Goodwillie Calculus in the category of algebras over a spectral operad

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The main purpose of this paper is to apply the theory developed in [26] to the specific case of functors between categories of the form $Alg_{\mathcal{O}}$.

Section 1 introduces the basic necessary definitions and notations about symmetric spectra Sp^{Σ}, model structures in that category, operads and algebras over them, and model structures in the categories of algebras.

Section 2 then begins the task of understanding (finitary) homogeneous functors between algebra categories by understanding their stabilization. The results here are Theorem 2.3 and its Corollary 2.9 showind that the stabilization of $Alg_{\mathcal{O}}$ is the module category $Mod_{\mathcal{O}(1)}$.

Section 3 completes the task of classifying (finitary) homogeneous functors between algebra categories by classifying homogeneous functors between their stabilizations, the module categories. The result here is Theorem 3.2, which generalizes the well known characterization in the case of spectra.

Finally, section 4 concludes the paper by establishing our main results about Goodwillie calculus in the $Alg_{\mathcal{O}}$ categories. In subsection 4.1 we finally establish Theorem 4.3, saying that the Goodwillie tower for $Id_{Alg_{\mathcal{O}}}$ is indeed the homotopy completion tower studied in [17]. Subsection 4.2 establishes Theorem 4.8, which is probably more surprising. It roughly says that, as far as (finitary) n-excisive functors are concerned, Goodwillie calculus can not distinguish the category $Alg_{\mathcal{O}}$ from $Alg_{\mathcal{O}_{\leq n}}$. Lastly, subsection 4.3 establishes Theorem 4.12, which shows the $Alg_{\mathcal{O}}$ categories satisfy at least a weak analogue of the chain rule from [1].

1 Basic definitions

The majority of the material in this section is adapted from [15], which should be consulted for details. We will only cover here the bare minimum necessary for the remainder of the paper.

1.1 Symmetric Spectra

Definition 1.1. The category Sp^{Σ} of symmetric spectra is the category such that

- objects X are sequences $X_n \in SSet_*^{\Sigma_n}$ (i.e., X_n is a pointed simplicial set with a Σ_n action), together with structure maps (compatible with the

$\Sigma_m \times \Sigma_n$ action)

$$S^m \wedge X_n \to X_{m+n}$$

satisfying appropriate compatibility conditions.

- maps $f\colon X \to Y$ are sequences of Σ_n maps $f_n\colon X_n \to Y_n$ compatible with the structure maps in the obvious way.

Also, we denote by $\wedge$ the standard monoidal structure on Sp^Σ with unit S, the canonical symmetric spectrum such that $S_n = S^n$, the n-sphere.

Definition 1.2 (Free symmetric spectra)**.** Let $H \subset \Sigma_m$ be a subgroup and $A \in SSet^H_*$.

The free spectrum $F^H_m(A)$ generated by A is the symmetric spectrum with spaces

$$(F^H_m(A))_n = \begin{cases} \Sigma_n \underset{\Sigma_{n-m}\times H}{\times} (S^{n-m} \wedge A) & \text{if } n \geq m \\ * & \text{if } n < m \end{cases}$$

with the natural maps.

Or, in other words, F^H_m is the left adjoint to the forgetful functor $Sp^\Sigma \to SSet^H_*$.

1.2 Stable model structures on Sp^Σ

Our interest on Sp^Σ is as a model category for spectra. The stable w.e.s used for this are somewhat technical to define, as the actual definition resorts to defining injective Ω-spectra first, hence we refer to [21] for the precise definition. For our purposes it is enough to view these stable equivalences as being those maps that induce isomorphisms on the stable homotopy groups π^s_n[1].

However, while we will only be interested in model structures on Sp^Σ which use the stable w.e.s as the notion of w.e.s, there are multiple such model structures, and it will be useful for us to be aware of several of them, which we list in the following definition.

Definition 1.3. We have the following stable model structures on Sp^Σ.

- **level cofibrations stable model structure**, where the cofibrations are levelwise cofibrations of the underlying symmetric sequences.

- **flat stable model structure**, with generating cofibrations given by $F^H_m((\delta\Delta_k)_+) \to F^H_m((\Delta_k)_+)$ for $m \geq 0$, $H \subset \Sigma_m$ any subgroup.

- **positive flat stable model structure**, with generating cofibrations given by $F^H_m((\delta\Delta_k)_+) \to F^H_m((\Delta_k)_+)$ for $m \geq 1$, $H \subset \Sigma_m$ any subgroup.

- **stable model structure**, with generating cofibrations given by

$$F^*_m((\delta\Delta_k)_+) \to F^*_m((\Delta_k)_+)$$

for $m \geq 0$, and where $* \subset \Sigma_m$ denotes the trivial subgroup.

[1] The catch here is that, when working with symmetric spectra, one can't usually simply define $\pi^s_n(X) = \varinjlim_m \pi_{n+m} X_m$, unless X already satisfies some fibrancy type condition.

- **positive stable model structure**, with generating cofibrations given by

$$F_m^*((\delta\Delta_k)_+) \to F_m^*((\Delta_k)_+)$$

for $m \geq 0$, and where $* \subset \Sigma_m$ denotes the trivial subgroup.

Remark 1.4. It is worth noting the hierarchy of these model categories: the *level cofibrations stable* structure has the most cofibrations, followed by the *flat stable model structure*, followed by either the *positive flat stable* or the *stable* model structures (which refine the flat stable model structure in different ways), and followed finally by the *positive stable* model structure, which has the least cofibrations of all.

Proposition 1.5. *The five model structures listed above are all left proper cellular model categories.*

Proof. We recall that left properness means that the pushout of a w.e. along a cofibration is again a w.e.. Hence it suffices to prove this property for the model category structure with the most cofibrations, namely the levelwise cofibration model structure. This result is then Lemma 5.4.3 part (1) of [21].

Cofibrant generation of each of these model structures is proved in several papers: for the level cofibration model structure and the stable model structure this is proved in [21]; for the flat stable model structure and the positive flat stable model structure this is proved in [32], and for the positive stable model structure this is proved in [24][2].

Cellularity of these model categories (see Definition A.1 of [20] for the definition) follows immediately from these being categories built out of simplicial sets.

Indeed, any set of objects A will be compact with respect to any set of level injections K by choosing a regular ordinal γ greater than the cardinality of all the simplicies appearing in A, as then a map from $a \in A$ into a relative K-cell complex will factor through the minimal complex containing the images of the simplices, and this subcomplex will have less than γ cells.

Hence indeed the domains of I are compact with respect to I, and the domains of J small relative to the cofibrations (by a similar but easier argument). Finally, it is clear that the cofibrations are categorical monomorphisms, since they are always levelwise monomorphisms.

□

1.3 Operads and algebras

Definition 1.6. An (spectral) operad $\mathcal{O}$ in Sp^Σ is a sequence of "spectra of n-ary operations" $\mathcal{O}(n) \in Sp^\Sigma$, for $n \geq 0$, together with

- Σ_n actions on $\mathcal{O}(n)$,

[2]A little care is needed here because the results proven in [24] are for topologically based spectra instead of for simplicial ones, so one needs to adapt the arguments present there. Alternatively, it is fairly straightforward to see that one has a positive level model structure in the simplicial case, and that this is a left proper cellular model category, hence the result can also be derived by applying the left Bousfield localization techniques of [18].

- multiplication maps

$$\mathcal{O}(n) \wedge \mathcal{O}(m_1) \wedge \cdots \wedge \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n)$$

  and unit map

$$S \to \mathcal{O}(1),$$

- together with associativity, unity and change of order of variables compatibility conditions.

Definition 1.7. The category $Alg_{\mathcal{O}}$ of algebras over the operad $\mathcal{O}$ is the category such that

- objects X are symmetric spectra plus algebra maps

$$\mathcal{O}(n) \wedge X^{\wedge n} \to X$$

  satisfying appropriate compatibility conditions.

- maps $f : X \to Y$ are maps $X \to Y$ of the underlying symmetric spectra which are compatible with the algebra structure maps.

Definition 1.8 (Free algebras)**.** The free $\mathcal{O}$-algebra functor (usually denoted $\mathcal{O}$) is the left adjoint to the forgetful functor $Alg_{\mathcal{O}} \to Sp^{\Sigma}$.

It assigns to a symmetric spectrum X the canonical algebra $\mathcal{O}(X)$ whose underlying spectrum is $\bigvee_{n=0}^{\infty}(\mathcal{O}(n) \wedge X^{\wedge n})_{\Sigma_n}$.

We will also occasionally use the notation $\mathcal{O} \circ X$ for this algebra, where $\circ$ is meant to evoke the composition product of symmetric sequences.

Notations 1.9. Note that for any $\mathcal{O}$-algebra X its structural multiplication maps can be packaged into a map

$$\mathcal{O}(X) \xrightarrow{\mu} X.$$

Note further that this is actually a map of $\mathcal{O}$-algebras.

We will be using the following standard result.

Proposition 1.10. *Let $\mathcal{O} \to \mathcal{O}'$ be a map of operads. Then there is an adjunction*

$$Alg_{\mathcal{O}} \underset{forget}{\overset{\mathcal{O}' \circ_{\mathcal{O}} -}{\rightleftarrows}} Alg_{\mathcal{O}} \ ,$$

where the functor $\mathcal{O}' \circ_{\mathcal{O}} -$ is defined by the natural coequalizer $coeq(\mathcal{O}'\mathcal{O}X \rightrightarrows \mathcal{O}'X)$.

Remark 1.11. As a particular case of the previous result, consider the obvious map[3] $\mathcal{O}(1) \to \mathcal{O}$. Then each spectrum $\mathcal{O}(n)$ has a natural left action of $\mathcal{O}$ together with n different right actions of $\mathcal{O}(1)$. Note that these actions are all compatible, and that the right actions are furthermore related by the symmetric group action. Altogether these make $\mathcal{O}(n)$ into a $(\mathcal{O}(1), \mathcal{O}(1)^{\wr n})$-bimodule[4].

[3]Notice that $\mathcal{O}(1)$ is itself a monoid, and can hence be viewed as an operad concentrated in degree 1.

[4]Here $\mathcal{O}(1)^{\wr n}$ denotes the wreath product $\Sigma_n \ltimes \mathcal{O}(1)^{\wedge n} = \vee_{\sigma \in \Sigma_n} \mathcal{O}(1)^{\wedge n}$. Multiplication in this ring spectrum is such that multiplying the σ component by the τ component lands in the $\sigma\tau$ component, and σ acts on the second $\mathcal{O}(1)^{\wedge n}$ copy before multiplying those coordinatewise.

A direct calculation then shows that

$$\mathcal{O} \underset{\mathcal{O}(1)}{\circ} X = \bigvee_{n\geq 0} (\mathcal{O}(n) \underset{\mathcal{O}(1)^{\wedge n}}{\wedge} X^{\wedge n})_{\Sigma_n} = \bigvee_{n\geq 0} \mathcal{O}(n) \underset{\mathcal{O}(1)^{\wr n}}{\wedge} X^{\wedge n}.$$

Proposition 1.12. *Let X be in $Alg_{\mathcal{O}}$, and consider the undercategory $(Alg_{\mathcal{O}})_{X/}$ of $\mathcal{O}$-algebras under X.*

Then there exists an **enveloping operad** $\mathcal{O}_X$ *such that*

$$(Alg_{\mathcal{O}})_{X/} = Alg_{\mathcal{O}_X}.$$

More specifically, one has

$$\mathcal{O}_X(n) = coeq(\coprod_{m\geq 0} \mathcal{O}(n+m) \wedge_{\Sigma_m} (\mathcal{O} \circ X)^{\wedge m} \rightrightarrows \coprod_{m\geq 0} \mathcal{O}(n+m) \wedge_{\Sigma_m} X^{\wedge m})$$

with the two maps induced by the operad structure and the algebra structure.

Proof. This is just [15, Prop. 4.7] reinterpreted (and restricted from left modules to algebras, i.e., left modules concentrated in degree 0.).

Indeed, what that proposition shows is that the forgetful functor $(Alg_{\mathcal{O}})_{X/}$ has its left adjoint given by $\mathcal{O}_X \circ \cdot$. The result then follows since the conditions of Beck's monadicity theorem are immediate. $\square$

1.4 Model structures on $Alg_{\mathcal{O}}$

The following is a particular case of the main result of [15] (as well as the current author's work[5] in [27, Thm. 1.5]), combined with the simplicial structure described in the follow-up paper [17, Sec. 6].

Theorem 1.13. *Suppose Sp^{Σ} is endowed with either the positive stable model structure or the positive flat stable model structure.*

Then, for any operad $\mathcal{O}$, the projective model structure[6] on $Alg_{\mathcal{O}}$ exists.

Furthermore, this is a simplicial model structure.

We will need a somewhat explicit description of the simplicial tensoring and cotensoring.

First, recall that in Sp^{Σ} those are given by

$$(K \otimes X)_n = K_+ \wedge X_n$$

and

$$Map(K, X)_n = (X_n)^K,$$

where $K \in SSet$ and $X \in Sp^{\Sigma}$.

In other words, this these reflect pointwise the tensoring and cotensoring of $SSet_*$ over $SSet$. Note that $K \otimes X$ can also be described as $F_0^* K \wedge X$.

[5]In fact, the purpose of [27] was in part to correct a key technical problem with the proof in [15]. Namely, certain "Σ_n-cofibrancy" claims of "n-fold smash products" used in that proof turn out to be false, with [27] establishing correct (laxer) "Σ_n-cofibrancy" that still suffice for the remainder of the proof to apply.

