EE 40 – Frequency Response

Reading Material:

Chapter 9

The Transfer Function

Transfer function of a circuit or system describes the output response to an input excitation as a function of the angular frequency ω.

$$\mathbf{H}(\omega) = \frac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)}$$ Voltage Gain

$$\mathbf{H}(\omega) = M(\omega)\, e^{j\phi(\omega)},$$

where by definition,

$$M(\omega) = |\mathbf{H}(\omega)|, \qquad \phi(\omega) = \tan^{-1}\left\{\frac{\mathfrak{Im}[\mathbf{H}(\omega)]}{\mathfrak{Re}[\mathbf{H}(\omega)]}\right\}$$

Magnitude

Phase

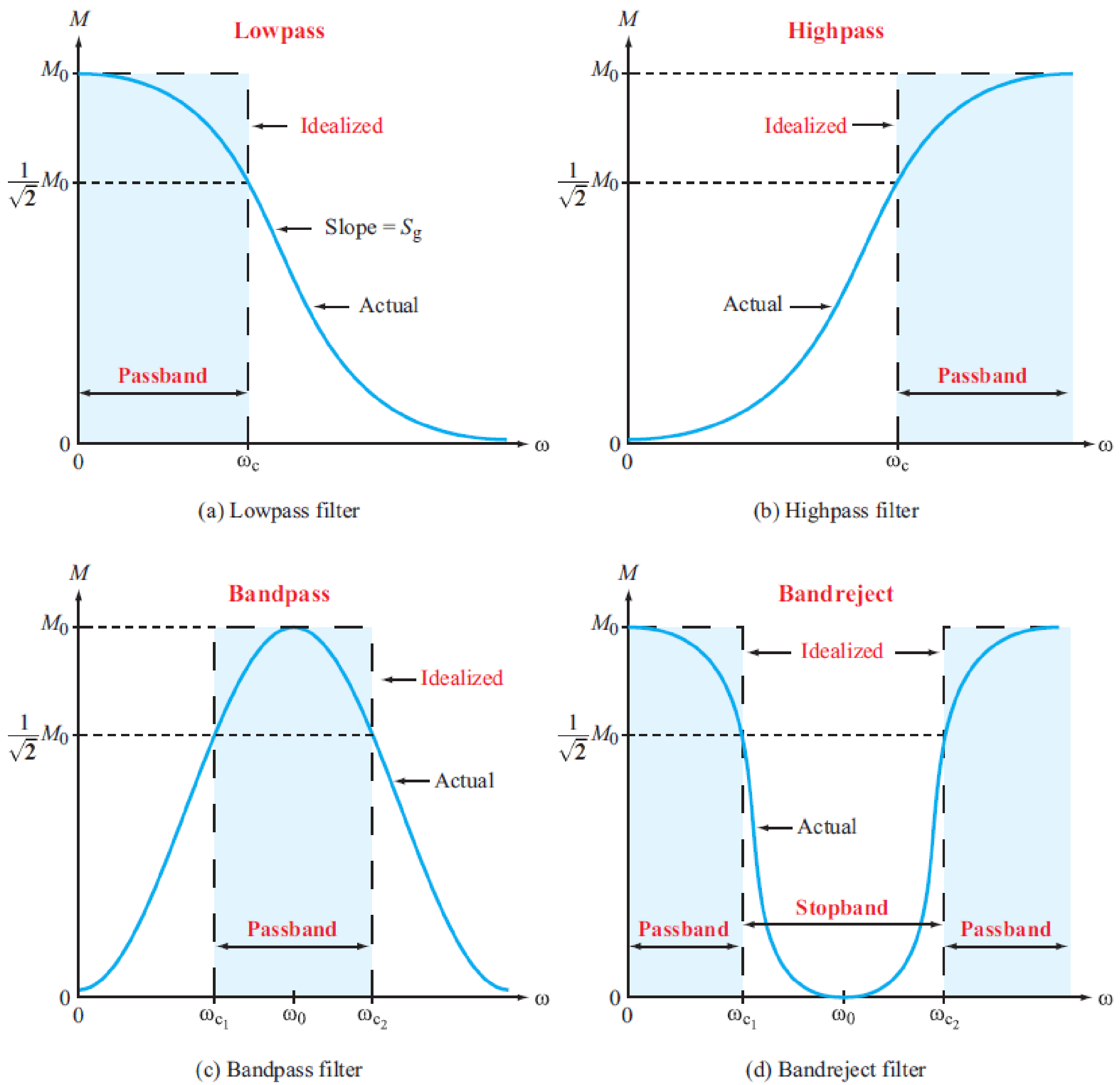
Other Transfer Functions

Current gain: $\mathbf{H}_I(\omega) = \dfrac{\mathbf{I}_{\text{out}}(\omega)}{\mathbf{I}_{\text{in}}(\omega)}$

Transfer impedance: $\mathbf{H}_{\mathbf{Z}}(\omega) = \dfrac{\mathbf{V}_{\text{out}}(\omega)}{\mathbf{I}_{\text{in}}(\omega)}$

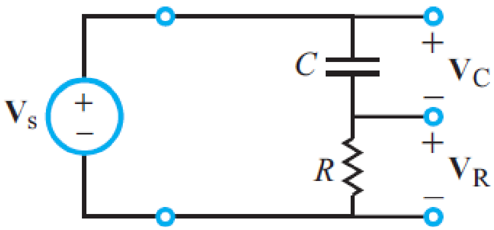
Transfer admittance: $\mathbf{H}_{\mathbf{Y}}(\omega) = \dfrac{\mathbf{I}_{\text{out}}(\omega)}{\mathbf{V}_{\text{in}}(\omega)}$

Filters

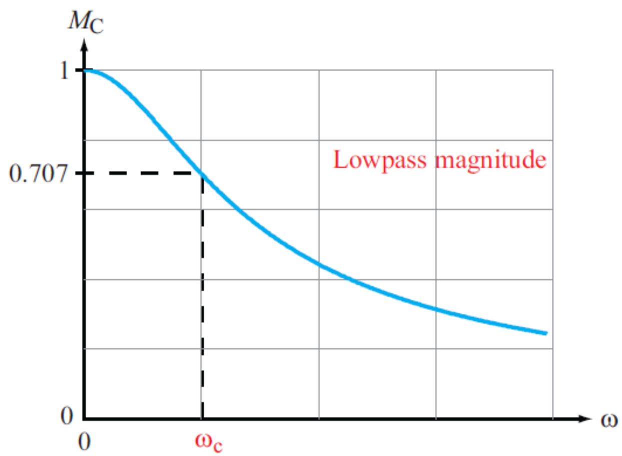


M
Lowpass
M_0
Idealized
$\frac{1}{\sqrt{2}}M_0$
Slope = S_g
Actual
Passband
0
0
ω_c
ω
(a) Lowpass filter
M
Highpass
M_0
Idealized
$\frac{1}{\sqrt{2}}M_0$
Actual
Passband
0
0
ω_c
ω
(b) Highpass filter
M
Bandpass
M_0
Idealized
$\frac{1}{\sqrt{2}}M_0$
Actual
Passband
0
0
ω_{c_1}
ω_0
ω_{c_2}
ω
(c) Bandpass filter
M
Bandreject
M_0
Idealized
$\frac{1}{\sqrt{2}}M_0$
Actual
Stopband
Passband
Passband
0
0
ω_{c_1}
ω_0
ω_{c_2}
ω
(d) Bandreject filter

RC Low-pass filter



(a) RC circuit



Lowpass Filter

Application of voltage division gives

$$\mathbf{V}_C = \frac{\mathbf{V}_s \mathbf{Z}_C}{R + \mathbf{Z}_C} = \frac{\mathbf{V}_s / j\omega C}{R + \frac{1}{j\omega C}}.$$

The transfer function corresponding to $\mathbf{V}_C$ is

$$\mathbf{H}_C(\omega) = \frac{\mathbf{V}_C}{\mathbf{V}_s} = \frac{1}{1 + j\omega RC},$$

Corner Frequency ω_c

The corner frequency ω_c is defined as the angular frequency at which $M(\omega)$ is equal to $1/\sqrt{2}$ of the reference peak value,

