ltimes V(T)$ denote the Grothendieck construction for the action of G on the non-equivariant vertices $V(T)$ (often called the action groupoid), it is immediate that an equivariant T-substitution datum is the same as a functor $G \ltimes V(T) \to \Omega$ whose restriction to $V(T) \subset G \ltimes V(T)$ is a (non-equivariant) substitution datum.

AUNDERPLANG PROP

Proposition 4.19. *Let $T \in \Omega_G$ be a G-tree. There are isomorphisms of categories*

$$\begin{array}{ccc} \mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & \Omega^{\mathsf{pt}}_{G,T/} \\ \{U_{v_{Ge}}\} & \longmapsto & \big(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\big) \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}_{\mathsf{r}}(T) & \rightleftarrows & \Omega^{\mathsf{rt}}_{G,T/} \\ \{U_{v_{Ge}}\} & \longmapsto & \big(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\big) \end{array} \tag{4.20}$$

SUBDATAUNDERPLANG EQ

SUBDATAUNDERPLANG PROP SUBDATAUNDERPLAN COR SUBSDATUMCONV REM

Proof. This is a minor adaptation of the non-equivariant analogues Proposition 4.19 and Corollary 3.45. Since $\mathsf{Sc}(T)$ inherits a G action, one can form the Grothendieck construction $G\ltimes\mathsf{Sc}(T)$ and by Remark 4.18 equivariant substitution data $\{U_{v_{Ge}}\}$ therefore induce functors $U_{(-)}: G \ltimes \mathsf{Sc}(T) \to \Omega$. It is then immediate that $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ inherits a G-action, provided it exists. The key observation is then that, since $\mathsf{Sc}(T)$ is now a disconnected poset, this colimit is to be interpreted as taken in the category Φ of forests rather than in Ω.

Additionally, we note that the need to use rooted data comes from the fact that rooted isomorphisms $T \to T'$ are in bijection with rooted substitution data that are given by isomorphisms, a statement that fails in the absence of the rooted condition. $\square$

Remark 4.21. We will need to know that in the planar case each of the maps

$$U_{v_{Ge}} \to U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$$

SUBDATAUNDERPLANG PROP

induced by the previous proof is a planar map of G-trees. This requires two observations: (i) the restrictions to each of the constituent non-equivariant trees $U_{ge^{\uparrow}\le ge}$ is planar by Proposition 4.19; (ii) the restriction to the roots of $U_{v_{Ge}}$ is injective and order preserving since it matches the inclusion of the roots of $T_{v_{Ge}}$, and the map $T \to U$ is a planar map of G-trees.

PULLCOMP REM

SUBDATAUNDERPLANG PROP

Remark 4.22. The isomorphisms in Proposition 4.19 are compatible with root pullback of trees. More concretely, any pullback $\pi: S = \varphi^* T \to T$ induces pullbacks $\pi_{Ge}: S_{v_{Ge}} \to T_{v_{Ge}}$ for $v_{Ge} \in V_G(S)$ and one has commutative diagrams

$$\begin{array}{ccc} \mathsf{Sub}_{\mathsf{p}}(S) & \rightleftarrows & \Omega^{\mathsf{pt}}_{G,S/} \\ {\scriptstyle(\pi_{Ge})}\uparrow & & \uparrow{\scriptstyle\pi^*} \\ \mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & \Omega^{\mathsf{pt}}_{G,T/} \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}_{\mathsf{r}}(S) & \rightleftarrows & \Omega^{\mathsf{rt}}_{G,S/} \\ {\scriptstyle(\pi_{Ge})}\uparrow & & \uparrow{\scriptstyle\pi^*} \\ \mathsf{Sub}_{\mathsf{r}}(T) & \rightleftarrows & \Omega^{\mathsf{rt}}_{G,T/} \end{array} \tag{4.23}$$

SUBDATAUNDERPLANG2 EQ

PLANARSTRING SEC

4.3 Planar strings

The leaf-root and vertex functors will allow us to reinterpret our results concerning substitution.

Definition 4.24. The category $\Omega_{G,n}$ of *substitution n-strings* is the category whose objects are strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} T_n$$

where $T_i \in \Omega_G$ and the f_i are tall planar maps, and arrows are commutative diagrams

$$\begin{tikzcd}
T_0 \arrow[r, "f_1"] \arrow[d, "q_0"'] & T_1 \arrow[r, "f_2"] \arrow[d, "q_1"'] & \cdots \arrow[r, "f_n"] & T_n \arrow[d, "q_n"'] \\
T'_0 \arrow[r, "f'_1"'] & T'_1 \arrow[r, "f'_2"'] & \cdots \arrow[r, "f'_n"'] & T'_n
\end{tikzcd} \tag{4.25}$$ PTNARROW EQ

where each q_i is a quotient map.

IMPOPERATORS NOT

Notation 4.26. Since compositions of planar tall arrows are planar tall and identity arrows are planar tall it follows that $\Omega_{G,\bullet}$ forms a simplicial object in Cat, with faces given by composing and degeneracies by inserting identities.

Noting that $\Omega_{G,0} = \Omega^q_G$ and setting $\Omega_{G,-1} = \Sigma_G$, the leaf-root functor $\Omega^q_G \xrightarrow{\mathsf{lr}} \Sigma_G$ makes $\Omega^q_{G,\bullet}$ into an augmented simplicial object and, furthermore, the maps $s_{-1}: \Omega^q_{G,n} \to \Omega^q_{G,n+1}$ sending $T_0 \to T_1 \to \cdots \to T_n$ to $\mathsf{lr}(T_0) \to T_0 \to T_1 \to \cdots \to T_n$ equip it with extra degeneracies.

Notation 4.27. We extend the vertex functor to a functor $V_G: \Omega_{G,n+1} \to \mathsf{F} \wr \Omega_{G,n}$ by

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)} \tag{4.28}$$ VGDEF EQ

where we abuse notation by writing $T_{i,v_{Ge}}$ for $T_{i,(f_i \circ \cdots \circ f_1)(v_{Ge})}$.

The following is a reinterpretation of Proposition 4.19.

SUBSASPULL PROP

Proposition 4.29. *The diagram*

$$\begin{tikzcd}
\Omega_{G,n+1} \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n+1}"'] & \mathsf{F} \wr \Omega_{G,n} \arrow[d, "\mathsf{F} \wr d_{0,\cdots,n}"] \\
\Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd} \tag{4.30}$$ PTPULL EQ

is a pullback diagram in Cat.

Proof. An object in the pullback (4.30) over $T \in \Omega_{G,0} = \Omega^q_G$ is precisely the same as a n-string in $\mathsf{Sub}(T)$, and thus by Proposition 4.19 equivalent to a $n+1$ planar tall string starting at T.

The case of arrows is slightly more subtle. A quotient map $\pi: T \to T'$ induces a G-equivariant poset map $\pi_*: \mathsf{Sc}(T) \to \mathsf{Sc}(T')$ (or equivalently, a map of Grothendieck constructions $G \ltimes \mathsf{Sc}(T) \to G \ltimes \mathsf{Sc}(T')$) and diagrams as on the left below (where v_{Ge} ranges over $V_G(T)$ and $e' = \varphi(e)$) induce diagrams (of functors $\mathsf{Sc}(T) \to \Omega$) as on the right below.

$$\begin{tikzcd}
T_{v_{Ge}} \arrow[r] \arrow[d] & T_{1,v_{Ge}} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_{n,v_{Ge}} \arrow[d] \\
T'_{v_{Ge'}} \arrow[r] & T'_{1,v_{Ge'}} \arrow[r] & \cdots \arrow[r] & T'_{n,v_{Ge'}}
\end{tikzcd}
\qquad
\begin{tikzcd}
T_{(-)} \arrow[r] \arrow[d] & T_{1,(-)} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_{n,(-)} \arrow[d] \\
T'_{(-)} \circ \pi_* \arrow[r] & T'_{1,(-)} \circ \pi_* \arrow[r] & \cdots \arrow[r] & T'_{n,(-)} \circ \pi_*
\end{tikzcd} \tag{4.31}$$ PTNARROWLOC EQ

Passing to colimits then gives the desired commutative diagram (4.25). Moreover, diagrams of the form (4.25) clearly induce diagrams as in (4.31) and it is straightforward to check that these are inverse processes. $\square$

DSCOM REM

Remark 4.32. The diagrams (with back and lower slanted faces instances of (PTPULL EQ 4.30))

$$\begin{tikzcd}
\Omega_{G,n+2} \arrow[rr] \arrow[dr, "d_{i+1}"] \arrow[dd] & & \mathsf{F} \wr \Omega_{G,n+1} \arrow[dr, "\mathsf{F}\wr d_i"] \arrow[dd] & \\
& \Omega_{G,n+1} \arrow[rr] \arrow[dl] & & \mathsf{F} \wr \Omega_{G,n} \arrow[dl] \\
\Omega_{G,0} \arrow[rr] & & \mathsf{F} \wr \Sigma_G &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_{G,n+1} \arrow[rr] \arrow[dr, "s_i"] \arrow[dd] & & \mathsf{F} \wr \Omega_{G,n} \arrow[dr, "\mathsf{F}\wr s_{i-1}"] \arrow[dd] & \\
& \Omega_{G,n+2} \arrow[rr] \arrow[dl] & & \mathsf{F} \wr \Omega_{G,n+1} \arrow[dl] \\
\Omega_{G,0} \arrow[rr] & & \mathsf{F} \wr \Sigma_G &
\end{tikzcd}$$

commute whenever defined (i.e. $0 \le i \le n+1$).

INDVNG NOT

Notation 4.33. We will let

$$V_{G,n} \colon \Omega_{G,n} \to \mathsf{F} \wr \Sigma_G$$

be inductively defined by $V_{G,n} = \sigma_0 \circ V_{G,n-1} \circ V_G$.

Remark 4.34. When $n = 2$, $V_{G,2}$ is thus the composite

$$\Omega_{G,2} \xrightarrow{V_G} \mathsf{F} \wr \Omega_{G,1} \xrightarrow{V_G} \mathsf{F} \wr \mathsf{F} \wr \Omega_{G,0} \xrightarrow{V_G} \mathsf{F} \wr \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F} \wr \Sigma_G$$

In light of Remarks VERTEXDECOMP REM VERTEXDECOMPG REM 3.47 and 4.14, $V_{G,n}(T_0 \to \cdots \to T_n)$ is identified with the tuple

$$(T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_n)}, \tag{4.35}$$ VGNISO EQ

though this requires changing the total order in $V_G(T_n)$. Rather than using the order induced by T_n, one instead equips $V_G(T_n)$ with the order induced lexicographically from the maps $V_G(T_n) \to V_G(T_{n-1}) \to \cdots \to V_G(T_0)$, i.e., for $v, w \in V_G(T_n)$ the condition $v < w$ is determined by the lowest i such that the images of $v, w \in V_G(T_i)$ are distinct.

4.4 A monad on spans

WSPAN DEF

Definition 4.36. We will write $\mathsf{WSpan}^l(\mathcal{C}, \mathcal{D})$ (resp. $\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})$), which we call the category of *left weak spans* (resp. *right weak spans*), to denote the category with objects the spans

$$\mathcal{C} \xleftarrow{k} A \xrightarrow{F} \mathcal{D},$$

arrows the diagrams as on the left (resp. right) below

$$\begin{tikzcd}
& A_1 \arrow[dl, "k_1"'] \arrow[dr, "F_1"] \arrow[dd, "i"] \arrow[dd, phantom, shift left=4ex, "\overset{\varphi}{\Swarrow}"] & \\
\mathcal{C} & & \mathcal{D} \\
& A_2 \arrow[ul, "k_2"] \arrow[ur, "F_2"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[dl, "k_1"'] \arrow[dr, "F_1"] \arrow[dd, "i"] \arrow[dd, phantom, shift left=4ex, "\overset{\varphi}{\Nearrow}"] & \\
\mathcal{C} & & \mathcal{D} \\
& A_2 \arrow[ul, "k_2"] \arrow[ur, "F_2"'] &
\end{tikzcd} \tag{4.37}$$ TWISTEDARROWRIGHT EQ

which we write as $(i, \varphi) \colon (k_1, F_1) \to (k_2, F_2)$, and composition given in the obvious way.

Remark 4.38. There are natural isomorphisms

$$\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D}) \simeq \mathsf{WSpan}^l(\mathcal{C}^{op}, \mathcal{D}^{op}). \tag{4.39}$$ LRSPANISO EQ

RANLANADJ REM

Remark 4.40. The terms *left/right* are motivated by the existence of adjunctions (which are seen to be equivalent by using (LRSPANISO EQ 4.39))

$$\mathsf{Lan} \colon \mathsf{WSpan}^l(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \colon \iota$$

$$\iota \colon \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})^{op} \colon \mathsf{Ran}$$

where the functors ι denote the obvious inclusions (note the need for the $(-)^{op}$ in the second adjunction) and $\mathsf{Lan}/\mathsf{Ran}$ denote the left/right Kan extension functors.

We will mainly be interested in the span categories $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \simeq \mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$.

OMEGAGNA NOT

Notation 4.41. Given a functor $\pi\colon A \to \Sigma_G$, we let $\Omega_{G,n}^{(A)}$ denote the pullback (in Cat)

$$\begin{tikzcd}
\Omega_{G,n}^{(A)} \arrow[r, "V_{G,n}^{(A)}"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_{G,n} \arrow[r, "V_{G,n}"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}$$

Explicitly, the objects of $\Omega_{G,n}^{(A)}$ are pairs

$$(T_0 \to \cdots \to T_n, (a_{e^\uparrow \le e})_{(e^\uparrow \le e) \in V_G(T_n)}) \tag{4.42}$$

OMEGAGNA EQ

such that $\pi(a_{e^\uparrow \le e}) = T_{n,e^\uparrow \le e}$.

Remark 4.43. Our primary interest here will be in the $\Omega_{G,0}^{(A)}$ construction. Importantly, the composite maps $\Omega_{G,0}^{(A)} \to \Omega_{G,0} \to \Sigma_G$ allow us to iterate the $\Omega_{G,0}^{(-)}$ construction. In practice, the role of higher strings $\Omega_{G,n}^{(A)}$ will then be to provide more convenient models for iterated $\Omega_{G,0}^{(-)}$ constructions.

Indeed, the content of Proposition SUBSASPULL PROP 4.29 is then that there are compatible identifications $\Omega_{G,0}^{(\Omega_{G,n})} \simeq \Omega_{G,n+1}$ which identify $V_G^{(\Omega_{G,n})}$ with V_G.

Moreover, since all squares in the diagram

$$\begin{tikzcd}
\Omega_{G,n+1}^{(A)} \arrow[r, "V_G^{(A)}"] \arrow[d] & \mathsf{F} \wr \Omega_{G,n}^{(A)} \arrow[r, "\mathsf{F} \wr V_{G,n}^{(A)}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_{G,n+1} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \Omega_{G,n} \arrow[r, "\mathsf{F} \wr V_{G,n}"] \arrow[d] & \mathsf{F} \wr \mathsf{F} \wr \Sigma_G \arrow[r, "\sigma^0"] & \mathsf{F} \wr \Sigma_G \\
\Omega_{G,0} \arrow[r] & \mathsf{F} \wr \Sigma_G & &
\end{tikzcd} \tag{4.44}$$

ALLSQUARES EQ

are pullback squares (the top center square is so by induction, the top right square by direct verification, the total top square by definition of $\Omega_{G,n+1}^{(A)}$ and the bottom left square by Proposition SUBSASPULL PROP 4.29), we likewise obtain identifications $\Omega_G^{\left(\Omega_{G,n}^{(A)}\right)} \simeq \Omega_{G,n+1}^{(A)}$.

Proposition 4.45. *For any $A \to \Sigma_G$ there are functors $d_0^{(A)}\colon \Omega_{G,1}^{(A)} \to \Omega_G^{(A)}$ and natural isomorphisms*

$$\begin{tikzcd}
\Omega_{G,1}^{(A)} \arrow[r, "V_G"] \arrow[d, "d_0^{(A)}"'] & \mathsf{F} \wr \Omega_G^{(A)} \arrow[r, "\mathsf{F} \wr V_G"] & \mathsf{F} \wr \mathsf{F} \wr A \arrow[d, "\sigma^0"] \arrow[dll, Rightarrow, "\pi^{(A)}"] \\
\Omega_G^{(A)} \arrow[rr, "V_G"'] & & \mathsf{F} \wr A,
\end{tikzcd} \tag{4.46}$$

SHUFFLEPERMA EQ

both natural in $A \to \Sigma$. Here naturality of $\pi^{(-)}$ means that for a functor $H\colon A \to B$ with *corresponding diagram*

$$\begin{tikzcd}
\Omega^{(A)}_{G,1} \arrow[rr, "V_G^{(A)}"] \arrow[dr, "d_0^{(A)}"] \arrow[dd, "\Omega^{(H)}_{G,1}"'] & & \mathsf{F}\wr\Omega^{(A)}_{G,0} \arrow[r, "\mathsf{F}\wr V_G^{(A)}"] \arrow[dd, dashed] & \mathsf{F}\wr\mathsf{F}\wr A \arrow[dr, "\sigma^0"] \arrow[dd, dashed] \arrow[dll, Rightarrow, "\pi^{(A)}"] & \\
& \Omega^{(A)}_{G,0} \arrow[rrr, "V_G^{(A)}"'] \arrow[dd] & & & \mathsf{F}\wr A \arrow[dd, "\mathsf{F}\wr H"] \\
\Omega^{(B)}_{G,1} \arrow[rr, dashed] \arrow[dr, "d_0^{(B)}"'] & & \mathsf{F}\wr\Omega^{(B)}_{G,0} \arrow[r, dashed] & \mathsf{F}\wr\mathsf{F}\wr B \arrow[dr, dashed] \arrow[dll, Rightarrow, dashed] & \\
& \Omega^{(B)}_{G,0} \arrow[rrr, "V_G^{(B)}"'] & & & \mathsf{F}\wr B
\end{tikzcd} \tag{4.47}$$

PICUBOIDAB EQ

one has an equality

$$(\mathsf{F}\wr H)\pi^{(A)} = \pi^{(B)}\Omega^{(H)}_{G,1}$$

(i.e. the two natural isomorphisms between the two distinct functors $\Omega^{(A)}_{G,1} \rightrightarrows \mathsf{F}\wr B$ coincide).

Proof. Informally, using the object description in (4.42), $d_0^{(A)}$ is simply given by the formula

$$d_0^{(A)}\left(T_0 \to T_1, (a_{e^\uparrow \le e})_{(e^\uparrow \le e)\in V_G(T_1)}\right) = \left(T_1, (a_{e^\uparrow \le e})_{(e^\uparrow \le e)\in V_G(T_1)}\right), \tag{4.48}$$

GENDO EQ

though one must note that since in (4.42) the order in $V_G(T_1)$ is induced lexicographically from the string, the two orders for $V_G(T_1)$ in each side of (4.48) do not coincide.

It now follows that the composites $\sigma^0 \circ (\mathsf{F}\wr V_G^{(A)}) \circ V_G^{(A)}$ and $V_G^{(A)} \circ d_0^{(A)}$ differ by the natural automorphism $\pi^{(A)}$ given by the tuple permutations interchanging the two orders in $V_G(T_1)$ for each $T_0 \to T_1$.

The commutativity of (4.47) is clear. □

Definition 4.49. Suppose $\mathcal{V}$ has finite products.

We define an endofunctor N of $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$ by letting $N(\Sigma_G \leftarrow A \to \mathcal{V}^{op})$ be the span $\Sigma_G \leftarrow \Omega_G^{(A)} \to \mathcal{V}^{op}$ given composition along the diagram

$$\begin{tikzcd}
\Omega^{(A)}_{G,0} \arrow[r] \arrow[d] & \mathsf{F}\wr A \arrow[r] \arrow[d] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r, "\Pi^{op}"] & \mathcal{V}^{op} \\
\Omega_{G,0} \arrow[r] \arrow[d] & \mathsf{F}\wr\Sigma_G & & \\
\Sigma_G & & &
\end{tikzcd}$$

and defined on maps of spans in the obvious way.

One has a multiplication $\mu\colon N \circ N \Rightarrow N$ given by the natural isomorphisms

$$\begin{tikzcd}
\Sigma \arrow[d, equal] & \Omega^{(A)}_{G,1} \arrow[l] \arrow[r, "V_G"] \arrow[d, "d_0^{(A)}"'] & \mathsf{F}\wr\Omega^{(A)}_G \arrow[r, "\mathsf{F}\wr V_G"] & \mathsf{F}\wr\mathsf{F}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dll, Rightarrow, "\pi^{(A)}"] & \mathsf{F}\wr\mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[dl, Rightarrow, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma & \Omega^{(A)}_G \arrow[l] \arrow[rr, "V_G"'] & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.50}$$

MULTDEFSPAN EQ

where α is an associativity isomorphism for the product Π. We we note that naturality of μ follows from the commutativity of (4.47).

Lastly, there is a unit $\eta\colon id \Rightarrow N$ given by the strictly commutative diagrams

$$\begin{tikzcd}
\Sigma \arrow[d, equal] & A \arrow[l] \arrow[r, equal] \arrow[d, "s_{-1}^{(A)}"'] & A \arrow[r] \arrow[d] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma & \Omega^{(A)}_G \arrow[l] \arrow[r, "V_G"'] & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}.
\end{tikzcd} \tag{4.51}$$

UNITSPAN EQ

MONSPAN PROP **Proposition 4.52.** *(N, μ, η) form a monad on* $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$.

Proof. The natural transformation component of $\mu \circ (N\mu)$ is given by the composite diagram

$$\begin{tikzcd}
\Omega_{G,2}^{(A)} \arrow[r] \arrow[d, "d_1^{(A)}"'] & \mathsf{F}\wr\Omega_{G,1}^{(A)} \arrow[r] \arrow[d] & \mathsf{F}^{\wr 2}\wr\Omega_G^{(A)} \arrow[r] \arrow[dl, phantom, "\mathsf{F}\wr\pi^{(A)}"] & \mathsf{F}^{\wr 3}\wr A \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 3}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\mathsf{F}\wr\alpha"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,1}^{(A)} \arrow[r] \arrow[d, "d_0^{(A)}"'] & \mathsf{F}\wr\Omega_G^{(A)} \arrow[rr] & & \mathsf{F}^{\wr 2}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dlll, phantom, "\pi^{(A)}"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[rr] \arrow[d, "\sigma^0"] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dlll, phantom, "\alpha"] \\
\Omega_G^{(A)} \arrow[rrr] & & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.53}$$

ASSOCSPAN1 EQ

whereas the natural transformation component of $\mu \circ (\mu N)$ is given by

$$\begin{tikzcd}
\Omega_{G,2}^{(A)} \arrow[r] \arrow[d, "d_0^{(A)}"'] & \mathsf{F}\wr\Omega_{G,1}^{(A)} \arrow[r] \arrow[dl, phantom, "\pi\left(\Omega_G^{(A)}\right)"] & \mathsf{F}^{\wr 2}\wr\Omega_G^{(A)} \arrow[r] \arrow[d] & \mathsf{F}^{\wr 3}\wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r] \arrow[d] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,1}^{(A)} \arrow[rr] \arrow[d, "d_0^{(A)}"'] & & \mathsf{F}\wr\Omega_G^{(A)} \arrow[r] & \mathsf{F}^{\wr 2}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dlll, phantom, "\pi^{(A)}"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op} \arrow[d, equal] \arrow[dlll, phantom, "\alpha"] \\
\Omega_G^{(A)} \arrow[rrr] & & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.54}$$

ASSOCSPAN2 EQ

That the rightmost sections of (4.53) and (4.54) coincide follows from compatibility of the associativity isomorphisms for Π^{op}.

For the leftmost sections, note first that, in either diagram, the top right and bottom left paths $\Omega_{G,2}^{(A)} \to \mathsf{F}\wr A$ differ only by the induced order on $V_G(T_2)$ for each string $T_0 \to T_1 \to T_2$. More explicitly, the top right paths use the order induced lexicographically from the string $T_0 \to T_1 \to T_2$ while the bottom left paths use the order induced exclusively by T_2. The two left sections then coincide since are both given by the permutation interchanging these orders, the only difference being that the intermediate stage of (4.53) uses the order induced lexicographically from $T_0 \to T_2$ while (4.54) uses the order induced lexicographically from $T_1 \to T_2$.

As for unit conditions, $\mu \circ (N\eta)$ is represented by

$$\begin{tikzcd}
\Omega_G^{(A)} \arrow[r] \arrow[d, "s_0^{(A)}"'] & \mathsf{F}\wr A \arrow[r, equal] \arrow[d] & \mathsf{F}\wr A \arrow[r] \arrow[d, "\delta^1"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^1"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,1}^{(A)} \arrow[r] \arrow[d, "d_0^{(A)}"'] & \mathsf{F}\wr\Omega_G^{(A)} \arrow[r] & \mathsf{F}^{\wr 2}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dll, phantom, "\pi^{(A)}"] & \mathsf{F}^{\wr 2}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dll, phantom, "\alpha"] \\
\Omega_G^{(A)} \arrow[rr] & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.55}$$

UNITSPAN1 EQ

while $\mu \circ (\eta N)$ is represented by

$$\begin{tikzcd}
\Omega_G^{(A)} \arrow[r, equal] \arrow[d, "s_{-1}^{(A)}"'] & \Omega_G^{(A)} \arrow[r] \arrow[d] & \mathsf{F}\wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,1}^{(A)} \arrow[r] \arrow[d, "d_0^{(A)}"'] & \mathsf{F}\wr\Omega_G^{(A)} \arrow[r] & \mathsf{F}\wr\mathsf{F}\wr A \arrow[r] \arrow[d, "\sigma^0"] \arrow[dll, phantom, "\pi^{(A)}"] & \mathsf{F}\wr\mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dll, phantom, "\alpha"] \\
\Omega_G^{(A)} \arrow[rr] & & \mathsf{F}\wr A \arrow[r] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.56}$$

UNITSPAN2 EQ

It is straightforward to check that the composites of the left and right sections of both (4.55) and (4.56) are strictly commutative diagrams, and thus that (4.55) and (4.56) coincide. $\square$

4.5 The free genuine operad monad

Recalling that $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op}) \simeq \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$, Proposition 4.52 and Remark 4.40 give an adjuntion

$$\mathsf{Lan}\colon \mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})\colon \iota \tag{4.57}$$

LANIOTAADJ EQ

together with a monad N in the leftmost category $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$. We now turn to showing that, under reasonable hypothesis on $\mathcal{V}$, the composite $\mathsf{Lan} \circ N \circ \iota$ inherits a monad structure from N. The key will be to show that under such conditions the map $\mathsf{Lan} \circ N \Rightarrow \mathsf{Lan} \circ N \circ \iota \circ \mathsf{Lan}$ is a natural isomorphism.

Recall that following Convention 3.18 our model for O_G consists of totally ordered sets. One therefore has *root functors*

$$\Omega_G^q \xrightarrow{\mathsf{r}} \mathsf{O}_G, \qquad \Sigma_G \xrightarrow{\mathsf{r}} \mathsf{O}_G$$

sending each planar G-tree to its ordered orbital G-set of roots.

