ANALYSIS I

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Contents

1 Introduction

1.1 Difference between Analysis and Calculus

Calculus is sth we fit the rules and use our intuition of the rules to solve physical problems. Analysis is the foundation lies under the Calculus. It helps us understand the limit of the theorems in Calculus we used.

1.2 The questions which will be answered in this book

- What is a real number?
  - What is the smallest real number after zero?
  - Why there are more real numbers than rational numbers?
  - What is the next real number?
  - Why 2 has a square root but -2 doesn't?
- How to take limit of a sequence of real numbers?
  - Add infinitely many real numbers ends up with a rational number?
  - Rearrange the infinite sum still has a same sum?
  - Which sequence has a limit and which doesn't?

- What is a function?
  - What does the properties of the functions mean?
  - Arithmatic of functions in infinite case.
  - Differentiablility of a infinite sequence of functions?
  - $f(1) = 1, f(3) = 3$ does it exist $f(x) = 2$? Why?

2 Why do Analysis? & Purpose of this book

2.1 Why do Analysis?

I'll list a few examples to establish the fail of serveral theorems.

Example 2.1 (Division by zero). $ab = cb \to a = c$, if $b = 0$ then $a \neq c$ also stands.

Example 2.2 (Divergent series). Suppose

$$S = 2 + 4 + ...$$

then

$$2S = 4 + 8 + ...$$

let $2S - S$ we have $S = -2$.

Example 2.3 (Limit values of functions).

$$\lim_{x\to\infty} sin(x) = \lim_{y+\pi\to\infty} sin(y+\pi) = \lim_{y+\pi\to\infty} (-sin(y)) = -\lim_{y\to\infty} sin(y)$$

.

Example 2.4 (Interchanging limits).

$$\lim_{x\to 0}\lim_{y\to 0} \frac{x^2}{x^2+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+0^2} = 1$$

however, we have

$$\lim_{y\to 0}\lim_{x\to 0} \frac{x^2}{x^2+y^2} = \lim_{y\to 0} \frac{0^2}{0^2+y^2} = 0.$$

Example 2.5 (L'Hôpital's rule).

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

one can apply this to $f(x) := x, g(x) := 1 + x, x_0 := 0$.

Example 2.6 (Limits and Length). We use staircase to approximate the phypotenuse, then we take the limit of this approximation and find that the hypotenuse has length 2 instead of length $\sqrt{2}$.

We will justified these rules and know when they are illegal.

3 Beginning: The Natural Numbers

Why do the rules of the algebra works well? Rules are not chosed arbitrarily! Explore complex properties from the simpler ones. Why an obvious statement really is obvious. **Fundamental Skill** mathematical induction

3.1 How does one define the natural numbers?

We will forget about everything we know to prove the fundamental result to avoid the circularity. **Rusume: formal document that summerize the individual's professional qualifications** Forget about the decimal system, because it doesn't matters and it is not as natral as you might think.

Example 3.1 (Unatral things about the decimal system)**.**

$$0.99999\cdots = 1$$

3.2 The Peano Axioms

Definition 3.1 (Informal definition of natural numbers)**.**

$$\mathcal{N} := \{0, 1, 2, \ldots\}$$

starting at 0 and counting to infinity.

Example 3.2 (Informal definition of operation on natural numbers)**.** exponentiation is nothing more than repeated multiplication, multiplication is nothing more than repeated addition, addition is nothing more than counting forward. Counting forward is one learns first about natural numbers.

In mathematics try not to define a variable more tha once which will cause confusion. We will use the zero number and the increment operation(the successor operation) to define the natural numbers. We use $n++$ to represent the successor of n.

We need to axioms of the two things we care about above.

Axiom 3.2. 0 *is a natural number.*

Axiom 3.3. n *is a natural number, then the sucessor of* n *is also a natural number.*

We denote the sucessor of n using $n++$

Definition 3.4. We define $1 := 0++, 2 := (0++)++, \ldots$

We might have the "wrap-around issue" which means 0 becomes the sucessor for some natural numbers.