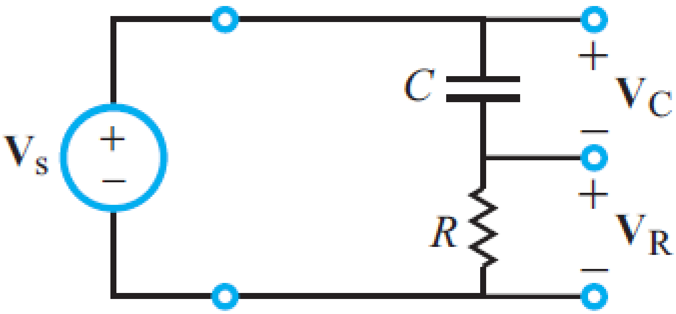
$$M(\omega_c) = \frac{M_0}{\sqrt{2}} = 0.707 M_0. \tag{9.5}$$

$$M_C^2(\omega_c) = \frac{1}{1 + \omega_c^2 R^2 C^2} = \frac{1}{2},$$

leads to

$$\omega_c = \frac{1}{RC}.$$

RC High-pass filter



(a) RC circuit

The output across R in Fig. 9-5(a) leads to

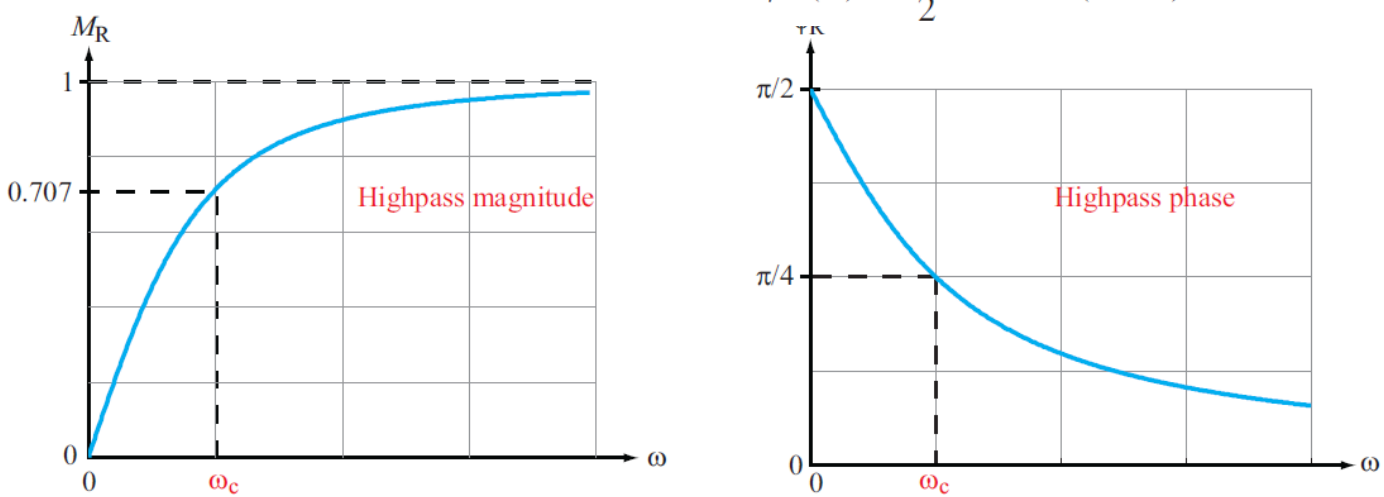
$$\mathbf{H}_R(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{j\omega RC}{1 + j\omega RC}.$$

The magnitude and phase angle of $\mathbf{H}_R(\omega)$ are given by

$$M_R(\omega) = |\mathbf{H}_R(\omega)| = \frac{\omega RC}{\sqrt{1 + \omega^2 R^2 C^2}}$$

and

$$\phi_R(\omega) = \frac{\pi}{2} - \tan^{-1}(\omega RC).$$

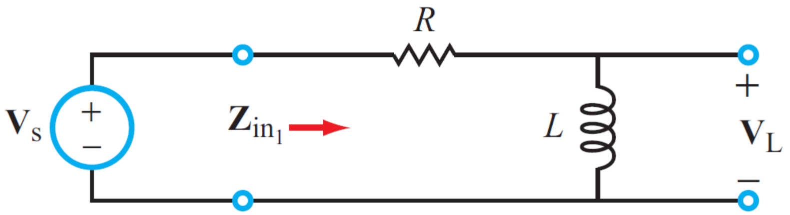


(c) Magnitude and phase angle of $\mathbf{H}_R(\omega) = \mathbf{V}_R / \mathbf{V}_s$

Resonance

Resonant Frequency ω_0

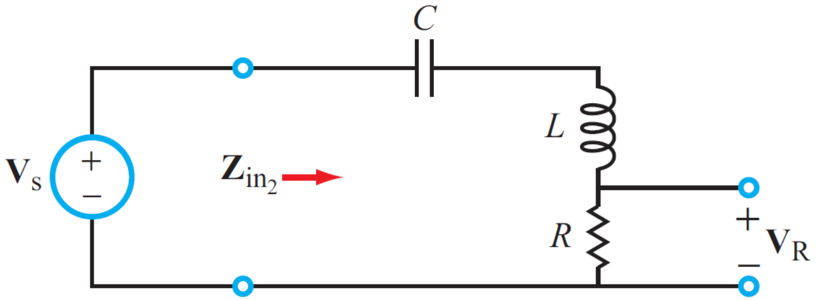
Resonance is a condition that occurs when the input impedance or input admittance of a circuit containing reactive elements is purely real, and the angular frequency at which it occurs is called the ***resonant frequency*** ω_0. Often (but not always) the



(a) First-order RL filter

$\mathbf{Z}_{in1} = R + j\omega L.$

$\text{Im}\,[\mathbf{Z}_{in1}] = 0$ when $\omega = 0$



(b) Series RLC circuit

$\text{Im}\,[\mathbf{Z}_{in2}] = 0$ requires that $\mathbf{Z}_L = -\mathbf{Z}_C$
or, equivalently, $\omega_2 = 1/LC$

The dB scale

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

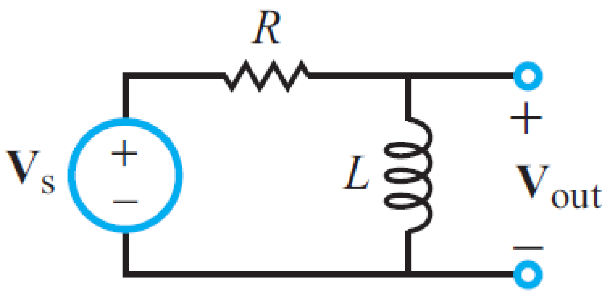
$$G \text{ [dB]} = 10 \log G = 10 \log \left(\frac{P}{P_0} \right) \qquad \text{(dB)}.$$

$$G \text{ [dB]} = 10 \log \left(\frac{\frac{1}{2}|\mathbf{V}|^2 R}{\frac{1}{2}|\mathbf{V}_0|^2 R} \right) = 20 \log \left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|} \right)$$

