Graph stable equivalences and operadic constructions in equivariant spectra

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Abstract

One of the key insights of the work of Blumberg and Hill in [2] is that, when dealing with operads on genuine G-spectra, the homotopy theory of algebras over an operad is not merely determined by the operad up to levelwise G-equivalence.

In this paper we introduce a notion of G-graph stable equivalence for operads in G-spectra, then show that the maps of operads inducing Quillen equivalences between the corresponding algebra categories are precisely the G-graph stable equivalences.

1 Main results

The following are our key results concerning operads in genuine G-spectra.

Theorem 1.1. *Let $\mathcal{O}$ be any operad in $(\mathsf{Sp}^\Sigma)^G$, and let $(\mathsf{Sp}^\Sigma)^G$, Sym^G be equipped with the respective positive S G-graph stable model structures.*

Then the respective induced projective model structures on $\mathsf{Alg}_\mathcal{O}$, $\mathsf{Mod}^l_\mathcal{O}$ exist, and these are simplicial model structures.

Further, if $\mathcal{O} \to \bar{\mathcal{O}}$ is a G-graph stable equivalence of operads then the induce-forget adjunctions

$$\bar{\mathcal{O}} \circ_\mathcal{O} (-) \colon \mathsf{Alg}_\mathcal{O} \rightleftarrows \mathsf{Alg}_{\bar{\mathcal{O}}} \colon \mathsf{fgt}, \qquad \bar{\mathcal{O}} \circ_\mathcal{O} (-) \colon \mathsf{Mod}^l_\mathcal{O} \rightleftarrows \mathsf{Mod}^l_{\bar{\mathcal{O}}} \colon \mathsf{fgt} \tag{1.2}$$

are Quillen equivalences.

We will refer to the model structures in Theorem 1.1 as the projective positive S G-graph stable model structures.

The following result provides a converse to the second part of Theorem 1.1 by showing that (1.2) is only a Quillen adjunction when $\mathcal{O} \to \bar{\mathcal{O}}$ is a G-graph stable equivalence.

Theorem 1.3. *Let $f\colon A \to B$ be a map in $(\mathsf{Sp}^\Sigma)^{G\times\Sigma_n}$. Then the maps*

$$A \wedge_{\Sigma_n} (-)^{\wedge n} \colon (\mathsf{Sp}^\Sigma)^G \to (\mathsf{Sp}^\Sigma)^G \quad , \quad B \wedge_{\Sigma_n} (-)^{\wedge n} \colon (\mathsf{Sp}^\Sigma)^G \to (\mathsf{Sp}^\Sigma)^G$$

are left derivable and the induced natural transformation

$$f \wedge^L_{\Sigma_n} (-)^{\wedge n} \colon A \wedge^L_{\Sigma_n} (-)^{\wedge n} \to B \wedge^L_{\Sigma_n} (-)^{\wedge n}$$

of left derived functors is a levelwise G-stable equivalence if and only if f is a G-graph stable equivalence.

Theorem 1.4. *Let $\mathcal{O}$ be an operad in $(\mathsf{Sp}^{\Sigma})^G$ and consider the relative composition product*

$$\mathsf{Mod}^r_{\mathcal{O}} \times \mathsf{Mod}^l_{\mathcal{O}} \xrightarrow{-\circ_{\mathcal{O}}-} \mathsf{Sym}^G.$$

Regard $\mathsf{Mod}^l_{\mathcal{O}}$ as equipped with the projective positive S G-graph stable model structure and Sym^G as equipped with the S G-graph stable model structure.

Suppose f_2 is a cofibration between cofibrant objects in $\mathsf{Mod}^l_{\mathcal{O}}$. Then if the map f_1 in $\mathsf{Mod}^r_{\mathcal{O}}$ is an underlying cofibration (respectively monomorphism) in Sym^G, then so is the pushout product

$$f_1 \square^{\circ_{\mathcal{O}}} f_2$$

with respect to $\cdot \circ_{\mathcal{O}} \cdot$. Further, this map is also a w.e. if either f_1 or f_2 are.

Proofs of Theorems 1.1, 1.3 and 1.4 can be found in subsection 4.

2 G-graph stable model structures

2.1 Definitions

We first set some notation. Throughout the paper we let Sp^{Σ} denote the category of simplicial symmetric spectra. For a more detailed discussion, see [17].

In this paper we will be interested in spectra which are acted on simultaneously by two groups G and H with differing roles. This is reflected in the notion of G-graph stable equivalence, which we now define.

Definition 2.1. Let $(\mathsf{Sp}^{\Sigma})^{G\times H}$ be the category of $G \times H$-spectra. A *G-graph stable equivalence* is a map which restricts to a genuine W-stable equivalence for any graph subgroup $W \subset G \times H$ associated to a homomorphism $\varphi: \bar{G} \to H$ with domain a subgroup $\bar{G} \subset G$.

Further, we will care about the following notions of cofibration in $(\mathsf{Sp}^{\Sigma})^{G\times H}$.