Root functors are compatible with the leaf-root functor and the inclusion, i.e. the following commute.

$$\begin{tikzcd} \Omega_G^q \arrow[r, "\mathsf{lr}"] \arrow[dr, "\mathsf{r}"'] & \Sigma_G \arrow[d, "\mathsf{r}"] \\ & \mathsf{O}_G \end{tikzcd} \qquad \begin{tikzcd} \Sigma_G \arrow[r, hook] \arrow[dr, "\mathsf{r}"'] & \Omega_G^q \arrow[d, "\mathsf{r}"] \\ & \mathsf{O}_G \end{tikzcd} \tag{4.58}$$

ROOTLEAFTROOTCOM EQ

Moreover, the diagrams (4.58) possess some extra structure we will need to make use of. Indeed, both functors are split Grothendieck fibrations: given a map $\varphi\colon A \to B$ in O_G and G-tree T such that $\mathsf{r}(T) = B$ we can build a cartesian arrow $\varphi^*(T) \to T$ by letting $\varphi^*(T)$ to be the pullback G-tree together with the planar structure on roots given by A and on non-equivariant nodes given by their image via $\varphi^*(T) \to T$.

It now follows that (4.58) are diagrams of split Grothendieck fibrations.

One advantage of split Grothendieck fibrations (in general) is the following initiality condition on overcategories.

Definition 4.59. Suppose we have two split fibrations $r : \mathcal{C} \to \mathcal{E}$ and $r : \mathcal{D} \to \mathcal{E}$ and a map of fibrations f as below.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "f"] \arrow[dr, "r"'] & & \mathcal{D} \arrow[dl, "r"] \\ & \mathcal{E} & \end{tikzcd}$$

For any $d \in \mathcal{D}$, let $d \downarrow_{\mathsf{r}} \mathcal{C}$ denote the subcategory of $d \downarrow \mathcal{C}$ of just those maps $d \to f(c)$ which map to the identity under r.

Lemma 4.60. *In the above scenario, $d \downarrow_{\mathsf{r}} \mathcal{C}$ is inital in $d \downarrow \mathcal{C}$.*

Proof. We must show that the appropriate iterated overcategories are non-empty and connected. To that end, suppose we have a map $\phi : d \to f(c)$ in $\mathcal{D}$, and let $q \coloneqq r(\phi) : r(d) \to r(c)$. Since $\mathcal{C}$ is fibrant over $\mathcal{E}$, we have a Cartesian arrow $q' : c' \to c$ lifting $q : r(d) = r(c') \to r(c)$. Thus we have diagrams in $\mathcal{E}$ and $\mathcal{D}$ of the form

$$\begin{tikzcd} r(d) \arrow[r, equal, "id"] \arrow[dr, "r(\phi)"'] & r(c') \arrow[d, "q"] \\ & r(c) \end{tikzcd} \qquad \begin{tikzcd} d \arrow[r, dotted, "\exists!"] \arrow[dr, "\phi"'] & f(c') \arrow[d, "f(q')"] \\ & f(c) \end{tikzcd}$$

As f preserves Cartesian arrows, $f(q')$ is Cartesian, and hence we have a unique lifting $p(\phi) : d \to f(c')$ of $r(d) = r(c')$. Thus the iterated overcategory is inhabited.

To show it is connected, any other factorization $d \xrightarrow{\psi} f(c'') \to f(c)$ such that $r(\psi) = id$, we can again use the fact that q' is Cartesian to produce a lift $f(c';) \to f(c')$ of $r(c'') = r(c')$; by uniqueness of the map $d \to f(c')$, we have that the diagram below commutes, finishing the proof.

$$\begin{tikzcd}
& d \arrow[dl, "\exists!"'] \arrow[dr, "\psi"] & \\
f(c') \arrow[dr, "f(q')"'] & & f(c'') \arrow[ll, dotted, "\exists!"'] \arrow[dl] \\
& f(c) &
\end{tikzcd}$$

□

Remark 4.61. The above proof can be easily modified to show that $d \downarrow_{\mathsf{r}} \mathcal{C}$ is in fact a coreflective subcategory of $d \downarrow \mathcal{C}$, with $\phi \mapsto p(\phi)$ the reflection.

Luis: to actually show this would take just as long - functoriality and the fact that the inclusion is a left adjoint both require unique liftings/factorizations

Definition 4.62. A split Grothendieck fibration $A \xrightarrow{\mathsf{r}} \mathsf{O}_G$ is called a *root fibration* and a split Grothendieck fibration diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[dr, "\mathsf{r}"'] & B \arrow[d, "\mathsf{r}"] \\
& \mathsf{O}_G
\end{tikzcd}$$

is called a *root fibration functor*.

The relevance of root fibrations is given by the following couple of lemmas.

ROOTFIBPULL LEM

Lemma 4.63. *If $A \to \Sigma_G$ is a root fibration functor then so is $\Omega_G^{(A)} \to \Omega_G$, naturally in A.*

Proof. We consider the pullback diagram below.

$$\begin{tikzcd}
\Omega_{G,0}^{(A)} \arrow[r, "V_G^{(A)}"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd} \tag{4.64}$$

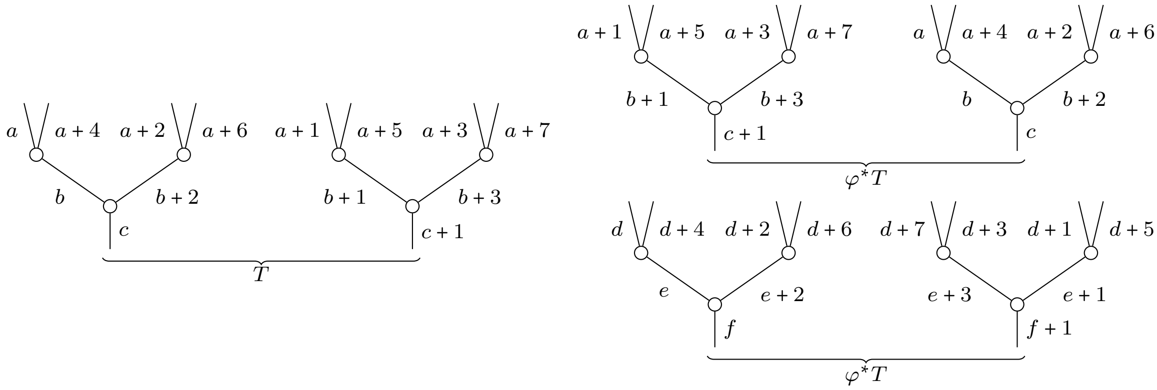
ROOTIMPLIESROOT EQ

The hypothesis that $A \to \Sigma_G$ is root fibration implies that the rightmost map in (4.64) is a map of split Grothendieck fibrations over $\mathsf{F} \wr \mathsf{O}_G$.

ROOTIMPLIESROOT EQ

Since the map V_G sends the chosen cartesian arrows in $\Omega_{G,0}$ (over O_G) to chosen cartesian arrows of $\mathsf{F} \wr \Sigma_G$ (over $\mathsf{F} \wr \mathsf{O}_G$), the result follows. □

Example 4.65. Let $G = \mathbb{Z}_{/8}$. The following exemplifies a pull back along the twist map $\varphi\colon G/2G \to G/2G$ (i.e., accounting for order, φ is the permutation (12)), with the topmost representation of $\varphi^* T$ maintaining the chosen generators for each edge orbit from T and the bottom representation choosing instead the generators to be minimal with regard to the planar structure.



(4.66)

We note that $(\varphi^*(T))_{v_{Ge}} = \psi^*(T_{v_{Gb}})$ for ψ the permutation $(13)(24)$ encoded by the composite identifications $\{1,2,3,4\} \simeq \{e, e+2, e+3, e+1\} \simeq \{b+1, b+3, b, b+2\} \simeq \{3,4,1,2\}$.

LANPULLCOMA LEM

Lemma 4.67. *Suppose that $\mathcal{V}$ is complete and that $A \to \Sigma_G$ is a root fibration. If the rightmost triangle in*

$$\begin{tikzcd} \Omega^{(A)}_{G,0} \arrow[r, "V^{(A)}_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathcal{V} \\ \Omega_{G,0} \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \arrow[ur] \arrow[u, phantom, shift left=4, "\Uparrow"] & \end{tikzcd} \tag{4.68}$$

is a right Kan extension diagram then so is the composite diagram.

Proof. Unpacking definitions using the pointwise formula for right Kan extensions (McL [4, X.3.1]), it suffices to check that for each $T \in \Omega_{G,0}$ the functor

$$T \downarrow \Omega^{(A)}_{G,0} \to V_G(T) \downarrow \mathsf{F} \wr A \tag{4.69}$$ LANPULLCOMA EQ

is initial. In the course of the proof of Lemma FINWREATPRODLIM LEM 4.3 it was shown that the subcategory

$$\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow A$$

is initial in the $V_G(T) \downarrow \mathsf{F} \wr A$.

On the other hand, since $\Omega^{(A)}_G \to \Omega_G$ is a root fibration functor, $T \downarrow \Omega^{(A)}_G$ has an initial subcategory $T \downarrow_{\mathsf{r},\simeq} \Omega^{(A)}_G$ with objects $(S \in \Omega^{(A)}_G, T \to u(S))$ such that $T \to u(S)$ is a quotient map that induces an ordered isomorphism on roots. Note that this can be restated as saying that $T \to u(S)$ is an isomorphism preserving the order of the roots.

The result now follows from the natural isomorphism

$$T \downarrow_{\mathsf{r},\simeq} \Omega^{(A)}_G \simeq \prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r},\simeq} A. \tag{4.70}$$ TDOWNISOA EQ

To see this, we focus first on the case $A = \Sigma_G$. In that case, the left hand side of (TDOWNISOA EQ 4.70) encodes replanarizations of T that preserve the root order. On the other hand, the right hand side encodes replanarizations of all the G-vertices that preserve the order of their roots, or, equivalently, replanarizations of the non-equivariant vertices of T. That these are equivalent is the content of Proposition PLANARIZATIONCHAR PROP 3.13.

Note that $(T \to S) \in (T \downarrow_{\mathsf{r},\simeq} \Omega_G)$ is then encoded by a tuple $\left(T_{v_{Ge}} \to \varphi^*_{v_{Ge}} S_{v_{Ge}}\right)_{v_{Ge} \in V_G(T)}$ where the pullbacks $\varphi^*_{v_{Ge}}$ are needed to correct the root order.

The case of general A follows likewise, using the corresponding pullbacks $\varphi^*_{v_{Ge}}$.

Note: an addendum is needed to show that (4.70) suffices, since $T \downarrow_{\mathsf{r},\simeq} \Omega_G^{(A)}$ is not sent directly to $\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r},\simeq} A$ □

Lemma 4.63 can be interpreted as saying that, if one defines a category $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$ of *rooted spans*

$$\Sigma_G^{op} \leftarrow A^{op} \to \mathcal{V}$$

where $A \to \Sigma_G$ is a root fibration functor, the monad N built in Proposition 4.52 lifts to a monad N_{r} in $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$, and likewise for the adjunction (4.57).

Corollary 4.71. *Suppose that finite products in $\mathcal{V}$ commute with colimits in each variable. The functors*

$$\mathsf{Lan} \circ N_{\mathsf{r}} \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan}, \qquad \mathsf{Lan} \circ \iota \Rightarrow id$$

are natural isomorphisms.

Proof. This follows by combining Lemma 4.67 with Lemma 4.3. □

THEMONAD DEF **Definition 4.72.** The *genuine equivariant operad monad* is the monad $\mathbb{F}_G$ on $\mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$ given by

$$\mathbb{F}_G = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

and with multiplication and unit given by the composites

$$\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ N_{\mathsf{r}} \circ N_{\mathsf{r}} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

$$id \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota.$$

Remark 4.73. The functor $\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$ is isomorphic to $\mathsf{Lan} \circ N \circ \iota$, and this isomorphism is compatible with the multiplication and unit in Definition 4.72, and we will henceforth simply write N rather than N_{r}.

From this point of view, the role of root fibrations is to guarantee that $\mathsf{Lan} \circ N \circ \iota$ is indeed a monad, but unnecessary to describe the monad structure itself.

REPACKAGERES REM **Remark 4.74.** Since a map

$$\mathbb{F}_G X = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota X \to X$$

is adjoint to a map

$$N_{\mathsf{r}} \circ \iota X \to \iota X$$

one easily verifies that X is a genuine equivariant operad, i.e. a $\mathbb{F}_G$-algebra, iff ιX is a N-algebra. Moreover, the bar resolution

$$\mathbb{F}_G^{\bullet+1} X$$

is isomorphic to

$$\mathsf{Lan}\left(N^{\bullet+1} \iota X\right).$$

4.6 Comparison with (regular) equivariant operads

We start by noting that in the case $G = *$, genuine operads simply recover the usual notion of symmetric operads, i.e. $\mathsf{Sym}_*(\mathcal{V}) \simeq \mathsf{Sym}(\mathcal{V})$ and $\mathsf{Op}_*(\mathcal{V}) \simeq \mathsf{Op}(\mathcal{V})$, and in what follows we will adopt the notations $\mathsf{Sym}^G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$ for the corresponding categories of G-objects. Our goal will be to relate these to the categories $\mathsf{Sym}_G(\mathcal{V})$ and $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant sequences and genuine operads.

We will throughout this section fix a total order of G such that the identity e is the first element, though we note that the exact order is unimportant, as any other such choice would lead to unique isomorphisms between the constructions in this section.

We thus have an inclusion functor

$$\begin{aligned} \iota\colon G \times \Sigma &\hookrightarrow \Sigma_G \\ C &\longmapsto G \cdot C \end{aligned}$$

where $G \cdot C$ is planarized so that the roots inherit the order of G and each of the individual copies of C inherits the planarization of C. Moreover, letting $\Sigma_G^{\mathrm{fr}} \hookrightarrow \Sigma_G$ denote the full subcategory of G-free corollas, there is an induced retraction $\rho\colon \Sigma_G^{\mathrm{fr}} \to G \times \Sigma$ defined by $\rho(\amalg_{1 \le i \le |G|} C_i) = G \cdot C_1$ together with isomorphisms $C \simeq \rho(C)$ uniquely determined by the condition that they are the identity on the first tree component C_1.

We now consider the associated adjunctions.

$$\begin{tikzcd} \mathsf{Sym}_G(\mathcal{V}) \arrow[r, "\iota^*"'] & \mathsf{Sym}^G(\mathcal{V}) \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"] \end{tikzcd} \tag{4.75}$$

TWOADJOINTS EQ

Explicitly, we have the formulas

$$\iota_! Y(\amalg_i C_i) = \begin{cases} Y(C_1), & \amalg_i C_i \in \Sigma_G^{\mathrm{fr}} \\ \varnothing, & \amalg_i C_i \notin \Sigma_G^{\mathrm{fr}} \end{cases}, \qquad \iota^* X(C) = X(G \cdot C), \qquad \iota_* Y(\amalg_i C_i) = \left(\prod_i Y(C_i)\right)^G,$$

where we note that in the formula for $\iota_*(-)$ the action of G interchanges factors according to the action on the indexing set, i.e. the root orbit of $\amalg_i C_i$. As a side note, we note that the formulas for $\iota_!$ and ι_* are independent of the chosen order of G.

REFLCOREFL REM

Remark 4.76. $\iota_!$ essentially identifies $\mathsf{Sym}^G(\mathcal{V})$ as the coreflexive subcategory of sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C) = \varnothing$ whenever C is not a free corolla.

By contrast, ι_* identifies $\mathsf{Sym}^G(\mathcal{V})$ with a far more interesting reflexive subcategory of sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C)$ for each C not a free corolla must satisfy a fixed point condition. Concretely, letting $\varphi\colon G \to \mathsf{r}(C)$ denote the unique map preserving the minimal element, one has

$$X(C) \xrightarrow{\simeq} X(\varphi^* C)^\Gamma$$

for $\Gamma \le \mathsf{Aut}(\varphi^* C)$ the subgroup preserving the quotient map $\varphi^* C \to C$ under precomposition.

There is an obvious natural transformation $\beta\colon \iota_! \Rightarrow \iota_*$ which for $\amalg_i C_i \in \Sigma_G^{\mathrm{fr}}$ sends $Y(C_1)$ to the "G-twisted diagonal" of $\prod_i Y(C_i)$. Moreover, letting $\eta_!, \epsilon_!$ (resp. η_*, ϵ_*) denote the unit and counit of the $(\iota_!, \iota^*)$ adjunction (resp. (ι^*, ι_*) adjunction) it is straightforward to check that the following diagram commutes.

$$\begin{tikzcd} \iota_! \iota^* \iota_* \arrow[r, Rightarrow, "\epsilon_!"] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota_* \arrow[d, Rightarrow, "\eta_!", "\simeq"'] \\ \iota_! \arrow[r, Rightarrow, "\eta_*"'] \arrow[ur, Rightarrow, "\beta"'] & \iota_* \iota^* \iota_! \end{tikzcd} \tag{4.77}$$

BETADEFSQUARE EQ

Remark 4.78. An exercise in adjunctions shows that any outer square as in (4.77) commutes provided at least one of the adjunctions in 4.75 is (co)reflexive, so that (4.77) can be regarded as an alternative definition of β.

BETADEFSQUARE EQ TWOADJOINTS EQ BETADEFSQUARE EQ

Proposition 4.79. *One has the following*

(i) the map $\iota^* \mathbb{F}_G \xrightarrow{\eta_*} \iota^* \mathbb{F}_G \iota_* \iota^*$ is an isomorphism, and thus (cf. Prop. 2.4) $\iota^* \mathbb{F}_G \iota_*$ is a monad; MONADADJ PROP

(ii) the map $\iota^* \mathbb{F}_G \iota_! \xrightarrow{\beta} \iota^* \mathbb{F}_G \iota_*$ is an isomorphism of monads;

(iii) the map $\iota_! \iota^* \mathbb{F}_G \iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G \iota_!$ is an isomorphism;

(iv) there is a natural isomorphism of monads $\alpha\colon \mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$.

Proof. We first show (i), starting with some notation. In analogy with Σ_G^{fr}, we write $\Omega_{G,0}^{\mathrm{fr}}$ for the subcategory of free trees and note that the leaf-root and vertex functors then restrict to functors $\mathsf{lr}\colon \Omega_{G,0}^{\mathrm{fr}} \to \Sigma_G^{\mathrm{fr}}$, $V_G\colon \Omega_{G,0}^{\mathrm{fr}} \to \mathsf{F} \wr \Sigma_G^{\mathrm{fr}}$. Moreover, for each $C \in \Sigma_G^{\mathrm{fr}}$ one has an equality of rooted undercategories between $C \downarrow_{\mathsf{r}} \Omega_{G,0}$ and $C \downarrow_{\mathsf{r}} \Omega_{G,0}^{\mathrm{fr}}$, and thus $\iota^*\mathbb{F}_G X$ is computed by the Kan extension of the following diagram.

$$\begin{tikzcd}
\Omega_{G,0}^{\mathrm{fr}} \arrow[r] \arrow[d] & \mathsf{F} \wr \Sigma_G^{\mathrm{fr}} \arrow[r, "\mathsf{F}\wr X"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma_G^{\mathrm{fr}}
\end{tikzcd} \tag{4.80}$$

IFGI EQ

(i) now follows by noting that $X \to \iota_*\iota^* X$ is an isomorphism when restricted to Σ_G^{fr}.

For (ii), to show that $\iota^*\mathbb{F}_G\iota_! \to \iota^*\mathbb{F}_G\iota_*$ is an isomorphism one just repeats the argument in the previous paragraph by noting that $\iota_! \to \iota_*$ is an isomorphism when restricted to Σ_G^{fr}. To check that this is a map of monads, we recall first that the monad structure on $\iota^*\mathbb{F}_G\iota_*$ is given as described in Proposition 2.4. Unpacking definitions, compatibility with multiplication reduces to showing that the composite $\iota_!\iota^* \xrightarrow{\epsilon_!} id \xrightarrow{\eta_*} \iota_*\iota^*$ coincides with $\beta\iota^*$ while compatibility with units reduces to showing that the composite $id \xrightarrow{\eta_!} \iota^*\iota_! \xrightarrow{\iota^*\beta} \iota^*\iota_* \xrightarrow{\epsilon_*} id$ is the identity. Both of these are a consequence of (4.77), following from the diagrams below (where the top composites are identities).

$$\begin{tikzcd}
\iota_!\iota^* \arrow[r, Rightarrow, "\iota_!\iota^*\eta_*"] \arrow[d, Rightarrow, "\epsilon_!"'] & \iota_!\iota^*\iota_*\iota^* \arrow[r, Rightarrow, "\iota_!\epsilon_*\iota^*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_!\iota_*\iota^*"'] & \iota_!\iota^* \arrow[dl, Rightarrow, "\beta\iota^*"] \\
id \arrow[r, Rightarrow, "\eta_*"'] & \iota_*\iota^*
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota^*\iota_* \arrow[r, Rightarrow, "\eta_!\iota^*\iota_*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota^*\iota_!\iota^*\iota_* \arrow[r, Rightarrow, "\iota^*\epsilon_!\iota_*"] \arrow[d, Rightarrow, "\iota^*\iota_!\epsilon_*"', "\simeq"] & \iota^*\iota_* \\
id \arrow[r, Rightarrow, "\simeq", "\eta_!"'] & \iota^*\iota_! \arrow[ur, Rightarrow, "\iota^*\beta"']
\end{tikzcd} \tag{4.81}$$

(iii) amounts to showing that if $X(C) = \varnothing$ whenever $C \notin \Sigma_G^{\mathrm{fr}}$ then it is also $\mathbb{F}_G X(C) = \varnothing$. But since for such $C \notin \Sigma_G^{\mathrm{fr}}$ the undercategory $C \downarrow \Omega_{G,0}$ consists of trees with at least one non-free vertex, the composite

$$C \downarrow \Omega_{G,0} \xrightarrow{V} \mathsf{F} \wr \Sigma_G \xrightarrow{\mathsf{F}\wr X} \mathsf{F} \wr \mathcal{V}^{op} \xrightarrow{\Pi} \mathcal{V}^{op}$$

is constant equal to $\varnothing$, and (iii) follows.

Finally, we show (iv). We will slightly abuse notation by writing $G \times \Sigma \hookrightarrow \Sigma_G$ for the image of ι and similarly $G \times \Omega_0 \hookrightarrow \Omega_{G,0}$ for the image of the obvious analogous functor $\iota\colon G \times \Omega_0 \to \Omega_{G,0}$. The map $\alpha\colon \mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$ is the adjoint to the map $\mathbb{F}\iota^* \to \iota^*\mathbb{F}_G$ encoded by the following diagram.

$$\begin{tikzcd}
G \times \Omega_0 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F} \wr (G \times \Sigma_0) \arrow[rr, "\iota^* X"] \arrow[dr] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dr, equal] & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega_{G,0} \arrow[rr] \arrow[dd] & & \mathsf{F} \wr \Sigma_G \arrow[rr, "X"'] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
G \times \Sigma \arrow[dr] & & & & & & \\
& \Sigma_G & & & & &
\end{tikzcd} \tag{4.82}$$

MONADEQUIV DEF

That α is a natural isomorphism follows by the previous identifications $C \downarrow_{\mathsf{r}} \Omega_{G,0} \simeq C \downarrow_{\mathsf{r}} \Omega_{G,0}^{\mathrm{fr}}$ for $C \in G \times \Sigma$ together with the fact that the retraction $\rho\colon \Omega_{G,0}^{\mathrm{fr}} \to G \times \Omega_0$ (built just as the retraction $\rho\colon \Sigma_G^{\mathrm{fr}} \to G \times \Sigma$) retracts $C \downarrow_{\mathsf{r}} \Omega_{G,0}^{\mathrm{fr}}$ to the undercategory $C \downarrow_{\mathsf{r}} G \times \Omega_0$, which is thus initial (as well as final), and the claim that α is an isomorphism follows.

Intuitively, the final claim that α is a map of monads follows from the fact that the composite $\mathbb{F}\mathbb{F} \to \iota^*\mathbb{F}_G\iota_!\iota^*\mathbb{F}_G\iota_! \to \iota^*\mathbb{F}_G\mathbb{F}_G\iota_!$ is encoded by the analogous natural transformation of diagrams for strings $G \times \Omega_1 \hookrightarrow \Omega_{G,1}^{\mathrm{fr}}$. However, since the presence of left Kan extensions in the definitions of $\mathbb{F}$, $\mathbb{F}_G$ can make a rigorous direct proof of this last claim fairly cumbersome, we sketch here a workaround argument. We first consider the adjunction $\iota_! \colon \mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V}) \rightleftarrows \mathsf{WSpan}^l(\Sigma_G^{op},\mathcal{V}) \colon \iota^*$ where $\iota_!$ is composition with ι and ι^* is the pullback of spans. Writing N, N_G for the monads on the span categories, mimicking (4.82) yields a map $\tilde{\alpha}\colon N \to \iota^* N_G \iota_!$ encoded by the diagram (where the front and back squares are pullbacks).

$$\begin{tikzcd}
(G\times\Omega)^{(\iota^*A)} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr\iota^*A \arrow[rr] \arrow[dr] \arrow[dd, dashed] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] \arrow[dr, equal] & & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega_{G,0}^{(A)} \arrow[rr] \arrow[dd] & & \mathsf{F}\wr A \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op} \\
G\times\Omega_0 \arrow[rr, dashed] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr(G\times\Sigma) \arrow[dr, dashed] & & & & & \\
& \Omega_{G,0} \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\Sigma_G & & & & \\
G\times\Sigma \arrow[dr] & & & & & & & \\
& \Sigma_G & & & & & &
\end{tikzcd}$$

The claim that $\tilde{\alpha}$ is a map of monads is then straightforward. Writing

$$\mathsf{Lan}\colon \mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V}) \rightleftarrows \mathsf{Fun}((G\times\Sigma)^{op},\mathcal{V})\colon j \qquad \mathsf{Lan}_G\colon \mathsf{WSpan}^l(\Omega_G^{op},\mathcal{V}) \rightleftarrows \mathsf{Fun}(\Omega_G^{op},\mathcal{V})\colon j_G$$

for the span functor adjunctions, $\alpha\colon \mathbb{F}\to\iota^*\mathbb{F}_G\iota_!$ can then be written as the composite

$$\mathsf{Lan} N j \to \mathsf{Lan}\iota^* N_G \iota_! j \to \iota^* \mathsf{Lan}_G N_G j_G \iota_!$$

where the first map is the isomorphism of monads induced by $\tilde{\alpha}$ and the second map can be shown directly to be a monad map by unpacking the monad structures in Propositions 2.3 and 2.4. ☐

Combining the previous result with Propositions 2.3 and 2.4 now gives the following.

Corollary 4.83. *The adjunctions (4.75) extends to adjunctions*

$$\begin{tikzcd}
\mathsf{Op}_G(\mathcal{V}) \arrow[r, "\iota^*"] & \mathsf{Op}^G(\mathcal{V}). \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"]
\end{tikzcd} \tag{4.84}$$ TWOADJOINTSOP EQ

In particular, ι_* identifies Op^G as a reflexive subcategory of Op_G.

