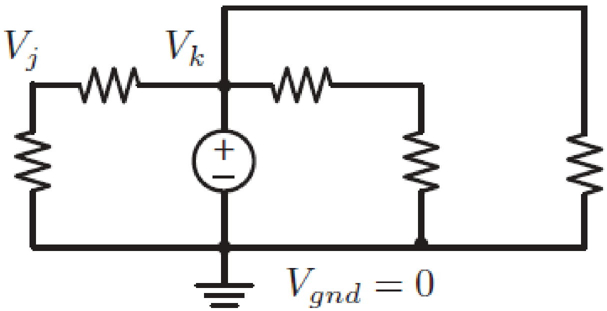
40 – Circuit Analysis Techniques

Reading Material:

Chapter 3

Specifying the Reference Node

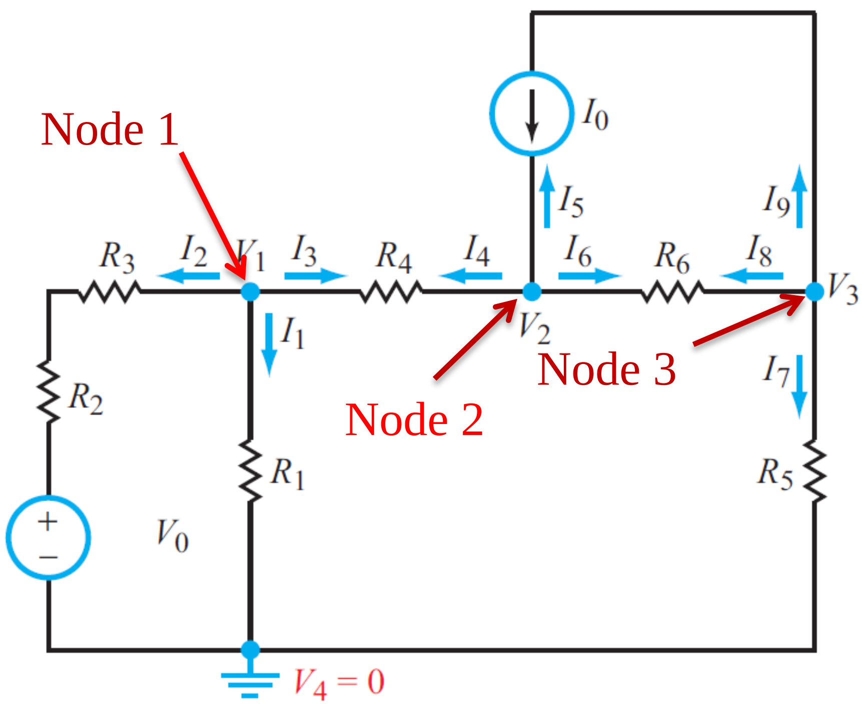


- Recall that voltage is defined as a quantity that measures the potential difference between two nodes in a circuit, V_{AB}.
- We can arbitrarily pick one node of the circuit and define all *node voltages* in reference to this node. Call this node *ground*, or node '0'. In other words, define V_k as the node voltage at node k which is the energy gained per unit charge as it moves from node gnd to node k, or in more cumbersome notation, $V_{k,gnd}$.
- If we subtract the two node voltages, we get

$$V_{j,gnd} - V_{k,gnd} = V_j - V_{gnd} - (V_k - V_{gnd}) = V_j - V_k = V_{j,k}$$

- In other words $V_{j,k} = V_{j,gnd} - V_{k,gnd}$, which makes sense since they are defined with respect to the same reference. Note that the reference potential is by definition at zero potential, $V_{gnd} = 0$.

Node-Voltage Method



Solution Procedure: Node Voltage

Step 1: Identify all extraordinary nodes, select one of them as a reference node (ground), and then assign node voltages to the remaining $(n_{ex} - 1)$ extraordinary nodes.

Step 2: At each of the $(n_{ex} - 1)$ extraordinary nodes, apply the form of KCL requiring the sum of all currents *leaving* a node to be zero.

Step 3: Solve the $(n_{ex} - 1)$ independent simultaneous equations to determine the unknown node voltages.

Node 1:

$$I_1 + I_2 + I_3 = 0.$$

$$\frac{V_1}{R_1} + \frac{V_1 - V_0}{R_2 + R_3} + \frac{V_1 - V_2}{R_4} = 0 \qquad \text{(node 1).}$$

Node 2

$$\frac{V_2 - V_1}{R_4} - I_0 + \frac{V_2 - V_3}{R_6} = 0 \qquad \text{(node 2),}$$

Node 3

$$\frac{V_3}{R_5} + \frac{V_3 - V_2}{R_6} + I_0 = 0 \qquad \text{(node 3).}$$

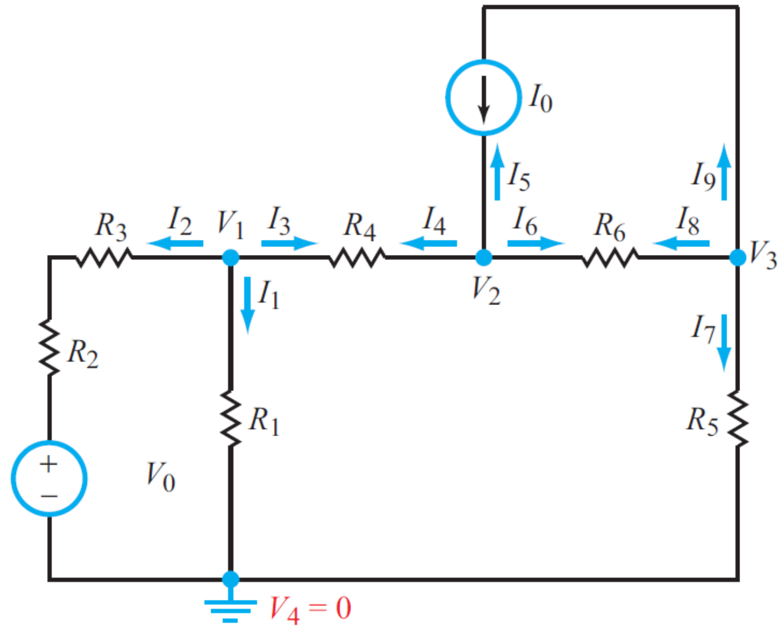
Node-Voltage Method

$$\left(\frac{1}{R_1} + \frac{1}{R_2 + R_3} + \frac{1}{R_4}\right) V_1 - \left(\frac{1}{R_4}\right) V_2 = \frac{V_0}{R_2 + R_3}, \tag{3.8a}$$

$$-\left(\frac{1}{R_4}\right) V_1 + \left(\frac{1}{R_4} + \frac{1}{R_6}\right) V_2 - \frac{V_3}{R_6} = I_0, \tag{3.8b}$$

and

$$-\left(\frac{1}{R_6}\right) V_2 + \left(\frac{1}{R_5} + \frac{1}{R_6}\right) V_3 = -I_0. \tag{3.8c}$$