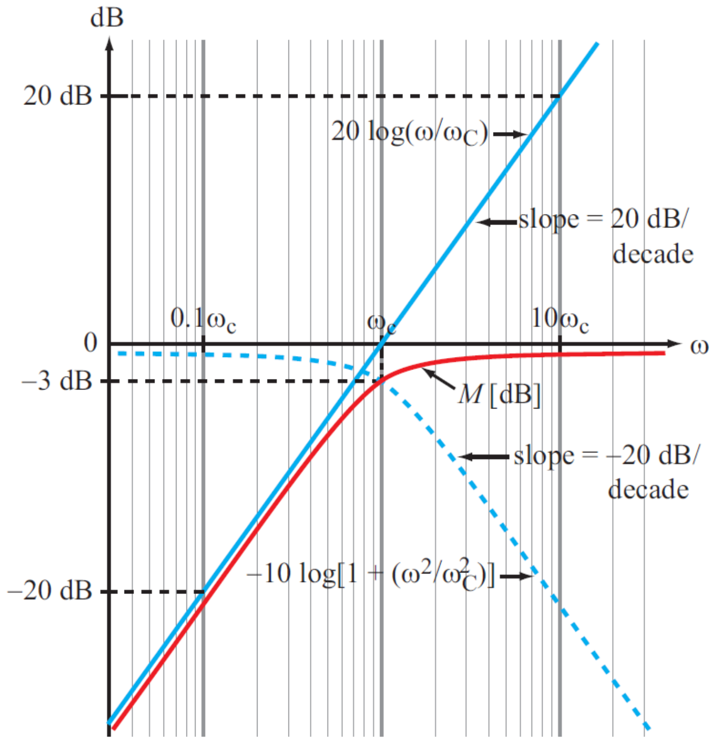
$$G = XY \rightarrow G \text{ [dB]} = X \text{ [dB]} + Y \text{ [dB]}.$$

$$G = \frac{X}{Y} \rightarrow G \text{ [dB]} = X \text{ [dB]} - Y \text{ [dB]}.$$

RL Filter - Magnitude



(a) RL circuit



(b) Magnitude plot

$$\mathbf{V}_{out} = \frac{j\omega L \mathbf{V}_s}{R + j\omega L},$$

which leads to

$$\mathbf{H} = \frac{\mathbf{V}_{out}}{\mathbf{V}_s} = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)}, \qquad (9.33)$$

with $\omega_c = R/L$.

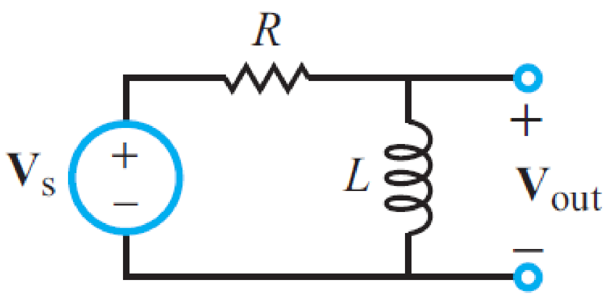
(b) The magnitude of **H** is given by

$$M = |\mathbf{H}| = \frac{(\omega/\omega_c)}{|1 + j(\omega/\omega_c)|} = \frac{(\omega/\omega_c)}{\sqrt{1 + (\omega/\omega_c)^2}}. \qquad (9.34)$$

Since H is a voltage ratio, the appropriate dB scaling factor is 20, so

$$\begin{aligned} M\ [\text{dB}] &= 20 \log M \\ &= 20 \log(\omega/\omega_c) - 20 \log[1 + (\omega/\omega_c)^2]^{1/2} \\ &= 20 \log(\omega/\omega_c) - 10 \log[1 + (\omega/\omega_c)^2]. \qquad (9.35) \end{aligned}$$

RL Filter - Phase

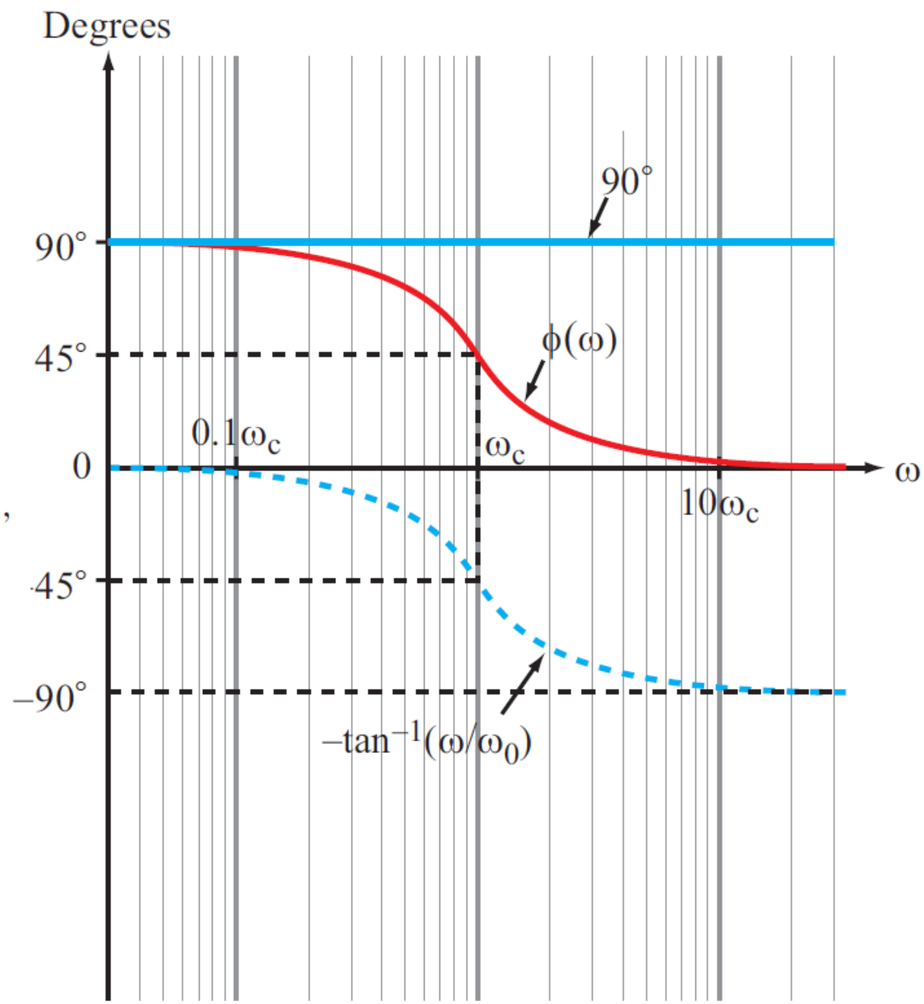


(a) RL circuit

$$\mathbf{H} = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{s}}} = \frac{j\omega L}{R + j\omega L} = \frac{j(\omega/\omega_c)}{1 + j(\omega/\omega_c)},$$

with $\omega_c = R/L$.

