Filtrations of colored operadic constructions and excisiveness of truncated operadic functors

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Abstract

We extend and refine well known filtrations for pushouts of free maps of operads to the setting of colored operads over a fixed set of colors $\mathfrak{C}$, resulting in a suitable $\mathfrak{C}$-fold filtration.

As a consequence, we obtain a direct proof that n-truncated operadic functors are n-excisive.

In this short note we extend and refine well known filtrations of diagrams of algebras over an operad (cf. for example [1], [2], [4]) to the case of colored operads, our main result here being Proposition 2.10. This result is preceded and inspired by a similar filtration that recently appeared in [6], but the filtration we present here (which in particular recovers the result in [6]), is substantially more powerful and flexible since rather than simply providing a simple infinite filtration it provides a "multidirectional infinite filtration", with a different direction for each color in $\mathfrak{C}$.

These filtrations and the extra flexibility they provide are important for upcoming work by the author on model categories for operads in G-spaces and (genuine) G-spectra.

In this write-up, however, we include only a single simple application: we show in Proposition 3.5 that n-truncated operadic functors on $Alg_{\mathcal{O}}$ are in fact n-excisive. This result originally appeared in the author's thesis, but the proof was fairly indirect, as one needed to first identify the stabilization of $Alg_{\mathcal{O}}$ and then classify multilinear functors on such stabilizations. Instead, our proof here is a short direct application of our new filtrations.

1 Definitions: colored operads, modules and algebras

The following is adapted from [6], although we warn the reader that our terminology differs on some key aspects, most notably in our usage of the term "$\mathfrak{C}$-symmetric sequence".

Throughout we will let $\mathfrak{C}$ be a fixed set, which we refer to as the *set of colors*. Further, Σ will denote the standard skeleton of the category of finite sets and bijections. Explicitly, the objects of Σ are the sets $\underline{r} = \{1, 2, \cdots, r\}$ for each $r \geq 0$.

Definition 1.1. Consider the obvious functor $\Sigma^{op} \to \mathsf{Set}$ given by $\mathsf{r} \mapsto \mathfrak{C}^{\times \underline{r}}$. We define $\Sigma_{\mathfrak{C}}$, the category of $\mathfrak{C}$-*words and shuffles* to be the **opposite category** of the Grothendieck construction of this functor (cf. for example [5, Construction 7.1.9]). Explicitly, the objects of $\Sigma_{\mathfrak{C}}$ are pairs

$$(\underline{r}, w \in \mathfrak{C}^{\times \underline{r}})$$

and a map $(\underline{r}, w) \to (\underline{r_*}, w_*)$ is a pair

$$(\sigma \colon \underline{r} \xrightarrow{\sim} \underline{r_*}, w = w_* \circ \sigma).$$

Notation 1.2. We will when convenient abbreviate objects of $\Sigma_{\mathfrak{C}}$ using only their second coordinate w and arrows using only their first coordinate σ.

For some intuition we note that one can think of each object w as a "word" in the colors in $\mathfrak{C}$ (the reader displeased with the notion of a "word of colors" is free to think of $\mathfrak{C}$ as a set of letters) and of each arrow $\sigma \colon w \to w_*$ as a shuffle of w into w_*. Note that with our conventions shuffles act on words on the left by pre-composition with the inverse.

Given $w \in \mathfrak{C}^{\times \underline{r}}$ and $w_* \in \mathfrak{C}^{\times \underline{r_*}}$ we then write $ww_* \in \mathfrak{C}^{\times \underline{r+r_*}}$ for the concatenation of w and w_*.

Further, for an object (r, w) we let $|w| = r$ denote the *length* of w and $\Sigma_w \leq \Sigma_r$ will denote the isotropy subgroup of w, i.e., those $\sigma \in \Sigma_r$ such that $w = w \circ \sigma$.

Definition 1.3. Let $(\mathcal{C}, \otimes, \mathbb{1})$ denote a closed symmetric monoidal category.

- The category of $\mathfrak{C}$-*sequences in* $\mathcal{C}$ is the category $\mathcal{C}^{\mathfrak{C}}$;
- The category $\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ of $\mathfrak{C}$-*symmetric sequences in* $\mathcal{C}$ is the category $\mathcal{C}^{\Sigma_{\mathfrak{C}}}$;
- The category $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ of $\mathfrak{C}$-*symmetric* $\mathfrak{C}$-*sequences in* $\mathcal{C}$ is the category $\mathcal{C}^{\Sigma_{\mathfrak{C}} \times \mathfrak{C}}$.

Remark 1.4. Note that $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}) \simeq (\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}))^{\mathfrak{C}} \simeq \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}^{\mathfrak{C}})$, i.e., one can alternatively regard $\mathfrak{C}$-symmetric $\mathfrak{C}$-sequences as $\mathfrak{C}$-sequences in $\mathfrak{C}$-symmetric sequences or as $\mathfrak{C}$-symmetric sequences in $\mathfrak{C}$-sequences.

Notation 1.5. Given $X \in \mathcal{C}^{\mathfrak{C}}$ and $c \in \mathfrak{C}$ we let $X^c \in \mathcal{C}$ denote its c component.

Further, given $w \in \mathfrak{C}^{\times \underline{r}}$ and $X \in \mathcal{C}^{\mathfrak{C}}$, we denote

$$X^{\otimes w} = X^{w(1)} \otimes X^{w(2)} \otimes \cdots \otimes X^{w(r)}.$$

When $\mathfrak{C}$ consists of a single color one has $\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}) = \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ and this is usually denoted by $\mathsf{Sym}(\mathcal{C})$ and called the *category of symmetric sequences in* $\mathcal{C}$. It is then well known that $\otimes$ induces two monoidal structures on $\mathsf{Sym}(\mathcal{C})$: the *tensor product* $\check{\otimes}$ and the *composition product* $\circ$.

The need to introduce both $\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ and $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ comes from the fact that the general colored analogues of $\check{\otimes}$ and $\circ$ are each defined on one those categories.

Definition 1.6. Given $X, Y \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ we define their *tensor product* $X \check{\otimes} Y \in \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ to be

$$(X \check{\otimes} Y)(w) = \coprod_{\varphi \colon \underline{|w|} \to \underline{2}} X\left(w|_{\varphi^{-1}(1)}\right) \otimes Y\left(w|_{\varphi^{-1}(2)}\right)$$

where $w|_{\varphi^{-1}(i)}$ is the word obtained by removing the letters of w not in $\varphi^{-1}(i)$.