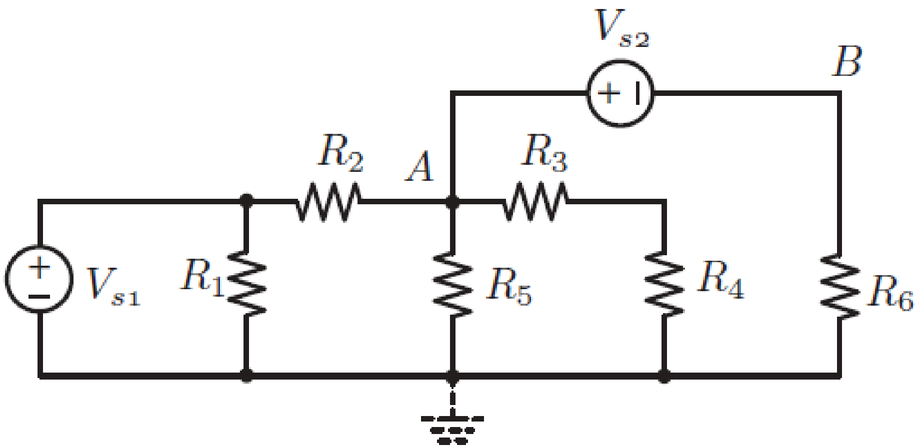
Three equations in 3 unknowns:
Solve using matrix math or MATLAB

$$a_{11} V_1 + a_{12} V_2 + a_{13} V_3 = b_1$$

$$a_{21} V_1 + a_{22} V_2 + a_{23} V_3 = b_2$$

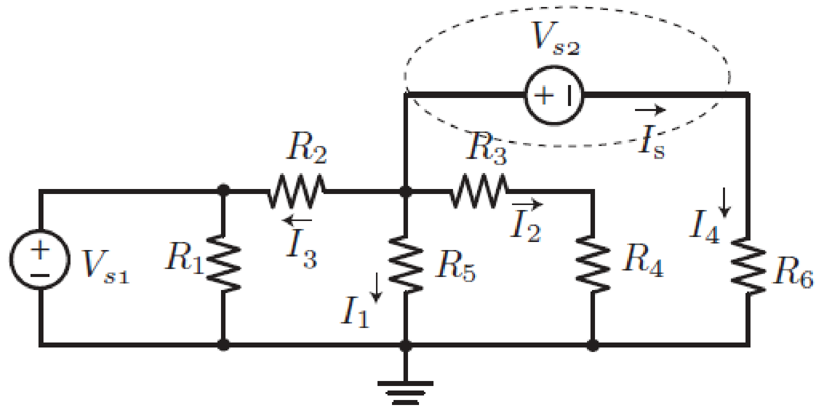
$$a_{31} V_1 + a_{32} V_2 + a_{33} V_3 = b_3$$

Dealing with floating sources



- In the above example, two voltage sources appear so that we cannot label both negative terminals as ground. So we pick one.
- In this circuit, the number of unknowns is smaller because if we known the voltage at one node connected to the voltage source, the voltage at the other has a fixed relation, since $V_{AB} = V_s$.
- But in writing down KCL at one of the nodes, we encounter a problem. The current through the voltage source can take on any value, which means that other circuit elements determine the current through it.

Dealing with floating sources



- But note that if we write the KCL equations for both terminals of the voltage source, this unknown current appears twice.

At the positive terminal: $I_1 + I_2 + I_3 + I_s = 0$

At the negative terminal: $I_4 - I_s = 0$

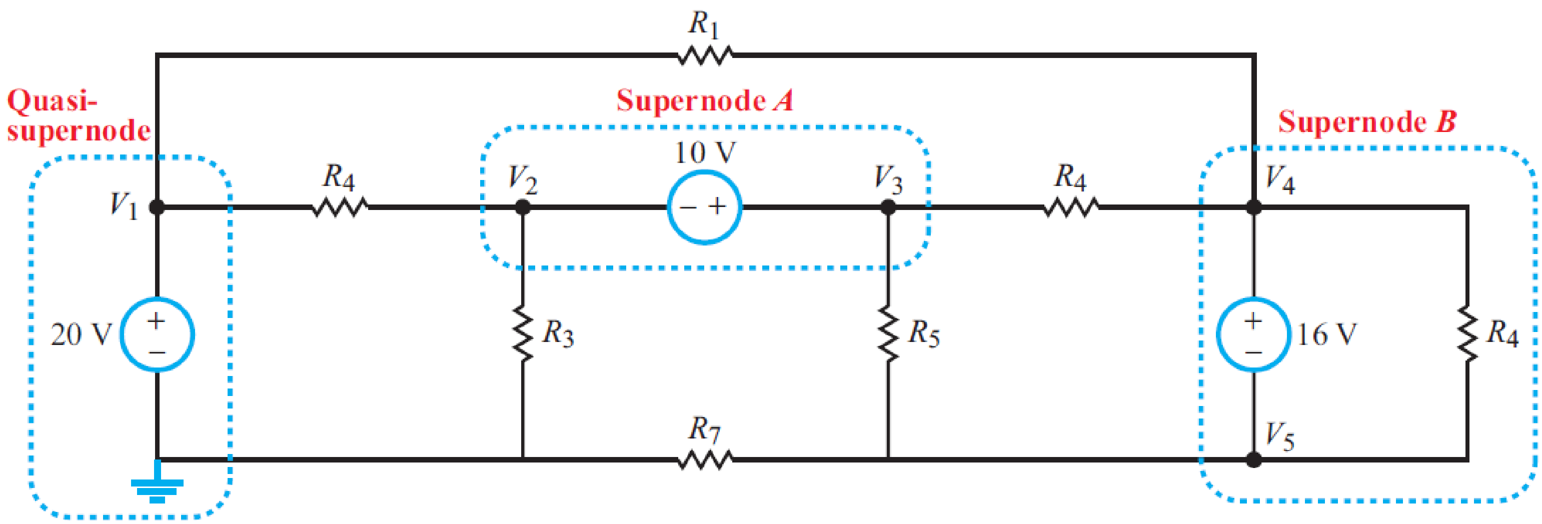
- If we add these two equations, I_s cancels out

$$I_1 + I_2 + I_3 + I_4 = 0$$

- This is an independent equation we can use. It's actually KCL for a "super node", which is what we call nodes a and b together. We have just shown that current continuity applies to a node and also to a super node.

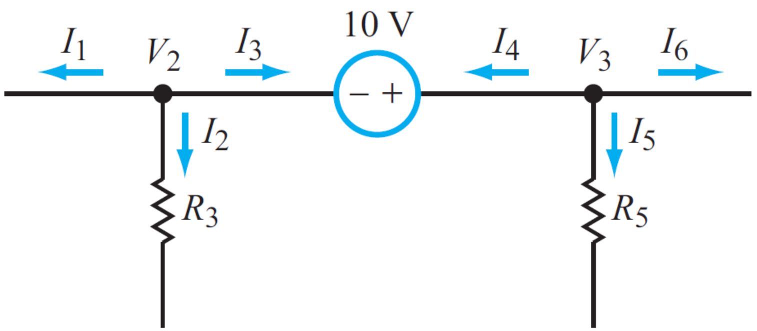
Supernodes

A supernode is formed when a voltage source connects two extraordinary nodes



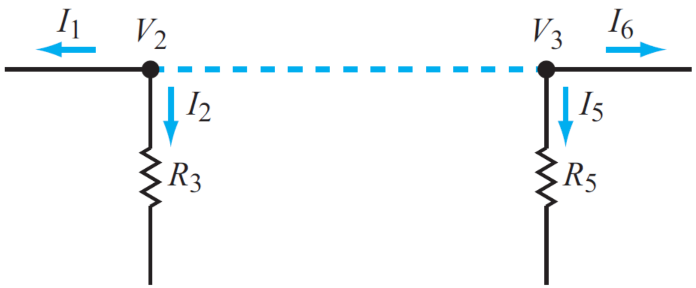
- Current through voltage source is unknown
- Less nodes to worry about, less work!
- Write KVL equation for supernode
- Write KCL equation for closed surface around supernode

KCL at a Supernode



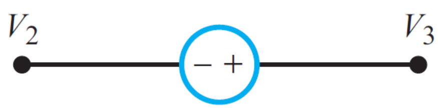
Supernode *A*

=



$I_1 + I_2 + I_5 + I_6 = 0$ (KCL)