$$\phi(\omega) = 90^\circ - \tan^{-1}\left(\frac{\omega}{\omega_c}\right)$$

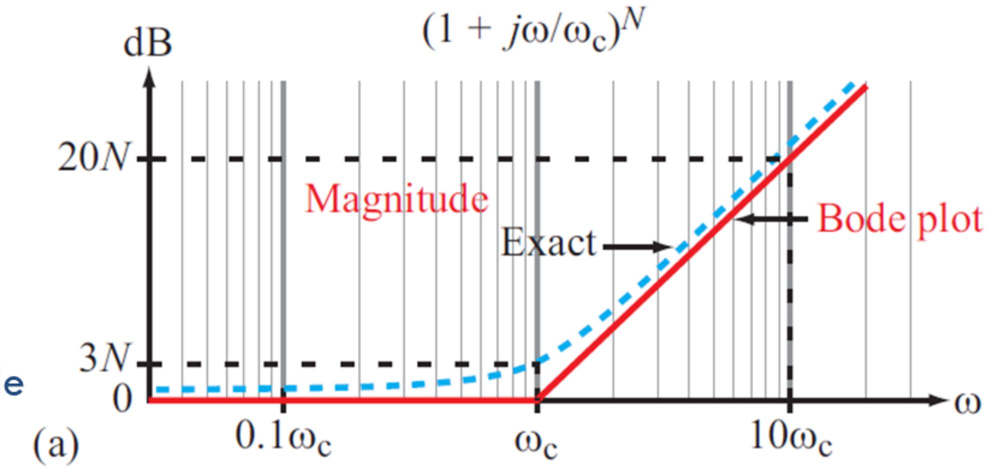


(c) Phase plot

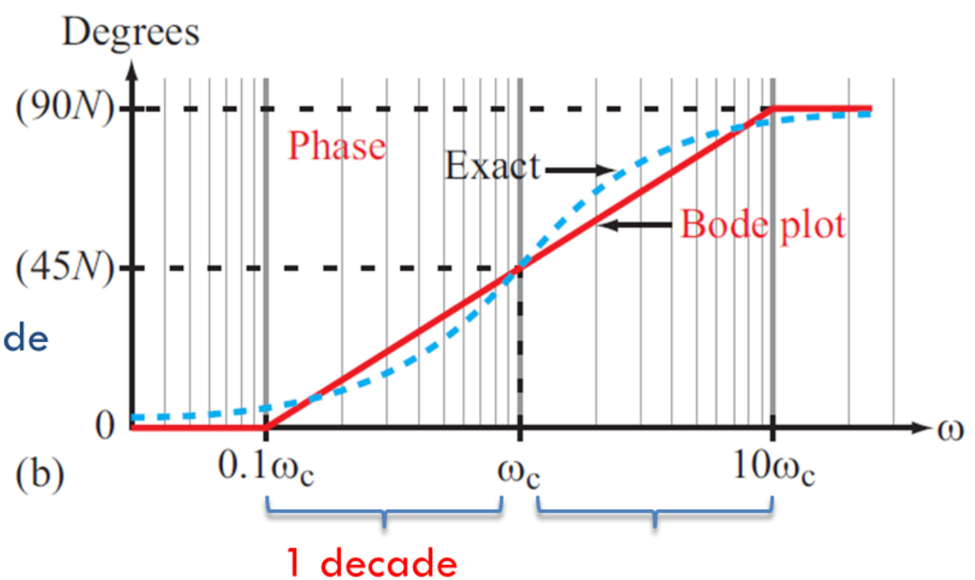
Bode Plots – Straight Line Approximations

Simple zero: $\mathbf{H} = (1 + j\omega/\omega_c)^N$

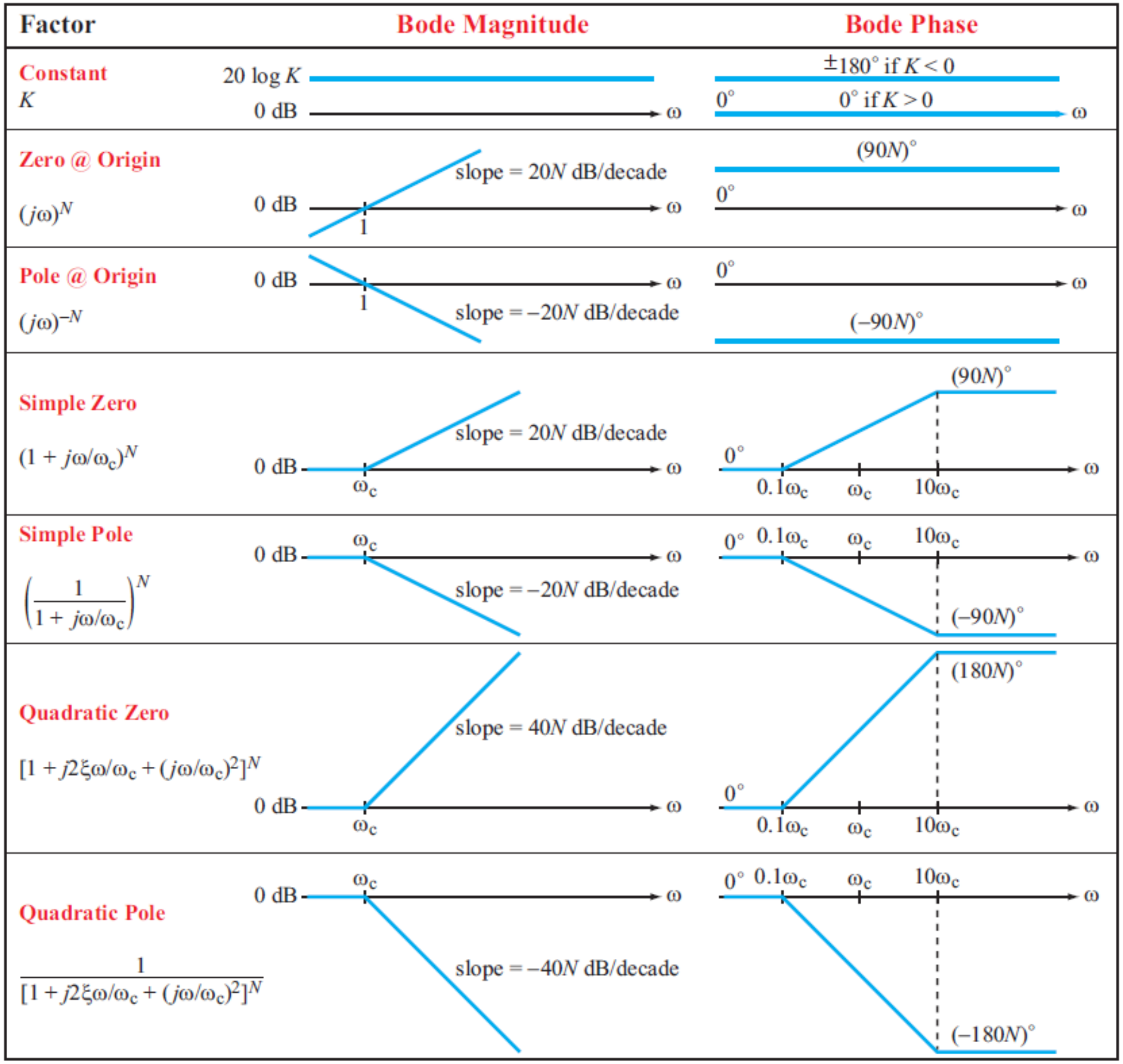
Bode Magnitude Slope= 20N dB per decade



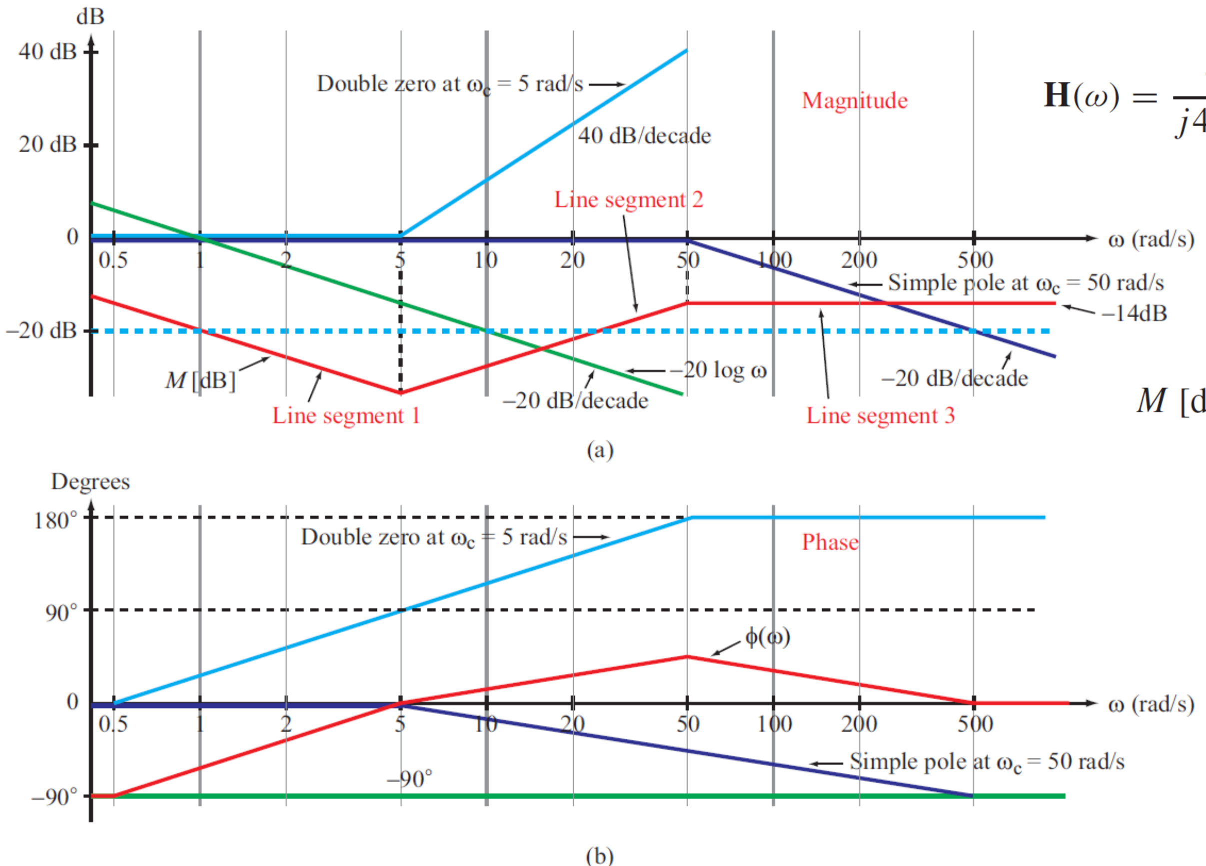
Bode Phase Slope= 45N degrees per decade



