Genuine equivariant operads

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Abstract

We build new algebraic structures, which we call genuine equivariant operads and which can be thought of as a hybrid between operads and coefficient systems. We then prove an Elmendorf-Piacenza type theorem stating that equivariant operads, with their graph model structure, are equivalent to genuine equivariant operads, with their projective model structure.

As an application, we build explicit models for the N_∞-operads of Blumberg and Hill.

Contents

1 Introduction

A surprising feature of topological algebra is that the category of (connected) topological commutative monoids is quite small, consisting only of products of Eilenberg-MacLane spaces (e.g. [15, 4K.6]). Instead, the more interesting structures are those monoids which are commutative and associative only up to homotopy and, moreover, up to "all higher homotopies". To capture these more subtle algebraic notions, Boardman-Vogt [4] and May [21] developed the theory of *operads*. Informally, an operad $\mathcal{O}$ consists of sets/spaces $\mathcal{O}(n)$ of "n-ary operations" carrying a Σ_n-action recording "reordering the inputs of the operations", and a suitable notion of "composition of operations". The purpose of the theory is then the study of "objects X with operations indexed by $\mathcal{O}$", referred to as *algebras*, with the notions of monoid, commutative monoid, Lie algebra, algebra with a module, and more, all being recovered as algebras over some fixed operad in an appropriate category. Of special importance are the E_∞-operads, introduced by May in [21], which are "homotopical replacements" for the commutative operad and encode the aforementioned "commutative monoids up to homotopy". In particular, while an E_∞-algebra structure on X does not specify unique maps $X^n \to X$, it nonetheless specifies such maps "uniquely up to homotopy".

E_∞-operads are characterized by the homotopy type of their levels $\mathcal{O}(n)$: $\mathcal{O}$ is E_∞ if and only if each $\mathcal{O}(n)$ is Σ_n-free and contractible, i.e., for each subgroup $\Gamma \leq \Sigma_n$ one has

$$\mathcal{O}(n)^\Gamma \sim \begin{cases} * & \text{if } \Gamma = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases}$$

Notably, when studying the homotopy theory of operads in topological spaces the preferred notion of weak equivalence is usually that of "naive equivalence", with a map of operads $\mathcal{O} \to \mathcal{O}'$ deemed a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a weak equivalence of spaces upon forgetting the Σ_n-actions (e.g. [1, 3.2]). In this context, E_∞-operads are then equivalent to the commutative operad Com and, moreover, any cofibrant replacement of Com is E_∞. However, naive equivalences differ from the equivalences in "genuine equivariant homotopy theory", where a map of G-spaces $X \to Y$ is deemed a G-equivalence only if the induced fix point maps $X^H \to Y^H$ are weak equivalences for all $H \leq G$. This contrast hints at a number of novel subtleties that appear in the study of equivariant operads, which we now discuss.

Firstly, noting that for any finite group G, a G-operad $\mathcal{O}$ (i.e. an operad $\mathcal{O}$ together with a G-action commuting with all the structure) the n-th level $\mathcal{O}(n)$ has a $G \times \Sigma_n$-action, one might guess that a map of G-operads $\mathcal{O} \to \mathcal{O}'$ should be called a weak equivalence if each of the maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ is a G-equivalence after forgetting the Σ_n-actions, i.e. if the maps

$$\mathcal{O}(n)^H \xrightarrow{\sim} \mathcal{O}'(n)^H, \qquad H \leq G \leq G \times \Sigma_n, \tag{1.1}$$

are weak equivalences of spaces. However, the notion of equivalence suggested in (1.1) turns out to not be "genuine enough". To see why, we first consider a homotopical replacement for Com using this theory: if one simply equips an E_∞-operad $\mathcal{O}$ with a trivial G-action, the resulting G-operad has fixed points for each subgroup $\Gamma \leq G \times \Sigma_n$ determined by

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \leq G, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.2}$$

However, as first noted by Costenoble-Waner [7] in their study of equivariant infinite loop spaces, the *G-trivial E_∞-operads* of (1.2) do not provide the correct replacement of Com in the G-equivariant context. Rather, that replacement is provided instead by the *G-E_∞-operads*, characterized by the fixed point conditions

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \cap \Sigma_n = \{*\}, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.3}$$

In contrasting (1.2) and (1.3), we note that the subgroups $\Gamma \leq G \times \Sigma_n$ such that $\Gamma \cap \Sigma_n = \{*\}$ are readily shown to be precisely the graphs of partial homomorphisms $G \geq H \to \Sigma_n$, and that $\Gamma \leq G$ if and only if Γ is the graph of a trivial homomorphism. As it turns out, the notion of weak equivalence described in (1.1) fails to distinguish (1.2) and (1.3), and indeed it is possible to build maps $\mathcal{O} \to \mathcal{O}'$ where $\mathcal{O}$ is a G-trivial E_∞-operad (as in (1.2)) and $\mathcal{O}'$ is a G-E_∞-operad (as in (1.3)). Therefore, in order to differentiate such operads, one needs to replace the notion of weak equivalence in (1.1) with the finer notion of *graph equivalence*, so that $\mathcal{O} \to \mathcal{O}'$ is considered a weak equivalence only if the maps

$$\mathcal{O}(n)^{\Gamma} \xrightarrow{\sim} \mathcal{O}'(n)^{\Gamma}, \qquad \Gamma \leq G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}. \tag{1.4}$$

are all weak equivalences.

As mentioned above, the original evidence [7] that (1.3), rather than (1.2), provides the best up-to-homotopy replacement for Com in the equivariant context comes from the study of equivariant infinite loop spaces. For our purposes, however, we instead focus on the perspective of Blumberg-Hill in [3], which concerns the Hill-Hopkins-Ravenel norm maps featured in the solution of the Kervaire invariant problem [16].

Given a G-spectrum R and finite G-set X with n elements, the corresponding *norm* is another G-spectrum $N^X R$ whose underlying spectrum is $R^{\wedge X} \simeq R^{\wedge n}$, but which is equipped with a "mixed" G-action that combines the actions on R and X in the natural way. Moreover, for any Com-algebra R, i.e. any strictly commutative G-ring spectrum, ring multiplication further induces so called *norm maps*

$$N^X R \to R. \tag{1.5}$$

Furthermore, by reducing structure on R the maps (1.5) are also defined when X is only a H-set for some subgroup $H \leq G$, and the maps (1.5) then satisfy a number of natural equivariance and associativity conditions. Crucially, we note that the more interesting of these associativity conditions involve H-sets for various H simultaneously (for an example packaged in operadic language, see (1.10) below).

The key observation at the source of the work in [3] is then that, operadically, norm maps are encoded by the graph fixed points appearing in (1.4). More explicitly, noting that a H-set X with n elements is encoded by a partial homomorphism $G \geq H \to \Sigma_n$, one obtains an associated graph subgroup $\Gamma_X \leq G \times \Sigma_n$, $\Gamma_X \cap \Sigma_n = \{*\}$, well defined up to conjugation. It then follows that for R an $\mathcal{O}$-algebra, maps of the form (1.5) are parametrized by the fixed point space $\mathcal{O}(n)^{\Gamma_X}$. The flaw of the G-trivial E_∞-operads described in (1.2) is then that they lack all norms maps other than those for H-trivial X, thus lacking some of the data encoded by Com. Further, from this perspective one may regard the more naive notion of weak equivalence in (1.1), according to which (1.2) and (1.3) are equivalent, as studying "operads without norm maps" (in the sense that equivalences ignore norm maps), while the equivalences (1.4) study "operads with norm maps".

Our first main result, Theorem I, establishes the existence of a model structure on G-operads with weak equivalences the graph equivalences of (1.4), though our analysis goes significantly further, again guided by Blumberg and Hill's work in [3].

The main novelty of [3] is the definition, for each finite group G, of a finite lattice of new types of equivariant operads, which they dub N_∞-operads. The minimal type of N_∞-operads is that of the G-trivial E_∞-operads in (1.2) while the maximal type is that of the G-E_∞-operads in (1.3). The remaining types, which interpolate between the two, can hence be thought of as encoding varying degrees of "up to homotopy equivariant commutativity". More concretely, each type of N_∞-operad is determined by a collection $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ where each $\mathcal{F}_n$ is itself a collection of graph subgroups of $G \times \Sigma_n$, with an operad $\mathcal{O}$ being called a *$N\mathcal{F}$-operad* if it satisfies the fixed point condition

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n, \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.6}$$

Such collections $\mathcal{F}$ are, however, far from arbitrary, with much of the work in [3, §3] spent cataloging a number of closure conditions that these $\mathcal{F}$ must satisfy. The simplest of these conditions state that each $\mathcal{F}_n$ is a *family*, i.e. closed under subgroups and conjugation. These first two conditions, which are common in equivariant homotopy theory, are a simple consequence of each $\mathcal{O}(n)$ being a space. However, the remaining conditions, all of which involve $\mathcal{F}_n$ for various n simultaneously and are a consequence of operadic multiplication, are both novel and subtle. In loose terms, these conditions, which are more easily described in terms of the H-sets X associated to the graph subgroups, concern closure of those under disjoint union, cartesian product, subobjects, and an entirely new key condition called *self-induction*. The precise conditions are collected in [3, Def. 3.22], which also introduces the term *indexing system* for an $\mathcal{F}$ satisfying all of those conditions. A main result of [3, §4] is then that whenever a $N\mathcal{F}$-operad $\mathcal{O}$ as in (1.6) exists, the associated collection $\mathcal{F}$ must be an indexing system. However, the converse statement, that given any indexing system $\mathcal{F}$ such an $\mathcal{O}$ can be produced, was left as a conjecture.

One of the key motivating goals of the present work was to verify this conjecture of Blumberg-Hill, which we obtain in Corollary IV. We note here that this conjecture has also been concurrently verified by Gutiérrez-White in [13] and by Rubin in [27], with each of their approaches having different advantages: Gutiérrez-White's model for $N\mathcal{F}$ is cofibrant while Rubin's model is explicit. Our model, which emerges from a broader framework, satisfies both of these desiderata.

To motivate our approach, we first recall the solution of a closely related but simpler problem: that of building universal spaces for families of subgroups. Given a family $\mathcal{F}$ of subgroups of G, a *universal space* X for $\mathcal{F}$, also called an *$E\mathcal{F}$-space*, is a space with fixed points X^H characterized just as in (1.6). In particular, whenever $\mathcal{O}$ is a $N\mathcal{F}$-operad, each $\mathcal{O}(n)$ is necessarily an $E\mathcal{F}_n$-space. The existence of $E\mathcal{F}$-spaces for any choice of the family $\mathcal{F}$ is best understood in light of Elmendorf's classical result from [10] (modernized by Piacenza in [25]) stating that there is a Quillen equivalence (recall that O_G is the *orbit* category, formed by the G-sets G/H)

$$\begin{array}{ccc} \mathsf{Top}^{\mathsf{O}_G^{op}} & \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} & \mathsf{Top}^G \\ (G/H \mapsto Y(G/H)) & \longmapsto & Y(G) \\ (G/H \mapsto X^H) & \longleftarrow\!\shortmid & X \end{array} \tag{1.7}$$

where the weak equivalences (and fibrations) on Top^G are detected on all fixed points and the weak equivalences (and fibrations) on the category $\mathsf{Top}^{\mathsf{O}_G^{op}}$ of *coefficient systems* are detected at each presheaf level. Noting that the fixed point characterization of $E\mathcal{F}$-spaces defines an obvious object $\delta_{\mathcal{F}} \in \mathsf{Top}^{\mathsf{O}_G^{op}}$ by $\delta_{\mathcal{F}}(G/H) = *$ if $H \in \mathcal{F}$ and $\delta_{\mathcal{F}}(G/H) = \varnothing$ otherwise,

$E\mathcal{F}$-spaces can then be built as $\iota^*(C\delta_{\mathcal{F}}) = C\delta_{\mathcal{F}}(G)$, where C denotes cofibrant replacement in $\mathsf{Top}^{\mathsf{O}_G^{op}}$. Moreover, we note that, as in [10, §3], these cofibrant replacements can be built via explicit simplicial realizations.

The overarching goal of this paper is then that of proving the analogue of Elmendorf-Piacenza's Theorem (1.7) in the context of operads with norm maps (i.e. with equivalences as in (1.4)), which we state as our main result, Theorem III. However, in trying to formulate such a result one immediately runs into a fundamental issue: it is unclear which category should take the role of the coefficient systems $\mathsf{Top}^{\mathsf{O}_G^{op}}$ in this context. This last remark likely requires justification. Indeed, it may at first seem tempting to simply employ one of the known formal generalizations of Elmendorf-Piacenza's result (see, e.g. [30, Thm. 3.17]) which simply replace Top on either side of (1.7) with a more general model category $\mathcal{V}$. However, if one applies such a result when $\mathcal{V} = \mathsf{Op}$ to establish a Quillen equivalence $\mathsf{Op}^{\mathsf{O}_G^{op}} \rightleftarrows \mathsf{Op}^G$ (the existence of this equivalence is due to upcoming work of Bergner-Gutiérrez), the fact that the levels of each $\mathcal{P} \in \mathsf{Op}^{\mathsf{O}_G^{op}}$ correspond only to those fixed-point spaces appearing in (1.1) would require working in the context of operads *without* norm maps, and thereby forgo the ability to distinguish the many types of $N\mathcal{F}$-operads.

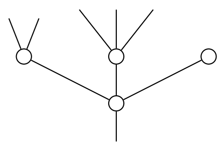
In order to work in the context of operads with norm maps we will need to replace $\mathsf{Top}^{\mathsf{O}_G^{op}}$ with a category Op_G of new algebraic objects we dub *genuine equivariant operads* (as opposed to (regular) equivariant operads Op^G). Each genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will consist of a list of spaces indexed in the same way as in (1.4) along with obvious restriction maps and, more importantly, suitable *composition maps.* Precisely identifying the required composition maps is one of the main challenges of this theory, and again we turn to [3] for motivation.

Analyzing the proofs of the results in [3, §4] concerning the closure properties for indexing systems $\mathcal{F}$, a common motif emerges: when performing an operadic composition

$$\begin{aligned} \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) &\longrightarrow \mathcal{O}(m_1 + \cdots + m_n), \\ (f, g_1, \cdots, g_n) &\longmapsto f(g_1, \cdots, g_n) \end{aligned} \tag{1.8}$$

careful choices of fixed point conditions on the operations $f, g_1, \cdots, g_n$ yield a fixed point condition on the composite operation $f(g_1, \cdots, g_n)$. The desired multiplication maps for a genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ will then abstract such interactions between multiplication and fixed points for an equivariant operad $\mathcal{O} \in \mathsf{Op}^G$. However, these interactions can be challenging to write down explicitly and indeed, the arguments in [3, §4] do not quite provide the sort of unified conceptual approach to these interactions needed for our purposes. The cornerstone of the current work was then the joint discovery by the authors of such a conceptual framework: equivariant trees.

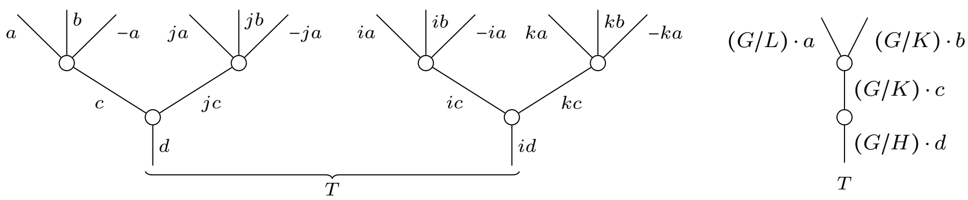
Non-equivariantly, it has long been known that the combinatorics of operadic composition is best visualized by means of tree diagrams. For instance, the tree



encodes the operadic composition

$$\mathcal{O}(3) \times \mathcal{O}(2) \times \mathcal{O}(3) \times \mathcal{O}(0) \to \mathcal{O}(5)$$

where the inputs $\mathcal{O}(3), \mathcal{O}(2), \mathcal{O}(3), \mathcal{O}(0)$ correspond to the nodes (i.e. circles) in the tree, with arity given by number of incoming edges (i.e. edges immediately above) and the output $\mathcal{O}(5)$ has arity given by counting leaves (i.e. edges at the top, not capped by a node). Similarly, the role of equivariant trees is, in the context of equivariant operads, to encode such operadic compositions together with fixed point compatibilities. A detailed introduction to equivariant trees can be found in [24, §4], where the second author develops the theory of equivariant dendroidal sets (which is a parallel approach to equivariant operads), though here we include only a single representative example. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternionic units and $G \geq H \geq K \geq L$ denote the subgroups $H = \langle j \rangle$, $K = \langle -1 \rangle$, $L = \{1\}$. There is then a G-tree T with *expanded representation* given by the two trees on the left below and *orbital representation* given by the (single) tree on the right.



(1.9)

We note that G acts on the expanded representation of T as indicated by the edge labels (so that the edges a, b, c, d have stabilizers L, K, K, H respectively), and the orbital representation is obtained by collapsing the edge orbits of the expanded representation. As explained in [24, Example 4.9], T then encodes the fact that for any equivariant operad $\mathcal{O} \in \mathsf{Op}^G$ the composition $\mathcal{O}(2) \times \mathcal{O}(3)^{\times 2} \to \mathcal{O}(6)$ restricts to a fixed point composition

$$\mathcal{O}(H/K)^H \times \mathcal{O}(K/L \amalg K/K)^K \to \mathcal{O}(H/L \amalg H/K)^H \tag{1.10}$$

where $\mathcal{O}(X)$ for an H-set (resp. K-set) X denotes $\mathcal{O}(|X|)$ together with a suitably mixed H-action (K-action). We note that the inputs $\mathcal{O}(H/K)^H$, $\mathcal{O}(K/L \amalg K/K)^K$ in (1.10) correspond to the nodes of the orbital representation in (1.9), though in contrast to the non-equivariant case arity is now determined by both incoming and outgoing *edge orbits*, while the output $\mathcal{O}(H/L \amalg H/K)^H$ is similarly determined by both the leaf and root edge orbits. The existence of maps of the form (1.10) is essentially tantamount to the subtlest closure property for indexing systems $\mathcal{F}$, self-induction (cf. [3, Def. 3.20]), and similar tree descriptions exist for all other closure properties, as detailed in [24, §9].

We can now at last give a full informal description of the category Op_G featured in our main result, Theorem III. A genuine equivariant operad $\mathcal{P} \in \mathsf{Op}_G$ has levels $\mathcal{P}(X)$ for each H-set X, $H \leq G$, that mimic the role of the fixed points $\mathcal{O}(X)^H \simeq \mathcal{O}(|X|)^{\Gamma_X}$ for $\mathcal{O} \in \mathsf{Op}^G$. More explicitly, there are restriction maps $\mathcal{P}(X) \to \mathcal{P}(X|_K)$ for $K \leq H$, isomorphisms $\mathcal{P}(X) \simeq \mathcal{P}(gX)$ where gX denotes the conjugate gHg^{-1}-set, and composition maps given by

$$\mathcal{P}(H/K) \times \mathcal{P}(K/L \amalg K/K) \to \mathcal{P}(H/L \amalg H/K)$$

in the case of the abstraction of (1.10), and more generally by

$$\begin{gathered}\mathcal{P}(H/K_1 \amalg \cdots \amalg H/K_n) \times \mathcal{P}(K_1/L_{11} \amalg \cdots \amalg K_1/L_{1m_1}) \times \cdots \times \mathcal{P}(K_n/L_{n1} \amalg \cdots \amalg K_n/L_{nm_n}) \\ \downarrow \\ \mathcal{P}(H/L_{11} \amalg \cdots \amalg H/L_{1m_1} \amalg \cdots \amalg H/L_{n1} \amalg \cdots \amalg H/L_{nm_n}).\end{gathered} \tag{1.11}$$

Lastly, these composition maps must satisfy associativity, unitality, compatibility with restriction maps, and equivariance conditions, as encoded by the theory of G-trees. Rather than making such compatibilities explicit, however, we will find it preferable for our purposes to simply define genuine equivariant operads intrinsically in terms of G-trees.

We end this introduction with an alternative perspective on the role of genuine equivariant operads. The Elmendorf-Piacenza theorem in (1.7) is ultimately a strengthening of the basic observation that the homotopy groups $\pi_n(X)$ of a G-space X are coefficient systems rather than just G-objects. Similarly, the generalized Elmendorf-Piacenza result [30, Thm.

3.17] applied to the category $\mathcal{V} = \mathsf{sCat}$ of simplicial categories strengthens the observation that for a G-simplicial category $\mathcal{C}$ the associated homotopy category $\mathrm{ho}(\mathcal{C})$ is a coefficient system of categories rather than just a G-category. Likewise, Theorem III strengthens the (not so basic) observation that for a G-simplicial operad $\mathcal{O}$ the associated homotopy operad $\mathrm{ho}(\mathcal{O})$ is neither just a G-operad nor just a coefficient system of operads but rather the richer algebraic structure that we refer to as a "genuine equivariant operad".

1.1 Main results

We now discuss our main results.

Fixing a finite group G, we recall that $\mathsf{Op}^G(\mathcal{V}) = (\mathsf{Op}(\mathcal{V}))^G$ denotes G-objects in $\mathsf{Op}(\mathcal{V})$.

Theorem I. *Let* $(\mathcal{V}, \otimes)$ *denote either* $(\mathsf{sSet}, \times)$ *or* $(\mathsf{sSet}_*, \wedge)$.

Then there exists a model category structure on $\mathsf{Op}^G(\mathcal{V})$ *such that* $\mathcal{O} \to \mathcal{O}'$ *is a weak equivalence (resp. fibration) if all the maps*

$$\mathcal{O}(n)^{\Gamma} \to \mathcal{O}'(n)^{\Gamma} \tag{1.12}$$

for $\Gamma \le G \times \Sigma_n, \Gamma \cap \Sigma_n = \{*\}$, *are weak equivalences (fibrations) in* $\mathcal{V}$.

More generally, for $\mathcal{F} = \{\mathcal{F}_n\}_{n \ge 0}$ *with* $\mathcal{F}_n$ *an arbitrary collection of subgroups of* $G \times \Sigma_n$ *there exists a model category structure on* $\mathsf{Op}^G(\mathcal{V})$, *which we denote* $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$, *with weak equivalences (resp. fibrations) determined by (1.12) for* $\Gamma \in \mathcal{F}_n$.

Lastly, analogous semi-model category structures $\mathsf{Op}^G(\mathcal{V})$, $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ *exist provided that* $(\mathcal{V}, \otimes)$*: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers.*

We note that a similar result has also been proven by Gutiérrez-White in [13].

Theorem I is proven in §5.4. Condition (i) can be found in [18, Def. 2.1.17] while (ii) can be found in [18, Def. 4.2.6]. The additional conditions (iii) and (iv), which are less standard, are discussed in §6.1 and §6.2, respectively. Further, by *semi-model category* we mean the notion introduced in [18] and [29], which relaxes the definition of model structure by requiring that some of the axioms need only apply if the domains of certain cofibrations are cofibrant. For further details, we recommend the discussion in [33, §2.2] or [11, §12.1].

Our next result concerns the model structure on the new category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads introduced in this paper. Before stating the result, we must first outline how $\mathsf{Op}_G(\mathcal{V})$ itself is built. Firstly, the levels of each $\mathcal{P} \in \mathsf{Op}_G(\mathcal{V})$, i.e. the H-sets in (1.11), are encoded by a category Σ_G of *G-corollas*, introduced in §3.3, which generalizes the usual category Σ of finite sets and isomorphisms. We then define G-symmetric sequences by $\mathsf{Sym}_G(\mathcal{V}) = \mathcal{V}^{\Sigma_G^{op}}$ and, whenever $\mathcal{V}$ is a closed symmetric monoidal category with diagonals (cf. Remark 2.18), we define in §4.2 a *free genuine equivariant operad* monad $\mathbb{F}_G$ on $\mathsf{Sym}_G(\mathcal{V})$ whose algebras form the desired category $\mathsf{Op}_G(\mathcal{V})$.

Moreover, inspired by the analogues $\mathsf{Top}^{\mathsf{O}^{op}_{\mathcal{F}}} \rightleftarrows \mathsf{Top}^G_{\mathcal{F}}$ of the Elmendorf-Piacenza equivalence where $\mathsf{Top}^{\mathsf{O}^{op}_{\mathcal{F}}}$ are partial coefficient systems determined by a family $\mathcal{F}$, we show in §4.4 that (a slight generalization of) Blumberg-Hill's indexing systems $\mathcal{F}$ give rise to sieves $\Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ and partial G-symmetric sequences $\mathsf{Sym}_{\mathcal{F}}(\mathcal{F}) = \mathcal{V}^{\Sigma^{op}_{\mathcal{F}}}$ which are suitably compatible with the monad $\mathbb{F}_G$, thus giving rise to categories $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ of *partial genuine equivariant operads.*

Theorem II. *Let* $(\mathcal{V}, \otimes)$ *denote either* $(\mathsf{sSet}, \times)$ *or* $(\mathsf{sSet}_*, \wedge)$. *Then the projective model structure on* $\mathsf{Op}_G(\mathcal{V})$ *exists. Explicitly, a map* $\mathcal{P} \to \mathcal{P}'$ *is a weak equivalence (resp. fibration) if all maps*

$$\mathcal{P}(C) \to \mathcal{P}'(C) \tag{1.13}$$

are weak equivalences (fibrations) in $\mathcal{V}$ *for each* $C \in \Sigma_G$.

More generally, for $\mathcal{F}$ *a weak indexing system, the projective model structure on* $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ *exists. Explicitly, weak equivalences (resp. fibrations) are determined by (1.13) for* $C \in \Sigma_{\mathcal{F}}$.

Lastly, analogous semi-model structures on $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ *exist provided that* $(\mathcal{V},\otimes)$*: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals.*

Theorem II is proven in §5.4 in parallel with Theorem I. We note that the condition (v) that $(\mathcal{V},\otimes)$ has diagonals (cf. Remark 2.18), which is not needed in Theorem I, is required to build the monad $\mathbb{F}_G$, and hence the categories $\mathsf{Op}_G(\mathcal{V})$, $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$.

The following is our main result.

Theorem III. *Let* $(\mathcal{V},\otimes)$ *denote either* $(\mathsf{sSet},\times)$ *or* $(\mathsf{sSet}_*,\wedge)$.

Then the adjunctions, where in the more general rightmost case $\mathcal{F}$ *is a weak indexing system,*

$$\mathsf{Op}_G(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G(\mathcal{V}), \qquad \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}). \tag{1.14}$$

are Quillen equivalences.

Morover, analogous Quillen equivalences of semi-model structures[1] $\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \simeq \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ *exist provided that* $(\mathcal{V},\otimes)$*: (i) is a cofibrantly generated model category; (ii) is a closed monoidal model category with cofibrant unit; (iii) has cellular fixed points; (iv) has cofibrant symmetric pushout powers; (v) has diagonals; (vi) has cartesian fixed points.*

Theorem III is proven in §6.4. Condition (vi), which is not needed in either of Theorems I,II is discussed in §6.2.

Lastly, our techniques also verify the main conjecture of [3], which we discuss in §6.5. Moreover, we note that our models for $N\mathcal{F}$-operads are given by explicit bar constructions.

Corollary IV. *For* $\mathcal{V} = \mathsf{sSet}$ *or* Top *and* $\mathcal{F} = \{\mathcal{F}_n\}_{n\ge 0}$ *any weak indexing system,* $N\mathcal{F}$*-operads exist. That is, there exist explicit operads* $\mathcal{O}$ *such that*

$$\mathcal{O}(n)^{\Gamma} \sim \begin{cases} * & \text{if } \Gamma \in \mathcal{F}_n \\ \varnothing & \text{otherwise.} \end{cases} \tag{1.15}$$

In particular, the map $\mathrm{Ho}(N_\infty\text{-}\mathsf{Op}) \to \mathcal{I}$ *in [3, Cor. 5.6] is an equivalence of categories.*

Moreover, if $\mathcal{O}'$ *has fixed points as in (1.15) for some collection of graph subgroups* $\mathcal{F} = \{\mathcal{F}_n\}_{n\ge 0}$*, then* $\mathcal{F}$ *must be a weak indexing system.*

1.2 Future Work

In order to simplify our discussion, this paper focuses exclusively on the theory of single colored (genuine) equivariant operads. Nonetheless, we conjecture that all three of Theorems I,II,III extend to the colored setting, and intend to show this in upcoming work. We note, however, that an important new subtlety emerges in the equivariant setting: while usual colored equivariant operads have G-sets of objects, colored genuine equivariant operads will instead have *coefficient systems* of objects.

This paper and [24] are the first pieces of a broader project aimed at understanding different models for equivariant operads. In the next major step of the project, we intend to connect the two papers by generalizing the main theorem of Cisinski and Moerdijk in [6] and showing the existence of a Quillen equivalence

$$\mathsf{dSet}^G \rightleftarrows \mathsf{sOp}^G, \tag{1.16}$$

where dSet^G is the category of equivariant dendroidal sets of [24] and sOp^G the category of equivariant colored simplicial operads with its (conjectural) "with norms" model structure, as discussed in the previous paragraph.

[1]See [11, §12.1.8] for a precise definition.

1.3 Outline

This paper is comprised of two major halves, with §3, §4 addressing the definition of the novel structure of genuine equivariant operads, and §5, §6 addressing the proofs of the main results, Theorems I,II,III. A more detailed outline follows.

§2 discusses some preliminary notions and notation that will be used throughout. Of particular importance are the notions of split Grothendieck fibrations, which we recall in §2.1, and the categorical wreath product defined in §2.2, which we use to define symmetric monoidal categories with diagonals (Remark 2.18).

§3 lays the groundwork for the definition of genuine equivariant operads in §4 by discussing the concept of node substitution (which is at the core of the definition of free operads) in the context of equivariant trees. The key idea, which is captured in diagram (3.37) and Proposition 3.85 is that such substitution data are encoded by special maps of G-trees that we call planar tall maps. The bulk of the section is spent studying these types of maps, culminating in the concept of planar strings in §3.4, which encode iterated substitution.

§4 then uses planar strings to provide the formal definition of the category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads in a two step process in §4.1 and §4.2. §4.3 then compares the genuine equivariant operad category $\mathsf{Op}_G(\mathcal{V})$ with the usual equivariant operad category $\mathsf{Op}^G(\mathcal{V})$, establishing the necessary adjunction to formulate Theorem III. §4.4 discusses the notion of partial genuine equivariant operads, which are very closely related to the indexing systems of Blumberg-Hill.

§5 proves Theorems I and II. As is often the case when proving existence of projective model structures, the key to this section is a careful analysis of the free extensions in Op_G as in diagram (5.1), with §5.1, §5.2, §5.3 dedicated to providing a suitable filtration of such free extensions, and §5.4 concluding the proofs.

§6 proves our main result, Theorem III. The core of the technical analysis is given in §6.1, §6.2 and §6.3, which carefully study the interplay between families of subgroups, fixed points, and pushout products, and provide the necessary ingredients for the characterization of the cofibrant objects in $\mathsf{Op}_G(\mathcal{V})$ given in Lemma 6.59, and from which Theorem III easily follows. §6.5 then establishes Corollary IV by using the theory of genuine equivariant operads to build explicit cofibrant models for $N\mathcal{F}$-operads.

Lastly, Appendix A provides the proof of a lengthy technical result needed when establishing the filtrations in §5.

2 Preliminaries

2.1 Grothendieck fibrations

Recall that a functor $\pi: \mathcal{E} \to \mathcal{B}$ is called a *Grothendieck fibration* [5, §8.1] if for every arrow $f: b' \to b$ in $\mathcal{B}$ and $e \in \mathcal{E}$ such that $\pi(e) = b$, there exists a *cartesian arrow* $f^*e \to e$ lifting f, i.e. an arrow such that for any choice of horizontal arrows

$$\begin{tikzcd} e'' \arrow[rr] \arrow[dr, dashed, "\exists!"'] & & e \\ & f^*e \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} b'' \arrow[rr] \arrow[dr] & & b \\ & b' \arrow[ur, "f"'] & \end{tikzcd}$$

for which the rightmost diagram commutes and $e'' \to e$ lifts $b'' \to b$, there exists a unique dashed arrow $e'' \to f^*e$ lifting $b'' \to b'$ and making the leftmost diagram commute.

In most contexts the cartesian arrows $f^*e \to e$ are assumed to be defined only up to unique isomorphism, but in all examples considered in this paper we will be able to identify preferred choices of cartesian arrows, and we will refer to those preferred choices as *pullbacks*. Moreover, pullbacks will be compatible with composition and units in the obvious way, i.e. $g^*f^*e = (fg)^*e$ and $id_b^*e = e$. On a terminological note, the data of a Grothendieck fibration together with such choices of pullbacks is sometimes called a *split fibration*, but we will have no need to distinguish the two concepts outside of the present discussion.

A map of Grothendieck fibrations (resp. split fibrations) is then a commutative diagram

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\delta"] \arrow[dr, "\pi"'] & & \bar{\mathcal{E}} \arrow[dl, "\bar{\pi}"] \\ & \mathcal{B} & \end{tikzcd} \tag{2.1}$$

such that δ preserves cartesian arrows (pullbacks).

There is a well known equivalence between Grothendieck fibrations over $\mathcal{B}$ and contravariant pseudo-functors $\mathcal{B}^{op} \to \mathsf{Cat}$ with split fibrations corresponding to (regular) contravariant functors. We recall how this works in the split case, starting with the covariant version.

Definition 2.2. Given a small category $\mathcal{B}$ and functor $\mathcal{E}_\bullet$

$$\begin{aligned} \mathcal{B} &\xrightarrow{\mathcal{E}_\bullet} \mathsf{Cat} \\ b &\longmapsto \mathcal{E}_b \end{aligned} \tag{2.3}$$

the *covariant Grothendieck construction* $\mathcal{B} \ltimes \mathcal{E}_\bullet$ (over $\mathcal{B}$) has objects pairs (b, e) with $b \in \mathcal{B}$, $e \in \mathcal{E}_b$ and arrows $(b, e) \to (b', e')$ given by pairs

$$(f: b \to b', g: f_*(e) \to e'),$$

where $f_*: \mathcal{E}_b \to \mathcal{E}_{b'}$ is a shorthand for the functor $\mathcal{E}_\bullet(f)$.

Note that the chosen pushforward of (b, e) along $f: b \to b'$ is then $(b', f_* e)$.

Further, for a contravariant functor $\mathcal{E}_\bullet: \mathcal{B}^{op} \to \mathsf{Cat}$, the *contravariant Grothendieck construction* is $(\mathcal{B}^{op} \ltimes \mathcal{E}_\bullet^{op})^{op}$ (over $\mathcal{B}$).

One useful property of Grothendieck fibrations is that right Kan extensions can be computed using fibers, i.e., given a functor $F: \mathcal{E} \to \mathcal{V}$ into a complete category $\mathcal{V}$ one has

$$\mathsf{Ran}_\pi F(b) \simeq \mathsf{lim} F|_{b \downarrow \mathcal{E}} \simeq \mathsf{lim} F|_{\mathcal{E}_b} \tag{2.4}$$

where the first identification is the usual pointwise formula for Kan extensions (cf. [20, X.3 Thm. 1]) and the second identification follows by noting that due to the existence of cartesian arrows the fibers $\mathcal{E}_b$ are initial (in the sense of [20, IX.3]) in the undercategories $b \downarrow \mathcal{E}$. In fact, a little more is true: a choice of cartesian arrows yields a right adjoint to the inclusion $\mathcal{E}_b \hookrightarrow b \downarrow \mathcal{E}$, so that $\mathcal{E}_b$ is a coreflexive subcategory of $b \downarrow \mathcal{E}$, a well known sufficient condition for initiality. In practice, we will also need a generalization of the Kan extension formula (2.4) for maps of Grothendieck fibrations as in (2.1). Keeping the notation therein, given an $\bar{e} \in \bar{\mathcal{E}}$ we will write $\bar{e} \downarrow_\pi \mathcal{E} \hookrightarrow \bar{e} \downarrow \mathcal{E}$ for the full subcategory of those pairs $(e, f: \bar{e} \to \delta(e))$ such that $\bar{\pi}(f) = id_{\bar{\pi}(\bar{e})}$.

Proposition 2.5. *Given a map of Grothendieck fibrations as in* (2.1)*, each subcategory* $\bar{e} \downarrow_\pi \mathcal{E}$ *for* $\bar{e} \in \bar{\mathcal{E}}$ *is an initial subcategory of* $\bar{e} \downarrow \mathcal{E}$ *and hence for each functor* $\mathcal{E} \to \mathcal{V}$ *with* $\mathcal{V}$ *complete one has*

$$\mathsf{Ran}_\delta F(\bar{e}) \simeq \mathsf{lim} F|_{\bar{e} \downarrow \mathcal{E}} \simeq \mathsf{lim} F|_{\bar{e} \downarrow_\pi \mathcal{E}}. \tag{2.6}$$

Proof. One readily checks that the assignment $(e, f: \bar{e} \to \delta(e)) \mapsto ((\pi(f)^* e, \bar{e} \to \delta\pi(f)^*(e)))$ (where $\delta\pi(f)^* = \bar{\pi}^*(f)\delta$) is right adjoint to the inclusion $\bar{e} \downarrow_\pi \mathcal{E} \hookrightarrow \bar{e} \downarrow \mathcal{E}$, so that the claim follows by coreflexivity (note that if we are not in the split case, pullbacks may be chosen arbitrarily). $\square$

We also record the following, the proof of which is straightforward.

Proposition 2.7. *Suppose that $\mathcal{E} \to \mathcal{B}$ is a (split) Grothendieck fibration. Then so is the map of functor categories $\mathcal{E}^{\mathcal{C}} \to \mathcal{B}^{\mathcal{C}}$ for any category $\mathcal{C}$, as well as the map $\bar{\mathcal{E}} \to \bar{\mathcal{B}}$ in any pullback of categories*

$$\begin{tikzcd} \bar{\mathcal{E}} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ \bar{\mathcal{B}} \arrow[r] & \mathcal{B}. \end{tikzcd}$$

2.2 Wreath product over finite sets

Throughout we will let F denote the usual skeleton of the category of (ordered) finite sets and all set maps. Explicitly, its objects are the finite sets $\{1, 2, \cdots, n\}$ for $n \geq 0$.

Definition 2.8. For a category $\mathcal{C}$, we write $\mathsf{F} \wr \mathcal{C} = (\mathsf{F}^{op} \ltimes (\mathcal{C}^{op})^{\times \bullet})^{op}$ for the contravariant Grothendieck construction (cf. Definition 2.2) of the functor

$$\begin{aligned} \mathsf{F}^{op} &\longrightarrow \mathsf{Cat} \\ I &\longmapsto \mathcal{C}^{\times I} \end{aligned}$$

Explicitly, the objects of $\mathsf{F} \wr \mathcal{C}$ are tuples $(c_i)_{i \in I}$ and a map $(c_i)_{i \in I} \to (d_j)_{j \in J}$ consists of a pair

$$(\phi: I \to J, (f_i: c_i \to d_{\phi(i)})_{i \in I}),$$

henceforth abbreviated as $(\phi, (f_i))$.

Remark 2.9. Let $(c_i)_{i \in I} \in \mathsf{F} \wr \mathcal{C}$ and write λ for the partition $I = \lambda_1 \amalg \cdots \amalg \lambda_k$ such that $1 \leq i_1, i_2 \leq n$ are in the same class iff $c_{i_1}, c_{i_2} \in \mathcal{C}$ are isomorphic. Writing $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$ and picking representatives $i_j \in \lambda_j$, the automorphism group of $(c_i)_{i \in I}$ is given by

$$\mathsf{Aut}\left((c_i)_{i \in I}\right) \simeq \Sigma_\lambda \wr \prod_i \mathsf{Aut}(c_i) \simeq \Sigma_{|\lambda_1|} \wr \mathsf{Aut}(c_{i_1}) \times \cdots \times \Sigma_{|\lambda_k|} \wr \mathsf{Aut}(c_{i_k}). \tag{2.10}$$

Notation 2.11. Using the coproduct functor $\mathsf{F}^{\wr 2} = \mathsf{F}^{\wr \{0,1\}} = \mathsf{F} \wr \mathsf{F} \xrightarrow{\amalg} \mathsf{F}$ (where $\coprod_{i \in I} J_i$ is ordered lexicographically) and the singleton $\{1\} \in \mathsf{F}$ one can regard the collection of categories $\mathsf{F}^{\wr n+1} \wr \mathcal{C} = \mathsf{F}^{\wr \{0, \cdots, n\}} \wr \mathcal{C}$ for $n \geq -1$ as a coaugmented cosimplicial object in Cat. As such, we will denote by

$$\delta^i: \mathsf{F}^{\wr n} \wr \mathcal{C} \to \mathsf{F}^{n+1} \wr \mathcal{C}, \qquad 0 \leq i \leq n$$

the cofaces obtained by inserting singletons $\{1\} \in \mathsf{F}$ and by

$$\sigma^i: \mathsf{F}^{n+2} \wr \mathcal{C} \to \mathsf{F}^{n+1} \wr \mathcal{C}, \qquad 0 \leq i \leq n$$

the codegeneracies obtained by applying the coproduct $\mathsf{F}^{\wr 2} \xrightarrow{\amalg} \mathsf{F}$ to adjacent F coordinates.

Further, note that there are identifications $\mathsf{F} \wr \delta^i = \delta^{i+1}$, $\mathsf{F} \wr \sigma^i = \sigma^{i+1}$.

Remark 2.12. If $\mathcal{V}$ has all finite coproducts then injections and fold maps assemble into a functor as on the left below. Dually, if $\mathcal{V}$ has all finite products then projections and diagonals assemble into a functor as on the right.

$$\begin{aligned} \mathsf{F} \wr \mathcal{V} &\xrightarrow{\ \amalg\ } \mathcal{V} \\ (v_i)_{i \in I} &\longmapsto \textstyle\coprod_{i \in I} v_i \end{aligned} \qquad\qquad \begin{aligned} (\mathsf{F} \wr \mathcal{V}^{op})^{op} &\xrightarrow{\ \Pi\ } \mathcal{V} \\ (v_i)_{i \in I} &\longmapsto \textstyle\prod_{i \in I} v_i \end{aligned} \tag{2.13}$$

Moreover, these functors satisfy a number of additional coherence conditions. Firstly, there is a natural isomorphism α as on the left below

$$\begin{tikzcd} \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F} \wr \amalg"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"] & \mathcal{V} \arrow[d, equal] \arrow[dll, Leftarrow, "\alpha"] \\ \mathsf{F} \wr \mathcal{V} \arrow[rr, "\amalg"'] & & \mathcal{V} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{V} \arrow[d, "\delta^0"'] \arrow[dr, equal] & \\ \mathsf{F} \wr \mathcal{V} \arrow[r, "\amalg"'] & \mathcal{V} \end{tikzcd} \tag{2.14}$$

that encodes both reparenthesizing of coproducts and removal of initial objects (note that the empty tuple $()_{i\in\varnothing} \in \mathsf{F} \wr \mathcal{V}$ is mapped under $\coprod$ to an initial object of $\mathcal{V}$). Additionally, we are free to assume that the triangle on the right of (2.14) strictly commutes, i.e. that "unary coproducts" of singletons (v) are given simply by v itself. α is then associative in the sense that the composite natural isomorphisms between the two functors $\mathsf{F}^{\wr 3} \wr \mathcal{V} \to \mathcal{V}$ in the diagrams below coincide.

$$
\begin{tikzcd}
\mathsf{F}^{\wr 3} \wr \mathcal{V} \arrow[r, "\mathsf{F}^{\wr 2}\wr\coprod"] \arrow[d, "\sigma^0"'] & \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\coprod"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] & \mathcal{V} \arrow[d, equal] \arrow[dll, Rightarrow, "\alpha"] \\
\mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\coprod"] \arrow[d, "\sigma^1"'] & \mathsf{F} \wr \mathcal{V} \arrow[rr, "\coprod"] & & \mathcal{V} \arrow[d, equal] \arrow[dlll, Rightarrow, "\alpha"] \\
\mathsf{F} \wr \mathcal{V} \arrow[rrr, "\coprod"'] & & & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F}^{\wr 3} \wr \mathcal{V} \arrow[r, "\mathsf{F}^{\wr 2}\wr\coprod"] \arrow[d, "\sigma^0"'] & \mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\coprod"] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] \arrow[d, equal] \arrow[dll, Rightarrow, "\mathsf{F}\wr\alpha"] & \mathcal{V} \arrow[d, equal] \\
\mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[rr, "\mathsf{F}\wr\coprod"] \arrow[d, "\sigma^0"'] & & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] & \mathcal{V} \arrow[d, equal] \arrow[dlll, Rightarrow, "\alpha"] \\
\mathsf{F} \wr \mathcal{V} \arrow[rrr, "\coprod"'] & & & \mathcal{V}
\end{tikzcd}
\tag{2.15}
$$

Similarly, α is unital in the sense that both of the following diagrams strictly commute or, more precisely, the composite natural transformation in either diagram is the identity for the functor $\coprod \colon \mathsf{F} \wr \mathcal{V} \to \mathcal{V}$.

$$
\begin{tikzcd}
\mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] \arrow[d, "\delta^0"'] & \mathcal{V} \arrow[r, equal] \arrow[d, "\delta^0"'] & \mathcal{V} \arrow[d, equal] \\
\mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\coprod"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] & \mathcal{V} \arrow[d, equal] \arrow[dll, Rightarrow, "\alpha"] \\
\mathsf{F} \wr \mathcal{V} \arrow[rr, "\coprod"'] & & \mathcal{V}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathsf{F} \wr \mathcal{V} \arrow[r, equal] \arrow[d, "\delta^1"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] \arrow[d, equal] & \mathcal{V} \arrow[d, equal] \\
\mathsf{F}^{\wr 2} \wr \mathcal{V} \arrow[r, "\mathsf{F}\wr\coprod"] \arrow[d, "\sigma^0"'] & \mathsf{F} \wr \mathcal{V} \arrow[r, "\coprod"] & \mathcal{V} \arrow[d, equal] \arrow[dll, Rightarrow, "\alpha"] \\
\mathsf{F} \wr \mathcal{V} \arrow[rr, "\coprod"'] & & \mathcal{V}
\end{tikzcd}
\tag{2.16}
$$

Remark 2.17. More generally, if $\mathcal{V}$ is an arbitrary symmetric monoidal category, one always has a functor $\Sigma \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ (where as usual $\Sigma \hookrightarrow \mathsf{F}$ denotes the skeleton of finite sets and isomorphisms) satisfying the obvious analogues of (2.14), (2.15), (2.16), as is readily shown using the standard coherence results for symmetric monoidal categories (moreover, we note that α itself encodes all associativity, unital and symmetry isomorphisms, with the right side of (2.14) and (2.16) being mere common sense desiderata for "unary products").

It is likely no surprise that the converse is also true, i.e. that a functor $\Sigma \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ satisfying the analogues of (2.14), (2.15), (2.16) endows $\mathcal{V}$ with a symmetric monoidal structure. We will however have no direct need to use this fact, and as such include only a few pointers concerning the associativity pentagon axiom (the hardest condition to check) that the interested reader may find useful. Firstly, it becomes convenient to write expressions such as $(A \otimes B) \otimes C$ instead as $(A \otimes B) \otimes (C)$, so as to encode notationally the fact that this is the image of $((A, B), (C)) \in \Sigma^{\wr 2} \wr \mathcal{V}$ under the top map in (2.14). The associativity isomorphisms are hence given by the composites $(A \otimes B) \otimes (C) \xrightarrow{\simeq} A \otimes B \otimes C \xleftarrow{\simeq} (A) \otimes (B \otimes C)$ obtained by combining $\alpha_{((A,B),(C))}$ and $\alpha_{((A),(B,C))}$. The pentagon axiom is then checked by combining *six* instances of each of the squares in (2.15) (i.e. twelve squares total), most of which are obvious except for the fact that the $(A \otimes B) \otimes (C \otimes D)$ vertex of the pentagon contributes two pairs of squares rather than just one, with each pair corresponding to the two alternate expressions $((A \otimes B)) \otimes ((C) \otimes (D))$ and $((A) \otimes (B)) \otimes ((C \otimes D))$.

Remark 2.18. In lieu of the two previous remarks, and writing $\mathsf{F}_s \hookrightarrow \mathsf{F}$ for the subcategory of surjections, we define a *symmetric monoidal category with fold maps* as a category $\mathcal{V}$ together with a functor $\mathsf{F}_s \wr \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ satisfying the analogues of (2.14), (2.15), (2.16). Further, the dual of such $\mathcal{V}$ is called a *symmetric monoidal category with diagonals*[2].

[2]These have also been called *relevant monoidal categories* [8].

Similarly, replacing F_s with the subcategory $\mathsf{F}_i \hookrightarrow \mathsf{F}$ of injections yields the notion of a *symmetric monoidal category with injection maps* or, dually, *symmetric monoidal category with projections*[3].

Finally, we note that if a symmetric monoidal category has both diagonals and projections, it must in fact be *cartesian monoidal* [9, IV.2].