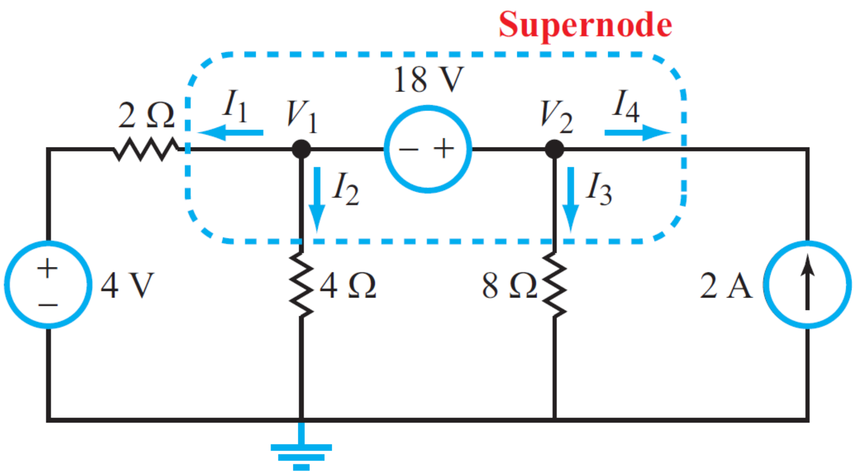
+



$V_3 - V_2 = 10$ V, (Auxiliary Eq.)

- Note that “internal” current in supernode cancels, simplifying KCL expressions
- Takes care of unknown current in a voltage source

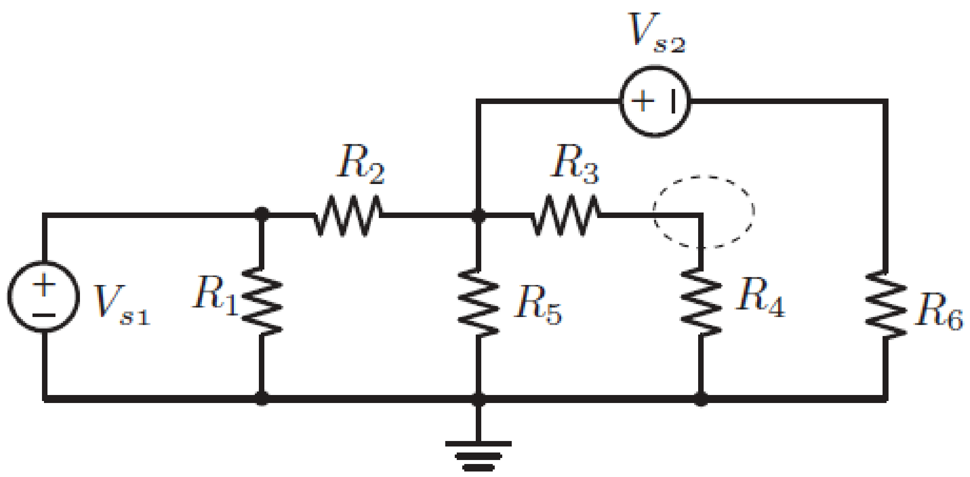
Example: Supernode



Solution:

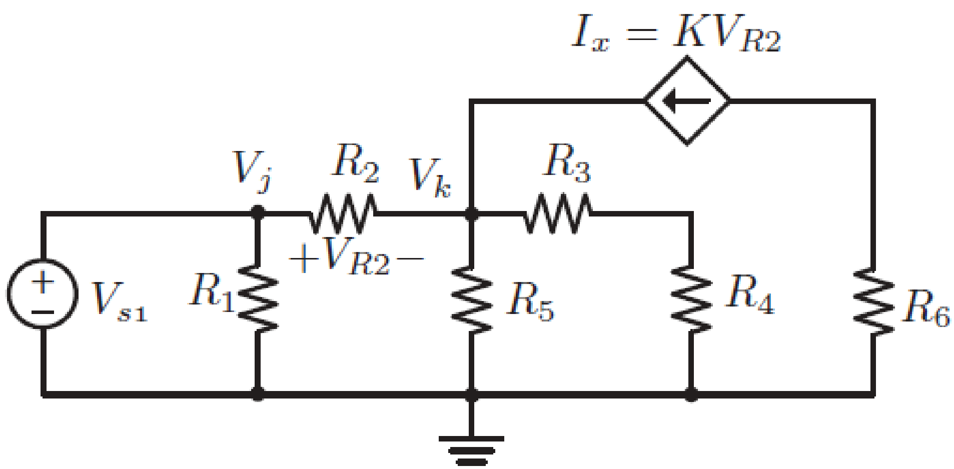
Determine: V1 and V2

Trivial Nodes



- Any node with less that three elements is in some sense trivial. That's because we can find the node voltage for such a node from the branch current. Thus, it's smart to avoid writing equations for these nodes and to deal with them later.
- In the above example the number of unknowns has been reduced from 2 to 1 by using this technique

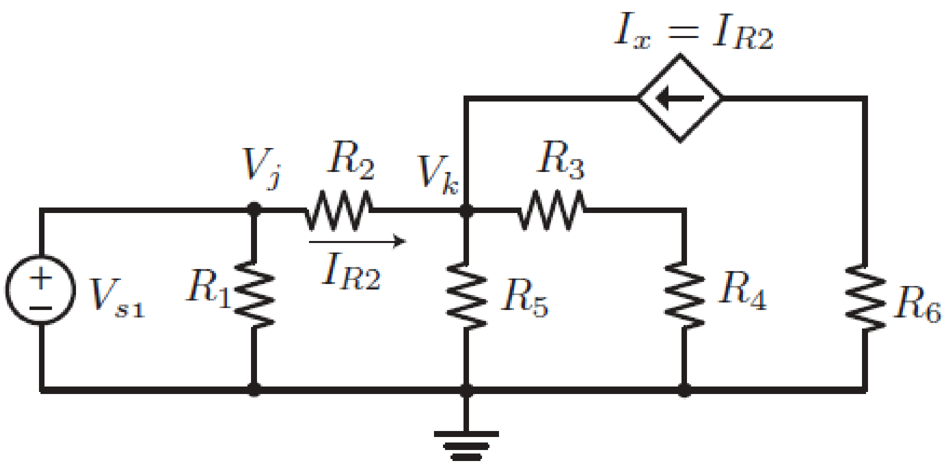
Dealing with Dependent Sources



- Dependent sources require a bit more work but are generally dealt with in the same way. In the above circuit we initially treat the current through the dependent source as an unknown in writing KCL.
- There are now more unknowns than equations. For each additional unknown (dependent current), we can write an additional equation which relates the dependent current to the node voltages. For instance, for a VCCS, this is trivial