Remark 4.85. Remark 4.76 extends to operads mutatis mutandis.

Moreover, the isomorphism $\iota_!\iota^*\mathbb{F}_G\iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G\iota_!$ then shows that $\mathbb{F}_G$ essentially preserves the image of $\iota_!$, and can thus be identified with $\mathbb{F}$ over it.

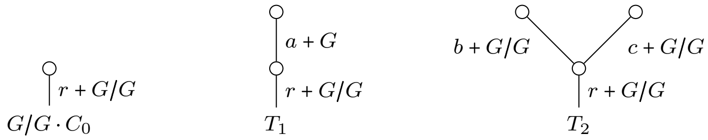
However, the analogous statement fails for ι_*, i.e., one does not always have that

$$\mathbb{F}_G\iota_* \xrightarrow{\eta_*} \iota_*\iota^*\mathbb{F}_G\iota_* \tag{4.86}$$ KEYNONISO EQ

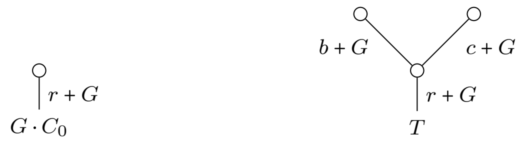
is an isomorphism. In fact, showing that (4.86) *does* become an isomorphism when restricted to suitably cofibrant objects is one of the key technical ingredients for our proof of the Quillen equivalence between $\mathsf{Op}_G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$, and will be the subject of §7.

For now, we end this section with a minimal counterexample to the more general claim. Let $G = \mathbb{Z}_{/2}$ and $Y = * \in \mathsf{Sym}^G(\mathcal{V})$ be the simpleton.

When evaluating $\mathbb{F}_G Y$ at the G-fixed stump corolla $G/G\cdot C_0$, the two G-trees T_1 and T_2 below encode two distinct points (since T_1, T_2 are not isomorphic as objects under $G/G\cdot T_0$).



However, when pulling these points back to the G-free stump corolla $G\cdot C_0$ one obtains the same point, namely that encoded by the G-tree T below.



Moreover, it is not hard to modify the example above to produce similar examples when evaluating $\mathbb{F}_G Y$ at non-empty corollas.

However, such counter-examples all require the use of trees with stumps. Indeed, it can be shown that (4.86) KEYNONISO EQ is an isomorphism whenever evaluated at a Y such that $Y(0) = \varnothing$.

5 Free extensions

Our overall goal in this section will be to produce a description of free genuine operad pushouts, i.e. pushouts of the form

$$\begin{tikzcd} \mathbb{F}_G A \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{F}_G B \arrow[r] & Y \end{tikzcd}$$

in the category Op_G of genuine equivariant operads.

EXTGENMON SEC

5.1 Extensions over general monads

Any monad T on $\mathcal{C}$ one obtains induced monads $T^{\times l}$ on $\mathcal{C}^{\times l}$, and we will make use of several standard relations between these. In particular, any map $\alpha\colon \underline{l} \to \underline{m}$ induces a forgetful functor such that for the forgetul functor $\alpha^*\colon \mathcal{C}^{\times l} \to \mathcal{C}^{\times n}$ one has $T^{\times l}\alpha^* \simeq \alpha^* T^{\times m}$.

Indeed, we will need to make use of a slightly more general setup. Letting I denote the identity monad on $\mathcal{C}$, and $K \subset \underline{m}$ be a subset, there is a monad $T^{\times K} \times I^{\times(\underline{m}-K)}$ on $\mathcal{C}^{\times m}$, which we abusively denote simply as $T^{\times K}$. Identities then determine maps of monads $T^J \to T^{\times K}$ whenever $J \subset K$ and, moreover, there are identifications $T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$. One then has the following.

MONADICFUN PROP

Proposition 5.1. *The functor*

$$T^{\times \alpha^{-1}(K)} \Rightarrow \alpha^* T^{\times K} \alpha_! \tag{5.2}$$

MONADFUNCTORALPHA EQ

adjoint to the identification $T^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ *is a map of monads on* $\mathcal{C}^{\times n}$*.*

Proof. We first note that there are identifications of functors $(FG)^{\times K} \simeq F^{\times K} G^{\times K}$ which are compatible with the identifications $F^{\times \alpha^{-1}(K)}\alpha^* \simeq \alpha^* F^{\times K}$ in the sense that the identification $(FG)^{\times \alpha^{-1}(K)} \circ \alpha^* \simeq \alpha^*(FG)^{\times K}$ matches the composite identification $F^{\times \alpha^{-1}(K)} G^{\times \alpha^{-1}(K)} \alpha^* \simeq F^{\times \alpha^{-1}(K)} \alpha^* G^{\times K} \simeq \alpha^* F^{\times K} G^{\times K}$.

Letting η, ϵ denote the unit and counit for the $(\alpha_!, \alpha^*)$ adjunction, (5.2) MONADFUNCTORALPHA EQ is then the composite

$$T^{\times \alpha^{-1}(K)} \xrightarrow{\eta} T^{\times \alpha^{-1}(K)} \alpha^* \alpha_! \simeq \alpha^* T^{\times K} \alpha_!.$$

That this is a monad map is the condition that the following multiplication and unit diagrams commute.

$$\begin{tikzcd} T^{\times \alpha^{-1}(K)} \circ T^{\times \alpha^{-1}(K)} \arrow[r] \arrow[d] & \alpha^* T^{\times K} \alpha_! \circ \alpha^* T^{\times K} \alpha_! \arrow[d] \\ T^{\times \alpha^{-1}(K)} \arrow[r] & \alpha^* T^{\times K} \alpha_! \end{tikzcd} \qquad \begin{tikzcd} I^{\times n} \arrow[d] \arrow[dr] & \\ T^{\times \alpha^{-1}(K)} \arrow[r] & \alpha^* T^{\times K} \alpha_! \end{tikzcd}$$

We argue only the case of the leftmost multiplication diagram, with commutativity of the unit diagram following by a similar but simpler argument. Since the precomposition $(-)\circ\alpha^*$ is the left adjoint to the precomposition $(-)\circ\alpha_!$ this follows from the following diagram.

$$\begin{tikzcd}
T^{\times\alpha^{-1}(K)}T^{\times\alpha^{-1}(K)}\alpha^* \arrow[r, "\simeq"] \arrow[dd] & T^{\times\alpha^{-1}(K)}\alpha^* T^{\times K} \arrow[r, "\eta"] \arrow[dr, equal] & T^{\times\alpha^{-1}(K)}\alpha^*\alpha_!\alpha^* T^{\times K} \arrow[r, "\simeq"] \arrow[d, "\epsilon"] & \alpha^* T^{\times K}\alpha_!\alpha^* T^{\times K} \arrow[d, "\epsilon"] \\
 & & T^{\times\alpha^{-1}(K)}\alpha^* T^{\times K} \arrow[r, "\simeq"] & \alpha^* T^{\times K} T^{\times K} \arrow[d] \\
T^{\times\alpha^{-1}(K)}\alpha^* \arrow[rrr, "\simeq"] & & & \alpha^* T^{\times K}
\end{tikzcd}$$

□

TALPHAKMOD REM **Remark 5.3.** Since $T^{\times K}\alpha_!$ is a right $\alpha^* T^{\times K}\alpha_!$-module, Proposition 5.1 implies that it is also a right $T^{\times\alpha^{-1}(K)}$-module or, moreover, a right $T^{\times J}$-module whenever $\alpha(J)\subset K$.

COMPPOSTCOMP REM **Remark 5.4.** Combining the precomposition and postcomposition adjunctions, the identification $T^{\times\alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ is then adjoint to a functor $\alpha_! T^{\times\alpha^{-1}(K)} \to T^{\times K}\alpha_!$ which is readily checked to be a map of right $T^{\times\alpha^{-1}(K)}$-modules.

More generally, for $\alpha(J)\subset K$, the composite $T^{\times J}\alpha^* \to T^{\times\alpha^{-1}(K)}\alpha^* \simeq \alpha^* T^{\times K}$ is thus adjoint to a map of right $T^{\times J}$-modules

$$\alpha_! T^{\times J} \to T^{\times K}\alpha_!. \tag{5.5}$$ RIGHTMODULETMAP EQ

We now unpack the content of (5.5) when $\alpha\colon \underline{l} \to *$ is the unique map to the simpleton $* = \underline{1}$. In this case we can instead write $\alpha_! = \coprod$, $\alpha^* = \Delta$, and we thus have commutative diagrams

$$\begin{tikzcd}
\coprod_J TTA_j \amalg \coprod_{\underline{n}-J} A_j \arrow[r] \arrow[d] & T\left(\coprod_J TA_j \amalg \coprod_{\underline{n}-J} A_j\right) \arrow[d] \\
\coprod_J TA_j \amalg \coprod_{\underline{n}-J} A_j \arrow[r] & T\left(\coprod_J A_j \amalg \coprod_{\underline{n}-J} A_j\right)
\end{tikzcd} \tag{5.6}$$ RIGHTMODULETMAPAUX EQ

where the vertical maps come from the right $T^{\times J}$-module structure. Writing $\amalg^a$ for the coproduct of T-algebras and recalling the canonical identifications $\coprod^a_K (TA_k) \simeq T\left(\coprod_K A_k\right)$, (5.6) in fact shows that the right $T^{\times J}$-module structure on $T\circ\coprod$ in fact codifies the multiplication maps

$$\coprod^a_J TTA_j \amalg^a \coprod^a_{\underline{l}-J} TA_j \to \coprod^a_J TA_j \amalg^a \coprod^a_{\underline{l}-J} TA_j.$$

5.2 Labeled planar strings

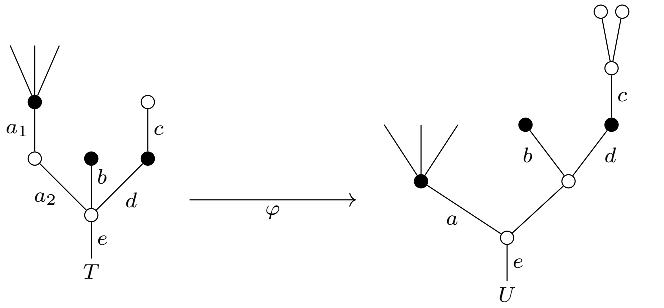
We now translate the results in the previous section to the context of the monad N on $\mathsf{WSpan}^l(\Sigma^{op},\mathcal{V})$. In analogy to the planar string models $\Omega^{(A)}_{G,n}$ for iterations $N^{\circ n+1}$ of the monad N, we will find it convenient to build similar string models $\Omega^{(\underline{A}_J)}_{G,n}$ for $N\circ\coprod\circ(N^{\times J})^{\circ n}$.

Definition 5.7. A *l-node labeled G-tree* (or just *l-labeled G-tree*) G-tree is a pair $(T, V_G(T)\to\{1,\cdots,l\})$ with $T\in\Omega_G$, which we think of as a G-tree together with G-vertices labels in $1,\cdots,l$.

Further, a tall map $\varphi\colon T\to S$ between l-labeled trees is called a *label map* if for each G-vertex v_{Ge} of T with label j, the vertices of the subtree $S_{v_{Ge}}$ are all labeled by j.

Lastly, given a subset $J\subset\underline{l}$, a planar label map $\varphi\colon T\to S$ is said to be J-inert if for every G-vertex v_{Ge} of T with label $j\in J$ it is $S_{v_{Ge}} = T_{v_{Ge}}$.

LABELEDTREES EX **Example 5.8.** Consider the 2-labeled trees below (for $G = *$ the trivial group), with black nodes ($\bullet$) denoting labels by the number 1 and white nodes ($\circ$) labels by the number 2. The planar map φ (sending $a_i \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto d$, $e \mapsto e$) is a label map which is $\{1\}$-inert.



(5.9) SUBSDATUMTREESLAB EQ

Definition 5.10. Let $0 \leq s \leq n$ and $J \subset \underline{l}$ be a subset.

We define $\Omega^J_{G,n,s}$ to have as objects n-planar strings

$$T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \xrightarrow{f_s} T_s \xrightarrow{f_{s+1}} T_{s+1} \xrightarrow{f_{s+2}} \cdots \xrightarrow{f_n} T_n \tag{5.11}$$

NSTRINGLAB EQ

together with l-labelings of $T_s, T_{s+1}, \cdots, T_n$ such that the $f_r, r > s$ are $(\underline{l} - J)$-inert label maps.

Arrows in $\Omega^J_{G,n,s}$ are quotients of strings $(q_r \colon T_r \to T'_r)$ such that $q_r, r \leq s$ are label maps.

Informally, $\Omega^{\underline{l}}_{G,n,s}$ consists of n-strings such that trees and maps after T_s are l-labeled.

Remark 5.12. Our main case of interest will that of $s = 0$, in which case we abbreviate $\Omega^J_{G,n} = \Omega^J_{G,n,0}$. Indeed, such strings will suffice to build models for $N \circ \coprod \circ (N^{\times J})^{\circ n}$.

However, to unpack the right $N^{\times J}$-module structure as in Remark ALPHAKMOD REM 5.3 one further needs to encode composites $NN \circ \coprod \circ (N^{\times J})^{\circ n-1}$, a role played by strings $\Omega^J_{G,n,1}$.

Notation 5.13. We will further write

$$\Omega^J_{G,n,-1} = \coprod_J \Omega_{G,n} \amalg \coprod_{\underline{l}-J} \Sigma_G, \qquad \Omega^J_{G,n,n+1} = \Omega_{G,n} \tag{5.14}$$

OMEGANMINUSONE EQ

To justify this convention, we note that a string as in NSTRINGLAB EQ (5.11) can be extended by prepending to it the map $\mathsf{lr}(T_0) = T_{-1} \xrightarrow{f_0} T_0$. If one then attempts to define $\Omega^J_{G,n,-1}$ by insisting that T_{-1} also be labeled, it follows that all node labels in each string must coincide, resulting in the coproduct decomposition in OMEGANMINUSONE EQ (5.14).

There are a number of obvious functors relating the $\Omega^J_{G,n,s}$ categories, which we now make explicit. Given $s \leq s'$ or $J \subset J'$ there are forgetful functors

$$\Omega^J_{G,n,s} \to \Omega^J_{G,n,s'} \qquad \Omega^J_{G,n,s} \to \Omega^{J'}_{G,n,s} \tag{5.15}$$

NKNFGT EQ

The simplicial operators in Notation SIMPOPERATORS NOT 4.26 generalize to operators (where $0 \leq i \leq n$, $-1 \leq j \leq n$)

$$\begin{array}{llll} d_i \colon \Omega^J_{G,n,s} \to \Omega^J_{G,n-1,s-1} & i < s & s_j \colon \Omega^J_{G,n,s} \to \Omega^J_{G,n+1,s+1} & j < s \\ d_i \colon \Omega^J_{G,n,s} \longrightarrow \Omega^J_{G,n-1,s} & s \leq i & s_j \colon \Omega^J_{G,n,s} \longrightarrow \Omega^J_{G,n+1,s} & s \leq j \end{array}$$

which are compatible with the forgetful functors in the obvious way.

Remark 5.16. For $J \subset J'$ the forgetful functor in NKNFGT EQ (5.15) is a fully faithful inclusion. However, and somewhat subtly, this is not the case the for the $s \leq s'$ forgetful functors. Indeed, regarding $T \to U$ in Examples LABELEDTREES EX 5.8 as an object in $\Omega^2_{*,n,0}$, changing the label of the $a_1 \leq a_2$ vertex of T from a $\circ$-label to a $\bullet$-label yields an alternate object $\bar{T} \to U$ of $\Omega^2_{*,n,0}$ forgetting to the same object of $\Omega^2_{*,n,1}$, yet $T \to U$ and $\bar{T} \to U$ are not isomorphic.

We note that this is a consequence of the fact that substitution data can replace unary nodes by stumps, which have no nodes.

Generalizing Notation 4.33 there is a commutative diagram

$$\begin{array}{ccc}
\Omega^J_{G,n,s} & \xrightarrow{V_{G,n}} & \mathsf{F} \wr \Sigma_G^{\amalg l} \\
\downarrow & & \downarrow \\
\Omega_{G,n} & \xrightarrow[V_{G,n}]{} & \mathsf{F} \wr \Sigma_G
\end{array}$$

where for a labeled string it is $V_G(T_0 \to \cdots \to T_n) = (T_{n,v_{Ge}})_{V_G(T_n)}$, where we regard $T_{n,v_{Ge}} \in \Sigma_G^{\amalg l} \simeq \Omega^{\underline{l}}_{G,-1,-1}$ by using the label in $1, \cdots, l$.

We now expand Notation 4.41.

Notation 5.17. Let $\underline{A}$ denote a $\underline{l}$-tuple $(\pi_j \colon A_j \to \Sigma_G)_{\underline{l}}$ of categories over Σ_G. We define $\Omega^{(\underline{A}),J}_{G,n,s}$ by the pullback diagram

$$\begin{array}{ccc}
\Omega^{(\underline{A}),J}_{G,n,s} & \xrightarrow{V^{(\underline{A})}_{G,n}} & \mathsf{F} \wr \coprod A_j \\
\downarrow & & \downarrow \\
\Omega^J_{G,n,s} & \xrightarrow[V_{G,n}]{} & \mathsf{F} \wr \Sigma_G^{\amalg l}
\end{array} \tag{5.18}$$

LTUPLEAPULL EQ

Explicitly, an object of $\Omega^{(\underline{A}),J}_{G,n,s}$ consists of a labeled string $T_0 \to \cdots T_n$ as in (5.11) together with a tuple $(a_{v_{Ge}})_{V_G(T_n)}$ such that $a_{v_{Ge}} \in A_j$ if v_{Ge} has label j and $\pi_j(a_{v_{Ge}}) = T_{n,v_{Ge}}$.

The reader may have noticed a certain asymmetry between our definition of the $V_{G,n}$ functors here versus their analogues in §4.3, where they were defined iteratively in terms of simpler functors V_G. This is because of the possibility that $s = -1$, in which case (5.14) applies and some caution is needed in that the following result fails.

ALLSQUARESJ PROP

Proposition 5.19. *Suppose $0 \le s \le n$. One has a diagram of pullback squares (generalizing (4.44))*

$$\begin{array}{ccccccc}
\Omega^{(\underline{A}),J}_{G,n,s} & \xrightarrow{V^{(A)}_G} & \mathsf{F} \wr \Omega^{(\underline{A}),J}_{G,n-1,s-1} & \xrightarrow{\mathsf{F}\wr V^{(A)}_{G,n}} & \mathsf{F} \wr \mathsf{F} \wr \coprod A_j & \xrightarrow{\sigma^0} & \mathsf{F} \wr \coprod A_j \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^J_{G,n,s} & \xrightarrow{V_G} & \mathsf{F} \wr \Omega^J_{G,n-1,s-1} & \xrightarrow{\mathsf{F}\wr V_{G,n}} & \mathsf{F} \wr \mathsf{F} \wr \Sigma_G^{\amalg l} & \xrightarrow{\sigma^0} & \mathsf{F} \wr \Sigma_G^{\amalg l} \\
\downarrow & & \downarrow & & & & \\
\Omega_{G,0} & \xrightarrow{V_G} & \mathsf{F} \wr \Sigma_G & & & &
\end{array} \tag{5.20}$$

ALLSQUARESJ EQ

such that the composite of the top squares is (5.18).

Proof. The V_G functors are defined just as in (4.28) via the formula

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)}$$

with the strings $T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}}$ inheriting the extra structure in the obvious way.

Since the top composite square, top center square and top right square are all pullback squares, it remains only to show that the bottom left square is a pullback. This last claim is simply a variation of Proposition 4.29, and follows from the same proof, since both labels and inertness conditions are inherited when assembling substitution data into trees via Proposition 3.42. □

5.3 Bar constructions on spans

We use the results in the previous sections to obtain a string description of the bar constructions

$$\coprod_J^a N^{\bullet+1} A_j \amalg^a \coprod_{\underline{l}-J}^a NA_j.$$

For simplicity, we discuss first the particular case $\coprod^a N^{\bullet+1} A$. Writing the span as $\Sigma_G \leftarrow A \xrightarrow{F} \mathcal{V}$ the identifications $\Omega_{G,0}^{\left(\Omega_{G,n}^{(A)}\right)} \simeq \Omega_{G,n+1}^{(A)}$ iteratively identify the operator in the bar construction $N^{\bullet+1} A$ as follows.

The top boundaries d_n have natural transformation given by

$$\begin{tikzcd}
\Omega_{G,n}^{(A)} \arrow[r, "V_G^{\circ n}"] \arrow[d] & \mathsf{F}^{\wr n} \wr \Omega_{G,0}^{(A)} \arrow[r, "\mathsf{F}^{\wr n} \wr F_1"] \arrow[d] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"] \arrow[d, equal] \arrow[dl, Rightarrow, "\mathsf{F}^{\wr n} \wr m"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,n-1}^{(A)} \arrow[r, "V_G^{\circ n}"'] & \mathsf{F}^{\wr n} \wr A \arrow[r, "\mathsf{F}^{\wr n} \wr F"'] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.21}$$

where m is the natural transformation component of the multiplication $NA \to A$, and the remaining differentials d_i for $0 \le i < n$ are given by

$$\begin{tikzcd}
\Omega_{G,n}^{(A)} \arrow[r, "V_G^{\circ n+1}"] \arrow[d, "d_i^{(A)}"'] \arrow[dr, phantom, "\Leftarrow \pi_i^{(A)}"] & \mathsf{F}^{\wr n+1} \wr A \arrow[r, "F"] \arrow[d, "\sigma^i"] & \mathsf{F}^{\wr n+1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n+1}"] \arrow[d, "\sigma^i"] \arrow[dr, phantom, "\Leftarrow \alpha_i"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,n-1}^{(A)} \arrow[r, "V_G^{\circ n}"'] & \mathsf{F}^{\wr n} \wr A \arrow[r, "F"'] & \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.22}$$

REMAINDIFF EQ

where $\pi_i^{(A)}$ interchanges lexicographic orders on the i-th F coordinate of $F^{\wr n}$ and α_i is the natural associativity isomorphism.

Maybe add degeneracies

Similarly, Proposition ALLSQUARESJ PROP 5.19 shows that $\Omega_{G,n}^{(\underline{A})} \simeq \Omega_{G,0}^{\left(\amalg \Omega_{G,n-1}^{(A_j)}\right)}$ so that the top boundaries d_n in the bar construction $N \circ \amalg \circ (N^{\times l})^{\circ n} \underline{A}$ are given by

$$\begin{tikzcd}
\Omega_{G,n}^{(\underline{A})} \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr \coprod \Omega_{G,n-1}^{(A_j)} \arrow[r, "V_G^{\circ n-1}"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \Omega_{G,0}^{(A_j)} \arrow[r, "F_1"] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"] \arrow[d, equal] \arrow[dl, Rightarrow, "\underline{m}"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,n-1}^{(\underline{A})} \arrow[r, "V_G"'] & \mathsf{F} \wr \coprod \Omega_{G,n-2}^{(A_j)} \arrow[r, "V_G^{\circ n-1}"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr A_j \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"'] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.23}$$

where $\underline{m}$ stands for the functor induced by the tuple of multiplication maps $m_j \colon NA_j \to A_j$, and the other boundaries d_i for $0 \le i < n$ are given by

$$\begin{tikzcd}
\Omega_{G,n}^{(\underline{A})} \arrow[r, "V_G"] \arrow[d, "d_i^{(\underline{A})}"'] & \mathsf{F} \wr \coprod \Omega_{G,n-1}^{(A_j)} \arrow[r, "V_G^{\circ n}"] \arrow[dl, phantom, "\Leftarrow \pi_i^{(\underline{A})}"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr A \arrow[r, "F"] \arrow[d, "\sigma^i"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n}"] \arrow[d, "\sigma^i"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"] \arrow[dl, phantom, "\Leftarrow \alpha_i"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega_{G,n-1}^{(\underline{A})} \arrow[r, "V_G"'] & \mathsf{F} \wr \coprod \Omega_{G,n-2}^{(A_j)} \arrow[r, "V_G^{\circ n-1}"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr A \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r, "\Pi^{\circ n-1}"'] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\Pi"'] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.24}$$

DISJBARDN EQ

where again $\pi_i^{(\underline{A})}$ interchanges lexicographic orders on the i-th F coordinate and α_i is again an associativity isomorphism. We note that (DISJBARDN EQ 5.24) follows directly from (REMAINDIFF EQ 5.22) for $0 < i < n$, but that the case $i = 0$, which uses the $N^{\times l}$ right action on $N \circ \amalg$ (cf. Remark TALPHAKMOD REM 5.3), which after unpacked leads to the composite diagram below.

$$\begin{tikzcd}
\Omega^{(\underline{A})}_{G,n} \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-1} \arrow[rr] \arrow[d] & & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} A_j \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \mathcal{V}^{op} \arrow[rr] \arrow[d] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n,1} \arrow[r] \arrow[d, "d_0^{(\underline{A})}"'] & \mathsf{F} \wr \Omega^{(\underline{A})}_{G,n-1} \arrow[r] \arrow[dl, Rightarrow, "\pi_0"] & \mathsf{F}^{\wr 2} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \coprod \mathsf{F}^{\wr n-2} A_j \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \coprod \mathsf{F}^{\wr n-2} \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, Rightarrow, "\alpha_0"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A})}_{G,n-1} \arrow[rr] & & \mathsf{F} \wr \coprod \Omega^{(A_j)}_{G,n-2} \arrow[r] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-2} A_j \arrow[r] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-2} \mathcal{V}^{op} \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{5.25}$$

ASSOCSPANJ1 EQ

Finally, using the inclusions $\Omega^{(\underline{A}),J}_{G,n} \hookrightarrow \Omega^{(\underline{A})}_{G,n}$, one obtains analogous descriptions of the bar constructions $N \circ \amalg \circ (N^{\times J})^{\circ n} \underline{A}$, depicted below.

$$\begin{tikzcd}
\Omega^{(\underline{A}),J}_{G,n} \arrow[r] \arrow[d] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr \Omega^{(A_j)}_{G,0} \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F_1"] \arrow[d] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] \arrow[dl, Rightarrow, "\underline{m}"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^{(\underline{A}),J}_{G,n-1} \arrow[r] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.26}$$

$$\begin{tikzcd}
\Omega^{(\underline{A}),J}_{G,n} \arrow[r] \arrow[d, "d_i^{\underline{A}}"'] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"] \arrow[d, "\sigma^i"] \arrow[dl, Rightarrow, "\pi_i^{(\underline{A})}"] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^i"] & \mathcal{V}^{op} \arrow[d, equal] \arrow[dl, Rightarrow, "\alpha_i"] \\
\Omega^{(\underline{A}),J}_{G,n-1} \arrow[r] & \mathsf{F} \wr \left(\coprod_J \mathsf{F}^{\wr n-1} \wr A_j \amalg \coprod_{\underline{l}-J} A_j\right) \arrow[r, "F"'] & \mathsf{F} \wr \coprod \mathsf{F}^{\wr n-1} \wr \mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op}
\end{tikzcd} \tag{5.27}$$

DISJBARDNJ EQ

5.4 Transferring simplicial colimits of left Kan extensions

TRANSFSIMP SEC

Given genuine equivariant operads $X, Y \in \mathsf{Op}_G$ one has an isomorphism

$$X \amalg^a Y \simeq \operatorname{colim}_{\Delta^{op}} \left(\mathbb{F}_G^{\bullet+1} X \amalg^a \mathbb{F}_G^{\bullet+1} Y\right)$$

so that combining Remarks REPACKAGERES REM 4.74 and Remark PRECOMPPOSTCOMP REM 5.4 with the results in the previous section one obtains isomorphisms

$$X \amalg^a Y \simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}\left(N^{\bullet+1} \iota X \amalg^a N^{\bullet+1} \iota Y\right)\right) \tag{5.28}$$

$$\simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}\left(N \circ \amalg \circ (N^{\times 2})^{\bullet}(\iota X, \iota Y)\right)\right) \tag{5.29}$$

$$\simeq \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}_{\Omega^{2,op}_{G,\bullet} \to \Sigma^{op}_G} N_{\bullet}^{(X,Y)}\right) \tag{5.30}$$

COLIMLAN EQ

where we write $N_{\bullet}^{(X,Y)} \colon \Omega^{2,op}_{G,\bullet} \to \mathcal{V}$ for the induced functor.