Remark 2.19. Extending Notation 2.11 one sees that $\mathsf{F}\wr(-)$, $\mathsf{F}_i\wr(-)$, $\mathsf{F}_s\wr(-)$, $\Sigma\wr(-)$ define monads in the category of categories.

We end this section by collecting some straightforward lemmas that will be used in §4.

Lemma 2.20. *If $\mathcal{E}\to\mathcal{B}$ a (split) Grothendieck fibration then so is $\mathsf{F}_s\wr\mathcal{E}\to\mathsf{F}_s\wr\mathcal{B}$.*

Moreover, if $\mathcal{E}\to\bar{\mathcal{E}}$ is a map of (split) Grothendieck fibrations over $\mathcal{B}$ then $\mathsf{F}_s\wr\mathcal{E}\to\mathsf{F}_s\wr\bar{\mathcal{E}}$ is a map of (split) Grothendieck fibrations over $\mathsf{F}_s\wr\mathcal{B}$.

Proof. Given a map $(\phi,(f_i))\colon(b'_i)_{i\in I}\to(b_j)_{j\in J}$ in $\mathsf{F}\wr\mathcal{B}$ and object $(e_j)_{j\in J}$ one readily checks that its pullback can be defined by $(f^*_{\phi(i)}e_{\phi(i)})_{i\in I}$. □

Lemma 2.21. *Suppose that $\mathcal{V}$ is a bicomplete monoidal category with fold maps such that the monoidal product commutes with limits in each variable. If the leftmost diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "k"'] & \mathcal{V} \\ \mathcal{D} \arrow[ur, "H"'] \arrow[u, phantom, shift left=4, "\Uparrow\eta"] & \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathsf{F}_s\wr\mathcal{C} \arrow[r, "\mathsf{F}_s\wr G"] \arrow[d, "\mathsf{F}_s\wr k"'] & \mathsf{F}_s\wr\mathcal{V} \arrow[r, "\otimes"] & \mathcal{V} \\ \mathsf{F}_s\wr\mathcal{D} \arrow[ur, "\mathsf{F}_s\wr H"'] \arrow[urr, bend right, "\otimes\circ\mathsf{F}_s\wr H"'] \arrow[u, phantom, shift left=6, "\Uparrow\mathsf{F}_s\wr\eta"] & & \end{tikzcd} \tag{2.22}$$

is a right Kan extension diagram then so is the composite of the rightmost diagram.

Dually, if $\mathcal{V}$ has diagonals, the monoidal product commutes with colimits in each variable, and the leftmost diagram

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "G"] \arrow[d, "k^{op}"'] & \mathcal{V} \\ \mathcal{D}^{op} \arrow[ur, "H"'] \arrow[u, phantom, shift left=4, "\Downarrow\epsilon"] & \end{tikzcd} \qquad\qquad \begin{tikzcd} (\mathsf{F}_s\wr\mathcal{C})^{op} \arrow[r, "(\mathsf{F}_s\wr G^{op})^{op}"] \arrow[d, "(\mathsf{F}_s\wr k)^{op}"'] & (\mathsf{F}_s\wr\mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\ (\mathsf{F}_s\wr\mathcal{D})^{op} \arrow[ur, "(\mathsf{F}_s\wr H^{op})^{op}"'] \arrow[urr, bend right, "\otimes\circ(\mathsf{F}_s\wr H^{op})^{op}"'] \arrow[u, phantom, shift left=6, "\Downarrow"] & & \end{tikzcd} \tag{2.23}$$

is a left Kan extension diagram then so is the composite of the rightmost diagram.

Proof. Unpacking definitions using the pointwise formula for Kan extensions (cf. [20, X.3 Thm. 1] or (2.4)), the claim concerning (2.22) amounts to showing that for each $(d_i)\in\mathsf{F}_s\wr\mathcal{D}$ one has natural isomorphisms

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow\mathsf{F}_s\wr\mathcal{C})}\left(\bigotimes_j G(c_j)\right)\simeq\bigotimes_i\lim_{(d_i\to kc_i)\in d_i\downarrow\mathcal{C}}(G(c_i)). \tag{2.24}$$

Proposition 2.5 now applies to the map $\mathsf{F}_s\wr\mathcal{C}\to\mathsf{F}_s\wr\mathcal{D}$ of Grothendieck fibrations over F_s and one readily checks that $(d_i)\downarrow_\pi\mathsf{F}_s\wr\mathcal{C}\simeq\prod_i(d_i\downarrow\mathcal{C})$ so that

$$\lim_{((d_i)\to(kc_j))\in((d_i)\downarrow\mathsf{F}_s\wr\mathcal{C})}\left(\bigotimes_j G(c_j)\right)\simeq\lim_{(d_i\to kc_i)\in\prod_i(d_i\downarrow\mathcal{D})}\left(\bigotimes_i G(c_i)\right)$$

and the isomorphisms (2.24) now follow from the assumption that the monoidal product commutes with limits in each variable. □

Remark 2.25. The previous results also hold if we replace F_s with F, F_i, Σ.

[3] These are equivalent to *semicartesian symmetric monoidal categories* [19].

2.3 Monads and adjunctions

In §4 we will make use of the following straightforward results concerning the transfer of monads along adjunctions (note that L (resp. R) denotes the left (right) adjoint).

Proposition 2.26. *Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be an adjunction and T a monad on $\mathcal{D}$. Then:*

(i) RTL is a monad and R induces a functor $R: \mathsf{Alg}_T(\mathcal{D}) \to \mathsf{Alg}_{RTL}(\mathcal{C})$;

(ii) if $LRTL \xrightarrow{\epsilon} TL$ is an isomorphism one further has an induced adjunction

$$L: \mathsf{Alg}_{RTL}(\mathcal{C}) \rightleftarrows \mathsf{Alg}_T(\mathcal{D}): R.$$

Proposition 2.27. *Let $L: \mathcal{C} \rightleftarrows \mathcal{D}: R$ be an adjunction, T a monad on $\mathcal{C}$, and suppose further that*

$$LR \xrightarrow{\epsilon} id_{\mathcal{D}}, \qquad LT \xrightarrow{\eta} LTRL$$

are natural isomorphisms (so that in particular $\mathcal{D}$ is a reflexive subcategory of $\mathcal{C}$). Then:

(i) LTR is a monad, with multiplication and unit given by

$$LTRLTR \xrightarrow{\eta^{-1}} LTTR \to LTR, \qquad id_{\mathcal{D}} \xrightarrow{\epsilon^{-1}} LR \to LTR;$$

(ii) $d \in \mathcal{D}$ is a LTR-algebra iff Rd is a T-algebra;

(iii) there is an induced adjunction

$$L: \mathsf{Alg}_T(\mathcal{C}) \rightleftarrows \mathsf{Alg}_{LTR}(\mathcal{D}): R.$$

Any monad T on $\mathcal{C}$ induces obvious monads $T^{\times l}$ on $\mathcal{C}^{\times l}$. More generally, and letting I denote the identity monad, a partition $\{1, \cdots, l\} = \lambda_a \amalg \lambda_i$, which we denote by λ, determines a monad $T^{\times \lambda} = T^{\times \lambda_a} \times I^{\times \lambda_i}$ on $\mathcal{C}^{\times l}$. Here "a" stands for "active" and "i" for "inert".

Such monads satisfy a number of compatibility conditions. Firstly, if $\lambda'_a \subseteq \lambda_a$ there is a monad map $T^{\times \lambda'} \Rightarrow T^{\times \lambda}$, and we write $\lambda' \leq \lambda$. Moreover, writing $\alpha^*: \mathcal{C}^{\times m} \to \mathcal{C}^{\times l}$ for the forgetful functor induced by a map $\alpha: \{1, \cdots, l\} \to \{1, \cdots, m\}$, one has an equality $T^{\times \alpha^* \lambda} \alpha^* = \alpha^* T^{\times \lambda}$, where $\alpha^* \lambda$ is the pullback partition. The following is straightforward.

Proposition 2.28. *Suppose $\mathcal{C}$ has finite coproducts and write $\alpha_!: \mathcal{C}^{\times l} \to \mathcal{C}^{\times m}$ for the left adjoint of α^*. Then the map*

$$T^{\times \alpha^* \lambda} \Rightarrow \alpha^* T^{\times \lambda} \alpha_! \tag{2.29}$$

adjoint to the identity $T^{\times \alpha^* \lambda} \alpha^* = \alpha^* T^{\times \lambda}$ is a map of monads on $\mathcal{C}^{\times l}$.

Hence, since $T^{\times \lambda} \alpha_!$ is a right $\alpha^* T^{\times \lambda} \alpha_!$-module, it is also a right $T^{\times \lambda'}$-module[4] *whenever $\lambda' \leq \alpha^* \lambda$. Finally, the natural map*

$$\alpha_! T^{\times \alpha^* \lambda} \Rightarrow T^{\times \lambda} \alpha_! \tag{2.30}$$

is a map of right $T^{\times \alpha^* \lambda}$-modules, and thus also a map of right $T^{\times \lambda'}$-modules whenever $\lambda' \leq \alpha^* \lambda$.

Remark 2.31. We unpack the content of (2.30) when $\alpha: \{1, \cdots, l\} \to *$ is the unique map to the singleton $*$, in which case we write $\alpha_! = \coprod$. We thus have commutative diagrams

$$\begin{array}{ccc} \coprod_{j \in \lambda_a} TTA_j \amalg \coprod_{j \in \lambda_i} A_j & \longrightarrow & T\left(\coprod_{j \in \lambda_a} TA_j \amalg \coprod_{j \in \lambda_i} A_j\right) \\ \downarrow & & \downarrow \\ \coprod_{j \in \lambda_a} TA_j \amalg \coprod_{j \in \lambda_i} A_j & \longrightarrow & T\left(\coprod_{j \in \lambda_a} A_j \amalg \coprod_{j \in \lambda_i} A_j\right) \end{array} \tag{2.32}$$

[4] Recall that a right (resp. left) module over a monad T on $\mathcal{C}$ is a functor $M: \mathcal{C} \to \mathcal{D}$ (resp. $N: \mathcal{D} \to \mathcal{C}$) together with an action natural transformation $M \circ T \Rightarrow M$ (resp. $T \circ N \Rightarrow N$) that is suitably associative and unital.

for each collection $(A_j)_{j\in\underline{l}}$ in $\mathcal{C}$, where the vertical maps come from the right $T^{\times\lambda}$-module structure. Writing $\check{\amalg}$ for the coproduct of T-algebras and recalling the canonical identifications $\check{\coprod}_{k\in K}(TA_k) \simeq T\left(\coprod_{k\in K} A_k\right)$, (2.32) shows that the right $T^{\times\lambda}$-module structure on $T \circ \coprod$ codifies the multiplication maps

$$\check{\coprod}_{j\in\lambda_a} TTA_j \mathbin{\check{\amalg}} \check{\coprod}_{j\in\lambda_i} TA_j \to \check{\coprod}_{j\in\lambda_a} TA_j \mathbin{\check{\amalg}} \check{\coprod}_{j\in\lambda_i} TA_j.$$

3 Planar and tall maps, and substitution

Throughout, we will assume that the reader is familiar with the category Ω of trees. A good introduction to Ω is given by [22, §3], where arrows are described both via the "colored operad generated by a tree" and by identifying explicit generating arrows, called faces and degeneracies. Alternatively, Ω can also be described using the algebraic model of *broad posets* introduced by Weiss in [31] and further worked out by the second author in [24, §5]. This latter will be our "official" model, though a detailed understanding of broad posets is needed only to follow our formal discussion of planar structures in §3.1. Otherwise, the reader willing to accept the results of §3.1 should need only an intuitive grasp of the notations $\underline{e} \le e$, $f \le_d e$ and $e^{\uparrow}$ to read the remainder of the paper. Such understanding can be obtained from [24, Example 5.10] and Example 3.3 below.

Given a finite group G, there is also a category Ω_G of G-trees, jointly discovered by the authors and first discussed by the second author in [24, §4.3,§5.3], which we now recall. Firstly, we let Φ denote the category of forests, i.e. "formal coproducts of trees". A broad poset description of Φ is found in [24, §5.2], but here we prefer the alternative definition $\Phi = \mathsf{F} \wr \Omega$. The category of G-forests is then Φ^G, i.e. the category of G-objects in Φ. Similarly writing F^G for the category of G-objects in F and identifying the G-orbit category as the subcategory $\mathsf{O}_G \hookrightarrow \mathsf{F}^G$ of those sets with transitive actions, Ω_G can be described as given by the pullback of categories

$$\begin{tikzcd} \Omega_G \arrow[r] \arrow[d, "\mathsf{r}"'] & \Phi^G \arrow[d, "\mathsf{r}"] \\ \mathsf{O}_G \arrow[r] & \mathsf{F}^G \end{tikzcd} \tag{3.1}$$

(where $\mathsf{r} : \Phi \to \mathsf{F}$ is the *root functor*, sending a forest to its set of roots), which is a repackaging of [24, Def. 5.44]. Explicitly, a G-tree T is then a tuple $T = (T_x)_{x\in X}$ with $X \in \mathsf{O}_G$ together with isomorphisms $T_x \to T_{gx}$ that are suitably associative and unital.

3.1 Planar structures

The specific model for the orbit category O_G used in (3.1) has extra structure not found in the usual model (i.e. that of the G-sets G/H for $H \le G$), namely the fact that each $X \in \mathsf{O}_G$ comes with a canonical total order (the underlying set of X being one of the sets $\{1, \cdots, n\}$).

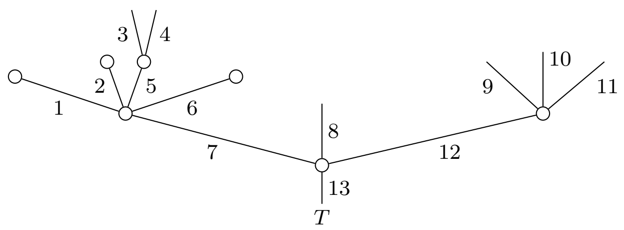
We will find it convenient to use a model of Ω with similar extra structure, given by planar structures on trees. Intuitively, a planar structure on a tree is the data of a planar representation of the tree, and definitions of *planar trees* along those lines are found throughout the literature. However, to allow for precise proofs of some key results concerning the interaction of planar structures with the maps in Ω (namely Propositions 3.21, 3.42) we will instead use a combinatorial definition of planar structures in the context of broad posets.

In what follows a tree will be a *dendroidally ordered broad poset* as in [31], [24, Def. 5.9].

Definition 3.2. Let $T \in \Omega$ be a tree. A *planar structure* of T is an extension of the descendancy partial order $\le_d$ to a total order $\le_p$ such that:

- *Planar*: if $e \le_p f$ and $e \nleq_d f$ then $g \le_d f$ implies $e \le_p g$.

Example 3.3. An example of a planar structure on a tree T follows, with $\leq_p$ encoded by the number labels.



Intuitively, given a planar depiction of a tree T, $e \leq_d f$ holds when the downward path from e passes through f and $e \leq_p f$ holds if either $e \leq_d f$ or if the downward path from e is to the left of the downward path from f (as measured at the node where the paths intersect).

It is visually clear that a planar depiction of a tree amounts to choosing a total order for each of the sets of *input edges* of each node (i.e. those edges immediately above that node).

While we will not need to make this last statement precise, we will nonetheless find it convenient to show that our Definition 3.2 of planarity is equivalent to such choices of total orders for each of the sets of input edges. To do so, we first introduce some notation.

Notation 3.4. Let $T \in \Omega$ be a tree and $e \in T$ an edge. We will denote

$$I(e) = \{f \in T : e \leq_d f\}$$

and refer to this poset as the *input path of* e.

We will repeatedly use the following, which is a consequence of [24, Cor. 5.26].

Lemma 3.5. *If $e \leq_d f$, $e \leq_d f'$, then f, f' are $\leq_d$-comparable.*

Proposition 3.6. *Let $T \in \Omega$ be a tree. Then*

(a) for any $e \in T$ the finite poset $I(e)$ is totally ordered;

(b) the poset $(T, \leq_d)$ has all joins, denoted $\vee$. In fact, $\bigvee_i e_i = \min(\bigcap_i I(e_i))$.

Proof. (a) is immediate from Lemma 3.5. To prove (b) we note that the root edge is in every input path, hence $\min(\bigcap_i I(e_i))$ exists by (a), and that this is clearly the join $\bigvee_i e_i$. □

Notation 3.7. Let $T \in \Omega$ be a tree and suppose that $e <_d b$. We will denote by $b_e^{\uparrow} \in T$ the predecessor of b in $I(e)$.

Proposition 3.8. *Suppose e, f are $\leq_d$-incomparable edges of T and write $b = e \vee f$. Then*

(a) $e <_d b$, $f <_d b$ and $b_e^{\uparrow} \neq b_f^{\uparrow}$;

(b) $b_e^{\uparrow}, b_f^{\uparrow} \in b^{\uparrow}$. In fact $\{b_e^{\uparrow}\} = I(e) \cap b^{\uparrow}$, $\{b_f^{\uparrow}\} = I(f) \cap b^{\uparrow}$;

(c) if $e' \leq_d e$, $f' \leq_d f$ then $b = e' \vee f'$ and $b_{e'}^{\uparrow} = b_e^{\uparrow}$, $b_{f'}^{\uparrow} = b_f^{\uparrow}$.

Proof. (a) is immediate: the condition $e = b$ (resp. $f = b$) would imply $f \leq_d e$ (resp. $e \leq_d f$) while the condition $b_e^{\uparrow} = b_f^{\uparrow}$ would provide a predecessor of b in $I(e) \cap I(f)$.

For (b), note that any relation $a <_d b$ factors as $a \leq_d b_a^* <_d b$ for some unique $b_a^* \in b^{\uparrow}$, where uniqueness follows from Lemma 3.5. Choosing $a = e$ implies $I(e) \cap b^{\uparrow} = \{b_e^*\}$ and letting a range over edges such that $e \leq_d a <_d b$ shows that b_e^* is in fact the predecessor of b.

To prove (c) one reduces to the case $e' = e$, in which case it suffices to check $I(e) \cap I(f') = I(e) \cap I(f)$. But if it were otherwise there would exist an edge a satisfying $f' \leq_d a <_d f$ and $e \leq_d a$, and this would imply $e \leq_d f$, contradicting our hypothesis. □

Proposition 3.9. *Let $c = e_1 \vee e_2 \vee e_3$. Then $c = e_i \vee e_j$ iff $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$.*
Therefore, all ternary joins in $(T, \leq_d)$ are binary, i.e.

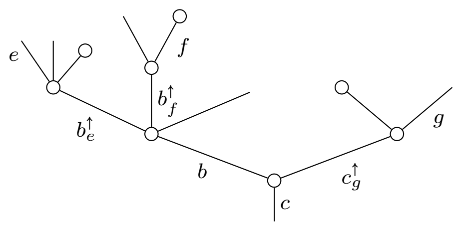
$$c = e_1 \vee e_2 \vee e_3 = e_i \vee e_j \tag{3.10}$$

for some $1 \leq i < j \leq 3$, and (3.10) fails for at most one choice of $1 \leq i < j \leq 3$.

Proof. If $c^{\uparrow}_{e_i} \neq c^{\uparrow}_{e_j}$ then $c = \min\left(I(e_i) \cap I(e_j)\right) = e_i \vee e_j$, whereas the converse follows from Proposition 3.8(a).

The "therefore" part follows by noting that $c^{\uparrow}_{e_1}$, $c^{\uparrow}_{e_2}$, $c^{\uparrow}_{e_3}$ can not all coincide, or else c would not be the minimum of $I(e_1) \cap I(e_2) \cap I(e_3)$. □

Example 3.11. In the following example $b = e \vee f$, $c = e \vee f \vee g$, $c^{\uparrow}_{e} = c^{\uparrow}_{f} = b$.



Given a set S of size n we write $\mathsf{Ord}(S) \simeq \mathsf{Iso}(S, \{1, \cdots, n\})$. We will also abuse notation by regarding its objects as pairs $(S, \leq)$ where $\leq$ is a total order on S.

Proposition 3.12. *Let $T \in \Omega$ be a tree. There is a bijection*

$$\begin{array}{ccc} \{\textit{planar structures } (T, \leq_p)\} & \xrightarrow{\simeq} & \prod_{(a^{\uparrow} \leq a) \in V(T)} \mathsf{Ord}(a^{\uparrow}) \\ \leq_p & \longmapsto & (\leq_p |_{a^{\uparrow}}) \end{array}$$

Proof. We will keep the notation of Proposition 3.8 throughout, i.e. e, f are $\leq_d$-incomparable edges and we write $b = e \vee f$.

We first show injectivity, i.e. that the restrictions $\leq_p |_{a^{\uparrow}}$ determine if $e <_p f$ holds or not. If $b^{\uparrow}_{e} <_p b^{\uparrow}_{f}$, the relations $e \leq_d b^{\uparrow}_{e} <_p b^{\uparrow}_{f} \geq_d f$ and Definition 3.2 imply it must be $e <_p f$. Dually, if $b^{\uparrow}_{f} <_p b^{\uparrow}_{e}$ then $f <_p e$. Thus $b^{\uparrow}_{e} <_p b^{\uparrow}_{f} \Leftrightarrow e <_p f$ and injectivity follows.

To check surjectivity, it suffices (recall that e, f are assumed $\leq_d$-incomparable) to check that defining $e \leq_p f$ to hold iff $b^{\uparrow}_{e} < b^{\uparrow}_{f}$ holds in $b^{\uparrow}$ yields a planar structure.

Antisymmetry and the total order conditions are immediate, and it thus remains to check the transitivity and planar conditions. Transitivity of $\leq_p$ in the case $e' \leq_d e <_p f$ and the planar condition, which is the case $e <_p f \geq_d f'$, follow from Proposition 3.8(c). Transitivity of $\leq_p$ in the case $e <_p f \leq_d f'$ follows since either $e \leq_d f'$ or else e, f' are $\leq_d$-incomparable, in which case one can apply Proposition 3.8(c) with the roles of f, f' reversed.

It remains to check transitivity in the hardest case, that of $e <_p f <_p g$ with $\leq_d$-incomparable f, g. We write $c = e \vee f \vee g$. By the "therefore" part of Proposition 3.9, either: (i) $e \vee f <_d c$, in which case Proposition 3.9 implies $c = e \vee g$, $c^{\uparrow}_{e} = c^{\uparrow}_{f}$ and transitivity follows; (ii) $f \vee g <_d c$, which follows just as (i); (iii) $e \vee f = f \vee g = c$, in which case $c^{\uparrow}_{e} < c^{\uparrow}_{f} < c^{\uparrow}_{g}$ in $c^{\uparrow}$ so that $c^{\uparrow}_{e} \neq c^{\uparrow}_{g}$ and by Proposition 3.9 it is also $c = e \vee g$ and transitivity follows. □

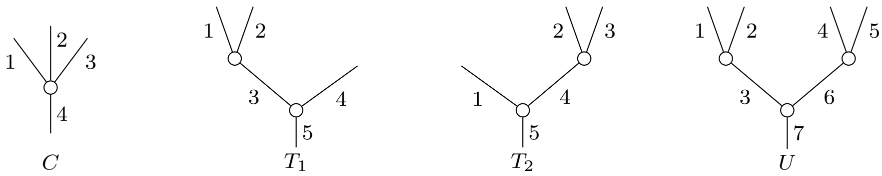
Remark 3.13. Proposition 3.12 states in particular that $\leq_p$ is the closure of the $\leq_d$ relations and the $\leq_p$ relations within each $a^{\uparrow}$ under the planar condition in Definition 3.2.

The discussion of the substitution procedure in §3.2 will be simplified by working with a model for the category Ω with exactly one representative of each possible planar structure on each tree or, more precisely, a model where the only isomorphisms preserving the planar structure are the identities. On the other hand, exclusively using such a model for Ω

throughout would, among other issues, make the discussion of faces in §3.2 rather awkward. We now describe our conventions to address such issues.

Let Ω^p denote the category of *planarized trees*, with objects pairs $T_{\leq_p} = (T, \leq_p)$ of trees together with a planar structure, and morphisms *underlying* maps of trees (i.e. ignoring the planar structures). There is a full subcategory $\Omega^s \hookrightarrow \Omega^p$, whose objects we call *standard models*, of those $T_{\leq_p}$ whose underlying set is one of the sets $\underline{n} = \{1, 2, \cdots, n\}$ and for which $\leq_p$ coincides with the canonical order.

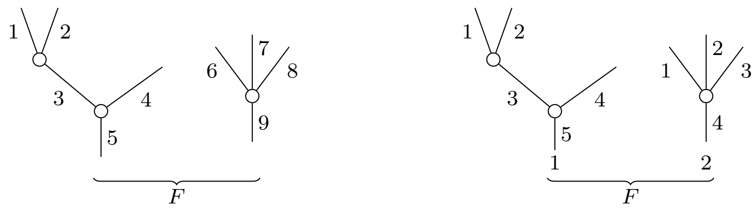
Example 3.14. Some examples of standard models, i.e. objects of Ω^s, follow (further, Example 3.3 can also be interpreted as such an example).



Here T_1 and T_2 are isomorphic to each other but not isomorphic to any other standard model in Ω^s while both C and U are the unique objects in their isomorphism classes.

Given $T_{\leq_p} \in \Omega^p$ there is an obvious standard model $T^s_{\leq_p} \in \Omega^s$ given by replacing each edge by its order following $\leq_p$. Indeed, this defines a retraction $(-)^s : \Omega^p \to \Omega^s$ and a natural transformation $\sigma : id \Rightarrow (-)^s$ given by isomorphisms preserving the planar structure (in fact, the pair $((-)^s, \sigma)$ is uniquely characterized by this property).

Remark 3.15. Definition 3.2 readily extends to the broad poset definition of forests $F \in \Phi$ in [24, Def. 5.27], with the analogue of Proposition 3.12 then stating that a planar structure is equivalent to total orderings of the nodes of F together with a total ordering of its set of roots. There are thus two competing notions of standard forests: the [24, Def. 5.27] model Φ^s whose objects are planar forest structures on one of the standard sets $\{1, \cdots, n\}$ and (following the discussion at the start of §3) the model $\mathsf{F} \wr \Omega^s$, whose objects are tuples, indexed by a standard set, of planar tree structures on standard sets. An illustration follows.



However, there is a *canonical* isomorphism $\Phi^s \simeq \mathsf{F} \wr \Omega^s$ (with both sides of the diagram above then depicting the same planar forest). Moreover, while the similarly defined categories Φ^p and $\mathsf{F} \wr \Omega^p$ are only equivalent (rather than isomorphic), their retractions onto $\Phi^s \simeq \mathsf{F} \wr \Omega^s$ are compatible, and we will thus henceforth not distinguish between Φ^s and $\mathsf{F} \wr \Omega^s$.

Convention 3.16. From now on we write simply Ω, Ω_G to denote the categories Ω^s, Ω^s_G of standard models (where planar structures are defined in the underlying forest as in Remark 3.15). Therefore, whenever a construction produces an object or diagram in Ω^p or Ω^p_G, we always implicitly reinterpret it by using the standardization functor $(-)^s$.

Similarly, any finite set (resp. orbital finite G-set) together with a total order is implicitly reinterpreted as an object of F (resp. O_G).

Example 3.17. To illustrate our convention, consider the trees in Example 3.14.

There are subtrees $F_1 \subset F_2 \subset U$ where F_1 is the subtree with edge set $\{1, 2, 6, 7\}$ and F_2 is the subtree with edge set $\{1, 2, 3, 6, 7\}$, both with inherited tree and planar structures.

Applying $(-)^s$ to the inclusion diagram on the left below then yields a diagram as on the right.

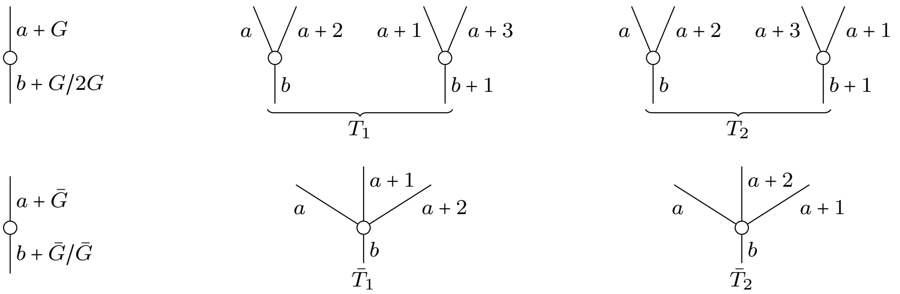
$$\begin{tikzcd}[column sep=small] F_1 \arrow[rr, hook] \arrow[dr, hook] & & U \\ & F_2 \arrow[ur, hook] & \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] C \arrow[rr] \arrow[dr] & & U \\ & T_1 \arrow[ur] & \end{tikzcd}$$

Similarly, let $\le_{(12)}$ and $\le_{(45)}$ denote alternate planar structures for U exchanging the orders of the pairs $1, 2$ and $4, 5$, so that one has objects $U_{\le_{(12)}}$, $U_{\le_{(45)}}$ in Ω^p. Applying $(-)^s$ to the diagram of underlying identities on the left yields the permutation diagram on the right.

$$\begin{tikzcd}[column sep=small] U \arrow[rr, "id"] \arrow[dr, "id"'] & & U_{\le_{(45)}} \\ & U_{\le_{(12)}} \arrow[ur, "id"'] & \end{tikzcd} \qquad\qquad \begin{tikzcd}[column sep=small] U \arrow[rr, "(45)"] \arrow[dr, "(12)"'] & & U \\ & U \arrow[ur, "(12)(45)"'] & \end{tikzcd}$$

Example 3.18. An additional reason to leave the use of $(-)^s$ implicit as described in Convention 3.16 is that when depicting G-trees it is preferable to choose edge labels that describe the action rather than the planarization (which is already implicit anyway).

For example, for the two groups $G = \mathbb{Z}_{/4}$ and $\bar{G} = \mathbb{Z}_{/3}$, in both diagrams below the orbital representation on the left represents the isomorphism class consisting only of the two trees $T_1, T_2 \in \Omega_G$ and $\bar{T}_1, \bar{T}_2 \in \Omega_{\bar{G}}$ on the right.



In general, isomorphism classes are of course far bigger. The interested reader may show that there are $3 \cdot 3! \cdot 2 \cdot 3! \cdot 3!$ trees in the isomorphism class of the tree depicted in (1.9).

Definition 3.19. A map $S \xrightarrow{\varphi} T$ in Ω preserving the planar structure $\le_p$ is called a *planar map*.

More generally, a map $F \to G$ in one of the categories Φ, Φ^G, Ω_G of forests, G-forests, G-trees is called a *planar map* if it is an independent map (cf. [24, Def. 5.28]) compatible with the planar structures $\le_p$.

Remark 3.20. The need for the independence condition is justified by [24, Lemma 5.33] and its converse, since non independent maps do not reflect $\le_d$-comparability.

However, we note that in the case of Ω_G independence admits simpler characterizations: φ is independent iff φ is injective on each edge orbit iff φ is injective on the root orbit.

Proposition 3.21. *Let $F \xrightarrow{\varphi} G$ be an independent map in Φ (or Ω, Ω_G, Φ_G). Then there is a unique factorization*

$$F \xrightarrow{\simeq} \bar{F} \to G$$

such that $F \xrightarrow{\simeq} \bar{F}$ is an isomorphism and $\bar{F} \to G$ is planar.

Proof. We need to show that there is a unique planar structure $\le_p^{\bar{F}}$ on the underlying forest of F making the underlying map a planar map. Simplicity of the broad poset G ensures that for any vertex $e^{\uparrow} \le e$ of F the edges in $\varphi(e^{\uparrow})$ are all distinct while independence of φ likewise ensures that the edges in $\varphi(\underline{r}_F)$ are distinct. By (the forest version of) Proposition 3.12 the only possible planar structure $\le_p^{\bar{F}}$ is the one which orders each set $e^{\uparrow}$ and the root tuple $\underline{r}_F$ according to their images. The claim that φ is then planar follows from Remark 3.13 together with the fact ([24, Lemma 5.33]) that φ reflects $\le_d$-comparability. $\square$

Remark 3.22. Proposition 3.21 says that planar structures can be pulled back along independent maps. However, they can not always be pushed forward. As a counter-example, in the setting of Example 3.14, consider the map $C \to T_1$ defined by $1 \mapsto 1$, $2 \mapsto 4$, $3 \mapsto 2$, $4 \mapsto 5$.

We end this section with a different type of pullback. Indeed, the reader may have noted that it follows from Proposition 2.7 that both vertical maps in (3.1) are split Grothendieck fibrations. We now introduce some terminology.

Definition 3.23. The map $\mathsf{r}\colon \Omega_G \to \mathsf{O}_G$ in (3.1) is called the *root functor*.

Further, fiber maps (i.e. maps inducing identities, i.e. ordered bijections, on $\mathsf{r}(-)$) are called *rooted maps* and pullbacks with respect to r are called *root pullbacks*.

To motivate the terminology, note first that unpacking definitions shows that $\mathsf{r}(T)$ is the ordered set of tree components of $T \in \Omega_G$, which coincides with the ordered set of roots. The exact name choice is meant to accentuate the connection with another key functor described in §3.3, which we call the *leaf-root functor*.

Further, unpacking the construction in (3.1), one sees that the pullback of the G-tree $T = (T_x)_{x\in X}$ with structure maps $T_x \to T_{gx}$ along the map $\varphi\colon Y \to X$ in O_G is simply the G-tree $(T_{\varphi(y)})_{y\in Y}$ with structure maps $T_{\varphi(y)} \to T_{g\varphi(y)} = T_{\varphi(gy)}$.

Example 3.24. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$, $H = \langle j \rangle$ and $K = \langle -1 \rangle$. Figure 1 illustrates the pullbacks of two G-trees, T and S, along the twist map $\tau\colon G/H \to G/H$ and the unique map $\pi\colon G/H \to G/G$, respectively (or, more precisely, noting that in our model the underlying set of G/H is actually $\{1,2\}$, τ is the permutation (12)). We note that the stabilizers of a, b, c are $\{1\}, K, H$ for T and K, H, G for S. The top depictions of $\tau^* T$, $\pi^* S$ then use the edge orbit generators suggested by T, S while the bottom depictions choose generators that are minimal with regard to the planar structure, so that in $\tau^* T$ it is $d = ic$, $e = ib$, $f = ia$ and in $\pi^* S$ it is $e = ib'$, $d = ia'$.

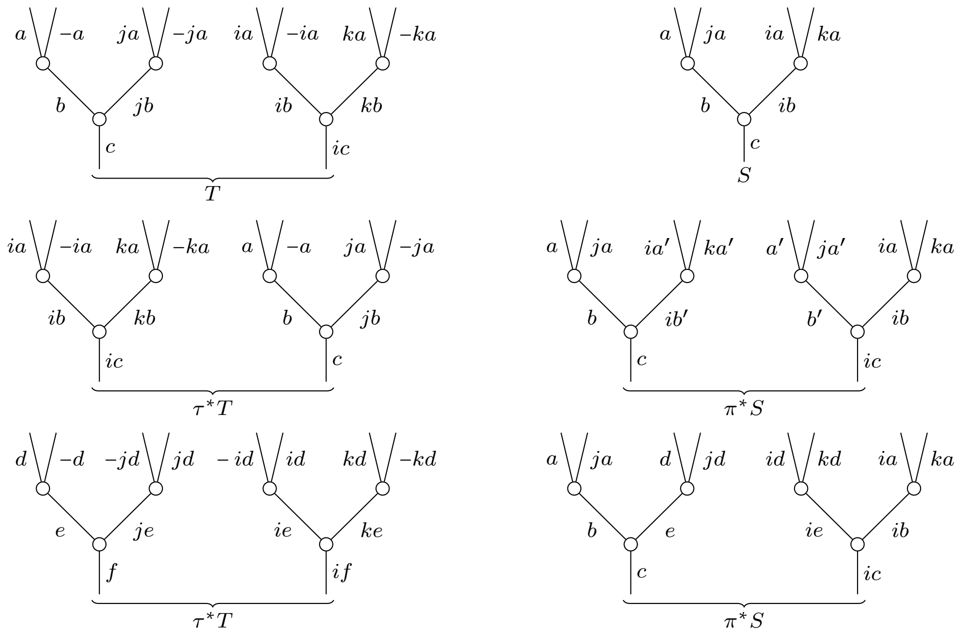


Figure 1: Root pullbacks

3.2 Outer faces, tall maps, and substitution

One of the key ideas needed to describe the free operad monad is the notion of *substitution* of tree nodes, a process that we will prefer to repackage in terms of maps of trees.

In preparation for that discussion, we first recall some basic definitions and results concerning outer subtrees and tree grafting, as in [24, §5].

Definition 3.25. Let $T \in \Omega$ be a tree and $e_1 \cdots e_n = \underline{e} \leq e$ a broad relation in T.

We define the *planar outer face* $T_{\underline{e} \leq e}$ to be the subtree with underlying set those edges $f \in T$ such that

$$f \leq_d e, \quad \forall_i f \nless_d e_i, \tag{3.26}$$

with generating broad relations the relations $f^{\uparrow} \leq f$ for those $f \in T$ satisfying $\forall_i f \neq e_i$ in addition to (3.26), and planar structure pulled back from T (in the sense of Remark 3.22).

Moreover, inclusions of the form $T_{\underline{e} \leq e} \hookrightarrow T$ are called *planar outer face maps.*

Remark 3.27. If one forgoes the requirement that $T_{\underline{e} \leq e}$ be equipped with the pulled back planar structure, the inclusion $T_{\underline{e} \leq e} \hookrightarrow T$ is usually called simply an *outer face map.*

We now recap some basic results.

Notation 3.28. We write $\eta \in \Omega$ for the *stick tree* consisting of a single edge and no vertices.

Proposition 3.29. *Let $T \in \Omega$ be a tree.*

(a) $T_{\underline{e} \leq e}$ is a tree with root e and leaf tuple $\underline{e}$;

(b) there is a bijection

$$\{\textit{planar outer faces of } T\} \leftrightarrow \{\textit{broad relations of } T\};$$

(c) if $R \to S$ and $S \to T$ are (planar) outer face maps then so is $R \to T$;

(d) any pair of broad relations $\underline{g} \leq v$, $\underline{f} v \leq e$ induces a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "v"] \arrow[d, "v"'] & T_{\underline{g} \leq v} \arrow[d] \\ T_{\underline{f} v \leq e} \arrow[r] & T_{\underline{f}\underline{g} \leq e}. \end{tikzcd} \tag{3.30}$$

Further, $T_{\underline{f}\underline{g} \leq e}$ is the unique choice of pushout that makes the maps in (3.30) planar.

Proof. We first show (a). That $T_{\underline{e} \leq e}$ is indeed a tree is the content of [24, Prop. 5.20]: more precisely, $T_{\underline{e} \leq e} = (T^{\leq e})_{<\underline{e}}$ in the notation therein. That the root of $T_{\underline{e} \leq e}$ is e is clear and that the leaf tuple is $\underline{e}$ follows from [24, Remark 5.23].

(b) follows from (a), which shows that $\underline{e} \leq e$ can be recovered from $T_{\underline{e} \leq e}$.

(c) follows from the definition of outer face together with [24, Lemma 5.33], which states that the $\leq_d$ relations on S, T coincide.

Since by (b) and (c) both $T_{\underline{g} \leq v}$ and $T_{\underline{f} v \leq e}$ are outer faces of $T_{\underline{f}\underline{g} \leq e}$, the first part of (d) is a restatement of [24, Prop. 5.15], while the additional planarity claim follows by Proposition 3.12 together with the vertex identification $V(T_{\underline{f}\underline{g} \leq e}) = V(T_{\underline{f} v \leq e}) \amalg V(T_{\underline{g} \leq v})$. □

Definition 3.31. A map $S \xrightarrow{\varphi} T$ in Ω is called a *tall map* if

$$\varphi(\underline{l}_S) = \underline{l}_T, \qquad \varphi(r_S) = r_T,$$

where $l_{(-)}$ denotes the (unordered) leaf tuple and $r_{(-)}$ the root.

The following is a restatement of [24, Cor. 5.24]

Proposition 3.32. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphism,*

$$S \xrightarrow{\varphi^t} U \xrightarrow{\varphi^u} T$$

as a tall map followed by an outer face (in fact, $U = T_{\varphi(l_S)\leq\varphi(r_S)}$).

We recall that a face $F \to T$ is called *inner* if it is obtained by iteratively removing inner edges, i.e. edges other than the root or the leaves. In particular, it follows that a face is inner if and only if it is tall. The usual degeneracy-face decomposition (cf. [22, Lemma 3.1] or [24, Prop. 5.37]) thus combines with Proposition 3.32 to give the following.

Corollary 3.33. *Any map $S \xrightarrow{\varphi} T$ in Ω has a factorization, unique up to unique isomorphisms,*

$$S \xrightarrow{\varphi^-} U \xrightarrow{\varphi^i} V \xrightarrow{\varphi^u} T$$

as a degeneracy followed by an inner face followed by an outer face.

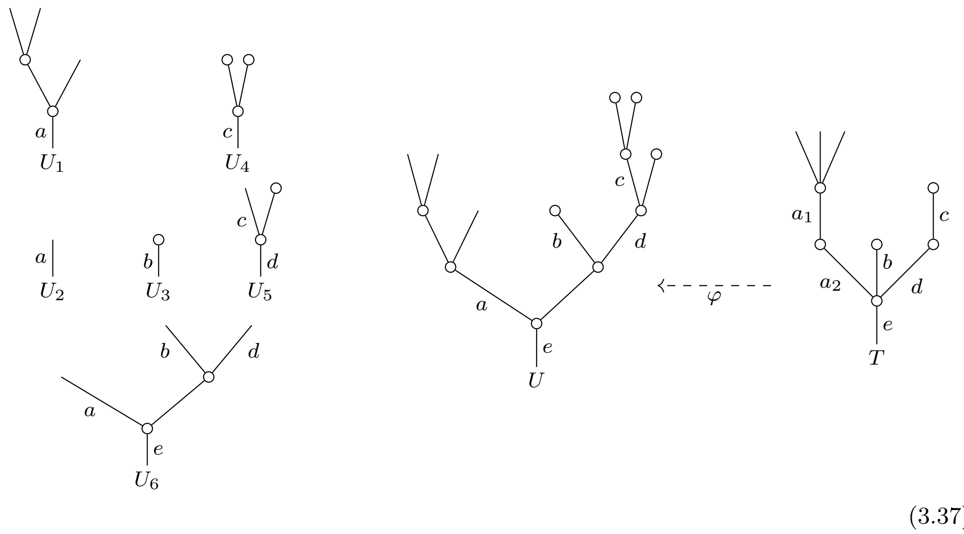
We will find it convenient throughout to regard the groupoid Σ of finite sets as the subcategory $\Sigma \hookrightarrow \Omega$ consisting of *corollas* (i.e. trees with a single vertex) and isomorphisms.

Notation 3.34. Given a tree $T \in \Omega$ there is a unique corolla $\mathsf{lr}(T) \in \Sigma$ and planar tall map $\mathsf{lr}(T) \to T$, which we call the *leaf-root* of T (this name is motivated by the equivariant analogue, discussed in §3.3). Explicitly, the number of leaves of $\mathsf{lr}(T)$ matches that of T, together with the inherited order.

We now turn to discussing the substitution operation. We start with an example focused on the closely related notion of iterated graftings of trees (as described in (3.30)).

Example 3.35. The trees $U_1, U_2, \cdots, U_6$ on the left below can be grafted to obtain the tree U in the middle. More precisely (among other possible grafting orders), one has

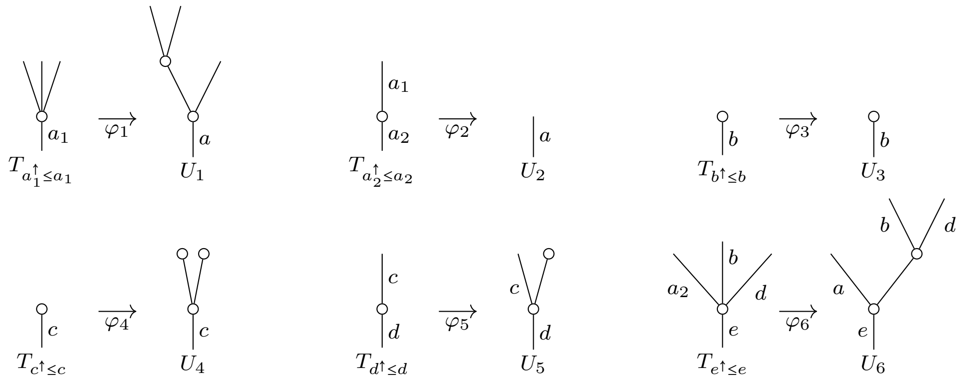
$$U = (((((U_6 \amalg_a U_2)) \amalg_a U_1) \amalg_b U_3) \amalg_d U_5) \amalg_c U_4 \tag{3.36}$$



(3.37)

We now consider the tree T, which is built by converting each U_i into the corolla $\mathsf{lr}(U_i)$, and then performing the same grafting operations as in (3.36). T can then be regarded as encoding the combinatorics of the iterated grafting in (3.36), with alternative ways to reparenthesize operations in (3.36) in bijection with ways to assemble T out of its nodes.

One can now therefore think of the iterated grafting (3.36) as being instead encoded by the tree T together with the (unique) planar tall maps φ_i below.



(3.38)

From this perspective, U can now be thought of as obtained from T by *substituting* each of its nodes with the corresponding U_i. Moreover, the φ_i assemble to a planar tall map $\varphi: T \to U$ (such that $a_i \mapsto a, b \mapsto b, \cdots, e \mapsto e$), which likewise encodes the same information.

One of the fundamental ideas shaping our perspective on operads is then that substitution data as in (3.38) can equivalently be repackaged using planar tall maps.

Definition 3.39. Let $T \in \Omega$ be a tree.

A *T-substitution datum* is a tuple $(U_{e^\uparrow \le e})_{(e^\uparrow \le e) \in V(T)}$ together with tall maps $T_{e^\uparrow \le e} \to U_{e^\uparrow \le e}$. Further, a map of T-substitution data $(U_{e^\uparrow \le e}) \to (V_{e^\uparrow \le e})$ is a tuple of tall maps $(U_{e^\uparrow \le e} \to V_{e^\uparrow \le e})$ compatible with the substitution maps.

Lastly, a substitution datum is called *planar* if the chosen maps are planar (so that $\mathsf{lr}(U_{e^\uparrow \le e}) = T_{e^\uparrow \le e}$), and a morphism between planar data is called a *planar morphism* if it consists of a tuple of planar maps.

We denote the category of (resp. planar) T-substitution data by $\mathsf{Sub}(T)$ (resp. $\mathsf{Sub}_{\mathsf{p}}(T)$).

Definition 3.40. Let $T \in \Omega$ be a tree. The *Segal core poset* $\mathsf{Sc}(T)$ is the poset with objects the single edge subtrees η_e and vertex subtrees $T_{e^\uparrow \le e}$, ordered by inclusion.

Remark 3.41. Note that the only arrows in $\mathsf{Sc}(T)$ are inclusions of the form $\eta_a \subset T_{e^\uparrow \le e}$. In particular, there are no pairs of composable non-identity arrows in $\mathsf{Sc}(T)$.

Given a T-substitution datum $\{U_{\{e^\uparrow \le e\}}\}$ we abuse notation by writing

$$U_{(-)}: \mathsf{Sc}(T) \to \Omega$$

for the functor $\eta_a \mapsto \eta$, $T_{e^\uparrow \le e} \mapsto U_{e^\uparrow \le e}$ and sending the inclusions $\eta_a \subset T_{e^\uparrow \le e}$ to the composites

$$\eta \xrightarrow{a} T_{e^\uparrow \le e} \to U_{e^\uparrow \le e}.$$

Proposition 3.42. *Let $T \in \Omega$ be a tree. There is an isomorphism of categories*

$$\begin{array}{ccc} \mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & T \downarrow \Omega^{\mathsf{pt}} \\ (U_{e^\uparrow \le e}) & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \\ \left(U_{\varphi(e^\uparrow) \le \varphi(e)}\right) & \longleftarrow\!\!\shortmid & (T \xrightarrow{\varphi} U) \end{array}$$

where $T \downarrow \Omega^{\mathsf{pt}}$ denotes the category of planar tall maps under T and $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is chosen in the unique way that makes the inclusions of the $U_{e^\uparrow \le e}$ planar.

Proof. We first show in parallel that: (i) $\operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, which we denote U_T, exists; (ii) for the datum $(T_{e^\uparrow \le e})$, it is $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$; (iii) $V(U_T) = \coprod_{V(T)} V(U_{e^\uparrow \le e})$; (iv) the induced map $T \to U_T$ is planar tall.