$$I_x = KV_{R2} = K(V_j - V_k)$$

Dealing with Dependent Sources

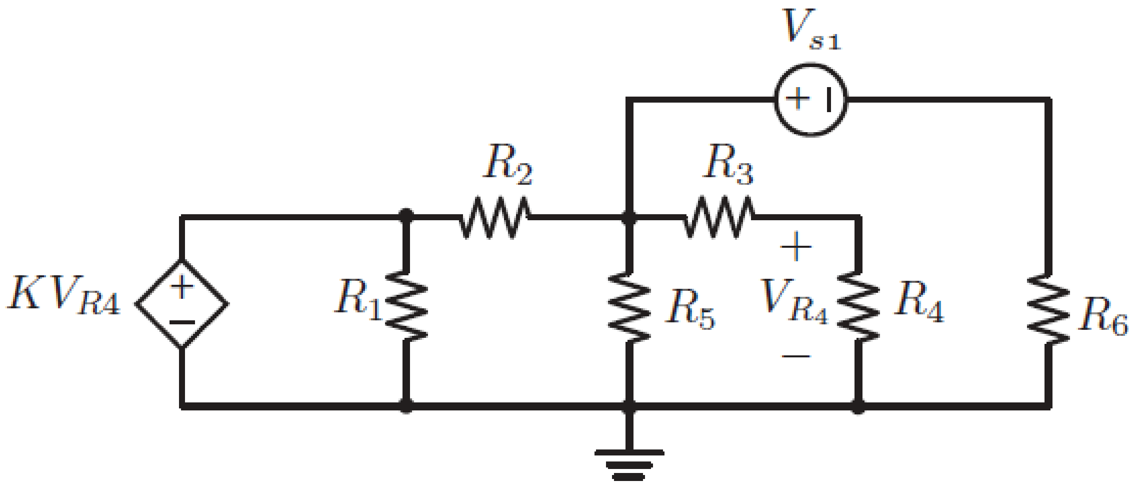


- For a CCCS, we simply need to calculate the current

$$I_x = JI_{R2} = J(V_j - V_k)/R_2$$

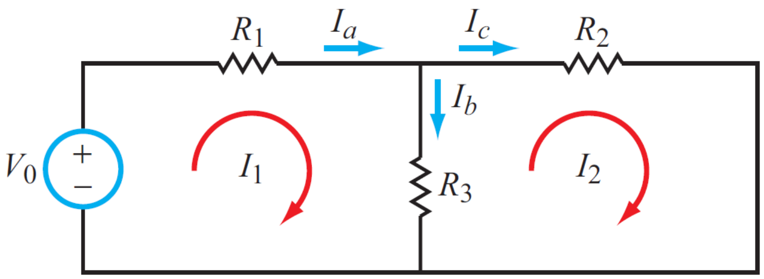
- Instead of making life more complicated by introducing new equations, we should actually just eliminate the unknown currents from our original set of equations. In other words, substitute the above equations for I_x and I_y, in our original $n-1$ node equations.

Dealing with Dependent Sources



- Nodes with dependent voltage sources are handled in the same way. They can also form supernodes and a KCL equation can be written for the supernode, just like independent sources. The only difference is that the potential difference between the floating dependent voltage source is not known until all the equations are solved.

Mesh-Current Method



Solution Procedure: Mesh Current

Step 1: Identify all meshes and assign each of them an unknown mesh current. For convenience, define the mesh currents to be clockwise in direction.

Step 2: Apply kirchhoff's voltage law (KVL) to each mesh.

Step 3: Solve the resultant simultaneous equations to determine the mesh currents.

$$-V_0 + I_1 R_1 + (I_1 - I_2) R_3 = 0 \qquad \text{(mesh 1)}$$

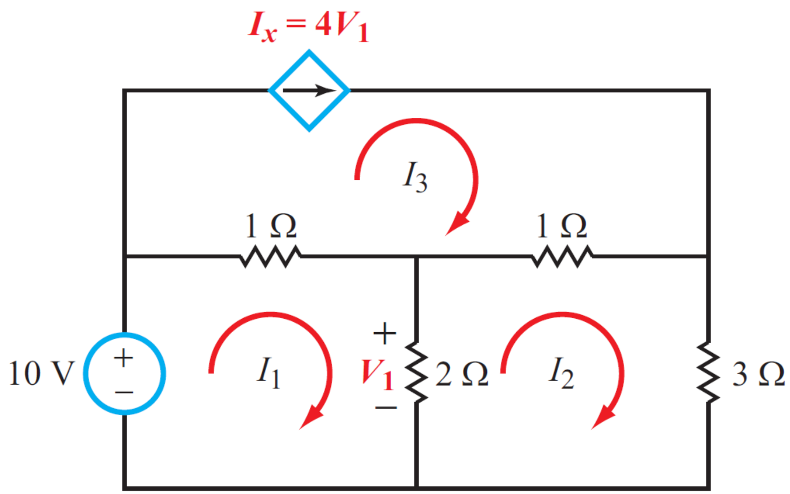
$$(I_2 - I_1) R_3 + I_2 R_2 = 0 \qquad \text{(mesh 2)}$$

Two equations in 2 unknowns:
Solve using matrix math or MATLAB

$$(R_1 + R_3) I_1 - I_2 R_3 = V_0 \qquad \text{(mesh 1)}$$

$$-R_3 I_1 + (R_2 + R_3) I_2 = 0 \qquad \text{(mesh 2)}$$

Example: Mesh Analysis



Solution Procedure: Mesh Current

Step 1: Identify all meshes and assign each of them an unknown mesh current. For convenience, define the mesh currents to be clockwise in direction.

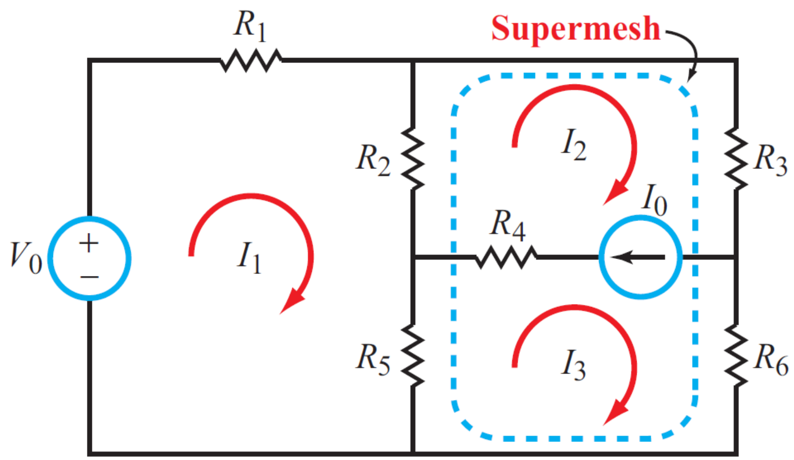
Step 2: Apply kirchhoff's voltage law (KVL) to each mesh.

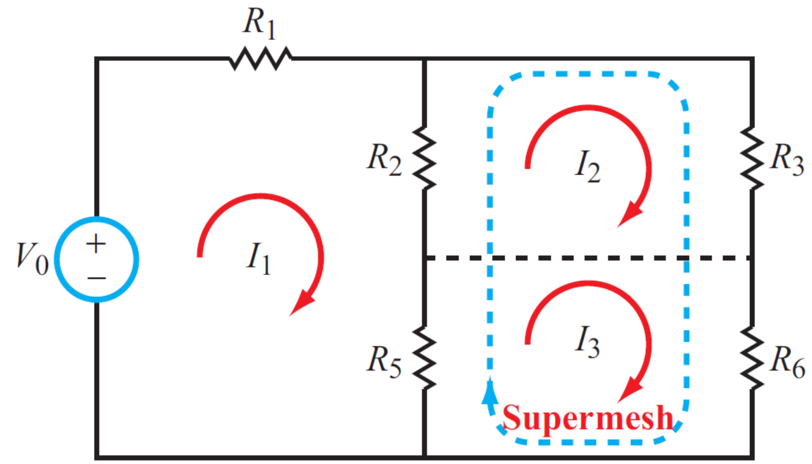
Step 3: Solve the resultant simultaneous equations to determine the mesh currents.