The purpose of this section will be show that one can repackage formulas such as (COLIMLAN EQ 5.30) with a single left Kan extension over a category $\Omega^2_G = |\Omega^2_{G,\bullet}|$ obtained from $\Omega^2_{G,\bullet}$ via realization in Cat.

We note that $\Omega^2_{G,\bullet}$ together with the corresponding functors to Σ_G, $\mathcal{V}^{op}$ can be viewed as a simplicial object $\Delta^{op} \to \mathsf{WSpan}^l(\Sigma, G^{op}, \mathcal{V})$, and our first task will be to repackage such functors in terms of Grothendieck constructions.

MPSPANREIN LEMMA

Lemma 5.31. *Functors $F \colon \mathcal{D} \ltimes \mathcal{I}_{\bullet} \to \mathcal{C}$ are in bijection with lifts*

$$\begin{tikzcd}
& \mathsf{WSpan}^l(*, \mathcal{C}) \arrow[d, "\mathsf{fgt}"] \\
\mathcal{D} \arrow[r, "\mathcal{I}_{\bullet}"'] \arrow[ur, dashed, "\mathcal{I}_{\bullet}^F"] & \mathsf{Cat}.
\end{tikzcd}$$

where fgt *is the functor forgetting the maps to* $*$ *and* $\mathcal{C}$*.*

Proof. This is a matter of unpacking notation. The restrictions $F|_{\mathcal{I}_d}$ to the fibers $\mathcal{I}_d \subset D \ltimes \mathcal{I}_\bullet$ are precisely the functors $\mathcal{I}^F_d : \mathcal{I}_d \to \mathcal{C}$ describing $\mathcal{I}^F_\bullet(d)$.

Furthermore, the images $F\left((d,i) \to (d', f_*(i))\right)$ of the pushout arrows over a fixed arrow $f : d \to d'$ of $\mathcal{D}$ assemble to a natural transformation

$$\begin{tikzcd} \mathcal{I}_d \arrow[d, "f_*"'] \arrow[dr, "I^F_d"] & \\ \mathcal{I}_{d'} \arrow[r, "I^F_{d'}"'] \arrow[ur, phantom, "\Downarrow"] & \mathcal{C} \end{tikzcd} \tag{5.32}$$

which describes $\mathcal{I}^F_\bullet(f)$. It is straightforward to check that the associativity and unitality conditions coincide. □

In the cases of interest we will have $\mathcal{D} = \Delta^{op}$, so that $\mathcal{I}_\bullet$ can be interpreted as an object $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. By recalling the standard cosimplicial object $[\bullet] \in \mathsf{Cat}^{\Delta}$ given by $[n] = (0 \to 1 \to \cdots \to n)$ one obtains the following definition.

Definition 5.33. The left adjoint

$$|-| : \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat} : (-)^{[\bullet]}$$

will be called the *realization* functor.

Remark 5.34. More explicitly, one has

$$|\mathcal{I}_\bullet| = coeq\left(\coprod_{[n]\to[m]} [n] \times \mathcal{I}_m \rightrightarrows \coprod_{[n]} [n] \times \mathcal{I}_n\right). \tag{5.35}$$ REALDEF EQ

Example 5.36. Any $\mathcal{I} \in \mathsf{Cat}$ induces objects $\mathcal{I}, \mathcal{I}_\bullet, \mathcal{I}^{[\bullet]} \in \mathsf{Cat}^{\Delta^{op}}$ where $\mathcal{I}$ is the constant simplicial object and $\mathcal{I}_\bullet$ is the nerve $N\mathcal{I}$ with each level regarded as a discrete category. It is straightforward to check that $|\mathcal{I}| = |\mathcal{I}_\bullet| = |\mathcal{I}^{[\bullet]}| = \mathcal{I}$.

OBJGENREL LEMMA **Lemma 5.37.** *Given* $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ *one has an identification* $ob(|\mathcal{I}_\bullet|) \simeq ob(\mathcal{I}_0)$*. Furthermore, the arrows of* $|\mathcal{I}_\bullet|$ *are generated by the image of the arrows in* $\mathcal{I}_0 \simeq \mathcal{I}_0 \times [0]$ *and the image of the arrows in* $[1] \times ob(\mathcal{I}_1)$*.*

For each $i_1 \in \mathcal{I}_1$, we will denote the arrow of $|\mathcal{I}_\bullet|$ induced by the arrow in $[1] \times \{i_1\}$ by

$$d_1(i_1) \xrightarrow{i_1} d_0(i_1).$$

Proof. We write $d_{\hat{k}}$, $d_{\hat{k},\hat{l}}$ for the simplicial operators induced by the maps $[0] \xrightarrow{0 \mapsto k} [n]$, $[1] \xrightarrow{0\mapsto k, 1 \mapsto l} [n]$ which can informally be thought of as the "composite of all faces other than d_k, d_l". Using (5.35) one has equivalence relations of objects

$$[n] \times \mathcal{I}_n \ni (k, i_n) \sim (0, d_{\hat{k}}(i_n)) \in [0] \times \mathcal{I}_0$$

and since for any generating relation $(k, i_n) \sim (l, i'_m)$ it is $d_{\hat{k}}(i_n) = d_{\hat{l}}(i'_m)$ the identification $ob(|\mathcal{I}_\bullet|) \simeq ob(\mathcal{I}_0)$ follows.

To verify the claim about generating arrows, note that any arrow of $[n] \times \mathcal{I}_n$ factors as

$$(k, i_n) \to (l, i_n) \xrightarrow{I_n} (l, i'_n) \tag{5.38}$$ FACTORIZATIONREAL EQ

for $I_n : i_n \to i'_n$ an arrow of $\mathcal{I}_n$. The $d_{\hat{l}}$ relation identifies the right arrow in (5.38) with $(0, d_{\hat{l}}(i_n)) \xrightarrow{d_{\hat{l}}(I_n)} (0, d_{\hat{l}}(i'_n))$ in $[0] \times \mathcal{I}_0$ while (if $k < l$) the $d_{\hat{k},\hat{l}}$ relation identifies the left arrow with $(0, d_{\hat{k},\hat{l}}(i_n)) \to (1, d_{\hat{k},\hat{l}}(i_n))$ in $[1] \times \mathcal{I}_1$. The result follows. □

Remark 5.39. Given $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$, $\mathcal{C} \in \mathsf{Cat}$, the isomorphisms

$$Hom_{\mathsf{Cat}}(|\mathcal{I}_\bullet|, \mathcal{C}) \simeq Hom_{\mathsf{Cat}^{\Delta^{op}}}(\mathcal{I}_\bullet, \mathcal{C}^{[\bullet]})$$

together with the fact that $\mathcal{C}^{[\bullet]}$ is always 2-coskeletal show that $|\mathcal{I}_\bullet|$ is determined by the categories $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ and maps between them, i.e. by the truncated version of formula (5.35) REALDEF EQ with $n, m \leq 2$.

Indeed, it can be shown that a sufficient set of generating relations in $|\mathcal{I}_\bullet|$ is given by (i) the relations in $\mathcal{I}_0$ (including relations stating that identities of $\mathcal{I}_0$ are identities of $|\mathcal{I}_\bullet|$); (ii) relations stating that for each $i_0 \in \mathcal{I}_0$ the arrow $i_0 = d_1(s_0(i_0)) \xrightarrow{s_0(i_0)} d_1(s_0(i_0)) = i_0$ is an identity; (iii) for each arrow $I_1 \colon i_1 \to i_1'$ in $\mathcal{I}_1$ the relation that the square below commutes

$$\begin{tikzcd}
d_1(i_1) \arrow[r, "i_1"] \arrow[d, "d_1(I_1)"'] & d_0(i_1) \arrow[d, "d_0(I_1)"] \\
d_1(i_1') \arrow[r, "i_1'"] & d_0(i_1')
\end{tikzcd}$$

and (iv) for each object $i_2 \in \mathcal{I}_2$ the relation that the following triangle commutes.

$$\begin{tikzcd}
d_{1,2}(i_2) \arrow[rr, "d_1(i_2)"] \arrow[dr, "d_2(i_2)"'] & & d_{0,1}(i_2) \\
& d_{0,2}(i_2) \arrow[ur, "d_0(i_2)"'] &
\end{tikzcd}$$

PLANARSTRING EX

Example 5.40. For $\Omega_{G,\bullet}$ the simplicial object of planar strings one has $|\Omega_{G,\bullet}| = \Omega_G^t$, the category of G-trees and tall maps. Indeed, arrows of $\Omega_{G,0}$ and objects of $\Omega_{G,1}$ are naturally identified with the quotient arrows and planar tall arrows of Ω_G^t, which are a generating set of arrows. And likewise, relations in $\Omega_{G,0}$, arrows in $\Omega_{G,1}$ and objects in $\Omega_{G,2}$ are identified with the relations of Ω_G^t.

Analogously, for $\Omega_{G,\bullet}^J$ the simplicial object of planar $\underline{l}$-labeled strings that are $(\{l\} - J)$-inert, one has $|\Omega_{G,\bullet}^J| = \Omega_G^{J,t}$, the category of $\underline{l}$-labeled G-trees and $(\{l\} - J)$-inert tall maps.

The following is the key result in this section.

SOURCEFINAL PROP

Proposition 5.41. *Let $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then there is a natural functor*

$$\Delta^{op} \ltimes \mathcal{I}_\bullet \xrightarrow{\ s\ } |\mathcal{I}_\bullet|. \tag{5.42}$$

Further, s is final.

Remark 5.43. The s in the result above stands for *source*. This is because, for any $\mathcal{I} \in \mathsf{Cat}$, the map $\Delta^{op} \ltimes \mathcal{I}^{[\bullet]} \to \left|\mathcal{I}^{[\bullet]}\right| \simeq \mathcal{I}$ is given by $s(i_0 \to \cdots \to i_n) = i_0$.

Proof. Recall that $|\mathcal{I}_\bullet|$ is the coequalizer (5.35) REALDEF EQ. Given $(k, g_m) \in [n] \times \mathcal{I}_m$, we will write $[k, g_m]$ for the corresponding object in $|\mathcal{I}_\bullet|$. To simplify notation, we will write objects of $\mathcal{I}_n$ as i_n and implicitly assume that $[k, i_n]$ refers to the class of the object $(k, i_n) \in [n] \times \mathcal{I}_n$.

We define s on objects by $s([n], i_n) = [0, i_n]$ and on an arrow $(\phi, I_m) \colon (n, i_n) \to (m, i_m')$ as the composite (note that $\phi \colon [m] \to [n]$ and $I_m \colon \phi^*(i_n) \to i_m$)

$$[0, i_n] \to [\phi(0), i_n] = [0, \phi^*(i_n)] \xrightarrow{I_m} [0, i_m']. \tag{5.44}$$ TARGETDEFINITON EQ

To check associativity, the cases of of a pair of either two fiber arrows (i.e. arrows where ϕ is the identity) or two pushforward arrows (i.e. arrows where I_m is the identity) are immediate from (5.44) TARGETDEFINITON EQ, hence we are left with the case $([n], i_n) \xrightarrow{I_n} ([n], i_n') \to ([m], \phi^*(i_n'))$ of a fiber arrow followed by a pushforward arrow. Noting that in $\Delta^{op} \ltimes \mathcal{I}_\bullet$ this composite can be rewritten as $([n], i_n) \to ([m], \phi^*(i_n)) \xrightarrow{\phi^*(I_n)} ([m], \phi^*(i'_n))$ this amounts to checking that

$$\begin{tikzcd}
{[0, i_n]} \arrow[r] \arrow[d, "I_n"'] & {[\phi(0), i_n)]} \arrow[r, equal] \arrow[d, "I_n"'] & {[0, \phi^*(i_n)]} \arrow[d, "\phi^*(I_n)"] \\
{[0, i'_n]} \arrow[r] & {[\phi(0), i'_n]} \arrow[r, equal] & {[0, \phi^*(i_n)]}
\end{tikzcd} \tag{5.45}$$

commutes in $|\mathcal{I}_\bullet|$, which is the case since the left square is encoded by a square in $[n] \times \mathcal{I}_n$ and the right square is encoded by an arrow in $[m] \times \mathcal{I}_n$.

We now turn to showing that s is final.

Fix $j \in \mathcal{I}_0$. We will show that $[0, j] \downarrow \Delta^{op} \ltimes \mathcal{I}_\bullet$ is indeed connected. By Lemma OBJGENREL LEMMA 5.37 any object in this undercategory has a description (not necessarily unique) as a pair

$$\Big(([n], i_n) \, , [0, j] \xrightarrow{f_1} \cdots \xrightarrow{f_r} s([n], i_n) \Big)$$

where each f_i is a generating arrow of $|\mathcal{I}_\bullet|$ induced by either an arrow I_0 of $\mathcal{I}_0$ or object $i_1 \in \mathcal{I}_1$.

We will connect this object to the canonical object $(([0], h), [0, h] = [0, h])$, arguing by induction on r. If $n \neq 0$, the map $d_{\hat{0}}: ([n], i_n) \to ([0], d^*_{\hat{0}}(i_n))$ and the fact that $s\big(d^*_{\hat{0}}\big) = id_{[0, d^*_{\hat{0}}(i_n)]}$ provides an arrow to an object with $n = 0$ without changing r. If $n = 0$, one can apply the induction hypothesis by lifting f_r to $\Delta^{op} \ltimes \mathcal{I}_\bullet$ according to one of two cases: (i) if f_r is induced by an arrow I_0 of $\mathcal{I}_0$, the lift of f_r is simply $([0], i'_0) \xrightarrow{I_0} ([0], i_0)$; (ii) if f_r is induced by $i_1 \in \mathcal{I}_1$ the lift is provided by the map $([1], i_1) \to ([0], d_0(i_1))$. □

In practice, we will need to know that s satisfies the following stronger finality condition with respect to left Kan extensions.

URCELANFINAL COR

Corollary 5.46. *Consider a map $\mathcal{I}_\bullet \to \mathcal{J}$ between $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ and a constant object $\mathcal{J} = \mathcal{J}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then the source map s*

$$\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "s"] \arrow[dr] & & {|\mathcal{I}_\bullet|} \arrow[dl] \\
& \mathcal{J} &
\end{tikzcd}$$

is Lan-*final over $\mathcal{J}$, i.e. the functors $s \downarrow j: (\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \to |\mathcal{I}_\bullet| \downarrow j$ are final for all $j \in \mathcal{J}$.*

Proof. It is clear that $(\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \simeq \Delta^{op} \ltimes (\mathcal{I}_\bullet \downarrow j)$ while Lemma UNDERLEFTADJ LEM 2.2 guarantees that, since $(-) \downarrow j$ is a left adjoint, $|\mathcal{I}_\bullet| \downarrow j \simeq |\mathcal{I}_\bullet \downarrow j|$. One thus reduces to Proposition SOURCEFINAL PROP 5.41. □

We end this section with two basic lemmas that will allows us to apply Corollary SOURCELANFINAL COR 5.46 to the tree categories we will be interested in.

TWISTING LEMMA

Lemma 5.47. *Let $\mathcal{I}^F_\bullet \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ be such that the diagrams*

$$\begin{tikzcd}
\mathcal{I}_n \arrow[dr, "F_n"] \arrow[d, "d_i"'] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] \arrow[ur, phantom, "\Nearrow \delta_i"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[dr, "F_n"] \arrow[d, "s_j"'] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] \arrow[ur, phantom, "\Nearrow \sigma_j"] & \mathcal{C}
\end{tikzcd} \tag{5.48}$$

IDENTSIMPRELSISO EQ

commute up to isomorphism for $0 < i \leq n$, $0 \leq j \leq n$.

Then the functors $\tilde{F}_n: \mathcal{I}_n \to \mathcal{C}$ given by the composites

$$\mathcal{I}_n \xrightarrow{d_{1,\cdots,n}} \mathcal{I}_0 \xrightarrow{F_0} \mathcal{C}$$

assemble to an object $\mathcal{I}^{\tilde{F}}_\bullet \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ which is isomorphic to $\mathcal{I}^F_\bullet$ and such that the corresponding diagrams IDENTSIMPRELSISO EQ *(5.48) for $0 < i \leq n$, $0 \leq j \leq n$ are strictly commutative.*

Proof. This follows by a straightforward verification. □

SOURCEFACT LEM **Lemma 5.49.** *A (necessarily unique) factorization*

$$\begin{tikzcd} \Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr] \arrow[dr, "s"'] & & \mathcal{C} \\ & |\mathcal{I}_\bullet| \arrow[ur, dashed] & \end{tikzcd} \tag{5.50}$$ SOURCEFACT EQ

exists iff for the associated object $\mathcal{I}_\bullet \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ (cf. Lemma 5.31) all faces d_i for $0 < i \le n$ and degeneracies s_j for $0 \le j \le n$ are strictly commutative, i.e. they are given by diagrams

$$\begin{tikzcd} \mathcal{I}_n \arrow[d, "d_0"'] \arrow[dr, "F_n"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] \arrow[ur, phantom, "\Leftarrow\varphi_n"] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] & \\ \mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] & \\ \mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C} \end{tikzcd} \tag{5.51}$$ IDENTSIMPRELS EQ

Proof. For the "if" direction, it suffices to note that s sends all pushout arrows of $\Delta^{op} \ltimes \mathcal{I}_\bullet$ for faces d_i, $0 \le i \le n$ and degeneracies s_j, $0 \le j \le n$ to identities and this yields the commutative diagrams (5.51).

For the "only if" direction, this will follow by building a functor $\mathcal{I}_\bullet \xrightarrow{\bar{F}} \mathcal{C}^{[\bullet]}$ together with the naturality of the source map s (recall that $|\mathcal{C}^{[\bullet]}| \simeq \mathcal{C}$). We define $\bar{F}_n|_{k \to k+1}$ as the map

$$F_{n-k} d_{0,\cdots,k-1} \xrightarrow{\varphi_{n-k} d_{0,\cdots,k-1}} F_{n-k-1} d_{0,\cdots,k}. \tag{5.52}$$ EQUIVALENCEDEF EQ

The claim that $s \circ (\Delta^{op} \ltimes \bar{F})$ recovers the horizontal map in (5.50) is straightforward, hence the real task is to prove that (5.52) indeed defines a map of simplicial objects.

$$\varphi_{n-1} d_i = \varphi_n, \ \ 1 < i \qquad \varphi_{n-1} d_1 = (\varphi_{n-1} d_0) \circ \varphi_n, \qquad \varphi_{n+1} s_i = \varphi_n, \ \ 0 < i, \qquad \varphi_{n+1} s_0 = id_{F_n} \tag{5.53}$$

Next, note that there is no ambiguity in writing simply $\varphi_{n-k} d_{0,\cdots,k-1}$ to denote the map (5.52). We now check that $\bar{F}_{n-1} d_i = d_i \bar{F}_n$, $0 \le i \le n$, which must be verified after restricting to each $k \to k+1$, $0 \le k \le n-2$. There are three cases, depending on i and k:

$(i < k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{0,\cdots,k}$;

$(i = k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_1 d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_0 \circ \varphi_{n-k}) d_{0,\cdots,k-1} = (\varphi_{n-k-1} d_{0,\cdots,k}) \circ (\varphi_{n-k} d_{0,\cdots,k-1})$;

$(i > k+1)$ $\varphi_{n-k-1} d_{0,\cdots,k-1} d_i = \varphi_{n-k-1} d_{i-k} d_{0,\cdots,k-1} = \varphi_{n-k} d_{0,\cdots,k-1}$.

The case of degeneracies is similar. □

DUALRESULTS REM **Remark 5.54.** One can twist all results by the opposite functor

$$\Delta \xrightarrow{(-)^{op}} \Delta$$

which sends $[n]$ to itself and d_i, s_i to d_{n-i}, s_{n-i}. In doing so, one obtains vertical isomorphisms

$$\begin{tikzcd} \Delta^{op} \ltimes (\mathcal{J}_\bullet \circ (-)^{op}) \arrow[r, "s"] \arrow[d, "\simeq"'] & |\mathcal{J}_\bullet \circ (-)^{op}| \arrow[d, "\simeq"] \\ \Delta^{op} \ltimes \mathcal{J}_\bullet \arrow[r, "t"'] & |\mathcal{J}_\bullet| \end{tikzcd}$$

which reinterpret the "source" functor as what one might call the "target" functor, with $t([n], i_n) = [n, i_n]$ rather than $s([n], i_n) = [0, i_n]$.

Corollary 5.46 now says that t is Lan-final and Lemmas 5.47, 5.49 generalize in the obvious way by replacing s with t and d_0 with d_n.

5.5 The category of extension trees

In this section we combine the previous sections to obtain a compact description of free extension pushouts

$$\begin{array}{ccc} \mathbb{F}_G A & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathbb{F}_G B & \longrightarrow & Y \end{array} \tag{5.55}$$ FREEEXT EQ

as a left Kan extension over a convenient category of trees.

For simplicity, we first explain how to obtain a similar description for the simpler case of a coproduct $X \amalg^a Y$. By (5.30), one has a description

$$\begin{aligned} X \amalg^a Y \simeq & \operatorname{colim}_{\Delta^{op}} \left(\mathsf{Lan}_{\Omega^{2,op}_{G,\bullet} \to \Sigma^{op}_G} N^{(X,Y)}_\bullet \right) \\ \simeq & \mathsf{Lan}_{\Delta^{op} \ltimes \Omega^{2,op}_{G,\bullet} \to \Sigma^{op}_G} N^{(X,Y)}_\bullet \end{aligned}$$

where the second identification follows from formal properties of Grothendieck constructions. DUALRESULTS REM

Combining the fact that (5.27) consists of natural isomorphisms with (the Remark 5.54 dual of) Lemma 5.47, yields an isomorphic twisted functor $\tilde{N}^{(X,Y)}_\bullet$ with strictly commutative s_i and d_i for $i \neq n$. The dual of Lemma 5.49 now says that $\tilde{N}^{(X,Y)}_\bullet$ factors via the target map t though $\Omega^{2,op}_{G,\bullet} \xrightarrow{\sim} |\Omega^{2,op}_{G,\bullet}|$ (writing $\tilde{N}^{(X,Y)}$ for the factorization) and thus the dual of Corollary 5.46 finally yields

$$X \amalg^a Y \simeq \mathsf{Lan}_{\Omega^{2,op}_G \to \Sigma^{op}_G} \tilde{N}^{(X,Y)}. \tag{5.56}$$

We recall that by Example 5.40, $\Omega^{\underline{2}}_G$ is simply the category of $\underline{2}$-labeled trees and tall label maps.

More generally, one has

$$\coprod^a_J X_j \amalg^a \coprod^a_{\underline{l}-J} \mathbb{F}_G X_j \simeq \mathsf{Lan}_{\Omega^{J,op}_G \to \Sigma^{op}_G} \tilde{N}^{(\underline{X})}. \tag{5.57}$$ LANCOPRODDESC

where Ω^J_G is the category of $\underline{l}$-labeled trees and tall $(\underline{l} - J)$-inert label maps.

Remark 5.58. We note that the twisting $\tilde{N}^{(X,Y)}_\bullet$ is fairly harmless. For explicitness, we focus on the simplest case of a "unary coprodut", in which case (5.57) is simply recovering the genuine equivariant operad X from its bar resolution. In that case $N^X_2 : \Omega^{op}_{G,2} \to \mathcal{V}$ is given by the top map in (4.53) or, equivalently, by the top map in (4.54) (we note that, in the notation therein, it is $A = \Sigma_G$). On the other hand, the twisted map $\tilde{N}^X_2 : \Omega^{op}_{G,2} \to \mathcal{V}$ is given by the left bottom composite in either of (4.53), (4.54). Informally, the role of this twisting is therefore simply that of replacing the order on $V_G(T_n)$ induced lexicographically by planar strings $T_0 \to \cdots \to T_n$ with the simpler order induced directly from T_n.

In what follows we will largely be able to ignore this technicality. Indeed, the role of lexicographic orders in building (5.57) is that of guaranteeing that $\tilde{N}_\bullet$ satisfies the necessary simplicial identities, which are ensured by appealing to the bar construction for the monad N.

We now turn to the task of building (5.55) as a left Kan extension. One has a colimit description

$$\mathbb{F}B \coprod_{\mathbb{F}A} X \simeq \operatorname{colim}_{\Delta^{op}} \Big(\mathbb{F}B \amalg \mathbb{F}A \amalg X \leftleftarrows \mathbb{F}B \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg X \Lleftarrow \mathbb{F}B \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg \mathbb{F}A \amalg X \quad \cdots \Big) \tag{5.59}$$ FREEEXTUSEFCOL EQ

where all differentials are fold maps of $\mathbb{F}A$ except to the n-th differential d_n, which is induced by the two maps $\mathbb{F}A \to X$, $\mathbb{F}A \to \mathbb{F}B$.

By the previous discussion each individual object $X \amalg (\mathbb{F}A)^{\amalg 2n+1} \amalg \mathbb{F}B$ in (5.59) can be described as a left Kan extension over the tree category $\Omega_G^{\{X\}}$ where $\{X\} \subset \{B, A, \cdots, A, X\}$ is a simpleton. The maps in (5.59) can themselves be encoded as span maps between the $\Omega_G^{\{X\}}$. To see this, we make (5.59) more precise. Firstly, we write $\langle n \rangle$ for the poset

$$-\infty \le -n \le -n+1 \le \cdots \le -1 \le 0 \le 1 \le \cdots \le n-1 \le n \le +\infty.$$

The posets $\langle n \rangle$ together with antisymmetric (i.e. such that $f(-x) = -f(x)$) poset maps preserving all three of $-\infty, 0, +\infty$ then form a simplicial object[1] $\langle - \rangle \colon \Delta^{op} \to \mathsf{F}$.