Definition 2.2. Let $(\mathsf{Sp}^{\Sigma})^{G\times H}$ be the category of $G \times H$-spectra.

- a *monomorphism* is a map which is a monomorphism of simplicial sets at all spectra levels;
- a *S cofibration* (respectively, positive S cofibration) is a map which forgets to a underlying S cofibration (respectively, positive S cofibration) in Sp^{Σ}.

Theorem 2.3. *There exist cofibrantly generated simplicial model categories on $(\mathsf{Sp}^{\Sigma})^{G\times H}$ with w.e.s the G-graph stable equivalences and cofibrations either the monomorphisms or the (possibly positive) S cofibrations.*

The proof of Theorem 2.3 can be found in Appendix A.

Remark 2.4. In proving our main results we also make use of an additional type of cofibrations. Namely, when dealing with the category $(\mathsf{Sp}^{\Sigma})^{G\times H\times T}$, we define S $\Sigma\times G\times H$-inj T-proj cofibrations to be those maps which after forgetting the $G \times H$ action become S Σ-inj T-proj cofibrations in the sense of [17]. One can readily modify the proof of Theorem 2.3 to apply to these cofibrations by using as generating (trivial) cofibrations the maps

$$S \otimes (A \to B) \times (G \times H \times T \times \Sigma_m)/W, \qquad W \cap \{*\} \times \{*\} \times T \times \{*\} = * \tag{2.5}$$

for $A \to B$ one of the standard generating (trivial) cofibrations of S_*.

The study of this subtle notion of cofibration was one of the main purposes of [17]. Briefly, their relevance is as follows. Given a positive cofibration $f: A \to B$, the n-fold pushout product $f^{\square n}: Q^n_{n-1} \to B^n$ exhibits properties one would expect from a Σ_n-projective cofibration even though it fails to be so in the strict sense. S Σ-inj Σ_n-proj cofibrancy then provides a slightly laxer notion making $f^{\square n}$ into a cofibration while keeping most of the key formal properties one expects from "real" Σ_n-projective cofibrancy.

Fortunately, since the cofibration notions in this paper are all defined by forgetting structure, we will not need a detailed discussion of these issues, and will instead be able to simply refer to the relevant results in [17].

2.2 Key properties of the G-graph stable model structures

We know list the key results concerning the G-graph stable model structures we will need. These are modeled after the results in [17, Sec. 4].

Proposition 2.6. *Let $\bar{H} \subset H$ be finite groups, and suppose each category is equipped with the respective S G-graph stable model structure. Then both adjunctions*

$$\mathsf{res}^{G\times H}_{G\times \bar{H}}: (\mathsf{Sp}^\Sigma)^{G\times H} \rightleftarrows (\mathsf{Sp}^\Sigma)^{G\times \bar{H}}: ((-)^{S\otimes H_+})^{\bar{H}}$$

$$H \times_{\bar{H}} (-): (\mathsf{Sp}^\Sigma)^{G\times \bar{H}} \rightleftarrows (\mathsf{Sp}^\Sigma)^{G\times H}: \mathsf{res}^{G\times H}_{G\times \bar{H}}$$

are Quillen adjunctions.

Proof. In both cases it is clear that the left adjoints preserve cofibrations. Further, by [11, Sec. 6.2], $\mathsf{res}^{G\times H}_{G\times \bar{H}}$ preserves all G-graph stable equivalences since the graph subgroups of $G \times \bar{H}$ are a subset of those in $G \times H$.

To deal with $H \times_{\bar{H}} (-)$, one fixes a graph subgroup W associated to a homomorphism $\varphi: \bar{G} \to H$ and notes that for any map $f: A \to B$ one has a decomposition of the W action as

$$H \times_{\bar{H}} f \cong \bigvee_{H/\bar{H}} f \cong \bigvee_{i\in W\backslash H/\bar{H}} W \times_{W_i} \varphi^*_{h_i} f \tag{2.7}$$

where W_i, the intersection of W with the isotropy $G \times h_i \bar{H} h_i^{-1}$ of the $h_i\bar{H}$ summand in the intermediate decomposition, is a graph subgroup associated to a homomorphism $\varphi_i: \bar{\bar{G}}_i \to h_i \bar{H} h_i^{-1}$ (and where $\varphi^*_{h_i}$ denotes the pullback of the action along the conjugation isomorphism $\varphi_{h_i}: h_i\bar{H}h_i^{-1} \to H$). The result is now clear by [11, Sec. 6.2]. □

Theorem 2.8. *For $f: A \to B$ a trivial cofibration in $(\mathsf{Sp}^\Sigma)^{G\times H}$ in the S G-graph stable model structure, the map*

$$f^{\square n}: Q^n_{n-1}(f) \to B^n$$

is a trivial cofibration in $(\mathsf{Sp}^\Sigma)^{G\times(\Sigma_n \wr H)}$ for the S G-graph stable model structure.