But

Supermesh

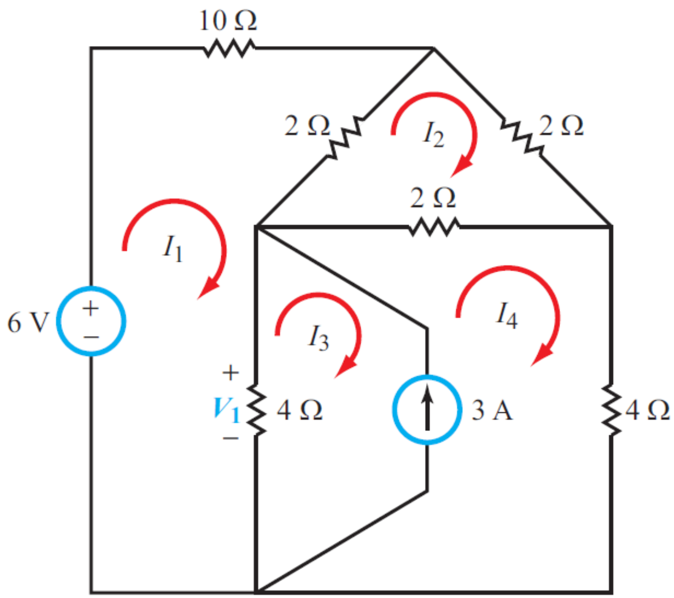
A supermesh results when two meshes have a current source(with or w/o a series resistor) in common





- Voltage across current source is unknown
- Write KVL equation for closed loop that ignores branch with current source
- Write KCL equation for branch with current source (auxiliary equation)

Example: Supermesh



(a) Original circuit

Nodal versus Mesh

When do you use one vs. the other?
What are the strengths of nodal versus mesh?

- Nodal Analysis
  - Node Voltages (voltage difference between each node and ground reference) are *UNKNOWNS*
  - KCL Equations at Each *UNKNOWN* Node Constrain Solutions (*N* KCL equations for *N* Node Voltages)
- Mesh Analysis
  - “Mesh Currents” Flowing in Each Mesh Loop are *UNKNOWNS*
  - KVL Equations for Each Mesh Loop Constrain Solutions (*M* KVL equations for *M* Mesh Loops)

Count nodes, meshes, look for supernode/supermesh

Nodal Analysis by Inspection

- Requirement: All sources are independent current sources

$$\begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & & & \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} i_{t_1} \\ i_{t_2} \\ \vdots \\ i_{t_n} \end{bmatrix}$$

$$\mathbf{GV} = \mathbf{I}_{\mathrm{t}},$$

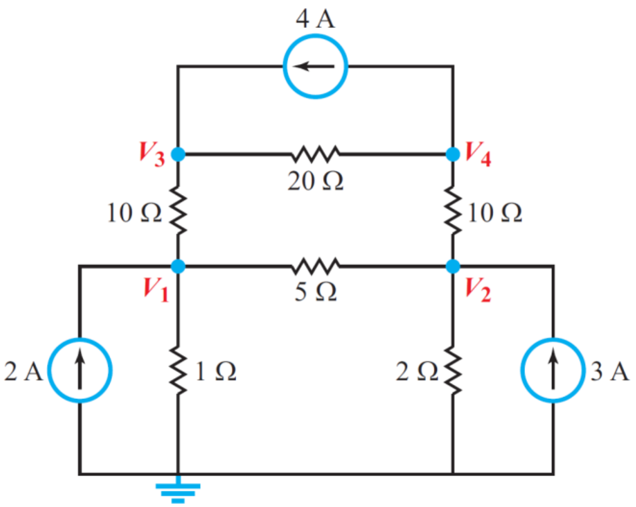
G_{kk} = sum of all conductances connected to node k

$G_{k\ell} = G_{\ell k}$ = *negative* of conductance(s) connecting nodes k and ℓ, with $k \neq \ell$

V_k = voltage at node k

I_{t_k} = total of current sources *entering* node k (a negative sign applies to a current source leaving the node).

Example: Nodal by Inspection

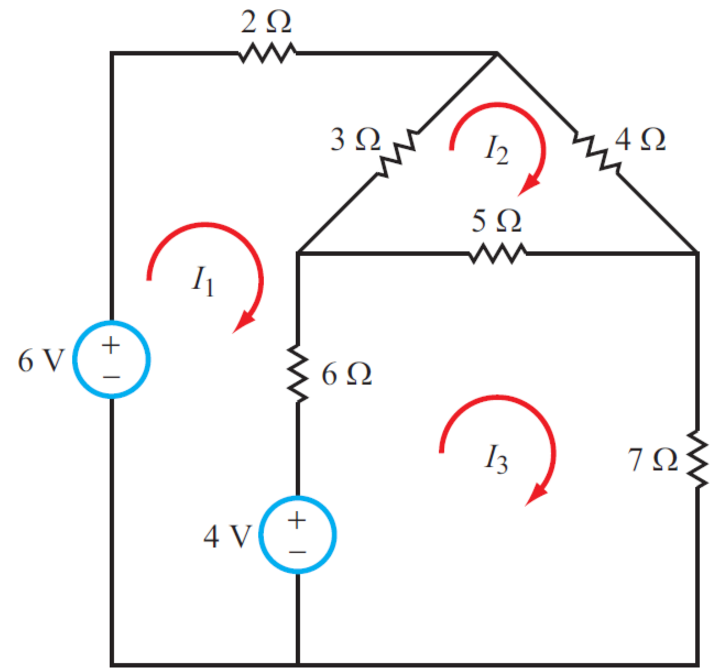


Mesh by Inspection

$$\mathbf{RI} = \mathbf{V}_t,$$

Requirement: All sources are independent voltage sources

$$\begin{bmatrix} R_{11} & R_{12} & \cdots & R_{1n} \\ R_{21} & R_{22} & \cdots & R_{2n} \\ \vdots & & & \\ R_{n1} & R_{n2} & \cdots & R_{nn} \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_n \end{bmatrix} = \begin{bmatrix} \upsilon_{t_1} \\ \upsilon_{t_2} \\ \vdots \\ \upsilon_{t_n} \end{bmatrix}, \qquad (3.29)$$



where

R_{kk} = sum of all resistances in mesh k,

$R_{k\ell} = R_{\ell k}$ = *negative* of the sum of all resistances shared between meshes k and ℓ (with $k \neq \ell$)

i_k = current of mesh k

υ_{t_k} = total of all independent voltage sources in mesh k, with positive assigned to a voltage rise when moving around the mesh in a clockwise direction.

$$\begin{bmatrix} (2+3+6) & -3 & -6 \\ -3 & (3+4+5) & -5 \\ -6 & -5 & (5+6+7) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 6-4 \\ 0 \\ 4 \end{bmatrix}$$

Linearity

A circuit is linear if output is proportional to input

- A function $f(x)$ is linear if $f(ax) = af(x)$
- All circuit elements will be assumed to be linear or can be modeled by linear equivalent circuits
  - Resistors $V = IR$
  - Linearly Dependent Sources
  - Capacitors
  - Inductors

We will examine theorems and principles that apply to linear circuits to simplify analysis

Superposition

If a circuit contains more than one independent source, the voltage (or current) response of any element in the circuit is equal to the algebraic sum of the individual responses associated with the individual independent sources, as if each had been acting alone.

Superposition trades off the examination of several simpler circuits in place of one complex circuit

Solution Procedure: Source Superposition

Step 1: Set all independent sources equal to zero (by replacing voltage sources with short circuits and current sources with open circuits), except for source 1.