Bode Factors



Example: Bode Plots



$$\mathbf{H}(\omega) = \frac{400(1 + j\omega/5)^2}{j4000\omega(1 + j\omega/50)} = \frac{-j0.1(1 + j\omega/5)^2}{\omega(1 + j\omega/50)}$$

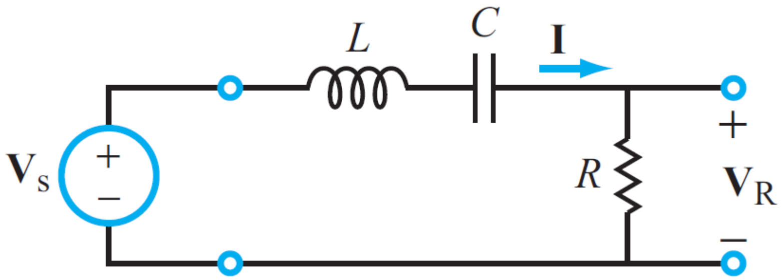
$$\begin{aligned} M\ [\text{dB}] &= 20\log|\mathbf{H}| \\ &= 20\log 0.1 + 40\log|1 + j\omega/5| \\ &\quad - 20\log\omega - 20\log|1 + j\omega/50| \\ &= -20\ \text{dB} + 40\log|1 + j\omega/5| \\ &\quad - 20\log\omega - 20\log|1 + j\omega/50|. \end{aligned}$$

$$\phi = -90^\circ + 2\tan^{-1}\frac{\omega}{5} - \tan^{-1}\frac{\omega}{50}$$

Bandpass RLC Filter

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + j(\omega L - \frac{1}{\omega C})}$$

$$= \frac{j\omega C\mathbf{V}_s}{(1 - \omega^2 LC) + j\omega RC}$$



$$\mathbf{H}_{BP}(\omega) = \frac{\mathbf{V}_R}{\mathbf{V}_s} = \frac{R\mathbf{I}}{\mathbf{V}_s} = \frac{j\omega RC}{(1 - \omega^2 LC) + j\omega RC}$$

$$M_{BP}(\omega) = |\mathbf{H}_{BP}(\omega)| = \frac{\omega RC}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 R^2 C^2}}$$

$$\phi_R(\omega) = 90° - \tan^{-1}\left[\frac{\omega RC}{1 - \omega^2 LC}\right]$$

$$\omega_{c_1} = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}},$$

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

$$\omega_{c_2} = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}.$$

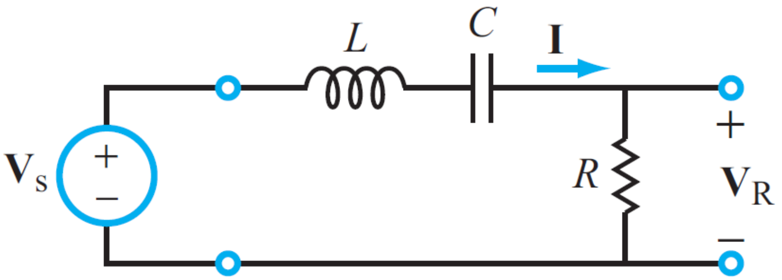
$$B = \omega_{c_2} - \omega_{c_1} = \frac{R}{L}.$$

Filter Q

Quality Factor *Q*: characterizes degree of selectivity of a circuit

$$Q = 2\pi \left(\frac{W_{\text{stor}}}{W_{\text{diss}}} \right) \Bigg|_{\omega=\omega_0},$$

where W_{stor} is *the maximum energy that can be* ***stored*** in the circuit at resonance ($\omega = \omega_0$)*, and* W_{diss} is *the* ***energy dissipated*** by the circuit during a single period *T.*