The argument is by induction on the number of vertices of T, with the base cases of T with 0 or 1 vertices being immediate, since then T is the terminal object of $\mathsf{Sc}(T)$. Otherwise, one can choose a non trivial grafting decomposition so as to write $T = R \amalg_e S$, resulting in identifications $\mathsf{Sc}(R) \subset \mathsf{Sc}(T)$, $\mathsf{Sc}(S) \subset \mathsf{Sc}(T)$ with $\mathsf{Sc}(R) \cup \mathsf{Sc}(S) = \mathsf{Sc}(T)$ and $\mathsf{Sc}(R) \cap \mathsf{Sc}(S) = \{\eta_e\}$. The existence of $U_T = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is thus equivalent to the existence of the pushout below (where the rightmost diagram merely simplifies notation).

$$\begin{tikzcd}
\eta \arrow[r, "e"] \arrow[d, "e"'] & \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)} \arrow[d, dashed] \\
\operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)} \arrow[r, dashed] & \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}
\end{tikzcd}
\qquad
\begin{tikzcd}
\eta \arrow[r, "e"] \arrow[d, "e"'] & U_R \arrow[d, dashed] \\
U_S \arrow[r, dashed] & U_T
\end{tikzcd}
\tag{3.43}$$

By induction, U_R and U_S exist for any $U_{(-)}$, equal R and S in the case $U_{(-)} = T_{(-)}$, $V(U_R) = \coprod_{V(R)} V(U_{e^\uparrow \le e})$ and likewise for S (so that there are unique choices of U_R, U_S making the inclusions of $U_{e^\uparrow \le e}$ planar), and the maps $R \to \operatorname{colim}_{\mathsf{Sc}(R)} U_{(-)}$, $S \to \operatorname{colim}_{\mathsf{Sc}(S)} U_{(-)}$ are planar tall. But it now follows that (3.43) is a grafting pushout diagram (cf. (3.30)), so that the pushout indeed exists. The conditions $T = \operatorname{colim}_{\mathsf{Sc}(T)} T_{(-)}$, $V(U_T) = \coprod_{V(T)} V(U_{e^\uparrow \le e})$, and that $T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$ is planar tall follow.

The fact that the two functors in the statement are inverse to each other is clear from the same inductive argument. □

Corollary 3.44. *Let $T \in \Omega$ be a tree. The formulas in Proposition 3.42 give an isomorphism of categories*

$$\mathsf{Sub}(T) \rightleftarrows T \downarrow \Omega^{\mathrm{t}}$$

where $T \downarrow \Omega^{\mathrm{t}}$ denotes the category of tall maps under T.

Proof. This is a consequence of Proposition 3.21 together with the previous result. Indeed, Proposition 3.12 can be restated as saying that isomorphisms $T \to T'$ are in bijection with substitution data consisting of isomorphisms, and thus bijectiveness of $\mathsf{Sub}(T) \to T \downarrow \Omega^{\mathrm{t}}$ reduces to that in the previous result. □

Remark 3.45. As noted in the proof of Proposition 3.42, writing $U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, one has

$$V(U) = \coprod_{(e^\uparrow \le e) \in V(T)} V(U_{e^\uparrow \le e}). \tag{3.46}$$

Alternatively, (3.46) can be regarded as a map $\varphi^* \colon V(U) \to V(T)$ induced by the planar tall map $\varphi \colon T \to U$. Explicitly, $\varphi^*(U_{u^\uparrow \le u})$ is the unique $T_{t^\uparrow \le t}$ such that there is an inclusion of outer faces $U_{u^\uparrow \le u} \hookrightarrow U_{t^\uparrow \le t}$, so that φ^* indeed depends contravariantly on the tall map φ.

Remark 3.47. Suppose that $e \in T$ has input path $I_T(e) = (e = e_n < e_{n-1} < \cdots < e_0)$. It is intuitively clear that for a tall map $\varphi \colon T \to U$ the input path of $\varphi(e)$ is built by gluing input paths in the $U_{t^\uparrow \le t}$. More precisely (and omitting φ for readability), one has

$$I_U(e_n) \simeq I_{n-1}(e_n) \amalg_{e_{n-1}} I_{n-2}(e_{n-1}) \amalg_{e_{n-2}} \cdots \amalg_{e_1} I_1(e_0).$$

where $I_k(-)$ denotes the input path in $U_{e_k^\uparrow \le e_k}$. More formally, this follows from the characterization of predecessors in Proposition 3.8(b).

We end this section with a couple of lemmas that will allow us to reverse the substitution procedure of Proposition 3.42 and will be needed in §5.2. Recall that the single edge tree $\eta \in \Omega$ is called the stick tree, cf. Notation 3.28.

Proposition 3.48. *Let $U \in \Omega$ be a tree. Then:*

(i) *given non-stick outer subtrees U_i such that $V(U) = \coprod_i V(U_i)$ there is a unique tree T and planar tall map $T \to U$ such that the sets $\{U_i\}$, $\{U_{e^\uparrow \leq e}\}$ coincide;*

(ii) *given multiplicities $m_e \geq 1$ for each edge $e \in U$, there is a unique planar degeneracy $\rho: T \to U$ such that $\rho^{-1}(e)$ has m_e elements;*

(iii) *planar tall maps $T \to U$ are in bijection with collections $\{U_i\}$ of outer subtrees such that $V(U) = \coprod_i V(U_i)$ and U_j is not an inner edge of any U_i whenever $U_j \simeq \eta$ is a stick.*

Proof. We first show (i) by induction on the number of subtrees U_i. The base case $\{U_i\} = \{U\}$ is immediate, setting $T = \mathsf{lr}(U)$. Otherwise, U must not be a corolla and letting e be an edge that is both an inner edge of U and a root of some U_i, and one can form a grafting pushout diagram

$$\begin{tikzcd} \eta \arrow[r, "e"] \arrow[d, "e"'] & U^{\leq e} \arrow[d] \\ U_{\not\leq e} \arrow[r] & U \end{tikzcd} \tag{3.49}$$

where $U^{\leq e}$ (resp. $U_{\not\leq e}$) are the outer faces consisting of the edges $u \leq_d e$ (resp. $u \not\leq_d e$). Since there is a unique U_i containing the vertex $e^\uparrow \leq e$, it follows from the definition of outer face that there is a nontrivial partition $\{U_i\} = \{U_i | U_i \hookrightarrow V\} \amalg \{U_i | U_i \hookrightarrow W\}$. Existence of $T \to U$ now follows from the induction hypothesis. For uniqueness, the condition that no U_i is a stick guarantees that T possesses a single inner edge mapping to e, and thus admits a compatible decomposition as in (3.49), so that uniqueness too follows from the induction hypothesis.

For (ii), we argue existence by nested induction on the number of vertices $|V(U)|$ and the sum of the multiplicities m_e. The base case $|V(U)| = 0$, i.e., $U = \eta$ is immediate. Otherwise, writing $m_e = m'_e + 1$, one can form a decomposition (3.49) where either $|V(V)|, |V(W)| < |V(U)|$ or one of V, W is η, so that $T \to U$ can be built via the induction hypothesis. For uniqueness, note first that by [24, Lemma 5.33] each pre-image $\rho^{-1}(e)$ is linearly ordered and by the "further" claim in [24, Cor. 5.39] the remaining broad relations are precisely the pre-image of the non-identity relations in U, showing that the underlying broad poset of the tree T is unique up to isomorphism. Strict uniqueness is then Proposition 3.21.

(iii) follows by combining (i) and (ii). Indeed, any planar tall map $T \to U$ has a unique factorization $T \twoheadrightarrow \bar{T} \hookrightarrow U$ as a planar degeneracy followed by a planar inner face, and each of these maps is classified by the data in (b) and (a). □

Lemma 3.50. *Suppose $T_1, T_2 \hookrightarrow T$ are two outer faces with at least one common edge e. Then there exists an unique outer face $T_1 \cup T_2$ such that $V(T_1 \cup T_2) = V(T_1) \cup V(T_2)$.*

Proof. The result is obvious if either T is a corolla or if one of T_1, T_2 is one of the root or leaf stick subtrees.

Otherwise, one can necessarily choose e to be an inner edge of T, in which case all three of T_1, T_2, T admit compatible decompositions as in (3.49) and the result follows by induction on $|V(T)|$. □

3.3 Equivariant leaf-root and vertex functors

This section introduces two functors that are central to our definition of the category Op_G of genuine equivariant operads: the leaf-root and vertex functors.

We start by recalling a key class of maps of G-trees.

Definition 3.51. Let $S = (S_y)_{y \in Y}$ and $T = (T_x)_{x \in X}$ be G-trees. A map of G-trees

$$\varphi = (\phi, (\varphi_y)): S \to T$$

is called a *quotient* if each of the constituent tree maps

$$\varphi_y : S_y \to T_{\phi(y)}$$

is an isomorphism of trees.

The category of G-trees and quotients is denoted Ω_G^0 (this notation is justified in §3.4).

Remark 3.52. Quotients can alternatively be described as the cartesian arrows for the Grothendieck fibration $\Omega_G \xrightarrow{\mathsf{r}} \mathsf{O}_G$. We note that this is strictly more general than the notion of root pullbacks (Figure 1), which are the *chosen* cartesian arrows: those quotients such that each $\varphi_y : S_y \to T_{\phi(y)}$ is a planar isomorphism, i.e., an identity.

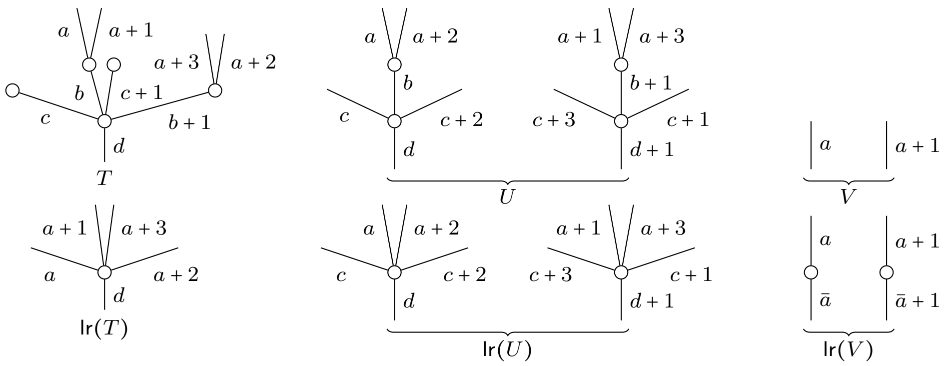
Definition 3.53. The *G-symmetric category*, whose objects we call *G-corollas*, is the full subcategory $\Sigma_G \hookrightarrow \Omega_G^0$ of those G-trees $C = (C_x)_{x\in X}$ such that some (and thus all) C_x is a corolla $C_x \in \Sigma \hookrightarrow \Omega$ (cf. Notation 3.34).

Definition 3.54. The *leaf-root functor* is the functor $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$ defined by

$$\mathsf{lr}\left((T_x)_{x\in X}\right) = \left(\mathsf{lr}(T_x)\right)_{x\in X}.$$

Remark 3.55. The leaf-root functor extends to a functor $\mathsf{lr}: \Omega_G^{\mathsf{t}} \to \Sigma_G$, where Ω_G^{t} is the category of tall maps, defined exactly as in Definition 3.51, but not to a functor defined on all arrows in Ω_G. Nonetheless, we will be primarily interested in the restriction $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$.

Remark 3.56. Generalizing the remark in Notation 3.34, $\mathsf{lr}(T)$ can alternatively be characterized as being the *unique* G-corolla which admits an also unique planar tall map $\mathsf{lr}(T) \to T$. Moreover, $\mathsf{lr}(T)$ can usually be regarded as the "smallest inner face" of T, obtained by removing all the inner edges, although this characterization fails when $T = (\eta_x)_{x\in X}$ is a stick G-tree. Some examples with $G = \mathbb{Z}_{/4}$ follow.



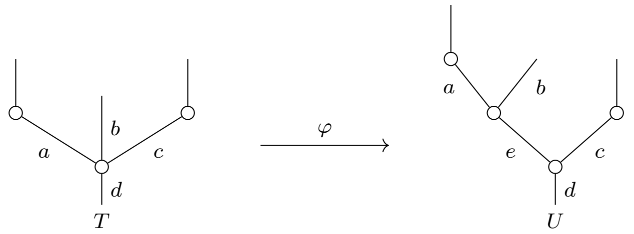
Remark 3.57. Since planarizations can not be pushed forward along tree maps (cf. Remark 3.22) the leaf-root functor $\mathsf{lr}: \Omega_G^0 \to \Sigma_G$ is not a Grothendieck fibration, but instead only a map of Grothendieck fibrations over O_G (for the obvious root functor $\mathsf{r}: \Sigma_G \to \mathsf{O}_G$).

Definition 3.58. Given $T = (T_x)_{x\in X} \in \Omega_G$ we define its set of *vertices* to be $V(T) = \coprod_{x\in X} V(T_x)$ and its set of *G-vertices* to be the orbit set $V(T)/G$.

Furthermore, we will regard $V(T)$ as an object of F by using the induced planar order (with $e^{\uparrow} \leq e$ ordered according to e) and likewise $V_G(T)$ will be regarded as an object of F by using the lexicographic order: i.e. vertex equivalence classes $[e^{\uparrow} \leq e]$ are ordered according to the planar order $\leq_p$ of the smallest representative ge, $g \in G$.

Remark 3.59. Following Remark 3.45, a tall map $\varphi: T \to U$ of G-trees induces a G-equivariant map $\varphi^*: V(U) \to V(T)$ and thus also a map of orbits $\varphi^*: V_G(U) \to V_G(T)$. We note, however, that φ^* is not in general compatible with the order on $V_G(-)$ even if φ is planar, as is indeed the case even in the non-equivariant setting.

A minimal example follows.



In $V(T)$ the vertices are ordered as $a < c < d$ while in $V(U)$ they are ordered as $a < e < c < d$ but the map $\varphi^*: V(U) \to V(T)$ is given by $a \mapsto a, c \mapsto c, d \mapsto d, e \mapsto d$.

Notation 3.60. Given $T = (T_x)_{x \in X} \in \Omega_G$ and $(e^\uparrow \le e) \in V(T)$ we write $T_{e^\uparrow \le e}$ as a shorthand for $T_{x, e^\uparrow \le e}$, where $e \in T_x$.

Further, each element of $V_G(T)$ corresponds to an unique edge orbit Ge for e not a leaf. We will prefer to write G-vertices as v_{Ge}, and write

$$T_{v_{Ge}} = (T_{f^\uparrow \le f})_{f \in Ge} \tag{3.61}$$

where Ge inherits the planar order.

We note that $T_{v_{Ge}}$ is always a G-corolla, leading to the following definition.

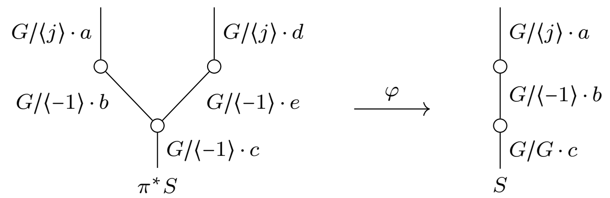
Definition 3.62. The *G-vertex functor* is the functor

$$\begin{aligned} \Omega_G^0 &\xrightarrow{V_G} \mathsf{F}_s \wr \Sigma_G \\ T &\longmapsto (T_{v_{Ge}})_{v_{Ge} \in V_G(T)}, \end{aligned}$$

where F_s is the category of finite sets and surjections of Remark 2.18.

Remark 3.63. Note that though the composite $\Omega_G^0 \to \mathsf{F}_s \wr \Sigma_G \to \mathsf{F}_s$ coincides on objects with the functor described in Remark 3.59, the variance is now reversed.

Remark 3.64. In the non-equivariant case the vertex functor can be defined to land instead in $\Sigma \wr \Sigma$. The need to introduce the $\mathsf{F}_s \wr (-)$ construction comes from the fact that in general quotient maps do not preserve the number of G-vertices. As an example, let $G = \{\pm 1, \pm i, \pm j, \pm k\}$ and consider the pullback map $\varphi: \pi^* S \to S$ of Example 3.24 determined by the assignments $a \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto ia$, $e \mapsto ib$, and presented below in orbital notation.



Note that $T = \pi^* S$ has three G-vertices v_{Gc}, v_{Gb}, v_{Ge} while S has only two G-vertices v_{Gc} and v_{Gb}. $V_G(\varphi)$ then maps the two G-corollas $T_{v_{Gb}}$ and $T_{v_{Ge}}$ isomorphically onto $S_{v_{Gb}}$ and the G-corolla $T_{v_{Gc}}$ by a non-isomorphism quotient onto $S_{v_{Gc}}$.

The following elementary statement will play an important auxiliary role.

Lemma 3.65. *The G-vertex functor*

$$\Omega_G^0 \xrightarrow{V_G} \mathsf{F}_s \wr \Sigma_G$$

sends pullbacks over O_G (i.e. root pullbacks) to pullbacks over $\mathsf{F}_s \wr \mathsf{O}_G$ (cf. Lemma 2.20).

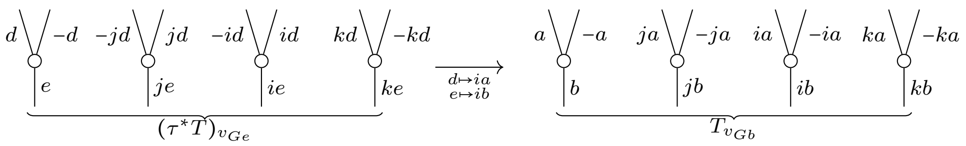
Proof. Note first that an arrow $(\phi, (\varphi_i)) \colon (C_i)_{i\in I} \to (C'_j)_{j\in J}$ is a pullback for the split fibration $\mathsf{F}_s \wr \Sigma_G \to \mathsf{F}_s \wr \mathsf{O}_G$ iff each of the constituent arrows $\varphi_i \colon C_i \to C'_{\phi(i)}$ are pullbacks for the split fibration $\Sigma_G \to \mathsf{O}_G$.

The pullback $\psi^* T \xrightarrow{\bar{\psi}} T$ of $T = (T_x)_{x\in X} \in \Omega^0_G$ over $\psi \colon Y \to X$ has the form $(T_{\psi(y)})_{y\in Y} \to (T_x)_{x\in X}$ and it now suffices to check that each of the vertex maps $(\psi^* T)_{v_{Ge}} \to T_{v_{G\bar{\psi}(e)}}$ is itself a pullback. By (3.61), this is the statement that for $f \in Ge$ the induced map

$$(\psi^* T)_{f^{\uparrow} \le f} \to T_{\bar{\psi}(f^{\uparrow}) \le \bar{\psi}(f)} \tag{3.66}$$

is an identity (i.e. planar isomorphism), and letting y be such that $f \in T_{\psi(y)}$ one sees that (3.66) is the identity $T_{\psi(y), f^{\uparrow} \le f} = T_{x, \bar{\psi}(f)^{\uparrow} \le \bar{\psi}(f)}$, where $x = \psi(y)$, finishing the proof. $\square$

Example 3.67. The following depicts one of the maps (3.66) for the pullback $\tau^* T \to T$ in Example 3.24.



Note that $(\tau^* T)_{v_{Ge}} = \rho^* T_{v_{Gb}}$ for ρ the map $\{e, je, ie, ke\} \to \{b, jb, ib, kb\}$ defined by $e \mapsto ib$ so that, accounting for orders, ρ is the block permutation $\rho = (13)(24)$.

We are now in a position to generalize Definition 3.39.

Definition 3.68. Let $T \in \Omega_G$ be a G-tree.

A *(resp. planar) T-substitution datum* is a tuple $\left(U_{f^{\uparrow} \le f}\right)_{V(T)}$ of trees together with

(i) associative and unital G-action maps $U_{f^{\uparrow} \le f} \to U_{g f^{\uparrow} \le g f}$;

(ii) (planar) tall maps $T_{f^{\uparrow} \le f} \to U_{f^{\uparrow} \le f}$ compatible with the G-action maps.

Further, a map of (planar) T-substitution data $\left(U_{f^{\uparrow} \le f}\right) \to \left(V_{f^{\uparrow} \le f}\right)$ is a compatible tuple of (planar) tall maps $\left(U_{f^{\uparrow} \le f} \to V_{f^{\uparrow} \le f}\right)$.

We denote the category of (planar) T-substitution data by $\mathsf{Sub}(T)$ (resp. $\mathsf{Sub}_{\mathsf{p}}(T)$).

Recall that a map of G-trees is called *rooted* if it induces an ordered isomorphism on the root orbit (cf. Definition 3.23), and we note that by Definition 3.19 planar tall maps of G-trees are always rooted.

Remark 3.69. Writing $U^{\mathsf{r}}_{v_{Ge}} = (U_{f^{\uparrow} \le f})_{f \in Ge}$ a T-substitution datum can equivalently be encoded by the tuple $\left(U^{\mathsf{r}}_{v_{Ge}}\right)_{V_G(T)}$ together with *rooted* tall maps $T_{v_{Ge}} \to U^{\mathsf{r}}_{v_{Ge}}$. The need to include r (which stands for "rooted") in the notation is explained by Remark 3.72.

Further, the T-substitution datum is planar iff the maps $T_{v_{Ge}} \to U^{\mathsf{r}}_{v_{Ge}}$ are as well.

Remark 3.70. Writing $T = (T_x)_{x\in X}$ as usual one obtains (non-equivariant) T_x-substitution data $U_{x,(-)}$ for each T_x. We again write $U_{x,(-)} \colon \mathsf{Sc}(T_x) \to \Omega$ and note that these are compatible with the G-action in the sense that the obvious diagram

$$\begin{tikzcd}
\mathsf{Sc}(T_x) \arrow[rr, "U_{x,(-)}"] \arrow[dr, "g"'] & & \Omega \\
& \mathsf{Sc}(T_{gx}) \arrow[ur, "U_{gx,(-)}"'] &
\end{tikzcd}$$

commutes. Writing $\mathsf{Sc}(T) = \coprod_x \mathsf{Sc}(T_x)$, these diagrams assemble into a functor $G \ltimes \mathsf{Sc}(T) \to \Omega$, where $G \ltimes \mathsf{Sc}(T)$ is the Grothendieck construction for the G-action (which, explicitly, adds arrows $\eta_a \to \eta_{ga}$, $T_{e^{\uparrow} \le e} \to T_{g e^{\uparrow} \le g e}$ to $\mathsf{Sc}(T)$ that satisfy obvious compatibilities).

In the following we write $\mathrm{colim}_{\mathsf{Sc}(T)} U_{(-)}$ to mean $(\mathrm{colim}_{\mathsf{Sc}(T_x)} U_{x,(-)})_{x\in X}$ or, in other words, we take the colimit in $\Phi = \mathsf{F} \wr \Omega$ rather than in Ω (as is needed since Ω lacks coproducts).

Corollary 3.71. *Let $T \in \Omega_G$ be a G-tree. There are isomorphisms of categories*

$$\begin{array}{ccc} \mathsf{Sub}_{\mathsf{p}}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{pt}} \\ \left(U_{f^{\uparrow} \leq f}\right)_{V(T)} & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \end{array} \qquad \begin{array}{ccc} \mathsf{Sub}(T) & \rightleftarrows & T \downarrow \Omega_G^{\mathsf{rt}} \\ \left(U_{f^{\uparrow} \leq f}\right)_{V(T)} & \longmapsto & \left(T \to \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}\right) \end{array}$$

where $T \downarrow \Omega_G^{\mathsf{pt}}$ (resp. $T \downarrow \Omega_G^{\mathsf{rt}}$) is the category of planar tall (resp. rooted tall) maps under T.

Proof. This is a direct consequence of the non-equivariant analogues Proposition 3.42 and Corollary 3.44 applied to each individual T_x together with the equivariance analysis in Remark 3.70. □

Remark 3.72. Writing $U = \operatorname{colim}_{\mathsf{Sc}(T)} U_{(-)}$, it follows from the non-equivariant results Proposition 3.42 and Corollary 3.44 that each inclusion map $U_{f^{\uparrow} \leq f} \to U$ is planar, so that there is no conflict with Notation 3.60.

However, some care is needed concerning the $U^{\mathsf{r}}_{v_{Ge}}$ appearing in the reformulation of substitution data given in Remark 3.69. Letting $\varphi: T \to U$ be the induced map, one sees that while $U^{\mathsf{r}}_{v_{Ge}}$ and $U_{v_{G\varphi(e)}}$ have the same constituent trees (with the latter defined by Notation 3.60), the roots of $U^{\mathsf{r}}_{v_{Ge}}$ are ordered by Ge while those of $U_{v_{G\varphi(e)}}$ are ordered by $G\varphi(e)$. More succinctly, it is then $U^{\mathsf{r}}_{v_{Ge}} = \varphi^*_{Ge} U_{v_{G\varphi(e)}}$ for $\varphi_{Ge}: Ge \to G\varphi(e)$ the induced map.

Lastly, we note that such distinctions are unnecessary for planar data, since then the φ_{Ge} are ordered isomorphisms (i.e. identities), so that $U^{\mathsf{r}}_{v_{Ge}} = U_{v_{G\varphi(e)}}$.

Remark 3.73. The isomorphisms in Corollary 3.71 are compatible with root pullbacks of trees. More concretely, as in the proof of Lemma 3.65, each pullback $\bar{\psi}: \psi^* T \to T$ determines pullback maps $\bar{\psi}_{Ge}: (\psi^* T)_{v_{Ge}} \to T_{v_{G\bar{\psi}(e)}}$, which we note are pullbacks over the maps $\bar{\psi}_{Ge}: Ge \to G\bar{\psi}(e)$ in O_G. The definition of pullback then allows us to uniquely fill any diagram (where we reformulate substitution data as in Remark 3.69)

$$\begin{tikzcd} (\psi^* T)_{v_{Ge}} \arrow[r, dashed] \arrow[d] & \bar{\psi}^*_{Ge} U^{\mathsf{r}}_{v_{G\bar{\psi}(e)}} \arrow[d] \\ T_{v_{G\bar{\psi}(e)}} \arrow[r] & U^{\mathsf{r}}_{v_{G\bar{\psi}(e)}} \end{tikzcd}$$

defining the left vertical functors (with the right functors defined analogously) in each of the commutative diagrams below.

$$\begin{tikzcd} \mathsf{Sub}_{\mathsf{p}}(\psi^* T) \arrow[r, shift left] & \psi^* T \downarrow \Omega_G^{\mathsf{pt}} \arrow[l, shift left] \\ \mathsf{Sub}_{\mathsf{p}}(T) \arrow[u, "(\bar{\psi}^*_{Ge})"] \arrow[r, shift left] & T \downarrow \Omega_G^{\mathsf{pt}} \arrow[l, shift left] \arrow[u, "\psi^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Sub}(\psi^* T) \arrow[r, shift left] & \psi^* T \downarrow \Omega_G^{\mathsf{rt}} \arrow[l, shift left] \\ \mathsf{Sub}(T) \arrow[u, "(\bar{\psi}^*_{Ge})"] \arrow[r, shift left] & T \downarrow \Omega_G^{\mathsf{rt}} \arrow[l, shift left] \arrow[u, "\psi^*"'] \end{tikzcd} \tag{3.74}$$

3.4 Planar strings

We now use the leaf-root and vertex functors to repackage our substitution results in a format that will be more convenient for our definition of genuine equivariant operads in §4.

Definition 3.75. The category Ω_G^n of *planar n-strings* is the category whose objects are strings

$$T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} T_n \tag{3.76}$$

where $T_i \in \Omega_G$ and the φ_i are planar *tall* maps, while arrows are commutative diagrams

$$\begin{tikzcd} T_0 \arrow[r, "\varphi_1"] \arrow[d, "\pi_0"'] & T_1 \arrow[r, "\varphi_2"] \arrow[d, "\pi_1"'] & \cdots \arrow[r, "\varphi_n"] & T_n \arrow[d, "\pi_n"'] \\ T'_0 \arrow[r, "\varphi'_1"'] & T'_1 \arrow[r, "\varphi'_2"'] & \cdots \arrow[r, "\varphi'_n"'] & T'_n \end{tikzcd} \tag{3.77}$$

where each π_i is a quotient map.

Notation 3.78. Since compositions of planar tall arrows are planar tall and identity arrows are planar tall it follows that $\Omega_G^{\bullet}$ forms a simplicial object in Cat, with faces given by composition and degeneracies by inserting identities.

Further setting $\Omega_G^{-1} = \Sigma_G$, the leaf-root functor $\Omega_G^0 \xrightarrow{\mathsf{lr}} \Sigma_G$ makes $\Omega_G^{\bullet}$ into an augmented simplicial object and, furthermore, the maps $s_{-1}\colon \Omega_G^n \to \Omega_G^{n+1}$ sending $T_0 \to T_1 \to \cdots \to T_n$ to $\mathsf{lr}(T_0) \to T_0 \to T_1 \to \cdots \to T_n$ equip it with extra degeneracies.

Remark 3.79. The identification $\Omega_G^{-1} = \Sigma_G$ can be understood by noting that a string as in (3.76) is equivalent to a string

$$T_{-1} \xrightarrow{\varphi_0} T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} T_n \tag{3.80}$$

where $T_{-1} = \mathsf{lr}(T_0) = \cdots = \mathsf{lr}(T_n)$.

Remark 3.81. Since for any planar n-string we have $\mathsf{r}(T_i) = \mathsf{r}(T_j)$ for any $1 \le i, j \le n$, there is a well defined functor $\mathsf{r}\colon \Omega_G^n \to \mathsf{O}_G$, which is readily seen to be a split Grothendieck fibration. Furthermore, generalizing Remark 3.57, all operators d_i, s_j are maps of split Grothendieck fibrations.

Notation 3.82. We extend the vertex functor to a functor $V_G\colon \Omega_G^n \to \mathsf{F}_s \wr \Omega_G^{n-1}$ by

$$V_G(T_0 \to T_1 \to \cdots \to T_n) = (T_{1,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_0)} \tag{3.83}$$

where we abuse notation by writing $T_{i,v_{Ge}}$ for $(T_{i,\bar{\varphi}_i(f)^{\uparrow} \le \bar{\varphi}_i(f)})_{f \in Ge}$, where $\bar{\varphi}_i = \varphi_i \circ \cdots \circ \varphi_1$.

Alternatively, regarding $T_0 \to \cdots \to T_n$ as a string of n arrows in $T_0 \downarrow \Omega_G^{\mathsf{pt}}$, the object $V_G(T_0 \to \cdots \to T_n)$ can be thought of as the image of the inverse functor in Corollary 3.71, written according to the reformulation in Remark 3.69 (where since we are in the planar case we need not distinguish between the $U^{\mathsf{r}}_{(-)}$ and $U_{(-)}$ notations (cf. Remark 3.72)). Note however that from this perspective functoriality needs to be addressed separately.

Notation 3.84. For $X \subseteq \{0, 1, \cdots, n\}$ we write $d_X\colon \Omega_G^n \to \Omega_G^{n-|X|}$ for the functor which sends $T_0 \to T_1 \to \cdots \to T_n$ to the string with $T_x, x \in X$ omitted.

Note that, in light of (3.80), this makes sense even when $X = \{0, 1, \cdots, n\}$.

We now obtain a key reinterpretation (and slight strengthening) of Corollary 3.71.

Proposition 3.85. *For any $n \ge 0$ the commutative diagram*

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n}"'] & \mathsf{F}_s \wr \Omega_G^{n-1} \arrow[d, "\mathsf{F} \wr d_{0,\cdots,n-1}"] \\
\Omega_G^0 \arrow[r, "V_G"'] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd} \tag{3.86}$$

is a pullback diagram in Cat.

Proof. Let us write $P = \Omega_G^0 \times_{\mathsf{F}_s \wr \Sigma_G} \mathsf{F}_s \wr \Omega_G^{n-1}$ for the pullback, so that our goal is to show that the canonical map $\Omega_G^n \to P$ is an isomorphism.

That $\Omega_G^n \to P$ is an isomorphism on objects follows by combining the alternative description of V_G in Notation 3.82 with the planar half of Corollary 3.71 (in fact, this yields isomorphisms of the fibers over Ω_G^0, but we will not directly use this fact). We will hence write $T_0 \to \cdots \to T_n$ to denote an object of P as well.

An arrow in P from $T_0 \to \cdots \to T_n$ to $T'_0 \to \cdots \to T'_n$ then consists of a quotient $\pi_0\colon T_0 \to T'_0$ together with a $V_G(T_0)$ indexed tuple of quotients of strings (where we write $e' = \pi_0(e)$)

$$\begin{tikzcd}
T_{0,v_{Ge}} \arrow[r] \arrow[d, "\pi_{0,e}"'] & T_{1,v_{Ge}} \arrow[r] \arrow[d, "\pi_{1,e}"'] & \cdots \arrow[r] & T_{n,v_{Ge}} \arrow[d, "\pi_{n,e}"] \\
T'_{0,v_{Ge'}} \arrow[r] & T'_{1,v_{Ge'}} \arrow[r] & \cdots \arrow[r] & T'_{n,v_{Ge'}}.
\end{tikzcd} \tag{3.87}$$

That $\Omega^n_G \to P$ is injective on arrows is then clear.

For surjectivity, note first that by Lemma 3.65 the composite $P \to \Omega^0_G \to \mathsf{O}_G$ is a split Grothendieck fibration and $P \to \Omega^0_G$ is a map of split Grothendieck fibrations. Indeed, pullbacks in P can be built explicitly as those arrows such that π_0 and all $\pi_{i,e}$ in (3.87) are pullbacks (alternatively, an abstract argument also works). The alternative description of V_G in Notation 3.82 combined with (3.74) then show that $\Omega^n_G \to P$ preserves pullback arrows, so that surjectivity needs only be checked for maps in the fibers over O_G, i.e. on rooted maps. Tautologically, a map in P is rooted iff $\pi_0 : T_0 \to T'_0$ is. But since a quotient is an isomorphism iff it is so on roots, we further have that a map in P is rooted iff $\pi_0 : T_0 \to T'_0$ is a rooted isomorphism and each $\pi_{i,e}$ in (3.87) is an isomorphism. But now reinterpreting (3.87) as a tuple of diagrams indexed by $f \in Ge$ one obtains a diagram in $\mathsf{Sub}(T_0)$ of the same shape which, once converted to a diagram in $T_0 \downarrow \Omega^{\mathrm{rt}}_G$ using the rooted half of Corollary 3.71, yields the desired rooted map (3.77) in Ω^n_G lifting the rooted map in P. □

Notation 3.88. For $0 \le k \le n$ we let

$$V^k_G : \Omega^n_G \to \mathsf{F}_s \wr \Omega^{n-k-1}_G$$

be inductively defined by setting $V^0_G = V_G$ and letting V^{k+1}_G be the composite

$$\Omega^n_G \xrightarrow{V_G} \mathsf{F}_s \wr \Omega^{n-1}_G \xrightarrow{V^k_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \Omega^{n-k-2}_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Omega^{n-k-2}_G.$$

Remark 3.89. When $n = 2$, V^2_G is thus the composite

$$\Omega^2_G \xrightarrow{V_G} \mathsf{F}_s \wr \Omega^1_G \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \Omega^0_G \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \mathsf{F}_s \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \mathsf{F}_s \wr \Sigma_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Sigma_G$$

while for $n = 4$, V^1_G is the composite

$$\Omega^4_G \xrightarrow{V_G} \mathsf{F}_s \wr \Omega^3_G \xrightarrow{V_G} \mathsf{F}_s \wr \mathsf{F}_s \wr \Omega^2_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Omega^2_G.$$

In light of Remarks 3.45 and 3.59, $V^n_G(T_0 \to \cdots \to T_n)$ is identified with the tuple

$$(T_{k,v_{Ge}} \to \cdots \to T_{n,v_{Ge}})_{v_{Ge} \in V_G(T_k)}, \tag{3.90}$$

where we note that strings are written in prepended notation as in (3.80), so that $T_{k,v_{Ge}}$ is superfluous unless $k = n$. Further, note that this requires changing the order of $V_G(T_k)$. Rather than using the order induced by T_k, one instead equips $V_G(T_k)$ with the order induced lexicographically from the maps $V_G(T_k) \to V_G(T_{k-1}) \to \cdots \to V_G(T_0)$ of Remark 3.45. I.e., for $v, w \in V_G(T_k)$ the condition $v < w$ is determined by the lowest l such that the images of $v, w \in V_G(T_l)$ are distinct.

Therefore, for each d_i with $i < k$ there are natural isomorphisms as on the left below which interchange the lexicographical order on the indexing set $V_G(T_k)$ induced by the string $V_G(T_k) \to V_G(T_{k-1}) \to \cdots \to V_G(T_0)$ with the one induced by the string that omits $V_G(T_i)$. For d_i with $i > k$ one has commutative diagrams as on the right below. Note that no such diagram is defined for d_k.

$$\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[d, equal] \arrow[dl, Rightarrow, "\pi_i"] \\
\Omega^{n-1}_G \arrow[r, "V^{k-1}_G"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[d, "d_{i-k-1}"] \\
\Omega^{n-1}_G \arrow[r, "V^k_G"'] & \mathsf{F}_s \wr \Omega^{n-k-2}_G
\end{tikzcd} \tag{3.91}$$

Similarly, for s_j with $j < k$ (resp. $j \geq k$) one has commutative diagrams as on the left (resp. right) below. Note that for s_k one uses the extra degeneracy $s_{k-k-1} = s_{-1}$.

$$\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[d, equal] \\
\Omega^{n+1}_G \arrow[r, "V^{k+1}_G"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "s_j"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[d, "s_{j-k-1}"] \\
\Omega^{n+1}_G \arrow[r, "V^k_G"'] & \mathsf{F}_s \wr \Omega^{n-k}_G
\end{tikzcd}
\tag{3.92}$$

The functors V^k_G and isomorphisms π_i satisfy a number of compatibilities that we now catalog.

Proposition 3.93. *(a) The composite*

$$\Omega^n_G \xrightarrow{V^k_G} \mathsf{F}_s \wr \Omega^{n-k-1}_G \xrightarrow{V^l_G} \mathsf{F}^{\wr 2}_s \wr \Omega^{n-k-l-2}_G \xrightarrow{\sigma^0} \mathsf{F}_s \wr \Omega^{n-k-l-2}_G$$

equals the functor V^{k+l+1}_G.

(b) The functors V^k_G send pullback arrows for the split Grothendieck fibration $\Omega^k_G \to \mathsf{O}_G$ to pullback arrows for $\mathsf{F}_s \wr \Omega^{n-k-1}_G \to \mathsf{F}_s$.

(c) The isomorphisms $\pi_i(T_0 \to \cdots \to T_n)$ are pullback arrows for the split Grothendieck fibration $\mathsf{F}_s \wr \Omega^{n-k-1}_G \to \mathsf{F}_s$. Moreover, the projection of $\pi_i(T_0 \to \cdots \to T_n)$ onto F_s depends only on $T_0 \to \cdots \to T_i$.

(d) The rightmost diagrams in both (3.91) and (3.92) are pullback diagrams in Cat.

(e) For $i < k$ the composite natural transformation in the diagram below is π_i.

$$\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[r, "\mathsf{F}_s \wr V^l_G"] \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_i"] & \mathsf{F}^{\wr 2}_s \wr \Omega^{n-k-l-2}_G \arrow[r, "\sigma^0"] \arrow[d, equal] & \mathsf{F}_s \wr \Omega^{n-k-l-2}_G \arrow[d, equal] \\
\Omega^{n-1}_G \arrow[r, "V^{k-1}_G"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[r, "\mathsf{F}_s \wr V^l_G"'] & \mathsf{F}^{\wr 2}_s \wr \Omega^{n-k-l-2}_G \arrow[r, "\sigma^0"'] & \mathsf{F}_s \wr \Omega^{n-k-l-2}_G
\end{tikzcd}
\tag{3.94}$$

For $k < i < k + l + 1$ the composite natural transformation in the diagram below is π_i.

$$\begin{tikzcd}
\Omega^n_G \arrow[r, "V^k_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega^{n-k-1}_G \arrow[r, "\mathsf{F}_s \wr V^l_G"] \arrow[d, "\mathsf{F}_s \wr d_{i-k-1}"'] & \mathsf{F}^{\wr 2}_s \wr \Omega^{n-k-l-2}_G \arrow[r, "\sigma^0"] \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \mathsf{F}_s \wr \pi_{i-k-1}"] & \mathsf{F}_s \wr \Omega^{n-k-l-2}_G \arrow[d, equal] \\
\Omega^{n-1}_G \arrow[r, "V^k_G"'] & \mathsf{F}_s \wr \Omega^{n-k-2}_G \arrow[r, "\mathsf{F}_s \wr V^{l-1}_G"'] & \mathsf{F}^{\wr 2}_s \wr \Omega^{n-k-l-2}_G \arrow[r, "\sigma^0"'] & \mathsf{F}_s \wr \Omega^{n-k-l-2}_G
\end{tikzcd}
\tag{3.95}$$

(f) Restricting to the case $k = n$, the pairs (d_i, π_i) and $(s_j, id_{V^n_G})$ satisfy all possible simplicial identities (i.e. those with $i \neq n$). Explicitly, for $0 \leq i' < i < n$ the composite natural transformations in the diagrams

$$\begin{tikzcd}
\Omega^n_G \arrow[r] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_i"] \\
\Omega^{n-1}_G \arrow[r] \arrow[d, "d_{i'}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i'}"] \\
\Omega^{n-2}_G \arrow[r] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega^n_G \arrow[r] \arrow[d, "d_{i'}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i'}"] \\
\Omega^{n-1}_G \arrow[r] \arrow[d, "d_{i-1}"'] & \mathsf{F}_s \wr \Sigma_G \arrow[d, equal] \arrow[dl, phantom, "\Leftarrow \pi_{i-1}"] \\
\Omega^{n-2}_G \arrow[r] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd}
\tag{3.96}$$

coincide, and similarly for the face-degeneracy relations.

Proof. (a) follows by induction on k, with $k = 0$ being the definition. More generally (and writing F for F_s) one has

$$\begin{aligned}
\sigma^0(\mathsf{F} \wr V_G^l)V_G^{k+1} =& \sigma^0(\mathsf{F} \wr V_G^l)\sigma^0(\mathsf{F} \wr V_G^k)V_G = \sigma^0\sigma^0(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G \\
=& \sigma^0\sigma^1(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G = \sigma^0(\mathsf{F} \wr \sigma^0)(\mathsf{F}^{\wr 2} \wr V_G^l)(\mathsf{F} \wr V_G^k)V_G \\
=& \sigma^0\left(\mathsf{F} \wr \left(\sigma^0(\mathsf{F} \wr V_G^l)V_G^k\right)\right)V_G = \sigma^0\left(\mathsf{F} \wr V_G^{k+l+1}\right)V_G = V_G^{k+l+2}.
\end{aligned}$$

(b) generalizes Lemma 3.65, and follows by induction using that result, Lemma 2.20, and the obvious claim that $\mathsf{F} \wr \mathsf{F} \wr A \xrightarrow{\sigma^0} \mathsf{F} \wr A$ sends pullbacks over $\mathsf{F} \wr \mathsf{F}$ to pullbacks over F.

(c) is clear. Also, (e) and (f) are easy consequences of (b) and (c): since all natural transformations involved consist of pullback arrows, one needs only check each claim after forgetting to the F_s coordinate, which is straightforward.

Lastly, we argue (d) by induction on k and n. The case $k = 0$ for the rightmost diagram in (3.91) follows by the diagram on the left below, combined with Proposition 3.85 applied to the bottom and total squares. The general case then follows from the right diagram, where the left square is in the case $k = 0$, the middle square is a pullback by induction (and since $\mathsf{F} \wr (-)$ preserves pullback squares), and the rightmost square is clearly a pullback.

$$\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-1} \arrow[d, "d_{i-1}"] \\
\Omega_G^{n-1} \arrow[r, "V_G"] \arrow[d, "d_{1,\cdots,n}"'] & \mathsf{F}_s \wr \Omega_G^{n-2} \arrow[d, "d_{0,\cdots,n-1}"] \\
\Omega_G^0 \arrow[r, "V_G"] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd}
\qquad
\begin{tikzcd}
\Omega_G^n \arrow[r, "V_G"] \arrow[d, "d_i"'] & \mathsf{F}_s \wr \Omega_G^{n-1} \arrow[r, "V_G^k"] \arrow[d, "\mathsf{F}_s \wr d_{i-1}"'] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-2} \arrow[r, "\sigma^0"] \arrow[d, "\mathsf{F}_s^{\wr 2} \wr d_{i-1}"'] & \mathsf{F}_s \wr \Omega_G^{n-k-2} \arrow[d, "\mathsf{F}_s \wr d_{i-1}"'] \\
\Omega_G^{n-1} \arrow[r, "V_G"'] & \mathsf{F}_s \wr \Omega_G^{n-3} \arrow[r, "V_G^k"'] & \mathsf{F}_s^{\wr 2} \wr \Omega_G^{n-k-3} \arrow[r, "\sigma^0"'] & \mathsf{F}_s \wr \Omega_G^{n-k-3}
\end{tikzcd}
\tag{3.97}$$

The claim for the rightmost square in (3.92) follows by the analogous diagrams with the d_i (but not $d_{1,\cdots,n}$, $d_{0,\cdots,n-1}$) replaced with s_j. $\square$

4 Genuine equivariant operads

In this section we now build the category $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant operads. We do so by building a monad $\mathbb{F}_G$ on the category $\mathsf{Sym}_G(\mathcal{V}) = \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$ of G-symmetric sequences on $\mathcal{V}$, for $\mathcal{V}$ a symmetric monoidal category with diagonals (cf. Remark 2.18). The underlying endofunctor of $\mathbb{F}_G$ is easy to describe: given $X \in \mathsf{Sym}_G(\mathcal{V})$, $\mathbb{F}_G X$ is given by the left Kan extension diagram

$$\begin{tikzcd}
(\Omega_G^0)^{op} \arrow[r, "V_G^{op}"] \arrow[d, "\mathsf{lr}"'] & (\mathsf{F}_s \wr \Sigma_G)^{op} \arrow[r, "(\mathsf{F}_s \wr X^{op})^{op}"] \arrow[dl, phantom, "\Leftarrow"] & (\mathsf{F}_s \wr \mathcal{V}^{op})^{op} \arrow[r, "\otimes"] & \mathcal{V} \\
\Sigma_G^{op} \arrow[urrr, "\mathbb{F}_G X"'] & & &
\end{tikzcd}
\tag{4.1}$$

Explicitly, using Proposition 2.5 and the fact that the rooted under categories $C \downarrow_{\mathsf{r}} \Omega_G^0$ are groupoids yields the formula

$$\mathbb{F}_G X(C) \simeq \coprod_{T \in \mathsf{Iso}(C \downarrow_{\mathsf{r}} \Omega_G^0)} \left(\bigotimes_{v \in V_G(T)} X(T_v) \right) \cdot_{\mathsf{Aut}(T)} \mathsf{Aut}(C), \tag{4.2}$$

though we will prefer to work with (4.1) throughout.

To intuitively motivate the monad structure of $\mathbb{F}_G X$, note that (4.2) roughly states that $\mathbb{F}_G X$ consists of "G-trees T with G-nodes suitably labeled by X", and thus that $\mathbb{F}_G \mathbb{F}_G X$

consists of "G-trees T_0 with G-nodes labeled by G-trees $T_{1,i}$ with G-nodes labeled by X". The substitution discussion in §3.2, §3.4 then says that $\mathbb{F}_G\mathbb{F}_G X$ roughly consists of "planar tall maps of G-trees $T_0 \to T_1$ with G-nodes of T_1 labeled by X" (for a precise statement, see Remark 4.33), so that the multiplication $\mathbb{F}_G\mathbb{F}_G \to \mathbb{F}_G$ is obtained by "forgetting T_0".

To rigorously describe the monad structure on $\mathbb{F}_G$, however, we will find it preferable to separate the left Kan extension step in (4.1) from the remaining construction. As such, we will build a monad N on a larger category $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$ in §4.1 (see Proposition 4.19), which we then transfer to $\mathsf{Sym}_G(\mathcal{V})$ in §4.2 by using the (Lan, ι) adjunction in Remark 4.5. §4.3 then compares genuine equivariant operads with regular equivariant operads, obtaining the pair of adjunctions in Corollary 4.40, which are required when formulating and proving our main results. Lastly, 4.4 shows that the indexing systems of Blumberg-Hill (or, more precisely, a slight generalization called "weak indexing systems"; see Remark 4.56) naturally give rise to notions of "partial genuine operads".

4.1 A monad on spans

Definition 4.3. We write $\mathsf{WSpan}^l(\mathcal{C}, \mathcal{D})$ (resp. $\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})$), which we call the category of *left weak spans* (resp. *right weak spans*), to denote the category with objects the spans

$$\mathcal{C} \xleftarrow{k} A \xrightarrow{X} \mathcal{D},$$

arrows the diagrams as on the left (resp. right) below

$$\begin{tikzcd}[column sep=small]
 & A_1 \arrow[dl, "k_1"'] \arrow[dd, "i"] \arrow[dr, "X_1"] & \\
\mathcal{C} & & \mathcal{D} \arrow[dl, phantom, "\Swarrow\varphi"] \\
 & A_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & A_1 \arrow[dl, "k_1"'] \arrow[dd, "i"] \arrow[dr, "X_1"] & \\
\mathcal{C} & & \mathcal{D} \arrow[dl, phantom, "\Nearrow\varphi"] \\
 & A_2 \arrow[ul, "k_2"] \arrow[ur, "X_2"'] &
\end{tikzcd}$$

which we write as $(i, \varphi): (k_1, X_1) \to (k_2, X_2)$, and composition given in the obvious way.

