

Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads, which can be thought of as a hybrid between equivariant operads and coefficient systems. We then prove an Elmendorf type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads with their projective model structure.

As an application, we build explicit models for the N_∞ -operads of Blumberg and Hill.

Contents

1	Introduction	1
2	Planar and tall maps	1
2.1	Planar structures	1

1 Introduction

No content yet.

2 Planar and tall maps

2.1 Planar structures

Throughout we will work with trees possessing *planar structures* or, more intuitively, trees embedded into the plane.

Our preferred model for trees will be that of broad posets first introduced by Weiss in [We12] and further worked out by the second author in [Pe16b]. We now define planar structures in this context.

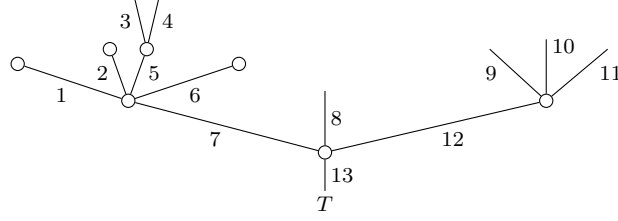
Definition 2.1. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descandancy partial order \leq_d to a total order \leq_p such that:

- *Planar*: if $e \leq_p f$ and $e \not\leq_d f$ then $g \leq_d f$ implies $e \leq_p g$.

Example 2.2. An example of a planar structure on a tree T follows, with \leq_r encoded by

PLANARIZE DEF

the number labels.



Intuitively, given a planar depiction of a tree T , $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

Intuitively, a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that Definition 2.1 is equivalent to such choosing total orders for each of the sets of input edges. To do so, we first introduce some notation.

Notation 2.3. Let $T \in \Omega$ be a tree and $e \in T$ and edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path* of e .

We will repeatedly use the following, which is a consequence of [Pe16b, Cor. 5.26].

Lemma 2.4. *If $e \leq_d f$, $e \leq_d f'$, then f, f' are \leq_d -comparable.*

Proposition 2.5. *Let $T \in \Omega$ be a tree. Then*

- (a) *for any $e \in T$ the finite poset $I(e)$ is totally ordered;*
- (b) *the poset (T, \leq_d) has all joins, denoted \vee . In fact, $\vee_i e_i = \min(\cap_i I(e_i))$.*

Proof. (a) is immediate from Lemma 2.4. To prove (b) we note that $\min(\cap_i I(e_i))$ exists by (a), and that this is clearly the join $\vee e_i$. \square

Notation 2.6. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^\dagger \in T$ the predecessor of b in $I(e)$.

Proposition 2.7. *Suppose e, f are \leq_d -incomparable edges of T and write $b = e \vee f$. Then*

- (a) *$e <_d b$, $f <_d b$ and $b_e^\dagger \neq b_f^\dagger$;*
- (b) *$b_e^\dagger, b_f^\dagger \in b^\dagger$. In fact $\{b_e^\dagger\} = I(e) \cap b^\dagger$, $\{b_f^\dagger\} = I(f) \cap b^\dagger$;*
- (c) *if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^\dagger = b_e^\dagger$, $b_{f'}^\dagger = b_f^\dagger$.*

Proof. (a) is immediate: the condition $e = g$ (resp. $f = g$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^\dagger = b_f^\dagger$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^\dagger$, where uniqueness follows from Lemma 2.4. Choosing $a = e$ implies $I(e) \cap b^\dagger = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b .

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. \square

Proposition 2.8. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c_{e_i}^\dagger \neq c_{e_j}^\dagger$.*

Therefore, all ternary joins in (T, \leq_d) are binary, i.e.

$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{2.9}$$

for some $1 \leq i < j \leq 3$, and (2.9) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c_{e_i}^\dagger \neq c_{e_j}^\dagger$, then $c = \min(I(e_i) \cap I(e_j)) = e_i \vee e_j$, whereas the converse follows from Proposition 2.7(a). INPUTPREDECESSORPROP PROP

The “therefore” part follows by noting that $c_{e_1}^\dagger, c_{e_2}^\dagger, c_{e_3}^\dagger$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

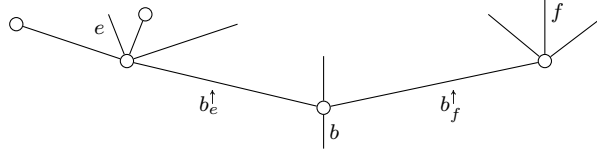
Notation 2.10. Given a set S of size n we write $\text{Ord}(S) \simeq \text{Iso}(S, \{1, \dots, n\})$. We will usually abuse notation by regarding its objects as pairs (S, \leq) where \leq is a total order in S .