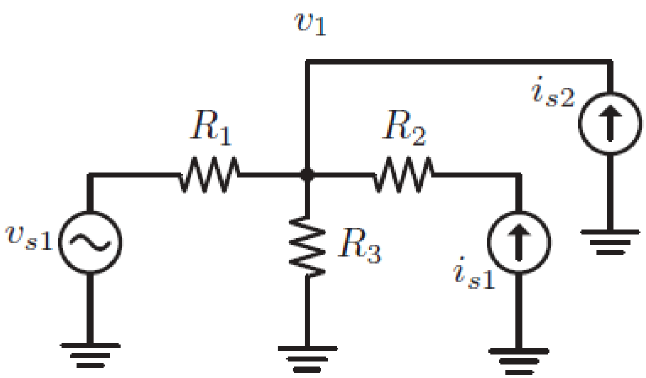
Step 2: Apply node-voltage, mesh-current, or any other convenient analysis technique to solve for the response v_1 due to source 1.

Step 3: Repeat the process for sources 2 through n, calculating in each case the response due to that one source acting alone.

Step 4: Use Eq. (3.30) to determine the total response v.

Alternatively, the procedure can be used to find currents i_1 to i_n and then to add them up algebraically to find the total current i using Eq. (3.31).

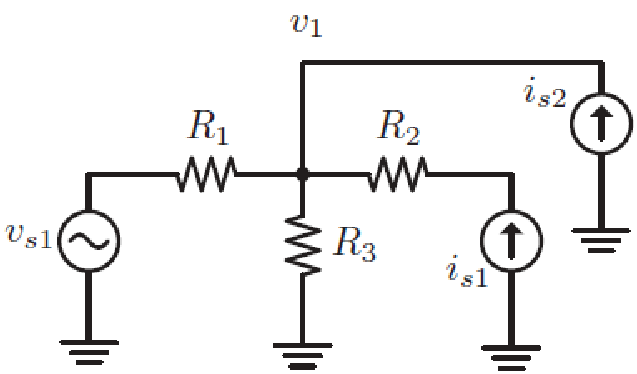
Example: Superposition



- In this example there are three independent sources. When we analyze the circuit source by source, the circuit is often simple enough that we can solve the equations directly by inspection.
- First turn of i_{s1} and i_{s2}. Zero current means that we replace these sources with open circuits. The node voltage v_1 is therefore by inspection

$$v_1^{v_{s1}} = \frac{R_2}{R_1 + R_2} v_{s1}$$

Example: Superposition



- Next turn off v_{s1} (short circuit) and i_{s2} (open circuit). The current i_{s1} will therefore divide between r_2 and r_1 and establish a voltage at node v_1 (equivalently, it see's a parallel combination of R_1 and R_2

$$v_1^{i_{s1}} = \frac{R_2 R_1}{R_1 + R_2} i_{s1}$$

- Finally, we turn off all sources except i_{s2}. Now only R_1 and R_2 remain (R_3 is dangling)

$$v_1^{i_{s2}} = \frac{R_2 R_1}{R_1 + R_2} i_{s1}$$

Example: Superposition

- By superposition, the node voltage v_1 is the sum of the three node voltages due to each source

$$v_1 = v_1^{i_{s1}} + v_1^{i_{s2}} + v_1^{v_{s1}} = \frac{R_2}{R_1 + R_2}(v_{s1} + R_1(i_{s1} + i_{s2}))$$

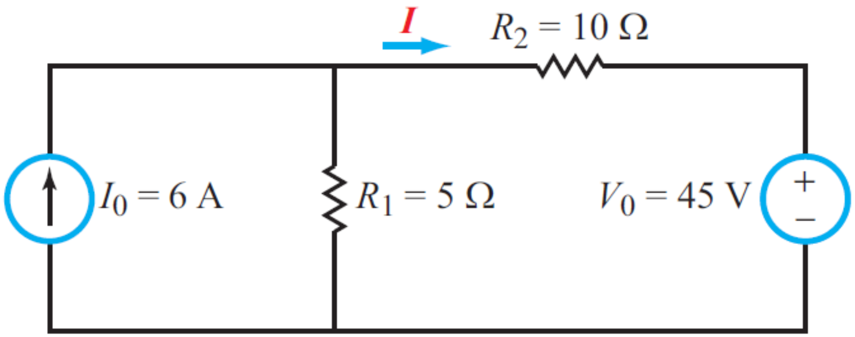
- We can verify the solution by performing KCL directly at node 1

$$(v_1 - v_{s1})G_1 + v_1 G_2 - i_{s1} - i_{s2} = 0$$

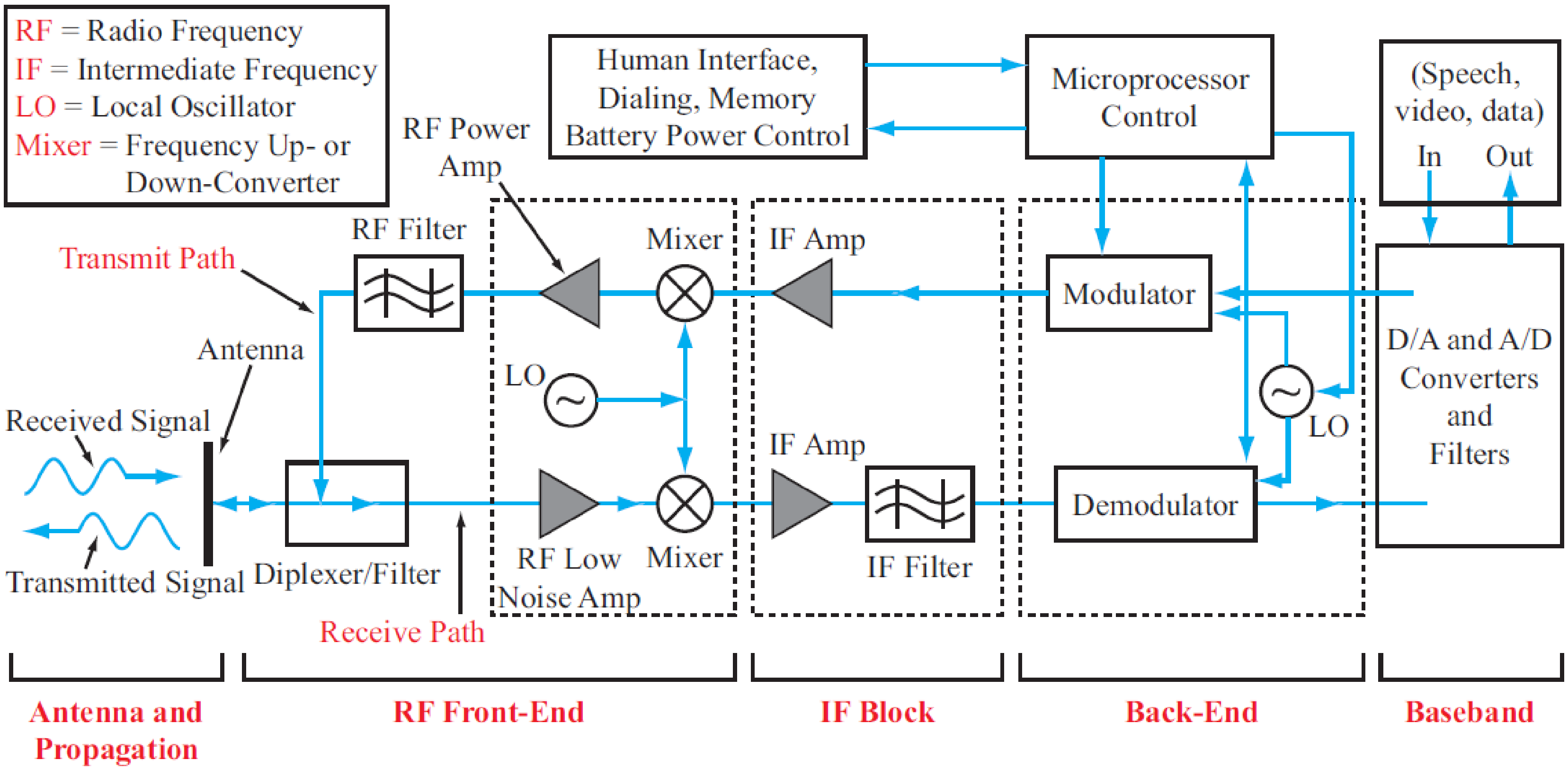
$$v_1(G_1 + G_2) = v_{s1}G_1 + i_{s1} + i_{s2}$$

- The answer here is just as fast but we don't have any intuition about the operation of the circuit. We're perhaps more likely to make an algebraic error if it's all math without any thinking.