Axiom 3.5. 0 *is not the successor of any natural number.*

Priori: one already knows to be true before begins a proof

We can also have number systems where $n++ = k, k \neq 0$, to avoid this we have the following

Axiom 3.6. *Different natural numbers must have different successors.*

Example 3.3 (Informal example)**.** Consider$\mathbb{N} : \{0, 0.5, 1, 1.5, \ldots\}$, this set satisfies the above axioms but that's not the integers we want.

We only want the integer to be obtained from 0 and the increment operation.

Axiom 3.7 (Principle of mathematical induction)**.** $P(n)$ *any property pertaining to a natural number* n*. Suppose* $P(0)$*is true, and suppose that whenever* $P(n)$ *is true,* $P(n++)$ *is also true. Then* $P(n)$ *is true for every natural number* n*.*

The axiom above prohibit any numbers other than the "True" natural numbers appear in $\mathbb{N}$. By considering the following: Let $P(n) = n$isn't a half-integer then we can show that $0, 1, \ldots$ all aren't half integers.

How to use Mathematical induction to the problem "$P(n)$is true for every natural number$n$"

1. Verify the base case $n = 0$.
2. Suppose n is a natural numberm and $P(n)$ has already been proven.
3. Show $P(n++)$ is also true.

The Axioms we introduce above is so-called **Peano Axioms** for the natural numbers

In this case we can build an equivalence relations on the natural number systems by Peano axioms. eg: Hindu-Arabic natural number system is the same as Roman system.

*We leave one thing here, we don't actually produce a natural number system"- come back later

One strange things about the natural numbers is that every natural number is finite, 0 is finite, n is finite, $n++$ is also finite. But there are infinitely many natural numbers.

There are number systems that admit "infinite" numbers like "p-adic, cardinal, ordinal,..."

Remark: we introduce natural numbers in a axiomatic way rather than constructive way, in this case we only know what can we do with them but don't know what is natural number and do they exist in the real world

And this is the core of mathematics-only care about what properties the objects have, don't care about what they are or what do they mean.

A little historical background Treat numbers axiomatically is very recent, not much more than a hundred years old. Before then, numbers were generally understood to be inextricably connected to some external concept, like the cardinality of a set...

The problem arise when we want to move to other number systems.

Definition 3.8 (Recursive definitions)**.** For each number n, consider function $f_n : \mathbb{N} \to \mathbb{N}$. Let $c \in \mathbb{N}$, then we can assign a unique a_n to each n, such that $a_0 = c, a_{n++} := f_n(a_n)$ for each natural number n.

To specify the question first, this means give a family of functions f and the starting value c, we can uniquely define a sequence.

. We use induction. Base case: $a_{0++} := f_0(a_0) = f_0(c)$ which is uniquely determined.
Notice that none of the other definitions will redefine a_0 **since** $n++ \neq 0$
Suppose a_n is uniquely determined, then $a_{n++} = f_n(a_n)$ is uniquely defined. **Notice that none of the other definitions will redefine** a_n **because** $n++ = m++$ **implies** $n = m$ ☐

Recursion definition doesn't works for the systems which has some sort of wrap-around.

We will now use recursion definition to define addition and multiplication.

3.3 Addition

Definition 3.9 (Addition for natural numbers)**.** Let m be a natural number.

- [add 0] $0 + m := m$
- Suppose we have define $n + m$
- [add $n++$ to m] $(n++) + m := (n + m)++$

In order to compare with the recursion definition, here $f_n(a_n)$ simply means a_n++.

Another thing worth notice is that the definition is symmetric, which means we can define right addition in the same way and the result will be the same.

Show the sum of two natural numbers is again a natural number.

.
- Base case: By definition above m is a natural number, then $0 + m := m$ is again a natural number.

- Suppose for every n, $m+n$ is again a natural number.
- Then $(m++)+n := (m+n)++$ is a natural number by the axiom.

$\square$

In order to deduce the fact that addition is commutative for the natural numbers we will deduce it from inside.

From a logical point of view, there is no difference between a lemma, proposition, theorem or corollary- they are all claims wait to be proved

Lemma 3.10. $n+0=n$ *for any natural number* n.

The proof is quite trivial, one can directly use induction on n to show.