Proof. It follows directly from monoidality of the non-equivariant S-model structure on symmetric spectra that $f^{\square n}$ is again a cofibration, so by [11, Prop. 4.2] it suffices to show that the quotient, which is S-cofibrant and isomorphic to

$(B/A)^{\wedge n}$, is G-graph stably contractible (as a $G\times(\Sigma_n \wr H)$-symmetric spectrum). Now fix a graph subgroup W associated to a homomorphism $\varphi : \bar{G} \to (\Sigma_n \wr H)$.

Postcomposing with the projection $G \times \Sigma_n \wr H \to \Sigma_n$ defines a W-action on $\underline{n}$ and one obtains a multiplicative analogue of (2.7)

$$(B/A)^{\wedge n} \cong \prod_{i \in W \backslash \underline{n}} N_{W_i}^{W}(B/A) \tag{2.9}$$

where W_i, the isotropy of the i summand, is the graph subgroup corresponding to a homomorphism $\varphi_i : \bar{\bar{G}}_i \to (\Sigma_n \wr H)$ (note that in this case φ_i is just a restriction of φ) and N denotes the multiplicative norm (cf. [**?**]).

Since B/A is W_i-stably contractible and S-cofibrant by assumption, it follows from [11, Thm. 7.8] that $N_{W_i}^{W}(B/A)$ is W-stably contractible, hence by [11, Prop. 4.13] so is the product in (2.9), finishing the proof. □

Remark 2.10. For the purposes of the present paper (namely in the proof of Theorem 1.4), Theorem 2.8 needs to be combined it with [17, Thm. 1.3] so as to obtain the "Σ_n- cofibrancy" properties (see Remark 2.4) of $f^{\square n}$ (for f a positive cofibration) needed for one to apply Theorem 2.11 bellow. This mimics the relationship between [17, Thm. 1.4] (after which Theorem 2.11 is modeled) and [17, Thm. 1.3].

Theorem 2.11. *Consider the functor*

$$(\mathsf{Sp}^{\Sigma})^{G\times H\times T} \times (\mathsf{Sp}^{\Sigma})^{G\times \hat{H}\times T} \xrightarrow{-\wedge_T -} (\mathsf{Sp}^{\Sigma})^{G\times H\times \hat{H}},$$

where the first category $(\mathsf{Sp}^{\Sigma})^{G\times H\times T}$ is regarded as equipped with the S $\Sigma{\times}G{\times}H$-inj T-proj G-graph stable model structure (see Remark 2.4).

Then $\wedge_T$ is a left Quillen bifunctor if either:

(a) Both $(\mathsf{Sp}^{\Sigma})^{G\times \hat{H}\times T}$ and the target $(\mathsf{Sp}^{\Sigma})^{G\times H\times \hat{H}}$ are equipped with the respective monomorphism G-graph stable model structures;

(b) Both $(\mathsf{Sp}^{\Sigma})^{G\times \hat{H}\times T}$ and the target $(\mathsf{Sp}^{\Sigma})^{G\times H\times \hat{H}}$ are equipped with the respective S G-graph stable model structures.

The proof requires a detailed analysis of the way the appearing model structures are built and is hence postponed to Appendix A.

3 Cofibrancy of operadic constructions

3.1 Terminology: operads, modules and algebras

We now recall some notation and terminology concerning operads. We refer to [17] for the full definitions.

Definition 3.1. Let $(\mathcal{C}, \otimes, \mathbb{1})$ denote a closed symmetric monoidal category.

Then the category $\mathsf{Sym}(\mathcal{C})$ of *symmetric sequences in $\mathcal{C}$* is the category of functors $\Sigma \to \mathcal{C}$.

$\mathsf{Sym}(\mathcal{C})$ can be equipped with two usual monoidal structures.

Definition 3.2. Given $X, Y \in \mathsf{Sym}(\mathcal{C})$ we define their *tensor product* to be

$$(X \check{\otimes} Y)(r) = \bigvee_{0 \le \bar{r} \le r} \Sigma_r \times_{\Sigma_{\bar{r}} \times \Sigma_{r-\bar{r}}} X(\bar{r}) \otimes Y(r - \bar{r})$$

and their *composition product* to be

$$(X \circ Y)(r) = \bigvee_{\bar{r} \ge 0} X(\bar{r}) \otimes_{\Sigma_{\bar{r}}} (Y^{\check{\otimes}\bar{r}})(r). \tag{3.3}$$

Definition 3.4. An *operad* $\mathcal{O}$ in $\mathcal{C}$ is a monoid object in $\mathsf{Sym}(\mathcal{C})$ with respect to $\circ$, i.e., a symmetric sequence $\mathcal{O}$ together with multiplication and unit maps

$$\mathcal{O} \circ \mathcal{O} \to \mathcal{O}, \qquad \mathcal{I} \to \mathcal{O}$$

satisfying the usual associativity and unit conditions.

