PRINCETON LECTURES IN REAL ANALYSIS

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Contents

1 Lebesgue Dominated Convergence Theorem

Definition 1.1 (Dominated Convergence Theorem)**.** If we have a sequence of measurable functions $\{f_n\}$ converges to f a.e., if $|f_n| \leq g$ this implies the absolute value of the difference of $f_n - f$ tends to 0. As a consequence the integral of f_n tends to the integral of f.

1.1 Almost everywhere

Definition 1.2 (Almost everywhere equal)**.** f, g defined on a set E are equal almost everywhere, we write

$$f(x) = g(x) \text{ a.e. } x \in E,$$

If the measure of the set which consists of the point p where $f(p) \neq g(p)$ has measure zero.

Definition 1.3 (Almost everywhere convergence)**.** We say a sequence of functions $\{f_n\}$ converges almost everywhere to f on set E if:

$$\{x \in E | \{f_n(x)\} \nrightarrow f(x)\}$$

has measure 0.

Remark:

- **a.e. = a.s. = p.a.e(Pointwise almost everywhere)**
- **Pointwise converges doesn't implies the measure, it just says that on every point of the domain we care, the corresponding sequence converges.**

Definition 1.4 (Measurable Function)**.** A function f on $\mathbb{R}^d$ is measurable if for every $a \in \mathbb{R}$:

$$\{x \in \mathbb{R}^d | f(x) < a\}$$

is measurable.

To find some intuitive examples to this definition of measurable funcitons, characteristic functions and step functions(simple functions) will be our choices.

1.2 Integration

2 Littlewood's Three Principle

Theorem 2.1 (Littlewood's principle)**.**

- *Every measurable function is almost continuous(Lusin);*
- *Every measurable set is almost a finite union of intervals;*
- *Every converges sequence is almost uniformly converges(Egorov Theorem).*

3 Egorov Theorem

Definition 3.1 (Egorov Theorem)**.** Suppose we have a sequence of measurable functions defined on set E with $m(E)$ is finite. Assume the sequence of measurable functions converges to some function f a.e. on E. Given a ϵ, we can find a closed subset A_ϵ of E s.t. A_ϵ is differ from E with a set which has measure ϵ and converges becomes uniform converges on this closed set.

3.1 Measurable Functions

The definition of measurable functions is a progressive definition, we first look at the simple function.

Definition 3.2 (Finite-value function is measurable)**.** A finite-value function is measurable iff preimage of any openset O is measurable iff preimage of any closed set F is measurable.

Remark: We use F to declare the closed sets since fermé.