Proposition 1.7. $(\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}), \check{\otimes}, \check{\mathbb{1}})$ *is a closed symmetric monoidal category with unit* $\check{\mathbb{1}}(\varnothing) = \mathbb{1}$*,* $\check{\mathbb{1}}(w) = \varnothing, w \neq \varnothing$*.*

Recalling that $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}) = (\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}))^{\mathfrak{C}}$ and combining Notation 1.5 with Proposition 1.7 one gets that for $X \in \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ one can define $X^{\check{\otimes}(-)}(-) \in \mathsf{Sym}_{\mathfrak{C}}(\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}))$ by

$$X^{\check{\otimes}w}(\bar{w}) = (X^{w(1)}\check{\otimes}\cdots\check{\otimes}X^{w(r)})(\bar{w}).$$

Definition 1.8. Given $X, Y \in \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ we define their *composition product* to be

$$(X \circ Y)^c(w) = \int^{\bar{w}\in\Sigma_{\mathfrak{C}}} X^c(\bar{w})\check{\otimes}Y^{\check{\otimes}\bar{w}}(w) = X^c(-)\check{\otimes}_{\Sigma_{\mathfrak{C}}}Y^{\check{\otimes}(-)}(w).$$

Here $\int^{\bar{w}\in\Sigma_{\mathfrak{C}}}$ denotes the coend construction, and the right hand side is merely alternative notation for that coend that we will use throughout the paper.

One has the following result (for a discussion of reflexive coequalizers see for example [2, Def. 3.26] and the propositions immediately following it).

Proposition 1.9. $(\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}), \circ, \mathcal{I})$ *is a (non-symmetric) monoidal category, with unit* $\mathcal{I}^c(c) = \mathbb{1}$*,* $\quad \mathcal{I} = \varnothing$*, otherwise.*

Further, $\circ$ *commutes with all colimits in the first variable and with filtered colimits and reflexive coequalizers in the second variable.*

Definition 1.10. A $\mathfrak{C}$-colored *operad* $\mathcal{O}$ in $\mathcal{C}$ is a monoid object in $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ with respect to $\circ$, i.e., a $\mathfrak{C}$-symmetric $\mathfrak{C}$-sequence $\mathcal{O}$ together with multiplication and unit maps

$$\mathcal{O} \circ \mathcal{O} \to \mathcal{O}, \qquad \mathcal{I} \to \mathcal{O}$$

satisfying the usual associativity and unit conditions.

Definition 1.11. Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad in $\mathcal{C}$. A *left module* N (resp. *right module* M) over $\mathcal{O}$ is an object in $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ together with a map

$$\mathcal{O} \circ N \to N \qquad (\text{resp. } M \circ \mathcal{O} \to M)$$

satisfying the usual associativity and unit conditions. The category of left modules (resp. right modules) over $\mathcal{O}$ is denoted $\mathsf{Mod}^l_{\mathcal{O}}$ (resp. $\mathsf{Mod}^r_{\mathcal{O}}$). Further, left modules X over $\mathcal{O}$ concentrated in degree 0 (i.e. such that $X^c(w) = \varnothing$ for $|w| \geq 1$) are called *algebras* over $\mathcal{O}$. The category of algebras over $\mathcal{O}$ is denoted $\mathsf{Alg}_{\mathcal{O}}$.

Proposition 1.12. *The categories* $\mathsf{Mod}^r_{\mathcal{O}}$*,* $\mathsf{Mod}^l_{\mathcal{O}}$ *and* $\mathsf{Alg}_{\mathcal{O}}$ *have all small limits and colimits.*

Further, all limits and colimits in $\mathsf{Mod}^r_{\mathcal{O}}$ *are underlying in* $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}\mathcal{C}$*, and likewise for all limits, filtered colimits and reflexive coequalizers in both* $\mathsf{Mod}^l_{\mathcal{O}}$ *and* $\mathsf{Alg}_{\mathcal{O}}$*.*

Definition 1.13. Given $M \in \mathsf{Mod}^r_{\mathcal{O}}$, $N \in \mathsf{Mod}^l_{\mathcal{O}}$, their *relative composition product* is the reflexive coequalizer

$$M \circ_{\mathcal{O}} N = \operatorname{colim}(M \circ \mathcal{O} \circ N \rightrightarrows M \circ N).$$

Lemma 1.14. *Consider the bifunctors*

$$\mathsf{Mod}^l_{\mathcal{O}} \times \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}) \xrightarrow{-\circ-} \mathsf{Mod}^l_{\mathcal{O}}, \qquad \mathsf{Mod}^r_{\mathcal{O}} \times \mathsf{Mod}^l_{\mathcal{O}} \xrightarrow{-\circ_{\mathcal{O}}-} \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}).$$

$\circ$ *preserves any colimit in the* $\mathsf{Mod}^l_{\mathcal{O}}$ *variable and* $\circ_{\mathcal{O}}$ *preserves reflexive coequalizers and filtered colimits in the* $\mathsf{Mod}^l_{\mathcal{O}}$ *variable.*

Proof. Since any $M \in \mathsf{Mod}^l_{\mathcal{O}}$ is a reflexive coequalizer $\mathrm{colim}(\mathcal{O} \circ \mathcal{O} \circ M \rightrightarrows \mathcal{O} \circ M)$ of free left modules, it suffices to verify the claim for diagrams of free left modules and free maps, and for those the result follows by Proposition 1.9. □

Noting again that $\mathsf{Sym}_{\mathfrak{C}}$ is itself a symmetric monoidal category allows one to iterate all the constructions defined so far. The upshot of this is given in Proposition 1.19, which will allow us to reduce the study of left modules to that of algebras.

Definition 1.15. The category $\mathsf{BSym}_{\mathfrak{C}}(\mathcal{C})$ of $\mathfrak{C}$-*bi-symmetric sequences in* $\mathcal{C}$ is the category $\mathsf{Sym}_{\mathfrak{C}}(\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}))$ of $\mathfrak{C}$-symmetric sequences of $\mathfrak{C}$-symmetric sequences in $\mathcal{C}$.

Further, the category $\mathsf{BSym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ of $\mathfrak{C}$-*bi-symmetric* $\mathfrak{C}$-*sequences in* $\mathcal{C}$ is the category $\mathsf{Sym}_{\mathfrak{C}}(\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}))$.

Remark 1.16. Since an object $X \in \mathsf{BSym}_{\mathfrak{C}}(\mathcal{C})$ is formed by objects $X(w,v) \in \mathcal{C}$, $w, v \in \Sigma_{\mathfrak{C}}$

$$(-)^{\mathsf{w}} \colon \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}) \hookrightarrow \mathsf{BSym}_{\mathfrak{C}}(\mathcal{C}), \qquad (-)^{\mathsf{v}} \colon \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}) \hookrightarrow \mathsf{BSym}_{\mathfrak{C}}(\mathcal{C})$$

defined by

$$X^{\mathsf{w}}(w,v) = \begin{cases} X(w), & \text{if } v = \varnothing \\ \varnothing, & \text{if } v \neq \varnothing \end{cases}, \qquad X^{\mathsf{v}}(w,v) = \begin{cases} X(v), & \text{if } w = \varnothing \\ \varnothing, & \text{if } r \neq \varnothing \end{cases}.$$

Remark 1.17. Note that $(-)^{\mathsf{w}}, (-)^{\mathsf{v}}$ naturally extend to inclusions $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}) \to \mathsf{BSym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$.

Following Definition 1.6 one can define a monoidal structure $\check{\check{\otimes}}$ in $\mathsf{BSym}_{\mathfrak{C}}(\mathcal{C})$ and following Definition 1.8 one can define a monoidal structure $\circ^{\mathsf{w}}$ on $\mathsf{BSym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$, where we mark the composition product $\check{\circ}^{\mathsf{w}}$ to indicate that w is kept as the operadic index.