[6]We recall that in a category $Alg_C(\mathcal{C})$ of algebras over some monad C in $\mathcal{C}$, the projective model structure (when it exists) on $Alg_C(\mathcal{C})$ is the one where w.e.s/fibrations are the maps which are underlying w.e.s/fibrations in $\mathcal{C}$.

In $Alg_{\mathcal{O}}$, the tensoring of a simplicial set K and a $\mathcal{O}$-algebra X is then given by the (algebraic) coequalizer

$$K \otimes^{alg} X = \mathcal{O}(K \otimes \mathcal{O}(X)) \underset{\mathcal{O}(\tau)}{\overset{\mathcal{O}(K \otimes \mu)}{\rightrightarrows}} \mathcal{O}(K \otimes X)$$

where $\mathcal{O}(X) \xrightarrow{\mu} X$ is the algebra structure map and $K \otimes \mathcal{O}(X) \xrightarrow{\tau} \mathcal{O}(K \otimes X)$ is the map

$$\bigvee_{n=0}^{\infty} K \otimes (\mathcal{O}(n) \wedge X^{\wedge n})_{\Sigma_n} \to \bigvee_{n=0}^{\infty} (K^{\times n} \otimes \mathcal{O}(n) \wedge X^{\wedge n})_{\Sigma_n}$$

which is induced at each level by the diagonal maps $K \to K^{\times n}$.

As for the cotensoring $Map_{alg}(K, X)$, the underlying symmetric spectrum is just $Map(K, X)$, with the algebra structures maps

$$\mathcal{O}(n) \wedge Map(K, X)^n \to Map(K, X)$$

being the adjoints to the composite

$$K \otimes \mathcal{O}(n) \wedge Map(K, X)^n \to K^n \otimes \mathcal{O}(n) \wedge Map(K, X)^n \cong \mathcal{O}(n) \wedge (K \otimes Map(K, X))^n \to X$$

where the first map is induced by the diagonal $K \to K^n$ and the last one by the counits $K \otimes Map(K, X) \to X$ and algebra structure map $\mathcal{O}(n) \wedge X^{\wedge n} \to X$.

1.5 Spectra on model categories

As shown in [26] (good) homogeneous functors between (good) model categories factor in an appropriate sense through the stabilization of those model categories, or in other words, spectra in those categories. We briefly recall here the notion of spectra used there.

Assume $\mathcal{C}$ a pointed simplicial model category, so that it also has a tensoring $\wedge$ over $SSet_*$.

Definition 1.14. The category $Sp(\mathcal{C})$

- objects X are sequences $X_n \in \mathcal{C}$, together with structure maps

$$S^1 \wedge X_n \to X_{1+n}$$

  where S^1 is the standard simplicial circle $\Delta^1/\partial\Delta^1$.

- maps $f \colon X \to Y$ are sequences of Σ_n maps $f_n \colon X_n \to Y_n$ compatible with the structure maps in the obvious way.

It is shown in [20] that $Sp(\mathcal{C})$ generally has a projective model structure.

Proposition 1.15 (Hovey)**.** *Suppose that $\mathcal{C}$ is a cofibrantly generated model category with I and J the generating cofibrations and trivial cofibrations.*

Then the **projective model structure** *on $Sp(\mathcal{C})$ exists and is cofibrantly generated with generating sets given by*[7]

$$I_{proj} = \bigcup_n F_n I, \quad J_{proj} = \bigcup_n F_n J.$$

[7]Here F_n denotes the free spectrum generated by an object in $\mathcal{C}$ at level n. The construction is entirely similar to that in 1.8, except one needs not care about Σ_n actions.

Definition 1.16. For $\mathcal{C}$ a cofibrantly generated model category the **stable model structure** on $Sp(\mathcal{C})$ is the left Bousfield localization (should it exist) of the projective model structure with respect to the set of maps

$$\{F_{n+1}(S^1 \wedge A_i) \to F_n A_i)\}$$

where A_i ranges over the (cofibrant replacements of, when necessary) domains and codomains of the maps in I.

Remark 1.17. For a detailed treatment of left Bousfield localizations, check [18, Chap. 3,4].

Notice that the existence of the **stable model structure** in $Sp(Alg_{\mathcal{O}})$ requires proof. Its existence does not follow for free from the result in [20], which shows that that model structure always exists when $\mathcal{C}$ is left proper cellular, because $Alg_{\mathcal{O}}$ is in fact almost never a left proper model category.

Notations 1.18. We denote by

$$\Sigma^\infty \colon \mathcal{C} \rightleftarrows Sp(\mathcal{C}) \colon \Omega^\infty$$

the standard Quillen adjoint functors.

Note that Σ^∞ is just what we previouly called F_0.

In fact, we will additionally denote by $\Omega^{\infty-n}$ the adjoint of F_n, which is naturally also denotable $\Sigma^{\infty-n}$.

Finally, by $\tilde{\Sigma}^{\infty-n}$ and $\tilde{\Omega}^{\infty-n}$ we denote the appropriate left and right derived functors[8] of $\Sigma^{\infty-n}$ and $\Omega^{\infty-n}$.

It will be useful to know that the conclusion of the following result of [20] holds when $\mathcal{C} = Sp^\Sigma$.

Theorem 1.19 (Hovey)**.** *Suppose $\mathcal{C}$ is a pointed simplicial almost finitely generated model category. Suppose further that in $\mathcal{C}$ sequential colimits preserve finite products, and that $Map(S^1, -)$ preserves sequential colimits.*

Then stable equivalences in $Sp(\mathcal{C})$ are detected by the $\tilde{\Omega}^{\infty-n}$ functors.

Proof. This is just a particular case of [20, Thm. 4.9]. □

The hypothesis of the result above do hold when Sp^Σ is given the flat stable model structure.

Indeed, in that case Sp^Σ is almost finitely generated (see [20] Chapter 4 for the definition), and in fact even finitely generated: searching the proofs given in [21] one sees that the generating cofibrations are maps $F_m^H((\delta\Delta^i)_+) \to F_m^H((\Delta_k)_+)$, with compact domains and codomains, and that the generating trivial cofibrations are the maps $F_m^H((\Lambda_k^i)_+) \to F_m^H((\Delta_k)_+)$ plus the simplicial mapping cylinders of the maps $F_{n+1}^*((S^1)_+) \to F_n^*((S^0)_+)$, which again have compact domains and codomains.

The remaining two conditions, that sequential and finite products commute, and that $Map(S^1, -)$ preserves sequential colimits, follow immediately from the fact that all these constructions are levelwise at the simplicial set level, where the statements are known to be true.

[8] As in [26] we assume that a set functorial choice of cofibrant and fibrant replacements has been made.

For the remaining model structures in Sp^{Σ} we listed it is not clear whether the hypotheses in Thm. 1.19 hold, but the **conclusion** of that result does nonetheless hold.

Indeed, to see this it suffices to check that the w.e.s on $Sp(Sp^{\Sigma})$ are exactly the same no matter what the base model category chosen on Sp^{Σ} is[9].

Now note that clearly the various possible *levelwise* model structures on $Sp(Sp^{\Sigma})$ are Quillen equivalent, and share the same weak equivalences. But since furthermore the notion of stable fibrant (i.e. local) objects in $Sp(Sp^{\Sigma})$ only changes with the base model structure on $Sp(\Sigma)$ up to levelwise w.e.s, it indeed follows that all the stable model structrures on $Sp(Sp^{\Sigma})$ do have the same w.e.s, as desired.

Remark 1.20. Note that for any of the choices of the underlying model structure in Sp^{Σ}, the generating trivial cofibrations in $Sp(Sp^{\Sigma})$ can be chosen to have cofibrant domains and codomains. This follows from the localization theory in [18], specifically [18, Prop. 4.5.1], together with the fact that the generating cofibrations in $Sp(Sp^{\Sigma})$ are directly seen to have cofibrant domains and codomains.

2 Model structure on Spectra of $Alg_{\mathcal{O}}$

As shown in [26] understanding homogeneous functors to/from a (good) model category is closely related to understanding its stabilization. Hence, our goal in this section will be to understand the (stable) model category $Sp(Alg_{\mathcal{O}})$.

Recall first of all that the underlying case model structures on $Alg_{\mathcal{O}}$ are those described in 1.4, i.e., the projective model structures built out of any of the "positive" stable model structures in Sp^{Σ}. We shall have no need to distinguish between these two model structures in our results.

Second, recall that in order to stabilize a model category $\mathcal{C}$, $\mathcal{C}$ must either be a pointed model category, else one must first replace $\mathcal{C}$ by its category $\mathcal{C}_*$ of pointed objects and then stabilize that category. In our context, we notice that $Alg_{\mathcal{O}}$ is pointed precisely when $\mathcal{O}(0) = *$ [10], hence we make that assumption for the rest of this section. We also point out that, by 1.12, $(Alg_{\mathcal{O}})_*$ is still always an operadic algebra category, hence our results still cover that case.

Our main result will be that, $Sp(Alg_{\mathcal{O}})$ is just $Mod_{\mathcal{O}(1)}(Sp^{\Sigma})$, up to a zig zag of Quillen equivalences.

However, some subtleties must be handled beforehand, like proving that the model structure on $Sp(Alg_{\mathcal{O}})$ actually exists. The reason one can't simply just use the theory developed in [20] is that $Alg_{\mathcal{O}}$ is rarely left proper.

Here's a sketch for why: it is easy (by appealing to universal properties), to check that in $Alg_{\mathcal{O}}$ it is $\mathcal{O}_X(A)\coprod_X Y = \mathcal{O}_Y(A)$. As the map $X \to \mathcal{O}_X(A)$ is a cofibration whenever A is cofibrant, one sees that left properness would require $X \mapsto \mathcal{O}_X$ to be a homotopical construction. But now recall that $\mathcal{O}_X(n)$ is constructed as a quotient of $\coprod_{m\geq 0} \mathcal{O}(n+m) \wedge_{\Sigma_m} X^{\wedge m}$, which is not expected to be a homotopical construction for general $\mathcal{O}$, and hence neither is $\mathcal{O}_X$. It is not too hard to craft specific examples where this can be seen explicitly.

[9] That these model structures always exist follows from Proposition 1.5 together with the first main theorem of [20].

[10] We shall also refer to this condition by saying that $\mathcal{O}$ is **non unital**.

In order to prove the existence of a model structure on $Sp(Alg_{\mathcal{O}})$ directly, we first notice that this category can also be regarded as $Alg_{\mathcal{O}_{sp}}(Sp(Sp^{\Sigma}))$, the algebras for a certain monad $\mathcal{O}_{sp}$ on the category $Sp(Sp^{\Sigma})$. We hence need only verify the hypotheses of [30, Lem. 2.3], a standard concerning existence of model structures in algebra categories.

We point out that this is exactly what is done in [15], where the pushouts necessary to apply [30, Lem. 2.3] are studied by means of an adequate filtration[11], and we will make use of a similar filtration adapted to the monad $\mathcal{O}_{sp}$.

Proposition 2.1. *$Sp(Alg_{\mathcal{O}})$ is the category $Alg_{\mathcal{O}_{sp}}(Sp(Sp^{\Sigma}))$ of algebras for the monad $\mathcal{O}_{sp}$ given by*

$$(\mathcal{O}_{sp}X)_n = \mathcal{O}(X_n)$$

with the structure maps given by the composite[12]

$$S^1 \wedge \mathcal{O}(X_n) \to \mathcal{O}(S^1 \wedge X_{n+1}) \to \mathcal{O}(X_{n+1}).$$

Proof. The first task is to show that $\mathcal{O}$ is actually a monad (and indeed a functor).

It is useful for this to consider the following auxiliary structures.

$$\wedge_{sp} \colon Sp(Sp^{\Sigma}) \times Sp(Sp^{\Sigma}) \to Sp(Sp^{\Sigma})$$

$$(X \wedge_{sp} Y)_n = X_n \wedge Y_n,$$

$$\wedge \colon Sp^{\Sigma} \times Sp(Sp^{\Sigma}) \to Sp(Sp^{\Sigma})$$

$$(X \wedge Y)_n = X \wedge Y_n,$$

where the structure maps for $X \wedge_{sp} Y$ are the natural composite $S^1 \wedge X_n \wedge Y_n \to (S^1)^{\wedge 2} \wedge X_n \wedge Y_n \cong S^1 \wedge X_n \wedge S^1 \wedge Y_n \to X_{n+1} \wedge Y_{n+1}$, with the first map induced by the diagonal map, and the structure maps for $X \wedge Y$ given by natural composite $S^1 \wedge X \wedge Y_n \cong X \wedge S^1 \wedge Y_n \to S^1 \wedge Y_{n+1}$.

It is then clear that $\wedge_{sp}$ is a **non unital** symmetric monoidal structure[13] on $Sp(Sp^{\Sigma})$.

Furthermore, $\wedge \colon Sp^{\Sigma} \times Sp(Sp^{\Sigma}) \to Sp(Sp^{\Sigma})$ behaves unitaly with respect to the unit of Sp^{Σ} and associatively with respect to both the monoidal structure[14] $\wedge$ on Sp^{Σ} and the monoidal structure $\wedge_{sp}$ on Sp^{Σ}.

Since furthermore each of these operations preserves colimits in each variable, it is formal to check that non unital operads in Sp^{Σ} induce monads on $Sp(Sp^{\Sigma})$, the monad associated to $\mathcal{O}$ being what we just called $\mathcal{O}_{sp}$.

It now remains to see that the categories $Sp(Alg_{\mathcal{O}})$ and $Alg_{\mathcal{O}_{sp}}(Sp(Sp^{\Sigma}))$ are indeed the same.

[11] Also, check [27, Sec. 5.2] for a slightly different approach.