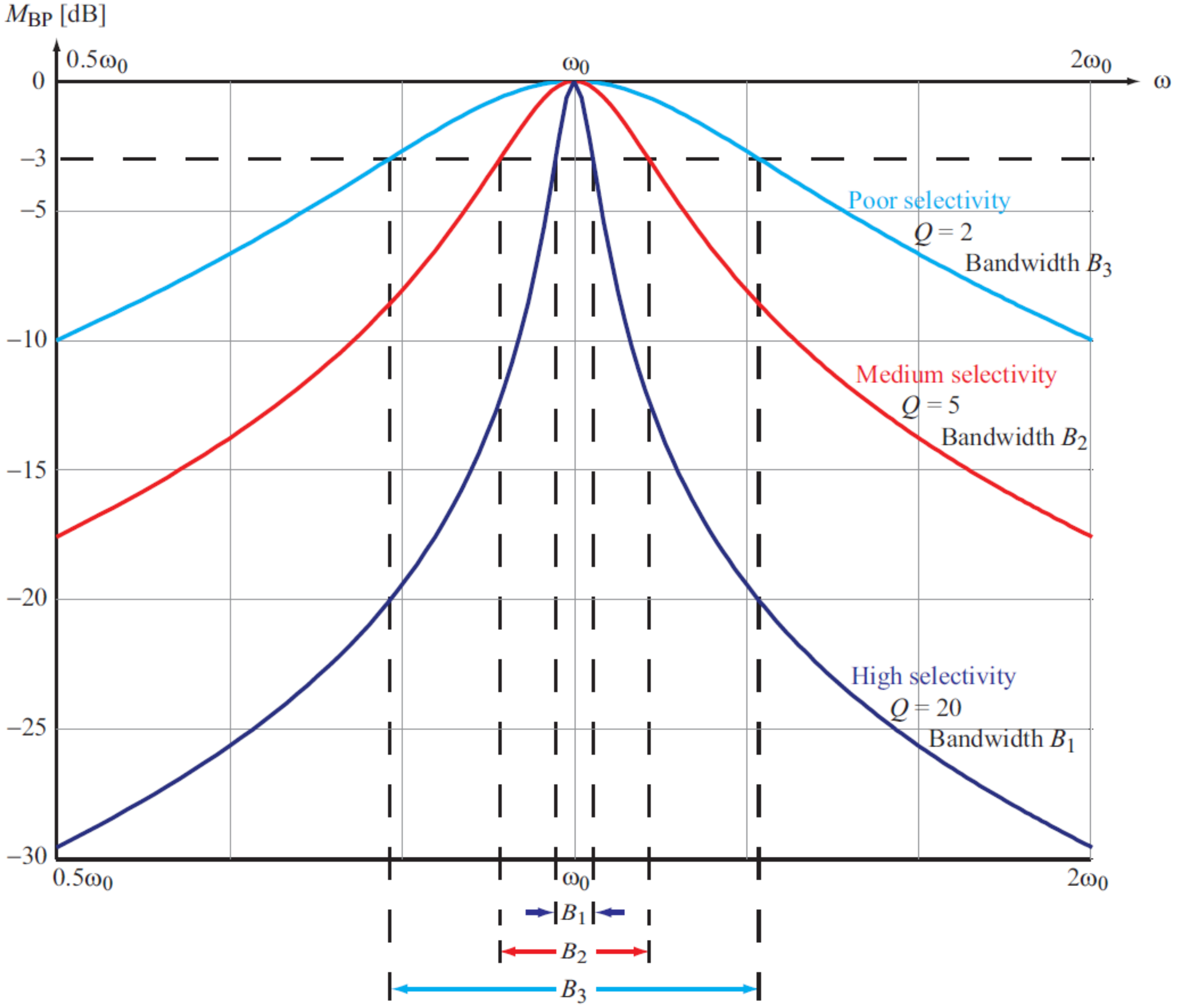


Resonant frequency

$$Q = \frac{\omega_0 L}{R} = \frac{\omega_0}{B},$$

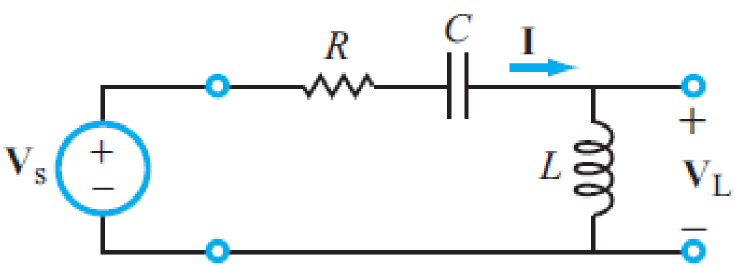
Bandwidth

Effect of Q

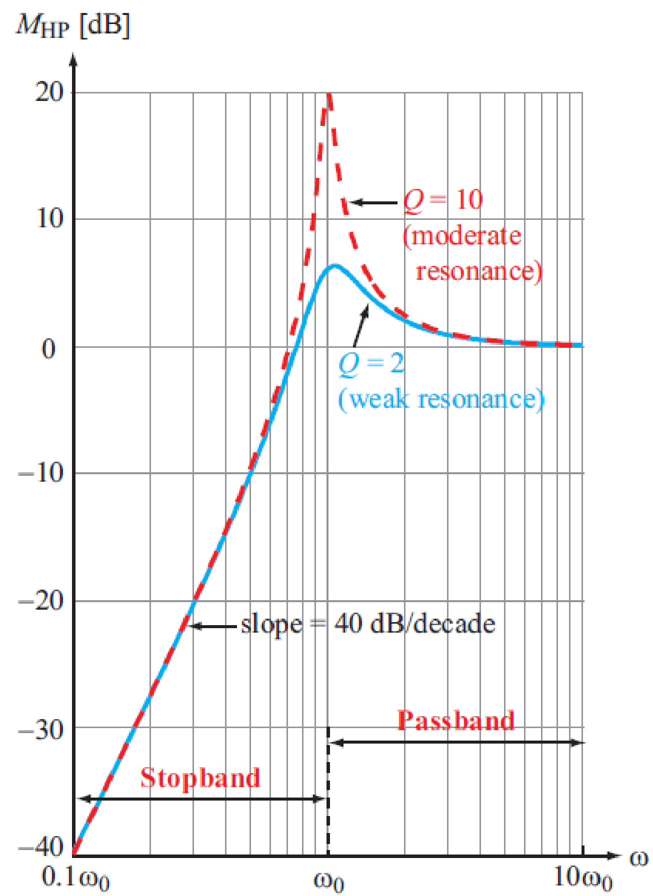


M_{BP} [dB]
$0.5\omega_0$
ω_0
$2\omega_0$
ω
0
–3
–5
–10
–15
–20
–25
–30
Poor selectivity
$Q = 2$
Bandwidth B_3
Medium selectivity
$Q = 5$
Bandwidth B_2
High selectivity
$Q = 20$
Bandwidth B_1
$0.5\omega_0$
ω_0
$2\omega_0$
B_1
B_2
B_3