Both of the following results follow by a straightforward calculation.

Proposition 1.18. $(-)^{\mathsf{w}}, (-)^{\mathsf{v}} \colon \mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}) \to \mathsf{BSym}_{\mathfrak{C}}(\mathcal{C})$ *are monoidal functors from the symmetric monoidal structure* $\check{\otimes}$ *to the symmetric monoidal structure* $\check{\check{\otimes}}$.

$(-)^{\mathsf{w}} \colon \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C}) \to \mathsf{BSym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ *is a monoidal functor from the monoidal structure* $\circ$ *to the monoidal structure* $\check{\circ}^{\mathsf{w}}$.

Proposition 1.19. *Let* $\mathcal{O}$ *be a* $\mathfrak{C}$-*colored operad in* $\mathcal{C}$. *There is a natural isomorphism of categories*

$$(-)^{\mathsf{v}} \colon \mathsf{Mod}^l_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\simeq} \mathsf{Alg}_{\mathcal{O}^{\mathsf{w}}}(\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})).$$

2 Filtrations

Our goal in this section is to establish a colored operad generalization and refinement of the filtrations in [4].

Definition 2.1. Let $\mathcal{O}$ be a $\mathfrak{C}$-colored operad in $\mathcal{C}$ and $A \in \mathsf{Alg}_{\mathcal{O}}$ regarded as an element of $\mathsf{Mod}^l_{\mathcal{O}}$. We define

$$\mathcal{O}_A = \mathcal{O} \amalg A, \tag{2.2}$$

were the coproduct is taken in $\mathsf{Mod}^l_{\mathcal{O}}$. Additionally, for $M \in \mathsf{Mod}^r_{\mathcal{O}}$ we define

$$M_A = M \circ_{\mathcal{O}} \mathcal{O}_A.$$

Remark 2.3. There are adjunctions

$$\iota\colon \mathsf{Alg}_{\mathcal{O}} \rightleftarrows \mathsf{Mod}^l_{\mathcal{O}}\colon (-)(\varnothing) \qquad (-)(\varnothing)\colon \mathsf{Mod}^l_{\mathcal{O}} \rightleftarrows \mathsf{Alg}_{\mathcal{O}}\colon \widetilde{(-)}$$

where ι is the inclusion and $\tilde{A}(\varnothing) = A$, $\tilde{A}^c(w) = *$ for $|w| \geq 1$. In particular, colimits in $\mathsf{Alg}_{\mathcal{O}}$ can be computed after the inclusion into $\mathsf{Mod}^l_{\mathcal{O}}$ and $\mathcal{O}_A(\varnothing) = A$.

Proposition 2.4. *Let $A \in \mathsf{Alg}_{\mathcal{O}}$ and $X \in \mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$. Then there is a natural isomorphism of $\mathsf{Mod}^l_{\mathcal{O}}$-valued functors*

$$(\mathcal{O} \circ X) \amalg A \simeq \mathcal{O}_A \circ X = \mathcal{O}_A(-) \otimes_{\Sigma_{\mathfrak{C}}} X^{\check{\otimes}(-)}.$$

Additionally, for $M \in \mathsf{Mod}^r_{\mathcal{O}}$ there is a natural isomorphism of $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$-valued functors

$$M \circ_{\mathcal{O}} ((\mathcal{O} \circ X) \amalg A) \simeq M_A \circ X = M_A(-) \otimes_{\Sigma_{\mathfrak{C}}} X^{\check{\otimes}(-)}.$$

Proof. We compute (applying Lemma 1.14 to the coproduct $\mathcal{O} \amalg A$)

$$\mathcal{O}_A \circ X = (\mathcal{O} \amalg A) \circ X \simeq (\mathcal{O} \circ X) \amalg (A \circ X) = (\mathcal{O} \circ X) \amalg A,$$

where $A \circ X = A$ since A is in degree 0. The additional claim is obvious. □

Proposition 2.5. *Given $M \in \mathsf{Mod}^r_{\mathcal{O}}$, $X \in \mathcal{C}^{\mathfrak{C}}$ and $A \in \mathsf{Alg}_{\mathcal{O}}$ one has natural isomorphisms of $\mathsf{Sym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$-valued functors*

$$M_{\mathcal{O}X \amalg A}(v) = (M \circ_{\mathcal{O}} (\mathcal{O} \amalg \mathcal{O} \circ X \amalg A))(v) \simeq M_A(wv) \otimes_{w \in \Sigma_{\mathfrak{C}}} X^{\otimes w}. \tag{2.6}$$

Proof. This follows formally using the $(-)^{\mathsf{v}}, (-)^{\mathsf{w}}$ functors. Combining Proposition 1.19 to change perspective to $\mathsf{Alg}_{\mathcal{O}^{\mathsf{w}}}(\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C}))$ with Proposition 2.4 yields

$$(M \circ_{\mathcal{O}} (\mathcal{O} \amalg \mathcal{O} \circ X \amalg A))^{\mathsf{v}} \simeq M^{\mathsf{w}} \circ^{\mathsf{w}}_{\mathcal{O}^{\mathsf{w}}} (\mathcal{O}^{\mathsf{v}} \amalg \mathcal{O}^{\mathsf{w}} \circ^{\mathsf{w}} X^{\mathsf{v}} \amalg A^{\mathsf{v}}) \simeq M^{\mathsf{w}}_{\mathcal{O}^{\mathsf{v}} \amalg A^{\mathsf{v}}} \circ^{\mathsf{w}} X^{\mathsf{v}}. \tag{2.7}$$

Applying Proposition 2.4 and noting $A^{\mathsf{v}} = A^{\mathsf{w}}$ (as A is an algebra) we compute

$$\begin{aligned} M^{\mathsf{w}}_{\mathcal{O}^{\mathsf{v}} \amalg A^{\mathsf{v}}} &= M^{\mathsf{w}} \circ^{\mathsf{w}}_{\mathcal{O}^{\mathsf{w}}} (\mathcal{O}^{\mathsf{w}} \amalg \mathcal{O}^{\mathsf{v}} \amalg A^{\mathsf{v}}) = M^{\mathsf{w}} \circ^{\mathsf{w}}_{\mathcal{O}^{\mathsf{w}}} (\mathcal{O}^{\mathsf{w}} \circ^{\mathsf{w}} (\mathcal{I}^{\mathsf{w}} \amalg \mathcal{I}^{\mathsf{v}}) \amalg A^{\mathsf{v}}) \simeq \\ &\simeq M^{\mathsf{w}}_{A^{\mathsf{v}}} \circ^{\mathsf{w}} (\mathcal{I}^{\mathsf{w}} \amalg \mathcal{I}^{\mathsf{v}}) = M^{\mathsf{w}}_{A^{\mathsf{w}}} \circ^{\mathsf{w}} (\mathcal{I}^{\mathsf{w}} \amalg \mathcal{I}^{\mathsf{v}}) \simeq (M_A)^{\mathsf{w}} \circ^{\mathsf{w}} (\mathcal{I}^{\mathsf{w}} \amalg \mathcal{I}^{\mathsf{v}}), \end{aligned}$$

showing $M^{\mathsf{w}}_{\mathcal{O}^{\mathsf{v}} \amalg A^{\mathsf{v}}}(w, v) \simeq M_A(wv)$. Plugging into (2.7) finishes the proof. □

Notation 2.8. Let I be a diagram category with an initial object $\varnothing$. We will denote by $I^{\oplus\mathfrak{C}}$ the full subcategory of $I^{\mathfrak{C}}$ of those tuples where all but finitely many entries equal $\varnothing$.