[12] Here the first map is the map we called τ in 1.4.

[13] This monoidal structure may look strange and unfamiliar at first. There is a very good reason for this, namely the fact that $\wedge_{sp}$ is always homotopically trivial, i.e., $X \wedge_{sp} Y$ is always nullhomotpic. This fact plays a crucial role in the proof of the main result of this section.

[14] We purposefully abuse notation here in using the symbol $\wedge$ to denote two different operations. We believe this should not cause confusion.

First, we show the objects are the same. It is immediately clear that $\mathcal{O}_{sp}$ algebras X are made out of $\mathcal{O}$-algebras X_n at each level. So it really only remains to see that having a map $\mathcal{O}_{sp}(X) \to X$ is equivalent to having maps[15] $S^1 \wedge_{\mathcal{O}} X_n \to X_{n+1}$ of algebras. To see this first rewrite the structure maps of spectra in adjoint form[16]. But it is now clear that both conditions are just the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{O}(X_n) \arrow[r] \arrow[d] & \mathcal{O}(Map(S^1, X_{n+1})) \arrow[r] & Map(S^1, \mathcal{O}(X_{n+1})) \arrow[d] \\
X_n \arrow[rr] & & Map(S^1, X_{n+1}),
\end{tikzcd}$$

the only difference being that the first condition concerns the squares obtained by ommiting $\mathcal{O}(Map(S^1, X_{n+1}))$ and the second those obtained by ommiting $Map(S^1, \mathcal{O}(X_{n+1}))$.

It remains only to see that maps in the two categories are the same, but this is clear: compatibility with spectra structure maps gives the same condition in both cases, and compatibility with the $\mathcal{O}_{sp}$ algebra structures is the same as compatibility with the $\mathcal{O}$-algebra structures at all the levels.

□

Lemma 2.2. *The class of maps in $Sp(Sp^{\Sigma})$ which are both levelwise monomorphisms stable equivalences is closed under pushouts, transfinite compositions and retracts.*

Proof. Recall that weak equivalences in $Sp(Sp^{\Sigma})$ are detected as equivalences at the hocolim $\tilde{\Omega}^n_h X_{n+k}$ level.

Now consider a pushout of a such a map. Levelwise all these pushouts are actually homotopy pushouts (since the level cofibration model structure on Sp^{Σ} is left proper), and since Sp^{Σ} is a stable, they are actually also levelwise homotopy pullbacks. But then applying $\tilde{\Omega}^n$ turns such (levelwise) squares into homotopy pullbacks. But then it is obvious that the original pushout square is a homotopy pushout after applying any of the $\tilde{\Omega}^{\infty-n}$ functor.

A similar easier argument deals with the case of transfinite compositions, and the statement for retracts is obvious.

□

Theorem 2.3. *The (monadic) projective model structures on*

$$Sp(Alg_{\mathcal{O}}) \cong Alg_{\mathcal{O}_{sp}}(Sp(Sp^{\Sigma}))$$

based on either the positive stable or the positive flat stable model structures on Sp^{Σ} exist (and are cofibrantly generated).

Proof. We verify the conditions of [30, Lem. 2.3].

It is immediate that $\mathcal{O}_{sp}$ commutes with filtered direct colimits, and the smallness conditions are obviously satisfied by adapting the argument made in the proof of Proposition 1.5.

[15] Here we use $\wedge_{\mathcal{O}}$ to denote the pointed simplicial tensoring of $Alg_{\mathcal{O}}$.

[16] I.e., $X_n \to Map(S^1, X_{n+1})$ rather than $S^1 \wedge X_n \to X_{n+1}$.

It hence remains to check that for J a set of generating trivial cofibrations in $Sp(Sp^{\Sigma},\Sigma)$ then $\{\mathcal{O}_{sp}(J)\}_{reg}$, the closure of $\mathcal{O}_{sp}(J)$ under pushouts and transfinite compositions, consists of w.e.s..

Suppose now that $X \to Y$ is in J, and consider any pushout diagram in $Sp(Alg_{\mathcal{O}})$ of the form

$$\begin{tikzcd} \mathcal{O}_{sp}(X) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathcal{O}_{sp}(Y) \arrow[r] & B. \end{tikzcd} \tag{2.4}$$

By Lemma 2.2, we will be done provided we can show that the map $A \to B$ is a level monomorphism and an underlying stable equivalence in $Sp(Sp^{\Sigma})$.

The strategy for this is to filter the map $A \to B$ (analogously to [15] or [27, Sec. 5.2]). Explicitly, we write B as $colim(A_0 \to A_1 \to A_2 \to \dots)$, where the successive A_n are built as pushouts

$$\begin{tikzcd} \mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Q^t_{t-1} \arrow[r] \arrow[d] & A_{t-1} \arrow[d] \\ \mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Y^{\wedge_{sp} t} \arrow[r] & A_t. \end{tikzcd} \tag{2.5}$$

A little care is needed in explaining the notation here. We do not quite define terms $\mathcal{O}_{sp,A}(t)$ in this context since the colimits used for the analogous definition in [15] would make no sense, as they would involve both objects in Sp^{Σ} and in $Sp(Sp^{\Sigma})$ simultaneously[17]. Rather, we define expressions $\mathcal{O}_{sp,A}(t) \wedge Z_1 \wedge_{sp} \cdots \wedge_{sp} Z_t$ as a whole by the coequalizers[18]

$$\mathcal{O}_{sp,A}(t) \wedge Z^T = coeq(\coprod_{p\geq 0} \mathcal{O}(p+t) \wedge_{\Sigma_p} (\mathcal{O}A)^{\wedge p} \wedge Z^T \rightrightarrows \coprod_{p\geq 0} \mathcal{O}(p+t) \wedge_{\Sigma_p} A^{\wedge p} \wedge Z^T), \tag{2.6}$$

with the equalized maps corresponding to the algebra structure of A and the structure maps of the operad[19].

Additionally, as in [15], the symbol Q^t_{t-1} is meant to suggest the "union" (i.e. colimit) of the terms in the t-cube $(X \to Y)^{\wedge_{sp} t}$ with the $Y^{\wedge_{sp} t}$ terminal vertex removed[20]. Finally, the subscripts $\wedge_{\Sigma_t}$ are meant to denote that one takes coinvariants with respect to the obvious (diagonal) Σ_t action.

The existence of the relevant maps in 2.5, along with the existence of the relevant maps $A_t \to B$, is then formally analogous to the treatment in [15] (or [27, Sec. 5.2]), merely replacing the monad $\mathcal{O}$ by the monad $\mathcal{O}_{sp}$.

It follows immediately that B is indeed the colimit of the A_t, since these colimits are levelwise on the (outer) spectra coordinates, and our filtration just restricts to the one in [15](or [27, Sec. 5.2]) for each level.

Hence by Lemma 2.2 it suffices to show that the maps

$$\mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Q^t_{t-1} \to \mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Y^{\wedge t}$$

[17] This is a reflection of the fact that the monoidal structure $\wedge_{sp}$ is non unital.

[18] We use the shorthand $Z^T = Z_1 \wedge_{sp} \cdots \wedge_{sp} Z_t$.

[19] Though note that here we only ever need to use the first p partial composition products.

[20] Though, again, one needs to view the expression $\mathcal{O}_{sp,A}(t) \wedge Q^t_{t-1}$ defined as a whole.

are all level monomorphisms and stable equivalences.

To see this, we note first that, since we are in the case $\mathcal{O}(0) = *$, the equalizer in 2.6 splits into two summands: that with $p = 0$, which is just $\mathcal{O}(t) \wedge Z^{\wedge T}$, and that with $p \geq 1$. We deal with these summands separately.

For the $p = 0$ summand, the $t = 1$ term is just $\mathcal{O}(1) \wedge X \to \mathcal{O}(1) \wedge Y$, and since Remark 1.20 ensures X, Y are levelwise cofibrant, this map is a levelwise monomorphism and, further, the levelwise smash products are homotopically meaningful, so that

$$\begin{aligned} \tilde{\Omega}^{\infty-n}(\mathcal{O}(1) \wedge Z) =& hocolim_k \left(\tilde{\Omega}^k(\mathcal{O}(1) \wedge Z_{k+n})\right) \sim \\ \sim& hocolim_k(\mathcal{O}(1) \wedge \tilde{\Omega}^k Z_{k+n}) \sim \\ \sim& \mathcal{O}(1) \wedge hocolim_k(\tilde{\Omega}^k Z_{k+n}) = \mathcal{O}(1) \wedge \tilde{\Omega}^{\infty-n} Z \end{aligned} \tag{2.7}$$

and by Thm. 1.19 one concludes $\mathcal{O}(1) \wedge X \to \mathcal{O}(1) \wedge Y$ is also a stable equivalence.

Still in the case of the $p = 0$ summand, the $t > 1$ terms are $\mathcal{O}(t) \wedge_{\Sigma_t} Q^t_{t-1} \to \mathcal{O}(t) \wedge_{\Sigma_t} Y^t$. Combining Remark 1.20, which implies $X \to Y$ is a levelwise positive cofibration, with [27, Thm. 1.3] and (part (a) of) [27, Thm. 1.4], one sees that those terms are indeed levelwise monomorphisms. To see that these are also stable equivalences, we claim that in fact both $\mathcal{O}(t) \wedge_{\Sigma_t} Q^t_{t-1}$ and $\mathcal{O}(t) \wedge_{\Sigma_t} Y^t$ are actually null homotopic spectra. Indeed, since stable equivalences are detected by the $\Omega^{\infty-n}$ functors, this will follow if we show that the structure maps of these spectra are null homotopic. The structure maps of $\mathcal{O}(t) \wedge_{\Sigma_t} Y^t$ are built by first considering the maps

$$\mathcal{S}^1 \wedge (Y_n)^t \to \mathcal{S}^t \wedge (Y_n)^t \cong (S^1 \wedge Y_n)^t \to (Y_{n+1})^t$$

and then applying the functor $\mathcal{O}(t) \wedge_{\Sigma_t} -$. The map above is clearly null homotopic if one forgets the Σ_t action (as it factors through a higher suspension), and it remains null homotopic after applying the functor $\mathcal{O}(t) \wedge_{\Sigma_t} -$ since Remark 1.20 and [27, Thm. 1.3] imply $Y^{\wedge t}$ is S Σ-inj Σ_n-proj cofibrant[21] so that [27, Thm. 1.4] guarantees that $\mathcal{O}(t) \wedge_{\Sigma_t} -$ is a homotopical construction. The exact same analysis works for $\mathcal{O}(t) \wedge_{\Sigma_t} Q^t_{t-1}$.

It remains to deal with the $p \geq 1$ summand. Showing that the desired maps are levelwise monomorphisms follows by repeating the arguments for the $p = 0$ case. To see that they are also stable equivalences, we again reduce to showing that both the domains and codomains are null homotopic. Indeed, we claim the structure maps for $(\mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Y^{\wedge t})_{p \geq 1}$ factor as

$$S^1 \wedge (\mathcal{O}_{A_n}(t) \wedge_{\Sigma_t} Y^t_n)_{p\geq 1} \to S^2 \wedge (\mathcal{O}_{A_n}(t) \wedge_{\Sigma_t} Y^t_n)_{p\geq 1} \to (\mathcal{O}_{A_{n+1}}(t) \wedge_{\Sigma_t} Y^{\wedge t}_{n+1})_{p\geq 1}.$$

This can be seen from the following commutative diagram between the defining

[21]The reader is encouraged to think of this as a "lax" form of (projective) Σ_t-cofibrancy.

equalizers (displayed vertically)

$$\begin{tikzcd}
\Sigma\left(\coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}(\mathcal{O}A_n)^{\wedge p}\wedge(Y_n)^{\wedge t}\right) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \Sigma\left(\coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}(A_n)^{\wedge p}\wedge(Y_n)^{\wedge t}\right) \arrow[d] \\
\coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}\Sigma(\mathcal{O}A_n)^{\wedge p}\wedge\Sigma(Y_n)^{\wedge t} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}\Sigma(A_n)^{\wedge p}\wedge\Sigma(Y_n)^{\wedge t} \arrow[d] \\
\coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}(\mathcal{O}A_{n+1})^{\wedge p}\wedge(Y_{n+1})^{\wedge t} \arrow[r, shift left] \arrow[r, shift right] & \coprod_{p\geq 1}\mathcal{O}(p+t)\underset{\Sigma_p\times\Sigma_t}{\wedge}(A_{n+1})^{\wedge p}\wedge(Y_{n+1})^{\wedge t}
\end{tikzcd} \tag{2.8}$$

Here the top vertical maps are induced by the diagonal maps $S^1 \to S^2$, and the bottom vertical ones are the induced by the previously described spectra structure maps. Note that the group actions are trivial on these sphere coordinates. That the diagram commutes follows from the fact that the bottom maps are themselves built using diagonal maps. As in the $p = 0, t > 1$ it follows that the $(\mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Y^t)_{p\geq 1}, (\mathcal{O}_{sp,A}(t) \wedge_{\Sigma_t} Q^t_{t-1})_{p\geq 1}$ spectra summands are null homotopic (note however that no form of Σ_t-cofibrancy is required in this case), concluding the proof.

□

Corollary 2.9. *The induced adjunctions $Sp(Mod_{\mathcal{O}[1]}) \rightleftarrows Sp(Alg_{\mathcal{O}})$ are Quillen equivalences.*

Proof. We need to show that, for $A \in Sp(Sp^{\Sigma})$ cofibrant and $D \in Sp(Alg_{\mathcal{O}})$ fibrant, a map $\mathcal{O}_{sp}A \to D$ is a w.e. iff the adjoint map $A \to D$ is. However, the second of these maps factors through the first as $A \xrightarrow{\mu} \mathcal{O}A \to D$, where μ is the unit of the free-forget adjunction. Hence it suffices to show that μ is a stable equivalence for any cofibrant A.