Proposition 2.11. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{aligned} \{\text{planar structures } (T, \leq_p)\} &\longrightarrow \prod_{(a^\dagger \leq a) \in V(T)} \text{Ord}(a^\dagger) \\ \leq_p &\longmapsto (\leq_p \mid a^\dagger) \end{aligned} \quad (2.12) \quad \text{PLANAR EQ}$$

Proof. We will keep the setup of Proposition 2.7 throughout: e, f are \leq_d -incomparable edges and we write $b = e \vee f$. INPUTPREDECESSORPROP PROP

We first show that (2.12) is injective, i.e. that the restrictions $\leq_p \mid a^\dagger$ determine if $e <_p f$ holds or not. If $b_e^\dagger <_p b_f^\dagger$, the relations $e \leq_d b_e^\dagger <_p b_f^\dagger \geq_d f$ and Definition 2.1 imply it must be $e <_p f$. PLANAR EQ



Dually, if $b_f^\dagger <_p b_e^\dagger$ then $f <_p e$. Thus $b_e^\dagger <_p b_f^\dagger \Leftrightarrow e <_p f$ and hence (2.12) is indeed injective. PLANAR EQ

To check that (2.12) is surjective, it suffices (recall that e, f are assumed \leq_d -incomparable) to check that defining $e \leq_p f$ to hold iff $b_e^\dagger < b_f^\dagger$ holds in b^\dagger yields a planar structure. PLANAR EQ

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of \leq_p in the case $e' \leq_d e \leq_d f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 2.7(c). Transitivity of \leq_p in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are \leq_d -incomparable, in which case one can apply 2.7(c) with the roles of f, f' reversed. INPUTPREDECESSORPROP PROP

It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with e, g incomparable f, g . We write $c = e \vee f \vee g$. By the “therefore” part of Proposition 2.8, either (i) $e \vee f <_d c$, in which case Proposition 2.8 implies $c_e^\dagger = c_f^\dagger$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c_e^\dagger < c_f^\dagger < c_g^\dagger$ in c^\dagger so that $c_e^\dagger \neq c_g^\dagger$ and by Proposition 2.8 it is also $e \vee g = c$ and transitivity follows. TERNARYJOIN PROP

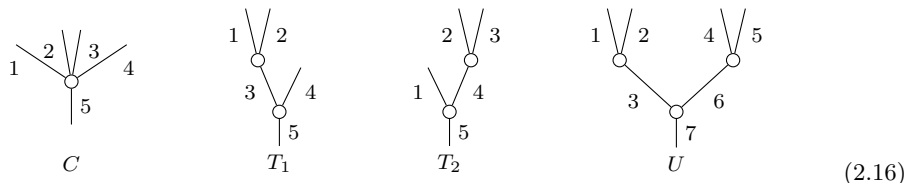
Remark 2.13. Definition 2.1 readily extends to forests $F \in \Phi$. The analogue of Proposition 2.11 then states that the data of a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. Indeed, this follows by either adapting the proof above or by noting that planar structures on F are clearly in bijection with planar structures on the join tree $F \star \eta$ (cf. [1, Def. 7.44]), which adds a single edge η to F , serving as the (unique) root of $F \star \eta$. PLANARIZE DEF

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Convention 2.14. From now on, we will write Ω (resp. Ω_G) to denote a model for the category of trees (resp. G -trees) where

- each object (i.e. tree) is equipped with a planar structure;
- morphisms ignore the planar structure;
- there is exactly one representative of each planarization, i.e. the identities are the only isomorphisms that preserve the planar structure.

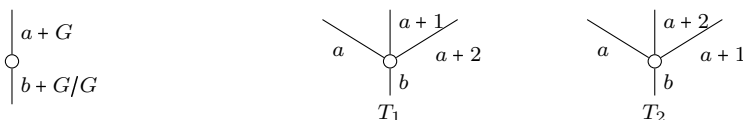
Remark 2.15. The reader desiring extra concreteness is welcome to think of the objects of Ω , Ω_G as consisting of planarized tree structures on one of the sets $\underline{n} = \{1, 2, \dots, n\}$ such that the planarization \leq_d is the canonical total order. Some trees depicted in this convention follow.



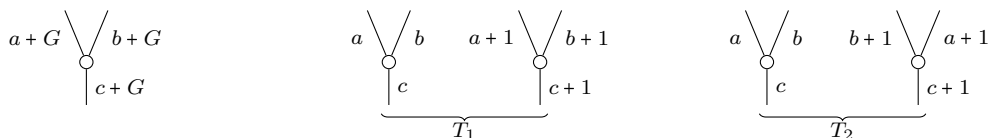
PLANAROMEGAEX1 EQ

We note that T_1 and T_2 are isomorphic and, moreover, they encode the only two isomorphism classes of planar structures on the their underlying dendroidal set, so that no other object of Ω is isomorphic to them. C and U , on the other hand, are isomorphic to no other object of Ω , since the planarizations of the underlying broad posets sets are unique up to isomorphism.

One drawback of the concrete convention illustrated in (2.16), however, is that discussion of subfaces of trees becomes awkward, since one can not then technically regard them as subobjects. To avoid this issue, we will often regard the objects of Ω as equivalence classes of trees with planarizations (with no ambiguity resulting since representatives are related via unique isomorphisms). Moreover, this is particularly convenient when discussing G -trees, as it otherwise the task of depicting the G -action becomes cumbersome. For some examples, (and recalling that the numbering of the edges as in (2.16) is superfluous, in the sense that it is already encoded in the planar picture itself), we note that for $G = \mathbb{Z}_3$ the orbital representation on the left below encodes the two isomorphic objects of Ω_G on the right (which are isomorphic to no other object of Ω_G).



Similarly, for $G = \mathbb{Z}_2$, the orbital representation on the left represents the two G -trees presented.



References

Pe16b

- [1] L. A. Pereira. Equivariant dendroidal sets. Available at: <http://www.faculty.virginia.edu/luisalex/>, 2016.

We12

- [2] I. Weiss. Broad posets, trees, and the dendroidal category. Available at: <https://arxiv.org/abs/1201.3987>, 2012.