For ease of notation, we will often denote an object $(i_c)_{c\in\mathfrak{C}}$ simply as $i_1 \cdots i_p$ where the i_j are the non $\varnothing$ components of (i_c). Further, when using this notation we convention that i_j is the component corresponding to the color $c_j \in \mathfrak{C}$.

Notation 2.9. Let I be a poset and consider a functor $F: I \to \mathcal{C}$. For each $e \in I$ we define the *latching map of F at e* to be

$$L_e(F) = \operatorname{colim}_{\bar{e}<e} F(\bar{e}) \xrightarrow{l_e(F)} F(e).$$

Note that if I is a poset then so is $I^{\oplus\mathfrak{C}}$.

Proposition 2.10. *Consider any pushout in $\mathsf{Alg}_{\mathcal{O}}$ of the form*

$$\begin{tikzcd} \mathcal{O} \circ X \arrow[r, "h"] \arrow[d, "\mathcal{O}\circ f"'] & A \arrow[d] \\ \mathcal{O} \circ Y \arrow[r] & B, \end{tikzcd} \tag{2.11}$$

and let $M \in \mathsf{Mod}^r_{\mathcal{O}}$. Then, in the underlying category $\mathcal{C}^{\mathfrak{C}}$,

$$M \circ_{\mathcal{O}} B \simeq \operatorname{colim}_{\omega^{\oplus\mathfrak{C}}} \mathcal{F}^M \tag{2.12}$$

where $\mathcal{F}^M((\varnothing)_{c\in\mathfrak{C}}) = M \circ_{\mathcal{O}} A$ and each latching map $l_{c_1^{r_1}\cdots c_p^{r_p}}$ is built as a pushout diagram (written as an arrow in the arrow category $(\mathcal{C}^{\mathfrak{C}})^{\to}$)

$$M_A(c_1^{r_1}\cdots c_p^{r_p}) \otimes_{\Sigma_{r_1}\times\cdots\times\Sigma_{r_p}} (f^{c_1})^{\square r_1} \square \cdots \square (f^{c_p})^{\square r_p} \to l_{c_1^{r_1}\cdots c_p^{r_p}}. \tag{2.13}$$

Proposition 2.14. *Further, the filtrations of Proposition 2.10 are compatible with decomposing the set of colors in the following sense.*

Suppose $\mathfrak{C} = \mathfrak{D} \amalg \bar{\mathfrak{D}}$ and consider the natural decomposition $f = f_{\mathfrak{D}} \amalg f_{\bar{\mathfrak{D}}}$. Then for the pushout diagram

$$\begin{tikzcd} \mathcal{O} \circ X_{\mathfrak{D}} \arrow[r, "h"] \arrow[d, "\mathcal{O}\circ f_{\mathfrak{D}}"'] & A \arrow[d] \\ \mathcal{O} \circ Y_{\mathfrak{D}} \arrow[r] & A_{\mathfrak{D}} \end{tikzcd} \tag{2.15}$$

the associated filtration is $\mathcal{F}^M_{\mathfrak{D}} = \operatorname{Lan}_{\omega^{\mathfrak{D}}\to\omega^{\mathfrak{C}}} \mathcal{F}^M|_{\omega^{\mathfrak{D}}}$, and the for the pushout diagram

$$\begin{tikzcd} \mathcal{O} \circ (X_{\bar{\mathfrak{D}}} \amalg Y_{\mathfrak{D}}) \arrow[r, "h"] \arrow[d, "\mathcal{O}\circ(f_{\bar{\mathfrak{D}}}\amalg id)"'] & A_{\mathfrak{D}} \arrow[d] \\ \mathcal{O} \circ Y \arrow[r] & B \end{tikzcd} \tag{2.16}$$

the associated filtration is $\mathcal{F}^M_{\mathfrak{D}\to\mathfrak{C}} = \operatorname{Lan}_{\omega^{\oplus\bar{\mathfrak{D}}}\to\omega^{\oplus\mathfrak{C}}} \operatorname{colim}_{\omega^{\oplus\mathfrak{D}}} \mathcal{F}^M$.

Collapsing the $\omega^{\oplus\mathfrak{C}}$-filtration to a ω-filtration by grouping terms according to total degree, one obtains the following result, generalizing [6, Prop. 4.3.18].

Corollary 2.17. *Consider a pushout as in (2.11). Then, in the underlying category $\mathcal{C}^{\mathfrak{C}}$,*

$$M \circ_{\mathcal{O}} B \simeq \operatorname{colim}\left(A_0^M \xrightarrow{i_1} A_1^M \xrightarrow{i_2} A_2^M \xrightarrow{i_3} A_3^M \xrightarrow{i_4} \cdots\right) \tag{2.18}$$

where $A_0^M = M \circ_{\mathcal{O}} A$ and each map i_r is built as a pushout diagram (written as an arrow in the arrow category $(\mathcal{C}^{\mathfrak{C}})^{\to}$)

$$M_A(w) \otimes_{w \in \Sigma_{\mathfrak{C}} \times \underline{r}} f^{\square w} \to i_r, \tag{2.19}$$

where $f^{\square w}$ is defined according to Notation 1.5 applied to the monoidal structure $\square$ in $\mathcal{C}^{\to}$.

Remark 2.20. It will be convenient to be able to apply Proposition 2.10 and Corollary 2.17 to the category $\mathsf{Mod}^l_{\mathcal{O}}(\mathcal{C})$, a move enabled by Proposition 1.19. This is mostly straightforward, with occurrences of $\mathcal{C}$ replaced by $\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ and $\otimes$ replaced by $\check{\otimes}$, though defining $M_N \in \mathsf{BSym}^{\mathfrak{C}}_{\mathfrak{C}}(\mathcal{C})$ when $N \in \mathsf{Mod}^l_{\mathcal{O}}$ requires some care. Analyzing Proposition 1.19 and Definition 2.1 leads to the definition

$$M_N = M^{\mathsf{w}}_{N^{\mathsf{v}}} = M^{\mathsf{w}} \,\check{\circ}^{\mathsf{w}}_{\mathcal{O}^{\mathsf{w}}} (\mathcal{O}^{\mathsf{w}} \amalg N^{\mathsf{v}}). \tag{2.21}$$

Note that this is compatible with Definition 2.1 when $N = A$ is an algebra since then $A^{\mathsf{v}} = A^{\mathsf{w}}$ so that $M^{\mathsf{w}}_{A^{\mathsf{v}}} = M^{\mathsf{w}}_{A^{\mathsf{w}}} = (M_A)^{\mathsf{w}}$.

The proof strategy follows similarly to the one used when proving the similar result in [4]. We start by briefly recalling the "diagram word category" used in that proof.