To see this, a retraction argument shows that one can reduce to the case of A being transfinitely built out as $A = colim_{\beta<\kappa} A_\beta$ where each successor ordinal map $A_\beta \to A_{\beta+1}$ is a pushout as in (2.4). It then suffices to inductively show that in the diagram

$$\begin{tikzcd}
\mathcal{O}_{sp}(X_\beta) \arrow[r] \arrow[d] & A_\beta \arrow[r] \arrow[d] & \mathcal{O}A_\beta \arrow[d] \\
\mathcal{O}_{sp}(Y_\beta) \arrow[r] & A_{\beta+1} \arrow[r] & \mathcal{O}A_{\beta+1}.
\end{tikzcd} \tag{2.10}$$

the right side "pushout corner map" $A_{\beta+1} \coprod_{A_\beta} \mathcal{O}A_\beta \to \mathcal{O}A_{\beta+1}$ is both a monomorphism and a stable equivalence (so that $A_\beta \to \mathcal{O}A_\beta$ is a trivial cofibration in the transfinite directed diagram category $(Sp(Sp^{\Sigma}))^{\kappa}$). But this now clearly follows from the analysis in the proof of Theorem 2.3, which shows that the $p = 0$, $t = 1$ part of the filtration of $\mathcal{O}A_\beta \to \mathcal{O}A_{\beta+1}$ is just the pushout of $A_\beta \to A_{\beta+1}$, and all the other parts of the filtration are null-homotopic.

□

Remark 2.11. We point out that Theorem 2.3 could have itself been proven by regarding $Sp(Alg_{\mathcal{O}})$ as the algebras for a certain monad over[22] $Sp(Mod_{\mathcal{O}[1]})$.

This would entail studying colimits of the form

$$\begin{tikzcd} \mathcal{O} \circ_{\mathcal{O}(1)} (X) \arrow[r] \arrow[d] & A \arrow[d] \\ \mathcal{O} \circ_{\mathcal{O}(1)} (Y) \arrow[r] & B, \end{tikzcd}$$

where X and Y are $\mathcal{O}(1)$-modules, by producing a filtration analogous to the one described in Diagram 2.5. Indeed, the only difference between the normal construction and this one is that $\wedge$ symbols get replaced by relative $\wedge_{\mathcal{O}(1)}$ symbols.

One would then obtain a sequence of $(\mathcal{O}(1), \mathcal{O}(1)^{\wr n})$-bimodules $\mathcal{O}_A^{\mathcal{O}(1)}(n)$, forming a "relative enveloping operad". We point out, however, that $\mathcal{O}_A^{\mathcal{O}(1)}(n)$ is just $\mathcal{O}_A(n)$ again[23]. Indeed, it is straightforward to see that the $\mathcal{O}_A(n)$ are themselves $(\mathcal{O}(1), \mathcal{O}(1)^{\wr n})$-bimodules (either by direct analysis of the formula, or by using the canonical map of operads $\mathcal{O} \to \mathcal{O}_A$), and one sees that these sequences must match since they are both codifying left adjoints to forgetful functors.

Remark 2.12. Notice that in the commutative diagram of Quillen adjunctions (with vertical adjunctions induced by the map of operads $\mathcal{O} \to \mathcal{O}(1)$)

$$\begin{tikzcd} Alg_{\mathcal{O}} \arrow[r, shift left] \arrow[d, shift right] & Sp(Alg_{\mathcal{O}}) \arrow[l, shift left] \arrow[d, shift right] \\ Mod_{\mathcal{O}(1)} \arrow[r, shift left] \arrow[u, shift right] & Sp(Mod_{\mathcal{O}(1)}) \arrow[l, shift left] \arrow[u, shift right] \end{tikzcd}$$

both the lower and right adjunctions are Quillen equivalences. Indeed, that the lower adjunction is a Quillen equivalence follows from Theorem 5.1 of [20], since $Mod_{\mathcal{O}(1)}$ was stable to start with. The right adjunction is also the right adjunction in the diagram of adjunctions

$$Sp(Mod_{\mathcal{O}(1)}) \rightleftarrows Sp(Alg_{\mathcal{O}}) \rightleftarrows Sp(Mod_{\mathcal{O}(1)})$$

induced by the maps of operads $\mathcal{O}(1) \to \mathcal{O} \to \mathcal{O}(1)$. Since these maps compose to the identity in $\mathcal{O}(1)$, so do the adjunctions, so that the right adjunction will be a Quillen equivalence if the first is, and this we showed in Corollary 2.9.

It then follows that we can think of the adjunction

$$Alg_{\mathcal{O}} \underset{forget}{\overset{\mathcal{O}(1)\circ_{\mathcal{O}}-}{\rightleftarrows}} Mod_{\mathcal{O}(1)}$$

as the stabilizing adjunction for $Alg_{\mathcal{O}}$, with $\mathcal{O}(1) \circ_{\mathcal{O}} -$ playing the role of Σ^∞ and $forget$ the role of Ω^∞, so that we will occasionally refer to these functors by those names.

[22] Note also that the theory from [20] does ensure that a stable model structure on $Sp(Mod_{\mathcal{O}[1]})$ exists, since $Mod_{\mathcal{O}[1]}$ is left proper.

[23] Note that this is not quite obvious from the defining formulae.

We notice that both of these functors also go by other names in the literature. Σ^∞ is often called the **indecomposables** functor, since it is obtained from an algebra X by killing elements that can be written as higher n-ary operations (i.e. $n \geq 2$). It is also costumary, at least for some operads, to denote its derived functor by **topological Andre-Quillen homology**. Ω^∞ is often called the **trivial extension**, since its image consists of those algebras where the higher n-ary operations identically vanish.

Finally, notice that we do not at this point yet know if the model structure in $Alg_{\mathcal{O}}$ we constructed fits the paradigm described on Subsection 1.5.

Proposition 2.13. *The model structure defined by Theorem 2.3 coincides with the left Bousfield localized model structure as required by Definition 1.16.*

Proof. We first show that the two model structures have the same cofibrations. Both sets of generating cofibrations are constructed based on the generating cofibrations I of Sp^Σ. For the model structure of Theorem 2.3 the generating cofibrations are then $\mathcal{O}_{sp}(\bigcup_n F_n I)$, while for the model structure in Definition 1.16 they are $\bigcup_n F_n \mathcal{O} I$. It is straightforward to check that these sets match (indeed, this just repeats the analysis in Proposition 2.1).

Now repeating the analysis above it also follows immediately that the generating trivial cofibrations for the projective model structure on $Sp(Alg_{\mathcal{O}})$ are trivial cofibrations for the model structure from Theorem 2.3, so that one concludes that the latter model structure is a left Bousfield localization of the former. Since left Bousfield localizations are completely determined by the classes of local objects [24], it now suffices to check that those are the same in both model structures.

By Proposition 3.2 of [20][25], the local objects as defined using Definition 1.16 are levelwise fibrant spectra $X_* \in Sp(Alg^{\mathcal{O}})$ such that the structure maps $X_n \to Map(S_1, X_{n+1})$ are weak equivalences. Since clearly these are also the fibrant objects for the model structure of Theorem 2.3, the proof is complete. $\square$

3 Homogeneous functors from Mod_A to Mod_B

Our goal in this section is to classify finitary homogeneous functors from Mod_A to Mod_B. We start by recalling the analogous result for finitary homogeneous functors from Sp^Σ to Sp^Σ (this can be found, for instance , as Corollary 2.5 in [22]).

Theorem 3.1 (Goodwillie)**.** *Let $F \colon Sp^\Sigma \to Sp^\Sigma$ be a finitary homogeneous functor. Then there exists a spectrum $\delta_n F$ with a Σ_n action such that F is homotopic to the functor $X \mapsto (\delta_n F \wedge X^{\wedge n})_{h\Sigma_n}$.*

The purpose of section is to show the following natural generalization.

[24] Note that this is not the same as saying that a model structure is determined by the cofibrations and the cofibrant objects.

[25] Note that though that the statement of that Proposition requires left properness, the remarks immediately after the proof point out that that condition is unnecessary when the domains of the generating cofibrations are themselves cofibrant, as is the case for $Alg_{\mathcal{O}}$.

Theorem 3.2. *Let $F\colon Mod_A \to Mod_B$ be a finitary homogeneous simplicial functor. Then there exists a $(B, A^{\wedge n})$-bimodule $\delta_n F$ with a Σ_n action interchanging the A-module structures (or, in other words, a $(B, A^{\wr n})$-bimodule) such that such that F is homotopic to the functor $X \mapsto (\delta_n F \wedge_{A^{\wedge n}} X^{\wedge n})_{h\Sigma_n}$.*

Here we assume the functor is already simplicial both for simplicity and because that suffices for the purpose we have in mind.

Naturally the bimodule $\delta_n F$ in the theorem has to be $F(A \wedge S_0)$, the value at the free module over the sphere spectrum. The hardest task is to prove that in general this object can actually be given the desired bimodule structure in a strict sense.

Recalling that a (strict) A-module structure on X is just a map of (strict) ring spectra $A \to End(X)$ it is clear that this problem would be immediatelly solved if one were dealing with spectral functors rather than mere simplicial ones. Further, since a heuristic argument shows that sprectral functors and linear simplicial functors share many homotopical properties, the case for $n = 1$ would essentially be solved were one to show that the homotopy theories of such functors are equivalent, and this is the essential goal of the next subsection.

Further, we note that the case $n > 1$ does not really present a bigger challenge. Indeed, though n-homogeneous functors can not, of course, be made spectral, they correspond to symmetric n-multilinear functors (when the target category is stable), and hence the $n > 1$ will also be solved if one generalizes the $n = 1$ result to show that the homotopy theories of multilinear multisimplicial symmetric functors and multispectral symmetric functors are equivalent.

3.1 Linear simplicial functors are spectral functors

In this subsection we shall be using several basic notions of enriched category theory, enriched representable functors and weighted enriched colimits, along with some basic results about those. We recomend [29] as a general reference for these.

Remark 3.3. Let Sp^{Σ} be given the flat stable model structure, and consider the induced projective model structure on Mod_A, for A any ring spectrum.

Then it follows from Sp^{Σ} being a monoidal category that Mod_A is a Sp^{Σ} model category with mapping spectra

$$Sp_A(X, Y) = eq(Sp(A \wedge X, Y) \rightrightarrows Sp(X, Y))$$

and the obvious (underlying) spectral tensoring and cotensoring.

Notice that Mod_A is also a simplicial model category by reduction of sctruture, with the simplicial mapping spaces induced given by $SSet_A(X, Y) = \Omega^{\infty} Sp_A(X, Y)$, and the obvious (underlying) simplicial tensoring and cotensoring.

We will have use for the following technical result concerning cofibrant/ fibrant replacements in these categories.

Proposition 3.4. *Mod_A with the model structure described above has simplicial cofibrant and fibrant replacement functors. Furthermore, it also has spectral fibrant replacement functors.*

Proof. The statement about simplicial cofibrant and fibrant replacements is well known (see, for example, Theorem 13.5.2 in [29]), as one needs only to perform the enriched version of the Quillen small object argument, which then works becuse all simplicial sets are cofibrant.

The claim about the existence of a spectral fibrant replacement functor is more delicate precisely because not all spectra are cofibrant, but a careful analysis of the enriched small object argument in that case still provides a fibrant replacement.

Indeed, note that a general generating trivial cofibration for Mod_A has the form $A \wedge Z \to A \wedge W$, for $Z \to W$ a generating trivial cofibration of Sp^Σ, and that the enriched argument requires, at the stage X_β for one to build $X_{\beta+1}$ as the pushout

$$\begin{tikzcd}
\displaystyle\bigvee_{\text{gen. triv. cof. } Z\to W} Sp^\Sigma(A\wedge Z, X_\beta)\wedge A\wedge Z \arrow[r] \arrow[d] & X_\beta \arrow[d] \\
\displaystyle\bigvee_{\text{gen. triv. cof. } Z\to W} Sp^\Sigma(A\wedge Z, X_\beta)\wedge A\wedge W \arrow[r] & X_{\beta+1}.
\end{tikzcd}$$

One now sees that the map $Sp^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge Z \to Sp^\Sigma(A \wedge Z, X_\beta) \wedge A \wedge W$ is still a trivial cofibration in the levelwise model structure (though, crucially, possibly not in Mod_A), so it still follows that the map $X \to X_\infty$ is an equivalence, and it is formal to check that X_∞ is fibrant, proving the result. $\square$

Proposition 3.5. *Let $C \subset Mod_A$ be a small subcategory.*

Consider the categories $Fun_{SSet}(C, Mod_B)$ and $Fun_{Sp^\Sigma}(C, Mod_B)$ of, respectively, enriched simplicial and enriched spectral functors from C to Mod_B.

Then the projective model structures on both of these functor categories exist.

Furthermore, these model categories are simplicial and cofibrantly generated, and $Fun_{Sp^\Sigma}(C, Mod_B)$ is also a spectral model category.