Remark 4.4. There are canonical natural isomorphisms

$$\mathsf{WSpan}^r(\mathcal{C}, \mathcal{D}) \simeq \mathsf{WSpan}^l(\mathcal{C}^{op}, \mathcal{D}^{op}).$$

Remark 4.5. The terms *left/right* are motivated by the existence of adjunctions (which are seen to be equivalent by the previous remark)

$$\mathsf{Lan}: \mathsf{WSpan}^l(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{Fun}(\mathcal{C}, \mathcal{D}): \iota$$
$$\iota: \mathsf{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \mathsf{WSpan}^r(\mathcal{C}, \mathcal{D})^{op}: \mathsf{Ran}$$

where the functors ι denote the obvious inclusions (note the need for the $(-)^{op}$ in the second adjunction) and $\mathsf{Lan}/\mathsf{Ran}$ denote the left/right Kan extension functors.

We will mainly be interested in the span categories $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \simeq \mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$.

Notation 4.6. Given a functor $\rho: A \to \Sigma_G$, $n \geq 0$, we let $\Omega_G^n \wr A$ denote the pullback in Cat

$$\begin{tikzcd}
\Omega_G^n \wr A \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F}_s \wr A \arrow[d] \\
\Omega_G^n \arrow[r, "V_G^n"'] & \mathsf{F}_s \wr \Sigma_G
\end{tikzcd} \tag{4.7}$$

We will write the top V_G^n functor as $V_G^n \wr A$ whenever we need to distinguish such functors.

Explicitly, by Remark 3.89 the objects of $\Omega_G^n \wr A$ are pairs

$$(T_0 \to \cdots \to T_n, (a_{v_{Ge}})_{v_{Ge} \in V_G(T_n)}) \tag{4.8}$$

such that $\rho(a_{v_{Ge}}) = T_{n, v_{Ge}}$, and where $V_G(T_n)$ is ordered lexicographically (cf. Remark 3.89) according to the string $T_0 \to \cdots \to T_n$.

Remark 4.9. Generalizing the notation $\Omega_G^{-1} = \Sigma_G$, we will also write $\Omega_G^{-1} \wr A = A$, in which case $V_G^{-1} \wr A \colon \Omega_G^{-1} \wr A \to \mathsf{F}_s \wr A$ is the obvious "singleton map" $\delta^0 \colon A \to \mathsf{F}_s \wr A$.

Remark 4.10. An alternative, and arguably more suggestive, notation for $\Omega_G^n \wr A$ would be $\Omega_G^n \wr_{\Sigma_G} A$, since we are really defining a "relative" analogue of the wreath product (so that in particular $\Omega_G^n \wr_{\Sigma_G} \Sigma_G \simeq \Omega_G^n$). However, we will prefer $\Omega_G^n \wr A$ due to space concerns.

Remark 4.11. The definition of $\Omega_G^n \wr A$ in (4.7) is unchanged by replacing F_s with F. As such, to avoid cluttering the larger diagrams in this section we will from now on often abuse notation by writing simply F instead of F_s.

Our primary interest here will be in the $\Omega_G^0 \wr (-)$ construction, which can be iterated thanks to the existence of the composite maps $\Omega_G^0 \wr A \to \Omega_G^0 \to \Sigma_G$. The role of the higher strings $\Omega_G^n \wr A$ will then be to provide more convenient models for iterated $\Omega_G^0 \wr (-)$ constructions. Indeed, Proposition 3.85 can be reinterpreted as providing a canonical identification $\Omega_G^0 \wr \Omega_G^n \simeq \Omega_G^{n+1}$, with the functor $V_G^0 \wr \Omega_G^n$ identified with the functor V_G as defined in Notation 3.82. Moreover, arguing by induction on n, the fact that the rightmost squares in (3.91) are pullbacks (Proposition 3.93) provides further identifications $\Omega_G^k \wr \Omega_G^n \simeq \Omega_G^{n+k+1}$ with $V_G^k \wr \Omega_G^n$ identified with V_G^k as defined by Notation 3.88.

Our first task is now to produce analogous identifications between $\Omega_G^k \wr \Omega_G^n \wr A = \Omega_G^k \wr (\Omega_G^n \wr A)$ and $\Omega_G^{n+k+1} \wr A$ (note that iterated wreath expressions should always be read as bracketed on the right, i.e. we do *not* define the expression $(\Omega_G^k \wr \Omega_G^n) \wr A$). We start by generalizing the key functors from §3.4.

Proposition 4.12. *There are functors*

$$\Omega_G^n \wr A \xrightarrow{V_G^k} \mathsf{F}_s \wr \Omega_G^{n-k-1} \wr A \qquad \Omega_G^n \wr A \xrightarrow{d_i} \Omega_G^{n-1} \wr A \qquad \Omega_G^n \wr A \xrightarrow{s_j} \Omega_G^{n+1} \wr A$$

where $i < n$, and natural isomorphisms

$$\pi_i \colon V_G^k \Rightarrow V_G^{k-1} \circ d_i$$

for $i < k$. Further, all of these are natural in A and they satisfy all the analogues of the properties listed in Proposition 3.93.

Proof. Though it is not hard to explicitly write formulas for V_G^k, d_i, s_j, π_i (see Remark 4.13 below), and then verify the desired properties, we here instead argue that the desiderata themselves can be used to uniquely, and coherently, define those functors.

Firstly, the functors $V_G = V_G^0$ are defined from the following diagram

$$\begin{tikzcd}
\Omega_G^{n+1} \wr A \arrow[r, dashed, "V_G"] \arrow[d] & \mathsf{F} \wr \Omega_G^n \wr A \arrow[r, "\mathsf{F} \wr V_G^n"] \arrow[d] & \mathsf{F}^{\wr 2} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_G^{n+1} \arrow[r, "V_G"] & \mathsf{F} \wr \Omega_G^n \arrow[r, "\mathsf{F} \wr V_G^n"] & \mathsf{F}^{\wr 2} \wr \Sigma_G \arrow[r, "\sigma^0"] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}$$

by noting that the middle and right squares are pullbacks, and choosing V_G to be the unique functor such that the top composite is V_G^{n+1}. The higher functors V_G^k are defined exactly as in (3.83), and the analogue of Proposition 3.93(a) follows by the same proof.

The analogue of Proposition 3.93(b) is tautological, as pullback arrows for $\Omega_G^n \wr A \to \mathsf{O}_G$ are defined as compatible pairs of pullbacks in Ω_G^n and $\mathsf{F} \wr A$.

To define d_i we consider the diagram below (for some $i < k$).

$$\begin{tikzcd}
\Omega_G^n \wr A \arrow[rr, "V_G^k"] \arrow[dr, dashed, "d_i"'] \arrow[dd] & & \mathsf{F} \wr \Omega_G^{n-k-1} \wr A \arrow[dr, equal] \arrow[dd] \arrow[dl, phantom, "\Leftarrow \pi_i"] & \\
& \Omega_G^{n-1} \wr A \arrow[rr, "V_G^{k-1}"'] \arrow[dd] & & \mathsf{F} \wr \Omega_G^{n-k-1} \arrow[dd] \\
\Omega_G^n \arrow[rr] \arrow[dr, "d_i"'] & & \mathsf{F} \wr \Omega_G^{n-k-1} \arrow[dr, equal] \arrow[dl, phantom, "\Leftarrow"] & \\
& \Omega_G^{n-1} \arrow[rr, "V_G^{k-1}"'] & & \mathsf{F} \wr \Omega_G^{n-k-1}
\end{tikzcd}$$

The desiderata that the top π_i consist of pullback arrows lifting the lower π_i implies that it is uniquely determined by the top V_G^k functor, and hence so is the top composite $V_G^{k-1}d_i$. But since the front face is a pullback square (by arguing via induction on k as in (3.97)), there is a unique choice for d_i. That this definition of $d_i \wr A$ is independent of k is a consequence of the fact that the composite natural transformation in (3.94) is π_i. Similarly, that the analogues of the left diagrams in (3.92) hold follows by an identical argument from the fact that the composites of (3.95) are π_{i+1}.

The definitions of the s_j are similar, except easier since there are no π_i to contend with.

The analogues of Proposition 3.93(c),(e),(f) are then tautological, and the analogue of Proposition 3.93(d) follows by an identical argument. $\square$

Remark 4.13. Explicitly, $V_G^k\colon \Omega_G^n \wr A \to \mathsf{F} \wr \Omega_G^{n-k-1} \wr A$ is defined by sending (4.8) to

$$\left(\left(T_{k,v_{Gf}} \to \cdots \to T_{n,v_{Gf}}, (a_{v_{Ge}})_{v_{Ge} \in V_G\left(T_{n,v_{Gf}}\right)}\right)\right)_{v_{Gf} \in V_G(T_k)}$$

where both $V_G(T_k)$ and $T_{n,v_{Gf}}$ are ordered lexicographically according to the associated planar strings.

Similarly, functors $d_i\colon \Omega_G^n \wr A \to \Omega_G^{n-1} \wr A$ for $0 \le i < n$ and $s_j\colon \Omega_G^n \wr A \to \Omega_G^{n+1} \wr A$ for $-1 \le j \le n$ are defined on the object in (4.8) by performing the corresponding operation on the $T_0 \to \cdots \to T_n$ coordinate and, in the d_i case, suitably reordering $V_G(T_n)$.

Remark 4.14. One upshot of Proposition 4.12 is that formally applying the symbol $(-)\wr A$ to the diagrams in Proposition 3.93 yields sensible statements. As such, we will simply refer to the corresponding part of Proposition 3.93 when using one of the generalized claims.

Corollary 4.15. *One has identifications $\Omega_G^k \wr \Omega_G^n \wr A \simeq \Omega_G^{n+k+1} \wr A$ which identify $V_G^k \wr \Omega_G^n \wr A$ with $V_G^k \wr A$. Further, these are associative in the sense that the identifications*

$$\Omega_G^k \wr \Omega_G^l \wr \Omega_G^n \wr A \simeq \Omega_G^{k+l+1} \wr \Omega_G^n \wr A \simeq \Omega_G^{k+l+n+2} \wr A$$

$$\Omega_G^k \wr \Omega_G^l \wr \Omega_G^n \wr A \simeq \Omega_G^k \wr \Omega_G^{l+n+1} \wr A \simeq \Omega_G^{k+l+n+2} \wr A$$

coincide. Lastly, one obtains identifications

$$d_i \wr \Omega_G^n \simeq d_i \quad \pi_i \wr \Omega_G^n \simeq \pi_i \quad s_j \wr \Omega_G^n \simeq s_j \quad \Omega_G^k \wr d_i \simeq d_{i+k+1} \quad \Omega_G^k \wr \pi_i \simeq \pi_{i+k+1} \quad \Omega_G^k \wr s_j \simeq s_{j+k+1}$$

Proof. The identification $\Omega_G^k \wr \Omega_G^n \wr A \simeq \Omega_G^{n+k+1} \wr A$ follows since by Proposition 3.93(a) both expressions compute the limit of the solid part of the diagram below.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F}^{\wr 2} \wr A \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega_G^{n+k+1} \arrow[r, "V_G^k"'] \arrow[d] & \mathsf{F} \wr \Omega_G^n \arrow[r, "\mathsf{F}\wr V_G^n"'] \arrow[d] & \mathsf{F}^{\wr 2} \wr \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \Sigma_G \\
\Omega_G^k \arrow[r, "V_G^k"'] & \mathsf{F} \wr \Sigma_G & &
\end{tikzcd}$$

Associativity follows similarly. The remaining identifications are obvious. $\square$

We now have all the necessary ingredients to define our monad on spans.

Definition 4.16. Suppose $\mathcal{V}$ has finite products or, more generally, that it is a symmetric monoidal category with diagonals in the sense of Remark 2.18.

We define an endofunctor N of $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$ by letting $N(\Sigma_G \leftarrow A \to \mathcal{V}^{op})$ be the span $\Sigma_G \leftarrow \Omega^0_G \wr A \to \mathcal{V}^{op}$ given by composition of the diagram

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] & \mathcal{V}^{op} \\
\Omega^0_G \arrow[r, "V_G"'] \arrow[d] & \mathsf{F} \wr \Sigma_G & & \\
\Sigma_G & & &
\end{tikzcd}$$

and defined on maps of spans in the obvious way.

One has a multiplication $\mu\colon N \circ N \Rightarrow N$ given by the natural isomorphism

$$\begin{tikzcd}
\Sigma_G \arrow[d, equal] & \Omega^1_G \wr A \arrow[l] \arrow[r, "V_G"] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \wr A \arrow[r, "\mathsf{F}\wr V_G"] \arrow[d, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"] \arrow[d, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma_G & \Omega^0_G \wr A \arrow[l] \arrow[rr, "V_G"'] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr, "\otimes^{op}"'] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.17}$$

where we note that the top right composite in the π_0 square is indeed V^1_G using the inductive description in (the $(-) \wr A$ analogue of) Notation 3.88.

Lastly, there is a unit $\eta\colon id \Rightarrow N$ given by the strictly commutative diagrams

$$\begin{tikzcd}
\Sigma_G \arrow[d, equal] & A \arrow[l] \arrow[r, equal] \arrow[d, "s_{-1}"'] & A \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Sigma_G & \Omega^0_G \wr A \arrow[l] \arrow[r, "V_G"'] & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, "\otimes^{op}"'] & \mathcal{V}^{op}.
\end{tikzcd} \tag{4.18}$$

Proposition 4.19. *(N, μ, η) is a monad on $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op})$.*

Proof. The natural transformation component of $\mu \circ (N\mu)$ is given by the composite diagram

$$\begin{tikzcd}[column sep=small]
\Omega^2_G \wr A \arrow[r] \arrow[d, "d_1"'] & \mathsf{F} \wr \Omega^1_G \wr A \arrow[r] \arrow[d, "\mathsf{F}\wr d_0"'] & \mathsf{F}^{\wr 2} \wr \Omega^0_G \wr A \arrow[r] \arrow[d, phantom, "\Leftarrow \mathsf{F}\wr\pi_0"] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^1"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\Leftarrow \mathsf{F}\wr\alpha"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^1_G \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \wr A \arrow[rr] \arrow[d, phantom, "\Leftarrow \pi_0"] & & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[rr] \arrow[d, "\sigma^0"] & & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^0_G \wr A \arrow[rrr] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.20}$$

whereas the natural transformation component of $\mu \circ (\mu N)$ is given by

$$\begin{tikzcd}[column sep=small]
\Omega^2_G \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^1_G \wr A \arrow[r] \arrow[d, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr \Omega^0_G \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 3} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d] & \mathsf{F}^{\wr} \mathcal{V}^{op} \arrow[r] \arrow[d, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^1_G \wr A \arrow[rr] \arrow[d, "d_0"'] & & \mathsf{F} \wr \Omega^0_G \wr A \arrow[r] \arrow[d, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] \arrow[d, phantom, "\Leftarrow \alpha"] & & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^0_G \wr A \arrow[rrr] & & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rrr] & & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.21}$$

That the rightmost sides of (4.20) and (4.21) coincide follows from the associativity of the isomorphisms α in (2.15). On the other hand, the leftmost sides coincide since they are instances of the "simplicial relation" diagrams in (3.96), as is seen by using (3.94) and (3.95) to reinterpret the top left sections.

As for the unit conditions, $\mu \circ (N\eta)$ is represented by

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r] \arrow[d, "s_0"'] & \mathsf{F} \wr A \arrow[r, equal] \arrow[d, "s_{-1}"'] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^1"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, equal] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^1_G \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \wr A \arrow[r] \arrow[dl, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^0_G \wr A \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.22}$$

while $\mu \circ (\eta N)$ is represented by

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r, equal] \arrow[d, "s_{-1}"'] & \Omega^0_G \wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F} \wr A \arrow[r] \arrow[d, "\delta^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[r, equal] \arrow[d, "\delta^0"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^1_G \wr A \arrow[r] \arrow[d, "d_0"'] & \mathsf{F} \wr \Omega^0_G \wr A \arrow[r] \arrow[dl, phantom, "\Leftarrow \pi_0"] & \mathsf{F}^{\wr 2} \wr A \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F}^{\wr 2} \wr \mathcal{V}^{op} \arrow[r] \arrow[d, "\sigma^0"] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[r] \arrow[dl, phantom, "\Leftarrow \alpha"] & \mathcal{V}^{op} \arrow[d, equal] \\
\Omega^0_G \wr A \arrow[rr] & & \mathsf{F} \wr A \arrow[r] & \mathsf{F} \wr \mathcal{V}^{op} \arrow[rr] & & \mathcal{V}^{op}
\end{tikzcd} \tag{4.23}$$

That (4.22) and (4.23) coincide follows analogously by the unital condition for α and the face-degeneracy relations in Proposition 3.93(f). $\square$

4.2 The genuine equivariant operad monad

Since $\mathsf{Wspan}^r(\Sigma_G, \mathcal{V}^{op}) \simeq \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$, Proposition 4.19 and Remark 4.5 give an adjunction

$$\mathsf{Lan}\colon \mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V}) \rightleftarrows \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})\colon \iota$$

together with a monad N in the leftmost category $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$.

We will now show that under reasonable conditions on $\mathcal{V}$ this monad can be transferred by using Proposition 2.27, i.e. we will show that the natural transformations $\mathsf{Lan} \circ N \Rightarrow \mathsf{Lan} \circ N \circ \iota \circ \mathsf{Lan}$ and $\mathsf{Lan} \circ \iota \Rightarrow id$ are isomorphisms.

This will require us to introduce a slight modification of the category of spans. For motivation, note that iterations $N^{\circ n+1} \circ \iota$ produce spans of the form $\Sigma_G \leftarrow \Omega^n_G \to \mathcal{V}^{op}$ (where we use the identification $\Omega^n_G \wr \Sigma_G \simeq \Omega^n_G$). As noted in Remark 3.81, the maps $\Omega^n_G \to \Sigma_G$ are maps of split fibrations over O_G, as are all other simplicial operators d_i, s_j.

Definition 4.24. The category $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$ of *rooted (left) spans* has as objects spans $\Sigma_G^{op} \leftarrow A^{op} \to \mathcal{V}$ together with a split Grothendieck fibration $\mathsf{r}\colon A \to \mathsf{O}_G$ such that $A \to \Sigma_G$ is a map of split fibrations.

Similarly, arrows are maps of spans that induce maps of split fibrations.

We refer to split fibrations $A \to \mathsf{O}_G$ as *root fibrations* and to maps between them as *root fibration maps.*

Remark 4.25. The condition that $A \to \mathsf{O}_G$ be a root fibration requires additional *choices* of root pullbacks. Therefore, the forgetful functor $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V}) \to \mathsf{Wspan}^l(\Sigma_G^{op}, \mathcal{V})$ is not quite injective on objects.

The relevance of rooted spans is given by the following couple of lemmas.

Lemma 4.26. *If $A \to \Sigma_G$ is a root fibration map then so is $\Omega^0_G \wr A \to \Omega^0_G$, naturally in A.*

Proof. The hypothesis that $A \to \Sigma_G$ is a root fibration map implies that the rightmost vertical map below is a map of split fibrations over $\mathsf{F} \wr \mathsf{O}_G$.

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[d] \\
\Omega^0_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G
\end{tikzcd}$$

Since by Lemma 3.65 the map V_G sends pullback arrows in Ω^0_G (over O_G) to pullback arrows in $\mathsf{F} \wr \Sigma_G$ (over $\mathsf{F} \wr \mathsf{O}_G$), the root pullback arrows in $\Omega^0_G \wr A$ can be defined as compatible pairs of pullback arrows in Ω^0_G and $\mathsf{F} \wr A$, and the result follows. □

Remark 4.27. Explicitly, if $\psi : Y \to X$ is a map in O_G, and $\tilde{T} = (T, (A_{v_{Ge}})_{V_G(T)}) \in \Omega^0_G \wr A$ lies over X, the pullback $\psi^* \tilde{T}$ is given by

$$\left(\psi^* T, (\bar{\psi}^*_{Ge} A_{v_{Ge}})_{V_G(\psi^* T)}\right)$$

where $\bar{\psi}$ is the map $\bar{\psi} : \psi^* T \to T$ and $\bar{\psi}_{Ge}$ denotes the restriction $\bar{\psi} : Ge \to G\bar{\psi}(e)$, as in Remark 3.73.

Lemma 4.28. *Suppose that $\mathcal{V}$ is complete and that $\rho : A \to \Sigma_G$ is a root fibration map. If the rightmost triangle in*

$$\begin{tikzcd}
\Omega^0_G \wr A \arrow[r, "V_G"] \arrow[d] & \mathsf{F} \wr A \arrow[r] \arrow[d] & \mathcal{V}^{op} \\
\Omega^0_G \arrow[r, "V_G"'] & \mathsf{F} \wr \Sigma_G \arrow[ur] \arrow[u, phantom, shift left=3, "\Nearrow"] &
\end{tikzcd}$$

is a right Kan extension diagram then so is the composite diagram.

Proof. Unpacking definitions using the pointwise formula for right Kan extensions (cf. [20, X.3 Thm. 1] or (2.4)), it suffices to check that for each $T \in \Omega^0_G$ the induced functor

$$T \downarrow \Omega^0_G \wr A \xrightarrow{V_G} V_G(T) \downarrow \mathsf{F} \wr A$$

is initial. We will slightly abuse notation by writing $(T \to U, (A_{v_{Gf}})_{V_G(U)})$ for the objects of $T \downarrow \Omega^0_G \wr A$, as well as $\left((T_{v_{Ge}} \to U_{\phi(v_{Ge})})_{v_{Ge} \in V_G(T)}, (A_v)_{v \in V}\right)$ for the objects of $V_G(T) \downarrow \mathsf{F} \wr A$, with the map $\phi : V_G(T) \to V$ and the condition $\rho(A_v) = U_v$ left implicit.

By Proposition 2.5, $T \downarrow \Omega^0_G \wr A$ has an initial subcategory $T \downarrow_{\mathsf{r}} \Omega^0_G \wr A$ of those objects such that $T \to U$ is the identity on roots. Similarly, again by Proposition 2.5, $V_G(T) \downarrow \mathsf{F} \wr A$ has an initial subcategory

$$\prod_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r}} A \tag{4.29}$$

of those objects inducing an identity on $\mathsf{F} \wr \mathsf{O}_G$. Moreover, (4.29) comes together with a right retraction r, i.e. a right adjoint to the inclusion i into $V_G(T) \downarrow \mathsf{F} \wr A$, which is built using pullbacks. Explicitly, unpacking the proof of Proposition 2.5 one has that r is given by the assignment

HERE

$$\left((T_{v_{Ge}})_{V_G(T)} \xrightarrow{\tau} (U_x)_X, (A_x)_X\right) \mapsto \left(\left(T_{v_{Ge}} \to (\mathsf{r}\tau_{v_{Ge}})^* U_{\tau(v_{Ge})}\right), \left((\mathsf{r}\tau_{v_{Ge}})^* A_{\tau(v_{Ge})}\right)\right)$$

HERE

We now compute the following composite (where we abbreviate expressions $T_{v_{Ge}}$ as T_{Ge} and implicitly assume that tuples with index Ge (resp. Gf) run over $V_G(T)$ (resp. $V_G(U)$)).

$$\begin{array}{ccccc} T \downarrow_{\mathsf{r}} \Omega_G^0 \wr A & \xrightarrow{\quad V_G \quad} & V_G(T) \downarrow \mathsf{F} \wr A & \xrightarrow{\qquad r \qquad} & \prod\limits_{v_{Ge} \in V_G(T)} T_{v_{Ge}} \downarrow_{\mathsf{r}} A \\ (T \xrightarrow{\psi} U, (A_{Gf})) & \longmapsto & \big((T_{Ge} \to U_{G\psi(e)}), (A_{Gf})\big) & \longmapsto & \big((T_{Ge} \to \psi_{Ge}^* U_{G\psi(e)}), (\psi_{Ge}^* A_{G\psi(e)})\big) \end{array}$$

Since rooted quotients are isomorphisms, the ψ and ψ_{Ge} appearing above are isomorphisms, and hence the natural transformation $i \circ r \circ V_G \Rightarrow V_G$ is a natural isomorphism. Therefore, V_G will be initial provided that so is $i \circ r \circ V_G$, and since the inclusion i is initial, it suffices to show that $r \circ V_G$ is an isomorphism.

But now note that an arbitrary choice of rooted isomorphisms $T_{v_{Ge}} \to U^{\mathsf{r}}_{v_{Ge}}$ uniquely determines a compatible planar structure on T, and thus a unique isomorphism $\psi: T \to U$. Therefore, arbitrary choices of $\psi_{Ge}^* U_{G\psi(e)}$, $\psi_{Ge}^* A_{G\psi(e)}$ uniquely determine U, A_{Gf}, finishing the proof. □

Lemma 4.26 implies that copying Definition 4.16 yields a monad N_{r} on $\mathsf{Wspan}^l_{\mathsf{r}}(\Sigma_G^{op}, \mathcal{V})$ lifting the monad N.

Corollary 4.30. *Suppose that finite products in $\mathcal{V}$ commute with colimits in each variable or, more generally, that $\mathcal{V}$ is a symmetric monoidal category with diagonals such that $\otimes$ preserves colimits in each variable. Then the natural transformations*

$$\mathsf{Lan} \circ N_{\mathsf{r}} \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan}, \qquad \mathsf{Lan} \circ \iota \Rightarrow id$$

are natural isomorphisms.

Proof. This follows by combining Lemma 4.28 with Lemma 2.21. □

Recalling Proposition 2.27 now leads to the following.

Definition 4.31. The *genuine equivariant operad monad* is the monad $\mathbb{F}_G$ on $\mathsf{Sym}_G(\mathcal{V}) = \mathsf{Fun}(\Sigma_G^{op}, \mathcal{V})$ given by

$$\mathbb{F}_G = \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

and with multiplication and unit given by the composites

$$\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \circ \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ N_{\mathsf{r}} \circ N_{\mathsf{r}} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$$

$$id \overset{\simeq}{\Leftarrow} \mathsf{Lan} \circ \iota \Rightarrow \mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota.$$

We will write $\mathsf{Op}_G(\mathcal{V})$ for the category $\mathsf{Alg}_{\mathbb{F}_G}(\mathsf{Sym}_G(\mathcal{V}))$ of *genuine equivariant operads.*

Remark 4.32. The functor $\mathsf{Lan} \circ N_{\mathsf{r}} \circ \iota$ is isomorphic to $\mathsf{Lan} \circ N \circ \iota$, and this isomorphism is compatible with the multiplication and unit in Definition 4.31, and as such we will henceforth simply write N rather than N_{r}.

From this point of view, root fibrations play an auxiliary role in verifying that $\mathsf{Lan} \circ N \circ \iota$ is indeed a monad, but are unnecessary to describe the monad structure itself.

Remark 4.33. Since a map

$$\mathbb{F}_G X = \mathsf{Lan} \circ N \circ \iota X \to X$$

is adjoint to a map

$$N \circ \iota X \to \iota X$$

one easily verifies that X is a genuine equivariant operad, i.e. a $\mathbb{F}_G$-algebra, iff ιX is a N-algebra (cf. Proposition 2.27(ii)).

Moreover, the bar resolution $\mathbb{F}_G^{\circ n+1} X$ is isomorphic to $\mathsf{Lan}\big(N^{\circ n+1} \iota X\big)$.

4.3 Comparison with (regular) equivariant operads

In the case $G = *$, genuine operads coincide with the usual notion of symmetric operads, i.e. $\mathsf{Sym}_*(\mathcal{V}) \simeq \mathsf{Sym}(\mathcal{V})$ and $\mathsf{Op}_*(\mathcal{V}) \simeq \mathsf{Op}(\mathcal{V})$, and in what follows we will adopt the notations $\mathsf{Sym}^G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$ for the corresponding categories of G-objects. Our goal in this section will be to relate these to the categories $\mathsf{Sym}_G(\mathcal{V})$ and $\mathsf{Op}_G(\mathcal{V})$ of genuine equivariant sequences and genuine equivariant operads.

We will throughout this section fix a total order of G such that the identity e is the first element, though we note that the exact order is unimportant, as any other such choice would lead to unique isomorphisms between the constructions described herein.

We now have an inclusion functor

$$\begin{aligned} \iota\colon G \times \Sigma &\hookrightarrow \Sigma_G \\ C &\longmapsto G \cdot C \end{aligned}$$

where $G \cdot C$ is the constant tuple $(C)_{g\in G}$, which we think of as $|G|$ copies of C, planarized according to C and the order on G. Moreover, letting $\Sigma_G^{\mathrm{fr}} \hookrightarrow \Sigma_G$ denote the full subcategory of G-free corollas, there is an induced retraction $\rho\colon \Sigma_G^{\mathrm{fr}} \to G\times\Sigma$ defined by $\rho\left((C_i)_{1\le i\le |G|}\right) = G\cdot C_1$ together with isomorphisms $C \simeq \rho(C)$ uniquely determined by the condition that they are the identity on the first tree component C_1.

We now consider the associated adjunctions.

$$\begin{tikzcd} \mathsf{Sym}_G(\mathcal{V}) \arrow[r, "\iota^*"'] & \mathsf{Sym}^G(\mathcal{V}) \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"] \end{tikzcd} \tag{4.34}$$

Explicitly, we have the formulas (where we write G-corollas as $(C_i)_I$ for $I \in \mathsf{O}_G$)

$$\iota_! Y\left((C_i)_I\right) = \begin{cases} Y(C_1), & (C_i)_I \in \Sigma_G^{\mathrm{fr}} \\ \varnothing, & (C_i)_I \notin \Sigma_G^{\mathrm{fr}} \end{cases}, \quad \iota^* X(C) = X(G\cdot C), \quad \iota_* Y((C_i)_I) = \left(\prod_I Y(C_i)\right)^G,$$

where in the formula for ι_* the action of G interchanges factors according to the action on the indexing set I. As a side note, we note that the formulas for $\iota_!$ and ι_* are independent of the chosen order of G.

Remark 4.35. $\iota_!$ essentially identifies $\mathsf{Sym}^G(\mathcal{V})$ as the coreflexive subcategory of sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C) = \varnothing$ whenever C is not a free corolla.

On the other hand, ι_* identifies $\mathsf{Sym}^G(\mathcal{V})$ with the more interesting reflexive subcategory of those sequences $X \in \mathsf{Sym}_G(\mathcal{V})$ such that $X(C)$ for each C not a free corolla must satisfy a fixed point condition. Explicitly, letting $\varphi\colon G \to \mathsf{r}(C)$ denote the unique map preserving the minimal element, one has

$$X(C) \xrightarrow{\simeq} X(\varphi^* C)^\Gamma$$

for $\Gamma \le \mathsf{Aut}(\varphi^* C)$ the subgroup preserving the quotient map $\varphi^* C \to C$ under precomposition (note that $\varphi^* C \in \Sigma_G^{\mathrm{fr}}$).

There is an obvious natural transformation $\beta\colon \iota_! \Rightarrow \iota_*$ which for $(C_i)_I \in \Sigma_G^{\mathrm{fr}}$ sends $Y(C_1)$ to the "G-twisted diagonal" of $\prod_I Y(C_i)$. Moreover, letting $\eta_!, \epsilon_!$ (resp. η_*, ϵ_*) denote the unit and counit of the $(\iota_!, \iota^*)$ adjunction (resp. (ι^*, ι_*) adjunction) it is straightforward to check that the following diagram commutes.

$$\begin{tikzcd} \iota_!\iota^*\iota_* \arrow[r, Rightarrow, "\epsilon_!"] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota_* \arrow[d, Rightarrow, "\eta_!", "\simeq"'] \\ \iota_! \arrow[r, Rightarrow, "\eta_*"'] \arrow[ur, Rightarrow, "\beta"'] & \iota_*\iota^*\iota_! \end{tikzcd} \tag{4.36}$$

Remark 4.37. An exercise in adjunctions shows the outer square in (4.36) commutes provided at least one of the adjunctions in (4.34) is (co)reflexive, so that (4.36) can be regarded as an alternative definition of β.

Proposition 4.38. *One has the following:*

(i) the map $\iota^*\mathbb{F}_G \xrightarrow{\eta_*} \iota^*\mathbb{F}_G\iota_*\iota^*$ *is an isomorphism, and thus (cf. Prop. 2.27)* $\iota^*\mathbb{F}_G\iota_*$ *is a monad;*

(ii) the map $\iota^*\mathbb{F}_G\iota_! \xrightarrow{\beta} \iota^*\mathbb{F}_G\iota_*$ *is an isomorphism of monads;*

(iii) the map $\iota_!\iota^*\mathbb{F}_G\iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G\iota_!$ *is an isomorphism;*

(iv) there is a natural isomorphism of monads $\alpha\colon\mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$.

Proof. We first show (i), starting with some notation. In analogy with Σ_G^{fr}, we write $\Omega_G^{0,\mathrm{fr}}$ for the subcategory of free trees and note that the leaf-root and vertex functors then restrict to functors $\mathsf{lr}\colon\Omega_G^{0,\mathrm{fr}} \to \Sigma_G^{\mathrm{fr}}$, $V_G\colon\Omega_G^{0,\mathrm{fr}} \to \mathsf{F}\wr\Sigma_G^{\mathrm{fr}}$. Moreover, for each $C \in \Sigma_G^{\mathrm{fr}}$ one has an equality of rooted undercategories between $C\downarrow_{\mathsf{r}}\Omega_G^0$ and $C\downarrow_{\mathsf{r}}\Omega_G^{0,\mathrm{fr}}$, and thus $\iota^*\mathbb{F}_G X$ is computed by the Kan extension of the following diagram.

$$\begin{tikzcd}
\Omega_G^{0,\mathrm{fr}} \arrow[r] \arrow[d] & \mathsf{F}\wr\Sigma_G^{\mathrm{fr}} \arrow[r, "\mathsf{F}\wr X"] & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
\Sigma_G^{\mathrm{fr}}
\end{tikzcd}$$

(i) now follows by noting that $X \to \iota_*\iota^*X$ is an isomorphism when restricted to Σ_G^{fr}.

For (ii), to show that $\iota^*\mathbb{F}_G\iota_! \to \iota^*\mathbb{F}_G\iota_*$ is an isomorphism of functors one just repeats the argument in the previous paragraph by noting that $\iota_! \to \iota_*$ is an isomorphism when restricted to Σ_G^{fr}. To check that this is a map of monads, we first recall that the monad structure on $\iota^*\mathbb{F}_G\iota_*$ is given as described in Proposition 2.27. Unpacking definitions, compatibility with multiplication reduces to showing that the composite $\iota_!\iota^* \xrightarrow{\epsilon_!} id \xrightarrow{\eta_*} \iota_*\iota^*$ coincides with $\beta\iota^*$ while compatibility with units reduces to showing that the composite $id \xrightarrow{\eta_!} \iota^*\iota_! \xrightarrow{\iota^*\beta} \iota^*\iota_* \xrightarrow{\epsilon_*} id$ is the identity. Both of these are a consequence of (4.36), following from the diagrams below (where the top composites are identities).

$$\begin{tikzcd}
\iota_!\iota^* \arrow[r, Rightarrow, "\iota_!\iota^*\eta_*"] \arrow[d, Rightarrow, "\epsilon_!"'] & \iota_!\iota^*\iota_*\iota^* \arrow[r, Rightarrow, "\iota_!\epsilon_*\iota^*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_!\iota_*\iota^*"'] & \iota_!\iota^* \arrow[dl, Rightarrow, "\beta\iota^*"] \\
id \arrow[r, Rightarrow, "\eta_*"'] & \iota_*\iota^* &
\end{tikzcd}
\qquad
\begin{tikzcd}
\iota^*\iota_* \arrow[r, Rightarrow, "\eta_!\iota^*\iota_*", "\simeq"'] \arrow[d, Rightarrow, "\epsilon_*"', "\simeq"] & \iota^*\iota_!\iota^*\iota_* \arrow[r, Rightarrow, "\iota^*\epsilon_!\iota_*"] \arrow[d, Rightarrow, "\iota^*\iota_!\epsilon_*"', "\simeq"] & \iota^*\iota_* \\
id \arrow[r, Rightarrow, "\simeq", "\eta_!"'] & \iota^*\iota_! \arrow[ur, Rightarrow, "\iota^*\beta"'] &
\end{tikzcd}$$

(iii) amounts to showing that if $X(C) = \varnothing$ whenever $C \notin \Sigma_G^{\mathrm{fr}}$ then we must also have that $\mathbb{F}_G X(C) = \varnothing$. Indeed, since for $C \notin \Sigma_G^{\mathrm{fr}}$ the undercategory $C\downarrow\Omega_G^0$ consists of trees with at least one non-free vertex (namely the root vertex), the composite

$$C\downarrow\Omega_G^0 \xrightarrow{V_G} \mathsf{F}\wr\Sigma_G \xrightarrow{\mathsf{F}\wr X} \mathsf{F}\wr\mathcal{V}^{op} \xrightarrow{\otimes} \mathcal{V}^{op}$$

is constant equal to $\varnothing$, and (iii) follows.

Finally, we show (iv). We will slightly abuse notation by writing $G\times\Sigma \hookrightarrow \Sigma_G$ for the image of ι and similarly $G\times\Omega^0 \hookrightarrow \Omega_G^0$ for the image of the obvious analogous functor $\iota\colon G\times\Omega^0 \to \Omega_G^0$. The map $\alpha\colon\mathbb{F} \to \iota^*\mathbb{F}_G\iota_!$ is the adjoint to the map $\tilde{\alpha} : \mathbb{F}\iota^* \to \iota^*\mathbb{F}_G$ encoded on spans by the following diagram.

$$
\begin{tikzcd}
G\times\Omega^0 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr(G\times\Sigma) \arrow[rr, "\iota^*X"] \arrow[dr] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[dr, equal] & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega^0_G \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\Sigma_G \arrow[rr, "X"'] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
G\times\Sigma \arrow[dr] & & & & & & \\
& \Sigma_G & & & & &
\end{tikzcd}
\tag{4.39}
$$

That α is a natural isomorphism follows by the previous identifications $C\downarrow_{\mathsf{r}}\Omega^0_G \simeq C\downarrow_{\mathsf{r}}\Omega^{0,\mathrm{fr}}_G$ for $C\in G\times\Sigma$ together with the fact that the retraction $\rho\colon\Omega^{0,\mathrm{fr}}_G\to G\times\Omega^0$ (built just as the retraction $\rho\colon\Sigma^{\mathrm{fr}}_G\to G\times\Sigma$) retracts $C\downarrow_{\mathsf{r}}\Omega^{0,\mathrm{fr}}_G$ to the undercategory $C\downarrow_{\mathsf{r}} G\times\Omega^0$, which is thus initial (as well as final).

Intuitively, the final claim that α is a map of monads follows from the fact that the composite $\mathbb{F}\mathbb{F}\to\iota^*\mathbb{F}_G\iota_!\iota^*\mathbb{F}_G\iota_!\to\iota^*\mathbb{F}_G\mathbb{F}_G\iota_!$ is encoded by the analogous natural transformation of diagrams for strings $G\times\Omega^1\hookrightarrow\Omega^{1,\mathrm{fr}}_G$. However, since the presence of left Kan extensions in the definitions of $\mathbb{F}$, $\mathbb{F}_G$ can make a rigorous direct proof of this last claim fairly cumbersome, we sketch here a workaround argument. We first consider the adjunction $\iota_!\colon\mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V})\rightleftarrows\mathsf{WSpan}^l(\Sigma^{op}_G,\mathcal{V})\colon\iota^*$ where $\iota_!$ is composition with ι and ι^* is the pullback of spans. Writing N, N_G for the monads on the span categories, mimicking (4.39) yields a map $\tilde{\alpha}\colon N\to\iota^*N_G\iota_!$ encoded by the diagram (where the front and back squares are pullbacks).

$$
\begin{tikzcd}
(G\times\Omega^0)\wr\iota^*A \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr\iota^*A \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] \arrow[dr, equal] & \mathcal{V}^{op} \arrow[dr, equal] & \\
& \Omega^0_G\wr A \arrow[rr] \arrow[dd] & & \mathsf{F}\wr A \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\mathcal{V}^{op} \arrow[r] & \mathcal{V}^{op} \\
G\times\Omega^0 \arrow[rr] \arrow[dr] \arrow[dd] & & \mathsf{F}\wr(G\times\Sigma) \arrow[dr] & & & & \\
& \Omega^0_G \arrow[rr] \arrow[dd] & & \mathsf{F}\wr\Sigma_G & & & \\
G\times\Sigma \arrow[dr] & & & & & & \\
& \Sigma_G & & & & &
\end{tikzcd}
$$

The claim that $\tilde{\alpha}$ is a map of monads is then straightforward. Writing

$$
\mathsf{Lan}\colon\mathsf{WSpan}^l((G\times\Sigma)^{op},\mathcal{V})\rightleftarrows\mathsf{Fun}((G\times\Sigma)^{op},\mathcal{V})\colon j \quad \mathsf{Lan}_G\colon\mathsf{WSpan}^l(\Omega^{op}_G,\mathcal{V})\rightleftarrows\mathsf{Fun}(\Omega^{op}_G,\mathcal{V})\colon j_G
$$

for the span-functor adjunctions, $\alpha\colon\mathbb{F}\to\iota^*\mathbb{F}_G\iota_!$ can then be written as the composite

$$
\mathsf{Lan}Nj\to\mathsf{Lan}\iota^*N_G\iota_!j\to\iota^*\mathsf{Lan}_GN_Gj_G\iota_!
$$

where the first map is the isomorphism of monads induced by $\tilde{\alpha}$ and the second map can be shown directly to be a monad map by unpacking the monad structures in Propositions 2.26 and 2.27. $\square$

Combining the previous result with Propositions 2.26 and 2.27 now yields the following.

Corollary 4.40. *The adjunctions (4.34) extend to adjunctions*

$$
\begin{tikzcd}
\mathsf{Op}_G(\mathcal{V}) \arrow[r, "\iota^*"'] & \mathsf{Op}^G(\mathcal{V}). \arrow[l, bend right, "\iota_!"'] \arrow[l, bend left, "\iota_*"]
\end{tikzcd}
$$

In particular, ι_* identifies $\mathsf{Op}^G(\mathcal{V})$ as a reflexive subcategory of $\mathsf{Op}_G(\mathcal{V})$.

Remark 4.41. Remark 4.35 extends to operads mutatis mutandis.

Moreover, the isomorphism $\iota_!\iota^*\mathbb{F}_G\iota_! \xrightarrow{\epsilon_!} \mathbb{F}_G\iota_!$ then shows that $\mathbb{F}_G$ essentially preserves the image of $\iota_!$, and can thus be identified with $\mathbb{F}$ over it.

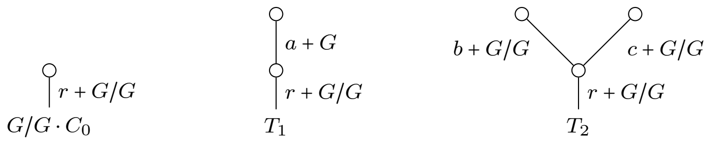
However, the analogous statement fails for ι_*, i.e., one does not always have that

$$\mathbb{F}_G\iota_* \xrightarrow{\eta_*} \iota_*\iota^*\mathbb{F}_G\iota_* \tag{4.42}$$

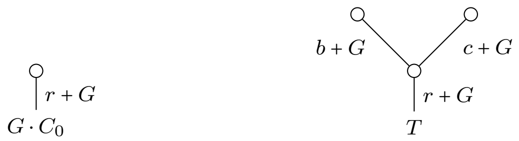
is an isomorphism. In fact, the claim that (4.42) *does* become an isomorphism when restricted to *cofibrant* objects is one of the key ingredients of our proof of the Quillen equivalence between $\mathsf{Op}_G(\mathcal{V})$ and $\mathsf{Op}^G(\mathcal{V})$ given by Theorem III, and will be the subject of §6.

For now, we end this section with a minimal counterexample to the more general claim. Let $G = \mathbb{Z}_{/2}$ and $Y = * \in \mathsf{Sym}^G(\mathcal{V})$ be the singleton.

When evaluating $\mathbb{F}_G\iota_*Y$ at the G-fixed stump corolla $G/G\cdot C_0$ (where $C_0 \in \Sigma$ denotes the 0-corolla), the two G-trees T_1 and T_2 below encode two distinct points (since T_1, T_2 are not isomorphic as objects under $G/G\cdot C_0$).



However, when pulling these points back to the G-free stump corolla $G\cdot C_0$ one obtains the same point in $\mathbb{F}_G\iota_*Y(G\cdot C_0)$, namely that encoded by the G-tree T below.



Moreover, it is not hard to modify the example above to produce similar examples when evaluating $\mathbb{F}_GY$ at non-empty corollas.

However, such counter-examples all require the use of trees with stumps. Indeed, it can be shown that (4.42) is an isomorphism whenever evaluated at a Y such that $Y(C_0) = \varnothing$.

4.4 Indexing systems and partial genuine operads

As discussed preceding Theorem II, the Elmendorf-Piacenza equivalence (1.7) has analogues

$$\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Top}^G_{\mathcal{F}}$$

for each family $\mathcal{F}$ of subgroups of G. Here $\mathsf{O}_{\mathcal{F}} \hookrightarrow \mathsf{O}_G$ consists of those G/H such that $H \in \mathcal{F}$ and thus the objects of $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}}$ are partial coefficient systems. These specialized equivalences provide an alternative approach to universal $E\mathcal{F}$-spaces: rather than cofibrantly replacing the object $\delta_{\mathcal{F}} \in \mathsf{Top}^{\mathsf{O}_G^{op}}$ as in the introduction, one builds an $E\mathcal{F}$-space by $\iota^*(C*) = (C*)(G)$ where now $* \in \mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}}$ is the terminal object and C the cofibrant replacement in $\mathsf{Top}^{\mathsf{O}_{\mathcal{F}}^{op}}$.

In keeping with the motivation that the Blumberg-Hill $N\mathcal{F}$ operads are the operadic analogues of universal $E\mathcal{F}$ spaces, we will now show that the closure conditions for indexing systems identified in [3, Def. 3.22] are (almost exactly) the necessary conditions to define categories $\mathsf{Op}_{\mathcal{F}}$ of partial genuine equivariant operads.

We start by recalling that in the classic setting $\mathcal{F}$ is a family of subgroups of G if and only if the associated subcategory $\mathsf{O}_{\mathcal{F}} \hookrightarrow \mathsf{O}_G$ is a sieve, defined as follows.

Definition 4.43. A *sieve* of a category $\mathcal{D}$ is a subcategory $\mathcal{S}$ such that for any arrow $f\colon d \to s$ in $\mathcal{D}$ with $s \in \mathcal{S}$ then both d and f are also in $\mathcal{S}$. In particular, sieves are full subcategories.

Definition 4.44. We call a sieve $\Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ a *family of G-corollas.*

Remark 4.45. A family of G-corollas $\Sigma_{\mathcal{F}}$ can equivalently be encoded by a collection $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ of families $\mathcal{F}_n$ of *graph subgroups* of $G \times \Sigma_n$, so that there is an equivalence of categories $\Sigma_{\mathcal{F}} \simeq \coprod \mathsf{O}_{\mathcal{F}_n}$ (see Lemma 6.43). As such, we abuse notation and abbreviate either set of data as $\mathcal{F}$.

Writing $\gamma\colon \Sigma_{\mathcal{F}} \hookrightarrow \Sigma_G$ for the inclusion and $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) = \mathcal{V}^{\Sigma_{\mathcal{F}}^{op}}$, we thus have a pair of adjunctions

$$\begin{tikzcd} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \arrow[r, bend left, "\gamma_!"] \arrow[r, bend right, "\gamma_*"'] & \mathsf{Sym}_G(\mathcal{V}) \arrow[l, "\gamma^*"'] \end{tikzcd} \tag{4.46}$$

Our focus will be on the $(\gamma_!, \gamma^*)$ adjunction. The requirement that $\Sigma_{\mathcal{F}}$ be a sieve then implies that $\gamma_!$ simply extends presheaves by the initial object $\varnothing \in \mathcal{V}$, so that $\gamma_!$ identifies $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ with a (coreflexive) subcategory of $\mathsf{Sym}_G(\mathcal{V})$. One may then ask for conditions on the family of corollas $\mathcal{F}$ such that the genuine operad monad $\mathbb{F}_G$ preserves this subcategory. The answer is almost exactly given by the Blumberg-Hill indexing systems.

Definition 4.47. Let $\mathcal{F}$ be a family of G-corollas.

We say that a G-tree T is a *$\mathcal{F}$-tree* if all of its G-vertices T_v, $v \in V_G(T)$ are in $\Sigma_{\mathcal{F}}$. We denote by $\Omega_{\mathcal{F}} \hookrightarrow \Omega_G$, $\Omega^0_{\mathcal{F}} \hookrightarrow \Omega^0_G$ the full subcategories spanned by the $\mathcal{F}$-trees.