Definition 2.22. Objects of $\mathcal{W}$ are pairs

$$(\underline{r}, \nu \in (x \to y)^{\underline{r}}),$$

and an arrow $(\underline{r}, \nu) \to (\underline{r_*}, \nu_*)$ is a pair

$$(\iota \colon \underline{r_*} \hookrightarrow \underline{r}, \nu \circ \iota \to \nu_*) \quad \text{such that } \nu\left(\underline{r} - \iota\left(\underline{r_*}\right)\right) \subset \{x\}$$

with composition defined in the obvious way.

Notation 2.23. To ease notation we will when convenient refer to an object of $\mathcal{G}$ by its second component w and to an arrow by its first component ι.

Remark 2.24. We think of each object $\nu \in \mathcal{W}$ as a word in the letters $\{x, y\}$. Further, we let $|\nu|$ denote the *length* of ν and $|\nu|_x$ (resp. $|\nu|_y$) denote the number of x's (resp. y's) in ν.

Notation 2.25. We recall certain named types of arrows in $\mathcal{W}$:

- a *shuffle* is an arrow $\sigma \colon \nu \to \nu \circ \sigma$ for $\sigma \in \Sigma^{op}_{|\nu|}$;
- a *tidy arrow* is an arrow $\pi \colon \nu = \bar{\nu} x^a \to \nu_*$ for π the inclusion $\underline{|\nu_*|} = \underline{|\bar{\nu}|} \subset \underline{|\nu|}$;
- a *removing arrow* is an arrow $\nu \to y^{|\nu|_y}$;
- a *replacing arrow* is an arrow $\nu \to y^{|\nu|}$.

Definition 2.26. $\bar{\mathcal{W}}$ is the subcategory of $\mathcal{W}$ with the same objects but only the shuffles, removing and replacing arrows.

Further, for each $r \geq 0$, let $\mathcal{W}_{\leq r}$ (resp. $\mathcal{W}_r$) denote the full subcategory of those $\nu \in \mathcal{W}$ satisfying $|\nu| \leq r$ (resp. $|\nu| = r$).

The following categorical lemmas were proven in [4].

Lemma 2.27. *The subcategory $\bar{\mathcal{W}}$ is final in $\mathcal{W}$.*

Lemma 2.28. *Let $\bar{\mathcal{W}}_{y^r}$ denote the full subcategory of $\bar{\mathcal{W}}$ of objects that admit arrows to y^r. The group Σ_{y^r} of shuffles of y^r is final in $\bar{\mathcal{W}}_{y^r}$.*

Lemma 2.29. *The subcategory $\mathcal{W}_{\leq(r-1)}$ is final in $\mathcal{W}_{\leq r} - y^r$.*

Lemma 2.30. *$\mathcal{W}_{\leq r} = (\mathcal{W}_{\leq r} - y^r) \cup \mathcal{W}_r$. In fact, $N(\mathcal{W}_{\leq r}) = N(\mathcal{W}_{\leq r} - y^r) \cup N(\mathcal{W}_r)$.*

For ease of notation, we will denote an element of $\mathcal{W}^{\oplus \mathcal{C}}$ as $\nu_1 \cdots \nu_p$ where the ν_i are the non initial components of the tuple.

Lemma 2.31. *The pushout diagram (**??**) and $M \in \mathsf{Mod}^r_{\mathcal{O}}(\mathcal{C})$ naturally induces a functor*

$$F^M : \mathcal{W}^{\oplus \mathfrak{C}} \to \mathcal{C}^{\mathfrak{C}}.$$

Proof. For convenience we throughout write $\nu_\bullet(x,y)$ for $\nu_1(x,y)\cdots\nu_p(x,y) \in \mathcal{W}^{\oplus \mathfrak{C}}$ and similarly for arrows, and likewise write $c_\bullet^{|w_\bullet|}$ for $c_1^{|\nu_1|}\cdots c_p^{|\nu_p|}$ and $\nu_\bullet^{\otimes}(X^{c_\bullet}, Y^{c_\bullet})$ for $\nu_1^{\otimes}(X^{c_1}, Y^{c_1}) \otimes \cdots \otimes \nu_p^{\otimes}(X^{c_p}, Y^{c_p})$.

We define F^M on objects as

$$F^M(\nu_\bullet(x,y)) = M_A(c_\bullet^{|\nu_\bullet|}) \otimes \nu_\bullet^{\otimes}(X^{c_\bullet}, Y^{c_\bullet}), \tag{2.32}$$

For arrows, we first declare that for a tuple of shuffles $\sigma_\bullet$, $\sigma_i : \nu_i \to \nu_i \circ \sigma_i$,

$$M_A(c_\bullet^{|\nu_\bullet|}) \otimes \nu_\bullet^{\otimes}(X^{c_\bullet}, Y^{c_\bullet}) \xrightarrow{F^M(\sigma_\bullet)} M_A(c_\bullet^{|\nu_\bullet|}) \otimes (\nu_\bullet \circ \sigma_\bullet)^{\otimes}(X^{c_\bullet}, Y^{c_\bullet})$$

to be the obvious map combining the actions of σ_i^{-1} on $M_A(c_1^{|\nu_1|}\cdots c_p^{|\nu_p|})$ with the shuffles of the ν_i.

Since any arrow in $\mathcal{W}$ can be made tidy by pre-composing with a shuffle, it remains to coherently define F^M on tuples of tidy arrows. For a tidy tuple $\pi_\bullet$, $\pi_i : \nu_i = \bar{\nu}_i(x,y)x^{a_i} \to \hat{\nu}_i(x,y)$, define $F^M(\pi_\bullet)$ via the diagram (with vertical maps the summand inclusions induced by Proposition 2.4 and writing $\mathcal{O}(-)$ for $\mathcal{O} \circ (-)$)

$$\begin{tikzcd}
M_A(c_\bullet^{|\nu_\bullet|}) \otimes \bar{\nu}_\bullet^{\otimes} \otimes (X^{c_\bullet})^{\otimes a_\bullet} \arrow[rr, "F^M(\pi_\bullet)"] \arrow[d, hook] & & M_A(c_\bullet^{|\hat{\nu}_\bullet|}) \otimes \hat{\nu}_\bullet^{\otimes} \arrow[d, hook] \\
M \circ_{\mathcal{O}} (\mathcal{O}(\bar{\nu}_\bullet^{\amalg}) \amalg \mathcal{O}(X^{c_\bullet})^{\amalg a_\bullet} \amalg A) \arrow[rr, "M \circ_{\mathcal{O}} (\mathcal{O} f_* \amalg h_* \amalg id_A)"] & & M \circ_{\mathcal{O}} (\mathcal{O}(\hat{\nu}_\bullet^{\amalg}) \amalg A).
\end{tikzcd} \tag{2.33}$$

$F^M(\pi_\bullet)$ is well defined since $\pi_i \sigma_i$ is tidy only for $\sigma_i \in \Sigma_{a_i} \subset \Sigma_{|w_i|}$ and such shuffles do not change (2.33). It follows that F^M is well defined on all arrows.