Definition 3.5. Let $\mathcal{O}$ be an operad in $\mathcal{C}$. A *left module* N (resp. *right module* M) over $\mathcal{O}$ is an object in $\mathsf{Sym}(\mathcal{C})$ together with a map

$$\mathcal{O} \circ N \to N \qquad (\text{resp. } M \circ \mathcal{O} \to M)$$

which satisfies the usual associativity and unit conditions. The category of left modules (resp right modules) over $\mathcal{O}$ is denoted $\mathsf{Mod}^l_{\mathcal{O}}$ (resp. $\mathsf{Mod}^r_{\mathcal{O}}$).

Further, left modules X over $\mathcal{O}$ concentrated[1] in degree 0 are called *algebras* over $\mathcal{O}$. The category of algebras over $\mathcal{O}$ is denoted $\mathsf{Alg}_{\mathcal{O}}$.

Definition 3.6. The category $\mathsf{BSym}(\mathcal{C})$ of *bi-symmetric sequences in* $\mathcal{C}$ is the category $\mathsf{Sym}(\mathsf{Sym}(\mathcal{C}))$ of symmetric sequences of symmetric sequences in $\mathcal{C}$.

Following Definition 3.2 one can hence build two monoidal structures on $\mathsf{BSym}(\mathcal{C})$ which we denote, respectively, by $\check{\check{\otimes}}$ and $\check{\circ}^{\mathsf{r}}$, where $\check{\circ}^{\mathsf{r}}$ is built by keeping r as the "operadic" index. Note that while $\check{\check{\otimes}}$ behaves symmetrically with respect to the two indexes r and s, $\check{\circ}^{\mathsf{r}}$ does not.

3.2 Model structures on Sym^G

Throughout the remainder of the paper we shall abbreviate $\mathsf{Sym}((\mathsf{Sp}^{\Sigma})^G)$ simply as Sym^G.

We now introduce the key model structures on Sym^G that we will need.

Definition 3.7. The *monomorphism G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is obtained by combining the monomorphism G-graph stable model structures in $(\mathsf{Sp}^{\Sigma})^{G\times H\times\Sigma_r}$ in all degrees.

Definition 3.8. The *S G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is obtained by combining the S G-graph stable model structures in $(\mathsf{Sp}^{\Sigma})^{G\times H\times\Sigma_r}$ in all degrees.

Definition 3.9. The *positive S G-graph stable model structure* on $\mathsf{Sym}^{G\times H}$ is the model structure obtained by combining the positive S G-graph stable model structure in $(\mathsf{Sp}^{\Sigma})^{G\times H}$ on degree $r = 0$ with the S G-graph stable model structures in $(\mathsf{Sp}^{\Sigma})^{G\times H\times\Sigma_r}$ in degrees $r \ge 1$.

[1] I.e. such that $X(r) = \varnothing$ for $r \ge 1$.

Remark 3.10. By analogy to Remark 2.4, we will also use briefly S $\Sigma{\times}\Sigma{\times}G{\times}H$-inj T-proj cofibrations in $\mathsf{Sym}^{G\times H\times T}$, which are those maps which are underlying S $\Sigma\times\Sigma$-inj T-proj cofibrations in the sense of [17, Sec. 5.3] after forgetting the $G\times H$ action.

These cofibrations are necessary for the formulation of Proposition 3.15 bellow.

Propositions 2.6 and Theorems 2.11 and 2.8 directly imply analogous results in the context of symmetric sequences, which we now state.

Proposition 3.11. *Let $\bar{H}\subset H$ be finite groups, and suppose each category is equipped with the respective S G-graph stable model structure. Then both adjuntions*

$$\mathsf{res}^{G\times H}_{G\times\bar{H}}\colon \mathsf{Sym}^{G\times H} \rightleftarrows \mathsf{Sym}^{G\times\bar{H}}\colon ((-)^{S\otimes H_+})^{\bar{H}}$$

$$H\times_{\bar{H}}(-)\colon \mathsf{Sym}^{G\times\bar{H}} \rightleftarrows \mathsf{Sym}^{G\times H}\colon \mathsf{res}^{G\times H}_{G\times\bar{H}}$$

are Quillen adjunctions.

Proof. This follows by applying Proposition 2.6 at each level. $\square$

Proposition 3.12. *For $f\colon A\to B$ a trivial cofibration in $\mathsf{Sym}^{G\times H}$ for the S G-graph stable model structure the map*

$$f^{\square\check{\wedge}n}\colon Q^n_{n-1}(f)\to B^n$$

is a trivial cofibration in $\mathsf{Sym}^{G\times(\Sigma_n\wr H)}$ for the S G-graph stable model structure.