Proof. Letting I and J denote the sets of generating cofibrations and generating cofibrations of Mod_B, one obtains the natural candidates for the generating cofibrations for the functor categories $Fun_{SSet}(C, Mod_B)$ and $Fun_{Sp^\Sigma}(C, Mod_B)$, the sets $\coprod_{c\in Ob(C)} SSet(c,-)\wedge I$, $SSet(c,-)\wedge J$ and $Sp^\Sigma(c,-)\wedge I$, $Sp^\Sigma(c,-)\wedge J$

To see this defines a cofibrantly generated model category one needs to check the conditions of Theorem 2.1.19 of [19]. The only non obvious condition is 4, the fact $J - cell \subset W \cap (I - cof)$ or, more specifically, proving $J - cell \subset W$. But this follows from the fact that the maps in $SSet(c,-)\wedge J$ and $Sp^\Sigma(c,-)\wedge J$ are levelwise monomorphisms and stable equivalences (since so is the smash of a stable trivial cofibration with <u>any</u> spectrum).

That these categories are simplicial is formal, with mapping spaces defined by

$$SSet(F, G) = eq(\prod_{c\in C} SSet(F(c), G(c)) \rightrightarrows \prod_{c,c'\in C} SSet(F(c) \wedge SSet(c, c'), G(c')))$$

with each map induced by adjointess using either the structure of either F or G as simplicial functors. It is also formal that one has simplicial tensoring and cotensorings, which are just pointwise, and it is then obvious that the model structure is simplicial (by verifying that condition using the cotensoring).

The proof that $Fun_{Sp^\Sigma}(C, Mod_B)$ is also a spectral model category is entirely analogous. $\square$

Proposition 3.6. *There is a simplicial Quillen adjuntction*

$$Spf\colon Fun_{SSet}(C, Mod_B) \rightleftarrows Fun_{Sp^\Sigma}(C, Mod_B)\colon fgt$$

Proof. The right adjoint is the natural "restriction" of spectral functors to simplicial ones obtained by applying Ω^∞ to the maps

$$Sp^\Sigma_C(c, c') \xrightarrow{G(c,c')} Sp^\Sigma_{Mod_B}(G(c), G(c'))$$

that compose a spectral functor G.

The left adjoint is defined freely by its value on the representable functors $SSet(c, -) \wedge X$, which one easily verifies are necessarily sent to $Sp^\Sigma(c, -) \wedge X$. Since any simplicial functor is canonically an enriched weighted colimit of representables, namely

$$F = colim^W_{C^{op}\times Mod_B}(SSet(c, -) \wedge X),$$

where the weight $W\colon C \times D^{op}$ is $W(c, x) = SSet(X, F(c))$, one obtains the explicit formula

$$SpfF = colim^W_{C^{op}\times Mod_B}(Sp^\Sigma(c, -) \wedge X).$$

$\square$

It is worth noting that the model categories we just defined are not the ones we are ultimately interested in, both because Goodwillie calculus deals only with homotopy functors and because we want to restrict to those topological functors which happen to be 1-homogeneous.

In order to do this, we need to localize the model structures we just defined, and being able to do this is the main purpose of the following result.

Lemma 3.7. *The model structures on $Fun_{SSet}(C, Mod_B)$ and $Fun_{Sp^\Sigma}(C, Mod_B)$ are left proper cellular.*

Proof. Reall that it was shown in the proof of 3.5 cofibrations in these categories are in particular pointwise monomorphic natural transformations.

Hence left properness follows immediately from the fact that in Mod_B the pushouts of weak equivalences by monomorphisms are weak equivalences.

As for cellularity, the proof is essentially a repeat of the proof of 1.5, but with a minor wrinkle, which we explain now. Suppose given a natural transformation $F \xrightarrow{\tau} G$, where G is a cellular functor. Since the generating cofibrations are pointwise monomorphisms, the argument from the proof of 1.5 applies to show that there is a small enough[26] subcellular functor $\tilde{G}$ of G such that the $\tau(c)$ factor uniquely as $F \xrightarrow{\tilde{\tau}(c)} \tilde{G} \xrightarrow{i(c)} G$. The wrinkle to verify is that these $\tilde{\tau}(c)$ assemble to an enriched natural transformation. This amounts to a straightforward diagram chase of diagrams of enriched mapping spaces, though one needs to point out that the $i(c)$ are monomorphisms in the enriched sense, rather than just categorically. $\square$

[26]The cardinal of cells being bounded by the union of the cardinals of all the simplices in the $F(c)$.

Lemma 3.7 means that, by the main Theorem of [18], one is free to localize these model structures, and hence to obtain appropriate model categories of homotopical, excisive, etc, functors. Doing this requires some technical care, however, since our source category C is a small subcategory of Mod_A. Hence, for instance, demanding a functor to be homotopical is not something one would expect meaningful if C is too small, so that w.e. objects can not actually be linked by (a zig zag of) weak equivalences in C.

Ensuring these type of problems do not occur is the goal of the following definition.

Definition 3.8. Let $C \subset Mod_A$ be a small full simplicial/spectral subcategory.

We say C is **Goodwillie closed**, if $* \in C$, C is closed under finite hocolimits, tensoring with finite spectra, a chosen spectral fibrant replacement functor (as is ensured to exist by Proposition 3.4), and a chosen simplicial cofibrant replacement functor.

Notice that any small C has a small Goodwillie closure $\bar{C}$, since the closure conditions do not increase the cardinal of any (infinite) C.

Definition 3.9. The **homotopical** model structures $Fun^h_{SSet}(C, Mod_B)$ and $Fun^h_{Sp^\Sigma}(C, Mod_B)$ are the left Bousfield localizations with respect to the maps $SSet(c, -) \to SSet(c', -)$ induced by weak equivalences in C.

Notice that the fibrant objects for these model structures are just the levelwise fibrant homotopical functors.

Remark 3.10. Notice that since the left adjoint clearly sends fibrant objects to fibrant objects, the adjunction from 3.6 descends to a Quillen adjunction between the homotopical model structures

$$Fun^h_{SSet}(C, Mod_B) \rightleftarrows Fun^h_{Sp^\Sigma}(C, Mod_B).$$

Definition 3.11. The linear model structure

$$Fun^{h,lin}_{SSet}(C, Mod_B)$$

is the further localization with respect to the maps

$$* \to SSet(*, -) \wedge X,$$

and, for each hococoartesian square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

in $\mathcal{C}$, the maps $SSet(B,-)\wedge X \coprod^h_{SSet(D,-)\wedge X} SSet(C,-)\wedge X \to SSet(A,-)\wedge X$. (Here in both cases X ranges over the domains/codomains X of the generating cofibrations of Mod_B.)

Remark 3.12. Notice that, by Proposition 3.2 of [20], which says that w.e.s in Mob_B can be detected by the mapping spaces out of such X, it follows that the fibrant objects of $Fun^{h,lin}_{SSet}(C, Mod_B)$ are precisely the levelwise fibrant, homotopy functors which are homotopically pointed[27] and sending pushout squares to pullback squares.

[27] I.e., sending $*$ to a contractible object.

Lemma 3.13. *Suppose C is Goodwillie closed.*

Then for F any homotopical spectral functor, the reduced topological functor $fgtF$ is pointed and 1-excisive.

Proof. To see that $fgtF$ is pointed, note that $*$ is the only object in either C or Mod_B where the identity map and the null self map coincide. But since any functor preserves identity maps and spectral functors preserve null maps it follows that $fgtF$ is pointed.

To check that such a functor is 1-excisive it suffices, by [26], to check that the natural map[28] $F \to \Omega F \Sigma$ is a weak equivalence (here we are free to assume that F is pointwise bifibrant, as necessary). Since the target category is stable this is equivalent to showing that the adjoint map $\Sigma F \to F\Sigma$ is a weak equivalence. We show this by proving the existence of left and right inverses up to homotopy. To do this, let Σ^{-1} denote $F_1 S^0$ (recall that Σ and Ω are defined based on $F_0 S^1$). Notice that there is a natural w.e. $\Sigma^{-1}\Sigma \xrightarrow{\epsilon} S^0$.

To prove the existence of a left inverse is suffices to do so for $\Sigma^{-1}\Sigma F \to \Sigma^{-1} F \Sigma$, and this inverse provided by the diagram

$$\Sigma^{-1}\Sigma F \to \Sigma^{-1} F\Sigma \to F\Sigma^{-1}\Sigma \to F$$

where the second map follows from F being a spectral functor, and the last map is induced by ϵ. The full composite is then a weak equivalence since it too is induced by ϵ.

The other side is analogous. To show a left inverse it suffices to do so for $\Sigma F \Sigma^{-1} \to F\Sigma\Sigma^{-1}$. This inverse is provided by the diagram

$$\Sigma\Sigma^{-1} F \to \Sigma F \Sigma^{-1} \to F\Sigma\Sigma^{-1} \to F$$

where the first map again follows by F being a spectral functor, the last map from ϵ, and again the full composite is a weak equivalence since it is induced by ϵ.

$\square$

Lemma 3.14. *Suppose C is Goodwillie closed.*

Then the fibrant replacement of $SSet(c,-)\wedge X$ in $Fun^{h,lin}_{SSet}(C, Mod_B)$, where X is stable cofibrant, is, up to levelwise fibrant replacement, given by

$$fgt(Sp^{\Sigma}((c)^c, (-)^f) \wedge X),$$

where $(-)^f$ denotes a (spectral) functorial fibrant replacement functor, and $(-)^c$ denotes any cofibrant replacement functor.

Proof. Since levelwise fibrant replacement does not affect the other fibrancy conditions in $Fun^{h,lin}_{SSet}(C, Mod_B)$ we will largely omit it so as to simplify notation.

First we note that $SSet(c,-) \wedge X$ a priori fails all fibrancy conditions, as it is neither homotopical, homotopically pointed, or 1-excisive. We deal with these conditions in succession.

First, we claim the "homotopification" of $SSet(c,-)\wedge X$ is $SSet((c)^c, (-)^f)\wedge X$. That the canonical map $SSet(c,-) \wedge X \to SSet((c)^c, -) \wedge X$ is w.e. in

[28] Here we drop fgt for simplicity of notation.

the homotopical model structure is immediate since this is precisely one of the maps being localized. Hence one needs only deal with the case of c a cofibrant object, which we assume in the remainder of the proof. Now notice that there is a canonical natural transformation $SSet(c,-) \wedge X \to SSet(c,(-)^f) \wedge X$, induced by the natural transformation $id \to (-)^f$, and hence, since clearly $SSet(c,(-)^f) \wedge X$ is homotopic, the claim will follow if we show that it is the universal homotopy functor with a map from $SSet(c,-) \wedge X$, where universality is read in the homotopy category of functors. Hence let $SSet(c,-) \wedge X \to F$ be a natural transformation with F a homotopy functor. Existence of the factorization then follows immediatly from the fact that the map on the right in the natural diagram

$$\begin{tikzcd}
SSet(c,-) \wedge X \arrow[r] \arrow[d] & F \arrow[d, "\sim"] \\
SSet(c,(-)^f) \wedge X \arrow[r] & F \circ (-)^f
\end{tikzcd}$$

is a levelwise equivalence. Similarly, uniqueness follows from the bottom left map in

$$\begin{tikzcd}
SSet(c,-) \wedge X \arrow[r] \arrow[d] & SSet(c,(-)^f) \wedge X \arrow[r] \arrow[d, "\sim"] & F \arrow[d, "\sim"] \\
SSet(c,(-)^f) \wedge X \arrow[r, "\sim"] & SSet(c,((-)^f)^f) \wedge X \arrow[r] & F \circ (-)^f
\end{tikzcd}$$

being a levelwise equivalence. Indeed, the claim is that the right upper map is determined by the upper composite. But for this is suffices for it to be determined by the lower composite, and this follows from knowing that the indicated maps are levelwise equivalences.

We next show that the homotopically pointed localization of $SSet(c,(-)^f) \wedge X$ is $SSet_*(c,(-)^f) \wedge X$, where we denote by $SSet_*(-,-)$ the enrichement over pointed simplicial sets $SSet_*$ of the simplical pointed category C. This may seem slightly confusing since $SSet(-,-)$ and $SSet_*(-,-)$ have the same underlying simplicial set, but the crucial point here is that the $\wedge$ operations are different depending on which context one is working in, so that $SSet(c,(-)^f) \wedge X$ becomes reinterpreted as $(SSet_*(c,(-)^f))_+ \wedge X$ in the pointed simplicial context. But now notice that the natural pushout diagram

$$\begin{tikzcd}
X \simeq (SSet_*(*,(-)^f))_+ \wedge X \arrow[r] \arrow[d] & (SSet_*(c,(-)^f))_+ \wedge X \arrow[d] \\
* \arrow[r] & SSet_*(c,(-)^f) \wedge X
\end{tikzcd}$$

is in fact a levelwise homotopy pushout (this follows from the properties of the level cofibration model structure on spectra, since the top map is an injection), and hence it is now clear that

$$Map(SSet(c,(-)^f) \wedge X, F) \simeq Map(SSet_*(c,(-)^f) \wedge X, F)$$

for any F a homotopically pointed functor.

Finally, it remains to show that the "1-excisification" of $SSet_*(c, (-)^f) \wedge X$ is $fgt(Sp^\Sigma(c, (-)^f))$. Recall that, as proven in [26], and following Goodwillie, the linearization of a pointed (pointwise fibrant) simplicial functor F can be computed by $hocolim_k(\Omega^k \circ F \circ \Sigma^k)$. Consider the natural morphism[29]

$$hocolim_k(\Omega^k \circ (SSet_*(c, (-)^f) \wedge X)^f \circ \Sigma^k) \to hocolim_k(\Omega^k \circ (Sp^\Sigma(c, (-)^f) \wedge X)^f \circ \Sigma^k).$$

By Lemma 3.13, it suffices to show that this map is a weak equivalence. But up to equivalence this map can be rewritten as

$$hocolim_k(\Omega^k \circ \Sigma^\infty SSet_*(c, (-)^f) \circ \Sigma^k) \wedge X \to hocolim_k(\Omega^k \circ Sp^\Sigma_*(c, (-)^f) \circ \Sigma^k) \wedge X,$$

and it hence suffices to show that $hocolim_k(\Omega^k \circ \Sigma^\infty SSet_*(c, (-)^f) \circ \Sigma^k) \to hocolim_k(\Omega^k \circ Sp^\Sigma_*(c, (-)^f) \circ \Sigma^k)$ is a weak equivalence, but this follows from the fact that a spectrum K can be described as $hocolim(\Sigma^{\infty - k}(\Sigma^k K))$, finishing the proof.