(5.55) thus induces a simplicial object $(B, A, X)_{\langle n \rangle} \in \mathsf{F} \wr \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$.

Each level of $(\iota B, \iota A, \iota X)_{\langle n \rangle}$ is then a $N^{\times \{+\infty\}}$-algebra on $\left(\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})\right)^{\times \langle n \rangle}$, compatibly with the simplicial maps. One thus obtains a *bisimplicial* object

$$\Sigma_G^{op} \leftarrow \Omega_{G,\bullet}^{\{+\infty\}_{\langle \bullet \rangle}, op} \xrightarrow{N_\bullet^{(B,A,X)_{\langle \bullet \rangle}}} \mathcal{V}$$

on $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$ whose realization along the string direction yields the spans

$$\Sigma_G^{op} \leftarrow \Omega_G^{\{+\infty\}_{\langle \bullet \rangle}, op} \xrightarrow{N^{(B,A,X)_{\langle \bullet \rangle}}} \mathcal{V} \tag{5.60}$$

PARTREALSPAN EQ

discussed above, except now assembled into a simplicial object in $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$.

All degeneracies s_i and differentials d_i of (5.60) other than the top differential d_n are induced by maps α^* described in §5.1 and thus given by strictly commutative diagrams, so that Lemma 5.49 and Corollary 5.46 can be applied (this time with no need to appeal to Lemma 5.47) so as to allow (5.59) to be repackaged as

$$\mathbb{F}B \coprod_{\mathbb{F}A} X \simeq \mathsf{Lan}_{\Omega_G^{e,op} \to \Sigma_G^{op}} N^{(B,A,X)} \tag{5.61}$$

FREEEXTUSEFCOLNEW EQ

where we write Ω_G^e for $|\Omega_G^{\{+\infty\}_{\langle \bullet \rangle}}|$. We now turn to the task of describing Ω_G^e, starting with by defining it directly.

EXTTREECAT DEF

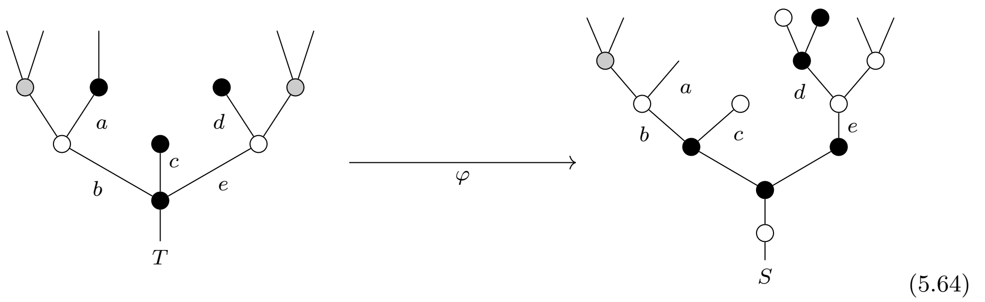
Definition 5.62. The *extension tree category* Ω_G^e is the category whose objects are $\{B, A, X\}$-labeled trees and whose maps $\varphi \colon T \to S$ are tall maps of trees such that

(i) if $T_{v_{Ge}}$ has an A-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has an A-label;

(ii) if $T_{v_{Ge}}$ has a B-label, then $S_{v_{Ge}} = T_{v_{Ge}}$ and $S_{v_{Ge}}$ has either an A-label or a B-label;

(iii) if $T_{v_{Ge}}$ has a X-label, then $S_{v_{Ge}}$ has only A and X-labels.

Example 5.63. The following is an example of a planar map in Ω_G^e, where black nodes represent X-labeled nodes, grey nodes represent B-labeled nodes and white nodes represent A-labeled nodes.



(5.64) REGALTERNMAP EQ

[1] Indeed, we recall that the opposite simplex category Δ^{op} can equivalently described as the category of *intervals*, i.e. finite ordered posets with distinct top and bottom, along with order maps preserving both top and bottom. $\langle n \rangle$ can then be regarded as obtained by gluing the interval $0 \le 1 \le \cdots \le n \le +\infty$ with its opposite.

Proposition 5.65. *One has an identification*

$$\Omega^e_G \simeq |\Omega^{\{+\infty\}(\bullet)}_G|.$$

Proof. We note first that Ω^e_G contains all label maps that are $\{A,B\}$-inert. In fact, any map of Ω^e_G clearly has a unique factorization as such a label map followed by an underlying planar isomorphism of trees that replaces some of the X and B labels with A labels. We will refer to the former as label maps and to the latter as relabel maps.

We recall that $\Omega^{\{+\infty\}(n)}_G$ consists of trees with $2n+3$ types of labels: X-labels, B-labels and $2n+1$ distinct types of A-labels. One can equivalently encode such a tree as a string $T_0 \to \cdots \to T_n$ of relabel maps. Indeed, the A-label nodes of T_n in such a string are partitioned into $2n+1$ types according to that node's labels one the T_i (which are either all A's, some X's and then A's or some B's and then A's). Moreover, a diagram

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d, "f_0"'] & T_1 \arrow[r] \arrow[d, "f_1"'] & \cdots \arrow[r] & T_n \arrow[d, "f_n"'] \\
T'_0 \arrow[r] & T'_1 \arrow[r] & \cdots \arrow[r] & T'_n
\end{tikzcd}$$

with f_i label maps of Ω^e_G is then equivalent to a label map $f_n \colon T_n \to T'_n$ respecting all $2n+3$ labels in $\Omega^{\{+\infty\}(n)}_G$. Since the string description above is also compatible with the simplicial structure maps in the obvious way, the result is now clear. $\square$

Our next task will be that of identifying a convenient Lan-final subcategory $\bar{\Omega}^e_G \hookrightarrow \Omega^e_G$. We first introduce the auxiliary notion of alternating trees. We recall the notion of input path (Notation INPUTPATH NOT 3.4) $I(e) = \{f \in T : e \leq_d f\}$ for an edge $e \in T$, which naturally extends to T in any of $\in \Omega, \Phi, \Omega_G, \Phi_G$.

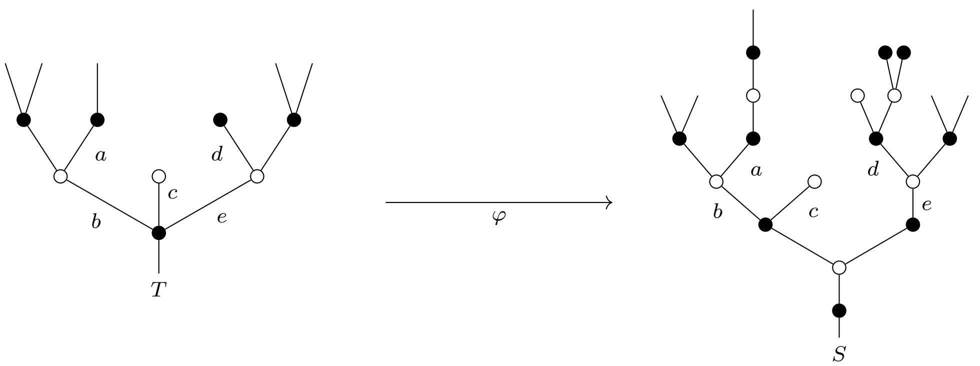
Definition 5.66. A G-tree $T \in \Omega_G$ is called *alternating* if, for all leafs $l \in T$ one has that the input path $I(l)$ has an even number of elements.

Further, a vertex $e^{\uparrow} \leq e$ is called *active* if $|I(e)|$ is odd and *inert* otherwise.

Finally, a tall map $T \xrightarrow{\varphi} T'$ between alternating G-trees is called a *tall alternating map* if for any inert vertex $e^{\uparrow} \leq e$ of T one has that $T'_{e^{\uparrow} \leq e)}$ is an inert vertex of T'.

We will denote the category of alternating G-trees and tall alternating maps by Ω^a_G.

Example 5.67. Two alternating trees (for $G = *$ the trivial group) and a planar tall alternating map between them follow, with active nodes in black ($\bullet$) and white nodes in white ($\circ$).



(5.68) REGALTERNMAPLR EQ

The term "alternating" comes from the fact that no adjacent nodes have the same color. We note, however, that there is additional restriction: the "outer" vertices, i.e. those immediately below a leaf or the one immediately above the root, are necessarily black/active (not, however, that this does *not* apply to stumps).

Remark 5.69. One can extend Definition 3.39 to the alternating context by defining a substitution datum to be alternating if it is given by isomorphisms for inert nodes and by alternating maps for active nodes. It is then straightforward to check that Proposition 3.42 and its equivariant analogue Proposition 4.19 extend to give alternating analogues.

Definition 5.70. $\bar{\Omega}^e_G \hookrightarrow \Omega^e_G$ is the full subcategory of (B, A, X)-labeled trees whose underlying trees is alternating, active nodes are labeled by X, and passive nodes are labeled by A or B.

We note that conditions (i) and (ii) in Definition 5.62 imply that maps in $\bar{\Omega}^e_G$ are underlying alternating maps.

The following establishes the required finality of $\bar{\Omega}^e_G$ in Ω^e_G.

LXP PROP

Proposition 5.71. *For each $U \in \Omega^e_G$ there exists a unique $\mathsf{lr}_X(U) \in \bar{\Omega}^e_G$ together with a unique planar label map of Ω^e_G*

$$\mathsf{lr}_X(U) \to U.$$

Furthermore, lr_X extends to a right retraction $\mathsf{lr}_X\colon \Omega^e_G \to \bar{\Omega}^e_G$.

Proof. Given U, we form a collection of outer faces $\{U^A_i\} \amalg \{U^B_j\} \amalg \{U^X_k\}$ where the U^A_i, U^b_j are simply the A, B-labeled nodes and the $\{U^X_k\}$ are the maximal outer subtrees whose nodes have only X-labels (we note that these may possible be sticks). Lemma 3.51 then guarantees that the $V_G(U^X_k)$ are disjoint, so that one can apply (the equivariant version of Proposition 3.49) to build

$$T = \mathsf{lr}(U) \to U \tag{5.72}$$

LRXDEF EQ

such that $\{U_{v_{Ge}}\} = \{U^A_i\} \amalg \{U^B_j\} \amalg \{U^X_k\}$. T has an obvious (B, A, X)-labeling making (5.72) into a label map, but we must still check $T \in \bar{\Omega}^e_G$, i.e. that T is alternating with the X-labeled vertices being precisely the X-labeled vertices. Let us now write any input path of T as $I(e) = (e = e_n \le e_{n-1} \le \cdots \le e_1 \le e_0)$. By Lemma 3.51 and maximality of the U^X_k, no pair of consecutive vertices v_{Ge_i} and $v_{Ge_{i+1}}$ can be both X-labeled. On the other hand, again by Lemma 3.51 any edge of U belongs to some U^X_k and therefore: (i) at least one of in each pair of consecutive vertices v_{Ge_i} and $v_{Ge_{i+1}}$ is X-labeled; (ii) if $r \in T$ is a root, v_{Gr} is X-labeled; (iii) if $l \in T$ is a leaf $v_{Gl_{n-1}}$ is X-labeled. This suffices to conclude $T \in \bar{\Omega}^e_G$, and uniqueness of T is immediate from the uniqueness in Lemma 3.51.

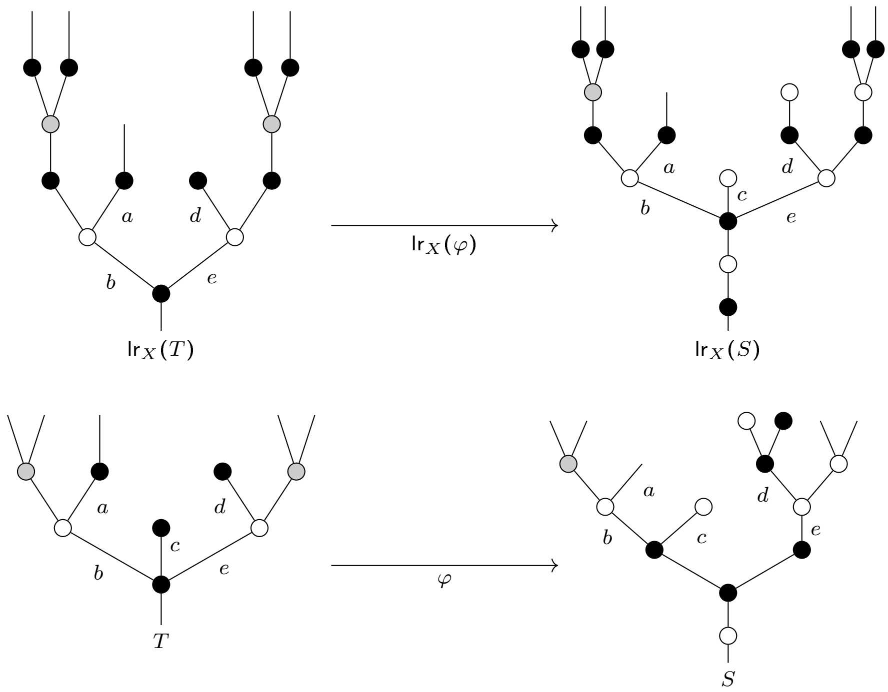
It remains to check that lr_X in fact defines a functor. We consider the following diagram.

$$\begin{tikzcd}
\mathsf{lr}_X(U) \arrow[r] \arrow[d, dashed, "\mathsf{lr}_X(f)"'] & U \arrow[d, "f"] \\
\mathsf{lr}_X(V) \arrow[r] & V
\end{tikzcd}$$

When f is a root pullback map, we define $\mathsf{lr}_X(f)$ to likewise be a root pullback map. When f is a rooted tall map, writing $T = \mathsf{lr}_X(U)$ one has a map of rooted T-substitution data $\{\mathsf{lr}_X(V_{v_{G_e}})\} \to \{V_{v_{G_e}}\}$, which after converted to a tree map yields the desired map $\mathsf{lr}_X(f)$. To check that lr_X respects composition of maps, the only non immediate case is that of a root pullback followed by a rooted map, in which case this follows from Remark 4.22. $\square$

Example 5.73. The following illustrates the lr_X construction when applied to the map φ

REGALTERNMAP EQ
in (5.64).



(5.74)

6 Model structures

Put together, the results in the previous section show that the free extension $\mathcal{P}[u]$ given by the pushout

$$\begin{tikzcd} \mathbb{F}_G X \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_G Y \arrow[r] & \mathcal{P}[u] \end{tikzcd} \tag{6.1}$$

CELLEXTPUSH EQ

is given by a left Kan extension along $(\bar{\Omega}^e_G)^{op} \xrightarrow{\mathsf{lr}} \Sigma^{op}_G$. So as to study the homotopical properties of the map $\mathcal{P} \to \mathcal{P}[u]$ we will identity a suitable filtration of this map, which will in turn be induced by a suitable filtration of the extension tree category $\bar{\Omega}^e_G$.

6.1 Filtration pieces

We now turn to the task of describing our filtration of $\bar{\Omega}^e_G$.

Firstly, we write $V_X(T)$ (resp. $V_Y(T)$) to denote the set of (non-equivariant) vertices of T with a X-label (resp. Y-label). We now define the *degree* of $T \in \bar{\Omega}^e_G$, denoted $|T|$, to be the sum $|T|_X + |T|_Y$, where $|T|_X$, $|T|_Y$ are defined by

$$|T|_X = \frac{|V_X(T)|}{|Gr|} = \sum_{Gv \in V_{G,X}(T)} \frac{|Gv|}{|Gr|}, \qquad |T|_Y = \frac{|V_Y(T)|}{|Gr|} = \sum_{Gv \in V_{G,Y}(T)} \frac{|Gv|}{|Gr|}$$

for Gr the root orbit of T.

Intuitively, $|T|_X$ counts the number of X-labeled vertices in each individual tree component of T.

Remark 6.2. One of the key properties of the degrees just defined is that they are invariant under root pullback.

IECES_DEFINITION

Definition 6.3. We define subcategories of $\bar{\Omega}^e_G$:

- $\bar{\Omega}^e_G[\leq k]$ (resp. $\Omega^e_G[k]$) is the full subcategory of trees $T \in \bar{\Omega}^e_G$ with $|T| \leq k$ (resp. $|T| = k$);
- $\bar{\Omega}^e_G[\leq k, -]$ (resp. $\bar{\Omega}^e_G[k, -]$) is the full subcategory of $\bar{\Omega}^e_G[\leq k]$ (resp. $\bar{\Omega}^e_G[k]$) of trees T with $|T|_Y \neq k$;
- $\bar{\Omega}^e_G[k, 0]$ is the full subcategory of $\bar{\Omega}^e_G[k]$ of trees T with $|T|_X = 0$ (or, equivalently, $|T|_Y = k$).

The above definitions still hold if we replace $\bar{\Omega}^e_G$ with Ω^a_G; in particular, we have vertical forgetful functors

$$\begin{tikzcd} \bar{\Omega}^e_G[k,-] \arrow[rr, hook] \arrow[dr, "\mathsf{fgt}"'] & & \bar{\Omega}^e_G[k] \arrow[dl, "\mathsf{fgt}"] \\ & \Omega^a_G[k] & \end{tikzcd}$$

LIMMOR REM

Remark 6.4. The categories $\bar{\Omega}^e_G[k]$ and $\bar{\Omega}^e_G[k,-]$ have only rather limited morphisms. In fact, all maps in these categories must be underlying quotients of trees. Indeed, it is clear from Definition EXTTREECAT DEF 5.62 that maps never lower degree and, moreover, degree is preserved iff $\mathcal{P}$-vertices are substituted by $\mathcal{P}$-vertices (rather than larger trees in $\bar{\Omega}^e_G$, which would necessarily possess X-vertices).

Moreover, we have a clear isomorphism of categories $\bar{\Omega}^e_G[k,0] \simeq \Omega^a_G[k]$.

_LAN_FINAL_LEMMA

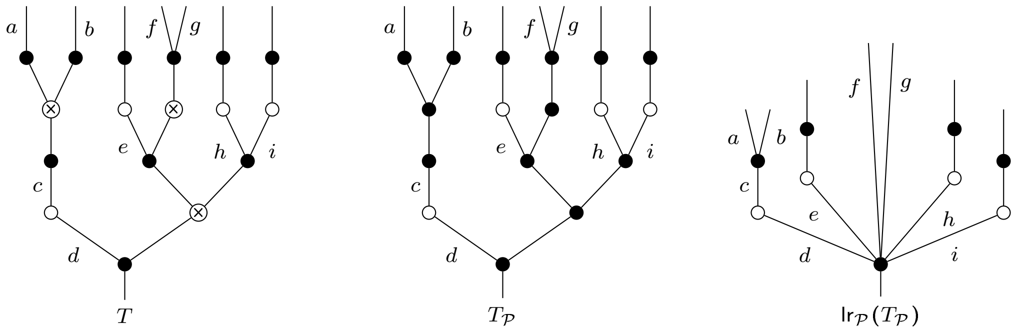
Lemma 6.5. $\bar{\Omega}^e_G[\leq k-1]$ *is* Ran*-initial in* $\bar{\Omega}^e_G[\leq k,-]$ *over* Σ_G.

In the proof we will make use of the following construction on $\Omega_{G,e}$: given $T \in \Omega_{G,e}$ we will let $T_{\mathcal{P}}$ denote the result of replacing all X-labeled nodes of T with $\mathcal{P}$-labeled nodes.

YINERT REM

Remark 6.6. Unlike the $\mathsf{lr}_{\mathcal{P}}$ construction of Proposition 5.71, which defines a functor $\mathsf{lr}_{\mathcal{P}}\colon \Omega^e_G \to \bar{\Omega}^e_G$, the construction $(-)_{\mathcal{P}}$ does not define a full functor $\Omega^e_G \to \Omega^e_G$, instead being functorial, and the obvious maps $T_{\mathcal{P}} \to T$ natural, only with respect to the Y-inert maps of Ω^e_G.

Example 6.7. Combining the $(-)_{\mathcal{P}}$ and $\mathsf{lr}_{\mathcal{P}}$ constructions one obtains a construction sending trees in $\bar{\Omega}^e_G$ to trees in $\bar{\Omega}^e_G$. We illustrate this for the tree $T \in \bar{\Omega}^e_G$ below, where black nodes are $\mathcal{P}$-labeled, white nodes filled with $\times$ are X-labeled, and empty white nodes are Y-labeled.



Proof of Lemma 6.5. Just as in the proof of Lemma 4.67, for each $C \in \Sigma_G$, the under categories $C \downarrow \bar{\Omega}^e_G[\le k-1]$, $C \downarrow \bar{\Omega}^e_G[\le k, -]$ have initial subcategories $C \downarrow_{\mathsf{r},\simeq} \bar{\Omega}^e_G[\le k-1]$. $C \downarrow_{\mathsf{r},\simeq} \bar{\Omega}^e_G[\le k, -]$ of those objects $(S, q\colon C \to \mathsf{lr}(S))$ such that q is an ordered isomorphism on roots, and thus an isomorphism in Σ_G.

It now suffices to show (cf. ([4, X.3.1])) that for each $(S, q\colon C \to \mathsf{lr}(S))$ in $C \downarrow_{\mathsf{r},\simeq} \bar{\Omega}^e_G[\le k, -]$ the undercategory

$$(S, q) \downarrow (C \downarrow_{\mathsf{r},\simeq} \bar{\Omega}^e_G[\le k-1]) \tag{6.8}$$

UNDERCATPR EQ

is non-empty and connected. Moreover, we note that an object in (6.8) is uniquely encoded by a map $T \to S$ inducing a rooted isomorphism on lr.

The case $S \in \Omega^e_G[\le k-1]$ is immediate. Otherwise, since $|S|_Y \neq k$ it is $|\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}})| < k$ and the map $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$, which is a rooted isomorphism on lr, shows that (6.8) is indeed non-empty.

Otherwise, given any rooted tall map $T \to S$ with $T \in \bar{\Omega}^e_G[k-1]$ (which gives a rooted isomorphism on lr and thus encodes a unique object of (6.8)). One can then form a diagram

$$\begin{tikzcd} & S & \mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \arrow[l] \\ T \arrow[ur] \arrow[r] & T' \arrow[u, "Y\text{-inert}"'] & \mathsf{lr}_{\mathcal{P}}(T'_{\mathcal{P}}) \arrow[l] \arrow[u] \end{tikzcd} \tag{6.9}$$

K-1LANFINAL EQ

where $T \to T' \to S$ is the natural factorization such that the second map is Y-inert, i.e., T' is obtained from T by simply relabeling to X those Y-labeled vertices of T that become X-vertices in S. Note that the existence of the right square in (6.9) follows from the map $T' \to S$ being Y-inert together with Remark 6.6. Since (6.9) becomes a diagram of rooted isomorphism on lr, it produces the necessary zigzag connecting the objects $T \to S$ and $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$ in (6.8), finishing the proof. □

Similarly to the $(-)_{\mathcal{P}}$ construction, there is also a construction T_Y which replaces all X-labels of $T \in \Omega^e_G$ with Y-labels. Moreover, in this case the construction restricts directly to a construction on $\bar{\Omega}^e_G$, which is easily seen to be functorial (and the $T_Y \to T$ maps natural) with regards to $\mathcal{P}$-inert maps. Remark 6.4 thus implies that $(-)_Y\colon \bar{\Omega}^e_G[k] \to \bar{\Omega}^e_G[k, 0]$ is a left retraction, resulting in the following.

N_FINALITY_LEMMA

Lemma 6.10. *$\bar{\Omega}^e_G[k, 0]$ is Ran-initial in $\bar{\Omega}^e_G[k]$ over Σ_G.*

In what follows we write $N^e\colon \Omega_G^{e,op} \to \mathcal{V}$ for the functor in (5.61), and abuse notation by likewise writing N^e for any of its restrictions to the subcategories in Definition 6.3.

We are now in a position to produce the desired filtration of the map $\mathcal{P} \to \mathcal{P}[u]$ in (6.1).

PK_DEFN **Definition 6.11.** Let $\mathcal{P}_k$ denote the left Kan extension

$$\begin{tikzcd}
\bar{\Omega}^e_G[\leq k]^{op} \arrow[r, "N^e"] \arrow[d, "\mathsf{lr}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[ur, "\mathcal{P}_k"'] &
\end{tikzcd}$$

Noting that $\bar{\Omega}^e_G[\leq 0] \simeq \Sigma_G$ (since $|T| = 0$ only if T is a G-corolla with $\mathcal{P}$-labeled vertex) and that $\bar{\Omega}^e_G$ is the union of (the nerves of) the $\bar{\Omega}^e_G[\leq k]$, one has a filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \operatorname{colim}_k \mathcal{P}_k = \mathcal{P}[u]. \tag{6.12}$$ FILT EQ

To analyze (6.12) homotopically we will further make use of a pushout description of each individual map $\mathcal{P}_{k-1} \to \mathcal{P}_k$. To do so, we note that the diagram of inclusions

$$\begin{tikzcd}
\bar{\Omega}^e_G[k,-] \arrow[r] \arrow[d] & \bar{\Omega}^e_G[\leq k,-] \arrow[d] \\
\bar{\Omega}^e_G[k] \arrow[r] & \bar{\Omega}^e_G[\leq k]
\end{tikzcd} \tag{6.13}$$ INCDIAG EQ

is a pushout of at the level of nerves. Indeed, this follows since

$$\bar{\Omega}^e_G[k] \cap \bar{\Omega}^e_G[\leq k,-] = \bar{\Omega}^e_G[k,-], \qquad \bar{\Omega}^e_G[k] \cup \bar{\Omega}^e_G[\leq k,-] = \bar{\Omega}^e_G[\leq k],$$

and since a map $T \to S$ in $\bar{\Omega}^e_G[\leq k]$ will be in one of subcategories in (6.13) iff T is.

Since Lemma 6.5 provides an identification $\operatorname{Lan}_{\bar{\Omega}_{G,e}[\leq k,-]^{op}} N^e \simeq \operatorname{Lan}_{\bar{\Omega}_{G,e}[\leq k-1]^{op}} N^e = \mathcal{P}_{k-1}$, applying left Kan extensions to (6.13) yields the pushout diagram below.

$$\begin{tikzcd}
\operatorname{Lan}_{\bar{\Omega}^e_G[k,-]^{op}} N^e \arrow[r] \arrow[d] & \mathcal{P}_{k-1} \arrow[d] \\
\operatorname{Lan}_{\bar{\Omega}^e_G[k]^{op}} N^e \arrow[r] & \mathcal{P}_k
\end{tikzcd} \tag{6.14}$$ FILTRATION_LAN_SQUARE_DIAGRAM

We will find it convenient for our purposes to have explicit levelwise descriptions for (6.14), which we now describe.