Lemma 3.11. *For any natural numbers* n *and* m, $n+(m++)=(n+m)++$

This lemma needs a little bit more effort. One might be confused of do induction on n or on m, but since $n+0=n$ is a special case, we should do induction on n and fix m. However, one shouldn't try to expand the terms since we don't have the associativity yet.

Choose to do induction on n or do induction on m If we are doing induction on m, we will try to show $n+((m++)++)=(n+(m++))++$. But unfortunately we don't have a claim to deal with it since we the only operation we can do is to expand the right hand side.

But if we do induction on n, we can use the addition we define earlier to composite the two terms.

Corollary 3.12. $n++=n+1$.

. $n++=(n+0)++=n+(0++)=n+1$ $\square$

Proposition 3.13 (Addition is commutative)**.** *For any natural numbers* a,b, *we have* $a+b=b+a$.

Proof by induction on a or b.

Proposition 3.14 (Addition is associative)**.** *For any natural number* a,b,c, *we have* $a+(b+c)=(a+b)+c$

Proof by induction on a, fix b,c.
Can we really do induction like this?

Proposition 3.15 (Cancellation law)**.** *For any natural numbers* a,b,c, *if* $a+b=a+c$, *then we have* $b=c$.

We use cancellation law to define subtraction Proof by induction and use the axiom to show $(a+b)++=(b+c)++$ **implies** $a+b=a+c$

Definition 3.16 (Positive natural number)**.** A natural number n is positive iff it is not equal to 0.

Proposition 3.17. *If* a *is a positive natural number and* b *is a natural number then* $a+b$ *is a positive number.*

Proof by induction and use axiom to show $(a+b)++$ is a natural number and not equals 0.

Corollary 3.18. *If* a,b *are natural numbers such that* $a+b=0$ *then* $a=0,b=0$.

Proof by contradiction.

Lemma 3.19 (Unique successor property)**.** *If* a *is a positive number. Then* $\exists! b$ *such that* $b++=a$.

Suppose there are b_1,b_2, show $b_1=b_2$.
Why we can define order when we can do addition?

Definition 3.20 (Ordering of the natural numbers)**.** For natural numbers m, n. $n \geq m$ or $m \leq n$, iff we have $n = m + a$ for some natural number a. $n > m$ or $m < n$, iff $n = m + a$ for some positive natural number a.

I think the reason is the order is a product of difference. Once we have addition we can then represent the difference between two objects.

Corollary 3.21. *There is no largest natural number n.*

Because $n + +$ is always larger.

Proposition 3.22 (Basic properties for natural numbers)**.** *Let a, b, c be natural numbers. Then*

- *reflexive: $a \geq a$*
- *transitive: If $a \geq b, b \geq c$, then $a \geq c$.*
- *antisymmetric: If $a \geq b$ and $b \geq a$, then $a = b$.*
- *addition preserves order: $a \geq b$ iff $a + c \geq b + c$.*
  Show by using cancellation law and commutative law.
- *$a < b$ iff $a + + \leq b$.*
- *$a < b$ iff $b = a + d$ for some positive number d.*

Proposition 3.23 (Trichotomy of order for natural numbers)**.** *Let a, b be natural numbers. Then one of the following statement:$a < b$ or $a = b$ or $a > b$ must stand.*

Trichotomy here means division into three parts.
For the proof part, we first show that there couldn't be more than one statement to be true.
Then we show by induction:

- Base case: Since $0 + b = b$, therefore $0 \leq b$.
- Induction Hypothesis: The statement.
- Goal: Show that $a + +, b$ must stand for one of the statement.
  - Discuss the three cases and use the property of $a + +$.

With the properties of order, we have a strong version of the principle of induction:

Proposition 3.24 (Strong principle of induction)**.** *Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all $m_0 \leq m' < m$ then $P(m)$ is also true. We can conclude that $P(m)$ is true for all $m \geq m_0$.*

Remark: We often use $m_0 = 0$ or $m_0 = 1$.
Vacuous: empty doesn't have any meaning Pertain: relate to

3.4 Multiplication

We admit all the things we proved for addition in the above section and use these rules of algebra.

This is the natural idea, we basically see multiplication as iterited addition where addition is just iteratied increment.