□

Theorem 3.15. *The adjunction*

$$Fun^{h,lin}_{SSet}(Mod^{fin}_A, Mod_B) \rightleftarrows Fun^h_{Sp^\Sigma}(Mod^{fin}_A, Mod_B)$$

is a Quillen equivalence.

Proof. That this is indeed a Quillen adjunction is just a restatement of Lemma 3.13.

Next we check that the (homotopy) unit of the adjunction is a w.e.. By the small object argument for the formation of cofibrant replacements it suffices to prove the statment for functors built cellularly from the generating cofibrations $SSet(c, -) \wedge X \to SSet(c, -) \wedge Y$. Obviously the left adjoint functor preserves homotopy colimits, but note further that so does the right adjoint fgt, since colimits of natural transformations are pointwise in the target. It hence suffices to show that the (homotopy) unit is a w.e. for functors of the form $SSet(c, -) \wedge X$. But this is just Lemma 3.14 together with the fact that the homotopical replacement of $Sp^\Sigma(c, -) \wedge X$ is $Sp^\Sigma((c)^c, (-)^f) \wedge X$, as is clear from the proof of that Lemma.

Finally, it suffices to check that the (right derived functor of the) right adjoint is conservative with respect to weak equivalences. But this is clear since in both categories weak equivalences between fibrant functors are given by pointwise weak equivalences, and clearly fgt preserves those.

□

The previous theorem naturally generalizes to the case of multilinear functors. We briefly indicate the changes that need to be made to the previous discussion to obtain that result.

Definition 3.16. Let $\mathcal{C}$, $\mathcal{D}$ be categories. A symmetric n-multifunctor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $\mathcal{C}^n \xrightarrow{F} \mathcal{D}$ together with natural isomorphisms

$$F \xrightarrow{\sim} F \circ \sigma,$$

where σ denotes the natural action of a permutation on $\mathcal{C}^n$, together with the obvious compatibility conditions.

[29]Here we omit the reduction fgt of spectral functors to simplicial functors for simplicity of notation.

Theorem 3.17. *Suppose C is Goodwillie closed.*

Then there is a Quillen equivalence

$$SymMFun^{h,multlin}_{SSet}(Mod^{fin}_{A}, Mod_B) \rightleftarrows SymMFun^{h}_{Sp^{\Sigma}}(Mod^{fin}_{A}, Mod_B).$$

Proof. First notice that in the category $MFun_{SSet}(C, D)$ of multifunctors without symmetry constraints, the role of representable functors is played by the functors of the form $SSet(c_1, -) \wedge \cdots \wedge SSet(c_n, -) \wedge X$, for $(c_1, \ldots, c_n) \in \mathcal{C}^n$. There is an obvious action of Σ^n on such functors (permuting the c_i), and hence one sees that the analog of 3.5 showing the existence of a cofibrantly generated projective model structure follows by considering generating cofibrations $I \wedge SSet(c_1, -) \wedge \cdots \wedge SSet(c_n, -) \wedge_{\Sigma_n} \Sigma_n$ and generating trivial cofibrations $J \wedge SSet(c_1, -) \wedge \cdots \wedge SSet(c_n, -) \wedge_{\Sigma_n} \Sigma_n$. The case of multispectral functors follows analogously.

The proof of the analog of Lemma 3.7, showing these are left proper cellular model structures, is entirely analogous, and the sets of localizing maps to obtain multihomotopical and multilinear model structures are induced from the $n = 1$ case by[30]

$$S_{multh} = S_h \wedge SSet(c_2, -) \wedge \cdots \wedge SSet(c_2, -) \wedge_{\Sigma_n} \Sigma_n$$

$$S_{multlin} = S_{lin} \wedge SSet(c_2, -) \wedge \cdots \wedge SSet(c_2, -) \wedge_{\Sigma_n} \Sigma_n.$$

The analog of Proposition 3.6, the existence of an adjunction is perhaps a little less clear, since it may not be obvious how to canonically express a symmetric functor as a colimit of appropriate representables. Rather than proving such a colimit expression, we notice instead that since such a technique does work for the non symmetric functor categories $MFun$, and since the right adjoint in that case is obviously compatible with the Σ_n action on $\mathcal{C}^n$, then so is the left adjoint, and hence that left adjoint naturally sends symmetric functors to symmetric functors, providing the desired adjuntion.

Finally, to finish the proof that this adjunction is indeed a Quillen equivalence one needs only the appropriate analogs of Lemmas 3.13 and 3.14, the proofs of which require no noteworthy alterations. □

3.2 Characterization of n-homogeneous finitary functors

In this section we finish the proof of Theorem 3.2, after a couple of additional Lemmas.

Here is a sketch of the main idea, ignoring some technicalities. As mentioned before, it suffices to prove the associated result saying that any symmetric G multilinear functor has the form

$$X_1, \ldots, X_n \mapsto \delta_n G \wedge_{A^{\wedge n}} X_1 \wedge \cdots \wedge X_n.$$

By a cofibrant replacement argument, one can essentially assume that the objects in the source category are cell complexes. Lemma 3.18 then says that any cell complex is the filtered hocolimit of its finite sucomplexes, meaning that G can be completely recovered from its values on the category of finite complexes

[30]Here we are abusing notation by noting that S_h and S_{lin} are built out of functors of the form $X^S Set(c, -)$, so that the notion of "permuting the c_i" makes sense.

C_0. But then letting C be a Goodwillie closure of C_0, Theorem 3.17 allows one to replace that restriction by a spectral functor $\tilde{G}$, so that $\tilde{G}(A,\ldots,A)$ then has a genuine (B, A^n) module structure, and Lemma 3.19 finally provides the desired map $\tilde{G}(A,\ldots,A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to \tilde{G}(X_1,\ldots,X_n)$, and it is then easy to finish the proof.

We now prove the aforementioned Lemmas.

Lemma 3.18. *Let X be any cellular object of Mod_A (based on any of the generating sets of cofibrations in Definition 1.3 other than those for the level cofibration model structure).*

Let further X^{fin} denote the category of finitesubcomplexes of X together with their inclusions.

Then one has a canonical equivalence

$$hocolim_{C\in X^{fin}} C \xrightarrow{\sim} X.$$

Proof. We first show that $colim_{C\in X^{fin}} C \xrightarrow{\simeq} X$ is an isomorphism. Since this colimit is filtered (as the union of finite subcomplexes is clearly a finite subcomplex), this is just equivalent to showing that any simplex of (any spectral level) of X belongs to some finite subcomplex, which we further claim can be chosen minimal. Call such a simplex s.

Now recall that X is presented as a transfinite colimit of monomorphisms

$$* = X_0 \to X_1 \to X_2 \to \ldots X$$

with $X_{\beta+1}$ obtained from X_β by attaching cells[31] $A \wedge F_n \Delta^m$ along their "boundary" $A \wedge F_n \delta\Delta^m$.

Notice that there must be a minimal $\beta+1$ for which $s \in X_{\beta+1}$, and a single specific cell e_s being attached to X_β for which $s \in e_s$ (abusing notation). The claim now follows by induction on $\beta+1$. Indeed, the "boundary" of e_s has the form $A \wedge F_n \delta\Delta^m$, and a map out this boundary is determined completely by the image of finitely many simplices $y_0,\ldots,y_m$ and, by induction, those images are contained in minimal finite subcomplexes $C_1,\ldots,C_m$, and it is then clear that $C_1 \cup \cdots \cup C_m \cup \{e_s\}$ is the minimal finite subcomplex containing s. Notice that from such such a complex being minimal it then follows that the intersection of finite subcomplexes is still a finite subcomplex.

We have now proven that $colim_{C\in X^{fin}} C \xrightarrow{\simeq} X$, so that the hocolim result will immediately follow from knowing that the identitiy diagram $X^{fin} \to Mod_A$ is projectively cofibrant. This ammounts to showing that for any C the canonical map $colim_{C' \subsetneq C} C' \to C$ is a cofibration. But this is clear since $colim_{C' \subsetneq C} C'$ is ensured to be a subcomplex of C due to the intersection of finite subcomplexes being a finite subcomplex.

□

Lemma 3.19. *Let $G\colon C^n \to Mod_B$ be a multispectral functor, where $C \subset Mod_A$ is a full subcategory closed under the free module construction on objects and containing A, the free module on the sphere spectrum S. Then there is a natural transformation*

$$G(A,\ldots,A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to G(X_1,\ldots,X_n)$$

[31] Note that though we present the argument for the generating cofibrations based on the stable model structure on Sp^Σ, the argument applies to the other model structures with no major alterations.

induced by the "identity" map at $(A, \cdots, A)$.

Furthermore, this natural transformation is compatible with Σ_n actions when F is a symmetric functor.

Proof. We first notice that $G(A, \cdots, A)$ does indeed have n commuting right module A structures, which are induced by the composites

$$A \to Sp_A^\Sigma(A, A) \xrightarrow{G_i} Sp_B^\Sigma(G(A, \ldots, A), G(A, \ldots, A))$$

where the first map describes the right A-module structure of A in Mod_A and the second map corresponds to spectrality of F in its i-th variable. For the remainder of the proof we deal only with the case $n = 1$ so as not to overbear the notation, but we note that the case $n > 1$ presents no further difficulties.

Now consider the diagram

$$G(A) \wedge A \wedge X \rightrightarrows G(A) \wedge X \to F(A \wedge X) \to F(X),$$

where the first two maps correspond to the left A-module structure on X and the right A-module structure on $G(A)$, the middle map is adjoint to $X \to Sp_A^\Sigma(A, A \wedge X) \to Sp_B^\Sigma(G(A), G(A \wedge X))$, and the final map is obtained by applying G to the structure multiplication map $A \wedge X \to X$, which we note is a map in Mod_A. Since $F(A) \wedge_A X$ is the coequalizer of the first two maps we will be finished by showing that the top and bottom compositions coindice.

Consider first the composition corresponding to the left module structure on X. By adjointness this corresponds to a composite

$$A \wedge X \to X \to Sp_A^\Sigma(A, A \wedge X) \to Sp_B^\Sigma(G(A), G(A \wedge X)) \to Sp_B^\Sigma(G(A), G(X)),$$

and by functoriality of F with respect to $A \wedge X \to X$ this is the same as a composite

$$A \wedge X \to X \to Sp_A^\Sigma(A, A \wedge X) \to Sp_A^\Sigma(A, X) \to Sp_B^\Sigma(G(A), G(X)).$$

On the other hand, the composition corresponding to the right module structure on $G(A)$ corresponds by adjointness to a composite

$$\begin{aligned} A \wedge X \to Sp_A^\Sigma(A, A) \wedge X \to Sp_B^\Sigma(G(A), G(A)) \wedge X \to Sp_B^\Sigma(G(A), G(A) \wedge X) \\ \to Sp_B^\Sigma(G(A), G(A \wedge X)) \to Sp_B^\Sigma(G(A), G(X)), \end{aligned} \tag{3.20}$$

which is rewritten using naturatily of F with respect to $X \to Sp_A^\Sigma(A, A \wedge X)$ as

$$\begin{aligned} A \wedge X \to & Sp_A^\Sigma(A, A) \wedge X \to Sp_A^\Sigma(A, A \wedge X) \to \\ \to & Sp_B^\Sigma(G(A), G(A \wedge X)) \to Sp_B^\Sigma(G(A), G(X)), \end{aligned} \tag{3.21}$$

which can be further rewritten using naturality with respect to the multiplication map $A \wedge X \to X$ as

$$A \wedge X \to Sp_A^\Sigma(A, A) \wedge X \to Sp_A^\Sigma(A, A \wedge X) \to Sp_A^\Sigma(A, X) \to Sp_B^\Sigma(G(A), G(X)). \tag{3.22}$$

Looking at both composites one sees that they both factor as a map $A \wedge X \to Sp_A^\Sigma(A, X)$, and unwinding definitions one sees that these two maps are precisely the maps that must match for X to be an A-module, finishing the proof. $\square$

Proof of Theorem 3.2. It suffices to show that any symmetric G multilinear multisimplicial functor is equivalent to one of the form

$$X_1, \dots, X_n \mapsto \delta_n G \wedge_{A^{\wedge n}} X_1 \wedge \cdots \wedge X_n.$$

First one replaces G by $G \circ Q^n$, where Q denotes the Quillen small object argument cofibrant replacement functor. Since $Q(f)$ for $f\colon X \to Y$ is always a cellular map, one has induced funtors[32]

$$hocolim_{((QX)^{fin})^n} G \to colim_{((QX)^{fin})^n} G \to G \circ Q^n,$$

where $(QX)^{fin}$ denotes the category of finite subcomplexes of QX. It then follows by Lemma 3.18 that the full composite is a weak equivalence for all X, since it is a weak equivalence on finite complexes and both functors commute with filtered colimits.

Furthermore, it is clear that one can replace the restriction of G to the category of finite complexes by any other equivalent functor $\bar{G}$ while still having a zig zag of weak equivalences between G and $hocolim_{((QX)^{fin})^n} \bar{G}$.