Proposition 6.15. *For each level $C \in \Sigma_G$, (6.14) is given by the following pushout in $\mathcal{V}^{\mathsf{Aut}(C)}$*

$$\begin{tikzcd}
\displaystyle\coprod_{[T]\in \mathsf{Iso}\left(C\downarrow_{\mathsf{r}}\Omega^a_G[k]\right)} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes Q^{in}_T[u] \right) \otimes_{\mathsf{Aut}(T)} \mathsf{Aut}(C) \arrow[r] \arrow[d] & \mathcal{P}_{k-1}(C) \arrow[d] \\
\displaystyle\coprod_{[T]\in \mathsf{Iso}\left(C\downarrow_{\mathsf{r}}\Omega^a_G[k]\right)} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \bigotimes_{v \in V^{in}_G(T)} Y(T_v) \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C) \arrow[r] & \mathcal{P}_k(C)
\end{tikzcd} \tag{6.16}$$ FILTRATION_LAN_LEVEL

where $\Omega^a_G[k]$ denotes alternating trees with exactly k passive vertices, $V^{ac}_G(T)$, $V^{in}_G(T)$ denote the active and passive vertices of T, and $Q^{in}_T[u]$ is the domain of the the iterated pushout product

$$\mathop{\square}_{v \in V^{in}_G(T)} u(T_v)\colon Q^{in}_T[u] \to \bigotimes_{v \in V^{in}_G(T)} Y(T_v).$$

Proof. We first note that, following Definition LIMMOR_REM 6.4, both $\Omega^e_G[k]^{op}$ and $\bar{\Omega}^e_G[k,-]^{op}$ are split Grothendieck constructions over $\Omega^a_G[k]^{op}$. The fibers of these Grothendieck constructions are the cube and punctured cube categories

$$(X \to Y)^{\times V^{in}_G(T)}, \qquad (X \to Y)^{\times V^{in}_G(T)} - Y^{\times V^{in}_G(T)}$$

and thus by computing the left Kan extensions on the leftmost map in (FILTRATION_LAN_SQUARE_DIAGRAM 6.14) iteratively by first left Kan extending to Ω^a_G, we can rewrite that map as

$$\mathrm{Lan}_{\Omega^a_G[k]^{op}}\left(\bigotimes_{v\in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v\in V^{in}_G(T)} u(T_v)\right).$$

The desired description of the leftmost map given in (FILTRATION_LAN_LEVEL 6.16) now follows by noting that the root undercategories $C \downarrow_{\mathsf{r}} \Omega^a_G[k]$ are groupoids. $\square$

6.2 Existence of (semi) model structures

replaced \ F with $\mathcal{F}$ (as opposed to $\mathbb{F}$)

In order to encode the homotopical information inspired by N_∞-operads dicussed in the introduction, we will introduce (semi) model structures on the categories $\mathcal{V}\mathsf{Op}_G$ and $\mathcal{V}\mathsf{Op}^G$ for a wide range of $\mathcal{V}$.

LUIS: these definition of "G-graph subgroup" and "G-vertex family" probably belong earlier in the paper, possibly the introduction. Though it could also make sense to mention them in the intro, and formally define them here

Definition 6.17. A *G-graph subgroup* of $G \times \Sigma_n$ is a subgroup $\Lambda \le G \times \Sigma_n$ such that $\Lambda \cap \Sigma_n = e$. Equivalently, $\Gamma = \Gamma(\phi)$ is the graph of some homomorphism $G \ge H \xrightarrow{\phi} \Sigma_n$.

Definition 6.18. A *G-vertex family* is a collection

$$\mathcal{F} = \coprod_{n\ge 0} \mathcal{F}_n$$

where each $\mathcal{F}_n$ is a *family* of G-graph subgroups of $G \times \Sigma_n$, closed under subgroups and conjugation. For a fixed $\mathcal{F}$, we call an H-set $A \in \mathsf{F}^H$ *$\mathcal{F}$-admissible* if for some (equivalently, any) choice of bijection $A \leftrightarrow \{1,\dots,|A|\}$, the graph subgroup of $G\times\Sigma_n$ encoding the induced H-action on $\{1,\dots,|A|\}$ is in $\mathcal{F}_n$.

Definition 6.19. For any G-vertex family $\mathcal{F}$, a G-tree $T \in \Omega_G$ is called *$\mathcal{F}$-admissible* if, for each vertex $e_1 \dots e_n \le e$ in $V(T)$, the set $\{e_1,\dots,e_n\}$ is an $\mathcal{F}$-admissible $\mathrm{Stab}_G(e)$-set. We let $\Omega_{\mathcal{F}} \subseteq \Omega_G$ and $\Sigma_{\mathcal{F}} \subseteq \Sigma_G$ denote the full subcategories spanned by the $\mathcal{F}$-admissible trees.

We will define $\mathcal{F}$-model structures on $\mathcal{V}\mathsf{Sym}_G$ and $\mathcal{V}\mathsf{Op}_G$ for any G-vertex family $\mathcal{F}$, given the following assumptions about our base category $\mathcal{V}$:

Definition 6.20. We say $\mathcal{V}$ satisfies ASSUMPTION 1 if the following hold:

1. $\mathcal{V}$ is a cofibrantly-generated Cartesian symmetric monoidal model category, and
2. $\mathcal{V}$ is has cellular fixed point functors for all finite grouops (c.f. Ste16 [7]).

Now, fix a G-vertex family $\mathcal{F} = \{\mathcal{F}_n\}$.

Definition 6.21. The *$\mathcal{F}$-projective model structure* on $\mathcal{V}\mathsf{Sym} = \mathcal{V}^{\Sigma^{op}_G}$ is the unique model structure induced by the adjunction

$$\mathcal{V}^{\Sigma^{op}_G} \leftrightarrows \mathcal{V}^{\Sigma^{op}_{\mathcal{F}}}.$$

Explicitly, a map f is a fibration (resp. weak equivalence) if $f(C)$ is one in $\mathcal{V}$ for all $C \in \Sigma_{\mathcal{F}}$.

Definition 6.22. A map $f : \emptyset \to \mathcal{P}$ in $\mathcal{V}\mathsf{Op}_G$ is called a

1. $\mathcal{F}$*-fibration* (resp. $\mathcal{F}$-weak equivalence) if $f(C) : \emptyset(C) \to \mathcal{P}(C)$ is one in $\mathcal{V}$ for all $\mathcal{F}$-admissible G-corollas $C \in \Sigma_{\mathcal{F}}$.
2. $\mathcal{F}$*-cofibration* if it has the left lifting property against all maps which are both $\mathcal{F}$-fibrations and $\mathcal{F}$-weak equivalences.
3. $\Sigma_{\mathcal{F}}$*-cofibration* if f is an $\mathcal{F}$-cofibration in $\mathcal{V}\mathsf{Sym}_G$.
4. *level* $\mathcal{F}$*-cofibration* if $f(C)$ is a cofibration in $\mathcal{V}_{gen}^{\mathrm{Aut}(C)}$ for all $C \in \Sigma_{\mathcal{F}}$.

In particular, $\mathcal{P} \in \mathcal{V}\mathsf{Op}_G$ will be called $\mathcal{F}$*-cofibrant* if $\varnothing \to \mathcal{P}$ is an $\mathcal{F}$-cofibration.

Definition 6.23. The $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}_G$, if it exists, is the unique model structure with the above specified weak equivalences and fibrations. Equivalently, it is the transfered model structure along the adjoints

$$\mathcal{V}\mathsf{Op}_G \underset{\mathsf{fgt}}{\overset{\mathbb{F}_G}{\rightleftarrows}} \mathcal{V}^{\Sigma_G^{op}} \rightleftarrows \prod_{\mathrm{Ob}(\Sigma_G)} \mathcal{V} \rightleftarrows \prod_{\mathrm{Ob}(\Sigma_{\mathcal{F}})} \mathcal{V}$$

Using general arguments of [3], this structure would be cofibrantly generated, with generating arrows

$$I_{\mathcal{F}} = \{\mathbb{F}_G(\Sigma_G(-,C)\cdot i) \mid C \in \Sigma_{\mathcal{F}},\ i \in I\}$$
$$J_{\mathcal{F}} = \{\mathbb{F}_G(\Sigma_G(-,C)\cdot j) \mid C \in \Sigma_{\mathcal{F}},\ j \in J\},$$

for I (resp. J) the generating (trivial) cofibrations of $\mathcal{V}$.

Definition 6.24. If $\mathcal{F}$ is the complete G-vertex family, so $\mathcal{F}_n$ is the family of **all** graph subgroups of $G \times \Sigma_n$, we refer to the $\mathcal{F}$-model structure as the *genuine* model structure.

FGTRIGHT REMARK **Remark 6.25.** We record the standard fact (generalized in 7.14) that if $\phi : \Pi \to \bar{\Pi}$ is a homomorphism of groups, then the induction map

$$\bar{\Pi} \cdot_{\Pi} (-) : \mathcal{V}^{\Pi} \to \mathcal{V}^{\bar{\Pi}}$$

is left Quillen for the genuine model structures.

E_SEMI_VOP_G_THM **Theorem 6.26.** *The genuine semi-model structure on* $\mathcal{V}\mathsf{Op}_G$ *exists for all* $\mathcal{V}$ *satisfying* Assumption 1.

The key technical result is a particular case of the stronger result Proposition 7.64.

X_COFIBRANT_PROP **Proposition 6.27.** *Fix* $T \in \Omega_G$*, and suppose we are given cofibrations* $f(Gv) \in \mathcal{V}_{gen}^{\mathrm{Aut}(T_{Gv})}$ *for all* $Gv \in V_G(T)$*, such that* $f(Gv) = f(\alpha(Gv))$ *for all* $\alpha \in \mathrm{Aut}(T)$*. Then the iterated box product*

$$f^{\square V_G(T)} = \mathop{\square}_{Gv \in V_G(T)} f(Gv)$$

is a cofibration in $\mathcal{V}_{gen}^{\mathrm{Aut}(T)}$.

Proof. Using grafting composition $T \simeq C \circ (T_1^1, \dots, T_1^{k_1}, T_2^1, \dots, T_r^{k_r})$, we go by induction on $|V_G(T)|$. The base cases of $|V_G(T)| = 0$ or 1 are trivial. Now, we note that we have a decomposition

$$f^{\square V_G(T)} = f(Gv_r) \,\square\, \mathop{\square}_{i=1}^{r} \left(\left(f^{\square V_G(T_i)} \right)^{\square k_i} \right)$$

where Gv_r is the root orbit. We will build this map in stages, preserving cofibrancy in each step.

- By induction, for each i, $f^{\square V_G(T_i)}$ is a cofibration in $\mathcal{V}_{gen}^{\mathrm{Aut}(T_i)}$.

- By Proposition 7.37, each $\left(f^{\square V_G(T_i)}\right)^{\square k_i}$ is a cofibration in $\mathcal{V}_{gen}^{\Sigma_{k_i} \wr \mathrm{Aut}(T_i)}$.
- By Remark 7.28, $\underset{i=1}{\square}\left(\left(f^{\square V_G(T_i)}\right)^{\square k_i}\right)$ is a cofibration in $\mathcal{V}_{gen}^{\Pi \Sigma_{k_i} \wr \mathrm{Aut}(T_i)}$.
- By Proposition 7.16, $f(Gv_r)$ is a cofibration in $\mathcal{V}_{gen}^{\mathrm{Aut}(T)}$.
- Finally, by the decomposotion on $\mathrm{Aut}(T)$ and Proposition 7.23, we have the full $f^{\square V_G(T)}$ is a cofibration in $\mathcal{V}_{gen}^{\mathrm{Aut}(T)}$, as desired.

If any $f(Gv)$ were trivial, then, during the appropriate step above, the resulting box product would again be trivial, by the same referenced results. □

Corollary 6.28. *For any cellular extension*

$$\begin{array}{ccc} \mathbb{F}_G X & \longrightarrow & \mathcal{P} \\ {\scriptstyle u}\downarrow & & \downarrow \\ \mathbb{F}_G Y & \longrightarrow & \mathcal{P}[u] \end{array}$$

where $u : X \to Y$ is a genuine level trivial cofibration and $\mathcal{P}$ is genuine level cofibrant in $\mathcal{V}_{gen}^{\Sigma_G^{op}}$, $\mathcal{P} \to \mathcal{P}[u]$ is a level cofibration, trivial if u is.

Proof. By using the levelwise filtration from (6.16), this follows from Proposition 6.27 and Remark 6.25. □

Proof of Theorem 6.26. This follows by general arguments of Kan [3, 11.6.1] from Proposition 6.28, as transfinite compositions are constructed levelwise, and all genuine (trivial) cofibrations in $\mathcal{V}^{\Sigma_G^{op}}$ and $\mathcal{V}\mathsf{Op}_G$ are so levelwise. □

Somewhat surprisingly, the existance of the $\mathcal{F}$-semi-model structure follows from the genuine.

Corollary 6.29. *For any G-vertex system $\mathcal{F}$, the $\mathcal{F}$-semi-model structure on $\mathcal{V}\mathsf{Op}_G$ exists.*

Proof. This follows from Corollary 6.28. Indeed, it suffices to show that if $u : X \to Y$ is a level trivial $\mathcal{F}$-cofibration and $\mathcal{P}$ is level $\mathcal{F}$-cofibrant in $\mathcal{V}^{\Sigma_G^{op}}$, then the cellular extension $\mathcal{P} \to \mathcal{P}[u]$ is a level trivial $\mathcal{F}$-cofibration. However, such u are level genuine cofibrations, and such $\mathcal{P}$ are level genuine cofibrant, and hence $\mathcal{P} \to \mathcal{P}[u]$ is a level trivial genuine cofibration by said corollary, and thus is a level trivial $\mathcal{F}$-cofibration. □

Moreover, if our base category $\mathcal{V}$ is sufficiently nice, these semi model structures are in fact Quillen model structures.

Theorem 6.30. *If genuine cofibrations in $\mathcal{V}^{\Pi}$ are underlying cofibrations in $\mathcal{V}$ for all finite groups Π, and every object in $\mathcal{V}$ is cofibrant, then the $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}_G$ exists.*

Proof. In these cases, the technical hypotheses on $\mathcal{P}$ in Corollary 6.28 are always satisfied. □

Example 6.31. The $\mathcal{F}$-model structure exists on sOp_G for any G-vertex family $\mathcal{F}$.

AUTTCOFPUSH PROP

For G-operads $\mathcal{V}\mathsf{Op}^G$, we will require the full strength and subtly of Proposition 7.64.

Again fixing a G-vertex family $\mathcal{F} = \{\mathcal{F}_n\}$, we identify the following maps in $\mathcal{V}\mathsf{Op}^G$.

_MAPS_DEFINITION

Definition 6.32. A map $f : \emptyset \to \mathcal{P}$ in $\mathcal{V}\mathsf{Op}^G$ is called a

- *$\mathcal{F}$-fibration* (resp. $\mathcal{F}$-weak equivalence) if for all n, $f(n) : \emptyset(n) \to \mathcal{P}(n)$ is one in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$; that is, $f(n)^\Gamma$ is a fibration (resp weak equivalence) in $\mathcal{V}$ for all $\Gamma \in \mathcal{F}_n$.
- *$\mathcal{F}$-cofibration* if it has the left lifting property against all map which are both $\mathcal{F}$-fibrations and $\mathcal{F}$-weak equivalences.
- *$\mathcal{F}$-level-cofibration* if $f(n)$ is a cofibration in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ for all n.

In particular, $\emptyset$ is *$\mathcal{F}$-cofibrant* if $\varnothing \to \emptyset$ is an $\mathcal{F}$-cofibration, where $\varnothing$ is the initial object of $\mathcal{V}$.

Definition 6.33. The *$\mathcal{F}$-model structure* is the unique model structure on $\mathcal{V}\mathsf{Op}^G$, if it exists, with (co)fibrations and weak equivalences as just defined above.

Equivalently, the $\mathcal{F}$-model structure is the unique transfered structure across the adjunction

$$\mathcal{V}\mathsf{Op}^G_{\mathcal{F}} \leftrightarrows \prod \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}.$$

Again, the machinery of Kan says the right hand side has generating (trivial) cofibrations

$$I_{\mathcal{F}} = \{G \times \Sigma(-,n)/\Gamma \cdot i \mid i \in I\}$$
$$J_{\mathcal{F}} = \{G \times \Sigma(-,n)/\Gamma \cdot j \mid j \in J\}$$

so $\mathcal{V}\mathsf{Op}^G_{\mathcal{F}}$ would have generating cofibrations $\mathbb{F}I_{\mathcal{F}} = \{\mathbb{F}(i_{\mathcal{F}}) \mid i_{\mathcal{F}} \in I_{\mathcal{F}}\}$, and similarly for trivial cofibrations.

_OP_SEMI_THEOREM

Theorem 6.34. *Let $\mathcal{V}$ satisfy* Assumption 1*, and let $\mathcal{F}$ be a weak indexing system. Then $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ can be endowed with the $\mathcal{F}$-semi-model structure, where the (co)fibrations are the $\mathcal{F}$-(co)fibrations, and the weak equivalences are the $\mathcal{F}$-weak equivalences. Moreover, $\mathcal{F}$-cofibrations with cofibrant domains are level $\mathcal{F}$-cofibrations.*

CELLULAR_EXTENSION_COFIBRATION_

Proof. Again by the machinery of Kan, this is an immediate corollary of Propositions 6.36 and 6.37, via 7.64. ☐

While this structure is weaker than a Quillen model structure, it is sufficient to give our first proof of the following.

ZATION_COROLLARY

Corollary 6.35. *For $\mathcal{V} = \mathsf{Top}$ and $\mathcal{F}$ any weak indexing system, there exists an operad $N\mathcal{F}$ such that $N\mathcal{F}(n)^\Gamma \simeq *$ if $\Gamma \in \mathcal{F}(n)$, and is empty otherwise. In particular,* $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathbb{I}$ *is an equivalence of categories.*

Proof. Recall that $\mathsf{Comm}(n) = *$ for all n. Consider the functorial factorization

$$\varnothing \to N\mathcal{F} \xrightarrow{\sim} \mathsf{Comm}$$

in $\mathcal{V}\mathsf{Op}^G$ with the $\mathcal{F}$-semi-model structure. Since the initial operad is cofibrant, Theorem G_OP_SEMI_THEOREM 6.34 implies that $\varnothing \to N\mathcal{F}$ is a level $\mathcal{F}$-cofibration, and hence each $N\mathcal{F}(n)$ is cofibrant in $\mathsf{Top}^{G\times\Sigma_n}_{\mathcal{F}_n}$; hence, for all $\Gamma \notin \mathcal{F}_n$, $N\mathcal{F}(n)^\Gamma = \varnothing$. Further, since $N\mathcal{F}$ is $\mathcal{F}$-equivalent to $*$, $N\mathcal{F}(n)^\Gamma \simeq *$ for all $\Gamma \in \mathcal{F}_n$. Hence, each $N\mathcal{F}(n)$ is a universal space for $\mathcal{F}_n$, as desired. ☐

6.2.1 cofibrancy stuff

IBRATION_THEOREM

Proposition 6.36. *Let $\mathcal{F}$ be a weak indexing system, and $\mathcal{V}$ a category satisfying* Assumption 1*. Further, let $\mathcal{P} \in \mathcal{V}\mathsf{Op}^G$ be level $\mathcal{F}$-cofibrant operad, $u : X \to Y$ a cofibration of symmetric sequences $\prod \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$, and $h : \mathbb{F}X \to Y$ a map of operads. Then the cellular extension $\mathcal{P} \to \mathcal{P}[u]$ given by the pushout*

$$\begin{tikzcd} \mathbb{F}X \arrow[r, "h"] \arrow[d, "\mathbb{F}(u)"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}Y \arrow[r] & \mathcal{P}[u] \end{tikzcd}$$

is a level $\mathcal{F}$-cofibration, trivial if u is so.

Proof. We first use the observation that $\mathcal{V}\mathsf{Op}^G_{\{*\}} = \mathcal{V}^G\mathsf{Op}_{\{*\}}$, and hence we can use the filtration (FILTRATION_LAN_LEVEL 6.16). Thus it suffices to show that, for any $T_0 \in \Omega$,

come back: update notations

$$\left(\mathop{\square}_{V_a(T)} \iota_{P(T_v)} \ \square \ [u]^{\square V_p(T)} \right) \otimes_{\mathrm{Aut}(T)} \Sigma_n$$

is a (trivial) cofibration in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$ given our assumptions.

By Lemma FGTRIGHT_PROP 7.14,

$$\Sigma_n \otimes_{\Sigma_{T_0}} (-) : \mathcal{V}^{G\times\Sigma_{T_0}}_{\mathcal{F}_{T_0}} \to \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$$

is left Quillen if and only if $\mathsf{lr}(\Gamma) \in \mathcal{F}_n$ whenever $\Gamma \leq G \times \Sigma_{T_0}$ defines an $\mathcal{F}$-admissible tree with n leaves. Thus, we see this holds precisely when $\mathcal{F}$ is a weak indexing system.

It remains to show that the given map in $\mathcal{V}^{G\times\Sigma_{T_0}}_{\mathcal{F}_{T_0}}$ is a (trivial) cofibration. We observe that this map is a large indexed box product, in particular over a list of cofibrations $f(v) \in \mathcal{V}^{G\times\Sigma_{T_v}}_{\mathcal{F}_{T_v}}$ such that $f(v) = f(\alpha(v))$ for all $v \in V(T)$ and $\alpha \in \mathrm{Aut}(T)$. The fact that this is again a (trivial) cofibration is the content of Proposition AUTTCOFPUSH_PROP 7.64. ☐

OFIBRANT_THEOREM

Corollary 6.37 (cf. [BM03 ?, Corollary 5.2, Proposition 4.3])**.** *The class of level $\mathcal{F}$-cofibrant operads in $\mathcal{V}\mathsf{Op}^G_{\{*\}}$ is closed under cellular extensions. Moreover, if $\emptyset \in \mathcal{V}\mathsf{Op}^G$ is $\mathcal{F}$-cofibrant, then the underlying symmetric sequence is level cofibrant.*

Proof. The main result is an immediate consequence of Theorem CELLULAR_EXTENSION_COFIBRATION_THEOREM 6.36. For the moreover, we recall that any $\mathcal{F}$-cofibrant operad $\emptyset$ can be built out of a retract of a composite of cellular extensions of generating $\mathcal{F}$-cofibrations $u : X \to Y$ of symmetric sequences, starting with the initial operad. As the initial operad is just the initial object of $\mathcal{V}$ in each level, it is level $\mathcal{F}$-cofibrant for all $\mathcal{F}$. Thus, the moreover follows from the main result and the observation that level $\mathcal{F}$-cofibrations are closed under retracts. ☐

7 Cofibrancy

COFIB SEC

HERE

test

7.1 Families of subgroups

This section establishes some useful properties of the model structures associated to families of subgroups. Throughout all groups will be assumed finite.

FAMILY DEF

Definition 7.1. A *family* $\mathcal{F}$ of subgroups of G is a collection of subgroups $H \leq G$ such that

- if $H \in \mathcal{F}$ then $H^g = gHg^{-1} \in \mathcal{F}$ for all $g \in G$;
- if $K \leq H$ and $H \in \mathcal{F}$ then $K \in \mathcal{F}$.

Remark 7.2. Any family determines a full subcategory $\mathsf{O}_{\mathcal{F}} \subset \mathsf{O}_G$ consisting of the orbital G-sets G/H for $H \in \mathcal{F}$.

Furthermore, $\mathsf{O}_{\mathcal{F}}$ has the property of being a sieve of O_G, i.e., for any map $G/K \to G/H$ in O_G such that $G/H \in \mathsf{O}_{\mathcal{F}}$ it is also $G/K \in \mathsf{O}_{\mathcal{F}}$. Moreover, it is straightforward to show that families as in Definition 7.1 are in fact in bijection with such sieves.

Remark 7.3. We note that for any fixed group G, families form a lattice, with the order given by inclusion, and meet and join given by intersection and union, respectively.

We now recall the following fundamental notion and result (cf. [7, Prop. 2.6]).

CELL DEF

Definition 7.4. $\mathcal{V}$ is said to have *cellular fixed points* if:

(i) fixed points $(-)^H$ preserve direct colimits;

(ii) fixed points $(-)^H$ preserve pushouts where one of the legs is $(G/K)\cdot f$, for f a cofibration;

(iii) for each object $A \in \mathcal{V}$, the natural map $(G/K)^H \cdot A \to ((G/K)\cdot A)^H$ is an isomorphism.

Proposition 7.5. *If $\mathcal{V}$ is a cofibrantly generated model category with cellular fixed points, then for any finite group G and family $\mathcal{F}$, there is a model structure $\mathcal{V}^G_{\mathcal{F}}$ on the category $\mathcal{V}^G$, called the* **$\mathcal{F}$-model structure** *such that both weak equivalences and fibrations are determined by the fixed points $(-)^H$ for $H \in \mathcal{F}$.*

We will also make use of the following strengthenings of conditions (ii) (iii) in Definition 7.4.

GENPIXEDPUSH REM

Remark 7.6. If $\mathcal{V}$ is cofibrantly generated, then (i) and (ii) in Definition 7.4 imply that fixed points $(-)^H$ preserve pushouts where one of the legs is a G-genuine cofibration.

Lemma 7.7. *If $\mathcal{V}$ is cofibrantly generated and has cellular fixed points then fixed points $(-)^H$ preserve any pushout diagram where one leg is a G-genuine cofibration.*

Proof. This is an immediate consequence of (i) and (ii) by writing the cofibration as a retract of a transfinite composition of pushouts of generating cofibrations. ☐

FIXEDDIAG LEM

Lemma 7.8. *Suppose $\mathcal{V}$ is strongly cofibrantly generated, has cellular fixed points and is a closed monoidal model category.*

Then for $C, D \in \mathcal{V}^G$ genuinely cofibrant the canonical map

$$C^H \otimes D^H \to (C \otimes D)^H \tag{7.9}$$

FIXEDTENSORISO EQ

is an isomorphism.

Proof. Considering first the case $C = G/H \cdot A$, where A is cofibrant in $\mathcal{V}$, the properties in Definition 7.4 ensure that the D satisfying the condition that (7.9) is an isomorphism are closed under cellular extension, and thus include all cofibrant objects. But now the same argument shows that the C satisfying that (7.9) is an isomorphism for each fixed cofibrant D are themselves closed under cellular extension, and the result follows. ☐

FIXCOF LEM

Lemma 7.10. *If $\mathcal{V}$ is cofibrantly generated and has cellular fixed points then each of the fixed point functors $(-)^H$ preserve cofibrations and pushouts where one of the legs is a cofibration.*

Proof. This follows by Definition 7.4: the case of generating cofibrations follows by (iii), their pushouts follow by (ii) and transfinite compositions by (i). □

PUSHFIX LEM

Lemma 7.11. *Suppose $\mathcal{V}$ is strongly cofibrantly generated, has cellular fixed points and is a closed monoidal model category.*

If f, g are G-genuine cofibrations between G-genuine cofibrant objects, then the canonical map

$$f^H \square g^H \to (f \square g)^H$$

is an isomorphism.

Proof. This follows by combining Lemmas 7.8 and 7.10. □

STROIII LEM

Lemma 7.12. *Suppose that $\mathcal{V}$ is cofibrantly generated and has cellular fixed points and that $A \in \mathcal{V}^H$ is genuinely cofibrant. Then the natural map*

$$(G/H)^K \cdot A^K \to (G \cdot_H A)^K \tag{7.13}$$

STROIII EQ

is an isomorphism.