Proof. Computing $X_1\check{\wedge}\cdots\check{\wedge}X_n$ iteratively and regrouping terms we get

$$(X_1\check{\wedge}\cdots\check{\wedge}X_n)(r) = \bigvee_{\phi\colon\underline{r}\to\underline{n}} X_1(\phi^{-1}(1))\wedge\cdots\wedge X_n(\phi^{-1}(n)).$$

The $\Sigma_n\times\Sigma_r$ action[2] interchanges these wedge summands via post- and pre-composition with ϕ. It is hence sufficient to prove the result for the subfunctor formed by the wedge summands corresponding to the $\Sigma_n\times\Sigma_r$ orbit of a single ϕ, and w.l.o.g. we can decompose $\underline{n}=\underline{n_0}\coprod\underline{n_1}\coprod\cdots\coprod\underline{n_k}$ where $\#\phi^{-1}(*)=i$ for any element $*\in\underline{n_i}$. It now follows that the $\Sigma_n\times\Sigma_r$ isotropy of ϕ is the subgroup $\Sigma_{n_0}\wr\Sigma_0\times\cdots\times\Sigma_{n_k}\wr\Sigma_k$.

Now letting $f\colon A\to B$ be a trivial cofibration in $\mathsf{Sym}^{G\times H}$, we conclude that the summand of $f^{\square n}$ corresponding to the ϕ subfunctor is isomorphic to

$$\Sigma_n\times\Sigma_r\times_{\Sigma_{n_0}\wr\Sigma_0\times\cdots\times\Sigma_{n_k}\wr\Sigma_k} f(0)^{\square n_0}\square f(1)^{\square n_1}\square\cdots\square f(k)^{\square n_k}. \tag{3.13}$$

By Theorem 2.8 each $f(i)^{\square n_i}$ is a S G-graph stable trivial cofibration, and by Theorem 2.11 (when $T=*$) so is $f(0)^{\square n_0}\square\cdots\square f(k)^{\square n_k}$. The proof is now complete by applying Proposition 2.6 (while noting that the induction from $\Sigma_{n_0}\wr\Sigma_0\times\cdots\times\Sigma_{n_k}\wr\Sigma_k$ to $\Sigma_n\times\Sigma_r$ in (3.13) is in fact an induction from $G\times\Sigma_{n_0}\wr(\Sigma_0\times H)\times\cdots\times\Sigma_{n_k}\wr(\Sigma_k\times H)$ to $G\times(\Sigma_n\wr H)\times\Sigma_r$). $\square$

Remark 3.14. Just as in Remark 2.10, Proposition 3.12 needs to be combined with [17, Prop. 5.60] so as to obtain the "lax Σ_n-projective cofibrancy" conditions needed for one to apply Proposition 3.15 below.

[2] Note that since we are not yet assuming all X_i are the same the Σ_n action actually corresponds to the structure symmetric monoidal structure isomorphisms of $\check{\wedge}$.

Proposition 3.15. *Consider the functor*

$$\mathsf{Sym}^{G\times H\times T}\times\mathsf{Sym}^{G\times\bar{H}\times T}\xrightarrow{-\check{\wedge}_T-}\mathsf{Sym}^{G\times H\times\bar{H}},$$

where the first category $\mathsf{Sym}^{G\times H\times T}$ *is regarded as equipped with the* S $\Sigma\times\Sigma\times G\times H$*-inj* T*-proj* G*-graph stable model structure (see Remark 3.10).*

Then $\check{\wedge}_T$ *is a left Quillen bifunctor if either:*

(a) Both $\mathsf{Sym}^{G\times\bar{H}\times T}$ *and the target* $\mathsf{Sym}^{G\times H\times\bar{H}}$ *are equipped with the respective monomorphism* G*-graph stable model structures;*

(b) Both $\mathsf{Sym}^{G\times\bar{H}\times T}$ *and the target* $\mathsf{Sym}^{G\times H\times\bar{H}}$ *are equipped with the respective* S G*-graph stable model structures.*

Proof. First recall that

$$(X\check{\otimes}Y)(r)=\bigvee_{0\leq\bar{r}\leq r}\Sigma_r\times_{\Sigma_{\bar{r}}\times\Sigma_{r-\bar{r}}}X(\bar{r})\wedge Y(r-\bar{r}).$$

Applying the second part of Proposition 2.6 we reduce to showing that pushout products for the bifunctors $(X,Y)\mapsto X(\bar{r})\wedge_T Y(r-\bar{r})$ are left biQuillen, and that follows by applying Theorem 2.11 in the case

$$(\mathsf{Sp}^{\Sigma})^{G\times H\times\Sigma_{\bar{r}}\times T}\times(\mathsf{Sp}^{\Sigma})^{G\times\bar{H}\times\Sigma_{r-\bar{r}}\times T}\xrightarrow{-\wedge_T-}(\mathsf{Sp}^{\Sigma})^{G\times H\times\bar{H}\times\Sigma_r\times\Sigma_{\bar{r}}}.$$

□

Remark 3.16. All of this subsection immediately generalizes to the category $\mathsf{BSym}^{G\times H}=\mathsf{Sym}(\mathsf{Sym}^{G\times H})$.