Example 2: Superposition



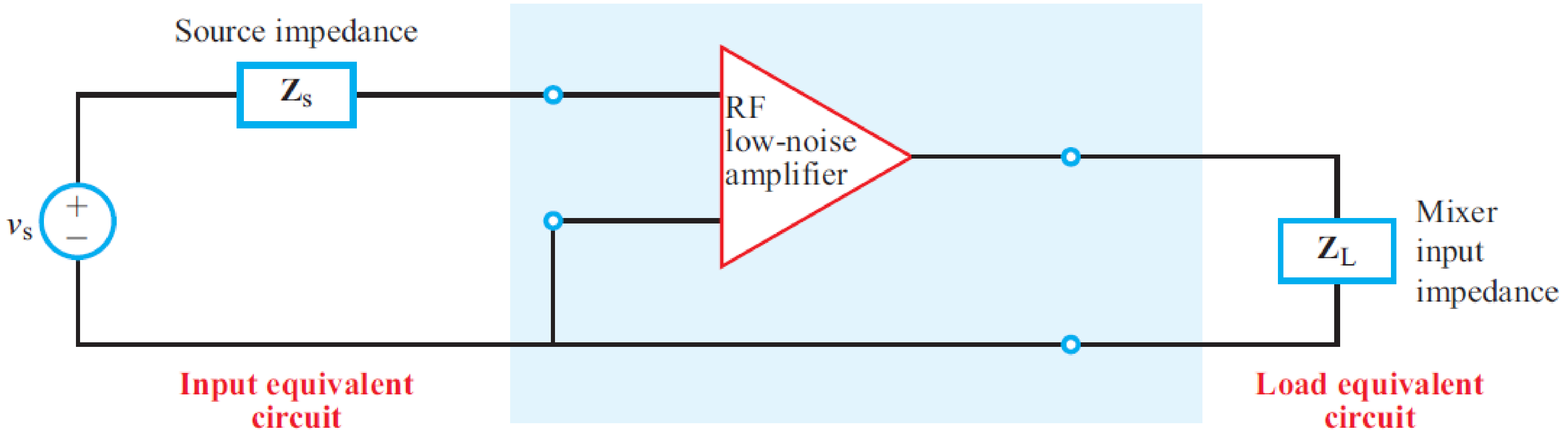
Cell Phone



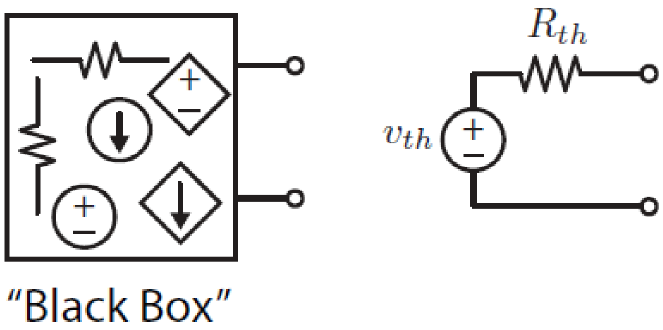
Today's systems are complex. We use a block diagram approach to represent circuit sections.

Equivalent Circuit Representation

- Fortunately, many circuits are linear
- Simple equivalent circuits may be used to represent complex circuits



Thévenin's Equivalent Circuit



- A powerful theorem in circuit analysis is the Thevenin equivalent theorem, which let's us replace a very complex circuit with a simple equivalent circuit model.
- In the black box there can be countless resistors, voltage sources (independent and dependent), current sourses (independent and dependent), and yet the *terminal* behavior of the circuit is captured by two elements.
- How can this be? Well, there is a big assumption in that all the resistors are linear (follow Ohm's Law) and all dependent sources are also linear.
- The equivalent circuit representation is often called a "black box", since the details of the circuitry are hidden.

Thévenin's Theorem

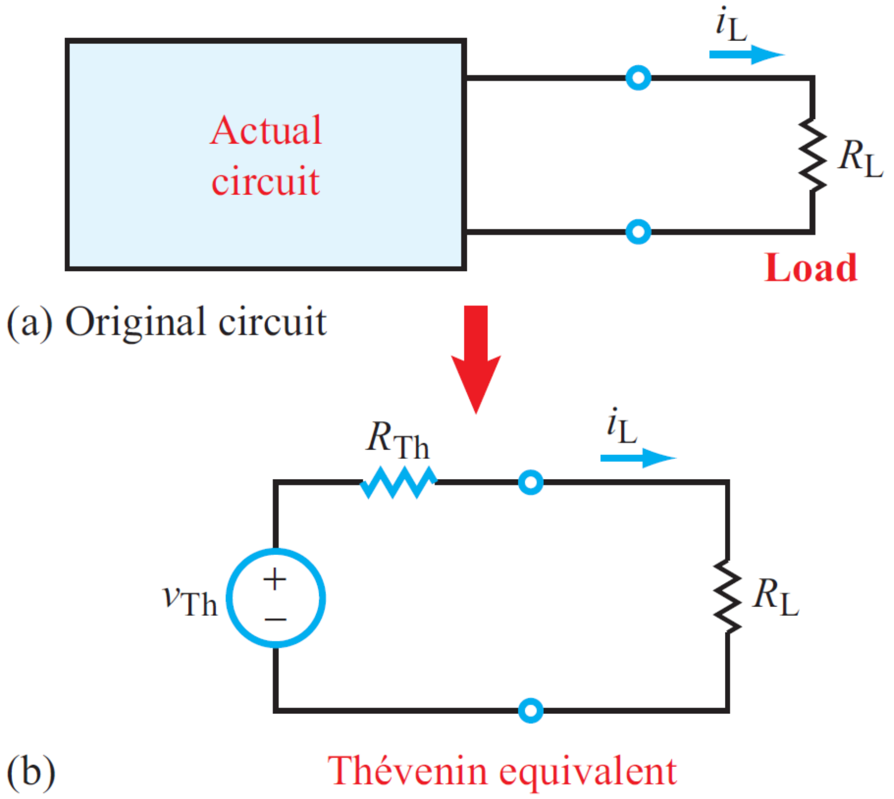
Linear two-terminal circuit can be replaced by an equivalent circuit composed of a voltage source and a series resistor

$$v_{\text{Th}} = v_{\text{oc}}$$

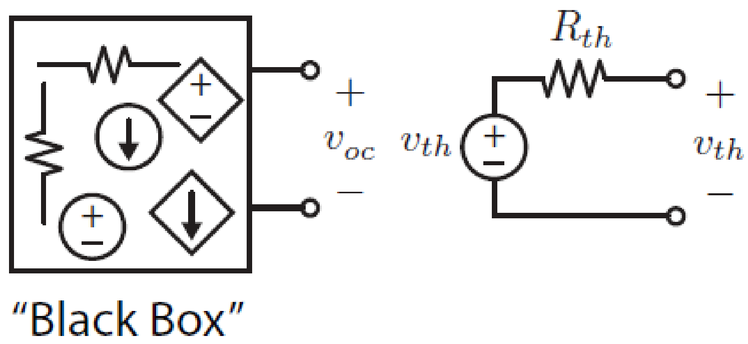
voltage across output with no load (open circuit)