Proof. The isomorphism (7.13) is obvious in the case of A a set, and thus follows from (iii) in Definition 7.4 for A either the domain or codomain of a generating cofibration $(H/L) \cdot f$. (i) and (ii) in Definition 7.4 now show that the property that (7.13) is an isomorphism is preserved by both pushouts of generating cofibrations and transfinite composition. The result follows by writing $\varnothing \to A$ as a retract of a transfinite composition of pushouts of generating cofibrations. □

We now list some key properties of $\mathcal{F}$-model structure that we will use throughout.

FGTRIGHT PROP

Proposition 7.14. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ as above.*

Then the adjunction

$$\bar{G} \cdot_G (-) : \mathcal{V}^G_{\mathcal{F}} \rightleftarrows \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} : \mathsf{fgt} \tag{7.15}$$

is a Quillen adjunction if for any $H \in \mathcal{F}$ it is $\phi(H) \in \bar{\mathcal{F}}$.

Proof. Since one has a canonical isomorphism of fixed points $(\mathsf{fgt}(X))^H \simeq X^{\phi(H)}$, it is immediate that the right adjoint preserves fibrations and trivial fibrations. □

FGTLEFT PROP

Proposition 7.16. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ as above.*

Then the adjunction

$$\mathsf{fgt}: \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{V}^G_{\mathcal{F}} : \mathsf{Hom}_G(\bar{G}, -) \tag{7.17}$$

is a Quillen adjunction if for any $H \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$.

Proof. Since the double coset formula yields that that

$$\mathsf{fgt}(\bar{G}/H \cdot f) \simeq \mathsf{fgt}(\bar{G}/H) \cdot f \simeq \left(\coprod_{[a] \in \phi(G) \backslash \bar{G}/H} G/\phi^{-1}(H^a) \right) \cdot f$$

it is immediate that the left adjoint preserves cofibrations and trivial cofibrations. □

Propositions 7.14 and 7.16 motivate the following definition.

Definition 7.18. Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{F}$ and $\bar{\mathcal{F}}$ families in G and $\bar{G}$. We define

$$\phi^*(\bar{\mathcal{F}}) = \{H \leq G : \phi(H) \in \bar{\mathcal{F}}\} \tag{7.19}$$

PHISTARDEF EQ

$$\phi_!(\mathcal{F}) = \{\phi(H)^{\bar{g}} \leq \bar{G} : \bar{g} \in \bar{G}, H \in \mathcal{F}\} \tag{7.20}$$

$$\phi_*(\mathcal{F}) = \{\bar{H} \leq \bar{G} : \forall_{\bar{g} \in \bar{G}} \left(\phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F}\right)\} \tag{7.21}$$

PHISTARDEF3 EQ

Lemma 7.22. *The $\phi^*(\bar{\mathcal{F}})$, $\phi_!(\mathcal{F})$, $\phi_*(\mathcal{F})$ just defined are themselves families. Furthermore*

(i) The "if" condition in Proposition FGTRIGHT PROP *7.14 holds iff $\mathcal{F} \subset \phi^*(\bar{\mathcal{F}})$ iff $\phi_!(\mathcal{F}) \subset \bar{\mathcal{F}}$.*

(ii) The "if" condition in Proposition FGTLEFT PROP *7.16 holds iff $\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F}$ iff $\bar{\mathcal{F}} \subset \phi_*(\mathcal{F})$.*

Proof. Since the result is elementary, we include only the proof of the second iff in (ii), which is the hardest step and illustrates the necessary arguments. This follows by the following equivalences.

$$\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F} \Leftrightarrow \left(\underset{H \le G, \phi(H) \in \bar{\mathcal{F}}}{\forall} H \in \mathcal{F} \right) \Leftrightarrow \left(\underset{\bar{H} \in \bar{\mathcal{F}}}{\forall} \phi^{-1}(\bar{H}) \in \mathcal{F} \right) \Leftrightarrow \left(\underset{\bar{H} \in \bar{\mathcal{F}}, \bar{g} \in \bar{G}}{\forall} \phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F} \right) \Leftrightarrow \phi^*(\bar{\mathcal{F}}) \subset \mathcal{F}$$

Note that the second equivalence follows since $H \le \phi^{-1}(\phi(H))$ and $\mathcal{F}$ is closed under subgroups while the third equivalence follows since $\bar{\mathcal{F}}$ is closed under conjugation. $\square$

BIQUILLENG PROP

Proposition 7.23. *Suppose that $\mathcal{V}$ is as above and also a closed monoidal model category. Then the bifunctor*

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^G_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^G_{\mathcal{F} \cap \bar{\mathcal{F}}} \tag{7.24}$$

BIQUILLENG EQ

is a left Quillen bifunctor.

Proof. The double coset formula now yields

$$(G/H \cdot f) \square (G/\bar{H} \cdot g) \simeq (G/H \times G/\bar{H}) \cdot (f \square g) \simeq \left(\coprod_{[a] \in H \backslash G / \bar{H}} G/H \cap \bar{H}^a \cdot (f \square g) \right) \tag{7.25}$$

and hence the result follows since families are closed under conjugation and subgroups. $\square$

EXTERINT DEF

Definition 7.26. Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be families in G and G, respectively.

We defined their *external intersection* to be the family of $G \times \bar{G}$ given by

$$\mathcal{F} \sqcap \bar{\mathcal{F}} = (\pi_G)^*(\mathcal{F}) \cap (\pi_{\bar{G}})^*(\bar{\mathcal{F}})$$

for $\pi_G: G \times \bar{G} \to G$, $\pi_{\bar{G}}: G \times \bar{G} \to \bar{G}$ the projections.

Remark 7.27. Combining Proposition FGTLEFT PROP 7.16 with Propositon BIQUILLENG PROP 7.23 yields that the following composite is a left Quillen bifunctor.

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \xrightarrow{\mathsf{fgt}} \mathcal{V}^{G \times \bar{G}}_{(\pi_G)^*(\mathcal{F})} \times \mathcal{V}^{G \times \bar{G}}_{(\pi_{\bar{G}})^*(\bar{\mathcal{F}})} \xrightarrow{\otimes} \mathcal{V}^{G \times \bar{G}}_{\mathcal{F} \sqcap \bar{\mathcal{F}}} \tag{7.28}$$

EXTERINTADJ EQ

7.2 Pushout powers

That (EXTERINTADJ EQ 7.28) is a left Quillen bifunctor (and its obvious higher order analogues) is one of the key properties of pushout products of $\mathcal{F}$ cofibrations when those cofibrations (and well as the group) are allowed to change. However, when those cofibrations (and hence also G) coincide there is an additional symmetric group action that must be considered.

To handle such actions we introduce the following axiom

Definition 7.29. We say $\mathcal{V}$ has *cofibrant symmetric pushout powers* if for each cofibration f the pushout product $f^{\square n}$ is a genuine Σ_n-cofibration.

Definition 7.30. Let $\mathcal{F}$ be a family of G, E a finite set and $e \in E$ any fixed element.

We define the *n-th semidirect power of* $\mathcal{F}$ to the family of $\Sigma_E \wr G = \Sigma_E \ltimes G^{\times E}$ given by

$$\mathcal{F}^{\ltimes E} = \left(\iota_{\Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G}\right)_* \left((\pi_G)^*(\mathcal{F}) \right), \tag{7.31}$$

FLTIMESN EQ

where ι is the inclusion $\Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to \Sigma_E \wr G$ and π is the projection $\Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to G$.

More explicitly, noting that in (7.21) one needs only consider conjugates by coset representatives of $\bar{G}/\phi(G)$, it follows that when computing ι one needs only consider conjugates by permutations of E not fixing e, so that one has that

$$K \in \mathcal{F}^{\ltimes E} \text{ iff } \underset{e\in E}{\forall}\, \pi_G\left(K \cap \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G\right) \in \mathcal{F}, \tag{7.32}$$

showing that in particular the definition in (7.31) is independent of the choice of $e \in E$.

Proposition 7.33. *One has*

$$\mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}} \subset \iota^*(\mathcal{F}^{\ltimes E \amalg \bar{E}}). \tag{7.34}$$

Hence, the following is a left Quillen bifunctor.

$$\Sigma_{E\amalg\bar{E}} \underset{\Sigma_E\times\Sigma_{\bar{E}}}{\cdot} (-\otimes -)\colon \mathcal{V}^{\Sigma_E \wr G} \times \mathcal{V}^{\Sigma_{\bar{E}}\wr G} \to \mathcal{V}^{\Sigma_{E\amalg\bar{E}}\wr G} \tag{7.35}$$

Proof. Let $K \in \mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}}$ and $e \in E$. Since

$$\pi_G\left(K \cap \Sigma_{\{e\}} \times G \times \Sigma_{E\amalg\bar{E}-e} \wr G\right) = \pi_G\left(\pi_{\Sigma_E \wr G}(K) \cap \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G\right), \tag{7.36}$$

it follows that K satisfies (7.32) for $\mathcal{F}^{\ltimes E\amalg\bar{E}}$ since $\pi_{\Sigma_E\wr G}(K)$ does so for $\mathcal{F}^{\ltimes E}$. The case of $e \in \bar{E}$ is identical.

(7.35) simply combines (7.28) and Proposition 7.14. □

Proposition 7.37. *Suppose that $\mathcal{V}$ has cellular fixed points and is a closed monoidal model category. Then, for all n and cofibration (resp. trivial cofibration) f of $\mathcal{V}^G_{\mathcal{F}}$ one has that $f^{\square n}$ is a cofibration (resp. trivial cofibration) of $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$.*

Our proof of Proposition 7.37 will be essentially a repetition of the main argument in the proof of [5, Thm. 1.2]. However, both for the sake of completeness and to stress that the argument is independent of the (fairly technical) model structures in [5], we include an abridged version of the proof below, the key ingredient of which is that (7.35) is a left Quillen bifunctor.

Proof. We first note that in the case of $i = (G/H)\cdot\bar{i}$, $H \in \mathcal{F}$, a generating (trivial) cofibration it is $i^{\square n} = (G/H)^{\times n}\cdot \bar{i}^{\square n}$, and thus this case follows since the $\Sigma_n \wr G$-orbits of $(G/H)^{\times n}$ are in $\mathcal{F}^{\ltimes n}$.

For the general case, we start by making the key observation that for composable arrows $\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ the n-fold pushout product $(hg)^{\square n}$ has a factorization

$$\bullet \xrightarrow{k_0} \bullet \xrightarrow{k_1} \cdots \xrightarrow{k_n} \bullet \tag{7.38}$$

where each k_i, $0 \le i \le n$, fits into a pushout product

$$\begin{array}{ccc} \bullet & \longrightarrow & \bullet \\ {\scriptstyle \Sigma_n \underset{\Sigma_{n-i}\times\Sigma_i}{\cdot}\left(g^{\square n-i}\square h^{\square i}\right)}\big\downarrow & \ulcorner & \big\downarrow {\scriptstyle k_i} \\ \bullet & \longrightarrow & \bullet. \end{array} \tag{7.39}$$

Briefly, (7.38) follows from suitable Σ_n-symmetric convex subposets $P_0 \subset P_1 \subset \cdots \subset P_n$ of the poset $P_n = (0 \to 1 \to 2)^{\times n}$ where P_0 consists of "tuples with at least one 0-coordinate" and P_i is obtained from P_{i-1} by adding the "tuples with $n-i$ 1-coordinates and i 2-coordinates". Additional details concerning this filtration appear in the proof of [5, Lemma 4.8].

The general proof now follows by writing f as a retract of a transfinite composition of pushouts of generating (trivial) cofibrations. As usual, retracts can be ignored, and we can hence assume that there is an ordinal κ and $X_\bullet\colon \kappa \to \mathcal{V}^G$ such that (i) $f_\beta\colon X_\beta \to X_{\beta+1}$ is the pushout of a (trivial) cofibration i_β; (ii) $\operatorname{colim}_{\alpha<\beta} X_\alpha \xrightarrow{\simeq} X_\beta$ for limit ordinals $\beta < \kappa$; (iii) setting $X_\kappa = \operatorname{colim}_{\beta<\kappa} X_\beta$, f equals the transfinite composite $X_0 \to X_\kappa$.

We argue by transfinite induction on κ. Writing $\bar{f}_\beta : X_0 \to X_\beta$ for the partial composites, it suffices to check that the natural transformation of κ-diagrams (rightmost map not included)

$$\begin{tikzcd}
Q^n(\bar{f}_1) \arrow[r] \arrow[d, "\bar{f}_0^{\square n}"'] & Q^n(\bar{f}_2) \arrow[r] \arrow[d, "\bar{f}_1^{\square n}"'] & Q^n(\bar{f}_3) \arrow[r] \arrow[d, "\bar{f}_2^{\square n}"'] & Q^n(\bar{f}_4) \arrow[r] \arrow[d, "\bar{f}_3^{\square n}"'] & \cdots \arrow[r] & Q^n(\bar{f}_\kappa) \arrow[d, "\bar{f}_\kappa^{\square n} = \mathrm{colim}_{\beta<\kappa} \bar{f}_\beta^{\square n}"] \\
X_1^{\otimes n} \arrow[r] & X_2^{\otimes n} \arrow[r] & X_3^{\otimes n} \arrow[r] & X_4^{\otimes n} \arrow[r] & \cdots \arrow[r] & X_\kappa^{\otimes n},
\end{tikzcd}$$

is κ-cofibrant, i.e. that the maps $Q^n(\bar{f}_\beta) \amalg_{\mathrm{colim}_{\alpha<\beta} Q^n(\bar{f}_\alpha)} \mathrm{colim}_{\alpha<\beta} X_\alpha^{\otimes n} \to X_\beta^{\otimes n}$ are cofibrations in $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$. Condition (ii) above implies that this map is an isomorphism for β a limit ordinal and the map $Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n} \to X_{\beta+1}^{\otimes n}$ for $\beta+1$ a successor. But since $Q^n(\bar{f}_{\beta+1}) \to Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n}$ is precisely the map k_0 of (7.38) for $g = \bar{f}_\beta$, $h = f_\beta$, this last map is the composite $k_n k_{n-1} \cdots k_1$ so that the result now follows from (7.39) combined with (7.35), the induction hypothesis applied to $\bar{f}_\beta$, the fact that $f_\beta^{\square k}$ is a pushout of $i_\beta^{\square k}$ (cf. [5, Lemma 4.11]) and the cofibrancy of $i_\beta^{\square k}$ proven at the beginning. $\square$

We will also need to understand the fixed points of $f^{\square n}$ for general subgroups $K \le \Sigma_n \wr G$.

To do so recall first that $f^{\square n}$ can be built from the composite

$$f^{\otimes n} : (0 \to 1)^{\times n} \xrightarrow{f^{\times n}} \mathcal{V}^{\times n} \xrightarrow{\otimes} \mathcal{V}$$

as the map

$$\mathrm{colim}_{(0\to 1)^{\times n} - (1,\cdots,1)} f^{\otimes n} \to Y^{\otimes n}.$$

It then follows that K acts on the diagram category $(0 \to 1)^{\times n}$ itself (via the composite $K \to \Sigma_n \wr G \to \Sigma_n$). The fixed diagram subcategory $\left((0 \to 1)^{\times n}\right)^K$ consists of those tuples in $\{0,1\}^n$ whose coordinates coincide if their indexes are in the same coset of n/K, i.e. there is an identification $\left((0 \to 1)^{\times n}\right)^K \simeq (0 \to 1)^{\times n/K}$.

Example 7.40. When $n = 3$ and $n/K = \{\{1,2\},\{3\}\}$ $(0 \to 1)^{n/K}$ is identified with the subposet on the right below.

$$\begin{tikzcd}[column sep=small]
& 000 \arrow[rr] \arrow[dl] \arrow[dd, dashed] & & 010 \arrow[dl] \arrow[dd] \\
100 \arrow[rr] \arrow[dd] & & 110 \arrow[dd] & \\
& 001 \arrow[rr, dashed] \arrow[dl, dashed] & & 011 \arrow[dl] \\
101 \arrow[rr] & & 111 &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
000 \arrow[dr] \arrow[dd] & \\
& 110 \arrow[dd] \\
001 \arrow[dr] & \\
& 111
\end{tikzcd}$$

It will be key for our purposes to know that fixed points $\left(f^{\square n}\right)^K$ can be computed by first restricting to the smaller cube $(0 \to 1)^{\times n/K}$, resulting in a cube of objects with K-actions, and then computing a pushout over that smaller cube (either before or after taking fixed points of each level). The formal result follows.

Proposition 7.41. *Suppose that $\mathcal{V}$ has cellular fixed points and is a closed monoidal model category with symmetric pushout powers. Let $K \le \Sigma_n \wr G$ be a subgroup, $f : X \to Y$ a map in $\mathcal{V}$ and consider the natural maps (in the arrow category)*

$$\mathop{\square}_{[i] \in n/K} \left(f^{\otimes [i]}\right)^K \to \left(f^{\square n}\right)^K. \tag{7.42}$$

FIXEDPUSH EQ

If f is a cofibration between cofibrant objects then all maps in (7.42) are isomorphisms.

Proof. The result will follow by induction on n. The base case $n = 1$ is obvious.

Moreover, it is obvious that (7.42), which is a map of arrows, is an isomorphism on the target objects, hence the real claim is that this map is also an isomorphism on sources.

We now note that by considering (7.38) for $g = \varnothing \to X$, $h = f$ and removing the last map k_n one obtains a filtration of the source of $f^{\square n}$. Applying $(-)^K$ to the leftmost map in (7.39) one thus obtains isomorphisms

$$\left(\Sigma_n \underset{\Sigma_{n-i}\times\Sigma_i}{\cdot} \left(X^{\otimes n-i} \otimes f^{\square i}\right)\right)^K \simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A} \otimes f^{\square B}\right)^K \simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A}\right)^K \otimes \left(f^{\square B}\right)^K$$

$$\simeq \coprod_{\substack{\underline{n}/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(\bigotimes_{[j]\in A/K} \left(X^{\otimes [j]}\right)^K\right) \otimes \left(\mathop{\square}_{[k]\in B/K} \left(f^{\otimes [k]}\right)^K\right)$$

where the first step is an instance of Lemma 7.12, the second step an instance of Lemma 7.11, and the last step follows by combining Lemma 7.8 with the induction hypothesis (since $|B| \le i < n$).

We have thus shown that the leftmost map in the pushouts (7.39) for $\left(f^{\square n}\right)^K$ is isomorphic to the leftmost map in the corresponding pushout for $\mathop{\square}_{[i]\in n/K} \left(f^{\otimes [i]}\right)^K$, and the result now follows. $\square$

7.3 G-graph families and trees

We note that in this section we use Σ to denote a general group.

Definition 7.43. A subgroup $\Gamma \le G \times \Sigma$ is called a *G-graph subgroup* if $\Gamma \cap \Sigma = *$.

Further, a family $\mathcal{F}$ of $G \times \Sigma$ is called a *G-graph family* if it consists only of G-graph subgroups.

Remark 7.44. One can show that Γ is a G-graph subgroup iff it can be written as

$$\Gamma = \{(k, \varphi(k)) : k \in K \le G\}$$

for some partial homomorphism $G \ge K \xrightarrow{\varphi} \Sigma$.

Remark 7.45. The collection of all G-graph subgroups is itself a family. Indeed, it coincides with $(\iota_\Sigma)_*(\{*\})$ for the inclusion homomorphism $\iota_\Sigma \colon \Sigma \to G \times \Sigma$.

Letting $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G \times \Sigma$ and $G \times \bar{\Sigma}$ we will write

$$\mathcal{F} \sqcap_G \bar{\mathcal{F}} = \Delta^*(\mathcal{F} \sqcap \bar{\mathcal{F}}) \qquad \mathcal{F}^{\ltimes_G n} = \Delta^*(\mathcal{F}^{\ltimes n})$$

where Δ denotes either of the diagonal inclusions $\Delta \colon G\times\Sigma\times\bar{\Sigma} \to G\times\Sigma\times G\times\bar{\Sigma}$ or $\Delta \colon G\times\Sigma_n\wr\Sigma \to \Sigma_n \wr (G\times\Sigma)$.

Remark 7.46. Unpacking Definition 7.26 one has that $\Gamma \in \mathcal{F} \sqcap_G \bar{\mathcal{F}}$ iff $\pi_{G\times\Sigma}(\Gamma) \in \mathcal{F}$, $\pi_{G\times\bar{\Sigma}}(\Gamma) \in \bar{\mathcal{F}}$.

Remark 7.47. Unpacking (7.32) and noting that

$$(G \times \Sigma_E \wr \Sigma) \cap \left(\Sigma_{\{e\}} \times (G\times\Sigma) \times \Sigma_{E-\{e\}} \wr (G\times\Sigma)\right) = G \times \Sigma_{\{e\}} \times \Sigma \times \Sigma_{E-\{e\}} \wr \Sigma$$

one has

$$K \in \mathcal{F}^{\ltimes_G E} \text{ iff } \underset{e\in E}{\forall}\, \pi_{G\times\Sigma}\left(K \cap G \times \Sigma_{\{e\}} \times \Sigma \times \Sigma_{E-e} \wr \Sigma\right) \in \mathcal{F}. \tag{7.48}$$

Combining either of (7.28) or Proposition 7.37 with Proposition 7.16 yields the following results.

Proposition 7.49. *Let $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G\times\Sigma$ and $G\times\bar{\Sigma}$. Then the following (with diagonal G-action on the images) is a left Quillen bifunctor.*

$$\mathcal{V}^{G\times\Sigma}_{\mathcal{F}} \times \mathcal{V}^{G\times\bar{\Sigma}}_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Sigma\times\bar{\Sigma}}_{\mathcal{F}\sqcap_G\bar{\mathcal{F}}} \tag{7.50}$$

POWERFG PROP **Proposition 7.51.** *Suppose $\mathcal{V}$ has cellular fixed points and is a closed monoidal model category.*

Let $\mathcal{F}$ be a G-graph family of $G \times \Sigma$. If f is a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma}_{\mathcal{F}}$ then so is $f^{\square n}$ a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma_n \wr \Sigma}_{\mathcal{F}^{\ltimes_G n}}$.

Remark 7.52. While it is easy to check that $\mathcal{F} \sqcap_G \bar{\mathcal{F}}$ is indeed a G-graph family of $G \times \Sigma \times \bar{\Sigma}$, note that it is *not* the case that $\mathcal{F}^{\ltimes_G n}$ is a G-graph family of $G \times \Sigma_n \wr \Sigma$, due to the need to consider the power Σ_n-action.

The G-graph families we will be interested in will encode certain families of G-trees. We start by with the case of corollas.

Definition 7.53. A *family* $\Sigma_{\mathcal{F}}$ of G-corollas is a sieve $\Sigma_{\mathcal{F}} \subset \Sigma_G$, i.e., a full subcategory such that for any morphism $T \to T'$ with $T' \in \Sigma_{\mathcal{F}}$ it is also $T \in \Sigma_{\mathcal{F}}$.

Equivalenty, the data of a family of corollas $\Sigma_{\mathcal{F}}$ consists of G-graph families $\mathcal{F}_n$ of $G \times \Sigma_n$ for each $n \geq 0$ with $C \in \Sigma_{\mathcal{F}}$ precisely if $C \simeq G \cdot_H C_H$ for $C_H \in \Sigma_n^H$ an H-equivariant corolla encoded by a partial homomorphism $G \geq H \to \Sigma_n$ encoding a subgroup in $\mathcal{F}_n$ (cf. Remark GRAPH REM 7.44).

Since $\Sigma_{\mathcal{F}}$ is determined by the families $\{\mathcal{F}_n\}_{n\geq 0}$, we will abuse notation and abbreviate both sets of data simply as $\mathcal{F}$ (alternatively, the reader can think of $\mathcal{F}$ as a "family in the groupoid Σ of finite sets").

Definition 7.54. Let $\Sigma_{\mathcal{F}}$ be a family of G-corollas.

We say that a G-tree T is an *$\mathcal{F}$-tree* if all of its G-vertices T_{v_G} are in $\Sigma_{\mathcal{F}}$.

VACUOUSNESS REM **Remark 7.55.** Note that by vacuousness the stick G-trees $G \cdot_H \eta$ are always $\mathcal{F}$-trees.

Proposition 7.56. *Let $\mathcal{F}$ be a family of G-corollas and $T \in \Omega$ a tree with automorphism group Σ_T.*

Let $\mathcal{F}_T$ be the collection of graph subgroups of $G \times \Sigma_T$ encoded by partial homomorphisms $G \leq H \to \Sigma_T$ such that the associated G-tree $G \cdot_H T$ is a $\mathcal{F}$-tree.

Then $\mathcal{F}_T$ is a G-graph family.

Proof. Closure under conjugation follows since conjugate graph subgroups produce isomorphic G-trees. As for subgroups, they are encoded by restrictions $K \leq H \to \Sigma_T$ which induce quotient maps $G \cdot_K T \to G \cdot_H T$, so that any vertex of $G \cdot_K T$ maps in Σ_G to some vertex of $G \cdot_H T$. ☐

UNPACKFTYPE REM **Remark 7.57.** Unpacking definitions, one sees that a partial homomorphism $G \geq H \to \Sigma_T$ encodes a subgroup in $\mathcal{F}_T$ iff, for each non-leaf edge $e \in T$ with H-isotropy $H_e \leq H$, the induced homomorphism

$$H_e \to \Sigma_{T_{e^{\uparrow}\leq e}} \simeq \Sigma_{|e^{\uparrow}|} \tag{7.58}$$ PARTIALHOMEDGE EQ

encodes a subgroup in $\mathcal{F}_{|e^{\uparrow}|}$.

TREEINDUCDESC REM **Remark 7.59.** Recall that any tree $T \in \Omega$ other than the stick η has an essentially unique grafting decomposition $T = (T_1, \cdots, T_n)$. Further, let λ be the partition $\{1, \cdots, n\} = \lambda_1 \amalg \cdots \amalg \lambda_k$ such that $1 \leq i_1, i_2 \leq n$ are in the same class iff $T_{i_1}, T_{i_2} \in \Omega$ are isomorphic.

Writing $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$ and picking representatives $i_j \in \lambda_j$ one then has isomorphisms

$$\Sigma_T \simeq \Sigma_\lambda \wr \prod_i \Sigma_{T_i} \simeq \Sigma_{|\lambda_1|} \wr \Sigma_{T_{i_1}} \times \cdots \times \Sigma_{|\lambda_k|} \wr \Sigma_{T_{i_k}} \tag{7.60}$$ TREEISOT EQ

where the second isomorphism, while not canonical (it depends on choices of isomorphisms $T_{i_j} \simeq T_l$ for each $i_j \neq l \in \lambda_j$) is nonetheless well defined up to conjugation.

The following, which is the core result in this section, is a reinterpretation of Remark UNPACKFTYPE REM 7.57 in light of the inductive description of trees in Remarks TREEINDUCDESC REM 7.59.