Indeed, one can define monomorphism G-graph stable, S G-graph stable and positive S G-graph stable model structures in $\mathsf{BSym}^{G\times H}=\mathsf{Sym}(\mathsf{Sym}^{G\times H})$ and S $\Sigma\times\Sigma\times\Sigma\times G\times H$-inj T-proj G-graph stable model structures in $\mathsf{BSym}^{G\times H\times T}$ by simply repeating Definitions 3.7, 3.8, 3.9 and Remark 3.10 except replacing the initial model structures in $(\mathsf{Sp}^{\Sigma})^{G\times H}$ by their analogues in $\mathsf{Sym}^{G\times H}$.

Analyzing the proofs of 3.11, 3.15 and 3.12 it is then clear that those results themselves imply their $\mathsf{BSym}^{G\times H}$ versions as well.

4 Proofs of the main theorems

Proof of Theorem 1.1. This follows by repeating the proof of [17, Thm. 1.5] by simply replacing instances of [17, Thm. 1.1] with Theorem 1.4. □

Proof of Theorem 1.3. The first claim, that the functors $A\wedge_{\Sigma_n}(-)^{\wedge n}$ and $B\wedge_{\Sigma_n}(-)^{\wedge n}$ are left derivable, follows by Ken Brown's Lemma and Theorem 1.4 applied to the unit operad $\mathcal{I}$, since it then follows that those functors send trivial cofibrations between cofibrant objects in the positive S G-stable model structure to G-stable equivalences.

For the second claim, the "if" part again follows by Ken Brown's Lemma combined with Theorem 1.4 applied to the unit operad $\mathcal{I}$, since it then follows that, for positive S cofibrant X, the functor $(-)\wedge_{\Sigma_n}X^{\wedge n}$ sends trivial cofibrations between cofibrant objects in the monomorphism G-graph stable model structure to G-stable equivalences.

For the "only if" claim, fix a G-graph subgroup W associated to $\varphi:\bar{G} \to \Sigma_n$. Now let $\bar{S}$ denote a positive cofibrant replacement for the sphere S and consider the positive S cofibrant spectrum

$$X = (G \times_{\bar{G}} \underline{n}) \cdot \bar{S},$$

where $\bar{G}$ acts on $\underline{n}$ via φ. Computing $X^{\wedge n}$ one obtains a natural decomposition

$$X^{\wedge n} = \bigvee_{G^{\times n} \times_{\bar{G}^{\times n}} \underline{n}^{\times n}} \bar{S}^{\wedge n}$$

such that the $\Sigma_n \wr G$ action, and hence also the diagonal $\Sigma_n \times G$ action, interchanges wedge summands.

Now recall that by hypothesis

$$A \wedge_{\Sigma_n} X^{\wedge n} \to B \wedge_{\Sigma_n} X^{\wedge n}$$

is a G-stable equivalence, and hence so must be each of its summands corresponding to a $\Sigma_n \times G$ orbit of $G^n \times_{\bar{G}^n} \underline{n}^n$. In particular, this must be the case for the (diagonal) orbit $G \times_{\bar{G}} \Sigma_n \subset G^n \times_{\bar{G}^n} \underline{n}^n$, whose summand is then

$$A \wedge_{\Sigma_n} \left((G \times_{\bar{G}} \times \Sigma_n) \cdot \bar{S}^{\wedge n} \right) \to B \wedge_{\Sigma_n} \left((G \times_{\bar{G}} \times \Sigma_n) \cdot \bar{S}^{\wedge n} \right)$$

and using the freeness of the Σ_n action this is in turn identified with

$$G \cdot_{\bar{G}} \varphi^*(A) \wedge \bar{S}^{\wedge n} \to G \cdot_{\bar{G}} \varphi^*(B) \wedge \bar{S}^{\wedge n},$$

where $\varphi^*(A), \varphi^*(B)$ denote A, B with the $\bar{G}$ actions obtained by pulling back the W action. Forgetting the full G action to a $\bar{G}$ action and focusing on the identity summand one sees that $\varphi^*(A) \to \varphi^*(B)$ must be a $\bar{G}$-stable equivalence, completing the proof. $\square$

Proof of Theorem 1.4. The result to be shown is an equivariant analogue of [17, Thm. 1.5] and it follows by repeating the same exact proof as in [17] except now using the G-graph stable results developed in this paper at certain key points. Rather than repeating the full proof, we list only the specific steps where the new results are used.

The case of regular (i.e. non trivial) cofibrations immediately reduces to [17, Thm. 1.5].

For the case where one of the cofibrations is trivial, assume first for simplicity (as in the proof in [17]) that f_2 is a map of algebras.

If f_2 is the trivial cofibration, the argument in [17] reduces to the case where f_2 is the pushout of a generating trivial cofibration $f: \mathcal{O} \circ X \to \mathcal{O} \circ Y$ and to checking that the "pushout corner map" in

$$\begin{tikzcd}
M_A(r) \wedge_{\Sigma_r} Q^r_{r-1} \arrow[r] \arrow[d] & M_A(r) \wedge_{\Sigma_r} Y^r \arrow[d] \\
N_A(r) \wedge_{\Sigma_r} Q^r_{r-1} \arrow[r] & N_A(r) \wedge_{\Sigma_r} Y^r
\end{tikzcd} \tag{4.1}$$

is a G-graph stable equivalence. This follows by Theorems 2.8, 2.11 and Remark 2.10.