Highpass Filter

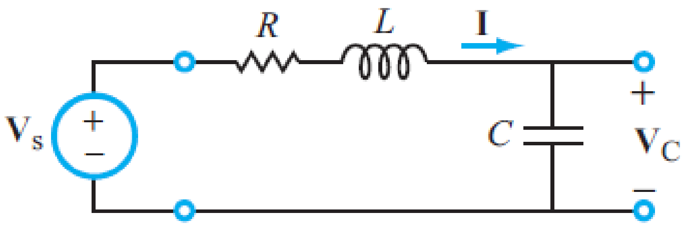


(a) $\mathbf{H}_{HP} = \mathbf{V}_L / \mathbf{V}_s$

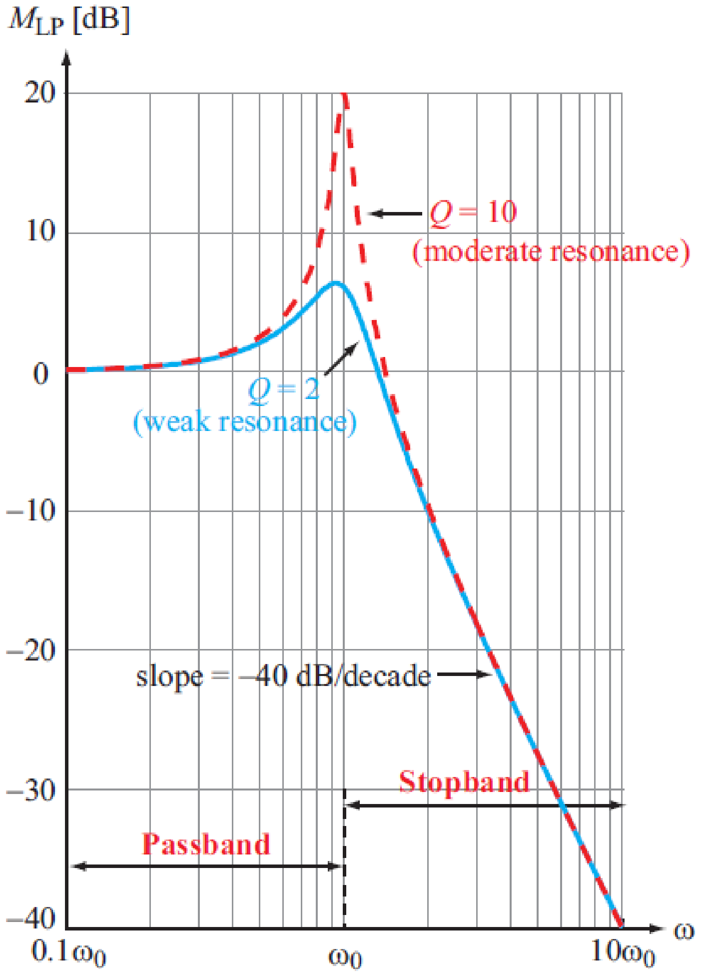


(b) Magnitude spectrum

Lowpass Filter

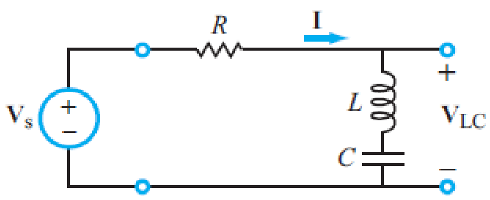


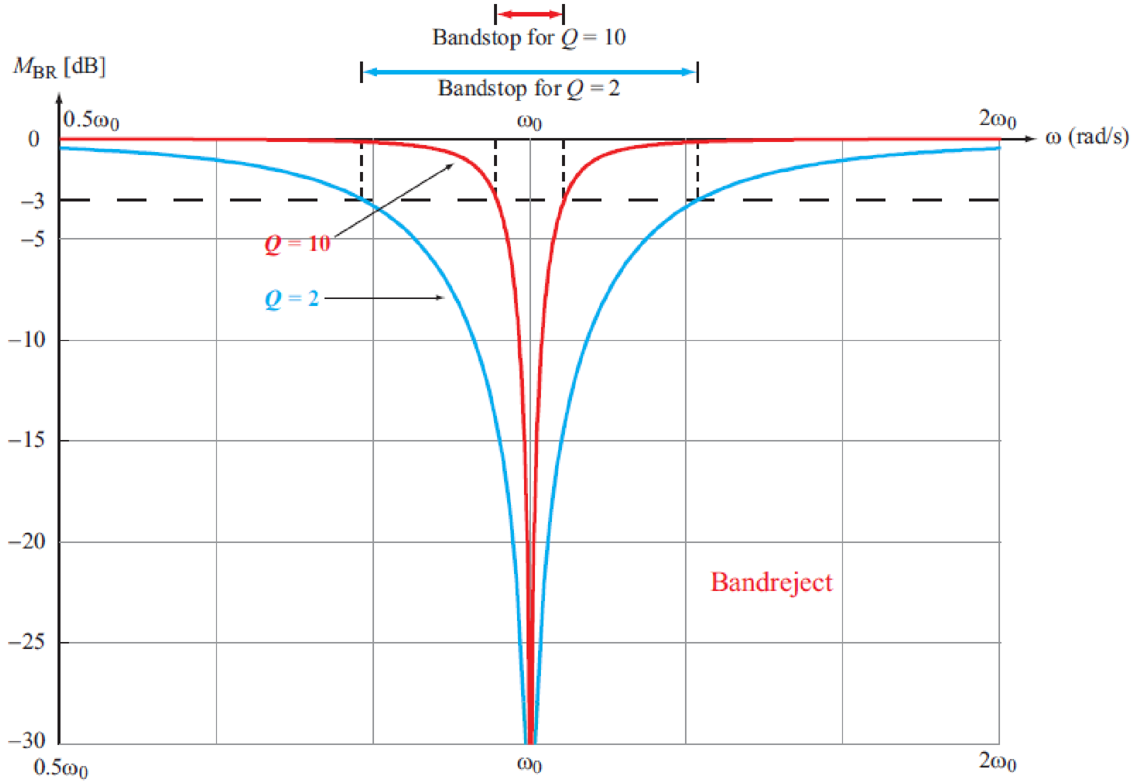
(a) $\mathbf{H}_{LP} = \mathbf{V}_C / \mathbf{V}_s$



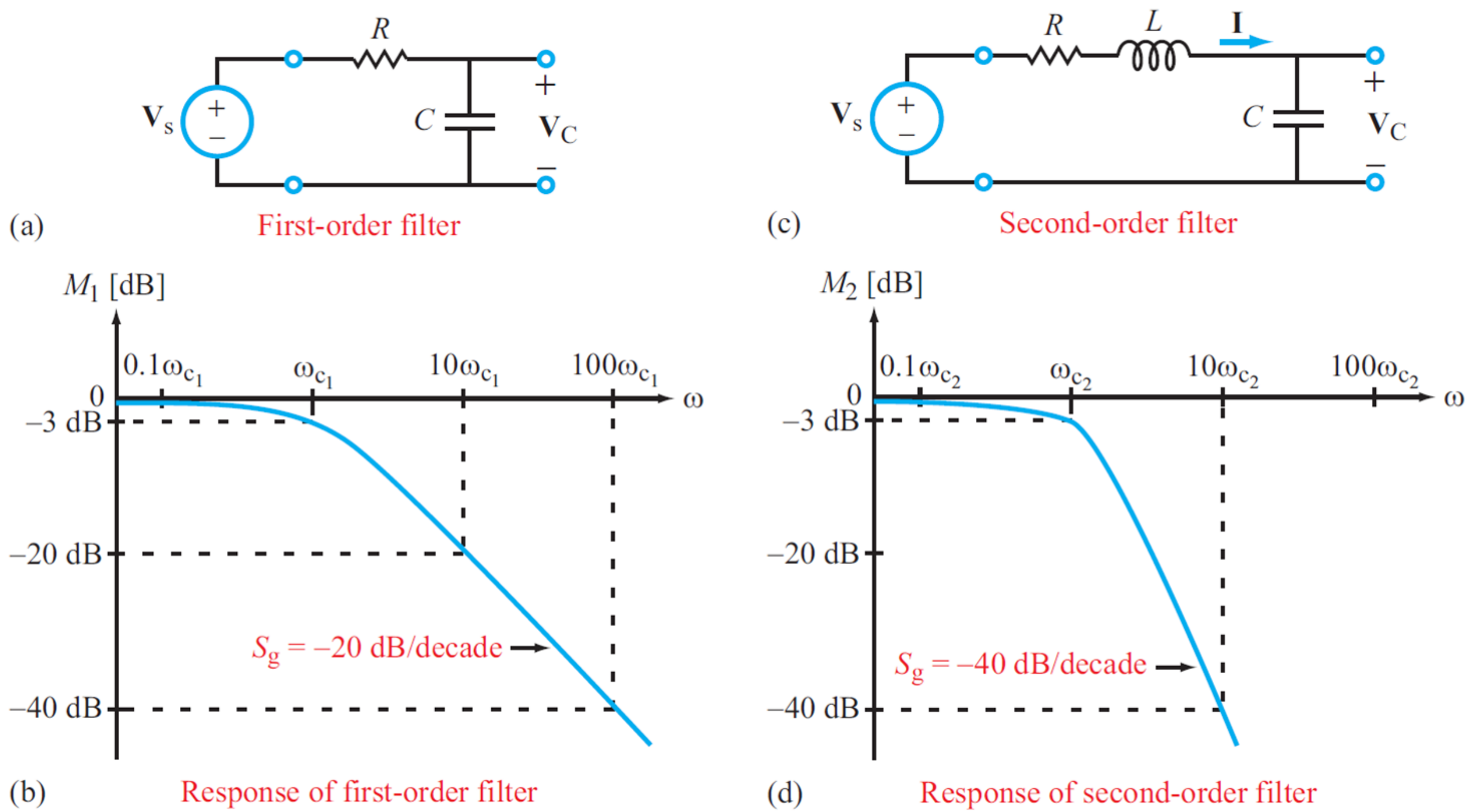
(b) Magnitude spectrum

Bandreject Filter





Filter Order



Active Filter - Lowpass

$$\mathbf{V}_{\text{out}} = -\frac{\mathbf{Z}_{\text{f}}}{\mathbf{Z}_{\text{s}}}\,\mathbf{V}_{\text{s}}$$

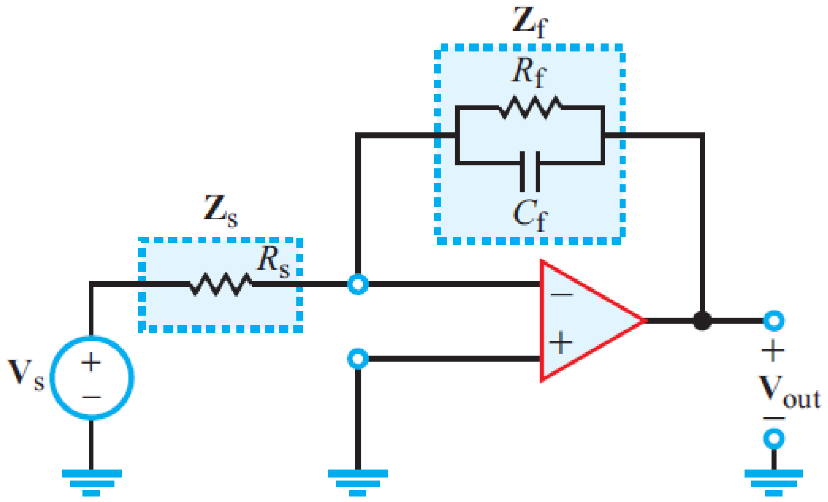
$$\mathbf{Z}_{\text{f}} = R_{\text{f}} \parallel \left(\frac{1}{j\omega C_{\text{f}}}\right) = \frac{R_{\text{f}}}{1 + j\omega R_{\text{f}} C_{\text{f}}}. \qquad (9.87\text{b})$$

The transfer function of the circuit, which we soon will recognize as that of a lowpass filter, is given by

$$\mathbf{H}_{\text{LP}}(\omega) = \frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{s}}} = -\frac{\mathbf{Z}_{\text{f}}}{\mathbf{Z}_{\text{s}}} = -\frac{R_{\text{f}}}{R_{\text{s}}}\left(\frac{1}{1 + j\omega R_{\text{f}} C_{\text{f}}}\right) = G_{\text{LP}}\left(\frac{1}{1 + j\omega/\omega_{\text{LP}}}\right),$$

where

$$\boxed{G_{\text{LP}} = -\frac{R_{\text{f}}}{R_{\text{s}}}, \qquad \omega_{\text{LP}} = \frac{1}{R_{\text{f}} C_{\text{f}}}.}$$