Definition 3.25 (Multiplication of natural numbers)**.** Let m be an natural number. We defin $0 \times m = 0$. Now suppose inductively that we have defined $n \times m$, then $(n++) \times m = n \times m + m$.

This definition use the exact same ways as how we define addition.

Check that the multiplication of two natural numbers is still a natural number Check by induction, then use **Sum of two natural numbers is a natural number**.

Lemma 3.26 (Multiplication is commutative)**.** *For natural numbers m, n, $m \times n = n \times m$.*

- First do induction on m.
  - One needs to show that $n \times 0 = 0$
    - use strong induction to show

Precedence: Sth comes first

To save the parentheses we will use ab to denote $a \times b$.

Lemma 3.27 (No zero divisors)**.** *Let n, m be natural numbers. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if n, m are positive natural numbers, then nm is also a positive natural number.*

There is no excuse to not use mathematical induction "If" Direction is easy to show. For the "only if" direction we use induction. We do induction on n.

- BC(Base case): $0 \times n = 0$ then one of n, m is equal to zero.
- Suppose the statement is true.
- Wts $(n++) \times m = 0$ implies $n++, m$ at least one zero.

Here's the question bother me at the beginning of the university In a ring, it is the mathematical object born with distributive law(i.e there is a natural mathematical distributive law presented on the mathematical object) or it is we define the distributive law on it.

Proposition 3.28 (Distributive law)**.** *For any natural numbers a, b, c, we have $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.*

Because of the commutivity we only need to show one identity. For the proof we do induction on c and consider to prove $(b + c)a = ba + ca$

- BC: $b + 0)a = ba, ba + 0a = ba + 0 = ba$
- Suppose the statement is true
- wts: $(b + (c++))a = ba + (c++)a$
  Just use the definition of the multiplication and Induction Hypothesis.

Proposition 3.29 (multiplication is associative)**.** *For any natural number a, b, c, we have $(a \times b) \times c = a \times (b \times c)$.*

Use induction on a.

- BC: $(0 \times b) \times c = 0, 0 \times (b \times c) = 0$
- Suppose the statement is true
- wts $(a++) \times b) \times c = (a++) \times (b \times c)$
  It's trivial to show this by using the distribution law.

Proposition 3.30 (Multiplication preserves order)**.** *$a < b$ for natural numbers a, b. Then for positive natural number c, $ac < bc$.*

Recall that $a < b$ implies $b = a + m$ for unique m. Proof: we should translate $a < b$ in identity form. Then using distribution law to calculate this identity, we will get the answer finally.

Corollary 3.31 (Cancellation law)**.** *For natural numbers a, b, c where c is nonzero. If $ab = ac$, then $b = c$.*

proof:

- By the trichotomy we know that b, c must have a relation. If $b = c$ then we proved, if not.
  - consider the relation and after multiplication we will lead a contradiction.

Remark: cancellation for addition will allow for "subtraction" and cancellation for multiplication will allow for "division".

Once we are familiar with addition and multiplication we define, then we can leave the notion of increment aside.

Notice that all the concept connecting with natural numbers is defined inductively, but what makes the induction principle right? That's because we admit it as a basic axiom!

Proposition 3.32 (Euclid's division lemma)**.** *n is a natural number, and q is a positive number. Then exist natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$.*

This algoritm marks the beginning of the number theory. Proof sketch: We use induction to show, do induction on n, fix q.

- BC: $0 = 0q + 0$
- suppose the statement stands
- $n + 1 = mq + r$
  Notice that $n = m_1 q + r_1$ and $r_1 + 1 \leq q$ and by trichotomy we know that $r_1 + 1 = q$ or $r_1 + 1 < q$.

Definition 3.33 (Definition for exponentiation for natural numbers)**.** Let m be a natural number, then we define $m^0 := 1$. Suppose we define m^n inductively, then $m^{n++} = m^n \times m$. In particular we define $0^0 := 1$.

This is the iterative definition, we can also give a recursive definition

Definition 3.34. For natural number $\{a_i\}$, consider functions $\{F_i\}$, $a_0 = m$. $a_{n++} = F_n(a_n) = a_n \times a_0$. We denote $m^n := a_n$.

4 Set Theory