LINEAR ALGEBRA

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Contents

1 Vector Spaces

Definition 1.1 (Vector Space)**.** For field K, K- vector space V is a set of objects closed under "addition" and "multiplication", where these two composition laws are connected by distribution law in the natural sense. Moreover both additive identity and multiplicative identity exist. $\forall x \in V$, x has a additive inverse but doesn't have a multiplicative inverse. $\forall c \in K, x \in V$, composition law between c and x, is just the scalar product in the natural sense. We call V the vector space over K.

2 Sum, direct sum, product of vector spaces

For convention we denote the vector space in preliminary w.r.t. V, W, the field F.

Definition 2.1 (Sum of vector spaces)**.** The sum of vector space V, W is defined as follows:

$$V + W := \{v + w | v \in V, w \in W\}.$$

Definition 2.2 (Direct Sum of vector spaces)**.** The direct sum of vector spaces V, W is defined as follows:

$$V \oplus W := \{v + w | v \in V, w \in W\}.$$

where $V \cap W = \emptyset$.

Unlike the definition of direct sum and sum of vector spaces relies on the operation are the same in the two vector spaces(Namely V, W are all subspaces in this case). We have a more general convetion of this progcess.

Definition 2.3 (Direct Product of vector spaces). We define the direct product of V, W as follows:

$$V \times W := \{(v, w) | v \in V, w \in W\}$$

we don't need V, W to be the subspaces(Namely has same operation).

3 Linear Mappings

3.1 Linear Mappings

3.2 Kernel and Image

4 Bilinear maps and its relation to Matrices

For convention let K be field, which can also be seen as one-dimensional vector space over itself. U, V, W to denote the K-vector spaces.

Definition 4.1 (Bilinear Map). Let $g : U \times V \to W$ be a map. g is a bilinear map if $\forall v \in V$, $g_v := g(u, v) : U \to W$ is a linear map; $\forall u \in U$, $g_u := g(u, v) : V \to W$ is a linear map. We say g is a Bilinear map.

We have an important theorem to characteristic the relationship between a bilinear map g and a matrix A.

Theorem 4.2 (Corresponding Theorem of bilinear map and matrix). *Given a bilinear map $g : K^m \times K^n \to K$, then exists a unique $A \in \mathbb{K}^{m \times n}$, s.t.*

$$g(x, y) = x^T A y, \ \forall x \in K^m, y \in K^n.$$

Proposition 4.3. *Let $Bli(K^m, K^n) := \{g | g : K^m \to K^n, g \text{ is bilinear}\}$. Then $Bli(K^m, K^n)$ is a K-vector space.*

The next theorem give a correspondence between space of bilinear maps and matrix, which is a baby-case to illustrate the universal property of tensor product.

Theorem 4.4 (Isomorphism theorem of Bilinear Maps). $Bli(K^m, K^n) \simeq Mat_{m \times n}(K)$.

We also have the following commutative diagram:

$$\begin{tikzcd}
K^m \times K^n \arrow[r] \arrow[d] & Bli(K^m, K^n) \\
Mat_{m \times n} \arrow[ur] &
\end{tikzcd}$$

5 Dual Space and Scalar Product

Definition 5.1 (Dual Space). Let $V^* := \mathbb{L}(V, K)$, we call V^* the dual space of V.

Remark: We call the elements in the dual space functionals.

Proposition 5.2. *Let V be a K-vector space of dimension n, then*

$$V \simeq K^n$$

With the above proposition we know that: Once the basis of the vector space B_V is given, we can associate $v \in V$ to $w \in K^n$, by the isomorphism.

Motivation: Now we substitute the isomorphism result into the definition of the dual space. Because in this way it is easier to express the morphisms.

We consider the following commutative diagram:

$$\begin{tikzcd} K^n \arrow[r, "\text{iso.}"] \arrow[dr, dashed, "\varphi"] & V \arrow[d] \\ & K \end{tikzcd}$$

This diagram arise the isomorphism:

$$K^n \to \mathbb{L}(V, K),\ A \mapsto L_A$$

Then we have the following result:

Proposition 5.3.

$$\dim(V) = \dim(V^*)$$