Active Filter - Highpass

$$\mathbf{Z}_s = R_s - \frac{j}{\omega C_s} \qquad \text{and} \qquad \mathbf{Z}_f = R_f, \tag{9.90}$$

as shown in Fig. 9-24, we would obtain the highpass-filter transfer function given by

$$\mathbf{H}_{HP}(\omega) = \frac{\mathbf{V}_{out}}{\mathbf{V}_s} = -\frac{\mathbf{Z}_f}{\mathbf{Z}_s} = -\frac{R_f}{R_s - j/\omega C_s} = G_{HP}\left[\frac{j\omega/\omega_{HP}}{1 + j\omega/\omega_{HP}}\right], \tag{9.91}$$

$$\boxed{G_{HP} = -\frac{R_f}{R_s} \qquad \text{and} \qquad \omega_{HP} = \frac{1}{R_s C_s}.}$$

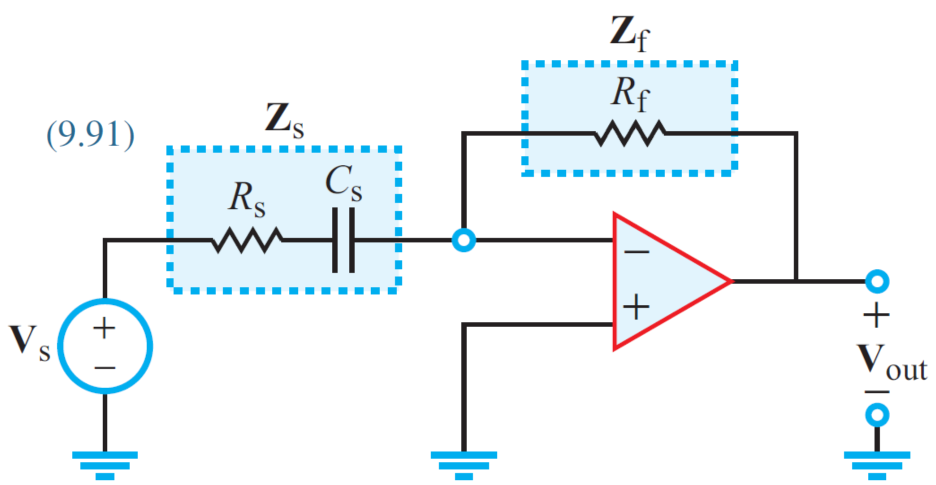
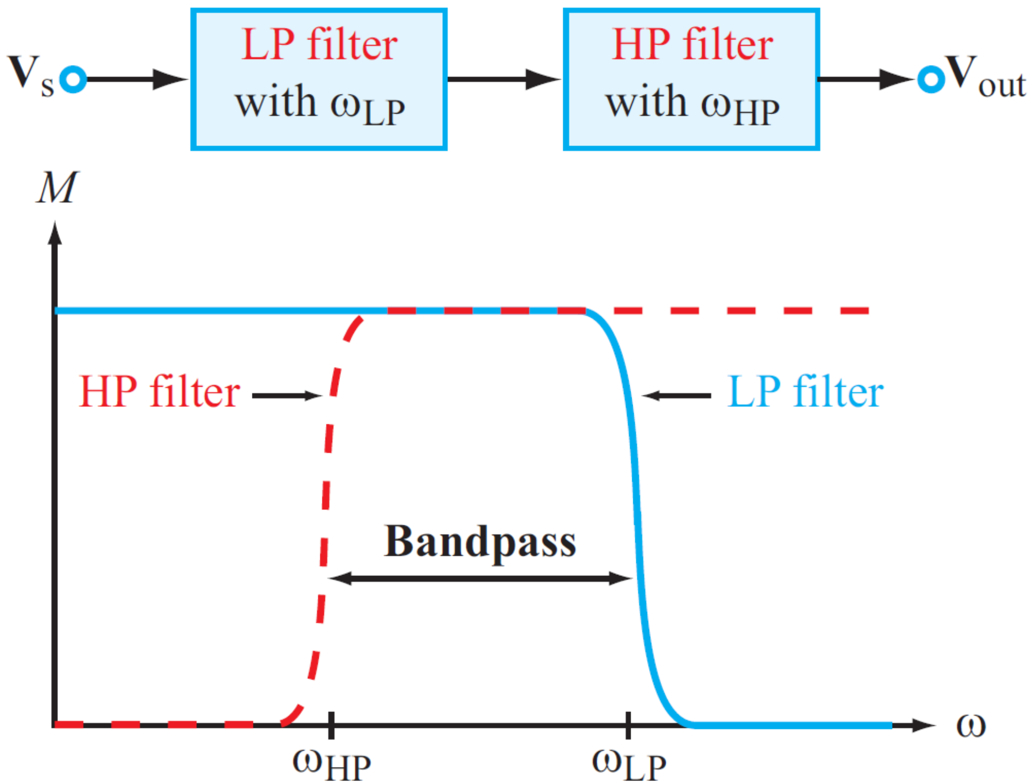


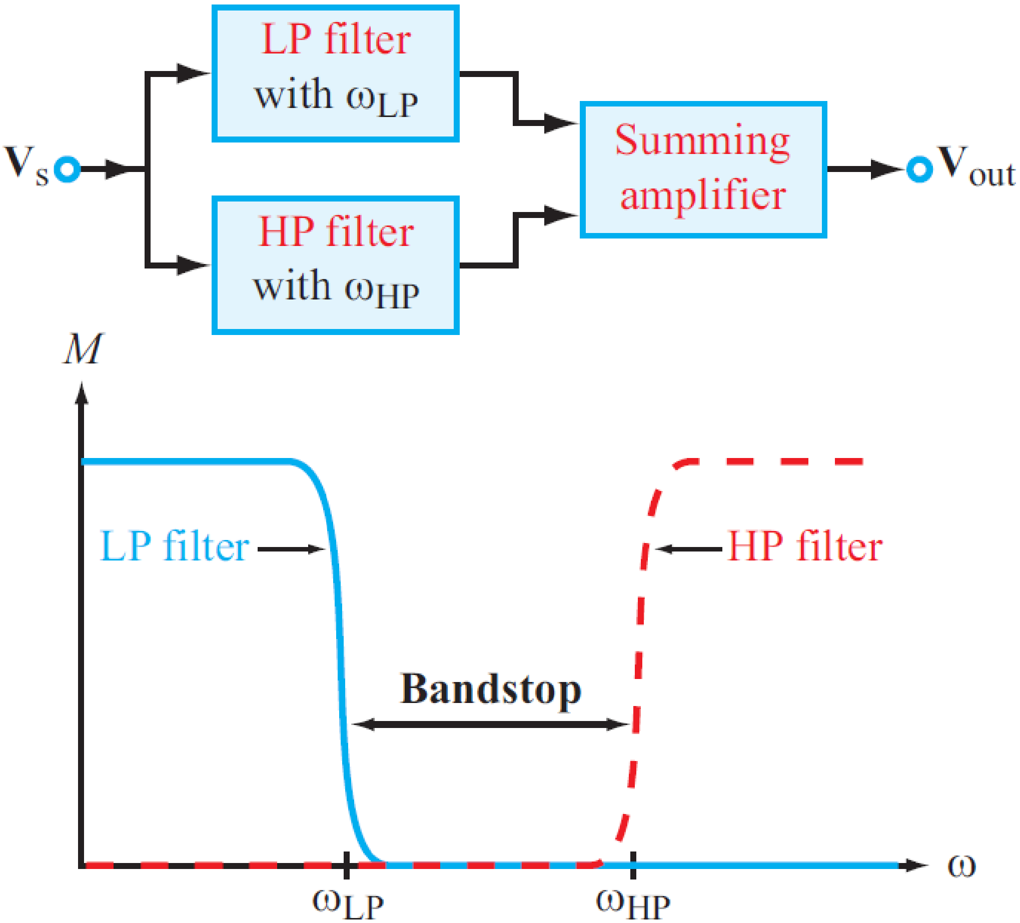
Figure 9-24: Single-pole active highpass filter.

Cascading Filters



(a) Bandpass filter

Cascading Filters



(b) Bandreject filter

Example: 3rd order lowpass filter