Remark 4.48. By vacuousness the stick G-trees $(G/H)\cdot\eta = (\eta)_{G/H}$ are always $\mathcal{F}$-trees.

Definition 4.49. A family $\mathcal{F}$ of G-corollas is called a *weak indexing system* if for any $\mathcal{F}$-tree $T \in \Omega^0_{\mathcal{F}}$ we have $\mathsf{lr}(T) \in \Sigma_{\mathcal{F}}$; that is, if the leaf-root functor restricts to a functor $\mathsf{lr}\colon \Omega^0_{\mathcal{F}} \to \Sigma_{\mathcal{F}}$. Moreover, $\mathcal{F}$ is called simply an *indexing system* if all trivial corollas $(G/H)\cdot C_n = (C_n)_{G/H}$ are in $\Sigma_{\mathcal{F}}$.

Remark 4.50. In light of Remark 4.48 any weak indexing system must contain the 1-corollas $(G/H)\cdot C_1 \simeq (C_1)_{G/H}$ for all $H \leq G$.

Remark 4.51. The notion of indexing system was first introduced in [3, Def. 3.22], though packaged quite differently. Moreover, a third definition of (weak) indexing systems as the sieves $\Omega_{\mathcal{F}} \hookrightarrow \Omega_G$ was presented by the second author in [24, §9]. The equivalence between the definitions in [3] and [24] is addressed in [24, Rmk. 9.7], hence here we address only the easier equivalence between Definition 4.49 and the sieve definition in [24, §9].

The existence of canonical maps $\mathsf{lr}(T) \to T$ shows that the sieve condition implies the lr condition in Definition 4.49. Conversely, as discussed immediately preceding [24, Def. 9.5], the sieve condition needs only be checked for inner faces and degeneracies, i.e. tall maps, and thus follows from Definition 4.49 since the subcategory $\Omega^1_{\mathcal{F}} \hookrightarrow \Omega^1_G$ of planar tall strings between $\mathcal{F}$-trees matches the pullback of $\Omega^0_{\mathcal{F}} \to \mathsf{F} \wr \Sigma_{\mathcal{F}} \leftarrow \mathsf{F} \wr \Omega^0_{\mathcal{F}}$.

The connection between weak indexing systems and $\mathbb{F}_G$ is given by the following, which generalizes Proposition 4.38.

Proposition 4.52. *Let $\mathcal{F}$ be a weak indexing system. Then:*

(i) the map $\gamma^*\mathbb{F}_G \xrightarrow{\eta_*} \gamma^*\mathbb{F}_G\gamma_*\gamma^*$ is an isomorphism, and thus (cf. Prop. 2.27) $\gamma^*\mathbb{F}_G\gamma_*$ is a monad;

(ii) the map $\gamma^*\mathbb{F}_G\gamma_! \xrightarrow{\beta} \gamma^*\mathbb{F}_G\gamma_*$ is an isomorphism of monads;

(iii) the map $\gamma_!\gamma^*\mathbb{F}_G\gamma_! \xrightarrow{\epsilon_!} \mathbb{F}_G\gamma_!$ is an isomorphism.

Proof. This follows just like the analogous parts of Proposition 4.38 by replacing $\mathsf{lr} : \Omega_G^{0,\mathrm{fr}} \to \Sigma_G^{\mathrm{fr}}$ with $\mathsf{lr} : \Omega_{\mathcal{F}}^0 \to \Sigma_{\mathcal{F}}$. For (i), note that if $C \in \Sigma_{\mathcal{F}}$ there is an identification between $C \downarrow_{\mathsf{r}} \Omega_G^0$ and $C \downarrow_{\mathsf{r}} \Omega_{\mathcal{F}}^0$, so that $\mathbb{F}_G X(C)$ only depends on the values of X on $\Sigma_{\mathcal{F}}$. (ii) is immediate. Lastly, (iii) follows since if $C \notin \Sigma_{\mathcal{F}}$ then any tree in $C \downarrow_{\mathsf{r}} \Omega_G^0$ must contain at least one G-vertex not in $\Sigma_{\mathcal{F}}$, so that indeed $\mathbb{F}_G \gamma_! Y(C) = \varnothing$. □

Notation 4.53. We write $\mathbb{F}_{\mathcal{F}} = \gamma^* \mathbb{F}_G \gamma_!$ for the induced monad on $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$, and $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ for the corresponding categories of algebras.

Corollary 4.54. *The adjunction (4.46) lifts to an adjunction on algebras*

$$\begin{tikzcd} \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \arrow[r, bend left, "\gamma_!"] \arrow[r, bend right, "\gamma_*"'] & \mathsf{Op}_G(\mathcal{V}) \arrow[l, "\gamma^*"'] \end{tikzcd} \tag{4.55}$$

Remark 4.56. Part (iii) of Proposition 4.52 states that if $\mathcal{F}$ is a weak indexing system then $\mathbb{F}_G$ essentially preserves the image of $\gamma_!$ (moreover, the converse is easily seen to also hold). As such, we will sometimes find it conceptually convenient to regard $\mathbb{F}_{\mathcal{F}}$ as "restricting $\mathbb{F}_G$".

Remark 4.57. The free corollas of §4.3 form a weak indexing system $\Sigma_G^{\mathrm{fr}} = \Sigma_{\mathcal{F}_{\mathrm{fr}}}$ and, moreover, there is an equivalence of categories $\mathsf{Op}^G \simeq \mathsf{Op}_{\mathcal{F}_{\mathrm{fr}}}$, so that Corollary 4.40 is a particular case of Corollary 4.54. However, while our discussion of Corollary 4.40 focuses on the (ι^*, ι_*)-adjunction, due to the fact that the intended model structures on $\mathsf{Op}^G(\mathcal{V})$ in Theorem I are defined via fixed point conditions, our discussion of Corollary 4.54 focuses on the $(\iota_!, \iota^*)$-adjunction, due to the model structures in Theorem II being projective.

Remark 4.58. In most cases, the rightmost (ι^*, ι_*)-adjunction appearing in Theorem III is induced by an inclusion $\iota : \Sigma_G^{\mathrm{fr}} \hookrightarrow \Sigma_{\mathcal{F}}$. However, it is possible for $\Sigma_G^{\mathrm{fr}} \not\subset \Sigma_{\mathcal{F}}$ (the most interesting case being that of $\Sigma_{\mathcal{F}} = \Sigma_G^{\geq 1}$ the corollas of arity ≥ 1, which model non-unital operads). In these cases (and compatibly with the $\Sigma_G^{\mathrm{fr}} \hookrightarrow \Sigma_{\mathcal{F}}$ case), we instead use the composite adjunction

$$\begin{tikzcd} \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \arrow[r, shift left, "\gamma_!"] & \mathsf{Op}_G(\mathcal{V}) \arrow[l, shift left, "\gamma^*"] \arrow[r, shift left, "\iota^*"] & \mathsf{Op}^G(\mathcal{V}) \arrow[l, shift left, "\iota_*"] \end{tikzcd} \tag{4.59}$$

Note that the right adjoint $\gamma^* \iota_*$ is still defined by computing fixed points while the left adjoint $\iota^* \gamma_!$ is still essentially a forgetful functor, with those levels not present in $\mathcal{F}$ declared to be $\varnothing$.

In practice, however, the use of the composite adjunction (4.59) is fairly benign, requiring only minor adjustments to the notation of the proofs in §6.4.

5 Free extensions and the existence of model structures

In order to prove all of our main theorems we will need to homotopically analyze free extensions of genuine equivariant operads, i.e. pushouts of the form

$$\begin{tikzcd} \mathbb{F}_G X \arrow[r] \arrow[d, "\mathbb{F}_G u"'] & \mathcal{P} \arrow[d] \\ \mathbb{F}_G Y \arrow[r] & \mathcal{P}[u] \end{tikzcd} \tag{5.1}$$

in the category $\mathsf{Op}_G(\mathcal{V})$. As is common in the literature (e.g. [28, 29, 1, 32, 23]), the key technical ingredient will be the identification of a suitable filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \mathcal{P}_\infty = \mathcal{P}[u] \tag{5.2}$$

of the map $\mathcal{P} \to \mathcal{P}[u]$ in the underlying category $\mathsf{Sym}_G(\mathcal{V})$. To explain how this filtration is obtained, and abbreviating $\mathbb{F}_G$ as $\mathbb{F}$, note first that $\mathcal{P}[u]$ is given by a coequalizer

$$\mathcal{P} \check{\amalg} \mathbb{F}X \check{\amalg} \mathbb{F}Y \underset{\longrightarrow}{\overset{\longrightarrow}{\dashleftarrow}} \mathcal{P} \check{\amalg} \mathbb{F}Y \tag{5.3}$$

where $\check{\amalg}$ denotes the algebraic coproduct, i.e. the coproduct in $\mathsf{Op}_G(\mathcal{V})$, and, a priori, the coequalizer is also calculated in $\mathsf{Op}_G(\mathcal{V})$. However, (5.3) is a so called *reflexive coequalizer*, meaning that the maps being coequalized have a common section, and standard arguments[5] show that it is hence also an underlying coequalizer in $\mathsf{Sym}_G(\mathcal{V})$.

In practice, we will need to enlarge (5.3) somewhat. Firstly, note that (5.3) corresponds to the two bottom levels of the bar construction $B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}Y) = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}Y$, whose colimit (over Δ^{op}) is again $\mathcal{P}[u]$. For technical reasons, we prefer the double bar construction

$$B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y) = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}X \check{\amalg} (\mathbb{F}X)^{\check{\amalg} l} \check{\amalg} \mathbb{F}Y = \mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} 2l+1} \check{\amalg} \mathbb{F}Y. \tag{5.4}$$

To actually describe the individual levels of (5.4) one further resolves $\mathcal{P}$ so as to obtain the bisimplicial object

$$B_l(\mathbb{F}^{n+1}\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y) = \mathbb{F}^{n+1}\mathcal{P} \check{\amalg} (\mathbb{F}X)^{\check{\amalg} 2l+1} \check{\amalg} \mathbb{F}Y \simeq \mathbb{F}\left(\mathbb{F}^n \mathcal{P} \amalg X^{\amalg 2l+1} \amalg Y\right), \tag{5.5}$$

where $\amalg$ denotes the coproduct in $\mathsf{Sym}_G(\mathcal{V})$. As in Remark 4.33, each level of (5.5) can then be described as

$$\mathsf{Lan} N(N^n \iota\mathcal{P} \amalg \iota X^{\amalg 2l+1} \amalg \iota Y) = \mathsf{Lan} N^{(\mathcal{P},X,Y)}_{n,l}, \tag{5.6}$$

for N the span monad (cf. Definition 4.16) and $\amalg$ now the coproduct of spans. In particular, each level of (5.5) is thus a left Kan extension over some category Ω_G^{n,λ_l}, which we explicitly identify in §5.1, giving the first identification below.

$$\mathcal{P} \check{\coprod}_{\mathbb{F}X} \mathbb{F}Y \simeq \operatorname{colim}_{(\Delta\times\Delta)^{op}} \left(\mathsf{Lan}_{\left(\Omega_G^{n,\lambda_l} \to \Sigma_G\right)^{op}} N^{(\mathcal{P},X,Y)}_{n,l} \right) \simeq \mathsf{Lan}_{\left(\Omega_G^{e} \to \Sigma_G\right)^{op}} \tilde{N}^{(\mathcal{P},X,Y)} \tag{5.7}$$

The second identification, which reduces the calculation to a single left Kan extension, is an instance of Proposition 5.37, a result whose proof is straightforward but lengthy, and thus postponed to the appendix. The category Ω_G^e of *extension trees* appearing on the right side is obtained as a categorical realization $\Omega_G^e = |\Omega_G^{n,\lambda_l}|$, which we explicitly describe and analyze in §5.2. In particular, we identify a smaller and more convenient subcategory $\widehat{\Omega}_G^e \hookrightarrow \Omega_G^e$ that is suitably initial, so that Ω_G^e can be replaced with $\widehat{\Omega}_G^e$ in (5.7).

The desired filtration (5.2) then follows from a filtration of the category $\widehat{\Omega}_G^e$ itself, and this discussion is the subject of §5.3.

Lastly, §5.4 concludes this section by using these filtrations to prove Theorems I and II.

5.1 Labeled planar strings

In this section, we explicitly identify the categories underlying the left Kan extensions in (5.6).

In the notation of Remark 2.31, letting $\langle\langle l \rangle\rangle = \{-\infty, -l, \cdots, -1, 0, 1, \cdots, l, \infty\}$ and writing λ_l for the partition $\lambda_{l,a} = \{-\infty\}$, $\lambda_{l,i} = \langle\langle l \rangle\rangle - \{-\infty\}$, (5.6) can be repackaged as an instance of the functor $\mathsf{Lan} \circ N \circ \coprod \circ (N^{\times \lambda_l})^{\circ n} \circ \iota^{\times \langle l \rangle}$. Our goal is thus to understand the underlying categories of the spans in the image of the functor $N \circ \coprod \circ (N^{\times\lambda_l})^{\circ n}$, though we will find it preferable and no harder to tackle the more general case of the functors $N^{s+1} \circ \coprod \circ (N^{\times\lambda})^{\circ n-s}$.

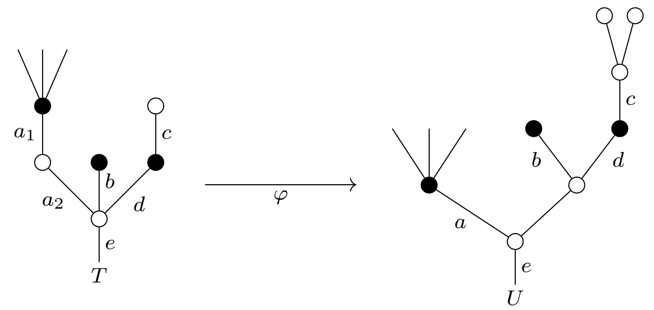
[5] For example, by the proof of [14, Prop. 3.27] it suffices to check that $\mathbb{F}_G$ preserves reflexive coequalizers. This follows from (4.1) and the fact that if $\otimes$ preserves colimits in each variable then it preserves reflexive coequalizers.

Definition 5.8. A *l-node labeled G-tree* (or just *l-labeled G-tree*) is a pair $(T, V_G(T) \to \{1,\cdots,l\})$ with $T \in \Omega_G$, which we think of as a G-tree together with G-vertices labels in $1,\cdots,l$.

Further, a tall map $\varphi: T \to S$ between l-labeled trees is called a *label map* if for each G-vertex v_{Ge} of T with label j, the vertices of the subtree $S_{v_{Ge}}$ are all labeled by j.

Lastly, given a subset $J \subset \underline{l}$, a planar label map $\varphi: T \to S$ is said to be J-inert if for every G-vertex v_{Ge} of T with label $j \in J$, we have $S_{v_{Ge}} = T_{v_{Ge}}$.

Example 5.9. Consider the 2-labeled trees below (for $G = *$ the trivial group), with black nodes ($\bullet$) denoting labels by the number 1 and white nodes ($\circ$) labels by the number 2. The planar map φ (sending $a_i \mapsto a$, $b \mapsto b$, $c \mapsto c$, $d \mapsto d$, $e \mapsto e$) is a label map which is $\{1\}$-inert.



Definition 5.10. Let $-1 \le s \le n$ and $\lambda = \lambda_a \amalg \lambda_i$ a partition of $\{1, 2, \cdots, l\}$.

We define $\Omega_G^{n,s,\lambda}$ to have as objects n-planar strings (where $T_{-1} = \mathsf{lr}(T_0)$ as in (3.80))

$$T_{-1} \xrightarrow{\varphi_0} T_0 \xrightarrow{\varphi_1} T_1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_s} T_s \xrightarrow{\varphi_{s+1}} T_{s+1} \xrightarrow{\varphi_{s+2}} \cdots \xrightarrow{\varphi_n} T_n \tag{5.11}$$

together with l-labelings of $T_s, T_{s+1}, \cdots, T_n$ such that the $\varphi_r, r > s$ are λ_i-inert label maps.

Arrows in $\Omega_G^{n,s,\lambda}$ are quotients of strings $(\pi_r: T_r \to T'_r)$ such that $\pi_r, r \ge s$ are label maps.

Further, for any $s < 0$ or $n < s'$ we write

$$\Omega_G^{n,s,\lambda} = \Omega_G^{n,-1,\lambda}, \qquad \Omega_G^{n,s',\lambda} = \Omega_G^{n}. \tag{5.12}$$

Intuitively, $\Omega_G^{n,s,\lambda}$ consists of strings that are labeled in the range $s \le r \le n$, with the extra cases (5.12) interpreted by infinitely prepending and postpending copies of T_{-1} and T_n to (5.11).

The main case of interest is that of $s = 0$, which we abbreviate as $\Omega_G^{n,\lambda} = \Omega_G^{n,0,\lambda}$, with the remaining $\Omega_G^{n,s,\lambda}$ playing an auxiliary role. The $s = -1$ case also deserves special attention.

Remark 5.13. For $s < 0$ there are identifications

$$\Omega_G^{n,s,\lambda} = \Omega_G^{n,-1,\lambda} \simeq \coprod_{\lambda_a} \Omega_G^n \amalg \coprod_{\lambda_i} \Sigma_G. \tag{5.14}$$

Indeed, since T_{-1} is a G-corolla, the label of its unique G-vertex determines all other labels.

Notation 5.15. We will write $(\Omega_G^n)^{\times\lambda}$ to denote the l-tuple with $(\Omega_G^n)_j^{\times\lambda} = \Omega_G^n$ if $j \in \lambda_a$ and $(\Omega_G^n)_j^{\times\lambda} = \Sigma_G$ if $j \in \lambda_i$. As such, (5.14) can be abbreviated as $\Omega_G^{n,-1,\lambda} = \coprod (\Omega_G^n)^{\times\lambda}$.

The $\Omega_G^{n,s,\lambda}$ categories are related by a number of obvious functors, which we now catalog.

Firstly, if $s \le s'$ there are forgetful functors

$$\Omega_G^{n,s,\lambda} \to \Omega_G^{n,s',\lambda} \tag{5.16}$$

and the simplicial operators in Notation 3.78 generalize to operators (for $0 \le i \le n$, $-1 \le j \le n$)

$$\begin{array}{llll} d_i: \Omega_G^{n,s,\lambda} \to \Omega_G^{n-1,s-1,\lambda} & i < s & s_j: \Omega_G^{n,s,\lambda} \to \Omega_G^{n+1,s+1,\lambda} & j < s \\ d_i: \Omega_G^{n,s,\lambda} \longrightarrow \Omega_G^{n-1,s,\lambda} & s \le i & s_j: \Omega_G^{n,s,\lambda} \longrightarrow \Omega_G^{n+1,s,\lambda} & s \le j \end{array} \tag{5.17}$$

which are compatible with the forgetful functors in the obvious way.

We will prefer to reorganize (5.16) and (5.17) somewhat. Defining functions $d_i:\mathbb{Z}\to\mathbb{Z}$ and $s_j:\mathbb{Z}\to\mathbb{Z}$ by

$$d_i(s)=\begin{cases}s-1, & i<s\\ s, & s\le i\end{cases}\qquad s_j(s)=\begin{cases}s+1, & j<s\\ s, & s\le j\end{cases}\tag{5.18}$$

(5.17) can be rewritten as maps $d_i:\Omega_G^{n,s,\lambda}\to\Omega_G^{n-1,d_i(s),\lambda}$ and $s_j:\Omega_G^{n,s,\lambda}\to\Omega_G^{n+1,s_j(s),\lambda}$. Therefore, we henceforth write simply $\Omega_G^{n,\bullet,\lambda}$ to denote the string of categories $\Omega_G^{n,s,\lambda}$ and forgetful functors, and abbreviate (5.17) as

$$d_i:\Omega_G^{n,\bullet,\lambda}\longrightarrow\Omega_G^{n-1,\bullet,\lambda}\qquad\qquad s_j:\Omega_G^{n,\bullet,\lambda}\longrightarrow\Omega_G^{n+1,\bullet,\lambda}$$

Remark 5.19. Considering the ordered sets $\langle n\rangle=\{0<1<\cdots<n<+\infty\}$, the formulas (5.18) define functions $d_i:\langle n\rangle\to\langle n-1\rangle$, $s_j:\langle n\rangle\to\langle n+1\rangle$ which preserve 0 and $+\infty$, except for s_{-1} which preserves only $+\infty$. This recovers the description of Δ^{op} as the category of intervals (i.e. ordered finite sets with a minimum and maximum and maps preserving them).

Next, the vertex functors V_G^k of (3.90) generalize to functors $V_G^k:\Omega_G^{n,s,\lambda}\to\mathsf{F}_s\wr\Omega_G^{n-k-1,s-k-1,\lambda}$ given by the same formula

$$(T_{k,v_{Ge}}\to\cdots\to T_{n,v_{Ge}})_{v_{Ge}\in V_G(T_k)},$$

as in (3.90), except with $T_{m,v_{Ge}}$ for $k\le m\le n$ inheriting the node labels from T_m (if any).

The diagrams in (3.91) for $i<k$ and $i>k$ now generalize to diagrams

$$\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, equal] \arrow[dl, Leftarrow, "\pi_i"] \\
\Omega_G^{n-1,\bullet,\lambda} \arrow[r, "V_G^{k-1}"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda}
\end{tikzcd}\qquad
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "d_i"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, "d_{i-k-1}"] \\
\Omega_G^{n-1,\bullet,\lambda} \arrow[r, "V_G^k"'] & \mathsf{F}_s\wr\Omega_G^{n-k-2,\bullet,\lambda}
\end{tikzcd}\tag{5.20}$$

while the diagrams in (3.92) for $j<k$ and $j>k$ generalize to diagrams

$$\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, equal] \\
\Omega_G^{n+1,\bullet,\lambda} \arrow[r, "V_G^{k+1}"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda}
\end{tikzcd}\qquad
\begin{tikzcd}
\Omega_G^{n,\bullet,\lambda} \arrow[r, "V_G^k"] \arrow[d, "s_j"'] & \mathsf{F}_s\wr\Omega_G^{n-k-1,\bullet,\lambda} \arrow[d, "s_{j-k-1}"] \\
\Omega_G^{n+1,\bullet,\lambda} \arrow[r, "V_G^k"'] & \mathsf{F}_s\wr\Omega_G^{n-k,\bullet,\lambda}
\end{tikzcd}\tag{5.21}$$

where we note that in all cases the s-index $\bullet$ varies according to (5.17).

Lastly, the $\Omega_G^{n,s,\lambda}$ are also functorial in λ. Explicitly, given $\alpha:\{1,\cdots,l\}\to\{1,\cdots,m\}$ and partitions such that $\lambda'\le\alpha^*\lambda$ one has forgetful functors

$$\Omega_G^{n,s,\lambda'}\to\Omega_G^{n,s,\lambda}\tag{5.22}$$

compatible with the forgetful functors (5.16), the simplicial operators d_i, s_j and the isomorphisms π_i.

Remark 5.23. When α is the identity and $\lambda'\le\lambda$ the forgetful functors in (5.22) are fully faithful inclusions. However, this is not the case for the forgetful functors in (5.16). Indeed, regarding the map $T\to U$ in Example 5.9 as an object in $\Omega_G^{1,0,\lambda}$ for $\lambda=\lambda_a\amalg\lambda_i=\{2\}\amalg\{1\}=\{\bullet\}\amalg\{\circ\}$, changing the label of $a_1\le a_2$ to a $\bullet$-label produces a non isomorphic object $\bar{T}\to U$ of $\Omega_G^{1,0,\lambda}$ that forgets to the same object of $\Omega_G^{1,1,\lambda}$.

We now extend Notation 4.6.

Notation 5.24. Let $(A_j) = (A_j \to \Sigma_G)_{1\le j\le l}$ be a l-tuple of categories over Σ_G. We define $\Omega_G^{n,s,\lambda} \wr (A_j)$ as the pullback

$$\begin{tikzcd} \Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[dd] & \mathsf{F} \wr \coprod A_j \arrow[d] \\ & \mathsf{F} \wr \coprod_l \Sigma_G \arrow[d] \\ \Omega_G^{n,s,\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \Omega_G^{-1,s-n-1,\lambda} \end{tikzcd} \tag{5.25}$$

Remark 5.26. To unpack (5.25), note first that by (5.12) $\Omega_G^{-1,r,\lambda}$ is simply either $\Sigma_G^{\amalg l}$ if $r < 0$ or Σ_G if $r \ge 0$, while $\Omega_G^{n,s,\lambda} = \coprod(\Omega_G^n)^{\times\lambda}$ if $s < 0$. We can thus break down (5.25) into the three cases $s < 0$, $0 \le s \le n$ and $n < s$, depicted below.

$$\begin{tikzcd} \Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\ \coprod(\Omega_G^n)^{\times\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \coprod_l \Sigma_G \end{tikzcd} \quad \begin{tikzcd} \Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\ \Omega_G^{n,s,\lambda} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \coprod_l \Sigma_G \end{tikzcd} \quad \begin{tikzcd} \Omega_G^{n,s,\lambda} \wr (A_j) \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\ \Omega_G^{n} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd} \tag{5.27}$$

Therefore, for $s > n$ (5.25) coincides with $\Omega_G^n \wr (\coprod_j A_j)$ as defined in Notation 4.6. Moreover, for $s < 0$ both squares in the diagram below are pullbacks and the bottom composite is V_G^n,

$$\begin{tikzcd} \coprod(\Omega_G^n)^{\times\lambda} \wr (A_j) \arrow[r, "\coprod(V_G^n)^{\times\lambda}"] \arrow[d] & \coprod \mathsf{F} \wr A_j \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod_j A_j \arrow[d] \\ \coprod(\Omega_G^n)^{\times\lambda} \arrow[r, "\coprod(V_G^n)^{\times\lambda}"'] & \coprod_l \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F} \wr \coprod_l \Sigma_G \end{tikzcd} \tag{5.28}$$

so that there is an identification $\Omega_G^{n,s,\lambda} \wr (A_j) \simeq \coprod(\Omega_G^n)^{\times\lambda} \wr (A_j)$, where in the right side $(-) \wr (-)$ is computed entry-wise.

Remark 5.29. The naturality of the $\Omega_G^{n,s,\lambda} \wr (A_j)$ constructions with regards to λ interacts with the tuple (A_j) in the obvious way, i.e., given $\alpha\colon \{1,\cdots,l\} \to \{1,\cdots,m\}$, $\lambda' \le \alpha^*\lambda$ and a map $(B_k) \to \alpha^*(A_j)$ one obtains a natural map

$$\Omega_G^{n,s,\lambda'} \wr (B_k) \to \Omega_G^{n,s,\lambda} \wr (A_j).$$

Proposition 5.30. *The analogue statements of Proposition 3.93 hold for the $\Omega_G^{n,s,\lambda}$ and the $\Omega_G^{n,s,\lambda} \wr (A_j)$ constructions, with the caveat that in the latter case we exclude the cases in Proposition 3.93(d)(e)(f) that involve d_n.*

Additionally, the natural squares (for $n \ge -1$)

$$\begin{tikzcd} \Omega_G^{n,n,\lambda} \arrow[r, "V_G^n"] \arrow[d] & \mathsf{F} \wr \coprod_l \Sigma_G \arrow[d] \\ \Omega_G^{n} \arrow[r, "V_G^n"'] & \mathsf{F} \wr \Sigma_G \end{tikzcd} \tag{5.31}$$

are also pullback squares.

Proof. Firstly, we note that the $\Omega_G^{n,s,\lambda}$ analogues, as well as the claim for (5.31), all follow from the previous results by keeping track of the labels on the strings, with the only non immediate part being the analogue of (d), stating that the right squares in (5.20) and (5.21) are pullbacks. Since in these diagrams the s-coordinate $\bullet$ is determined by the top left corner, a direct analysis shows that compatible choices of labels for strings on the top right and bottom left corners do assemble into the required labels on the top left corner, hence the result follows.

For the more general $\Omega^{n,s,\lambda}_G \wr (A_j)$ constructions, one can either build the general V^k_G, d_i, s_j, π_i explicitly, or mimic the argument in Proposition 4.12, reducing to the $\Omega^{n,s,\lambda}_G$ case. ☐

Corollary 5.32. *For $-1 \leq s \leq n$ there are natural identifications*

$$\Omega^k_G \wr \Omega^{n,s,\lambda}_G \wr (A_j) \simeq \Omega^{n+k+1,s+k+1,\lambda}_G \wr (A_j) \qquad \Omega^{n,s,\lambda}_G \wr (\Omega^k_G)^{\times\lambda} \wr (A_j) \simeq \Omega^{n+k+1,s,\lambda}_G \wr (A_j)$$

which identify $V^k_G \wr \Omega^{n,s,\lambda}_G \wr (A_j)$ with $V^k_G \wr (A_j)$ and $V^n_G \wr (\Omega^k_G)^{\times\lambda} \wr (A_j)$ with $V^n_G \wr (A_j)$.

Further, these identifications are compatible with each other and associative in the obvious ways, and they induce identifications

$$\begin{array}{lll} d_i \wr (\Omega^n_G)^{\times\lambda} \simeq d_i & \pi_i \wr (\Omega^n_G)^{\times\lambda} \simeq \pi_i & s_j \wr (\Omega^n_G)^{\times\lambda} \simeq s_j \\ \Omega^k_G \wr d_i \simeq d_{i+k+1} & \Omega^k_G \wr \pi_i \simeq \pi_{i+k+1} & \Omega^k_G \wr s_j \simeq s_{j+k+1} \end{array}$$

as well as obvious identifications of forgeful functors.

Proof. This is analogous to Corollary 4.15. For the first identification, the case $s \geq 0$ follows from the diagram below, where we note that the bottom arrow is $V^k_G \colon \Omega^k_G \to \mathsf{F} \wr \Sigma_G$.

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F}^{\wr 2} \wr \coprod(A_j) \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr \coprod(A_j) \arrow[d] \\
\Omega^{n+k+1,s+k+1,\lambda}_G \arrow[r, "V^k_G"'] \arrow[d, "d_{k+1,\cdots,n+k+1}"'] & \mathsf{F} \wr \Omega^{n,s,\lambda}_G \arrow[r, "\mathsf{F}\wr V^n_G"'] \arrow[d, "d_{0,\cdots,n}"] & \mathsf{F}^{\wr 2} \wr \coprod_l \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \coprod_l \Sigma_G \\
\Omega^{k,k+1,\lambda}_G \arrow[r, "V^k_G"'] & \mathsf{F} \wr \Omega^{-1,0,\lambda}_G & &
\end{tikzcd}$$

In the $s = -1$ case, the bottom arrow is instead $V^k_G \colon \Omega^{k,k,\lambda}_G \to \mathsf{F} \wr \Omega^{-1,-1,\lambda}_G = \mathsf{F} \wr \coprod_l \Sigma_G$, in which case one further attaches (5.31) to the diagram above.

The second identification is analogous, using the pullback diagram below, with the composite of the central horizontal arrows reinterpreted using (5.28).

$$\begin{tikzcd}
\bullet \arrow[r, dashed] \arrow[d, dashed] & \bullet \arrow[r, dashed] \arrow[d, dashed] & \mathsf{F} \wr \coprod \mathsf{F} \wr A_j \arrow[r] \arrow[d] & \mathsf{F}^{\wr 2} \wr \coprod A_j \arrow[r, "\sigma^0"] \arrow[d] & \mathsf{F} \wr \coprod A_j \arrow[d] \\
\Omega^{n+k+1,s,\lambda}_G \arrow[r, "V^n_G"'] \arrow[d, "d_{n+1,\cdots,n+k+1}"'] & \mathsf{F} \wr \coprod(\Omega^k_G)^{\times\lambda} \arrow[r, "\mathsf{F}\wr\coprod(V^n_G)^{\times\lambda}"'] \arrow[d, "d_{0,\cdots,k}"] & \mathsf{F} \wr \coprod_l \mathsf{F} \wr \Sigma_G \arrow[r] & \mathsf{F}^{\wr 2} \wr \coprod \Sigma_G \arrow[r, "\sigma^0"'] & \mathsf{F} \wr \coprod_l \Sigma_G \\
\Omega^{n,s,\lambda}_G \arrow[r, "V^n_G"'] & \mathsf{F} \wr \coprod_l \Sigma_G & & &
\end{tikzcd}$$

The additional claims are straightforward. ☐

Remark 5.33. The identifications in Corollary 5.32 do allow for the case $n = -1$, which is non-trivial due to the existence of $\Omega^{-1,-1,\lambda}_G = \coprod_l \Sigma_G$, in which case $\Omega^{-1,-1,\lambda}_G \wr (A_j) \simeq \coprod A_j$. For $-1 \leq s \leq n$ the identifications

$$\Omega^{n,s,\lambda}_G = \Omega^s_G \wr \Omega^{-1,-1}_G \wr (\Omega^{n-s-1}_G)^{\times\lambda}$$

then show that $\Omega^{n,s,\lambda}_G \wr (-)$ encodes (the underlying category of) the functor $N^{\circ s+1} \coprod (N^{\times\lambda})^{\circ n-s}$.

Furthermore, the left commutative square below, where vertical arrows are forgetful functors, the bottom square is one of the pullback squares (5.31), and the right diagram merely unpacks notation

$$\begin{tikzcd}
\Omega_G^{0,-1,\lambda} \arrow[r, "\coprod(V_G^0)^{\times\lambda}"] \arrow[d] & \coprod \mathsf{F} \wr (\Omega_G^{-1})^{\times\lambda} \arrow[r] & \mathsf{F} \wr \Omega_G^{-1,-2,\lambda} \arrow[d, equal] \\
\Omega_G^{0,0,\lambda} \arrow[rr, "V_G^0"] \arrow[d] & & \mathsf{F} \wr \Omega_G^{-1,-1,\lambda} \arrow[d] \\
\Omega_G^{0,1,\lambda} \arrow[rr, "V_G^0"] & & \mathsf{F} \wr \Omega_G^{-1,0,\lambda}
\end{tikzcd}
\qquad
\begin{tikzcd}
\coprod(\Omega_G^0)^{\times\lambda} \arrow[r] \arrow[d] & \coprod \mathsf{F} \wr \Sigma_G \arrow[d] \\
\Omega_G^{0,0,\lambda} \arrow[r] \arrow[d] & \mathsf{F} \wr \coprod \Sigma_G \arrow[d] \\
\Omega_G^0 \arrow[r] & \mathsf{F} \wr \Sigma_G
\end{tikzcd} \tag{5.34}$$

shows that the forgetful functor $\Omega_G^{0,-1,\lambda} \wr (A_j) \to \Omega_G^{0,0,\lambda} \wr (A_j)$ encodes the natural map $\coprod \circ N \Rightarrow N \circ \coprod$ of (2.30).

5.2 The category of extension trees

The purpose of this section is to make (5.7) explicit. We start by discussing realizations of simplicial objects in Cat.

Recalling the standard cosimplicial object $[\bullet] \in \mathsf{Cat}^{\Delta}$ given by $[n] = (0 \to 1 \to \cdots \to n)$ yields the following definition.

Definition 5.35. The left adjoint below is called the *realization* functor.

$$|-|\colon \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat}\colon (-)^{[\bullet]}$$

Remark 5.36. Suppose that $\mathcal{C} \in \mathsf{Cat}$ contains subcategories $\mathcal{C}_h$, $\mathcal{C}^v$ whose arrows span those of $\mathcal{C}$. Defining $\mathcal{C}^v_{h,\bullet} \in \mathsf{Cat}^{\Delta^{op}}$ so that the objects of $\mathcal{C}^v_{h,n}$ are n-strings in $\mathcal{C}_h$ and the arrows are compatible n-tuples of arrows in $\mathcal{C}^v$, it is straightforward to show that it is $|\mathcal{C}^v_{h,\bullet}| = \mathcal{C}$.

An immediate example is given by the planar strings in Definition 3.75. Writing $\mathcal{C} = \Omega_G^{\mathsf{t}}$ the category of tall maps, $\mathcal{C}_h = \Omega_G^{\mathsf{pt}}$ the category of planar tall maps and $\mathcal{C}^v = \Omega_G^0$ the category of quotients, one has $\mathcal{C}^v_{h,\bullet} = \Omega_G^{\bullet}$ and thus $|\Omega_G^{\bullet}| = \Omega_G^{\mathsf{t}}$.

Similarly, noting that the $\Omega_G^{n,\lambda} = \Omega_G^{n,0,\lambda}$ categories of §5.1 form a simplicial object, we have that the $|\Omega_G^{\bullet,\lambda}| = \Omega_G^{\mathsf{t},\lambda}$ is the category of tall label maps between l-labeled trees that induce quotients on nodes with λ-inert labels.

In the following statement, whose proof is delayed to the appendix, we note that it is shown in Lemma A.3 that $\mathrm{Ob}(|A_\bullet|) \simeq \mathrm{Ob}(A_0)$ and that arrows in $|A_\bullet|$ are generated by the arrows in A_0 together with arrows $d_1(a) \to d_0(a)$ for each $a \in A_1$.

Proposition 5.37. *Given a simplicial object $\Sigma_G \leftarrow A_\bullet \xrightarrow{N_\bullet} \mathcal{V}^{op}$ in $\mathsf{WSpan}^r(\Sigma_G, \mathcal{V}^{op})$ such that the natural transformation components of the differential operators d_i, $0 \le i < n$ and s_j, $0 \le j \le n$ are isomorphisms, there is an identification*

$$\lim_{\Delta} (\mathsf{Ran}_{A_n \to \Sigma_G} N_n) \simeq \mathsf{Ran}_{|A_\bullet| \to \Sigma_G} \tilde{N}$$

where $\tilde{N}\colon |A_\bullet| \to \mathcal{V}^{op}$ is given by N_0 on objects and generating arrows in A_0, and on generating arrows $d_1(a) \to d_0(a)$ for $a \in A_1$ as the composite

$$\begin{tikzcd}
A_0 \arrow[dr] & A_1 \arrow[l, "d_1"'] \arrow[r, "d_0"] \arrow[d] \arrow[dl, phantom, "\Rightarrow"] \arrow[dr, phantom, "\Leftarrow"] & A_0 \arrow[dl] \\
& \mathcal{V}^{op} &
\end{tikzcd}$$

Proposition 5.37 applies to both simplicial directions of the bisimplicial object

$$N^{(\mathcal{P},X,Y)}_{n,l} = N(N^{\circ n} \iota\mathcal{P} \amalg \iota X^{\amalg 2l+1} \amalg \iota Y)$$

in (5.6), whose underlying categories are Ω_G^{n,λ_l} for λ_l the partitions described at the beginning of §5.1. Indeed, in the n direction all d_i with $0 < i < n$ are induced by the multiplication $NN \to N$ defined in (4.17) while d_0 is induced by the composite $N \circ \coprod \circ N \to NN \circ \coprod \to N \circ \coprod$, with the second map again given by composition and the first induced by the natural map $\coprod \circ N \to N \circ \coprod$, which is encoded by a strictly commutative diagram of spans, as seen using the top part of (5.34) (or, more abstractly, it also suffices to note that N preserves arrows in $\mathsf{WSpan}^l(\Sigma_G^{op}, \mathcal{V})$ given by strictly commutative diagrams). Degeneracies are similar. Moreover, that the functor component of d_n matches the functor defined in (5.17) follows from the presence of the ι in (5.6).

As for the l direction, we note that our convention on the double bar construction $B_l(\mathcal{P}, \mathbb{F}X, \mathbb{F}X, \mathbb{F}X, \mathbb{F}Y)$, is symmetric, with d_l given by combining the maps $\mathbb{F}X \to \mathbb{F}Y$ and $\mathbb{F}X \to \mathcal{P}$ and the remaining differentials given by fold maps. Or, more precisely, the action of the differential operators on the sets of labels $\langle\langle l \rangle\rangle = \{-\infty, -l, \cdots -1, 0, 1, \cdots, l, +\infty\}$ is given by extending the functions in Remark 5.19 anti-symmetrically. But then the differential operators d_i, s_j for $0 \le i < l$ and $0 \le j \le l$ correspond to instances of the naturality in Remark 5.29 when $(B_k) = \alpha^*(A_j)$, and are hence given by strictly commutative maps of spans.

Our next task is thus that of identifying the category of extension trees Ω_G^e appearing in (5.7), i.e. to produce an explicit model for the double realization $|\Omega_G^{n,\lambda_l}|$. By Remark 5.36 we can first perform the realization in the n direction, so as to obtain $|\Omega_G^{n,\lambda_l}| = |\Omega_G^{\mathrm{t},\lambda_l}|$, where we recall that $\Omega_G^{\mathrm{t},\lambda_l}$ consists of $\langle\langle l \rangle\rangle$-labeled trees together with tall maps that induce quotients on all nodes not labeled by $-\infty$.

We now identify Ω_G^e directly.

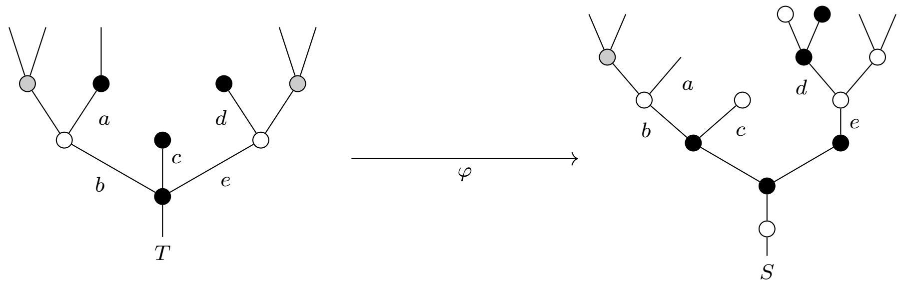
Definition 5.38. The *extension tree category* Ω_G^e has as objects $\{\mathcal{P}, X, Y\}$-labeled trees and as arrows tall maps $\varphi: T \to S$ such that:

(i) if $T_{v_{Ge}}$ has a X-label, then $S_{v_{Ge}} \in \Sigma_G$ and $S_{v_{Ge}}$ has a X-label;

(ii) if $T_{v_{Ge}}$ has a Y-label, then $S_{v_{Ge}} \in \Sigma_G$ and $S_{v_{Ge}}$ has either a X-label or a Y-label;

(iii) if $T_{v_{Ge}}$ has a $\mathcal{P}$-label, then $S_{v_{Ge}}$ has only X and $\mathcal{P}$-labels.

Example 5.39. The following is an example of a planar map in Ω_G^e for $G = *$, where black nodes represent $\mathcal{P}$-labeled nodes, grey nodes represent Y-labeled nodes and white nodes represent X-labeled nodes.



Remark 5.40. By changing any X-labels in $S_{v_{Ge}}$ into Y-labels (resp. $\mathcal{P}$-labels) whenever T_{v_G} has a Y-label (resp. $\mathcal{P}$-label), one obtains a factorization

$$T \to \bar{S} \to S$$

such that $T \to \bar{S}$ is a label map (cf. Definition 5.8) and $\bar{S} \to S$ is an underlying identity of trees that merely changes some of the Y and $\mathcal{P}$ labels into X-labels. We refer to the latter kind of map as a *relabel map*. It is clear that the label-relabel factorization is unique.

Proposition 5.41. *There is an identification* $\Omega_G^e \simeq |\Omega_G^{\mathrm{t},\lambda_l}|$.

Proof. We will show that Remark 5.36 applies to $\mathcal{C} = \Omega^e_G$, with $\mathcal{C}_h$ and $\mathcal{C}^v$ the categories of relabel and label maps. More precisely, we claim that there is an isomorphism $\mathcal{C}^v_{h,\bullet} \simeq \Omega^{\mathrm{t},\lambda_\bullet}_G$ of objects in $\mathsf{Cat}^{\Delta^{op}}$. Unpacking notation, one must first show that strings

$$T_0 \to T_1 \to \cdots \to T_l \tag{5.42}$$

of relabel arrows in Ω^e_G are in bijection with objects of $\Omega^{\mathrm{t},\lambda_l}_G$, i.e., with trees labeled by $\langle\langle l \rangle\rangle = \{-\infty, -l, \cdots, -1, 0, 1, \cdots, l, +\infty\}$. Noting that the maps in (5.42) are simply underlying identities on some fixed tree T that convert some of the $\mathcal{P}$, Y labels into X labels, we label a vertex $T_{v_{Ge}}$ by: (i) j such that $0 < j \leq +\infty$ if the last j labels of $T_{v_{Ge}}$ in (5.42) are Y labels (where $+\infty = l+1$); (ii) $-j$ such that $-\infty \leq -j < 0$ if the last j labels of $T_{v_{Ge}}$ in (5.42) are $\mathcal{P}$ labels; (iii) $j = 0$ if all labels in (5.42) are X-labels. This process clearly establishes the desired bijection on objects.

The compatibilities with arrows and with the simplicial structure are straightforward. $\square$

Letting $\tilde{N}^{(\mathcal{P},X,Y)}$ be built from $N^{(\mathcal{P},X,Y)}_{\bullet,\bullet}$ via a double application of Proposition 5.37 thus yields the following, establishing (5.7).

Corollary 5.43. $\mathcal{P} \coprod_{\mathbb{F}X} \mathbb{F}Y \simeq \mathrm{Lan}_{(\Omega^e_G \to \Sigma_G)^{op}} \tilde{N}^{(\mathcal{P},X,Y)}$.

Our next task is that of identifying a convenient initial subcategory $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$. We first introduce the auxiliary notion of alternating trees. Recall the notion of input path (Notation 3.4) $I(e) = \{f \in T : e \leq_d f\}$ for an edge $e \in T$, which naturally extends to T in any of $\Omega, \Phi, \Omega_G, \Phi_G$.

Definition 5.44. A G-tree $T \in \Omega_G$ is called *alternating* if, for all leafs $l \in T$ one has that the input path $I(l)$ has an even number of elements.

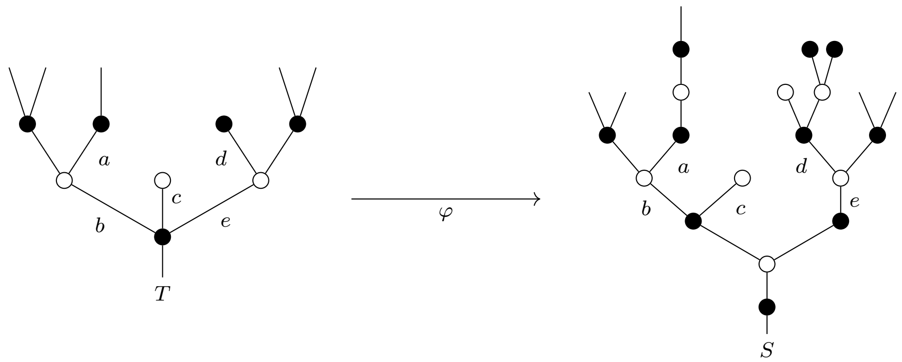
Further, a vertex $e^\uparrow \leq e$ is called *active* if $|I(e)|$ is odd and *inert* otherwise.

Finally, a tall map $T \xrightarrow{\varphi} S$ between alternating G-trees is called a *tall alternating map* if for any inert vertex $e^\uparrow \leq e$ of T one has that $S_{e^\uparrow \leq e}$ is an inert vertex of S.

We will denote the category of alternating G-trees and tall alternating maps by Ω^a_G.

Remark 5.45. A G-tree (resp. map) is alternating iff each component is.

Example 5.46. Two alternating trees (for $G = *$ the trivial group) and a planar tall alternating map between them follow, with active nodes in black ($\bullet$) and white nodes in white ($\circ$).



The term "alternating" reflects the fact that adjacent nodes have different colors, though there is an additional restriction: the "outer vertices", i.e. those immediately below a leaf or above the root, are necessarily black/active (this does not, however, apply to stumps).

Remark 5.47. If $T \in \Omega$ is alternating, it follows from Remark 3.47 that a tall map $\varphi: T \to U$ is an alternating map iff the corresponding substitution datum under Proposition 3.42 is given by the identity $U_{e^\uparrow \leq e} = T_{e^\uparrow \leq e}$ when $e^\uparrow \leq e$ is inert and by an alternating tree $U_{e^\uparrow \leq e}$ when $e^\uparrow \leq e$ is active.

Definition 5.48. $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$ is the full subcategory of $(\mathcal{P}, X, Y)$-labeled trees whose underlying tree is alternating, active nodes are labeled by $\mathcal{P}$, and inert nodes are labeled by X or Y.

Note that conditions (i) and (ii) in Definition 5.38 imply that for any map in $\widehat{\Omega}^e_G$ the underlying map is an alternating map.

The following is the key to establishing the desired initiality of $\widehat{\Omega}^e_G$ in Ω^e_G.