It is clear that F^M respects compositions of either shuffle or tidy tuples, and since general two-fold compositions in $\mathcal{W}^{\oplus \mathfrak{C}}$ factor as $\nu_\bullet \xrightarrow{\sigma_\bullet} \nu_\bullet \circ \sigma_\bullet \xrightarrow{\pi} \hat{\nu}_\bullet \xrightarrow{\hat{\sigma}_\bullet} \hat{\nu}_\bullet \circ \hat{\sigma}_\bullet \xrightarrow{\hat{\pi}_\bullet} \hat{\hat{\nu}}_\bullet$

We now verify F^M respects compositions. This is clear when composing either two shuffles tuples or two tidy tuples, and since general two-fold compositions in $\mathcal{W}^{\oplus\mathfrak{C}}$ factor as $\nu_\bullet \xrightarrow{\sigma_\bullet} \nu_\bullet \circ \sigma_\bullet \xrightarrow{\pi} \hat{\nu}_\bullet \xrightarrow{\hat{\sigma}_\bullet} \hat{\nu}_\bullet \circ \hat{\sigma}_\bullet \xrightarrow{\hat{\pi}_\bullet} \hat{\hat{\nu}}_\bullet$ with $\sigma_\bullet, \hat{\sigma}_\bullet$ shuffles and $\pi_\bullet, \hat{\pi}_\bullet$ tidy, it remains to show $F^M(\hat{\sigma}_\bullet \pi_\bullet) = F^M(\hat{\sigma}_\bullet) F^M(\pi_\bullet)$. Identifying $\sigma_i \in \Sigma_{|\hat{\nu}_i|} \subset \Sigma_{|\nu_i|}$, one has $\hat{\sigma}_i \pi_i \hat{\sigma}_i^{-1}$ tidy, so that by definition $F^M(\hat{\sigma}_\bullet \pi_\bullet) = F^M(\hat{\sigma}_\bullet \pi_\bullet \hat{\sigma}_\bullet^{-1}) F^M(\hat{\sigma}_\bullet)$. The claim now follows since (2.33) respects the action of $\hat{\sigma}_\bullet$. $\square$

Lemma 2.34.

$$M \circ_{\mathcal{O}} B \simeq \operatorname{colim}_{\mathcal{W}^{\oplus\mathfrak{C}}} F^M.$$

Proof. Note first that by Lemma 2.27 it suffices to show $M \circ_{\mathcal{O}} B \simeq \operatorname{colim}_{\bar{\mathcal{W}}^{\oplus\mathfrak{C}}} F^M$.

By general considerations one can describe B as a $\mathfrak{C}$-fold reflexive coequalizer

$$B \simeq \operatorname{colim}_{(\Delta^{op}_{\leq 1})^{\times\mathfrak{C}}} \mathcal{R}$$

where $\mathcal{R}\colon (\Delta^{op}_{\leq 1})^{\times\mathfrak{C}} \to \mathsf{Alg}_{\mathcal{O}}$ is defined on objects by

$$(\varphi\colon \mathfrak{C} \to \{0,1\}) \mapsto A \amalg \left(\coprod_{c \in \varphi^{-1}(0)} \mathcal{O}X^c \right) \amalg \left(\coprod_{c \in \mathfrak{C}} \mathcal{O}Y^c \right) \tag{2.35}$$

and on arrows in the obvious way. Note that by Lemma 1.14 it is hence also

$$M \circ_{\mathcal{O}} B \simeq \operatorname{colim}_{(\Delta^{op}_{\leq 1})^{\times\mathfrak{C}}} M \circ_{\mathcal{O}} \mathcal{R}.$$

Now note that by Proposition 2.4 for the constant zero tuple one has

$$\begin{aligned} M \circ_{\mathcal{O}} \mathcal{R}((0)) &= M \circ_{\mathcal{O}} \left(A \amalg (\coprod_{c \in \mathfrak{C}} \mathcal{O}X^c) \amalg (\coprod_{c \in \mathfrak{C}} \mathcal{O}Y^c) \right) \simeq M \circ_{\mathcal{O}} \left(A \amalg \mathcal{O}(\coprod_{c \in \mathfrak{C}} X^c \amalg Y_j) \right) \simeq \\ &\simeq \bigvee_{r_\bullet, s_\bullet \in \omega^{\oplus\mathfrak{C}}} M_A\,(c_\bullet^{r_\bullet + s_\bullet}) \underset{\Sigma_{r_\bullet} \times \Sigma_{s_\bullet}}{\otimes} (X^{c_\bullet})^{\otimes r_\bullet} \otimes (Y^{c_\bullet})^{\otimes s_\bullet}. \end{aligned} \tag{2.36}$$

Further, it is clear from (2.35) that all $\mathcal{R}(\varphi)$ are subwedge summands of $M \circ_{\mathcal{O}} \mathcal{R}((0))$ and that the reflexive maps in $(\Delta^{op}_{\leq 1})^{\times\mathfrak{C}}$ correspond to wedge summand inclusions. Noting that the naturality in Propositions 2.4 and 2.5 implies the maps in $\mathcal{R}$ all respect wedge summands, repackaging universal properties allows one to rewrite

$$M \circ_{\mathcal{O}} B = \operatorname{colim}_{\mathcal{M}^{\oplus\mathfrak{C}}} \bar{F}^M. \tag{2.37}$$

Here $\mathcal{M}$ is the diagram category whose objects we denote by monomials $x^i y^j$, $i, j \geq 0$ together with unique non identity arrows $x^i y^j \to y^{i+j}$, $x^i y^j \to y^j$ for $i \neq 0$ (note that non identity arrows can never be composed). $\bar{F}^M$ is defined on objects by

$$\bar{F}^M(x^{i_\bullet} y^{j_\bullet}) = M_A(c_\bullet^{i_\bullet + j_\bullet}) \otimes_{\Sigma_{i_\bullet} \times \Sigma_{j_\bullet}} (X^{c_\bullet})^{\otimes i_\bullet} \otimes (Y^{c_\bullet})^{\otimes j_\bullet}, \tag{2.38}$$

is induced on arrow components of the form $x^i y^j \to y^{i+j}$ by the map f_* in (**??**) and on arrow components $x^i y^j \to y^j$ by the map h_*.

There is an obvious functor $\bar{\mathcal{W}} \to \mathcal{M}$ defined by $w \mapsto x^{|w|_x} y^{|w|_y}$ (arrows are mapped in the only possible way and functoriality is trivial since non identity arrows in $\mathcal{M}$ can not be composed). We claim $\bar{F}^M = \operatorname{Lan}_{\bar{\mathcal{W}}^{\oplus\mathfrak{C}} \to \mathcal{M}^{\oplus\mathfrak{C}}} F^M$. By [3, X.3.1]

$$(\operatorname{Lan}_{\bar{\mathcal{W}}^{\oplus\mathfrak{C}} \to \mathcal{M}^{\oplus c}} F^M)(x^i y^j) = \operatorname{colim}_{\bar{\mathcal{W}} \downarrow x^i y^j} F^M|_{\bar{\mathcal{W}} \downarrow x^i y^j}. \tag{2.39}$$

Now note that $\bar{\mathcal{W}^{\oplus\mathfrak{C}}} \downarrow x^\bullet y^\bullet \simeq \Pi_\bullet(\bar{\mathcal{W}} \downarrow x^\bullet y^\bullet)$, that for $i \neq 0$ $\bar{\mathcal{W}} \downarrow x^i y^j$ is the groupoid of words w with $|w|_x = i, |w|_y = j$, while for y^r it is the category $\bar{\mathcal{W}}_{y^r}$ of Lemma 2.28 containing the final group Σ_{y^r}. In either case, the formula (2.39) computes the quotient of the terms in (2.32) by the obvious shuffle groupoid action and hence coincides with $\bar{F}^M$ on objects.