KEYLEMMAGECO LEM **Lemma 7.61.** *Let $\Sigma_{\mathcal{F}}$ be a family of G-corollas and $T \in \Omega$ a tree other than η. Then*

$$\mathcal{F}_T = (\pi_{G\times\Sigma_n})^* (\mathcal{F}_n) \cap \left(\mathcal{F}_{T_{i_1}}^{\ltimes_G |\lambda_1|} \sqcap_G \cdots \sqcap_G \mathcal{F}_{T_{i_k}}^{\ltimes_G |\lambda_k|} \right), \tag{7.62}$$ KEYLEMMAGECO EQ

where $\pi_{G\times\Sigma_n}$ denotes the composite $G \times \Sigma_T \to G \times \Sigma_\lambda \to G \times \Sigma_n$.

Proof. The argument is by induction on the decomposition $T = (T_1, \cdots, T_n)$ with the base case, that of a corolla, being immediate.

Consider a partial homomorphism $G \geq H \to \Sigma_T$ encoding a G-graph subgroup Γ of $G \times \Sigma_T$.

The condition that $\Gamma \in (\pi_{G\times\Sigma_n})^*(\mathcal{F}_n)$ states that the composite $H \to \Sigma_T \to \Sigma_n$ is in $\mathcal{F}_n$, and this is precisely (7.58) when $e = r$ is the root of T.

As for the condition $\Gamma \in \left(\mathcal{F}^{\ltimes_G |\lambda_1|}_{T_{i_1}} \sqcap_G \cdots \sqcap_G \mathcal{F}^{\ltimes_G |\lambda_k|}_{T_{i_k}}\right)$, by unpacking it by combining Remark 7.46 and (7.48), this translates to the condition that, for each $i \in \{1, \cdots, n\}$, one has

$$\pi_{G\times\Sigma_{T_i}} \left(\Gamma \cap G \times \Sigma_{\{i\}} \times \Sigma_{T_i} \times \Sigma_{\lambda - \{i\}} \wr \prod_{j \neq i} \Sigma_{T_j} \right) \in \mathcal{F}_{T_i} \tag{7.63}$$ KEYLEMMAGECOR EQ

where $\lambda - \{i\}$ denotes the induced partition of $\{1, \cdots, n\} - \{i\}$. Noting that the intersection subgroup appearing inside $\pi_{G\times\Sigma_{T_i}}$ in (7.63) can be rewritten as $\Gamma \cap \pi^{-1}_{\Sigma_n}(\Sigma_{\{i\}} \times \Sigma_{\{1,\cdots,n\}-\{i\}})$, we see that this is the graph subgroup encoded by the restriction $H_i \leq H \to \Sigma_T$, where H_i is the isotropy subgroup of the root r_i of T_i (equivalently, this is also the subgroup sending T_i to itself). But since for any edge $e \in T_i$ its isotropy H_e (cf. (7.58) is a subgroup of H_i, the induction hypothesis implies that (7.63) is equivalent to condition (7.58) across all non-leaf edges other than the root $r \in T$.

The paragraphs above show that (7.62) indeed holds when restricted to G-graph subgroups. However, it still remains to show that any group Γ in the right family in (7.62) indeed satisfies $\Gamma \cap \Sigma_T = *$, or in other words, that any element $\gamma \in \Gamma \leq G \times \Sigma_\lambda \wr \prod_i \Sigma_{T_i}$ with G-coordinate $\gamma_G = e$ is indeed the identity. But the condition $\pi_{G\times\Sigma_n}(\Gamma) \in \mathcal{F}_n$ now implies that the Σ_λ-coordinate is $\gamma_{\Sigma_\lambda} = e$ and thus (7.63) in turn implies that the Σ_{T_i}-coordinates are $\gamma_{\Sigma_{T_i}} = e$, finishing the proof. □

The results in this section now combine to yield the following.

AUTTCOFPUSH PROP **Proposition 7.64.** *Let $\Sigma_{\mathcal{F}}$ be a family of corollas and suppose that $f\colon A \to B$ is a $\mathcal{F}$-cofibration (resp. trivial cofibration) in $\mathsf{Sym}^G(\mathcal{V})$, i.e. that $f(r)\colon A(r) \to B(r)$ are cofibrations (resp. trivial cofibrations) in $\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$. Then for any tree $T \in \Omega$ the map*

$$f^{\square V(T)} = \mathop{\square}_{v \in V(T)} f(v)$$

is a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T}$.

Proof. This follows by induction on the decomposition $T = (T_1, \cdots, T_n)$ with the base cases of corollas and η being immediate. (7.62) combined with (7.50), (7.23) and Proposition 7.16 yields that

$$\mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n} \times \mathcal{V}^{G\times\Sigma_{|\lambda_1|}\wr\Sigma_{T_{i_1}}}_{\mathcal{F}^{\ltimes_G|\lambda_1|}_{T_{i_1}}} \times \cdots \times \mathcal{V}^{G\times\Sigma_{|\lambda_k|}\wr\Sigma_{T_{i_k}}}_{\mathcal{F}^{\ltimes_G|\lambda_k|}_{T_{i_k}}} \xrightarrow{\otimes} \mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T}$$

is a left Quillen multifunctor. The result now follows by Proposition 7.51. □

7.4 Indexing systems

The primary purpose of the notion of $\mathcal{F}$-tree is to classify notions of "partial genuine operads", by which we mean genuine operads whose mapping objects are only defined for some of corollas in Σ_G.

In practice, the fact that genuine operads possess a unit and multiplication requires $\Sigma_{\mathcal{F}}$ to satisfy an additional closure condition that we now introduce.

Luis: don't we need that $\Sigma_{\mathcal{F}}$ is a *sieve*? I.e. that it has restrictions?

INDEXSYS DEF **Definition 7.65.** $\Sigma_{\mathcal{F}}$ is called a *weak indexing system* if for any $\mathcal{F}$-tree T it is $\mathsf{lr}(T) \in \Sigma_{\mathcal{F}}$.

Additionally, $\Sigma_{\mathcal{F}}$ is called simply an *indexing system* if all trivial corollas $(G/H)\cdot C_n$ are in $\Sigma_{\mathcal{F}}$.

Remark 7.66. In light of Remark 7.55 any weak coefficient system must contain the 1-corollas $(G/H) \cdot C_1$.

The notion of indexing system in Definition 7.65 matches the eponymous notion first introduced in [2, Def. 3.22], though packaged quite differently. To justify this claim, we need the following lemma.

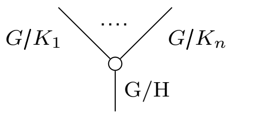
Lemma 7.67. *The condition in Definition 7.65 holds for all equivariant trees T with at least one inner edge orbit provided that it holds for those T with precisely one inner edge orbit.*

Proof. If T containing $n > 0$ edge orbits, one can build a sequence of inner face inclusions

$$\mathsf{lr}(T) = T_0 \to T_1 \to \cdots \to T_n = T$$

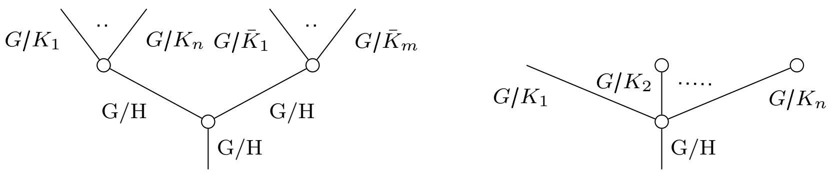
where each inclusion $T_i \to T_{i+1}$ misses exactly one inner edge orbit. The condition in the statement implies by downwards induction that all T_i will be $\mathcal{F}$-trees whenever T is. □

Remark 7.68. In the language of [2], the isomorphism classes $\Sigma_{\mathcal{F}}$ are instead encoded by H-sets. Namely, a H-set with orbital decomposition $K_1/H \amalg \cdots \amalg K_n/H$ corresponds to a G-tree with orbital representation as follows.



(7.69)

Under the assumption that all trivial H-sets are admissible (including the empty ones), the closure condition in Definition 7.65 implies that such H-sets are closed under both disjoint unions and subobjects. Indeed, this follows by taking lr of G-trees of the following forms.



(7.70)

It hence follows that, when in the presence of all trivial H-sets and if assuming closure under disjoint unions and subobjects, the third condition in Definition 7.65 needs only be stated for trees with a single root orbit, a single inner edge orbit (in fact, we can further assume a single leaf edge orbit), showing that under those assumptions the closure condition in Definition 7.65 is indeed equivalent to the self-induction condition of [2].

Remark 7.71. The closure condition in Definition 7.65 can be translated in terms of families as saying that for any tree $T \in \Omega$ and letting $\phi \colon \Sigma_T \to \Sigma_{\mathsf{lr}(T)}$ be the natural homomorphism, one has $(G \times \phi)(\Gamma) \in \mathcal{F}_{\mathsf{lr}(T)}$ for any $\Gamma \in \mathcal{F}_T$. Proposition 7.14 then says that

$$\Sigma_{\mathsf{lr}(T)} \cdot_{\Sigma_T} (-) \colon \mathcal{V}^{G \times \Sigma_T}_{\mathcal{F}_T} \to \mathcal{V}^{G \times \mathsf{lr}(T)}_{\mathcal{F}_{\mathsf{lr}(T)}}$$

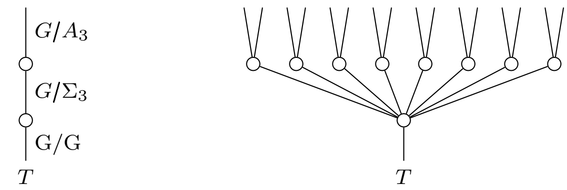
is a left Quillen functor.

Remark 7.72. When $G = *$, Proposition 7.64 coincides with [1, Lemma 5.9]. Moreover, it is not hard to leverage that non-equivariant result to provide a proof of Proposition 7.64 for the universal family Σ_G of all G-corollas. However, the reader of [1] may note that the proof therein is technically less involved, using no analogue of the rather subtle $\mathcal{F}^{\ltimes G^n}$ families. Indeed, this is reflected on the last paragraph of our proof of Lemma 7.61, which effectively uses the subtle condition (7.63) to deduce the much simpler condition $\Gamma \cap \prod \Sigma_{T_i} = *$ that would suffice for the direct generalization of [1, Lemma 5.9] suggested above. It it thus natural to wonder if a suitably simpler analogue of (7.63) might have been used, with the most natural candidate being the condition

$$\pi_{G\times\Sigma_{T_i}}(\Gamma \cap \prod \Sigma_{T_i}) \in \mathcal{F}. \tag{7.73}$$

WRONGCONJ

However, the following example shows that (7.73) is insufficient. Let $G = \Sigma_3 \wr \mathbb{Z}_{/2}$ and $\mathcal{F}$ be the indexing system generated (described in the H-set language of [2]) by the G-set G/Σ_3. Explicitly, all orbital sets in $\mathcal{F}$ can be built via self inductions of restrictions of G/Σ_3. Now consider the G-tree T (with unlabeled expanded representation on the right) below.



(7.74)

T easily satisfies (7.73) since no element of G fixes the lower corolla. On the other hand, $\Sigma_3/A_3 \notin \mathcal{F}$ since it is not a restriction of G/Σ_3 and indecomposable sets of size 2 can not be inductions. Indeed, it must also be $G/A_3 \notin \mathcal{F}$ since then it would be possible to build G/A_3^g (for some possibly non trivial $g \in G$) using a G-tree with lower corolla as in T, and it is now clear that T is the only possible such tree.

8 Model Structures on Genuine Operads

come back: this is all disorganized, internally, externally, everything

replaced \ F with $\mathcal{F}$ (as opposed to $\mathbb{F}$)

In order to encode the homotopical information inspired by N_∞-operads dicussed in the introduction, we will introduce (semi) model structures on the categories $\mathcal{V}\mathsf{Op}_G$ and $\mathcal{V}\mathsf{Op}^G$ for a wide range of $\mathcal{V}$, which are true model categories in the cases we are most interested in (i.e. for $\mathcal{V} = \mathsf{sSet}$). These model structures will be determined by a choice of weak indexing system, a generalization of the notion defined in [2].

8.1 Weak Indexing Systems

We recall certain constructions found in [6] relating graph subgroups, finite H-sets, and systems of categories.

Definition 8.1. A *G-graph subgroup* of $G \times \Sigma_n$ is a subgroup $\Lambda \leq G \times \Sigma_n$ such that $\Lambda \cap \Sigma_n = e$. Equivalently, $\Gamma = \Gamma(\phi)$ is the graph of some homomorphism $G \geq H \xrightarrow{\phi} \Sigma_n$.

Definition 8.2. A *G-vertex family* is a collection

$$\mathcal{F} = \coprod_{n \geq 0} \mathcal{F}_n$$

where each $\mathcal{F}_n$ is a *family* of G-graph subgroups of $G \times \Sigma_n$, closed under subgroups and conjugation. For a fixed $\mathcal{F}$, we call an H-set $A \in \mathsf{F}^H$ *$\mathcal{F}$-admissible* if for some (equivalently, any) choice of bijection $A \leftrightarrow \{1, \dots, |A|\}$, the graph subgroup of $G \times \Sigma_n$ encoding the induced H-action on $\{1, \dots, |A|\}$ is in $\mathcal{F}_n$.

Definition 8.3. For any G-vertex family $\mathcal{F}$, a G-tree $T \in \Omega_G$ is called *$\mathcal{F}$-admissible* if, for each vertex $e_1 \dots e_n \leq e$ in $V(T)$, the set $\{e_1, \dots, e_n\}$ is an $\mathcal{F}$-admissible $\mathrm{Stab}_G(e)$-set. We let $\Omega_\mathcal{F} \subseteq \Omega_G$ and $\Sigma_\mathcal{F} \subseteq \Sigma_G$ denote the full subcategories spanned by the $\mathcal{F}$-admissible trees.

Definition 8.4. A G-vertex family is called a *weak indexing system* if $\Omega_\mathcal{F}$ is a *sieve* of Ω_G; that is, for any map $f : S \to T$ with $T \in \Omega_\mathcal{F}$, we have that S (and the map f) are in $\Omega_\mathcal{F}$.

We note that this always holds for any $\mathcal{F}$ is f is an outer face or a quotient (for the latter, this follows from each $\mathcal{F}_n$ being closed under subgroups). However, closure under degeneracies implies that all trivial orbits H/H are $\mathcal{F}$-admissible, while closure under inner faces implies that the $\mathcal{F}$-admissible sets are closed under "broad self-induction": if $A \amalg H/K$ and B are $\mathcal{F}$-admissible H- and K-sets, respectively, then $A \amalg H \times_K B$ is also an $\mathcal{F}$-admissible H-set. It also implies, in particular, the following:

NG_VALENCE_LEMMA

Lemma 8.5. *The valence map* lr *restricts to a map* $\mathsf{lr} : \Omega_\mathcal{F} \to \Sigma_\mathcal{F}$*; that is, if T is $\mathcal{F}$-admissible, so is* $\mathsf{lr}(T)$*.* □

Lemma 8.6. *A weak indexing system $\mathcal{F}$ is an indexing system (in the sense of [2]) if and only if all trivial H-sets are $\mathcal{F}$-admissible, for all $H \leq G$.*

Proof. come back

□

We will define $\mathcal{F}$-model structures on $\mathcal{V}\mathsf{Op}_G$ for any weak indexing system $\mathcal{F}$; the $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Sym}_G$ will exist for any G-vertex family $\mathcal{F}$, but transfering requires the additional closure properties.

8.2 Semi Model Structures on Genuine Operads

Fix a G-vertex family $\mathcal{F} = \{\mathcal{F}_n\}$. For the following section, we fix the following about our base category $\mathcal{V}$:

Definition 8.7. We say $\mathcal{V}$ satisfies ASSUMPTION 1 (for $\mathcal{F}$) if the following hold:

1. $\mathcal{V}$ is a cofibrantly-generated Cartesian symmetric monoidal model category, and
2. $\mathcal{V}$ is has cellular fixed point functors for all finite grouops (c.f. [7]).

Definition 8.8. The *$\mathcal{F}$-projective model structure* on $\mathcal{V}\mathsf{Sym} = \mathcal{V}^{\Sigma_G^{op}}$ is the unique model structure induced by the adjunction

$$\mathcal{V}^{\Sigma_G^{op}} \leftrightarrows \mathcal{V}^{\Sigma_{\mathcal{F}}^{op}}.$$

Explicitly, a map f is a fibration (resp. weak equivalence) if $f(C)$ is one in $\mathcal{V}$ for all $C \in \Sigma_{\mathcal{F}}$.

Definition 8.9. A map $f : \mathcal{O} \to \mathcal{P}$ in $\mathcal{V}\mathsf{Op}_G$ is called a

1. *$\mathcal{F}$-fibration* (resp. $\mathcal{F}$-weak equivalence) if $f(C) : \mathcal{O}(C) \to \mathcal{P}(C)$ is one in $\mathcal{V}$ for all $\mathcal{F}$-admissible G-corollas $C \in \Sigma_{\mathcal{F}}$.
2. *$\mathcal{F}$-cofibration* if it has the left lifting property against all maps which are both $\mathcal{F}$-fibrations and $\mathcal{F}$-weak equivalences.
3. *$\Sigma_{\mathcal{F}}$-cofibration* if f is an $\mathcal{F}$-cofibration in $\mathcal{V}\mathsf{Sym}_G$.

In particular, $\mathcal{P} \in \mathcal{V}\mathsf{Op}_G$ will be called *$\mathcal{F}$-cofibrant* if $\varnothing \to \mathcal{P}$ is an $\mathcal{F}$-cofibration.

Definition 8.10. The $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}_G$, if it exists, is the unique model structure with the above specified weak equivalences and fibrations. Equivalently, it is the transfered model structure along the adjoints

$$\mathcal{V}\mathsf{Op}_G \underset{\mathsf{fgt}}{\overset{\mathbb{F}_G}{\leftrightarrows}} \mathcal{V}^{\Sigma_G^{op}} \leftrightarrows \prod_{\mathrm{Ob}(\Sigma_G)} \mathcal{V} \leftrightarrows \prod_{\mathrm{Ob}(\Sigma_{\mathcal{F}})} \mathcal{V}$$

Using general arguments of [3], this structure would be cofibrantly generated, with generating arrows

$$\begin{aligned} I_{\mathcal{F}} &= \{\mathbb{F}_G(\Sigma_G(-,C)\cdot i) \mid C \in \Sigma_{\mathcal{F}},\ i \in I\} \\ J_{\mathcal{F}} &= \{\mathbb{F}_G(\Sigma_G(-,C)\cdot j) \mid C \in \Sigma_{\mathcal{F}},\ j \in J\}, \end{aligned}$$

for I (resp. J) the generating (trivial) cofibrations of $\mathcal{V}$.

Definition 8.11. If $\mathcal{F}$ is the complete G-vertex family - so $\mathcal{F}_n$ is the family of **all** graph subgroups of $G \times \Sigma_n$ - we refer to the $\mathcal{F}$-model structure as the *genuine* model structure.

Theorem 8.12. *The genuine semi-model structure on $\mathcal{V}\mathsf{Op}_G$ exists for all $\mathcal{V}$ satisfying* ASSUMPTION 1.

Proof. By general arguments of Kan [3, 11.6.1], it suffices to show that any transfinite composite of cellular extensions $\mathcal{P} \to \mathcal{P}[u]$, each built by pushouts

$$\begin{tikzcd} \mathbb{F}_G X \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_G Y \arrow[r] & \mathcal{P}[u] \end{tikzcd}$$

where $u : X \to Y$ is a generating trivial cofibration and $\mathcal{P}$ is cofibrant in $\mathcal{V}_{gen}^{\Sigma_G^{op}}$, is itself a weak equivalence in $V_{gen}^{\Sigma_G^{op}}$.

In particular, it suffices to show that, for u a genuine level trivial cofibration and $\mathcal{P}$ genuine level cofibrant, any map $\mathcal{P} \to \mathcal{P}[u]$ is itself a genuine level trivial cofibration; since these are weaker conditions on u and $\mathcal{P}$, and transfinite composition is levelwise, this implies the above result. This case follows from the levelwise filtration from Theorem ??, by applying Propositions 8.14 and 7.14. □

Remark 8.13. In proving the above, we will often use the fact that, given families $\mathcal{F} \subseteq \bar{\mathcal{F}}$ of subgroups of a group Π, the identity map

$$\mathcal{V}^G_{\mathcal{F}} \to \mathcal{V}^G_{\bar{\mathcal{F}}}$$

is left Quillen (and in particular preserves cofibrations). In particular, any $\mathcal{F}$-cofibration is a genuine cofibration.

X_COFIBRANT_PROP

Proposition 8.14. *Fix $T \in \Omega_G$, and suppose we are given cofibrations $f(Gv) \in \mathcal{V}^{\mathrm{Aut}(T_{Gv})}_{gen}$ for all $Gv \in V_G(T)$, such that $f(Gv) = f(\alpha(Gv))$ for all $\alpha \in \mathrm{Aut}(T)$. Then the iterated box product*

$$f^{\square V_G(T)} = \mathop{\square}_{Gv \in V_G(T)} f(Gv)$$

is a cofibration in $\mathcal{V}^{\mathrm{Aut}(T)}_{gen}$.

Proof. Using grafting composition $T \simeq C \circ (T^1_1, \ldots, T^{k_1}_1, T^1_2, \ldots, T^{k_r}_r)$, we go by induction on $|V_G(T)|$. The base cases of $|V_G(T)| = 0$ or 1 are trivial. Now, we note that we have a decomposition

$$f^{\square V_G(T)} = f(Gv_r) \square \mathop{\square}_{i=1}^{r} \left(\left(f^{\square V_G(T_i)} \right)^{\square k_i} \right)$$

where Gv_r is the root orbit. We will build this map in stages, preserving cofibrancy in each step.

- By induction, for each i, $f^{\square V_G(T_i)}$ is a cofibration in $\mathcal{V}^{\mathrm{Aut}(T_i)}_{gen}$.
- By Proposition POWERF_PROP 7.37, each $\left(f^{\square V_G(T_i)} \right)^{\square k_i}$ is a cofibration in $\mathcal{V}^{\Sigma_{k_i} \wr \mathrm{Aut}(T_i)}_{gen}$.
- By Remark EXTERINTADJ_EQ 7.28, $\mathop{\square}_{i=1} \left(\left(f^{\square V_G(T_i)} \right)^{\square k_i} \right)$ is a cofibration in $\mathcal{V}^{\Pi \Sigma_{k_i} \wr \mathrm{Aut}(T_i)}_{gen}$.
- By Proposition FGTLEFT_PROP 7.16, $f(Gv_r)$ is a cofibration in $\mathcal{V}^{\mathrm{Aut}(T)}_{gen}$.
- Finally, by the decomposotion on $\mathrm{Aut}(T)$ and Proposition BIQUILLENG_PROP 7.23, we have the full $f^{\square V_G(T)}$ is a cofibration in $\mathcal{V}^{\mathrm{Aut}(T)}_{gen}$, as desired.

If any $f(Gv)$ were trivial, then, during the appropriate step above, the resulting box product would again be trivial, by the same referenced results. □

Corollary 8.15. *For any G-vertex system $\mathcal{F}$, the $\mathcal{F}$-semi-model structure on $\mathcal{V}\mathsf{Op}_G$ exists.*

Proof. This follows from the proof of Theorem GENUINE_SEMI_VOP_G_THM 6.26. In particular, it suffices to show that if $u : X \to Y$ is a level trivial $\mathcal{F}$-cofibration and $\mathcal{P}$ is level $\mathcal{F}$-cofibrant in $\mathcal{V}^{\Sigma^{op}_G}$, then the cellular extension $\mathcal{P} \to \mathcal{P}[u]$ is a level trivial $\mathcal{F}$-cofibration. However, such u are level genuine cofibrations, and such $\mathcal{P}$ are level genuine cofibrant, and hence $\mathcal{P} \to \mathcal{P}[u]$ is a level trivial genuine cofibration, and hence a level trivial $\mathcal{F}$-cofibration. □

8.3 True Model Structures

come back

8.4 Preservation of Cofibrant Objects

come back

8.5 Cofibrant Symmetric Collections

come back: can do all of this for $\mathcal{V}^{\Omega_G^{op}}$ and $\mathcal{V}^{\Sigma_G^{op}}$, don't need these crazy categories.

We first classify cofibrant objects in $\mathcal{V}\mathsf{Sym}_G$ by breaking $\Omega_{G,q}$ apart along the free G-trees..

Again, we fix an ordering on G.

Definition 8.16. Given a tree $T_0 \in \Omega$, let $\Omega_G[T_0]$ denote the full subcategory of $\Omega_{G,q}$ spanned by those trees which recieve a (quotient) map from $G \cdot T_0$. Further, we observe that $\Omega_{G,q}$ is isomorphic to the disjoint union

$$\coprod_{T_0 \in \mathrm{Iso}(\Omega)} \Omega_G[T_0].$$

_GAMMA_T_0_LEMMA **Lemma 8.17.** *$\Omega_G[T_0]$ and the orbit category $O_{\Gamma_{T_0}}$ are equivalent, where Γ_{T_0} is the family of all graph subgroups of $G \times \Sigma_{T_0}$.* □

The two above categories are not isomorphic: $O_{\Gamma_{T_0}}$ only records the planar structure on (say) the *first* tree component, and ignores all the rest.

¡¡¡¡¡¡ HEAD Modeled on the proof of [?, Theorem 2.10], we have the following result. ======= We can now prove a proposition, modeled on the proof of [7, Theorem 2.10]. ¿¿¿¿¿¿¿ refs/remotes/origin/master

Proposition 8.18. *If $X \in \mathcal{V}\mathsf{Sym}_G$ is cofibrant, then $\eta_X : X \to i_* i^* X$ is an isomorphism.*

Proof. Using Lemma 8.17 and the fact that pushouts are underlying and levelwise, this follows from cellularity precisely as in *loc cite*. Indeed, analogous arguments show that η is an isomorphism on representable sheaves $\Omega_G(-,T) \cdot A$, and then the result can be extended to transfinite composite of free extensions exactly as before.

□

8.5.1 G-Operads

Definition 8.19. A map $f : \varnothing \to \mathcal{P}$ in $\mathcal{V}\mathsf{Op}^G$ is called a

1. $\mathcal{F}$*-fibration* (resp. $\mathcal{F}$-weak equivalence) if ...

come back

Definition 8.20. The $\mathcal{F}$-model structure on $\mathcal{V}\mathsf{Op}^G$, ...

come back

Lemma 8.21. *The $\mathcal{F}$-semi-model structure, if it exists, is the transfered model structure along the adjunction*

$$\mathcal{V}\mathsf{Op}^G_{\mathcal{F}} \leftrightarrows \mathcal{V}\mathsf{Op}^{\mathcal{F}}_G$$

Corollary 8.22. *The $\mathcal{F}$(-semi)-model structure exists on $\mathcal{V}\mathsf{Op}^G$ whenever it exists on $\mathcal{V}\mathsf{Op}_G$.*

Theorem 8.23. *$\mathcal{V}\mathsf{Op}^G_{\mathcal{F}}$ and $\mathcal{V}\mathsf{Op}^{\mathcal{F}}_G$ are Quillen equivalent.*

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