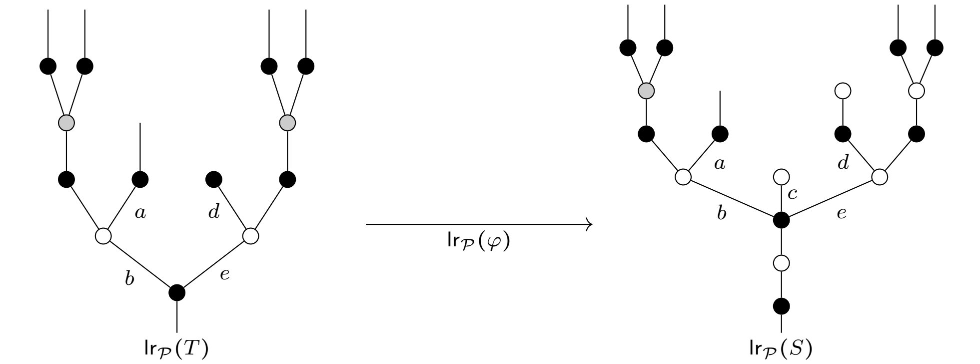
Proposition 5.49. *For each $U \in \Omega^e_G$ there exists a unique $\mathsf{lr}_{\mathcal{P}}(U) \in \widehat{\Omega}^e_G$ together with a unique planar label map in Ω^e_G*

$$\mathsf{lr}_{\mathcal{P}}(U) \to U. \tag{5.50}$$

Furthermore, $\mathsf{lr}_{\mathcal{P}}$ extends to a right retraction $\mathsf{lr}_{\mathcal{P}}\colon \Omega^e_G \to \widehat{\Omega}^e_G$.

Formally, the map (5.50) in Proposition 5.49 will be built using Proposition 3.48(iii), which loosely says that planar tall maps $T \to U$ are determined by certain collections $\{U_i\}$ of outer faces of U, with T obtained by replacing U_i with $\mathsf{lr}(U_i)$ (for the pictorial intuition, see Example 3.35). For the sake of intuition, we first present an example.

Example 5.51. The following illustrates the $\mathsf{lr}_{\mathcal{P}}$ construction applied to the map φ in Example 5.39. Intuitively, for each of the maximal $\mathcal{P}$-labeled outer subtrees $T^{\mathcal{P}}_k, S^{\mathcal{P}}_k$ of T, S, the functor $\mathsf{lr}_{\mathcal{P}}$ replaces $T^{\mathcal{P}}_k, S^{\mathcal{P}}_k$ with the corresponding leaf-root $\mathsf{lr}(T^{\mathcal{P}}_k), \mathsf{lr}(S^{\mathcal{P}}_k)$, which is again $\mathcal{P}$-labeled. Pictorially, this results in the following two effects: when $T^{\mathcal{P}}_k, S^{\mathcal{P}}_k$ are single edge subtrees of T, S (necessarily not adjacent to a $\mathcal{P}$-vertex) one degenerates that edge, adding a new $\mathcal{P}$-vertex of degree 1; when $T^{\mathcal{P}}_k, S^{\mathcal{P}}_k$ have vertices, so that they are subtrees composed of adjacent $\mathcal{P}$-vertices of T, S, those vertices are collapsed into a single $\mathcal{P}$-vertex.



Proof of Proposition 5.49. We first address the non-equivariant case $U \in \Omega^e$.

To build $\mathsf{lr}_{\mathcal{P}}(U)$, consider the collection of outer faces $\{U^X_i\} \amalg \{U^Y_j\} \amalg \{U^{\mathcal{P}}_k\}$ where the U^X_i, U^Y_j are simply the X, Y-labeled nodes, and the $\{U^{\mathcal{P}}_k\}$ are the maximal outer subtrees whose nodes have only $\mathcal{P}$-labels (these may possibly be sticks). Lemma 3.50 guarantees that each edge and each $\mathcal{P}$-labeled node belong to exactly one of the $U^{\mathcal{P}}_k$, and applying Proposition 3.48(iii) yields a planar tall map

$$T = \mathsf{lr}_{\mathcal{P}}(U) \to U \tag{5.52}$$

such that $\{U_{e^{\uparrow} \leq e}\}_{(e^{\uparrow} \leq e) \in V(T)} = \{U^X_i\} \amalg \{U^Y_j\} \amalg \{U^{\mathcal{P}}_k\}$. T has an obvious $(\mathcal{P}, X, Y)$-labeling making (5.52) into a label map, but we must still check $T \in \widehat{\Omega}^e_G$, i.e. that T is alternating with active vertices precisely those labeled by $\mathcal{P}$. But since the image of each $e \in T$ belongs to precisely one $U^{\mathcal{P}}_k$, e belongs to precisely one of the $\mathcal{P}$-labeled nodes of T, so that any leaf input path $I(l) = (l = e_n \leq e_{n-1} \leq \cdots \leq e_1 \leq e_0)$ must start with, end with, and alternate between $\mathcal{P}$-nodes, and thus have even length.

To check uniqueness, note that for any other planar label map $S \to U$ with $S \in \widehat{\Omega}^e_G$ and $e^{\uparrow} \leq e$ a $\mathcal{P}$ vertex of S the outer face $U_{e^{\uparrow} \leq e}$ must be a maximal $\mathcal{P}$-labeled outer face since the vertices adjacent to its root and leaves are labeled by either X or Y. The condition $V(U) = \coprod_{V(S)} V(U_{e^\uparrow \le e})$ thus guarantees that the collection of outer faces determined by S matches that determined by T except perhaps in the number of stick faces, so that the degeneracy-face factorizations $S \to F \to U$, $T \to F \to U$ factor through the same planar inner face F, with the unique labeling that makes the inclusion a label map. S, T are thus both trees in $\widehat{\Omega}^e_G$ obtained from F by adding degenerate $\mathcal{P}$ vertices, and since this can be done in at most one way, we conclude $S = T$.

To check functoriality, consider the diagram below, where $T \to U$ is the map defined above and $\varphi\colon U \to V$ any map in Ω^e_G.

$$\begin{tikzcd} T \arrow[r] \arrow[d, dashed] & U \arrow[d, "\varphi"] \\ S \arrow[r, dashed] & V \end{tikzcd} \tag{5.53}$$

The composite $T \to V$ is encoded by a substitution datum $\{T_{e^\uparrow \le e} \to V_{e^\uparrow \le e}\}$ which is given by an isomorphism if $e^\uparrow \le e$ has label X or Y (possibly changing a Y label to a X label), and by some $(X, \mathcal{P})$-labeled tree $V_{e^\uparrow \le e}$ if $e^\uparrow \le e$ has a $\mathcal{P}$-label. We now consider the factorization problem in (5.53), where we want $S \in \widehat{\Omega}^e_G$ and for the map $S \to V$ to the a planar label map. Combining Remark 5.47 with the uniqueness of the $\mathsf{lr}_{\mathcal{P}}(V_{e^\uparrow \le e})$, the only possibility is for S to be defined using the T substitution datum that replaces $T_{e^\uparrow \le e} \to V_{e^\uparrow \le e}$ with $T_{e^\uparrow \le e} \to \mathsf{lr}_{\mathcal{P}}(V_{e^\uparrow \le e})$ whenever $e^\uparrow \le e$ has a $\mathcal{P}$-label. Uniqueness of $\mathsf{lr}_{\mathcal{P}}(V)$ then implies $S = \mathsf{lr}_{\mathcal{P}}(V)$, and one sets $\mathsf{lr}_{\mathcal{P}}(\varphi)$ to be the map $T \to S$. Associativity and unitality are automatic from the uniqueness of the factorization of (5.53).

For $T = (T_x)_{x \in X}$ in Ω^e_G with G a general group, one sets $\mathsf{lr}_{\mathcal{P}}(T) = (\mathsf{lr}_{\mathcal{P}}(T_x))_{x \in X}$. □

Corollary 5.54. *The inclusion $\widehat{\Omega}^e_G \hookrightarrow \Omega^e_G$ is* Ran*-initial over Σ_G. In other words, for $\mathcal{C}$ any complete category and functor $N\colon \Omega^e_G \to \mathcal{C}$ it is*

$$\mathsf{Ran}_{\Omega^e_G \to \Sigma_G} N \simeq \mathsf{Ran}_{\widehat{\Omega}^e_G \to \Sigma_G} N.$$

Proof. Since $\mathsf{lr}_{\mathcal{P}}$ is a right retraction over Σ_G, the undercategories $C \downarrow \widehat{\Omega}^e_G$ are right retractions of $C \downarrow \Omega^e_G$ for any $C \in \Sigma_G$. □

5.3 Filtrations of free extensions

Summarizing §5.2, the discussion after Proposition 5.37 establishes (5.7), and hence Corollary 5.54 gives the alternative formula (the use of opposite categories turns Ran into Lan)

$$\mathcal{P}[u] \simeq \mathcal{P} \check{\coprod}_{\mathbb{F}X} \mathbb{F}Y \simeq \mathsf{Lan}_{\left(\widehat{\Omega}^e_G \to \Sigma_G\right)^{op}} \tilde{N}^{(\mathcal{P},X,Y)}, \tag{5.55}$$

which we will now use to filter the map $\mathcal{P} \to \mathcal{P}[u]$ in the underlying category $\mathsf{Sym}_G(\mathcal{V})$.

First, given $T = (T_i)_{i \in I} \in \Omega^e_G$, we write $V^X(T_i)$ (resp. $V^Y(T_i)$) to denote the set of X-labeled (Y-labeled) vertices of T_i. We define *degrees* of T by

$$|T|_X = |V^X(T_i)|, \qquad |T|_Y = |V^Y(T_i)|, \qquad |T| = |T|_X + |T|_Y,$$

which we note do not depend on the choice of $i \in I$.

Similarly, for $T = (T_i)_{i \in I} \in \Omega^a_G$ we write $V^{in}(T_i)$ for the inert vertices and $|T| = |V^{in}(T_i)|$.

Remark 5.56. One key property of the degrees $|T|$, $|T|_X$, $|T|_Y$ is that they are invariant under root pullbacks, which are defined by generalizing Definition 3.23 in the obvious way.

Definition 5.57. We specify some rooted (i.e. closed under root pullbacks) full subcategories of $\widehat{\Omega}^e_G$:

(i) $\widehat{\Omega}^e_G[\le k]$ (resp. $\widehat{\Omega}^e_G[k]$) is the subcategory of T with $|T| \le k$ ($|T| = k$);

(ii) $\widehat{\Omega}^e_G[\le k \backslash Y]$ (resp. $\widehat{\Omega}^e_G[k \backslash Y]$) is the subcategory of $\widehat{\Omega}^e_G[\le k]$ ($\widehat{\Omega}^e_G[k]$) of T with $|T|_Y \ne k$.

Similarly, we define subcategories $\Omega_G^a[\leq k]$, $\Omega_G^a[k]$ of Ω_G^a by the conditions $|T| \leq k$, $|T| = k$.

Remark 5.58. The categories $\widehat{\Omega}_G^e[k]$, $\widehat{\Omega}_G^e[k \backslash Y]$ and $\Omega_G^a[k]$ have rather limited morphisms.

Indeed, it is clear from Definitions 5.38 and 5.44 that maps never lower degree, and Remark 5.47 further ensures that degree is preserved iff $\mathcal{P}$-vertices are substituted by $\mathcal{P}$-vertices (rather than larger trees which would necessarily have inert vertices, thus increasing degree). Therefore, all maps in $\Omega_G^a[k]$ are quotients while maps in $\widehat{\Omega}_G^e[k]$, $\widehat{\Omega}_G^e[k \setminus Y]$ are underlying quotients of G-trees that relabel some Y-vertices to X-vertices. Moreover, this can be repackaged as saying that the diagonal forgetful functors in

$$\begin{tikzcd}
\widehat{\Omega}_G^e[k \setminus Y] \arrow[rr, hook] \arrow[dr] & & \widehat{\Omega}_G^e[k] \arrow[dl] \\
& \Omega_G^a[k] &
\end{tikzcd}$$

are Grothendieck fibrations whose fibers over $T \in \Omega_G^a[k]$ are the punctured cube and cube categories

$$(Y \to X)^{\times V_G^{in}(T)} - Y^{\times V_G^{in}(T)}, \qquad (Y \to X)^{\times V_G^{in}(T)}$$

for $V_G^{in}(T)$ the set of inert G-vertices.

Note that though $|V^{in}(T_i)| = k$ for each of the T_i that constitute $T = (T_i)_{i \in I}$, one can only guarantee $|V_G^{in}(T)| \leq k$.

Lemma 5.59. *The horizontal inclusion below*

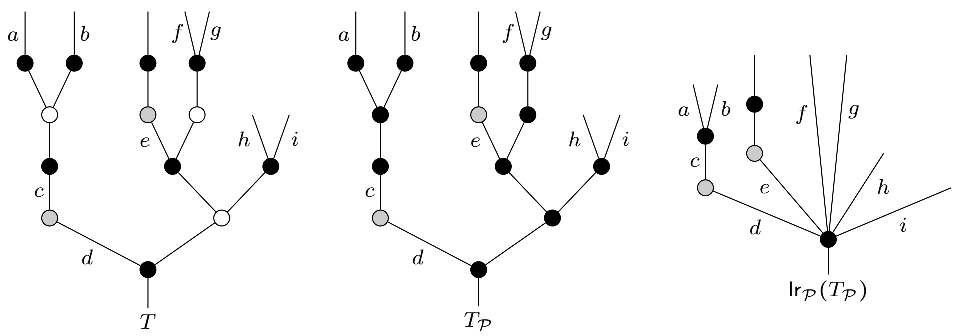
$$\begin{tikzcd}
\widehat{\Omega}_G^e[\leq k-1] \arrow[rr, hook] \arrow[dr, "\mathsf{lr}"'] & & \widehat{\Omega}_G^e[\leq k \setminus Y] \arrow[dl, "\mathsf{lr}"] \\
& \Sigma_G &
\end{tikzcd}$$

is Ran*-initial (in the sense of Corollary 5.54) over* Σ_G.

The proof will make use of an additional construction on Ω_G^e: given $T \in \Omega_G^e$ let $T_{\mathcal{P}}$ denote the result of replacing all X-labeled nodes of T with $\mathcal{P}$-labeled nodes.

Remark 5.60. In contrast to the functor $\mathsf{lr}_{\mathcal{P}}\colon \Omega_G^e \to \widehat{\Omega}_G^e$ of Proposition 5.49, the $(-)_{\mathcal{P}}$ construction does not define a full functor $\Omega_G^e \to \Omega_G^e$, instead only being functorial, and the obvious maps $T_{\mathcal{P}} \to T$ only being natural, with respect to the maps of Ω_G^e that preserve Y-labels.

Example 5.61. Combining the $(-)_{\mathcal{P}}$ and $\mathsf{lr}_{\mathcal{P}}$ constructions one obtains a construction sending trees in $\widehat{\Omega}_G^e$ to trees in $\widehat{\Omega}_G^e$. We illustrate this for the tree $T \in \widehat{\Omega}^e$ below (so that $G = *$), where black nodes are $\mathcal{P}$-labeled, white nodes are X-labeled, and grey nodes are Y-labeled.



Proof of Lemma 5.59. By Proposition 2.5 it suffices to show that for each $C \in \Sigma_G$ the map of rooted undercategories

$$C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\leq k-1] \to C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\leq k \setminus Y]$$

is initial, i.e. (cf. [20, IX.3]) that for each $(S, \pi: C \to \mathsf{lr}(S))$ in $C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\leq k \setminus Y]$ the overcategory

$$(C \downarrow_{\mathsf{r}} \widehat{\Omega}^e_G[\leq k-1]) \downarrow (S, \pi) \tag{5.62}$$

is non-empty and connected. By definition of rooted undercategory, π is the identity on roots and thus an isomorphism on Σ_G, so that objects of (5.62) correspond to maps $T \to S$ that induce a rooted isomorphism on lr, i.e. rooted tall maps.

The case $S \in \widehat{\Omega}^e_G[\leq k-1]$ is immediate, since then the identity $S = S$ is terminal in (5.62). Otherwise, since $|S|_Y \neq k$ we have $|\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}})| < k$ and the map $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$, which is a rooted tall, shows that (5.62) is indeed non-empty.

Now, consider a rooted tall map $T \to S$ with $T \in \widehat{\Omega}^e_G[\leq k-1]$. One can form a diagram

$$\begin{tikzcd} & S & \mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \arrow[l] \\ T \arrow[ur] \arrow[r] & T' \arrow[u, "Y\text{-pres}"'] & \mathsf{lr}_{\mathcal{P}}(T'_{\mathcal{P}}) \arrow[l] \arrow[u] \end{tikzcd} \tag{5.63}$$

where $T \to T' \to S$ is the natural factorization such that $T' \to S$ preserves Y-labels, i.e., T' is obtained from T by simply relabeling to X those Y-labeled vertices of T that become X-vertices in S. Note that by Remark 5.60, the existence of the right square relies on $T' \to S$ preserving Y-labels. Since all maps in (5.63) are rooted tall, this produces the necessary zigzag connecting the objects $T \to S$ and $\mathsf{lr}_{\mathcal{P}}(S_{\mathcal{P}}) \to S$ in the category (5.62), finishing the proof. ☐

In what follows we write $\tilde{N}: \widehat{\Omega}^{e,op}_G \to \mathcal{V}$ for the functor in (5.55) and any of its restrictions.

We are now in a position to produce the filtration (5.2) of the map $\mathcal{P} \to \mathcal{P}[u]$ in (5.1).

Definition 5.64. Let $\mathcal{P}_k$ denote the left Kan extension

$$\begin{tikzcd} \widehat{\Omega}^e_G[\leq k]^{op} \arrow[r, "\tilde{N}"] \arrow[d, "\mathsf{lr}"'] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{V} \\ \Sigma^{op}_G \arrow[ur, "\mathcal{P}_k"'] & \end{tikzcd}$$

Noting that $\widehat{\Omega}^e_G[\leq 0] \simeq \Sigma_G$ (since $|T| = 0$ only if T is a G-corolla with $\mathcal{P}$-labeled vertex) and that $\widehat{\Omega}^e_G$ is the union of (the nerves of) the $\widehat{\Omega}^e_G[\leq k]$, we obtain the desired filtration

$$\mathcal{P} = \mathcal{P}_0 \to \mathcal{P}_1 \to \mathcal{P}_2 \to \cdots \to \mathrm{colim}_k\, \mathcal{P}_k = \mathcal{P}[u]. \tag{5.65}$$

To analyze (5.65) homotopically we will further need a pushout description of each map $\mathcal{P}_{k-1} \to \mathcal{P}_k$. To do so, note that the diagram of inclusions

$$\begin{tikzcd} \widehat{\Omega}^e_G[k \setminus Y] \arrow[r] \arrow[d] & \widehat{\Omega}^e_G[\leq k \setminus Y] \arrow[d] \\ \widehat{\Omega}^e_G[k] \arrow[r] & \widehat{\Omega}^e_G[\leq k] \end{tikzcd} \tag{5.66}$$

is a pushout of at the level of nerves. Indeed, this follows since

$$\widehat{\Omega}^e_G[k] \cap \widehat{\Omega}^e_G[\leq k \setminus Y] = \widehat{\Omega}^e_G[k \setminus Y], \qquad \widehat{\Omega}^e_G[k] \cup \widehat{\Omega}^e_G[\leq k \setminus Y] = \widehat{\Omega}^e_G[\leq k],$$

and since a map $T \to S$ in $\widehat{\Omega}^e_G[\leq k]$ is in one of subcategories in (5.66) if and only if T is.

Since Lemma 5.59 provides an identification $\mathsf{Lan}_{\widehat{\Omega}^e_G[\preceq k\setminus Y]^{op}}\tilde{N} \simeq \mathsf{Lan}_{\widehat{\Omega}^e_G[\preceq k-1]^{op}}\tilde{N} = \mathcal{P}_{k-1}$, applying left Kan extensions to (5.66) yields the pushout diagram below.

$$\begin{tikzcd} \mathsf{Lan}_{\widehat{\Omega}^e_G[k\setminus Y]^{op}}\tilde{N} \arrow[r] \arrow[d] & \mathcal{P}_{k-1} \arrow[d] \\ \mathsf{Lan}_{\widehat{\Omega}^e_G[k]^{op}}\tilde{N} \arrow[r] & \mathcal{P}_k \end{tikzcd} \tag{5.67}$$

We will also make use of an explicit levelwise description of (5.67).

Proposition 5.68. *For each G-corolla $C \in \Sigma_G$, (5.67) is given by the following pushout in $\mathcal{V}^{\mathsf{Aut}(C)}$*

$$\begin{tikzcd} \coprod\limits_{[T]\in\mathsf{Iso}\left(C\downarrow_{\mathsf{r}}\Omega^a_G[k]\right)} \left(\bigotimes\limits_{v\in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes Q^{in}_T[u] \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C) \arrow[r] \arrow[d] & \mathcal{P}_{k-1}(C) \arrow[d] \\ \coprod\limits_{[T]\in\mathsf{Iso}\left(C\downarrow_{\mathsf{r}}\Omega^a_G[k]\right)} \left(\bigotimes\limits_{v\in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \bigotimes\limits_{v\in V^{in}_G(T)} Y(T_v) \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C) \arrow[r] & \mathcal{P}_k(C) \end{tikzcd} \tag{5.69}$$

where $V^{ac}_G(T)$, $V^{in}_G(T)$ denote the active and inert vertices of $T \in \Omega^a_G[k]$, and $Q^{in}_T[u]$ is the domain of the iterated pushout product

$$\mathop{\square}_{v\in V^{in}_G(T)} u(T_v) \colon Q^{in}_T[u] \to \bigotimes_{v\in V^{in}_G(T)} Y(T_v).$$

Proof. This is a consequence of Remark 5.58. Iteratively computing left Kan extensions by first left Kan extending to $\Omega^a_G[k]$, we can rewrite the leftmost map in (5.67) as

$$\mathsf{Lan}_{(\Omega^a_G[k]\to\Sigma_G)^{op}} \left(\bigotimes_{v\in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v\in V^{in}_G(T)} u(T_v) \right). \tag{5.70}$$

The desired description of the leftmost map given in (5.69) now follows by noting that the rooted undercategories $C \downarrow_{\mathsf{r}} \Omega^a_G[k]$ are groupoids (compare with (4.2)). ☐

5.4 Proof of Theorems I and II

In this section, we use the filtrations just developed to prove our first two main results, Theorems I and II, concerning the existence of model structures on $\mathsf{Op}^G(\mathcal{V})$ and $\mathsf{Op}_G(\mathcal{V})$.

Recall that for a group Σ, the genuine model structure (if it exists) on $\mathcal{V}^\Sigma$, which we denote $\mathcal{V}^\Sigma_{\text{gen}}$, has weak equivalences (resp. fibrations) those maps $X \to Y$ such that $X^H \to Y^H$ is a weak equivalence (fibration) for all $H \leq \Sigma$.

Our main proof will require some auxiliary results concerning genuine model structures. However, since these results are particular instances of subtler results from §6 which will require a far more careful analysis, we defer their proofs to those of the stronger results in §6.

Remark 5.71. The genuine model structure $\mathcal{V}^\Sigma_{\text{gen}}$ exists whenever $\mathcal{V}$ has *cellular fixed points.* The exact condition, originally from [12] and updated in [30], can be found in Definition 6.2. Moreover, note that this is condition (iii) in our main theorems. For our immediate purposes, however, we will only need to know that $\mathcal{V}^\Sigma_{\text{gen}}$ is then cofibrantly generated with generating (trivial) cofibrations the maps $\Sigma/H \cdot i$ for $H \leq \Sigma$ and i a generating (trivial) cofibration of $\mathcal{V}$.

More generally, given a family $\mathcal{F}$ (or even collection of subgroups) of Σ, there then also exists a model structure $\mathcal{V}^{\Sigma}_{\mathcal{F}}$ with weak equivalences, fibrations and generating (trivial) cofibrations all described by restricting H to $\mathcal{F}$.

Remark 5.72. A skeletal filtration argument shows that all objects in $\mathsf{sSet}^{\Sigma}_{\mathrm{gen}}$, $\mathsf{sSet}^{\Sigma}_{*,\mathrm{gen}}$ are cofibrant.

Remark 5.73. Suppose $\mathcal{V}$ has cellular fixed points and is a closed monoidal model category.

(i) Propositions 6.5 and 6.6 imply that for a group homomorphism $\phi: \Sigma \to \bar{\Sigma}$ the functors

$$\bar{\Sigma} \cdot_{\Sigma} (-) \colon \mathcal{V}^{\Sigma}_{\mathrm{gen}} \longrightarrow \mathcal{V}^{\bar{\Sigma}}_{\mathrm{gen}} \qquad \mathsf{res}^{\bar{\Sigma}}_{\Sigma} \colon \mathcal{V}^{\bar{\Sigma}}_{\mathrm{gen}} \longrightarrow \mathcal{V}^{\Sigma}_{\mathrm{gen}}$$

are left Quillen functors.

(ii) (6.15) says that the monoidal product on $\mathcal{V}$ lifts to a left Quillen bifunctor

$$\mathcal{V}^{\Sigma}_{\mathrm{gen}} \times \mathcal{V}^{\bar{\Sigma}}_{\mathrm{gen}} \xrightarrow{\otimes} \mathcal{V}^{\Sigma \times \bar{\Sigma}}_{\mathrm{gen}}.$$

The following lemma is the key to our main proof. Here, a map f in $\mathsf{Sym}_G(\mathcal{V})$ is called a *level genuine (trivial) cofibration* if each of the maps $f(C)$ for $C \in \Sigma_G$ are genuine trivial cofibrations in $\mathcal{V}^{\mathsf{Aut}(C)}_{\mathrm{gen}}$.

Lemma 5.74. *Suppose $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers (cf. Proposition 6.24).*

Let $\mathcal{P} \in \mathsf{Sym}_G(\mathcal{V})$ be level genuine cofibrant and $u : X \to Y$ in $\mathsf{Sym}_G(\mathcal{V})$ a level genuine cofibration. Then for each $T \in \Omega^a_G[k]$ and writing $C = \mathsf{lr}(T)$, the map

$$\left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v \in V^{in}_G(T)} u(T_v) \right) \underset{\mathsf{Aut}(T)}{\otimes} \mathsf{Aut}(C). \tag{5.75}$$

is a genuine cofibration in $\mathcal{V}^{\mathsf{Aut}(C)}_{gen}$, which is trivial if $k \geq 1$ and u is trivial.

Proof. Combining the homomorphism $\mathsf{Aut}(T) \to \mathsf{Aut}(C)$ with the leftmost left Quillen functor in Remark 5.73(i), it suffices to check that the parenthesized expression in (5.75) is a (trivial) genuine $\mathsf{Aut}(T)$-cofibration.

Furthermore, the homomorphism $\mathsf{Aut}(T) \to \mathsf{Aut}\left((T_v)_{v \in V^{ac}_G(T)}\right) \times \mathsf{Aut}\left((T_v)_{v \in V^{in}_G(T)}\right)$ combined with the rightmost left Quillen functor in Remark 5.73(i) and Remark 5.73(ii) then yield that it suffices to check that the two maps

$$\left(\varnothing \to \bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \right) = \mathop{\square}_{v \in V^{ac}_G(T)} (\varnothing \to \mathcal{P})(T_v), \qquad \mathop{\square}_{v \in V^{in}_G(T)} u(T_v)$$

are, respectively, $\mathsf{Aut}\left((T_v)_{v \in V^{ac}_G(T)}\right)$ and $\mathsf{Aut}\left((T_v)_{v \in V^{in}_G(T)}\right)$ genuine cofibrations, with the latter trivial if u is. Here, the automorphism groups are taken in the category in $\mathsf{F} \wr \Sigma_G$, and thus admit a product description of the form $\Sigma_{|\lambda_1|} \wr \mathsf{Aut}(T_{v_1}) \times \cdots \times \Sigma_{|\lambda_k|} \wr \mathsf{Aut}(T_{v_k})$ as in Remark 2.9. A further application of Remark 5.73(ii) yields that the required conditions need only be checked independently for the partial pushout product indexed by each λ_i, thus reducing to Proposition 6.24 (when $\mathcal{F}$ is the family of all subgroups). $\square$

Remark 5.76. If $T \in \Omega^a[k]$ is a non-equivariant alternating tree, $\mathcal{P}$ is level genuine cofibrant in $\mathsf{Sym}^G(\mathcal{V})$, and $u \colon X \to Y$ is a level genuine (trivial) cofibration in $\mathsf{Sym}^G(\mathcal{V})$, the previous result applied to $G \cdot T = (T)_{g \in G}$, $\iota_! \mathcal{P}$, $\iota_! u$, yields that the analogue of the map (5.75) is an $\mathsf{Aut}(G \cdot C_n) \simeq G \times \mathsf{Aut}(C_n) = G \times \Sigma_n$ level genuine (trivial) cofibration, where $C_n = \mathsf{lr}(T)$.

proof of Theorems I and II. We first build a seemingly unrelated model structure. Consider the composite adjunction below, with right adjoints on the bottom, and where the rightmost right adjoint simply forgets structure and the leftmost right adjoint is given by evaluation.

$$\prod_{C\in\Sigma_G} \mathcal{V}^{\mathsf{Aut}(C)}_{\mathrm{gen}} \underset{(\mathrm{ev}_C(-))}{\overset{}{\rightleftarrows}} \mathsf{Sym}_G(\mathcal{V}) \underset{}{\overset{\mathbb{F}_G}{\rightleftarrows}} \mathsf{Op}_G(\mathcal{V}) \tag{5.77}$$

We claim that $\mathsf{Op}_G(\mathcal{V})$ admits a (semi-)model structure with weak equivalences and fibrations defined by the composite right adjoint in (5.77). Noting that the left adjoint to $(\mathrm{ev}_C(-))$ is given by $(X_D) \mapsto \coprod_{D\in\Sigma_G} \mathsf{Hom}_{\Sigma_G}(-,D) \cdot_{\mathsf{Aut(D)}} X_D$ and using either [17, Thm. 11.3.2] in the model structure case $\mathcal{V} = \mathsf{sSet}, \mathsf{sSet}_*$ or [33, Thm. 2.2.2] in the semi-model category structure case, one must analyze free $\mathbb{F}_G$-extension diagrams of the form

$$\begin{tikzcd}
\mathbb{F}_G\left(\mathsf{Hom}_{\Sigma_G}(-,D)/H\cdot A\right) \arrow[r] \arrow[d, "u"'] & \mathcal{P} \arrow[d] \\
\mathbb{F}_G\left(\mathsf{Hom}_{\Sigma_G}(-,D)/H\cdot B\right) \arrow[r] & \mathcal{P}[u]
\end{tikzcd}$$

where $D \in \Sigma_G$, $H \leq \mathsf{Aut}(D)$, and $u\colon A \to B$ is a generating (trivial) cofibration in $\mathcal{V}$.

The map $\mathcal{P} \to \mathcal{P}[u]$ is then filtered as in (5.65), and since $\mathsf{Hom}_{\Sigma_G}(C,D)/H\cdot u$ is a (trivial) cofibration in $\mathcal{V}^{\mathsf{Aut}(C)}_{\mathrm{gen}}$ for all $C \in \Sigma_G$ (cf. Remark 5.71), combining the inductive description of the filtration in (5.69) with Lemma 5.74 shows that if $\mathcal{P}$ is level genuine cofibrant then $\mathcal{P} \to \mathcal{P}[u]$ is a level genuine cofibration, trivial whenever u is.

In the model structure case $\mathcal{V} = \mathsf{sSet}, \mathsf{sSet}_*$, Remark 5.72 guarantees that any $\mathcal{P}$ is level genuine cofibrant, and thus the conditions in [17, Thm. 11.3.2] are met (since transfinite composites of trivial cofibrations are again trivial cofibrations), showing the existence of the model structure. In the semi-model structure case, the condition that $\mathcal{P}$ is level genuine cofibrant does not quite coincide with the cell complex condition in [33, Thm. 2.2.2]. However, the regular (i.e. not trivial) cofibration case in the previous paragraph together with a routine induction argument over the cell decomposition of cellular $\mathcal{P}$ shows that cellular $\mathcal{P}$ are indeed level genuine cofibrant. Thus, the semi-model structure case also follows.

We now turn to showing the existence of the (semi-)model structures appearing in Theorems I and II, which are essentially corollaries of the existence of that defined by (5.77).

Firstly, consider the projective (semi-)model structure on $\mathsf{Op}_G(\mathcal{V})$. This model structure is transferred from the exact same adjunction (5.77), except equipping the leftmost $\mathcal{V}^{\mathsf{Aut}(C)}$ with their naive model structures, where weak equivalences and fibrations are defined by forgetting the $\mathsf{Aut}(C)$-action, and ignoring fixed point conditions. The desired projective model structure thus has both fewer generating (trivial) cofibrations and more weak equivalences than the "genuine projective" model structure defined by (5.77). Therefore, transfinite composites of pushouts of generating projective trivial cofibrations are genuine projective equivalences and hence also projective equivalences, showing that the condition in [17, Thm. 11.3.2(2)] holds, establishing the existence of the projective model structure . In the semi-model structure case, one replaces [17, Thm. 11.3.2(2)] with the obvious analogue (unfortunately, we know of no direct reference for this analogue, but its proof is identical).

To address the remaining cases in Theorems I and II, note first that by replacing the model structure in the leftmost category of (5.77) with $\prod_{C\in\Sigma_G} \mathcal{V}^{\mathsf{Aut}(C)}_{\mathcal{F}_C}$ for an arbitrary choice of collections of subgroups $\mathcal{F}_C$ of $\mathsf{Aut}(C)$ for $C \in \Sigma_G$, the exact same argument as in the previous paragraph yields a transferred $\{\mathcal{F}_C\}$ model structure in $\mathsf{Op}_G(\mathcal{V})$.

Letting $\mathcal{F}$ now denote a weak indexing system as in Theorem II and Corollary 4.54, one concludes in particular that there exists a "$\mathcal{F}$-projective" (semi-)model structure on $\mathsf{Op}_G(\mathcal{V})$, with weak equivalences and fibrations determined by evaluation at C for $C \in \Sigma_{\mathcal{F}}$. This does not quite coincide with the $\mathcal{F}$-projective model structure on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ appearing in Theorem II, since $\mathsf{Op}_G(\mathcal{V})$ is a larger category. But, since the inclusions $\gamma_!\colon \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Sym}_G(\mathcal{V})$, $\gamma_!\colon \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Op}_G(\mathcal{V})$ in (4.46), (4.55) preserve colimits and the monad $\mathbb{F}_{\mathcal{F}}$

defining $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ can be regarded as a restriction of $\mathbb{F}_G$, the desired condition in [17, Thm. 11.3.2(2)] when applied to the intended model structure on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ turns out to coincide with the corresponding condition for the $\mathcal{F}$-projective model structure on $\mathsf{Op}_G(\mathcal{V})$. The existence of the projective (semi-)model structures on $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ follows, finishing the proof of Theorem II.

We now turn to Theorem I. Should it be the case that $(\mathcal{V}, \otimes)$ has diagonals (which is not a requirement of Theorem I), one can simply use the inclusion $\iota_!: \mathsf{Op}^G(\mathcal{V}) \to \mathsf{Op}_G(\mathcal{V})$ of (4.34) and repeat the argument in the previous paragraph, except now for an arbitrary collection $\{\mathcal{F}_C\}$. Otherwise, one instead adapts the entire proof, starting with the obvious $\mathsf{Op}^G(\mathcal{V})$ analogue of (5.77) and using Remark 5.76 instead of Lemma 5.74 (as in this case, we may still use the filtration (5.69) with $G = *$ and $\mathcal{V} = \mathcal{V}^G$ by Remarks 3.64 and 2.17). $\square$

6 Cofibrancy and Quillen equivalences

In this final section we prove our main result, Theorem III. I.e. we show that there are Quillen equivalences

$$\mathsf{Op}_G(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G(\mathcal{V}) \qquad\qquad \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$$

In contrast to the existence of model structure results shown in §5.4, this will require a far more careful analysis of the genuine model structures $\mathcal{V}^G_{\mathcal{F}}$ mentioned in Remark 5.71. This analysis is the subject of §6.1 and §6.2, the results of which are converted to the setup of G-trees in §6.3, and culminate in the characterization of cofibrant objects in Op_G, $\mathsf{Op}_{\mathcal{F}}$ given by Lemma 6.59 in §6.4, with this final lemma tantamount to Theorem III.

Lastly, §6.5 discusses our models for the $N\mathcal{F}$-operads of Blumberg-Hill.

6.1 Families of subgroups

Throughout $\mathcal{F}$ denotes a *family* of subgroups of a finite group G, i.e. a collection of subgroups closed under conjugation and inclusion, or, equivalently (cf. §4.4), a sieve $\mathsf{O}_{\mathcal{F}} \hookrightarrow \mathsf{O}_G$.

Remark 6.1. For fixed G families form a lattice, ordered by inclusion, with meet and join given by intersection and union.

As mentioned in Remark 5.71, when $\mathcal{V}$ is cofibrantly generated and has cellular fixed points, [30, Prop. 2.6] shows that there exists a model structure $\mathcal{V}^G_{\mathcal{F}}$ on the G-object category $\mathcal{V}^G$ whose fibrations and weak equivalences are determined by the fixed points $(-)^H$ for $H \in \mathcal{F}$. Our analysis will require an explicit understanding of this cellularity condition, which we now recall.

Definition 6.2. A model category $\mathcal{V}$ is said to have *cellular fixed points* if for all finite groups G and subgroups $H, K \leq G$ one has that:

(i) fixed points $(-)^H: \mathcal{V}^G \to \mathcal{V}$ preserve direct colimits;

(ii) fixed points $(-)^H$ preserve pushouts where one leg is $(G/K) \cdot f$, for f a cofibration;

(iii) for each object $X \in \mathcal{V}$, the natural map $(G/K)^H \cdot X \to ((G/K) \cdot X)^H$ is an isomorphism.

This section will establish some simple useful properties of the $\mathcal{V}^G_{\mathcal{F}}$ model structures. We start by strengthening the cellularity conditions in Definition 6.2.

Proposition 6.3. *Let $\mathcal{V}$ be a cofibrantly generated model category with cellular fixed points. Then:*

(i) $(-)^H: \mathcal{V}^G \to \mathcal{V}$ preserves cofibrations and pushouts where one leg is a genuine cofibration;

(ii) if X is genuine cofibrant, the map $(G/K)^H \cdot X^H \to (G \cdot_K X)^H$ is an isomorphism.

Proof. Since both conditions are compatible with retracts, we are free to assume each cofibration $f: X \to Y$ (or, for Y cofibrant, the map $\varnothing \to Y$) is a transfinite composition

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \xrightarrow{f_3} \cdots \to Y = X_\beta = \operatorname{colim}_{\alpha<\beta} X_\alpha \tag{6.4}$$

where each $f_\alpha: X_\alpha \to X_{\alpha+1}$ is the pushout of a generating cofibration $(G/H) \cdot i_\alpha$. Both (i) and (ii) now follow by transfinite induction on α in the partial composite map $X_0 \to X_\alpha$, with the successor ordinal case following by Def. 6.2 (ii), (iii) and the limit ordinal case by Def. 6.2 (i). We note that (ii) also includes an obvious base case $X_0 = \varnothing$. □

Proposition 6.5. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ be cofibrantly generated with cellular fixed points. Then the adjunction*

$$\phi_! = \bar{G} \cdot_G (-): \mathcal{V}^G_{\mathcal{F}} \rightleftarrows \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}}: \mathsf{res}^{\bar{G}}_G = \phi^*$$

is a Quillen adjunction provided that for any $H \in \mathcal{F}$ we have $\phi(H) \in \bar{\mathcal{F}}$.

Proof. Since one has a canonical isomorphism of fixed points $(\mathsf{res}(X))^H \simeq X^{\phi(H)}$, it is immediate that the right adjoint preserves fibrations and trivial fibrations. □

Proposition 6.6. *Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{V}$ be cofibrantly generated with cellular fixed points. Then the adjunction*

$$\phi^* = \mathsf{res}^{\bar{G}}_G: \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \rightleftarrows \mathcal{V}^G_{\mathcal{F}}: \mathsf{Hom}_G(\bar{G}, -) = \phi_*$$

is a Quillen adjunction provided that for any $H \in \bar{\mathcal{F}}$ it is $\phi^{-1}(H) \in \mathcal{F}$.

Proof. A choice $\{a\}$ of double coset representatives of $\phi(G)\backslash\bar{G}/H$ gives G-orbit representatives of $\bar{G}/H$, yielding the formula $\mathsf{res}(\bar{G}/H) \simeq \coprod_{[a]\in\phi(G)\backslash\bar{G}/H} G/\phi^{-1}(H^a)$. Hence

$$\mathsf{res}\left(\bar{G}/H \cdot f\right) \simeq \mathsf{res}\left(\bar{G}/H\right) \cdot f \simeq \left(\coprod_{[a]\in\phi(G)\backslash\bar{G}/H} G/\phi^{-1}(H^a) \right) \cdot f$$

from which it follows that the left adjoint res preserves generating (trivial) cofibrations. □

Propositions 6.5 and 6.6 motivate the following definition.

Definition 6.7. Let $\phi: G \to \bar{G}$ be a homomorphism and $\mathcal{F}$ and $\bar{\mathcal{F}}$ families in G and $\bar{G}$. We define

$$\phi^*(\bar{\mathcal{F}}) = \{H \le G : \phi(H) \in \bar{\mathcal{F}}\} \tag{6.8}$$

$$\phi_!(\mathcal{F}) = \{\phi(H)^{\bar{g}} \le \bar{G} : \bar{g} \in \bar{G}, H \in \mathcal{F}\} \tag{6.9}$$

$$\phi_*(\mathcal{F}) = \{\bar{H} \le \bar{G} : \forall_{\bar{g}\in\bar{G}} \left(\phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F}\right)\} \tag{6.10}$$

Lemma 6.11. *The $\phi^*(\bar{\mathcal{F}})$, $\phi_!(\mathcal{F})$, $\phi_*(\mathcal{F})$ just defined are themselves families. Furthermore*

(i) The "provided that" condition in Proposition 6.5 holds iff $\mathcal{F} \subset \phi^*(\bar{\mathcal{F}})$ iff $\phi_!(\mathcal{F}) \subset \bar{\mathcal{F}}$.

(ii) The "provided that" condition in Proposition 6.6 holds iff $\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F}$ iff $\bar{\mathcal{F}} \subset \phi_*(\mathcal{F})$.

Proof. Since the result is elementary, we include only the proof of the second iff in (ii), which is the hardest step and illustrates the necessary arguments. This follows by the following equivalences.

$$\phi^*(\bar{\mathcal{F}}) \subset \mathcal{F} \Leftrightarrow \left(\forall_{\substack{H\le G \\ \phi(H)\in\bar{\mathcal{F}}}} H \in \mathcal{F} \right) \Leftrightarrow \left(\forall_{\bar{H}\in\bar{\mathcal{F}}} \phi^{-1}(\bar{H}) \in \mathcal{F} \right) \Leftrightarrow \left(\forall_{\substack{\bar{H}\in\bar{\mathcal{F}} \\ \bar{g}\in\bar{G}}} \phi^{-1}(\bar{H}^{\bar{g}}) \in \mathcal{F} \right) \Leftrightarrow \bar{\mathcal{F}} \subset \phi_*(\mathcal{F})$$

Here the second equivalence follows since $H \le \phi^{-1}(\phi(H))$ and $\mathcal{F}$ is closed under subgroups while the third equivalence follows since $\bar{\mathcal{F}}$ is closed under conjugation. □

Proposition 6.12. *Suppose that $\mathcal{V}$ is cofibrantly generated, has cellular fixed points, and is also a closed monoidal model category. Then the bifunctor*

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^G_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^G_{\mathcal{F}\cap\bar{\mathcal{F}}}$$

is a left Quillen bifunctor.

Proof. A choice $\{a\}$ of double coset representatives of $H\backslash G/\bar{H}$ gives orbit representatives $\{([e],[a])\}$ of $G/H \times G/\bar{H}$, yielding the formula $G/H \times G/\bar{H} \simeq \coprod_{[a]\in H\backslash G/\bar{H}} G/H \cap \bar{H}^a$. Hence

$$(G/H\cdot f)\,\square\,\big(G/\bar{H}\cdot g\big) \simeq \big(G/H \times G/\bar{H}\big)\cdot(f\,\square\, g) \simeq \left(\coprod_{[a]\in H\backslash G/\bar{H}} \big(G/H\cap \bar{H}^a\big)\cdot(f\,\square\, g)\right)$$

and the result follows since families are closed under conjugation and subgroups. $\square$

Definition 6.13. Let $\mathcal{F}$ and $\bar{\mathcal{F}}$ be families of G and $\bar{G}$, respectively.

We define their *external intersection* to be the family of $G\times\bar{G}$ given by

$$\mathcal{F}\sqcap\bar{\mathcal{F}} = (\pi_G)^*(\mathcal{F}) \cap (\pi_{\bar{G}})^*(\bar{\mathcal{F}})$$

for $\pi_G\colon G\times\bar{G}\to G$, $\pi_{\bar{G}}\colon G\times\bar{G}\to\bar{G}$ the projections.

Remark 6.14. Combining Proposition 6.6 with Propositon 6.12 yields that the following composite is a left Quillen bifunctor.

$$\mathcal{V}^G_{\mathcal{F}} \times \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \xrightarrow{\mathsf{res}} \mathcal{V}^{G\times\bar{G}}_{(\pi_G)^*(\mathcal{F})} \times \mathcal{V}^{G\times\bar{G}}_{(\pi_{\bar{G}})^*(\bar{\mathcal{F}})} \xrightarrow{\otimes} \mathcal{V}^{G\times\bar{G}}_{\mathcal{F}\sqcap\bar{\mathcal{F}}} \tag{6.15}$$

6.2 Pushout powers

That (6.15) is a left Quillen bifunctor (and its obvious higher order analogues) is one of the key properties of pushout products of $\mathcal{F}$ cofibrations when those cofibrations (and the group) are allowed to change. However, when those cofibrations (and hence G) coincide there is an additional symmetric group action that we will need to consider.

To handle these actions we will need two new axioms, which will concern cofibrancy and fixed point properties. We start by discussing the cofibrancy axiom.

Definition 6.16. We say that a symmetric monoidal model category $\mathcal{V}$ has *cofibrant symmetric pushout powers* if for each (trivial) cofibration f the pushout product power $f^{\square n}$ is a Σ_n-genuine (trivial) cofibration.

Remark 6.17. When $\mathcal{V}$ is cofibrantly generated the condition in Definition 6.16 needs only be checked for generating cofibrations. However, the argument needed is harder than usual (see, e.g., [18, Lemma 2.1.20]) due to $(-)^{\square n}$ not preserving composition of maps: one instead follows the argument in the proof of Proposition 6.24 below when $G = *$.

Example 6.18. Both $(\mathsf{sSet},\times)$ and $(\mathsf{sSet}_*,\wedge)$ have cofibrant symmetric pushout powers. To see this, we note first that the case of (non-trivial) cofibrations is immediate since genuine cofibrations are precisely the monomorphisms. For the case of $f\colon X\to Y$ a trivial cofibration, it is easier to first show directly that $f^{\otimes n}\colon X^{\otimes n}\to Y^{\otimes n}$ is a trivial cofibration, and then use the factorizations (6.25) for $h = f$, $g = (\varnothing\to X)$, in which case $f^{\otimes n} = k_n\cdots k_1$ and $f^{\square n} = k_n$, to show by induction on n that $f^{\square n}$ is also a trivial cofibration.

We now turn to describing the symmetric power analogue of Definition 6.13.

We start with notation. Letting λ be a partition $E = \lambda_1 \amalg\cdots\amalg \lambda_k$ of a finite set E, we write $\Sigma_\lambda = \Sigma_{\lambda_1}\times\cdots\times\Sigma_{\lambda_k} \le \Sigma_E$ for the subgroup of permutations preserving λ. In addition, given any $e\in E$ we write λ_e for the partition $E = \{e\}\amalg(E-e)$, so that Σ_{λ_e} is then the isotropy of e.

Definition 6.19. Let $\mathcal{F}$ be a family of G, E a finite set and $e \in E$ any fixed element.

We define the *n-th semidirect power of* $\mathcal{F}$ to be the family of $\Sigma_E \wr G = \Sigma_E \ltimes G^{\times E}$ given by

$$\mathcal{F}^{\ltimes E} = \left(\iota_{\Sigma_{\lambda_e} \wr G}\right)_* \left(\left(\pi_G\right)^* \left(\mathcal{F}\right)\right),$$

where ι is the inclusion $\Sigma_{\lambda_e} \wr G \to \Sigma_E \wr G$ and π the projection $\Sigma_{\lambda_e} \wr G = \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to G$.

More explicitly, since in (6.10) one needs only consider conjugates by coset representatives of $\bar{G}/\phi(G)$, when computing $\left(\iota_{\Sigma_{\lambda_e} \wr G}\right)_*$ one needs only conjugate by coset representatives of $\Sigma_E \wr G/\Sigma_{\lambda_e} \wr G \simeq \Sigma_E/\Sigma_{\lambda_e}$, so that

$$K \in \mathcal{F}^{\ltimes E} \text{ iff } \underset{e \in E}{\forall} \pi_G \left(K \cap \left(\Sigma_{\lambda_e} \wr G\right)\right) \in \mathcal{F}, \tag{6.20}$$

showing that in particular $\mathcal{F}^{\ltimes E}$ is independent of the choice of $e \in E$.