If instead f_1 is the trivial cofibration then it suffices to show that, in (4.1), $M_A \to N_A$ is a G-graph equivalence in Sym^G, and following along the proof in [17], namely the inductive argument along $\beta \le \kappa$, it suffices for the "pushout corner map" in the diagram

$$\begin{tikzcd} M_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Q^s_{s-1,\beta} \arrow[r] \arrow[d] & M_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Y^s_\beta \arrow[d] \\ N_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Q^s_{s-1,\beta} \arrow[r] & N_{\mathcal{O} \amalg A_\beta}(s) \check{\wedge}_{\Sigma_s} Y^s_\beta \end{tikzcd}$$

to be a G-graph stable equivalence in Sym^G. Hence by Propositions 3.12, 3.15 and Remark 3.14 it suffices to show that $M_{\mathcal{O} \amalg A_\beta}(r,s) \to N_{\mathcal{O} \amalg A_\beta}(r,s)$ is a G-graph stable equivalence in $(\mathsf{Sp}^\Sigma)^{G \times \Sigma_r \times \Sigma_s}$. But by [17, Prop. 5.21] this map can be identified $M_{A_\beta}(r+s) \to N_{A_\beta}(r+s)$, which is a G-graph stable equivalence in $(\mathsf{Sp}^\Sigma)^{G \times \Sigma_{r+s}}$ by induction on β and the result follows by Proposition 3.11.

In the more general case where f_2 is a general map of left modules one proceeds as described in the last paragraph of the proof in [17], except now using the bisymmetric sequence analogues of Theorems 2.8 and 2.11 described in Remark 3.16. □

A Existence of model structures

proof of Theorem 2.3. In either the monomorphism or the S cofibration case one starts by building a levelwise G-graph model category structure, i.e. a model structure where a map $X \to Y$ of $G \times H$-spectra is a w.e. if all maps

$$X_n^W \to Y_n^W$$

are w.e.s of simplicial sets for all n and graph subgroup W associated to a morphism $\varphi\colon \bar{G} \to H \times \Sigma_n$.

In the S cofibration case the existence of such a model structure follows either from [11, Prop. 2.30] or by repeating the proof of [17, Prop. 3.4]. For the monomorphism case, one instead repeats the arguments in [17, Thm. 3.8]. The only non trivial part of this is to show the analogue adaptation of [15, Lem. 5.1.7], and again the proof of [15] can be adapted directly by making sure that the spectrum FC built in the proof kills off relative homotopy groups of fixed points with respect to all G-graph subgroups W.

To produce the G-graph stable model structure, consider the commuting square of left Bousfield localizations

$$\begin{tikzcd} \left((\mathsf{Sp}^\Sigma)^{G\times H}\right)_{G\times H\text{-lv}} \arrow[r] \arrow[d] & \left((\mathsf{Sp}^\Sigma)^{G\times H}\right)_{G\times H\text{-st}} \arrow[d] \\ \left((\mathsf{Sp}^\Sigma)^{G\times H}\right)_{G\text{-gr-lv}} \arrow[r] & \left((\mathsf{Sp}^\Sigma)^{G\times H}\right)_{G\text{-gr-st}} \end{tikzcd} \tag{A.1}$$

where the horizontal arrows localize the maps

$$G \times H \ltimes_W (F_{U \amalg V} S^V \to F_U S^0) \tag{A.2}$$

for U, V representations of $W \subset H \times W$ and the vertical maps localize the maps[3]

$$F_n(E\mathcal{F}_n)_+ \to F_n S^0 \tag{A.3}$$

for $\mathcal{F}_n$ the family of G-graph subgroups of $G \times H \times \Sigma_n$.

We claim that the w.e.s in the bottom right corner are precisely the G-graph stable equivalences of Definition 2.1. The latter contains the former since $\mathsf{res}_W^{G\times H}$ is both a left and right Quillen functor for the genuine model structures [[11], 6.2] and the maps in (A.3) restrict to w.e.s when W is a graph subgroup. For the converse, it now suffices to check that the two classes of equivalences coincide when restricted to fibrant objects in the bottom right corner model structure. These are precisely those fibrant Ω-spectra X such that $X_n^W \simeq *$ for all n and W any non G graph subgroup of $G \times H \times \Sigma_n$. The first notion of w.e. then says that a map of Ω-spectra $X \to Y$ is a w.e. if

$$X_n^W \to Y_n^W$$

is a w.e. of simplicial sets for all subgroups W, while the second demands that condition only for W a G-graph subgroup, but these conditions are clearly the same. □

Remark A.4. In the proof of Theorem 2.11 below we will use the fact that in the lower horizontal localization in diagram (A.1) it in fact suffices to localize those maps of the form

$$\lambda_{U,V}^{(W)}: G \times H \ltimes_W (F_{U \amalg V} S^V \to F_U S^0) \tag{A.5}$$

for U, V representations of a **G-graph subgroup** $W \subset G \times H$.