Now let C be the smallest Goodwillie closed category that contains (a skeleton of) all finite complexes, and that is further closed under forming the free algebra over its objects. Then Theorem 3.17 provides an equivalent spectral functor $\bar{G}$, and Lemma 3.19 then provides a map of functors $G(A, \dots, A) \wedge_{A^n} X_1 \wedge \cdots \wedge X_n \to G(X_1, \dots, X_n)$. While it is not necessarily obvious whether this natural transformation is a weak equivalence over the whole of C, since it is so on $(A, \cdots, A)$ it is also so for any tuple with coordinates of the form $A \wedge F_n \delta \Delta^m$ (as this is a suspension of A and both sides are linear functors), and hence also on any finite complexes.

It hence finally follows that F is equivalent to $hocolim_{((QX)^{fin})^n} \bar{G}$, and since this last one is clearly (the left derived functor of)

$$X_1, \dots, X_n \mapsto \delta_n \bar{G} \wedge_{A^{\wedge n}} X_1 \wedge \cdots \wedge X_n,$$

the proof is concluded.

□

4 Goodwillie Calculus in the $Alg_{\mathcal{O}}$ categories

In this section we present our main results concerning Goodwillie Calculus as it relates to the $Alg_{\mathcal{O}}$ categories. We point out that we will throughout deal only with the case where those categories are pointed or, equivalently, $\mathcal{O}(0) = *$.

In subsection 4.1 we finally assemble the results proven so far to characterize the Goodwillie tower of the identity in $Alg_{\mathcal{O}}$ as being the homotopy completion tower studied in [17] and associated to the truncated operads $\mathcal{O}_{\leq n}$.

In subsection 4.2 we show that, when studying n-excisive functors either to or from $Alg_{\mathcal{O}}$ one can equivalently study n-excisive functors to or from $Alg_{\mathcal{O}_{\leq n}}$. Combining this with subsection 4.1 this effectively says that the category $Alg_{\mathcal{O}_{\leq n}}$ can be recovered from the category $Alg_{\mathcal{O}}$ purely in Goodwillie

[32] Here we use for $hocolim_{((QX)^{fin})^n} G$ the models $B(*, (QX)^{fin}, G) \simeq N(-/(QX)^{fin}) \otimes_{(QX)^{fin}} G$, as described in Theorem 6.6.1 of [29], and were we make a (simplicial) pointwise cofibrant replacement of G if necessary.

calculus theoretic terms. One might then wonder whether an arbitrary homotopical category $\mathcal{C}$ admits an analogous "truncation" $\mathcal{C}_{\leq n}$, a question the author would like to examine in future work.

Finally, subsection 4.3 shows that for finitary functors between categories of the form $Alg_{\mathcal{O}}$ one does have at least a weak version of the chain rule as proved for spaces and spectra (and conjectured in general) in [1].

4.1 The Goodwillie tower of $Id_{Alg_{\mathcal{O}}}$

In this subsection we characterize the Goodwillie tower of $Id_{Alg_{\mathcal{O}}}$. First we introduce some notation.

Definition 4.1. Let $\mathcal{O}$ be an operad such that $\mathcal{O}(0) = *$. Then the truncation operad $\mathcal{O}_{\leq n}$ is the operad[33] whose spaces are

$$\mathcal{O}_{\leq n}(m) = \begin{cases} \mathcal{O}(m) & \text{if } m \leq n \\ * & \text{if } n < m \end{cases}$$

Note that there is a canonical map of operads $\mathcal{O} \to \mathcal{O}_{\leq n}$.

Definition 4.2. Notice that putting together Theorem 3.2, the results of [26] and Remark 2.12 it follows that any n-homogeneous finitary functor $F\colon Alg_{\mathcal{O}} \to Alg_{\tilde{\mathcal{O}}}$ has the form[34]

$$X \mapsto \Omega^{\infty}_{\tilde{\mathcal{O}}}(\delta_n F \wedge_{\mathcal{O}(1)^n} (\Sigma^{\infty}_{\mathcal{O}} X)^n)_{h\Sigma_n}$$

where $\delta_n F$ is a $(\tilde{\mathcal{O}}(1), \mathcal{O}(1)^{\wr n})$-bimodule, which we call the n-derivative of F.

The following is the main theorem of this section.

Theorem 4.3. *The Goodwillie tower of the identity for $Alg_{\mathcal{O}}$ is given by the (left derived) truncation functors $\mathcal{O}_{\leq n} \circ_{\mathcal{O}} -$.*

Furthermore, the n-derivative is $\mathcal{O}(n)$ itself with its canonical $(\mathcal{O}(1), \mathcal{O}(1)^{\wr n})$-bimodule structure.

We first show that the proposed Goodwillie tower is indeed composed of n-excisive functors. This will follow by combining the results of [27] with the main result of the previous section.

Lemma 4.4. *The left derived functors of $\mathcal{O}_{\leq n} \circ_{\mathcal{O}} -\colon Alg_{\mathcal{O}} \to Alg_{\mathcal{O}}$ are n-excisive.*

Proof. In order to avoid the need to introduce cofibrant replacements everywhere we instead deal only with the restrictions of these functors to the cofibrant objects in $Mod_{\mathcal{O}}$. Further, note that by [27, Thm. 1.1] the functor $\mathcal{O}_{\leq n} \circ_{\mathcal{O}} -$ sends trivial cofibrations between cofibrant objects to w.e.s so that, by Ken Brown's Lemma, it is in fact a homotopic functor when restricted to cofibrant objects.

The proof follows by induction on n. The case $n = 1$ follows since $\mathcal{O}_{\leq 1} \circ_{\mathcal{O}} -$ is just $\Omega^{\infty}\Sigma^{\infty}$.

[33] We notice that the fact that this forms an operad depends on the fact that $\mathcal{O}(0) = *$.
[34] Here we use the notation of Remark 2.12.

Letting $\mathcal{O}_n$ denote the $\mathcal{O}$-bimodule with values $\mathcal{O}_n(n) = \mathcal{O}(n)$ and $\mathcal{O}_n(m) = *$ for $n \neq m$, by [27, Thm. 1.9] one has a hofiber sequence of functors

$$\mathcal{O}_n \circ_{\mathcal{O}} - \to \mathcal{O}_{\leq n} \circ_{\mathcal{O}} - \to \mathcal{O}_{\leq (n-1)} \circ_{\mathcal{O}} -, \tag{4.5}$$

where the first functor is $\mathcal{O}_n \circ_{\mathcal{O}} X = \mathcal{O}(n) \wedge_{\Sigma_n} (\mathcal{O}(1) \wedge_{\mathcal{O}} X)^{\wedge n}$, hence an n-homogeneous functor. To finish the argument, note that since holims in algebra categories are underlying, it suffices to prove that $\mathcal{O}_{\leq n} \circ_{\mathcal{O}}$ is n-excisive as a functor landing in spectra, and the result now follows by induction and the two-out-of-three property for n-excisive functors landing in a stable category.

□

Proof of Theorem 4.3. In order to avoid the need to introduce cofibrant replacements everywhere we instead deal only with the restrictions of these functors to the cofibrant objects in $Mod_{\mathcal{O}}$.

First note that since holimits in $Alg_{\mathcal{O}}$ are underlying one is free to just prove that the $\mathcal{O}_n \circ_{\mathcal{O}} -$ form the Goodwillie tower of the identity when viewed as functors landing in Sp^{Σ}. Thanks to Lemma 4.4 it remains only to show that the natural map $Id \to \mathcal{O}_{\leq n} \circ_{\mathcal{O}} -$ induces an isomorphism on the homogeneous layers 1 through n.

Now consider the natural left Quillen adjoints

$$Mod_{\mathcal{O}(1)} \xrightarrow{\mathcal{O} \circ_{\mathcal{O}(1)} -} Alg_{\mathcal{O}} \xrightarrow{\mathcal{O}(1) \circ \mathcal{O} -} Mod_{\mathcal{O}(1)}$$

which, as noticed in Remark 2.12, compose to the identity. Now, since $\mathcal{O} \circ_{\mathcal{O}(1)} -$ is a left Quillen functor it preserves homotopy colimits and hence precomposition with it commutes with the process of forming Goodwillie towers and layers. On the other hand, by [26] and Remark 2.12 any such homogeneous functors factors through $\mathcal{O}(1) \circ \mathcal{O}-$, and it hence now follows that one can just as well verify the theorem after precomposing with $\mathcal{O} \circ_{\mathcal{O}(1)} -$.

We have hence reduced the result to the claim that $\coprod_{k=1}^{n} \mathcal{O}(n) \wedge_{\Sigma n} X^n$ is the n-th excisive approximation to $\coprod_{k=1}^{\infty} \mathcal{O}(n) \wedge_{\Sigma n} X^n$. This is a well known fact about analytic functors, finishing the proof.

□

Remark 4.6. Notice that as a particular case of Theorem 4.3 it follows that for any truncated operad $\mathcal{O}_{\leq n}$ the category of algebras $Alg_{\mathcal{O}_{\leq n}}$ has the property that its identity functor is n-excisive.

Remark 4.7. Theorem 4.3 asserts that the n-stage of the Goodwillie tower for $Id_{Alg_{\mathcal{O}}}$ is the monad associated to the adjunction

$$Alg_{\mathcal{O}} \leftrightarrows Alg_{\mathcal{O}_{\leq n}},$$

hence describing it as a left Quillen functor, followed by $Id_{\mathcal{O}_{\leq n}}$, followed by a right Quillen functor, and from this perspective the n-excisiveness of this composite is then a consequence of the n-excisiveness of $Id_{Alg_{\mathcal{O}_{\leq n}}}$.

Now suppose that A is a $(\mathcal{O}', \mathcal{O})$-bimodule (in symmetric sequences), and consider the associated functor

$$Alg_{\mathcal{O}} \xrightarrow{F_A = A \circ_{\mathcal{O}} -} Alg_{\mathcal{O}'}.$$

It is then not hard to see that the proof of Theorem 4.3 can be adapted to show that the Goodwillie tower for F_A is given by the functor $F_{A_{\leq n}}$, where $A_{\leq n}$ denotes the obvious truncated $(\mathcal{O}', \mathcal{O})$-bimodule.

Notice that we then have a factorization[35]

$$\begin{tikzcd} Alg_{\mathcal{O}} \arrow[r, "F_{A_{\leq n}}"] \arrow[d, "\mathcal{O}_{\leq n}\circ_{\mathcal{O}} -"'] & Alg_{\mathcal{O}'} \\ Alg_{\mathcal{O}_{\leq n}} \arrow[r, "F_{A_{\leq n}}"] & Alg_{\mathcal{O}'_{\leq n}} \arrow[u, "fgt"'] \end{tikzcd}$$

associating to the n-excisive functor $F_{A_{\leq n}}: Alg_{\mathcal{O}} \to Alg_{\mathcal{O}'}$ an n-excisive functor $F_{A_{\leq n}}: Alg_{\mathcal{O}_{\leq n}} \to Alg_{\mathcal{O}'_{\leq n}}$ between algebras over the truncated operads. Perhaps more surprising is the fact that any finitary n-excisive functor admits a similar factorization. That is the content of the next subsection.

4.2 n-excisive finitary functors can be "truncated"

The objective of this subsection is to prove the following result.

Theorem 4.8. *Let $Alg_{\mathcal{O}} \xrightarrow{F} Alg_{\mathcal{O}'}$ be a finitary n-excisive functor.*

Then there is a finitary n-excisive $Alg_{\mathcal{O}_{\leq n}} \xrightarrow{\tilde{F}} Alg_{\mathcal{O}'_{\leq n}}$ such that one has a factorization (up to homotopy)

$$\begin{tikzcd} Alg_{\mathcal{O}} \arrow[r, "F"] \arrow[d, "\mathcal{O}_{\leq n}\circ_{\mathcal{O}} Q"'] & Alg_{\mathcal{O}'} \\ Alg_{\mathcal{O}_{\leq n}} \arrow[r, "\tilde{F}"] & Alg_{\mathcal{O}'_{\leq n}} \arrow[u, "fgt"'] \end{tikzcd}.$$

(here Q denotes a fixed cofibrant replacement functor)

Furthermore, any two such $\tilde{F}$ are equivalent.

The existence part of Theorem 4.8 is a fairly straightforward consequence of Theorem 4.3 and the following Lemma, which is a fairly direct adaptation of Proposition 3.1 of [1], while the uniquess part will use the additional Lemma 4.10.

Lemma 4.9. *Let F, G be pointed simplicial homotopy functors between categories of the form $Alg_{\mathcal{O}}$ (allowing for different operads for the source and target categories). Assume F and G are composable. Then*

1. *The natural map $P_n(FG) \to P_n((P_nF)G)$ is an equivalence.*
2. *If F is finitary, then the natural map $P_n(FG) \to P_n(F(P_nG))$ is an equivalence.*

Proof. The proof is a straightforward adaptation of that of Proposition 3.1 of [1], and we hence indicate only the main differences.

[35] Here we abuse notation by also viewing $A_{\leq n}$ as a $(\mathcal{O}'_{\leq n}, \mathcal{O}_{\leq n})$-bimodule, and by hence also denoting by $F_{A_{\leq n}}$ the functor associated to that $(\mathcal{O}'_{\leq n}, \mathcal{O}_{\leq n})$-bimodule.

First, and as was usual in the proofs of [26], note that one is free to restrict oneself to cofibrant objects and to assume that F and G take bifibrant values.

The proof of part (1) is then essentially unchanged, with the map

$$P_n(P_n(F)G) \xrightarrow{v_n} P_n(FG)$$

built using the enrichment of these functors over pointed simplicial sets, and it's key properties being retained.