Remark 6.21. The previous definition is likely to seem mysterious at first sight. Ultimately, the origin of this definition is best understood by working through this section backwards: the study of the interactions between equivariant trees and graph families, namely Lemma 6.49, requires the study of the families $\mathcal{F}^{\ltimes_G n}$ in Notation 6.37, which are variants of the $\mathcal{F}^{\ltimes n}$ construction for graph families. It then suffices, and is notationally far more convenient, to establish the required results first for the $\mathcal{F}^{\ltimes n}$ families and then translate them to the $\mathcal{F}^{\ltimes_G n}$ families.

Proposition 6.22. *Writing* $\iota: \Sigma_E \times \Sigma_{\bar{E}} \to \Sigma_{E \amalg \bar{E}}$ *for the inclusion, one has*

$$\mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}} \subset \iota^* \left(\mathcal{F}^{\ltimes E \amalg \bar{E}}\right).$$

Hence, the following is a left Quillen bifunctor for $\mathcal{V}$ *as in Proposition 6.12.*

$$\Sigma_{E \amalg \bar{E}} \underset{\Sigma_E \times \Sigma_{\bar{E}}}{\cdot} (- \otimes -): \mathcal{V}^{\Sigma_E \wr G} \times \mathcal{V}^{\Sigma_{\bar{E}} \wr G} \to \mathcal{V}^{\Sigma_{E \amalg \bar{E}} \wr G} \tag{6.23}$$

Proof. Let $K \in \mathcal{F}^{\ltimes E} \sqcap \mathcal{F}^{\ltimes \bar{E}}$ and $e \in E$. We write λ_e for the partition of $E \amalg \bar{E}$ and λ_e^E for the partition of E. One then has

$$\pi_G \left(K \cap \left(\Sigma_{\lambda_e} \wr G\right)\right) = \pi_G \left(\pi_{\Sigma_E \wr G}(K) \cap \left(\Sigma_{\lambda_e^E} \wr G\right)\right),$$

where on the right we write $\pi_{\Sigma_E \wr G}: \Sigma_E \wr G \times \Sigma_{\bar{E}} \wr G \to \Sigma_E \wr G$ and $\pi_G: \Sigma_{\lambda_e^E} \wr G = \Sigma_{\{e\}} \times G \times \Sigma_{E-e} \wr G \to G$. Therefore K satisfies (6.20) for $\mathcal{F}^{\ltimes E \amalg \bar{E}}$ since $\pi_{\Sigma_E \wr G}(K)$ does so for $\mathcal{F}^{\ltimes E}$. The case of $e \in \bar{E}$ is identical.

(6.23) simply combines the left Quillen bifunctor (6.15) with Proposition 6.5. $\square$

Proposition 6.24. *Suppose that* $\mathcal{V}$ *is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Then, for every n *and cofibration (resp. trivial cofibration)* f *of* $\mathcal{V}^G_{\mathcal{F}}$ *one has that* $f^{\square n}$ *is a cofibration (trivial cofibration) of* $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$.

Our proof of Proposition 6.24 will essentially repeat the main argument in the proof of [23, Thm. 1.2]. However, both for the sake of completeness and to stress that the argument is independent of the (fairly technical) model structures in [23], we include an abridged version of the proof below, the key ingredient of which is that (6.23) is a left Quillen bifunctor.

Proof. We first note that in the case of a generating (trivial) cofibration $i = (G/H) \cdot \bar{\imath}$, $H \in \mathcal{F}$, it is

$$i^{\square n} = (G/H)^{\times n} \cdot \bar{\imath}^{\square n} \simeq \Sigma_n \wr G \underset{\Sigma_n \wr H}{\cdot} \bar{\imath}^{\square n}.$$

But $\bar{\imath}^{\square n}$ is now a $\Sigma_n \wr H$-genuine (trivial) cofibration by combining the cofibrant symmetric pushout powers hypothesis with Proposition 6.6 and hence $i^{\square n}$ is a $\mathcal{F}^{\ltimes n}$ (trivial) cofibration by Proposition 6.5 since $\Sigma_n \wr H \in \mathcal{F}^{\ltimes n}$.

For the general case, we start by making the key observation that for composable arrows $\bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ the n-fold pushout product $(hg)^{\square n}$ has a factorization

$$\bullet \xrightarrow{k_0} \bullet \xrightarrow{k_1} \cdots \xrightarrow{k_n} \bullet \tag{6.25}$$

where each k_r, $0 \le r \le n$, fits into a pushout diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\Sigma_n \underset{\Sigma_{n-r}\times\Sigma_r}{\cdot} (g^{\square n-r} \square h^{\square r})"'] \arrow[dr, phantom, "\ulcorner"] & \bullet \arrow[d, "k_r"] \\ \bullet \arrow[r] & \bullet. \end{tikzcd} \tag{6.26}$$

Briefly, (6.25) follows from a filtration $P_0 \subset P_1 \subset \cdots \subset P_n$ of the poset $P_n = (0 \to 1 \to 2)^{\times n}$ where P_0 consists of "tuples with at least one 0-coordinate" and P_r is obtained from P_{r-1} by adding the "tuples with $n-r$ 1-coordinates and r 2-coordinates". Additional details concerning this filtration appear in the proof of [23, Lemma 4.8].

The general proof now follows by writing f as a retract of a transfinite composition of pushouts of generating (trivial) cofibrations as in (6.4). As usual, retracts preserve weak equivalences, and we can hence assume that there is an ordinal κ and $X_\bullet \colon \kappa \to \mathcal{V}^G$ such that (i) $f_\beta \colon X_\beta \to X_{\beta+1}$ is the pushout of a (trivial) cofibration i_β; (ii) $\operatorname{colim}_{\alpha<\beta} X_\alpha \xrightarrow{\simeq} X_\beta$ for limit ordinals $\beta < \kappa$; (iii) setting $X_\kappa = \operatorname{colim}_{\beta<\kappa} X_\beta$, f equals the transfinite composite $X_0 \to X_\kappa$.

We argue by transfinite induction on κ. Writing $\bar{f}_\beta \colon X_0 \to X_\beta$ for the partial composites, it suffices to check that the natural transformation of κ-diagrams (rightmost map not included)

$$\begin{tikzcd} Q^n(\bar{f}_1) \arrow[r] \arrow[d, "\bar{f}_1^{\square n}"'] & Q^n(\bar{f}_2) \arrow[r] \arrow[d, "\bar{f}_2^{\square n}"'] & Q^n(\bar{f}_3) \arrow[r] \arrow[d, "\bar{f}_3^{\square n}"'] & Q^n(\bar{f}_4) \arrow[r] \arrow[d, "\bar{f}_4^{\square n}"'] & \cdots \arrow[r] & Q^n(\bar{f}_\kappa) \arrow[d, "\bar{f}_\kappa^{\square n} = \operatorname{colim}_{\beta<\kappa} \bar{f}_\beta^{\square n}"] \\ X_1^{\otimes n} \arrow[r] & X_2^{\otimes n} \arrow[r] & X_3^{\otimes n} \arrow[r] & X_4^{\otimes n} \arrow[r] & \cdots \arrow[r] & X_\kappa^{\otimes n}, \end{tikzcd}$$

is (trivial) κ-cofibrant, i.e. that the maps $Q^n(\bar{f}_\beta) \amalg_{\operatorname{colim}_{\alpha<\beta} Q^n(\bar{f}_\alpha)} \operatorname{colim}_{\alpha<\beta} X_\alpha^{\otimes n} \to X_\beta^{\otimes n}$ are (trivial) cofibrations in $\mathcal{V}^{\Sigma_n \wr G}_{\mathcal{F}^{\ltimes n}}$. Condition (ii) above implies that this map is an isomorphism for β a limit ordinal while for $\beta+1$ a successor ordinal it is the map $Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n} \to X_{\beta+1}^{\otimes n}$. But since $Q^n(\bar{f}_{\beta+1}) \to Q^n(\bar{f}_{\beta+1}) \amalg_{Q^n(\bar{f}_\beta)} X_\beta^{\otimes n}$ is precisely the map k_0 of (6.25) for $g = \bar{f}_\beta$, $h = f_\beta$, this last map is the composite $k_n k_{n-1} \cdots k_1$ so that the result now follows from (6.26) together with the left Quillen bifunctor (6.23) since: (i) the induction hypothesis shows the cofibrancy of $\bar{f}_\beta^{\square n-r}$; (ii) the cofibrancy of $i_\beta^{\square r}$ together with the fact that $f_\beta^{\square r}$ is a pushout of $i_\beta^{\square r}$ (cf. [23, Lemma 4.11]) imply the cofibrancy of $f_\beta^{\square r}$. $\square$

We now turn to discussing the fixed points of pushout powers $f^{\square n}$.

Firstly, we assume throughout the following discussion that $(\mathcal{V}, \otimes)$ has diagonal maps, as in Remark 2.18. In particular, one has compatible Σ_n-equivariant maps $X \to X^{\otimes n}$.

Consider now a K-object $(X_e)_{e\in E}$ in $(\mathsf{F}_s \wr \mathcal{V})^K$ for some finite group K. Explicitly, this consists of an action of K on the indexing set E together with suitably associative and unital isomorphisms $X_e \to X_{ke}$ for each $(e,k) \in E \times K$. Moreover, writing K_e for the isotropy of $e \in E$, note that the induced fixed point isomorphism $X_e^{K_e} \to X_{ke}^{K_{ke}}$ does not depend on the choice of coset representative $k \in kK_e$, and we will thus abuse notation by writing $X_{[e]}^{K_{[e]}} = X_f^{K_f}$ for an arbitary choice of representative $f \in [e] = Ke$ (more formally, we mean that $X_{[e]}^{K_{[e]}} = \left(\coprod_{f\in[e]} X_f^{K_f}\right)/\Sigma_{[e]}$).

Diagonal maps then induce canonical composites (generalizing the twisted diagonals discussed following Remark 4.35)

$$X_{[e]}^{K_{[e]}} \to \left(X_{[e]}^{K_{[e]}}\right)^{\otimes [e]} \simeq \bigotimes_{f\in[e]} X_f^{K_f} \to \bigotimes_{f\in[e]} X_f,$$

leading to the following axiom.

Definition 6.27. We say that a symmetric monoidal category with diagonals $\mathcal{V}$ has *cartesian fixed points* if the canonical maps

$$\bigotimes_{[e]\in E/K} X_{[e]}^{K_{[e]}} \xrightarrow{\simeq} \left(\bigotimes_{e\in E} X_e\right)^K \tag{6.28}$$

are isomorphisms for all $(X_e)_{e\in E}$ in $(\mathsf{F}_s \wr \mathcal{V})^K$ for all finite groups K.

Remark 6.29. As the name implies, the condition in the previous definition is automatic for cartesian $\mathcal{V}$. Moreover, this condition is easily seen to hold for $\mathcal{V} = \mathsf{sSet}_*$.

The condition (6.28) naturally breaks down into two conditions.

The first condition, which makes sense in the absence of diagonals, corresponds to the case where K acts trivially on E and says that $X^K \otimes Y^K \xrightarrow{\simeq} (X \otimes Y)^K$, for $X, Y \in \mathcal{V}^K$.

The second condition, corresponding to the case where K acts transitively, concerns the fixed points of what is often called the norm object $N_{K_e}^K X_e \simeq \bigotimes_{e\in E} X_e$.

These two conditions roughly correspond to the two parts of Proposition 6.3, though now without cofibrancy requirements. In fact, if one modifies Definition 6.27 by requiring that (6.28) be an isomorphism only when the X_e are K_e-cofibrant, it is not hard to show that this modified condition can be deduced from the requirement that $\mathcal{V}$ be strongly cofibrantly generated (i.e. that the domains/codomains of the (trivial) generating cofibrations be cofibrant) together with isomorphisms $X^{\otimes (G/H)^K} \xrightarrow{\simeq} \left(X^{\otimes G/H}\right)^K$ for $X \in \mathcal{V}$ (i.e. a power analogue of Definition 6.2 (iii)).

Proposition 6.30. *Suppose that $\mathcal{V}$ is as in Proposition 6.24, and also has diagonals and cartesian fixed points. Let $K \leq \Sigma_n \wr G$ be a subgroup, $f\colon X \to Y$ a map in $\mathcal{V}^G$ and consider the natural maps (in the arrow category)*

$$\mathop{\square}_{[i]\in n/K} f_{[i]}^{K_{[i]}} \to \left(f^{\square n}\right)^K. \tag{6.31}$$

If f is a genuine cofibration between genuine cofibrant objects then (6.31) is an isomorphism.

At first sight, it may seem that the desired isomorphism (6.31) should be an immediate consequence of (6.28). However, the real content here is that the two pushout products in (6.31) are computed over cubes of different sizes. Namely, while the right hand side is computed using the cube $(0 \to 1)^{\times n}$, the left hand side is computed over the fixed point cube $\left((0 \to 1)^{\times n}\right)^K \simeq (0 \to 1)^{\times n/K}$ formed by those tuples whose coordinates coincide if their indices are in the same coset of n/K.

Example 6.32. When $n = 3$ and $n/K = \{\{1,2\},\{3\}\}$ the fixed subposet $(0 \to 1)^{\times n/K}$ is displayed on the right below.

$$\begin{tikzcd}
 & 000 \arrow[rr] \arrow[dl] \arrow[dd] & & 010 \arrow[dl] \arrow[dd] \\
100 \arrow[rr] \arrow[dd] & & 110 \arrow[dd] & \\
 & 001 \arrow[rr] \arrow[dl] & & 011 \arrow[dl] \\
101 \arrow[rr] & & 111 &
\end{tikzcd}
\qquad
\begin{tikzcd}
000 \arrow[dr] \arrow[dd] & \\
 & 110 \arrow[dd] \\
001 \arrow[dr] & \\
 & 111
\end{tikzcd}$$

proof of Proposition 6.30. The result will follow by induction on n. The base case $n = 1$ is obvious.

Moreover, it is clear from (6.28) that (6.31), which is a map of arrows, is an isomorphism on the target objects, hence the real claim is that this map is also an isomorphism on sources.

We now note that by considering (6.25) for $g = (\varnothing \to X)$, $h = f$ and removing the last map k_n one obtains a filtration of the source of $f^{\square n}$. Applying $(-)^K$ to the leftmost map in

(6.26) one has isomorphisms

$$\left(\Sigma_n \underset{\Sigma_{n-i}\times\Sigma_i}{\cdot} X^{\otimes n-i} \otimes f^{\square i}\right)^K \simeq \coprod_{\substack{n/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A} \otimes f^{\square B}\right)^K \simeq \coprod_{\substack{n/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(X^{\otimes A}\right)^K \otimes \left(f^{\square B}\right)^K$$
$$\simeq \coprod_{\substack{n/K = A/K \amalg B/K \\ |A|=n-i, |B|=i}} \left(\bigotimes_{[j]\in A/K} X_{[j]}^{K_{[j]}}\right) \otimes \left(\mathop{\square}_{[k]\in B/K} f_{[k]}^{K_{[k]}}\right)$$

Here the first step is an instance of Proposition 6.3(ii), with the required cofibrancy conditions following from Proposition 6.24. The second step follows from (6.28). Lastly, the third step follows by (6.28) together with the induction hypothesis, which applies since $|B| = i < n$.

Noting that Proposition 6.24 guarantees that all required maps are cofibrations so that fixed points $(-)^K$ commute with pushouts by Proposition 6.3(i), we have just shown that the leftmost maps in the pushout diagrams (6.26) for $\left(f^{\square n}\right)^K$ are isomorphic to the leftmost maps in the pushout diagrams for the corresponding filtration of $\mathop{\square}_{[i]\in n/K} f_{[i]}^{K_{[i]}}$. $\square$

Corollary 6.33. *Given a partition λ given by $\{1, 2, \cdots, n\} = \lambda_1 \amalg \cdots \amalg \lambda_k$, cofibrations between cofibrant objects f_i in $\mathcal{V}^{G_i}$, $1 \le i \le k$ and a subgroup $K \le \Sigma_{\lambda_1} \wr G_1 \times \cdots \times \Sigma_{\lambda_k} \wr G_k$, the natural map*

$$\mathop{\square}_{1\le i\le k} \mathop{\square}_{[j]\in\lambda_i/K} f_{i,[j]}^{K_{[j]}} \to \left(\mathop{\square}_{1\le i\le k} f_i^{\square\lambda_i}\right)^K.$$

is an isomorphism.

Proof. This combines Proposition 6.30 with the easier isomorphisms $f^K \square g^K \xrightarrow{\simeq} (f \square g)^K$, which follow by (6.28) together with the observation that $(-)^K$ commutes with pushouts thanks to the cofibrancy conditions and Proposition 6.3(i). $\square$

6.3 G-graph families and G-trees

We now convert the results in the previous sections to the context we are truly interested in: graph families. Throughout this section Σ will denote a general group, usually meant to be some type of permutation group.

Definition 6.34. A subgroup $\Gamma \le G \times \Sigma$ is called a *G-graph subgroup* if $\Gamma \cap \Sigma = *$.

Further, a family $\mathcal{F}$ of $G\times\Sigma$ is called a *G-graph family* if it consists of G-graph subgroups.

Remark 6.35. Γ is a G-graph subgroup iff it can be written as

$$\Gamma = \{(h, \varphi(h)) : h \in H \le G\}$$

for some partial homomorphism $G \ge H \xrightarrow{\varphi} \Sigma$, thus motivating the terminology.

Remark 6.36. The collection of all G-graph subgroups is itself a family $\mathcal{F}^{\Gamma}$. Indeed, this family coincides with $(\iota_\Sigma)_*(\{*\})$ for the inclusion homomorphism $\iota_\Sigma : \Sigma \to G \times \Sigma$.

Notation 6.37. Letting $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G \times \Sigma$ and $G \times \bar{\Sigma}$ we will write

$$\mathcal{F} \sqcap_G \bar{\mathcal{F}} = \Delta^*(\mathcal{F} \sqcap \bar{\mathcal{F}}) \qquad \mathcal{F}^{\ltimes_G n} = \Delta^*(\mathcal{F}^{\ltimes n})$$

where Δ denotes either of the diagonal inclusions $\Delta : G\times\Sigma\times\bar{\Sigma} \to G\times\Sigma\times G\times\bar{\Sigma}$ or $\Delta : G\times\Sigma_n\wr\Sigma \to \Sigma_n \wr (G \times \Sigma)$.

Remark 6.38. Unpacking Definition 6.13 one has that $\Gamma \in \mathcal{F} \sqcap_G \bar{\mathcal{F}}$ iff $\pi_{G\times\Sigma}(\Gamma) \in \mathcal{F}$ and $\pi_{G\times\bar{\Sigma}}(\Gamma) \in \bar{\mathcal{F}}$.

Remark 6.39. Unpacking (6.20) and noting that, as subgroups of $\Sigma_n \wr (G \times \Sigma)$,

$$(G \times \Sigma_E \wr \Sigma) \cap (\Sigma_{\lambda_e} \wr (G \times \Sigma)) = G \times \Sigma_{\lambda_e} \wr \Sigma$$

one has

$$K \in \mathcal{F}^{\ltimes_G E} \text{ iff } \underset{e \in E}{\forall} \pi_{G \times \Sigma}\left(K \cap (G \times \Sigma_{\lambda_e} \wr \Sigma)\right) \in \mathcal{F}.$$

Combining either the left Quillen bifunctor (6.15) or Proposition 6.24 with Proposition 6.6 yields the following results.

Proposition 6.40. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points. Let $\mathcal{F}$, $\bar{\mathcal{F}}$ be G-graph families of $G \times \Sigma$ and $G \times \bar{\Sigma}$. Then the following (with diagonal G-action on the images) is a left Quillen bifunctor.*

$$\mathcal{V}^{G \times \Sigma}_{\mathcal{F}} \times \mathcal{V}^{G \times \bar{\Sigma}}_{\bar{\mathcal{F}}} \xrightarrow{\otimes} \mathcal{V}^{G \times \Sigma \times \bar{\Sigma}}_{\mathcal{F} \sqcap_G \bar{\mathcal{F}}}$$

Proposition 6.41. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Let $\mathcal{F}$ be a G-graph family of $G \times \Sigma$. If f is a cofibration (resp. trivial cofibration) in $\mathcal{V}^{G \times \Sigma}_{\mathcal{F}}$, then so is $f^{\square n}$ in $\mathcal{V}^{G \times \Sigma_n \wr \Sigma}_{\mathcal{F}^{\ltimes_G n}}$.

Remark 6.42. It is straightforward to check that $\mathcal{F} \sqcap_G \bar{\mathcal{F}}$ is in fact also a G-graph family of $G \times \Sigma \times \bar{\Sigma}$. However, $\mathcal{F}^{\ltimes_G n}$ is *not* a G-graph family of $G \times \Sigma_n \wr \Sigma$, due to the need to consider the power Σ_n-action.

The G-graph families we will be interested in encode the families of G-corollas $\Sigma_{\mathcal{F}}$ of Definition 4.44 and, more generally, the families of G-trees $\Omega_{\mathcal{F}}$ of Definition 4.47.

First, note that a partial homomorphism $G \geq H \to \Sigma_n$ defines a H-action on the n-corolla $C_n \in \Sigma$ and hence, by choosing an arbitrary order of G/H and coset representatives g_i for G/H, a G-corolla $(g_i C_n)_{[g_i] \in G/H}$ in Σ_G. The following is then elementary.

Lemma 6.43. *Writing $\mathcal{F}^{\Gamma}_n$ for the family of G-graph subgroups of $G \times \Sigma_n$, there is an equivalence of categories (for any arbitrary choice of order of the G/H and of coset representatives)*

$$\coprod_{n \geq 0} \mathsf{O}_{\mathcal{F}^{\Gamma}_n} \xrightarrow{\simeq} \Sigma_G.$$

Hence, families of corollas $\Sigma_{\mathcal{F}}$ are in bijection with collections $\{\mathcal{F}_n\}_{n \geq 0}$ of G-graph families $\mathcal{F}_n \subset \mathcal{F}^{\Gamma}_n$.

We will hence abuse notation and use $\mathcal{F}$ to denote either $\{\mathcal{F}_n\}_{n \geq 0}$ or $\Sigma_{\mathcal{F}}$.

Note that a G-corolla $(C_i)_{i \in I}$ is in $\Sigma_{\mathcal{F}}$ iff for some (and thus all) $i \in I$ the action of the stabilizer H_i on C_i is given by a partial homomorphism $G \geq H_i \to \Sigma_n$ encoding a group in $\mathcal{F}_n$.

In what follows, given a tree with a H-action $T \in \Omega^H$, we will abbreviate $G \cdot_H T = (g_i T)_{[g_i] \in G/H}$ for some arbitrary (and inconsequential for the remaining discussion) choice of order on G/H and of coset representatives.

Proposition 6.44. *Let $\mathcal{F}$ be a family of G-corollas and $T \in \Omega$ a tree with automorphism group Σ_T. Write $\mathcal{F}_T$ for the collection of G-graph subgroups of $G \times \Sigma_T$ encoded by partial homomorphisms $G \geq H \to \Sigma_T$ such that the associated G-tree $G \cdot_H T$ is a $\mathcal{F}$-tree (cf. Definition 4.47).*

Then $\mathcal{F}_T$ is a G-graph family.

Proof. Closure under conjugation follows since conjugate graph subgroups produce isomorphic G-trees. As for subgroups, they correspond to restrictions $K \leq H \to \Sigma_T$, as thus also restrict the stabilizer actions on each vertex $T_{e^{\uparrow} \leq e}$. □

Remark 6.45. The closure condition defining weak indexing systems in Definition 4.49 can be translated in terms of families as saying that for any tree $T \in \Omega$ with $\mathsf{lr}(T) = C_n$ and $\phi: \Sigma_T \to \Sigma_n$ the natural homomorphism, one has $(id_G \times \phi)(\Gamma) \in \mathcal{F}_n$ for any $\Gamma \in \mathcal{F}_T$. Hence, by Proposition 6.5

$$\phi_!: \mathcal{V}^{G\times\Sigma_T}_{\mathcal{F}_T} \to \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$$

is a left Quillen functor.

Remark 6.46. Unpacking definitions, a partial homomorphism $G \geq H \to \Sigma_T$ encodes a subgroup in $\mathcal{F}_T$ iff, for each vertex $v = (e^\uparrow \leq e)$ of T with $H_e \leq H$ the H-isotropy of the edge e, the induced homomorphism

$$H_e \to \Sigma_{T_v} \simeq \Sigma_{|v|} \tag{6.47}$$

encodes a subgroup in $\mathcal{F}_{|v|}$, where $|v| = |e^\uparrow|$.

Remark 6.48. Recall that any tree $T \in \Omega$ other than the stick η has an essentially unique grafting decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$ where C_n is the root corolla and the leaves of C_n are grafted to the roots of the T_i. We now let λ be the partition $\{1, \cdots, n\} = \lambda_1 \amalg \cdots \amalg \lambda_k$ such that $1 \leq i_1, i_2 \leq n$ are in the same class iff $T_{i_1}, T_{i_2} \in \Omega$ are isomorphic.

Writing $\Sigma_\lambda = \Sigma_{\lambda_1} \times \cdots \times \Sigma_{\lambda_k}$ and picking representatives $i_j \in \lambda_j$ one then has isomorphisms

$$\Sigma_T \simeq \Sigma_\lambda \wr \prod_i \Sigma_{T_i} \simeq \Sigma_{|\lambda_1|} \wr \Sigma_{T_{i_1}} \times \cdots \times \Sigma_{|\lambda_k|} \wr \Sigma_{T_{i_k}}$$

where the second isomorphism, while not canonical (it depends on choices of isomorphisms $T_{i_j} \simeq T_l$ for each $i_j \neq l \in \lambda_j$) is nonetheless well defined up to conjugation.

The following, which is the key motivation behind the families defined in the last sections, reinterprets Remark 6.46 in light of the inductive description of trees in Remark 6.48.

Lemma 6.49. *Let $\Sigma_{\mathcal{F}}$ be a family of G-corollas and $T \in \Omega$ a tree other than η. Then*

$$\mathcal{F}_T = (\pi_{G\times\Sigma_n})^* (\mathcal{F}_n) \cap \left(\mathcal{F}^{\ltimes_G |\lambda_1|}_{T_{i_1}} \sqcap_G \cdots \sqcap_G \mathcal{F}^{\ltimes_G |\lambda_k|}_{T_{i_k}}\right), \tag{6.50}$$

where $\pi_{G\times\Sigma_n}$ denotes the composite $G \times \Sigma_T \to G \times \Sigma_\lambda \to G \times \Sigma_n$.

Proof. The argument is by induction on the decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$ with the base case, that of a corolla, being immediate.

Consider now a partial homomorphism $G \geq H \to \Sigma_T$ encoding a G-graph subgroup $\Gamma \leq G \times \Sigma_T$. The condition that $\Gamma \in (\pi_{G\times\Sigma_n})^* (\mathcal{F}_n)$ states that the composite $H \to \Sigma_T \to \Sigma_n$ is in $\mathcal{F}_n$, and this is precisely the condition (6.47) in Remark 6.46 for $e = r$ the root of T.

As for the condition $\Gamma \in \left(\mathcal{F}^{\ltimes_G |\lambda_1|}_{T_{i_1}} \sqcap_G \cdots \sqcap_G \mathcal{F}^{\ltimes_G |\lambda_k|}_{T_{i_k}}\right)$, by unpacking it by combining Remarks 6.38 and 6.39, this translates to the condition that, for each $i \in \{1, \cdots, k\}$, one has

$$\pi_{G\times\Sigma_{T_i}} \left(\Gamma \cap \left(G \times \Sigma_{\{i\}} \times \Sigma_{T_i} \times \Sigma_{\lambda-\{i\}} \wr \prod_{j\neq i} \Sigma_{T_j}\right)\right) \in \mathcal{F}_{T_i} \tag{6.51}$$

where $\lambda - \{i\}$ denotes the induced partition of $\{1, \cdots, n\} - \{i\}$. Noting that the intersection subgroup inside $\pi_{G\times\Sigma_{T_i}}$ in (6.51) can be rewritten as $\Gamma \cap \pi^{-1}_{\Sigma_n}(\Sigma_{\{i\}} \times \Sigma_{\{1,\cdots,n\}-\{i\}})$, we see that this is the graph subgroup encoded by the restriction $H \geq H_i \to \Sigma_T$, where H_i is the isotropy subgroup of the root r_i of T_i (equivalently, this is also the subgroup sending T_i to itself). But since for any edge $e \in T_i$ its isotropy H_e (cf. (6.47)) is a subgroup of H_i, the induction hypothesis implies that (6.51) is equivalent to condition (6.47) across all vertices other than the root vertex.

The previous paragraphs show that (6.50) indeed holds when restricted to G-graph subgroups. However, it still remains to show that any group Γ in the rightmost family in (6.50) is indeed a G-graph subgroup, i.e. $\Gamma \cap \Sigma_T = *$. In other words, we need to show that any element $\gamma \in \Gamma \leq G \times \Sigma_\lambda \wr \prod_i \Sigma_{T_i}$ whose G-coordinate is $\gamma_G = e$ is indeed the identity. But the condition $\pi_{G\times\Sigma_n}(\Gamma) \in \mathcal{F}_n$ now implies that for such γ the Σ_λ-coordinate is $\gamma_{\Sigma_\lambda} = e$ and thus (6.51) in turn implies that the Σ_{T_i}-coordinates are $\gamma_{\Sigma_{T_i}} = e$, finishing the proof. ∎

In preparation for our discussion of cofibrant objects in $\mathsf{Op}_G(\mathcal{V})$ in the next section, we end the current section by applying the results in the previous sections to study the leftmost map in the key pushout diagrams (5.69). More concretely, and writing $p(T_v)\colon \varnothing \to \mathcal{P}(T_v)$, we analyze the cofibrancy of the maps

$$\bigotimes_{v \in V_G^{ac}(T)} \mathcal{P}(T_v) \otimes \mathop{\square}_{v \in V_G^{in}(T)} u(T_v) \qquad \text{or} \qquad \mathop{\square}_{v \in V_G^{ac}(T)} p(T_v) \,\square \mathop{\square}_{v \in V_G^{in}(T)} u(T_v)$$

that constitute the inner part of (5.70), and where we recall that $T \in \Omega_G^a$ is an alternating tree. This analysis will consist of two parts, to be combined in the next section: (i) a $\mathcal{F}_{T_e}$-cofibrancy claim when $T = G \cdot T_e$ is free and; (ii) a fixed point claim for non free trees, as in Remark 4.35.

For both the sake of generality and to simplify notation in the proofs, we will state the following results using the labeled trees of Definition 5.8, and write $\Omega_G^{\underline{l}}$ for the category of l-labeled trees and quotients (we will have no need for string categories at this point). Moreover, l-labeled $\mathcal{F}$-trees $\Omega_{\mathcal{F}}^{\underline{l}}$ are defined exactly as in Definition 4.47, so that a labeled G-tree is a $\mathcal{F}$-tree if and only if the underlying G-tree is. Lastly, note that Remarks 6.46, 6.48 and Lemma 6.49 then extend to the l-labeled context, by now writing Σ_T for the group of label isomorphisms and defining the partition λ in Remark 6.48 by using label isomorphism classes.

Proposition 6.52. *Suppose that $\mathcal{V}$ is a cofibrantly generated closed monoidal model category with cellular fixed points and with cofibrant symmetric pushout powers.*

Let $\mathcal{F}$ be a family of corollas and suppose that $f_s\colon A_s \to B_s$, $1 \le s \le l$ are level $\mathcal{F}$-cofibrations (resp. trivial cofibrations) in $\mathsf{Sym}^G(\mathcal{V})$, i.e. that $f_s(r)\colon A_s(r) \to B_s(r)$ are cofibrations (trivial cofibrations) in $\mathcal{V}_{\mathcal{F}_n}^{G\times\Sigma_n}$. Then for any l-labeled tree $T \in \Omega^{\underline{l}}$ the map

$$f^{\square V(T)} = \mathop{\square}_{1\le s\le l} \mathop{\square}_{v \in V_s(T)} f_s(v)$$

(where $V_s(T)$ denotes vertices with label s) is a cofibration (resp. trivial cofibration) in $\mathcal{V}_{\mathcal{F}_T}^{G\times\Sigma_T}$.

Proof. This follows by induction on the decomposition $T = C_n \amalg_{n\cdot\eta} (T_1 \amalg \cdots \amalg T_n)$, with the base cases of corollas and η being immediate. Otherwise, note first that

$$f^{\square V(T)} \simeq f_{s_r}(n) \,\square \mathop{\square}_{1\le i\le k} \left(f^{\square V(T_{i_j})}\right)^{\square \lambda_i}$$

where we use the notation in Remark 6.48 and s_r is the root vertex label.

The description of $\mathcal{F}_T$ in (6.50) combined with the left Quillen functors in Propositions 6.40, 6.12 and 6.6 then yield that

$$\mathcal{V}_{\mathcal{F}_n}^{G\times\Sigma_n} \times \mathcal{V}_{\mathcal{F}_{T_{i_1}}^{\ltimes_G |\lambda_1|}}^{G\times\Sigma_{|\lambda_1|}\wr\Sigma_{T_{i_1}}} \times \cdots \times \mathcal{V}_{\mathcal{F}_{T_{i_k}}^{\ltimes_G |\lambda_k|}}^{G\times\Sigma_{|\lambda_k|}\wr\Sigma_{T_{i_k}}} \xrightarrow{\otimes} \mathcal{V}_{\mathcal{F}_T}^{G\times\Sigma_T}$$

is a left Quillen multifunctor. The result now follows by Proposition 6.41 together with the induction hypothesis. $\square$

Remark 6.53. When $G = *$, Proposition 6.52 matches [2, Lemma 5.9]. Moreover, it is not hard to modify the proof of [2, Lemma 5.9] to show Proposition 6.52 for the universal family Σ_G of all G-corollas. However, our arguments are more subtle than those in [2], which need no analogue of the $\mathcal{F}^{\ltimes_G n}$ families. Indeed, this is reflected at the end of our proof of Lemma 6.49, where (6.51) is used to deduce the simpler condition $\Gamma \cap \prod \Sigma_{T_i} = *$, a condition that would suffice if directly adapting [2, Lemma 5.9] to obtain the Σ_G case.

One might thus hope for similarly easier proofs of the general $\Sigma_{\mathcal{F}}$ case and, reverse engineering our arguments, the most natural such attempt would replace (6.51) with

$$\pi_{G\times\Sigma_{T_i}}(\Gamma \cap \prod \Sigma_{T_i}) \in \mathcal{F},$$

which is tantamount to replacing the families $\mathcal{F}^{\ltimes n}$ of Definition 6.19 with the families $(\iota_{G^{\times n}})_*(\mathcal{F} \sqcap \cdots \sqcap \mathcal{F})$. However, one can build indexing systems $\Sigma_{\mathcal{F}}$ (other than Σ_G) for which these simpler families do not satisfy the analogue of Lemma 6.49, and thus for which the analogue of Remark 6.45 fails.

Proposition 6.54. *Let $\mathcal{V}$ be as in Proposition 6.52, and suppose additionally that $\mathcal{V}$ has diagonal maps and cartesian fixed points.*

Let $f_s: A_s \to B_s$, $1 \le s \le l$ be genuine cofibrations between genuine cofibrant objects in $\mathsf{Sym}^G(\mathcal{V})$. For each $T \in \Omega^l_G$ define

$$f^{\square V_G(T)} = \mathop{\square}_{1\le s\le l} \mathop{\square}_{v\in V_{G,s}(T)} \iota_* f_s(v).$$

Then the canonical natural transformation

$$f^{\square V_G(-)} \to \iota_*\iota^* f^{\square V_G(-)} \tag{6.55}$$

is a natural isomorphism in $\mathcal{V}^{\Omega^{l,op}_G}$ (with $G \times \Omega^l \xrightarrow{\iota} \Omega^l_G$ the inclusion).

Proof. Note first that there is a coproduct decomposition

$$\Omega^l_G \simeq \coprod_{U\in \mathsf{Iso}(\Omega^l)} \Omega^l_G[U]$$

where $\Omega^l_G[U]$ is the full subcategory formed by the quotients of $G \cdot U$. It thus suffices to establish (6.55) for each subcategory $\Omega^l_G[U]$.

All such G-trees can be written as $T = G \cdot_H U_H$, where U_H denotes the underlying tree $U \in \Omega^l$ together with a H-action. By induction on $|G|$ we are free to assume $H = G$. Indeed, otherwise there are identifications $V_G(T) \simeq V_H(U_H)$ and $f^{\square V_G(T)} \simeq (\mathsf{res}^G_H f)^{\square V_H(U_H)}$ from which the desired isomorphism follows by induction.

We have thus reduced to the case $T = U_G$. Consider now the quotient map $(U)_{g\in G} = G\cdot U \to U_G$ given by the identity on the e component. The automorphisms of $G\cdot U$ compatible with the quotient map $G \cdot U \to U_G$ are the elements of the G-graph subgroup $K \le G \times \Sigma_U$ encoding the action $G \to \Sigma_U$ of G on U_G.

We now have identifications (recall that $V_G(U_G) = V(U)/G$)

$$f^{\square V_G(U_G)} \simeq \mathop{\square}_{[v]\in V_G(U_G)} \iota_* f_\bullet([v]) \simeq \mathop{\square}_{[v]\in V(U)/G} f^{G_{[v]}}_{\bullet,[v]} \simeq \left(\mathop{\square}_{v\in V(U)} f_\bullet(v)\right)^G \simeq \left(\mathop{\square}_{Gv\in V_G(G\cdot U)} \iota_* f_\bullet(Gv)\right)^K$$

Here the second identification combines the formula for ι_* in §4.3 with the cartesian fixed point formula (6.28), which always holds for the product. The third step follows by Corollary 6.33 (this is the step requiring the cofibrancy of the f_s). The last step repackages notation, again using the cartesian fixed point formula for ι_*. Noting that this last term is $\left(\iota_*\iota^* f^{\square V_G(-)}\right)(U_G)$ finishes the proof. $\square$

6.4 Cofibrancy and the proof of Theorem III

Propositions 6.52 and 6.54 will now allow us to prove Lemma 6.59, which provides a characterization of cofibrant objects in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$, and from which our main result Theorem III readily follows. We start by refining the key argument in the proof of [30, Thm. 2.10].

Proposition 6.56. *Let $\mathcal{V}$ be a cofibrantly generated model category with cellular fixed points, $\mathcal{F}$ a non-empty family of subgroups of G, and consider the reflexive adjunction*

$$\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathcal{V}^G_{\mathcal{F}}.$$

Then the cofibrant objects of $\mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}}}$ are precisely the essential image under ι_* of the cofibrant objects of $\mathcal{V}^G_{\mathcal{F}}$. Moreover, the analogous statement for cofibrations between cofibrant objects also holds.

Proof. Note first that since ι_* identifies $\mathcal{V}^G$ as a reflexive subcategory of $\mathcal{V}^{\mathsf{O}_{\mathcal{F}}^{op}}$, it is $X \simeq \iota_* Y$ for some $Y \in \mathcal{V}^G$ (i.e. $X \in \mathcal{V}^{\mathsf{O}_{\mathcal{F}}^{op}}$ is in the essential image of ι_*) iff both $\iota^* X \simeq Y$ and the unit map $X \xrightarrow{\simeq} \iota_* \iota^* X$ is an isomorphism.

Letting $C_{\mathcal{F}}$ (resp. $C^{\mathcal{F}}$) denote the classes of cofibrant objects in $\mathcal{V}^{\mathsf{O}_{\mathcal{F}}^{op}}$ (resp. $\mathcal{V}^G_{\mathcal{F}}$) we need to show $C_{\mathcal{F}} = \iota_*(C^{\mathcal{F}})$, where we slightly abuse notation by writing $\iota_*(-)$ for the essential image rather than the image. Since $C_{\mathcal{F}}$ is characterized as being the smallest class closed under retracts and transfinite composition of cellular extensions that contains the initial presheaf $\varnothing$, it suffices to show that $\iota_*(C^{\mathcal{F}})$ satisfies this same characterization.

It is immediate that $\iota_*(\varnothing) = \varnothing$. Further, the characterization in the first paragraph yields that $X \in \iota_*(C^{\mathcal{F}})$ iff $\iota^*(X) \in C^{\mathcal{F}}$ and $X \xrightarrow{\simeq} \iota_* \iota^* X$ is an isomorphism, showing that $\iota_*(C^{\mathcal{F}})$ is closed under retracts.

The crux of the proof will be to compare cellular extensions in $C_{\mathcal{F}}$ with the images under ι_* of the cellular extensions in $C^{\mathcal{F}}$. Firstly, note that the generating cofibrations in $\mathcal{V}^{\mathsf{O}_{\mathcal{F}}^{op}}$ have the form $\mathsf{Hom}(-, G/H) \cdot f$, and that by the cellularity axiom (iii) in Definition 6.2 this map is isomorphic to the map $\iota_*(G/H \cdot f)$. We now claim that the cellular extensions of objects in $\iota_*(C^{\mathcal{F}})$, i.e. pushout diagrams as on the left below

$$
\begin{tikzcd}
\iota_* X \arrow[r] \arrow[d, "\iota_* u"'] & \iota_* V \arrow[d, dashed] \\
\iota_* Y \arrow[r, dashed] & \tilde{W}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & V \arrow[d, dashed] \\
Y \arrow[r, dashed] & W
\end{tikzcd}
\tag{6.57}
$$

are precisely the essential image under ι_* of the cellular extensions of objects in $C^{\mathcal{F}}$, i.e., pushout diagrams as on the right above. That the solid subdiagrams in either side of (6.57) are indeed in bijection up isomorphism is simply the claim that ι^* is fully faithful, hence the real claim is that $\tilde{W} \simeq \iota_* W$. But this follows since by the cellularity axiom (ii) in Definition 6.2 the map ι_* preserves the rightmost pushout in (6.57) (recall that $u\colon X \to Y$ is assumed to be a generating cofibration of $\mathcal{V}^G_{\mathcal{F}}$).

Noting that the cellularity axiom (i) in Definition 6.2 implies that ι_* preserves filtered colimits finishes the proof that $C_{\mathcal{F}} = \iota_*(C^{\mathcal{F}})$.

The additional claim concerning cofibrations between cofibrant objects follows by the same argument. □

Corollary 6.58. *Let $\mathcal{V}$ be as above, $\phi\colon G \to \bar{G}$ a homomorphism, and $\mathcal{F}$, $\bar{\mathcal{F}}$ families of G, $\bar{G}$ such that $\phi_! \mathcal{F} \subset \mathcal{F}$. Then the diagram*

$$
\begin{tikzcd}
\mathcal{V}^{\mathsf{O}_{\mathcal{F}}^{op}} \arrow[d, "\phi_!"'] & \mathcal{V}^G_{\mathcal{F}} \arrow[l, "\iota_*"'] \arrow[d, "\phi_!"] \\
\mathcal{V}^{\mathsf{O}_{\bar{\mathcal{F}}}^{op}} & \mathcal{V}^{\bar{G}}_{\bar{\mathcal{F}}} \arrow[l, "\iota_*"]
\end{tikzcd}
$$

commutes up to isomorphism when restricted to cofibrant objects of $\mathcal{V}^G_{\mathcal{F}}$.

Proof. It is straightforward to check that the left adjoints commute, i.e. that there is a natural isomorphism $\iota^* \phi_! \simeq \phi_! \iota^*$ which by adjunction induces a natural transformation $\phi_! \iota_* \to \iota_* \phi_!$. More explicitly, this natural transformation is the composite

$$\phi_! \iota_* \to \iota_* \iota^* \phi_! \iota_* \xrightarrow{\simeq} \iota_* \phi_! \iota^* \iota_* \xrightarrow{\simeq} \iota_* \phi_!$$

where the last two maps are always isomorphisms. But when restricting to cofibrant objects the previous result guarantees both that $\phi_! \iota_*$ lands in cofibrant objects and that cofibrant objects are in the essential image of the bottom ι_*. The result follows. □

The following is the main lemma. We note that the operad half of (6.61) was also obtained by Gutiérrez-White in [13].

Lemma 6.59. *Let $\mathcal{V}$ be as in Theorem III and let $\mathcal{F}$ be a weak indexing system. Then in both of the adjunctions*

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \qquad\qquad \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \tag{6.60}$$

the cofibrant objects in the leftmost category are the essential image under ι_* of the cofibrant objects in the rightmost category. Moreover, both forgetful functors

$$\mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \qquad\qquad \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \xrightarrow{\mathsf{fgt}} \mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \tag{6.61}$$

preserve cofibrant objects.

Before starting our proof we recall that, as in Remark 4.58, we do not require that $\mathcal{F}$ contain all free corollas, in which case the adjunctions in (6.60) are officially composite adjunctions as in (4.59). To avoid cumbersome notation, and noting that the inclusions $\gamma_!\colon \mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Sym}_G(\mathcal{V})$, $\gamma_!\colon \mathsf{Op}_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Op}_G(\mathcal{V})$ of §4.4 are compatible with colimits and that the monad $\mathbb{F}_{\mathcal{F}}$ is simply a restriction of $\mathbb{F}_G$, we will simply work in the $\mathsf{Sym}_G(\mathcal{V})$, $\mathsf{Op}_G(\mathcal{V})$ categories throughout, with the implicit understanding that objects lie in the required subcategories. In particular, ι^*, ι_* will denote functors from/to $\mathsf{Sym}_G(\mathcal{V})$, $\mathsf{Op}_G(\mathcal{V})$.

Proof. We first observe that the claim concerning the symmetric sequence adjunction in (6.60) is not really new. Indeed, by Lemma 6.43 there are equivalences of categories $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V}) \simeq \prod_{n\geq 0} \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}_n}}$, $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V}) \simeq \prod_{n\geq 0} \mathcal{V}^{G\times\Sigma_n}_{\mathcal{F}_n}$, compatible with both the model structures and the (ι^*, ι_*) adjunctions, and hence the symmetric sequence statement merely repackages Proposition 6.56 (with an obvious empty family case if $\mathcal{F}_n = \varnothing$ for some n).

For the operad adjunction in (6.60), most of the argument in the proof of Proposition 6.56 applies mutatis mutandis except for the claim that $\mathbb{F}_G(\varnothing) \simeq \iota_*\mathbb{F}(\varnothing)$, which is readily checked directly, and the comparison of cellular extensions, which is the key claim.

Further, we will argue the forgetful functor claim (6.61) in parallel over the same cellular extensions (note that the underlying cofibrancy of $\mathbb{F}(\varnothing)$, $\mathbb{F}_G(\varnothing)$ follows from the cofibrancy of the unit $I \in \mathcal{V}$).

Explicitly, and borrowing the notation $C_{\mathcal{F}}$ (resp. $C^{\mathcal{F}}$) used in Proposition 6.56 for the classes of cofibrant objects in $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ (resp. $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$), we need to show that cellular extensions of objects in $\iota_*(C^{\mathcal{F}})$, such as on the left below

$$\begin{tikzcd}
\mathbb{F}_G\iota_*X \arrow[r] \arrow[d, "\iota_*u"'] & \iota_*\mathcal{O} \arrow[d, dashed] \\
\mathbb{F}_G\iota_*Y \arrow[r, dashed] & (\iota_*\mathcal{O})[\iota_*u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{F}X \arrow[r] \arrow[d, "u"'] & \mathcal{O} \arrow[d, dashed] \\
\mathbb{F}Y \arrow[r, dashed] & \mathcal{O}[u]
\end{tikzcd} \tag{6.62}$$

are precisely the essential image under ι_* of cellular extensions of objects in $C^{\mathcal{F}}$, as on the right above. Moreover, we can assume by induction that $\iota_*\mathcal{O}$, $\mathcal{O}$ are underlying cofibrant in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$, $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$. Now, recalling that Proposition 4.38(ii)(iv) gives natural isomorphisms

$$\iota^*\mathbb{F}_G\iota_* \simeq \iota^*\mathbb{F}_G\iota_! \simeq \mathbb{F}$$

we see that the two solid subdiagrams in (6.62) are in fact adjoint up to isomorphism, so that there is a bijection between such data. We now claim that the leftmost diagram in (6.62) will indeed be the image under ι_* of the rightmost diagram provided that all four objects are in the essential image of ι_*. Indeed, if that is the case then

$$\mathbb{F}_G\iota_*Z \simeq \iota_*\iota^*\mathbb{F}_G\iota_*Z \simeq \iota_*\mathbb{F}Z$$

for $Z = X, Y$ and since ι_* reflects colimits[6], it must indeed be that $(\iota_*\mathcal{O})[\iota_*u] \simeq \iota_*(\mathcal{O}[u])$.

[6] I.e. any diagram that becomes a colimit upon applying ι_* must have already been a colimit diagram.