To see (2.39) also coincides with $\bar{F}^M$ on arrows consider the commutative diagrams (with vertical maps induced by codiagonals and writing $\mathcal{O}(-)$ for $\mathcal{O} \circ (-)$)

$$\begin{tikzcd}
\mathcal{O}((Y^{c_\bullet})^{\amalg j_\bullet} \amalg (X^{c_\bullet})^{\amalg i_\bullet}) \amalg A \arrow[r, "f_*(\text{resp. } h_*)"] \arrow[d, "\nabla_*"'] & \mathcal{O}(Y^{c_\bullet})^{\amalg(i_\bullet+j_\bullet)} \amalg A \arrow[d, "\nabla_*"] \\
\mathcal{O}(Y^{c_\bullet} \amalg X^{c_\bullet}) \amalg A \arrow[r, "f_*(\text{resp. } h_*)"] & \mathcal{O}(Y^{c_\bullet}) \amalg A.
\end{tikzcd}$$

Since F^M is defined using (shuffles) of the top maps, and $\bar{F}^M$ is defined using the bottom maps, we conclude (2.39) indeed equals $\bar{F}^M$ on maps. Noting that left Kan extensions have the same colimit finishes the proof. □

Proof of Proposition 2.17. By the previous lemma $M \circ_{\mathcal{O}} B \simeq \operatorname{colim}_{\mathcal{W}^{\oplus\mathfrak{C}}} F^M$. We define $\mathcal{F}^M(c_1^{r_1} \cdots c_p^{r_p}) = \operatorname{colim}_{\mathcal{W}_{\leq(r_1,\cdots,r_p)}} F^M$, where $\mathcal{W}_{\leq(r_1,\cdots,r_p)} \simeq \mathcal{W}_{\leq r_1} \times \cdots \times \mathcal{W}_{\leq r_p} \subset \mathcal{W}^{\oplus\mathfrak{C}}$ denotes the obvious subcategory where the non $\varnothing$ entries of the tuples are in the colors $c_1, \cdots, c_p$. (2.18) is immediate since the $\mathcal{W}_{\leq r_1} \times \cdots \times \mathcal{W}_{\leq r_p}$ filter $\mathcal{W}^{\oplus\mathfrak{C}}$. It is straightforward to check that by iterating Lemma 2.30 one has pushout diagrams

$$\begin{tikzcd}
\operatorname{colim}_{\mathcal{W}_{=(r_1,\cdots,r_p)} - y_1^{r_1} \cdots y_p^{r_p}} F^M \arrow[r] \arrow[d] & \operatorname{colim}_{\mathcal{W}_{\leq(r_1,\cdots,r_p)} - y_1^{r_1} \cdots y_p^{r_p}} F^M \arrow[d] \\
\operatorname{colim}_{\mathcal{W}_{=(r_1,\cdots,r_p)}} F^M \arrow[r] & \operatorname{colim}_{\mathcal{W}_{\leq(r_1,\cdots,r_p)}} F^M,
\end{tikzcd}$$

and it hence suffices to verify these diagrams have the form (2.13). The two diagrams coincide on the bottom right corner by definition and on the top right corner by an iteration of Lemma 2.29 (which shows that $\mathcal{W}_{<(r_1,\cdots,r_p)}$, the subcategory where at least one degree is $< r_i$ is final in $\mathcal{W}_{\leq(r_1,\cdots,r_p)} - y_1^{r_1} \cdots y_p^{r_p}$). The left hand maps of the two diagrams are seen to coincide by direct computation. □

Proof of Proposition 2.14. The claim concerning (2.15) is immediate by the construction of F^M.

For the case of (2.16), it suffices to show that the natural transformation $F^M_{\mathfrak{D}\to\mathfrak{C}} \to \operatorname{Lan}_{\mathcal{W}^{\oplus\bar{\mathfrak{D}}} \to \mathcal{W}^{\oplus\mathfrak{C}}} \operatorname{colim}_{\mathcal{W}^{\oplus\mathfrak{D}}} F^M$ induced by naturality of Proposition 2.31 is a natural isomorphism, and it is clear that this only needs to be verified at $\mathcal{W}^{\oplus\bar{\mathfrak{D}}}$. For each $\bar{\nu}_1 \cdots \bar{\nu}_{\bar{p}} \in \mathcal{W}^{\oplus\bar{\mathfrak{D}}}$ this amounts to showing that

$$\operatorname*{colim}_{\nu_1\cdots\nu_p \in \mathcal{W}^{\oplus\mathfrak{D}}} M_A(d_1^{\nu_1} \cdots d_p^{\nu_p} \bar{d}_1^{\bar{\nu}_1} \cdots \bar{d}_{\bar{p}}^{\bar{\nu}_{\bar{p}}}) \otimes \nu_1^{\otimes} \otimes \cdots \otimes \nu_p^{\otimes} \otimes \bar{\nu}_1^{\otimes} \otimes \cdots \otimes \bar{\nu}_{\bar{p}}^{\otimes} \simeq M_{A_{\mathfrak{D}}}(\bar{d}_1^{\bar{\nu}_1} \cdots \bar{d}_{\bar{p}}^{\bar{\nu}_{\bar{p}}}) \otimes \bar{\nu}_1^{\otimes} \otimes \cdots \otimes \bar{\nu}_{\bar{p}}^{\otimes}$$

and that in turn reduces to showing that

$$\operatorname*{colim}_{\nu_1\cdots\nu_p \in \mathcal{W}^{\oplus\mathfrak{D}}} M_A(d_1^{\nu_1} \cdots d_p^{\nu_p}(-)) \otimes \nu_1^{\otimes} \otimes \cdots \otimes \nu_p^{\otimes} \simeq M_{A_{\mathfrak{D}}}(-)$$

is an isomorphism of $\mathfrak{C}$-symmetric $\mathfrak{C}$-sequences. But this now follows directly by Lemma 2.34 applied to the category $\mathsf{Sym}_{\mathfrak{C}}(\mathcal{C})$ and the diagram

$$\begin{tikzcd} \mathcal{O} \circ X_{\mathfrak{D}} \arrow[r, "h"] \arrow[d, "\mathcal{O} \circ f_{\mathfrak{D}}"'] & \mathcal{O}_A \arrow[d] \\ \mathcal{O} \circ Y_{\mathfrak{D}} \arrow[r] & \mathcal{O}_{A_{\mathfrak{D}}} \end{tikzcd} \tag{2.40}$$

since, as shown in the proof of Proposition 2.5, $M^{\mathsf{w}}_{\mathcal{O}^{\mathsf{v}}_A}(w, v) \simeq M_A(wv)$.