To see this is the case it suffices to show that localizing with respect to this smaller set produces the same local objects. Now note that any G-graph level fibrant object that is local with respect to the maps in (A.5) becomes W-genuine fibrant after applying $\mathsf{res}_W^{G\times H}$, and hence level equivalences between such objects are the same as G-graph stable equivalences. The desired conclusion follows.

Theorem A.6. *Let $\mathcal{O}$ be any operad in $(\mathsf{Sp}^\Sigma)^{G\times H}$.*

There exists a cofibrantly generated model structure on $\mathsf{Mod}_{\mathcal{O}}^r$, which we call the **monomorphism G-graph stable model structure***, such that*

- *cofibrations are the maps $X \to Y$ such that $X_n \to Y_n$ is a monomorphism of pointed simplicial sets for each $n \geq 0$.*
- *weak equivalences are the maps are the maps $X \to Y$ which are underlying G-graph stable equivalences of spectra.*

Further, this is a left proper cellular simplicial model category.

Proof. This result is an analogue of [17, Thm. 5.65] and essentially the same proof applies. The only differences worth mentioning are that in the levelwise stage of the proof one uses levelwise G-graph equivalences as defined in the proof of Theorem 2.3 above and that when proving the analogue of [15, Lem. 5.1.7] one needs the spectrum FC to kill relative homotopy groups with respect to all G-graph subgroups W. □

[3]Notational note: $F_n(E\mathcal{F}_n)_+$ is actually a semi-free spectrum.

proof of Theorem 2.11. Since cofibrancy is defined by forgetting structure, the case of regular cofibrations reduces to that of [17, Thm. 1.4], so that we need only worry about the case involving trivial cofibrations. It hence suffices to prove part (a).

Further, by fixing a graph subgroup W associated to a homomorphism $\varphi\colon \bar{G} \to H \times \hat{H}$ we further reduce to the case $H = \hat{H} = *$.

The remainder of the proof closely follows that of [17, Thm. 1.4], by first proving a level G-graph equivalence version of the result (see the proof of Theorem 2.3), then showing the left biQuillen functor localizes to stabilizations.

For the level structure result, the fact that all generating (trivial) cofibrations in the $\Sigma \times G$-inj H-proj level model structure have the form $S \otimes f$ (cf. (2.5)) one reduces to the analogous statement for the bifunctor

$$\mathsf{S}_*^{G\times T\times\Sigma} \times (\mathsf{Sp}^\Sigma)^{G\times T} \xrightarrow{-\otimes_T -} (\mathsf{Sp}^\Sigma)^G.$$

For $A_m \xrightarrow{i} B_m$ a $\Sigma \times G$-inj H-proj cofibration in $\mathsf{S}_*^{G\times T\times\Sigma}$ and $C \xrightarrow{f} D$ any monomorphism one then has

$$(i \,\square^{\otimes} f)_{\bar{m}} = \Sigma_{\bar{m}} \times_{\Sigma_m \times \Sigma_{\bar{m}-m}} i \,\square^{\wedge} f_{\bar{m}-m}.$$

We need to check this is a levelwise G-graph equivalence if either i or all f_m are. Fixing the graph subgroup W associated to $\varphi\colon \bar{G} \to \Sigma_{\bar{m}}$ one sees that the domains and codomains of the fixed point map $((i \,\square^{\otimes} f)_{\bar{m}})^W$ can be non trivial only if φ factors through $\Sigma_m \times \Sigma_{\bar{m}-m}$ up to conjugation by some $\sigma \in \Sigma_{\bar{m}}$. It then follows that one can forget the $\Sigma_m, \Sigma_{\bar{m}-m}$ actions, so that the claim follows from the Quillen biadjunction $\mathsf{S}_*^{G\times T} \times \mathsf{S}_*^{G\times T} \xrightarrow{-\wedge_T -} \mathsf{S}_*^G$ where one of the $\mathsf{S}_*^{G\times T}$ has the projective G-graph projective model structure and the other categories their respective (projective) G-genuine model structures.

To localize the level structure result to the desired stable version, we repeat the argument in the last paragraph of the proof of [17, Thm. 1.4] to conclude that, by [17, Lem. 4.3] and Remark A.4 above, it suffices to check that

$$X \wedge_T (G \times T \ltimes_W \lambda_{U,V}^{(W)})$$

is a G-stable equivalence for U, V representations of any G-graph subgroup W associated to $\varphi\colon \bar{G} \to H$. But since $W \cap T = \{*\}$ the map above can be rewritten G-equivariantly as (abusing notation by writing φ for the isomorphism $\bar{G} \xrightarrow{\cong} W$)

$$G \ltimes_{\bar{G}} \varphi^*(X \wedge \lambda_{U,V}^{(W)})$$

so that the result now follows by [11, Prop. 7.1] together with [11, Sec. 6.2]. $\square$

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