For part (2), the claim that the proof from [5] works when the middle category is spectra is replaced with the claim that that proof works whenever that middle category is Mod_A, with the use of Proposition 6.10 in [14] replaced by its generalization in Proposition 3.7 of [26] . The remainder of the proof equally follows, as after reducing to the case where F is n-homogeneous one can also write $F\tilde{F}'\Sigma^\infty_\mathcal{O}$, where $\Sigma^\infty_\mathcal{O}$ denotes the $\mathcal{O}(1) \circ_\mathcal{O} -$ functor, and since $\Sigma^\infty_\mathcal{O}$ shares all the relevant properties of Σ^∞, the result follows.

□

Lemma 4.10. *Let C denote the composite of (derived functors)*

$$Alg_{\mathcal{O}_{\leq n}} \xrightarrow{fgt} Alg_\mathcal{O} \xrightarrow{\mathcal{O}_{\leq n}\circ_\mathcal{O} Q} Alg_{\mathcal{O}_{\leq n}}.$$

(here Q denotes a cofibrant replacement functor)

Then the canonical counit map $C \to Id_{Alg_{\mathcal{O}_n}}$ becomes a weak equivalence after applying P_n.

Proof. Notice that since fgt is conservative (i.e. it reflects w.e.s) and it commutes with forming P_n, one may has well prove the result after postcomposition with fgt.

But now consider the canonical composite

$$fgt \circ Q \to fgt \circ \mathcal{O}_{\leq n} \circ_\mathcal{O} Q \circ fgt \to fgt.$$

This full composite is a weak equivalence, and it hence suffices to check that the first map is a weak equivalence after applying P_n. But this follows by Lemma 4.9, since the first map is induced by the unit $Id_\mathcal{O} \to fgt \circ O_{\leq n} \circ_\mathcal{O} -$, which is just the n-excisive approximation of $Id_\mathcal{O}$ by Theorem 4.3.

□

Proof of Theorem 4.8. Set $\tilde{F}'$ to be the composite

$$Alg_{\mathcal{O}_{\leq n}} \xrightarrow{ftg_{\mathcal{O}_{\leq n}}} Alg_\mathcal{O} \xrightarrow{F} Alg_{\mathcal{O}'} \xrightarrow{\mathcal{O}'_{\leq n}\circ_{\mathcal{O}'} Q} Alg_{\mathcal{O}'_{\leq n}}$$

and let $\tilde{F} = P_n(\tilde{F}')$.

Since $fgt_{\mathcal{O}'_{\leq n}}$ commutes with filtered hocolims and with holims, and since $\mathcal{O}_{\leq n} \circ_\mathcal{O} Q$ commutes with hocolimits, one has (a zig zag of) an equivalence $fgt_{\mathcal{O}'_{\leq n}} \circ P_n(\tilde{F}') \circ \mathcal{O}_{\leq n} \circ_\mathcal{O} Q \sim P_n(fgt_{\mathcal{O}'_{\leq n}} \circ \tilde{F}' \circ \mathcal{O}_{\leq n} \circ_\mathcal{O} Q)$.

But this is then just $P_n(P_n(Id_{Alg_\mathcal{O}}) \circ F \circ P_n(Id_{Alg_\mathcal{O}}))$, and Lemma 4.9 then applies to show that, since F was assumed finitary and n-excisive, this functor is just equivalent to F itself, proving the existence of factorization.

For uniqueness, the claim is then that any such $\tilde{F}$ is weak equivalent to $\mathcal{P}_n(\mathcal{O}'_{\leq n} \circ_{\mathcal{O}'} Q \circ F \circ fgt_{\mathcal{O}_{\leq n}})$, and our hypothesis says that this is weak equivalent to $P_n(C \circ \tilde{F} \circ C)$, where C is the composite appearing in Lemma 4.10. One then finishes the proof by applying Lemmas 4.9 and 4.10.

□

Remark 4.11. Theorem 4.8 roughly proves an equivalence of homotopy categories

$$Ho(Fun^{fin,\leq n}(Alg_{\mathcal{O}}, Alg_{\mathcal{O}'})) \simeq Ho(Fun^{fin,\leq n}(Alg_{\mathcal{O}_{\leq n}}, Alg_{\mathcal{O}'_{\leq n}})).$$

(here the notation $Fun^{fin,\leq n}$ is meant to denote n-excisive finitary functors)

We note however that this should underlie an actual Quillen equivalence of model categories $Fun^{fin,\leq n}(Alg_{\mathcal{O}}, Alg_{\mathcal{O}'})$ and $Fun^{fin,\leq n}(Alg_{\mathcal{O}_{\leq n}}, Alg_{\mathcal{O}'_{\leq n}})$.

Indeed, ignoring excisiveness for the moment, the $O_{\leq n} \circ_{\mathcal{O}} -, fgt_{\mathcal{O}}$ type adjunctions should indude a combined pre and post-composition adjunction between the functor categories, and this adjunction descends to n-excisive functors, which one should be able to view as a localization of the "model category of all functors". The proof of Theorem 4.8 is then essentially checking that the unit and counit in this hypothethical Quillen adjunction are weak equivalences.

Unfortunately it seems hard to construct appropriate $Fun^{fin,\leq n}$ model categories. For instance, the techniques of Section 3.1 essentially break down due to $Alg_{\mathcal{O}}$ generally not being left proper[36], and one hence does not have direct access to the localization machinery of [18], and so we are at current reduced to the weaker form of the result presented here.

4.3 Proto chain rule

Throughout this section we assume that each level $\mathcal{O}(n)$ is flat cofibrant for $n \geq 1$ after forgetting the Σ_n actions. Note that in this case $\mathcal{O} \circ -$ is an homotopy functor when restricted to positive flat cofibrant spectra and it sends positive flat cofibrant spectra to positive flat cofibrant spectra (both claims follow from[37] [27, Thm.1.1] when $\mathcal{O} = I$ and $f_2 = * \to \mathcal{O}$). These properties hence ensure that the functors appearing in Theorem 4.12 are homotopically meaningful.

Our goal in this subsection is to provide some evidence that a result analogous to the Chain Rule proved in [1] should also hold for functors between the $Alg_{\mathcal{O}}$ categories.

Firstly, recall that in that paper it was conjectured that for $\mathcal{C}$ a category in which one can do Goodwillie Calculus one should expect that the derivatives[38] $\delta_*(Id_{\mathcal{C}})$ form in some sense an operad, and one can hence view Theorem 4.3 as evidence of this, since as a symmetric sequence the derivatives $\delta_* Id_{Alg_{\mathcal{O}}}$ are just $\mathcal{O}$ itself. Unfortunately we have no intrisic construction of an operad structure on the $\delta_* Id_{Alg_{\mathcal{O}}}$ with which to compare the operad structure on $\mathcal{O}$, but we can offer at least some heuristic in that sense. Namely, given a homotopy functor $F\colon Alg_{\mathcal{O}} \to Algo_{\mathcal{O}'}$ consider the composite

$$Sp^{\Sigma} \xrightarrow{\mathcal{O} \circ Q} Alg_{\mathcal{O}} \xrightarrow{F} Alg_{\mathcal{O}'} \xrightarrow{fgt} Sp^{\Sigma}.$$

[36] In the rare case where $Alg_{\mathcal{O}}$ is indeed left proper, the main cases of this being when $\mathcal{O}$ is a monoid (i.e. concentrated in degree 1) or an enveloping operad of the form Com_A, we do however expect such a treatment to be viable.

[37] Note that in [27] positive flat cofibrancy is refered to as positive S cofibrancy.

[38] We note though that also part of this conjecture is that there is a sensible way to generally define such objects in the first place.

As was argued in the proof of Theorem 4.3 one can read the derivatives of F as the derivatives of this composite[39]. Applying this when $F = Id_{Alg_{\mathcal{O}}}$, and using the already known chain rule for functors in Sp^{Σ}, one gets a hypothetical operad structure map

$$\delta_*(Id_{Alg_{\mathcal{O}}}) \circ \delta_*(Id_{Alg_{\mathcal{O}}}) \simeq \delta_*(\mathcal{O} \circ -) \circ \delta_*(\mathcal{O} \circ -) \to \delta_*(\mathcal{O} \circ -) \simeq \delta_*(Id_{Alg_{\mathcal{O}}})$$

with the middle map[40] induced by the natural map of functors $fgt \circ \mathcal{O} \circ fgt \circ \mathcal{O} \to fgt \circ \mathcal{O}$.

Taking this idea further one can also construct "module structures" over $\mathcal{O}$ for the derivatives of other functors to or from $Alg_{\mathcal{O}}$. For instance, in the case of F a functor to $Alg_{\mathcal{O}}$ one can write

$$\delta_*(Id_{Alg_{\mathcal{O}}}) \circ \delta_*(F) \simeq \delta_*(\mathcal{O} \circ -) \circ \delta_*(fgt \circ F \circ \mathcal{O}' \circ -) \to \delta_*(fgt \circ F \circ \mathcal{O}' \circ -) \simeq \delta_*(F),$$

where the middle map is induced by the natural transformation $fgt \circ \mathcal{O} \circ fgt \circ F \to fgt \circ F$. The case of functors from $Alg_{\mathcal{O}}$ is similar.

One has the following result, which according to the previous remarks can be viewed as a weak version of the chain rule. We note that this is very closely in form (and proof) related to Theorem 16.1 of [1].

Theorem 4.12. *Consider finitary homotopy functors*

$$Alg_{\mathcal{O}'} \xrightarrow{F} Alg_{\mathcal{O}} \xrightarrow{G} Alg_{\mathcal{O}''}$$

Then the canonical maps

1. $|P_n B_\bullet(fgt \circ G \circ \mathcal{O}, fgt \circ \mathcal{O}, fgt \circ F \circ \mathcal{O}')| \xrightarrow{\sim} P_n(fgt \circ G \circ F \circ \mathcal{O}')$
2. $|D_n B_\bullet(fgt \circ G \circ \mathcal{O}, fgt \circ \mathcal{O}, fgt \circ F \circ \mathcal{O}')| \xrightarrow{\sim} D_n(fgt \circ G \circ F \circ \mathcal{O}')$

are equivalences.

Here the geometric realizations are taken in the homotopical sense (i.e. they are performed after making a Reedy cofibrant replacement).

Proof. The overall strategy of the proof mimics that of Theorem 16.1 of [1], and we hence focus on the points at which specific properties of $Alg_{\mathcal{O}}$ need to be used.

Notice that part (2) will follow immediatelly once we know (1), as homotopy fibers commute with geometric realization.

We first deal with part (1). The first step is to notice that, by 4.9, the map $G \to P_n G$ induces levelwise equivalences between the relevant augmented simplicial objects, so it suffices to prove the result assuming G is $n-excisive$.

Consider now a hofiber sequence of functors $G' \to G \to G''$, so that one wants to check that the result will follow for G if one knows it for both G' and G''. Notice that one then obtains an associated levelwise hofiber sequence of augmented simplicial objects. But since both weak equivalences and (homotopically meaningful) geometric realizations in $Alg_{\mathcal{O}}$ are computed in Sp^{Σ}, the

[39]Strickly speaking the argument used in that proof precoomposed with the maps $Mod_{\mathcal{O}(1)} \xrightarrow{\mathcal{O} \circ_{\mathcal{O}(1)} Q} Alg_{\mathcal{O}}$ and $Alg_{\mathcal{O}'} \xrightarrow{fgt} Mod_{\mathcal{O}'(1)}$, so now we are actually disregarding the $(\mathcal{O}'(1), \mathcal{O}(1)^{\wedge n})$-bimodule structures.

[40]Tecnichally speaking this map actually requires "inverting" the equivalence

result follows immediately from noting that the hofiber sequence of augmented simplicial objects is also an underlying hocofiber sequence.

We have now hence reduced to proving the claim in the case that G is a finitary n-homogeneous functor. At this point notice that P_n actually commutes with the geometric realization (since geometric realizations commute with both filtered hocolimits and the punctured cube holimits), so it sufficed to check that in this case one has an equivalence as in (1) with the P_n removed. Recall that G has the form $X \mapsto triv_{\mathcal{O}''}(\delta_n G \wedge_{h\Sigma_n} TAQ_{\mathcal{O}}(X)^{\wedge n})$, where $\delta_n G$ is a $(\mathcal{O}''(1), \mathcal{O}(1)^{\wr n})$-bimodule. This means that in the terms of the the simplicial augmented object one has a $fgt \circ triv_{\mathcal{O}''}$ on the leftmost side, and since this composite is just forgetting from $\mathcal{O}''(1)$-modules to Sp^{Σ}, one sees that the homotopy coinvariants $h\Sigma_n$ commute with the whole construction, and can hence be removed, so that one can deal instead with a functor of the form $X \mapsto triv_{\mathcal{O}''}(\delta_n G \wedge TAQ_{\mathcal{O}}(X)^{\wedge n})$. But in this case the augmented simplicial object being discussed is naturally the diagonal of a n multisimplical object[41], so that one can replace the geometric realization by a n-multigeometric realization. Since this amounts to moving the $\delta_n G \wedge -$ out of the whole thing, we have reduced to the case $G = TAQ$. But TAQ is a left Quillen functor, hence commuting with the (homotopical) realization, and one can hence take G as the identity. The result now follows by Theorem 1.8 of [16], saying that any $\mathcal{O}$-algebra is canonically the (homotopical) realization of its bar construction.

□

[41] By considering the multilinear functor associated to G, $X_1, \cdots, X_n \mapsto \delta_n G \wedge TAQ(X_1, \cdots, X_n)$.

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