□

3 Excisiveness of truncated operadic functors

In this section we use our main filtration result, Proposition 2.11, to provide a direct proof of the excisiveness of truncated operadic functors.

The first step is to note that one can encode diagrams in $\mathsf{Alg}_{\mathcal{O}}$ as algebras over a single operad for a different set of colors (cf. [6]).

Definition 3.1. Let $\mathcal{O}$ be any operad on colors $\mathfrak{C}$ and let I be a diagram category. Then $(\mathsf{Alg}_{\mathcal{O}})^I \simeq \mathsf{Alg}_{\mathcal{O}\otimes I}$, where $\mathcal{O} \otimes I$ is the colored operad with colors $\mathfrak{C} \times ob(I)$ and

$$(\mathcal{O} \otimes I)^{(c,d)}\left((c_1, d_1) \cdots (c_r, d_r)\right) = I(i_1, i) \times \cdots \times I(i_r, i) \cdot \mathcal{O}^c(c_1 \cdots c_r).$$

Lemma 3.2. *Let* $\mathsf{ctt}\colon \mathsf{Alg}_{\mathcal{O}} \to \mathsf{Alg}_{\mathcal{O}\otimes I}$ *denote the "constant diagram algebra functor". Then*

$$(\mathcal{O} \otimes I)_{\mathsf{ctt}(A)} = \mathsf{ctt}(\mathcal{O}_A).$$

Our interest in Lemma 3.2 comes mainly from the following case. Let $C_n = \mathcal{P}(\{1, \cdots, n\})$ be the diagram n-cube category. Giving a strongly cartesian n-cube in $B\colon C_n \to \mathcal{C}$ whose initial edges are pushouts of *generating cofibrations* is then equivalent to giving a pushout diagram

$$\begin{tikzcd} (\mathcal{O} \otimes C_n) \circ X \arrow[r, "h"] \arrow[d, "\mathcal{O} \circ f"'] & \mathsf{ctt}(B(\varnothing)) \arrow[d] \\ (\mathcal{O} \otimes C_n) \circ Y \arrow[r] & B \end{tikzcd} \tag{3.3}$$

where $X \to Y$ is a C_n-sequence concentrated in those colors that correspond to singletons of $\{1, \cdots, n\}$.

Unpacking the filtration in Proposition 2.11 in this case, and reading it at the color $\{1, \cdots, n\}$, one obtains an "n-fold infinite filtration of $M \circ_{\mathcal{O}} B$". Explicitly, one gets a diagram $\mathcal{B}^M\colon \omega^{\oplus\{1, \cdots, n\}} \to \mathcal{C}$ such that

- $\operatorname{colim}_{\omega^{\oplus S}} \mathcal{B}^M = (M \circ_{\mathcal{O}} B)(S)$ for each subset $S \subset \{1, \cdots, n\}$;
- the latching map $l_{1^{r_1} \cdots n^{r_n}}$ is a pushout

$$M_A(r_1 + \cdots + r_n) \otimes_{\Sigma_{r_1} \times \cdots \times \Sigma_{r_n}} (f^{\{1\}})^{\square r_1} \square \cdots \square (f^{\{n\}})^{\square r_n} \to l_{1^{r_1} \cdots n^{r_n}}.$$

Lemma 3.4. *Suppose that* M *is a* n*-truncated* $\mathcal{O}$*-bimodule (i.e.* $\mathcal{O}(m) = *$ *for* $m > n$*). Then* M_A *is also* n*-truncated for any* $\mathcal{O}$ *algebra* A.

Proof. Recalling that any algebra A is canonically described by a reflexive coequalizer, one has a reflexive coequalizer

$$M \circ_{\mathcal{O}} (\mathcal{O} \circ (\mathcal{O} \amalg \mathcal{O}A)) \rightrightarrows M \circ_{\mathcal{O}} (\mathcal{O} \amalg \mathcal{O}A) \to M \circ_{\mathcal{O}} (\mathcal{O} \amalg A) = M_A.$$

The maps being equalized can then be rewritten as

$$M \circ (\mathcal{O} \amalg \mathcal{O}A) \rightrightarrows M \circ (\mathcal{I} \amalg A).$$

Since $\mathcal{I}$ lives in operadic degree 1 it follows that $M \circ (\mathcal{I} \amalg A)$ is n-truncated and hence so is the coequalizer M_A. $\square$

Proposition 3.5. *Suppose that $\mathcal{C}$ is a stable cofibrantly category model category such that a projective model structure on $Alg_{\mathcal{O}}$ exists. Then, for any n truncated bimodule M the associated functor*

$$M \circ_{\mathcal{O}} (-) \colon Alg_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$$

is n-excisive.

Proof. Let $B \colon C_{n+1} \to \mathcal{C}$ be a strongly co-cartesian cube. We need to show that then $M \circ_{\mathcal{O}} B$ is a cartesian cube, and since we are assuming $\mathcal{C}$ stable it suffices to instead verify that this is a co-cartesian cube.

Now recall that the map $B(\varnothing) \to B(\{1\})$ can be written as a retract of some map in $Cell(\mathcal{O}(I))$, where I is a set of generating cofibrations in $\mathcal{C}$. Since retracts respect (strong) co-cartesian cubes, we reduce to the case where $B(\varnothing) \to B(\{1\})$ is a map in $Cell(\mathcal{O}(I))$. Again, since transfinite compositions of (strong) co-cartesian cubes are again (strong) co-cartesian, one further reduces to the case of $B(\varnothing) \to B(\{1\})$ the pushout of a map in $\mathcal{O}(I)$. Repeating the same argument for the remainder initial maps one reduces to the case where all maps $B(\varnothing) \to B(\{i\})$ are pushouts of maps in $\mathcal{O}(I)$, i.e., to the scenario described after Lemma 3.2, so that one obtains a n-fold filtration $\mathcal{B}^M$ of $M \circ_{\mathcal{O}} B$ as described. Thinking of $\omega^{\oplus\{1,\cdots,n+1\}}$ as an "infinitely filtered $(n+1)$-cube", it then follows that all non trivial latching maps occur at the faces of $\omega^{\oplus\{1,\cdots,n\}}$, so that it must indeed be the case that $B(\{1,\cdots,n+1\})$ is the colimit of $B|_{\mathcal{P}(\{1,\cdots,n+1\})-\{1,\cdots,n+1\}}$. $\square$

Example 3.6. As an illustration of the previous proof, consider the case of M 2-truncated. Then all non trivial latching maps for $\mathcal{B}^M \colon \omega^{\oplus\{1,2,3\}} \to \mathcal{C}$ occur at the diagram

$$\begin{tikzcd}
 & & 3^2 & & \\
 & & 3 \arrow[u] \arrow[r] \arrow[dl] & 23 & \\
 & 13 & \varnothing \arrow[u] \arrow[r] \arrow[dl] & 2 \arrow[u] \arrow[r] \arrow[dl] & 2^2 \\
 & 1 \arrow[u] \arrow[r] \arrow[dl] & 12 & & \\
1^2, & & & &
\end{tikzcd} \tag{3.7}$$

i.e. the full diagram $\mathcal{B}^M$ is the left Kan extension of its restriction to the subdiagram above.

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