To establish the remaining claim that the objects in the leftmost diagram in (6.62) are in the essential image of ι_*, we claim it suffices to show this for the bottom right corner $(\iota_*\mathcal{O})[\iota_* u]$ when $u: X \to Y$ is a general cofibration between cofibrant objects in $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$. Indeed, setting $X = \varnothing$ and $\mathcal{O} = \mathbb{F}(\varnothing)$, one has $(\iota_*\mathcal{O})[\iota_* u] = \mathbb{F}_G \iota_* Y$, and similarly for $\mathbb{F}_G \iota_* X$.

Now, writing $\mathcal{P} = \iota_*\mathcal{O}$, so that $(\iota_*\mathcal{O})[\iota_* u] = \mathcal{P}[\iota_* u]$, the condition that $\mathcal{P}[\iota_* u] \to \iota_*\iota^*\mathcal{P}[\iota_* u]$ is an isomorphism can be checked by forgetting to $\mathsf{Sym}_G(\mathcal{V})$. Moreover, and tautologically, the same is true for the underlying cofibrancy condition in (6.61). We can thus appeal to the filtration (5.65) of $\mathcal{P} \to \mathcal{P}[\iota_* u]$, and by Proposition 6.56 it suffices to verify by induction on k that the maps $\mathcal{P}_{k-1} \to \mathcal{P}_k$ are cofibrations between cofibrant objects in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$.

Using the iterative description of the $\mathcal{P}_k$ in (5.69) it now suffices to check that the leftmost map in (5.69) is a cofibration between cofibrant objects in $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$. We now recall that that map can also be described (cf. (5.70)) as

$$\mathsf{Lan}_{(\Omega^a_G[k] \to \Sigma_G)^{op}} \left(\bigotimes_{v \in V^{ac}_G(T)} \mathcal{P}(T_v) \otimes \mathop{\Box}_{v \in V^{in}_G(T)} u(T_v) \right). \tag{6.63}$$

Now consider the left square below, which is equivalent to the right square and thus, by Corollary 6.58, commutative on cofibrant objects.

$$\begin{tikzcd}
\mathcal{V}^{\Omega^a_{\mathcal{F}}[k]^{op}} \arrow[d, "\phi_!"'] & \mathcal{V}_{\mathcal{F}}^{G \times \Omega^a[k]^{op}} \arrow[l, "\iota_*"'] \arrow[d, "\phi_!"] \\
\mathcal{V}^{\Sigma^{op}_{\mathcal{F}}} & \mathcal{V}_{\mathcal{F}}^{G \times \Sigma^{op}} \arrow[l, "\iota_*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\prod_{T \in \mathsf{Iso}(\Omega^a[k])} \mathcal{V}^{\mathsf{O}^{op}_{\mathcal{F}_T}} \arrow[d, "\phi_!"'] & \prod_{T \in \mathsf{Iso}(\Omega^a[k])} \mathcal{V}_{\mathcal{F}_T}^{G \times \Sigma_T} \arrow[l, "\iota_*"'] \arrow[d, "\phi_!"] \\
\prod_{n \geq 0} \mathcal{V}^{\Sigma^{op}_{\mathcal{F}_n}} & \prod_{n \geq 0} \mathcal{V}_{\mathcal{F}}^{G \times \Sigma^{op}_n} \arrow[l, "\iota_*"]
\end{tikzcd}$$

Propositions 6.52 and 6.54 now show that the inner map inside the left Kan extension in (6.63) is in the essential image of the cofibrations between cofibrant objects under the top ι_* map. But since the Lan in (6.63) is the leftmost $\phi_!$ functor, the result, including the underlying cofibrancy claims in (6.61), now follows by Corollary 6.58. □

Remark 6.64. The previous proof in fact establishes the slightly more general claim that operads (in either $\mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ or $\mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$) that forget to cofibrant symmetric sequences (in either $\mathsf{Sym}_{\mathcal{F}}(\mathcal{V})$ or $\mathsf{Sym}^G_{\mathcal{F}}(\mathcal{V})$) are closed under cellular extensions of operads.

Morever, and as mentioned in Remark 4.41, it now follows that (4.42) is an isomorphism when restricted to cofibrant G-symmetric sequences.

proof of Theorem III. It suffices to show that both the derived unit and derived counit for the adjunction are given by weak equivalences.

For the counit, it is immediate from Lemma 6.59 that if $X \in \mathsf{Op}^G(\mathcal{V})$ is bifibrant the functor $\iota^*\iota_* X$ is already derived, and hence the derived counit is identified with the counit isomorphism $\iota^*\iota_* X \xrightarrow{\simeq} X$.

For the unit, note first that it is immediate from the definitions that $\iota_*: \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V}) \to \mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ detects fibrations (as well as weak equivalences), and thus by Lemma 6.59 $Y \in \mathsf{Op}_{\mathcal{F}}(\mathcal{V})$ is bifibrant iff $Y \simeq \iota_* X$ for $X \in \mathsf{Op}^G_{\mathcal{F}}(\mathcal{V})$ bifibrant. But then the functor $\iota_*\iota^* Y$ is also already derived (since $\iota^* Y \simeq \iota^*\iota_* X \simeq X$ is fibrant) and the derived unit is thus the isomorphism $Y \xrightarrow{\simeq} \iota_*\iota^* Y$. □

6.5 Realizing N_∞-operads

We now explain how the $N\mathcal{F}$-operads of Blumberg-Hill can be built from the theory of genuine equivariant operads, thus proving Corollary IV.

We start with an abstract argument, which has also been used by Gutiérrez-White in [13]. Writing $\mathcal{I} = \mathbb{F}(\varnothing)$ for the initial equivariant operad in $\mathsf{Op}^G(\mathsf{sSet})$, i.e. the operad consisting of a single operation at level 1, consider any "cofibration followed by trivial fibration" factorization (as given by the Quillen small object argument)

$$\mathcal{I} \rightarrowtail \mathcal{O}_{\mathcal{F}} \xrightarrow{\sim} \mathsf{Com} \tag{6.65}$$

in the model structure $\mathsf{Op}^G_{\mathcal{F}}(\mathsf{sSet})$. We claim that $\mathcal{O}_{\mathcal{F}}$ is a $N\mathcal{F}$-operad, i.e. that it has fixed points as described in Corollary IV. That $\mathcal{O}_{\mathcal{F}}(n)^{\Gamma} \sim *$ whenever $\Gamma \in \mathcal{F}_n$ follows from the fact that the map $\mathcal{O}_{\mathcal{F}} \xrightarrow{\sim} \mathsf{Com}$ is a $\mathcal{F}$-equivalence. On the other hand, by Lemma 6.59 the map $\mathcal{I} \rightarrowtail \mathcal{O}_{\mathcal{F}}$ is also an underlying cofibration in $\mathsf{Sym}^G_{\mathcal{F}}(\mathsf{sSet})$, and thus $\mathcal{O}_{\mathcal{F}}$ is underlying cofibrant in $\mathsf{Sym}^G_{\mathcal{F}}(\mathsf{sSet})$. The required condition that $\mathcal{O}_{\mathcal{F}}(n)^{\Gamma} = \varnothing$ whenever $\Gamma \notin \mathcal{F}_n$ now follows since this holds for any cofibrant object in $\mathsf{Sym}^G_{\mathcal{F}}(\mathsf{sSet})$, as can readily be checked via a cellular argument.

One drawback of the $N\mathcal{F}$-operad $\mathcal{O}_{\mathcal{F}}$ built in (6.65), however, is that it is not explicit, due to the need to use the small object argument. To obtain a more explicit model, we make use of the theory of genuine equivariant operads.

Firstly, any weak indexing system $\mathcal{F}$ gives rise to a genuine equivariant operad $\partial_{\mathcal{F}} \in \mathsf{Op}_G(\mathsf{Set})$ such that $\partial_{\mathcal{F}}(C) = *$ if $C \in \Sigma_{\mathcal{F}}$ and $\partial_{\mathcal{F}}(C) = \varnothing$ if $C \notin \Sigma_{\mathcal{F}}$. Alternatively, $\partial_{\mathcal{F}}$ can also be regarded as the terminal object of $\mathsf{Op}_{\mathcal{F}}(\mathsf{Set}) \hookrightarrow \mathsf{Op}_G(\mathsf{Set})$. The characterization of the cofibrant objects in $\mathsf{Op}_G(\mathsf{sSet})$ given by Lemma 6.59 now shows that the unique map $\iota_* \mathcal{O}_{\mathcal{F}} \xrightarrow{\simeq} \delta_{\mathcal{F}}$ is a cofibrant replacement in $\mathsf{Op}_G(\mathsf{sSet})$ and, moreover, it is clear from the argument in the previous paragraph that for any other cofibrant replacement $C\delta_{\mathcal{F}} \xrightarrow{\simeq} \delta_{\mathcal{F}}$ the equivariant operad $\iota^*(C\delta_{\mathcal{F}}) \in \mathsf{Op}^G(\mathsf{sSet})$ is a $N\mathcal{F}$-operad. We will now build an explicit model for such $C\delta_{\mathcal{F}}$. We start by considering the following adjunctions, where both of the right adjoints, which we write at the bottom, are forgetful functors.

$$\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)} \underset{\longleftarrow}{\xrightarrow{(X_C) \mapsto \amalg_C \mathsf{Hom}(-,C) \times X_C}} \mathsf{Sym}_G(\mathsf{Set}) \underset{\longleftarrow}{\xrightarrow{\mathbb{F}_G}} \mathsf{Op}_G(\mathsf{Set}) \tag{6.66}$$

We will find it convenient in the following discussion to abuse notation by omitting occurrences of the forgetful functors. As such, we write $\delta_{\mathcal{F}}$ not only for the object in $\mathsf{Op}_G(\mathsf{Set})$, but also for any of the underlying objects in $\mathsf{Sym}_G(\mathsf{Set})$, $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$. Similarly, $\mathbb{F}_G$ will denote both the functor in (6.66) and the monad on $\mathsf{Sym}_G(\mathsf{Set})$ while $\widetilde{\mathbb{F}}_G$ will denote both the top composite functor in (6.66) and the composite monad on $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$.

Since both adjunctions in (6.66) restrict to their $\mathcal{F}$ versions, in which case $\delta_{\mathcal{F}}$ denotes the terminal object of any of the $\mathcal{F}$ analogue categories, it follows that $\delta_{\mathcal{F}} \in \mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$ is a $\widetilde{\mathbb{F}}_G$-algebra, and we now consider the bar construction

$$B_n(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}}) = \widetilde{\mathbb{F}}_G \circ \widetilde{\mathbb{F}}_G^{\circ n}(\partial_{\mathcal{F}}),$$

where we regard the outer $\widetilde{\mathbb{F}}_G$ as the top composite functor in (6.66). We thus have $B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}}) \in \mathsf{Op}_{\mathcal{F}}(\mathsf{Set})^{\Delta^{op}} \hookrightarrow \mathsf{Op}_G(\mathsf{Set})^{\Delta^{op}} \simeq \mathsf{Op}_G(\mathsf{sSet})$ and, moreover, the unique genuine operad map $B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}}) \to \partial_{\mathcal{F}}$ is a weak equivalence in $\mathsf{Op}_G(\mathsf{sSet})$ thanks to the usual extra degeneracy argument [26, §4.5] (which applies after forgetting to $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$). Therefore, the following result suffices to show that $B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}})$ is a $N\mathcal{F}$-operad.

Proposition 6.67. $B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}}) \in \mathsf{Op}_G(\mathsf{sSet})$ *is cofibrant.*

Proposition 6.67 will follow by analyzing the skeletal filtration of $B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}})$ and showing that the corresponding latching maps, which are built using cubical diagrams, are cofibrations.

Recall that a *n-cube* on sSet is a functor $\mathcal{X}_{(-)} \colon \mathsf{P}_n \to \mathsf{sSet}$ for P_n the poset of subsets of $\underline{n} = \{1, \cdots, n\}$. We call a n-cube a *monomorphism n-cube* if the latching maps

$$\mathrm{colim}_{V \subsetneq U} \mathcal{X}_V = L_U \mathcal{X} \xrightarrow{l_U \mathcal{X}} \mathcal{X}_U$$

are monomorphisms for all $U \in \mathsf{P}_n$. Cubes and monomorphism cubes in $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$ are defined identically.

Remark 6.68. Using model category language, monomorphism n-cubes are the cofibrant objects for the projective model structure on n-cubes. As such, they are characterized as the n-cubes with the left lifting property against maps of n-cubes $\mathcal{Y}_{(-)} \to \mathcal{Z}_{(-)}$ that are levelwise trivial fibrations.

Lemma 6.69. *(a) The monad $\widetilde{\mathbb{F}}_G\colon \mathsf{Set}^{\times Ob(\Sigma_G)} \to \mathsf{Set}^{\times Ob(\Sigma_G)}$ sends monomorphism n-cubes to monomorphism n-cubes.*

(b) Letting $\eta\colon id \to \widetilde{\mathbb{F}}_G$ denote the unit and $A \to B$ be a monomorphism in $\mathsf{Set}^{\times Ob(\Sigma_G)}$, the square

$$\begin{array}{ccc} A & \longrightarrow & \widetilde{\mathbb{F}}_G A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \widetilde{\mathbb{F}}_G f} \\ B & \longrightarrow & \widetilde{\mathbb{F}}_G B \end{array}$$

is a monomorphism square (i.e monomorphism 2-cube).

Proof. Combining (4.2) with the top left functor in (6.66) yields the formula

$$\widetilde{\mathbb{F}}_G X(C) \simeq \coprod_{T \in \mathsf{Iso}(C \downarrow_{\mathsf{r}} \Omega^0_G)} \left(\prod_{v \in V_G(T)} \left(\coprod_{D \in \Sigma_G} \mathsf{Hom}(T_v, D) \times X(D) \right) \right) \cdot_{\mathsf{Aut}(T)} \mathsf{Aut}(C). \tag{6.70}$$

Distributing the inner $\coprod$ over the $\prod$ shows that $\widetilde{\mathbb{F}}_G f$ is a coproduct of monomorphisms with the map $f\colon A \to B$ corresponding to the summand with $C = T = D$, and hence (b) follows.

To show (a), note first that there are three types of operations in (6.70): coproducts, inductions and products. Since coproducts and inductions preserve both colimits and monomorphisms, they preserve monomorphism cubes, and it thus remains to show that so do products. Given monomorphism n-cubes $\mathcal{Y}_{(-)}, \mathcal{Z}_{(-)}$ consider first the $2n$-cube $(\mathcal{Y} \times \mathcal{Z})_{(U,V)} = \mathcal{Y}_U \times \mathcal{Z}_V$. It is straightforward to check that this $2n$-cube has latching maps $l_{(U,V)}\mathcal{Y} \times \mathcal{Z} = l_U \mathcal{Y} \square l_V \mathcal{Z}$, and is thus a monomorphism $2n$-cube. It remains to check that the diagonal n-cube $\Delta^*(\mathcal{Y} \times \mathcal{Z})$ is a monomorphism n-cube. Considering the adjuntion $\Delta^*\colon \mathsf{sSet}^{\mathsf{P}_n \times \mathsf{P}_n} \rightleftarrows \mathsf{sSet}^{\mathsf{P}_n}\colon \Delta_*$ and Remark 6.68 it suffices to check that Δ_* preserves level trivial fibrations of cubes. But this is obvious from the formula $(\Delta_* \mathcal{X})_{(U,V)} = \mathcal{X}_{U \cup V}$. □

proof of Proposition 6.67. We start by analyzing the latching maps for $B_\bullet = B_\bullet(\widetilde{\mathbb{F}}_G, \widetilde{\mathbb{F}}_G, \partial_{\mathcal{F}})$. To describe the n-th latching map, we start with the natural n-cube in $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$ given by $\mathcal{X}^n_U = \widetilde{\mathbb{F}}^{\circ U}_G(\partial_{\mathcal{F}})$ and where maps are induced by the unit $\eta\colon id \to \widetilde{\mathbb{F}}_G$. For example, in $\mathcal{X}^5_{(-)}$, the map $\mathcal{X}^5_{\{1,4\}} \to \mathcal{X}^5_{\{1,3,4,5\}}$ is

$$\widetilde{\mathbb{F}}^{\circ 2}_G(\partial_{\mathcal{F}}) \xrightarrow{\widetilde{\mathbb{F}}_G \eta \widetilde{\mathbb{F}}_G \eta} \widetilde{\mathbb{F}}^{\circ 4}_G(\partial_{\mathcal{F}}).$$

Since degeneracies of $B_\bullet$ are also induced by η, and recalling the notation $\underline{n} = \{1, \cdots, n\}$ for the maximum in P_n, one has that the n-th latching map of $B_\bullet$ is given by

$$\check{l}_n B_\bullet = \check{l}_{\underline{n}}(\widetilde{\mathbb{F}}_G \mathcal{X}^n) \simeq \widetilde{\mathbb{F}}_G(l_{\underline{n}} \mathcal{X}^n) \tag{6.71}$$

where the check decoration on $\check{l}$ for the two leftmost latching maps indicates that the colimits defining those latching maps are taken in $\mathsf{Op}_G(\mathsf{Set})$, while the rightmost latching map is computed in $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$.

The key to the proof is the claim that the maps $l_{\underline{n}} \mathcal{X}^n$ are monomorphisms. This will follow from the stronger claim that the $\mathcal{X}^n$ are monomorphim n-cubes, which we argue by induction on n. When $n = 0$ there is nothing to show. Otherwise, for any $U \subsetneq \{1, \cdots, n, n+1\}$ the restriction of $\mathcal{X}^{n+1}$ to subsets of U is isomorphic to the cube $\mathcal{X}^{|U|}$, so that we need only analyze the top latching map $l_{\underline{n+1}} \mathcal{X}^{n+1}$. We now write $\mathcal{X}^{n+1} = (\mathcal{X}^n \to \widetilde{\mathbb{F}}_G \mathcal{X}^n)$, regarding the $(n+1)$-cube as a map of n-cubes. The top latching map $l_{\underline{n+1}} \mathcal{X}^{n+1}$ is then the latching map of the composite square (the check decoration $\check{L}$ again denotes a latching object computed in $\mathsf{Op}_G(\mathsf{Set})$)

$$\begin{tikzcd}
L_{\underline{n}}\mathcal{X}^n \arrow[r, equal] \arrow[d] & L_{\underline{n}}\mathcal{X}^n \arrow[r] \arrow[d] & \mathcal{X}^n_{\underline{n}} \arrow[d] \\
\check{L}_{\underline{n}}(\widetilde{\mathbb{F}}_G \mathcal{X}^n) \arrow[r] & \widetilde{\mathbb{F}}_G(L_{\underline{n}}\mathcal{X}^n) \arrow[r] & \widetilde{\mathbb{F}}_G \mathcal{X}^n_{\underline{n}}
\end{tikzcd} \tag{6.72}$$

The latching map in the rightmost square (6.72) is a monomorphism since it is an instance of Lemma 6.69(b) applied to the map $l_{\underline{n}}\mathcal{X}^n : L_{\underline{n}}\mathcal{X}^n \to \mathcal{X}^n_{\underline{n}}$, which is a monomorphism by the induction hypothesis. On the other hand, writing $\tilde{\mathcal{X}}^n$ for the cube obtained from $\mathcal{X}^n$ by replacing the top level $\mathcal{X}^n_{\underline{n}}$ with $L_{\underline{n}}\mathcal{X}^n$, the left bottom horizontal map in (6.72) can be described as $\check{l}_{\underline{n}}(\widetilde{\mathbb{F}}_G \tilde{\mathcal{X}}^n) \simeq \widetilde{\mathbb{F}}_G(l_{\underline{n}}\tilde{\mathcal{X}}^n)$ (compare with (6.71)), which is a monomorphism by Lemma 6.69(a). Hence the latching maps in both squares in (6.72) are monomorphisms, and thus so is the latching map of the composite square, showing that $l_{\underline{n+1}}\mathcal{X}^{n+1}$ is a monomorphism, as desired.

To finish the proof, one now simply notes that the skeletal filtration of $B_\bullet$ is then iteratively described by the pushouts in $\mathsf{Op}_G(\mathsf{sSet})$ below, where the vertical maps are cofibrations in $\mathsf{Op}_G(\mathsf{sSet})$ since the maps $l_{\underline{n}}\mathcal{X}^n : L_{\underline{n}}\mathcal{X}^n \to \mathcal{X}^n_{\underline{n}}$ are monomorphisms.

$$\begin{tikzcd}
\widetilde{\mathbb{F}}_G(L_{\underline{n}}\mathcal{X}^n \times \Delta^n \amalg_{L_{\underline{n}}\mathcal{X}^n \times \partial\Delta^n} \mathcal{X}^n_{\underline{n}} \times \partial\Delta^n) \arrow[r] \arrow[d] & \mathsf{sk}_{n-1}B_\bullet \arrow[d] \\
\widetilde{\mathbb{F}}_G(\mathcal{X}^n_{\underline{n}} \times \Delta^n) \arrow[r] & \mathsf{sk}_n B_\bullet
\end{tikzcd}$$

□

Remark 6.73. We now address the "moreover" claim in Corollary IV. For any $\mathcal{O} \in \mathsf{Op}^G(\mathsf{sSet})$ one has $\pi_0(\iota_*\mathcal{O}) \in \mathsf{Op}_G(\mathsf{Set})$. Therefore, if $\mathcal{O}$ has fixed points as in (1.15) then $\pi_0(\iota_*\mathcal{O}) = \delta_{\mathcal{F}}$ for $\mathcal{F} = \{\mathcal{F}_n\}_{n\geq 0}$ a collection of families of graph subgroups. But the condition that $\delta_{\mathcal{F}} \in \mathsf{Op}_G(\mathsf{Set})$ simply repackages Definition 4.49.

Remark 6.74. If one appends the adjunction $\iota^* : \mathsf{Op}_G(\mathsf{Set}) \rightleftarrows \mathsf{Op}^G(\mathsf{Set}) : \iota_*$ to (6.66) one obtains an additional composite monad $\widehat{\mathbb{F}}_G$ on $\mathsf{Set}^{\times \mathrm{Ob}(\Sigma_G)}$. Moreover, Lemma 6.59 guarantees that the top composite in (6.66) lands in the essential image of ι_*, so that the monads $\widetilde{\mathbb{F}}_G$ and $\widehat{\mathbb{F}}_G$ are in fact isomorphic. This observation now hints at how one can build a model for $N\mathcal{F}$-operads directly in terms of (regular) equivariant operads, i.e. without making use of genuine equivariant operads. Namely, consider the adjunctions

$$\prod_{n\geq 0} \mathsf{Set}^{\times \mathrm{Ob}\left(\mathsf{O}^{op}_{\mathcal{F}^\Gamma_n}\right)} \rightleftarrows \prod_{n\geq 0} \mathsf{Set}^{\mathsf{O}^{op}_{\mathcal{F}^\Gamma_n}} \underset{\iota_*}{\overset{\iota^*}{\rightleftarrows}} \mathsf{Sym}^G(\mathsf{Set}) \overset{\mathbb{F}}{\rightleftarrows} \mathsf{Op}^G(\mathsf{Set}) \tag{6.75}$$

Abusing notation by again writing $\widehat{\mathbb{F}}_G$ for the composite monad and $\delta_{\mathcal{F}}$ for the obvious object on the leftmost category, it is not hard to use the equivalence in Lemma 6.43 to leverage our analysis so as to conclude that the bar construction $B_\bullet(\widehat{\mathbb{F}}_G, \widehat{\mathbb{F}}_G, \partial_{\mathcal{F}})$ built using (6.75) is also a cofibrant $N\mathcal{F}$-operad.

This latter model may seem deceptively simple. However, it is not easy to prove directly that $B_\bullet(\widehat{\mathbb{F}}_G, \widehat{\mathbb{F}}_G, \partial_{\mathcal{F}})$ is a $N\mathcal{F}$-operad, since as it turns out the required claim that $\partial_{\mathcal{F}}$ is a $\widehat{\mathbb{F}}_G$-algebra is itself not obvious. More precisely, the issue is that in building $\widehat{\mathbb{F}}_G$ one must compute fixed points of free operads, which is a non trivial task. In the present paper, this fixed point analysis is built into Lemma 6.59. Alternatively, a more direct fixed point analysis is given by Rubin in [27] and, in fact, the key technical analysis therein is tantamount to the claim that $\partial_{\mathcal{F}}$ is indeed a $\widehat{\mathbb{F}}_G$-algebra.

A Transferring Kan extensions

The purpose of this appendix is to provide the somewhat long proof of Proposition 5.37, which is needed when repackaging free extensions of genuine equivariant operads in (5.7).

We start with a more detailed discussion of the realization functor $|-|$ defined by the adjunction

$$|-|\colon \mathsf{Cat}^{\Delta^{op}} \rightleftarrows \mathsf{Cat}\colon (-)^{[\bullet]}$$

in Definition 5.35. More explicitly, one has

$$|\mathcal{I}_\bullet| = coeq\left(\coprod_{[n]\to[m]} [n]\times\mathcal{I}_m \rightrightarrows \coprod_{[n]} [n]\times\mathcal{I}_n \right). \tag{A.1}$$

Example A.2. Any $\mathcal{I}\in\mathsf{Cat}$ induces objects $\mathcal{I},\mathcal{I}_\bullet,\mathcal{I}^{[\bullet]}\in\mathsf{Cat}^{\Delta^{op}}$ where $\mathcal{I}$ is the constant simplicial object and $\mathcal{I}_\bullet$ is the nerve $N\mathcal{I}$ with each level regarded as a discrete category. It is straightforward to check that $|\mathcal{I}|\simeq|\mathcal{I}_\bullet|\simeq|\mathcal{I}^{[\bullet]}|\simeq\mathcal{I}$.

Lemma A.3. *Given $\mathcal{I}_\bullet\in\mathsf{Cat}^{\Delta^{op}}$ one has an identification $Ob(|\mathcal{I}_\bullet|)\simeq Ob(\mathcal{I}_0)$. Furthermore, the arrows of $|\mathcal{I}_\bullet|$ are generated by the image of the arrows in $\mathcal{I}_0\simeq\mathcal{I}_0\times[0]$ and the image of the arrows in $[1]\times Ob(\mathcal{I}_1)$.*

For each $i_1\in\mathcal{I}_1$, we will denote the arrow of $|\mathcal{I}_\bullet|$ induced by the arrow in $[1]\times\{i_1\}$ by

$$d_1(i_1)\xrightarrow{i_1} d_0(i_1).$$

Proof. We write $d_{\hat{k}}$, $d_{\hat{k},\hat{l}}$ for the simplicial operators induced by the maps $[0]\xrightarrow{0\mapsto k}[n]$, $[1]\xrightarrow{0\mapsto k,1\mapsto l}[n]$ which can informally be thought of as the "composite of all faces other than d_k, d_l". Using (A.1) one has equivalence relations between the objects $(k,i_n)\in[n]\times\mathcal{I}_n$ and $(0,d_{\hat{k}}(i_n))\in[0]\times\mathcal{I}_0$ and since for any generating relation $(k,i_n)\sim(l,i'_m)$ it is $d_{\hat{k}}(i_n)=d_{\hat{l}}(i'_m)$ the identification $\mathrm{Ob}(|\mathcal{I}_\bullet|)\simeq\mathrm{Ob}(\mathcal{I}_0)$ follows.

To verify the claim about generating arrows, note that any arrow of $[n]\times\mathcal{I}_n$ factors as

$$(k,i_n)\to(l,i_n)\xrightarrow{I_n}(l,i'_n) \tag{A.4}$$

for $I_n\colon i_n\to i'_n$ an arrow of $\mathcal{I}_n$. The $d_{\hat{l}}$ relation identifies the right arrow in (A.4) with $(0,d_{\hat{l}}(i_n))\xrightarrow{d_{\hat{l}}(I_n)}(0,d_{\hat{l}}(i'_n))$ in $[0]\times\mathcal{I}_0$ while (if $k<l$) the $d_{\hat{k},\hat{l}}$ relation identifies the left arrow with $(0,d_{\hat{k},\hat{l}}(i_n))\to(1,d_{\hat{k},\hat{l}}(i_n))$ in $[1]\times\mathcal{I}_1$. The result follows. ☐

Remark A.5. Given $\mathcal{I}_\bullet\in\mathsf{Cat}^{\Delta^{op}}$, $\mathcal{C}\in\mathsf{Cat}$, the isomorphisms

$$\mathsf{Hom}_{\mathsf{Cat}}(|\mathcal{I}_\bullet|,\mathcal{C})\simeq\mathsf{Hom}_{\mathsf{Cat}^{\Delta^{op}}}\left(\mathcal{I}_\bullet,\mathcal{C}^{[\bullet]}\right)$$

together with the fact that $\mathcal{C}^{[\bullet]}$ is 2-coskeletal show that $|\mathcal{I}_\bullet|$ is determined by the categories $\mathcal{I}_0,\mathcal{I}_1,\mathcal{I}_2$ and maps between them, i.e. by the truncation of formula $(A.1)$ for $n,m\le 2$.

Indeed, one can show that a sufficient set of generating relations for $|\mathcal{I}_\bullet|$ is given by: (i) the relations in $\mathcal{I}_0$ (including relations stating that identities of $\mathcal{I}_0$ are identities of $|\mathcal{I}_\bullet|$); (ii) relations stating that for each $i_0\in\mathcal{I}_0$ the arrow $i_0=d_1(s_0(i_0))\xrightarrow{s_0(i_0)}d_1(s_0(i_0))=i_0$ is an identity; (iii) for each arrow $I_1\colon i_1\to i'_1$ in $\mathcal{I}_1$ the relation that the square below commutes

$$\begin{tikzcd}
d_1(i_1) \arrow[r, "i_1"] \arrow[d, "d_1(I_1)"'] & d_0(i_1) \arrow[d, "d_0(I_1)"] \\
d_1(i'_1) \arrow[r, "i'_1"] & d_0(i'_1)
\end{tikzcd}$$

and; (iv) for each object $i_2 \in \mathcal{I}_2$ the relation that the following triangle commutes.

$$\begin{tikzcd} d_{1,2}(i_2) \arrow[rr, "d_1(i_2)"] \arrow[dr, "d_2(i_2)"'] & & d_{0,1}(i_2) \\ & d_{0,2}(i_2) \arrow[ur, "d_0(i_2)"'] & \end{tikzcd}$$

We now relate diagrams in the span categories of §4.3 with the Grothendieck constructions of Definition 2.2.

Lemma A.6. *Functors $F: \mathcal{D} \ltimes \mathcal{I}_\bullet \to \mathcal{C}$ are in bijection with lifts*

$$\begin{tikzcd} & \mathsf{WSpan}^l(*, \mathcal{C}) \arrow[d, "\mathsf{fgt}"] \\ \mathcal{D} \arrow[ur, dashed, "\mathcal{I}_\bullet^F"] \arrow[r, "\mathcal{I}_\bullet"'] & \mathsf{Cat}. \end{tikzcd}$$

where fgt *is the functor forgetting the maps to* $*$ *and* $\mathcal{C}$*.*

Proof. This is a matter of unpacking notation. The restrictions $F|_{\mathcal{I}_d}$ to the fibers $\mathcal{I}_d \hookrightarrow \mathcal{D} \ltimes \mathcal{I}_\bullet$ are precisely the functors $\mathcal{I}_d^F: \mathcal{I}_d \to \mathcal{C}$ describing $\mathcal{I}_\bullet^F(d)$.

Furthermore, the images $F((d,i) \to (d', f_*(i)))$ of the pushout arrows over a fixed arrow $f: d \to d'$ of $\mathcal{D}$ assemble to a natural transformation

$$\begin{tikzcd} \mathcal{I}_d \arrow[d, "f_*"'] \arrow[dr, "I_d^F"] \arrow[d, phantom, shift left=4ex, "\swArrow"] & \\ \mathcal{I}_{d'} \arrow[r, "I_{d'}^F"'] & \mathcal{C} \end{tikzcd}$$

which describes $\mathcal{I}_\bullet^F(f)$. One readily checks that the associativity and unitality conditions coincide. □

In the cases of interest we have $\mathcal{D} = \Delta^{op}$. The following is the key result in this section.

Proposition A.7. *Let $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then there is a natural functor*

$$\Delta^{op} \ltimes \mathcal{I}_\bullet \xrightarrow{\ s\ } |\mathcal{I}_\bullet|.$$

Further, s is final.

Remark A.8. The s in the result above stands for *source*. This is because, for $\mathcal{I} \in \mathsf{Cat}$, the map $\Delta^{op} \ltimes \mathcal{I}^{[\bullet]} \to |\mathcal{I}^{[\bullet]}| \simeq \mathcal{I}$ is given by $s(i_0 \to \cdots \to i_n) = i_0$.

Proof. Recall that $|\mathcal{I}_\bullet|$ is the coequalizer (A.1). Given $(k, g_m) \in [n] \times \mathcal{I}_m$, we write $[k, g_m]$ for the corresponding object in $|\mathcal{I}_\bullet|$. To simplify notation, we write objects of $\mathcal{I}_n$ as i_n and implicitly assume that $[k, i_n]$ refers to the class of the object $(k, i_n) \in [n] \times \mathcal{I}_n$.

We define s on objects by $s([n], i_n) = [0, i_n]$ and on an arrow $(\phi, I_m): (n, i_n) \to (m, i'_m)$ as the composite (note that $\phi: [m] \to [n]$ and $I_m: \phi^* i_n \to i_m$)

$$[0, i_n] \to [\phi(0), i_n] = [0, \phi^* i_n] \xrightarrow{I_m} [0, i'_m]. \tag{A.9}$$

To check compatibility with composition, the cases of a pair of either two fiber arrows (i.e. arrows where ϕ is the identity) or two pushforward arrows (i.e. arrows where I_m is the identity) are immediate from (A.9), hence we are left with the case $([n], i_n) \xrightarrow{I_n} ([n], i'_n) \to ([m], \phi^* i'_n)$ of a fiber arrow followed by a pushforward arrow. Noting that in $\Delta^{op} \ltimes \mathcal{I}_\bullet$ this composite can be rewritten as $([n], i_n) \to ([m], \phi^* i_n) \xrightarrow{\phi^* I_n} ([m], \phi^* i'_n)$ this amounts to checking that

$$\begin{tikzcd}
{[0, i_n]} \arrow[r] \arrow[d, "I_n"'] & {[\phi(0), i_n)]} \arrow[r, equal] \arrow[d, "I_n"'] & {[0, \phi^* i_n]} \arrow[d, "\phi^* I_n"] \\
{[0, i'_n]} \arrow[r] & {[\phi(0), i'_n]} \arrow[r, equal] & {[0, \phi^* i_n]}
\end{tikzcd}$$

commutes in $|\mathcal{I}_\bullet|$, which is the case since the left square is encoded by a square in $[n] \times \mathcal{I}_n$ and the right square is encoded by an arrow in $[m] \times \mathcal{I}_n$.

We now show that s is final. Fix $h \in \mathcal{I}_0$. We must check that $[0, h] \downarrow \Delta^{op} \ltimes \mathcal{I}_\bullet$ is connected. By Lemma A.3 any object in this undercategory has a description (not necessarily unique) as a pair

$$\left(([n], i_n), [0, h] \xrightarrow{f_1} \cdots \xrightarrow{f_r} s([n], i_n) \right) \tag{A.10}$$

where each f_i is a generating arrow of $|\mathcal{I}_\bullet|$ induced by either an arrow I_0 of $\mathcal{I}_0$ or object $i_1 \in \mathcal{I}_1$. We will connect (A.10) to the canonical object $(([0], h), [0, h] = [0, h])$, arguing by induction on r. If $n \neq 0$, the map $d_{\hat{0}}: ([n], i_n) \to ([0], d^*_{\hat{0}}(i_n))$ and the fact that $s\left(d^*_{\hat{0}}\right) = id_{[0, d^*_{\hat{0}}(i_n)]}$ provides an arrow to an object with $n = 0$ without changing r. If $n = 0$, one can apply the induction hypothesis by lifting f_r to $\Delta^{op} \ltimes \mathcal{I}_\bullet$ according to one of two cases: (i) if f_r is induced by an arrow I_0 of $\mathcal{I}_0$, the lift of f_r is simply $([0], i'_0) \xrightarrow{I_0} ([0], i_0)$; (ii) if f_r is induced by $i_1 \in \mathcal{I}_1$ the lift is provided by the map $([1], i_1) \to ([0], d_0(i_1))$. $\square$

Remark A.11. The involution

$$\Delta \xrightarrow{\tau} \Delta$$

which sends $[n]$ to itself and d_i, s_i to d_{n-i}, s_{n-i} induces vertical isomorphisms

$$\begin{tikzcd}
\Delta^{op} \ltimes (\mathcal{I}_\bullet \circ \tau) \arrow[r, "s"] \arrow[d, "\simeq"'] & {|\mathcal{I}_\bullet \circ \tau|} \arrow[d, "\simeq"] \\
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[r, "t"'] & {|\mathcal{I}^{op}_\bullet|^{op}}
\end{tikzcd}$$

which reinterpret the "source" functor as what one might call the "target" functor, with $t([n], i_n) = [n, i_n]$ rather than $s([n], i_n) = [0, i_n]$. The target functor is thus also final.

Moreover, the source/target formulations of all the results that follow are equivalent.

In practice, we will need to know that the source s and target t satisfy a stronger finality condition with respect to left Kan extensions.

Lemma A.12. *Let $\mathcal{J} \in \mathsf{Cat}$ be a small category and $j \in \mathcal{J}$. Then the under and over category functors*

$$\mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{(-)\downarrow j} \mathsf{Cat}, \qquad \mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{j \downarrow (-)} \mathsf{Cat}$$

are left adjoints, and hence preserve colimits.

Proof. The right adjoint to $(-) \downarrow j$, which we denote $(-)^{\downarrow j}: \mathsf{Cat} \to \mathsf{Cat} \downarrow \mathcal{J}$, is given on a category $\mathcal{C} \in \mathsf{Cat}$ by the Grothendieck construction $\mathcal{C}^{\downarrow j} = \mathcal{J} \ltimes \mathcal{C}^{\times \mathcal{J}(-, j)}$ for the functor

$$\begin{aligned} \mathcal{J} &\longrightarrow \mathsf{Cat} \\ k &\longmapsto \mathcal{C}^{\times \mathcal{J}(k, j)}. \end{aligned}$$

Given $(\mathcal{I} \xrightarrow{\pi} \mathcal{J}) \in (\mathsf{Cat} \downarrow \mathcal{J})$ and $\mathcal{C} \in \mathsf{Cat}$ we will show that functors $F: (\mathcal{I} \downarrow j) \to \mathcal{C}$ are in bijection with functors $\hat{F}: \mathcal{I} \to \mathcal{C}^{\downarrow j}$ over $\mathcal{J}$. Given F, we now describe the corresponding $\hat{F}$.

First, F associates to each object $(i, J: \pi(i) \to j)$ of $\mathcal{I} \downarrow j$ an object $F(i, J) \in \mathcal{C}$. Write $F_i \in \mathcal{C}^{\times \mathcal{J}(\pi(i), j)}$ for the assignment $J \mapsto F(i, J)$, i.e. $F_i(J) = F(i, J)$. On objects $i \in \mathcal{I}$ one then sets $\hat{F}(i) = (\pi(i), F_i)$.

Next, recall that arrows in $\mathcal{I} \downarrow j$ have the form $(i', J\circ\pi(I)) \to (i, J)$ for some arrow $I: i' \to i$ in $\mathcal{I}$. To each such arrow, F associates an arrow $F_{i'}(J\circ\pi(I)) \to F_i(J)$. Fixing I and allowing $J \in \mathcal{J}(\pi(i), j)$ to vary these arrows form a natural transformation $F_I: F_{i'} \circ \pi(I)^* \Rightarrow F_i$, where $\pi(I)^*: \mathcal{J}(\pi(i), j) \to \mathcal{J}(\pi(i'), j)$ denotes precomposition with $\pi(I)$. On arrows $I: i' \to i$ one now sets $\hat{F}(I): (\pi(i'), F_{i'}) \to (\pi(i), F_i)$ to be $(\pi(I): \pi(i') \to \pi(i), F_I: F_{i'} \circ \pi(I)^* \Rightarrow F_i)$.

It is clear that the procedures above relating the values of $F, \hat{F}$ on objects and arrows are invertible. One can readily check that the functoriality requirements on $F, \hat{F}$ match.

Noting that $j \downarrow (-)$ is the composite $\mathsf{Cat} \downarrow \mathcal{J} \xrightarrow{(-)^{op}} \mathsf{Cat} \downarrow \mathcal{J}^{op} \xrightarrow{(-)\downarrow j} \mathsf{Cat} \xrightarrow{(-)^{op}} \mathsf{Cat}$ yields that its right adjoint is the composite $\mathsf{Cat} \xrightarrow{(-)^{op}} \mathsf{Cat} \xrightarrow{(-)^{\downarrow j}} \mathsf{Cat} \downarrow \mathcal{J}^{op} \xrightarrow{(-)^{op}} \mathsf{Cat} \downarrow \mathcal{J}$. $\square$

Corollary A.13. *Consider a map $\mathcal{I}_\bullet \to \mathcal{J}$ between $\mathcal{I}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$ and a constant object $\mathcal{J} = \mathcal{J}_\bullet \in \mathsf{Cat}^{\Delta^{op}}$. Then the source and target maps*

$$
\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "s"] \arrow[dr] & & {|\mathcal{I}_\bullet|} \arrow[dl] \\
& \mathcal{J} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "t"] \arrow[dr] & & {|\mathcal{I}_\bullet^{op}|^{op}} \arrow[dl] \\
& \mathcal{J} &
\end{tikzcd}
$$

are Lan*-final over $\mathcal{J}$, i.e. the functors $s \downarrow j: (\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \to |\mathcal{I}_\bullet| \downarrow j$ are final for all $j \in \mathcal{J}$, and similarly for t.*

Proof. It is clear that $(\Delta^{op} \ltimes \mathcal{I}_\bullet) \downarrow j \simeq \Delta^{op} \ltimes (\mathcal{I}_\bullet \downarrow j)$ while Lemma A.12 guarantees that, since $(-) \downarrow j$ is a left adjoint, $|\mathcal{I}_\bullet| \downarrow j \simeq |\mathcal{I}_\bullet \downarrow j|$. One thus reduces to Proposition A.7. $\square$

We will require two additional straightforward lemmas.

Lemma A.14. *Let $\mathcal{I}_\bullet^F \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ be such that the diagrams*

$$
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] \arrow[dr, phantom, shift right=3ex, "\Uparrow \delta_i"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] \arrow[dr, phantom, shift right=3ex, "\Uparrow \sigma_j"] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C}
\end{tikzcd}
\tag{A.15}
$$

are given by natural isomorphisms for $0 < i \le n$, $0 \le j \le n$. Then the functors $\tilde{F}_n: \mathcal{I}_n \to \mathcal{C}$ given by the composites

$$\mathcal{I}_n \xrightarrow{d_{1,\cdots,n}} \mathcal{I}_0 \xrightarrow{F_0} \mathcal{C}$$

assemble to an object $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ which is isomorphic to $\mathcal{I}_\bullet^F$ and such that the corresponding diagrams (A.15) for $0 < i \le n$, $0 \le j \le n$ are strictly commutative.

Dually, if (A.15) are natural isomorphisms for $0 \le i < n$ and $0 \le j \le n$ one can form $\mathcal{I}_\bullet^{\tilde{F}} \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ such that the corresponding diagrams are strictly commutative.

Proof. This follows by a straightforward verification. $\square$

Lemma A.16. *A (necessarily unique) factorization*

$$
\begin{tikzcd}
\Delta^{op} \ltimes \mathcal{I}_\bullet \arrow[rr, "F_\bullet"] \arrow[dr, "s"'] & & \mathcal{C} \\
& {|\mathcal{I}_\bullet|} \arrow[ur, dashed, "F"'] &
\end{tikzcd}
\tag{A.17}
$$

exists iff for the associated object $\mathcal{I}_\bullet \in \mathsf{Span}(*, \mathcal{C})^{\Delta^{op}}$ (cf. Lemma A.6) all faces d_i for $0 < i \le n$ and degeneracies s_j for $0 \le j \le n$ are strictly commutative, i.e. they are given by diagrams

$$
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_0"'] \arrow[dr, "F_n"] \arrow[dr, phantom, shift right=3ex, "\Uparrow \varphi_n"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "d_i"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n-1} \arrow[r, "F_{n-1}"'] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{I}_n \arrow[d, "s_j"'] \arrow[dr, "F_n"] & \\
\mathcal{I}_{n+1} \arrow[r, "F_{n+1}"'] & \mathcal{C}
\end{tikzcd}
\tag{A.18}
$$

Dually, a factorization through the target $t\colon \Delta^{op} \ltimes \mathcal{I}_\bullet \to |\mathcal{I}_\bullet^{op}|^{op}$ exists iff the faces d_i and degeneracies s_j are strictly commutative for $0 \le i < n$, $0 \le j \le n$.

Proof. For the "only if" direction, it suffices to note that s sends all pushout arrows of $\Delta^{op} \ltimes \mathcal{I}_\bullet$ for faces d_i, $0 < i \le n$ and degeneracies s_j, $0 \le j \le n$ to identities, yielding the required commutative diagrams in (A.18).

For the "if" direction, this will follow by building a functor $\mathcal{I}_\bullet \xrightarrow{\bar{F}_\bullet} \mathcal{C}^{[\bullet]}$ together with the naturality of the source map s (recall that $|\mathcal{C}^{[\bullet]}| \simeq \mathcal{C}$). We define $\bar{F}_n|_{k\to k+1}$ as the map

$$F_{n-k}d_{0,\cdots,k-1} \xrightarrow{\varphi_{n-k}d_{0,\cdots,k-1}} F_{n-k-1}d_{0,\cdots,k}. \tag{A.19}$$

The claim that $s \circ (\Delta^{op} \ltimes \bar{F})$ recovers the horizontal map in (A.17) is straightforward, hence the real task is to prove that (A.19) defines a map of simplicial objects. First, functoriality of the original $F_\bullet$ yields identities

$$\varphi_{n-1}d_i = \varphi_n,\ \ 1 < i \qquad \varphi_{n-1}d_1 = (\varphi_{n-1}d_0) \circ \varphi_n, \qquad \varphi_{n+1}s_i = \varphi_n,\ \ 0 < i, \qquad \varphi_{n+1}s_0 = id_{F_n}$$

Next, note that there is no ambiguity in writing simply $\varphi_{n-k}d_{0,\cdots,k-1}$ to denote the map (A.19). We now check that $\bar{F}_{n-1}d_i = d_i\bar{F}_n$, $0 \le i \le n$, which must be verified after restricting to each $k \to k+1$, $0 \le k \le n-2$. There are three cases, depending on i and k:

$(i < k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_{0,\cdots,k}$;

$(i = k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_1d_{0,\cdots,k-1} = (\varphi_{n-k-1}d_0 \circ \varphi_{n-k})d_{0,\cdots,k-1} = (\varphi_{n-k-1}d_{0,\cdots,k}) \circ (\varphi_{n-k}d_{0,\cdots,k-1})$;

$(i > k+1)$ $\varphi_{n-k-1}d_{0,\cdots,k-1}d_i = \varphi_{n-k-1}d_{i-k}d_{0,\cdots,k-1} = \varphi_{n-k}d_{0,\cdots,k-1}$.

The case of degeneracies is similar. □

proof of Proposition 5.37. The result follows from the following string of identifications.

$$\begin{aligned}\lim_{\Delta}\left(\mathsf{Ran}_{A_n\to\Sigma_G}N_n\right) \simeq& \mathsf{Ran}_{\Delta\times\Sigma_G\to\Sigma_G}\left(\mathsf{Ran}_{A_n\to\Sigma_G}N_n\right) \simeq\\ \simeq& \mathsf{Ran}_{\Delta\times\Sigma_G\to\Sigma_G}\left(\mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Delta\times\Sigma_G}N_\bullet\right) \simeq\\ \simeq& \mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Sigma_G}N_\bullet \simeq \mathsf{Ran}_{(\Delta^{op}\ltimes A_\bullet^{op})^{op}\to\Sigma_G}\tilde{N}_\bullet \simeq \mathsf{Ran}_{|A_\bullet|\to\Sigma_G}\tilde{N}\end{aligned}$$

The first step simply rewrites $\lim_\Delta$. The second step follows from Proposition 2.5 applied to the map $(\Delta^{op} \ltimes A_\bullet^{op})^{op} \to \Delta \times \Sigma_G$ of Grothendieck fibrations over Δ, since for each $(n, C) \in \Delta \times \Sigma_G$ one has a natural identification between $(n, a) \downarrow_\pi (\Delta^{op} \ltimes A_\bullet^{op})^{op}$ and $C \downarrow A_n$. The third step follows since iterated Kan extensions are again Kan extensions. The fourth step twists $N_\bullet$ as in Lemma A.14 to obtain $\tilde{N}_\bullet$ such that the d_i, s_j are given by strictly commutative diagrams for $0 \le i < n$, $0 \le j \le n$. Lastly, the final step uses Lemma A.16 to conclude that $\tilde{N}_\bullet$ factors through the target functor t, obtaining $\tilde{N}$, and then uses Corollary A.13 to conclude that the Kan extensions indeed coincide. □

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