$$R_{\text{Th}} = R_{\text{in}}$$

Resistance at terminals with all independent circuit sources set to zero



Thévenin's Equivalent Circuit Derivation

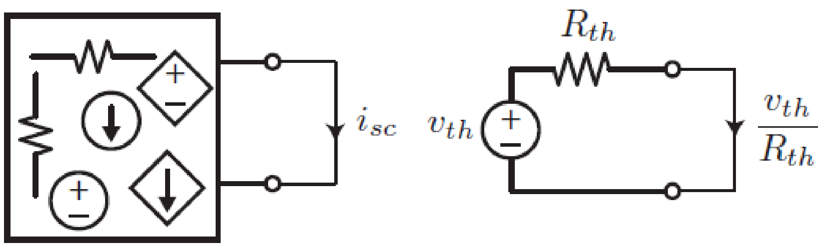


- Since a circuit is linear, then no matter how complicated it is, it's response to a stimulus at some terminal pair must be linear. It can therefore be represented by a linear equivalent resistor and a a fixed constant source voltage due to the presence of independent sources in the circuit.
- To find the equivalent source value, called the Thevenin voltage source v_{th}, simply observe that the open-circuit voltage of both the "black box" and the original circuit must equal, which means

$$v_{th} = v_{oc}$$

- In other words, open-circuit the original circuit, find its equivalent output voltage at the terminals of interest, and that's v_{th}

Thévenin's Circuit Source Resistance



"Black Box"

- To find equivalent Thevenin source resistance R_{th}, notice that in order for the terminal behavior of the two circuits to match, the current flow into a load resistor has to be the same for any load value. In particular, take the load as a short circuit.
- The output current of the Thevenin equivalent under a short circuit is given by

$$\frac{v_{th}}{R_t}$$

- Equating this to the short-circuit current of the original circuitry, we have

$$i_{sc} = \frac{v_{th}}{R_t}$$

or equivalently

$$R_{th} = \frac{v_{th}}{i_{sc}} = \frac{v_{oc}}{i_{sc}}$$

Norton's Theorem

Linear two-terminal circuit can be replaced by an equivalent circuit composed of a current source and parallel resistor

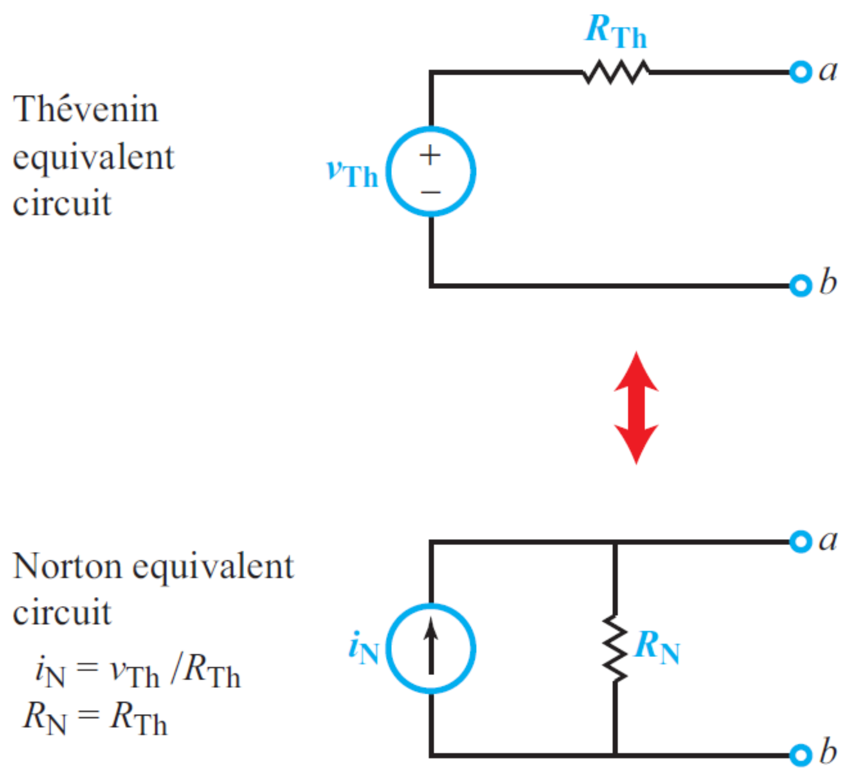
$$i_N = \frac{v_{Th}}{R_{Th}}$$

Current through output with short circuit

$$R_N = R_{Th}.$$

Resistance at terminals with all circuit sources set to zero

Thévenin and Norton Equivalency



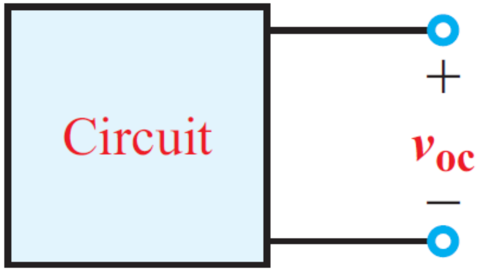
How Do We Find Thévenin/Norton Equivalent Circuits ?

- Method 1: Open circuit/Short circuit
  1. Analyze circuit to find v_{oc}
  2. Analyze circuit to find i_{sc}

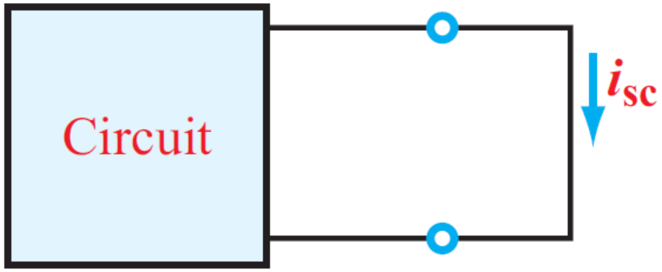
$$v_{Th} = v_{oc}$$

$$R_{Th} = \frac{v_{Th}}{i_{sc}}$$

Note: This method is applicable to any circuit, whether or not it contains dependent sources.

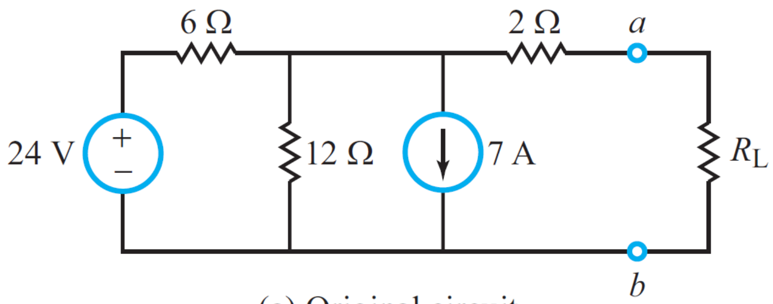


(a) $v_{Th} = v_{oc}$



(b) $R_{Th} = v_{oc} / i_{sc}$

Example: Thévenin Equivalent



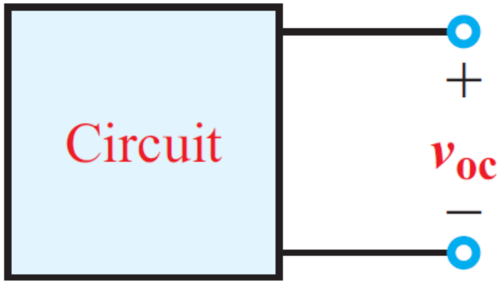
(a) Original circuit

How Do We Find Thévenin/Norton Equivalent Circuits?

Method 2: Equivalent Resistance

1. Analyze circuit to find either

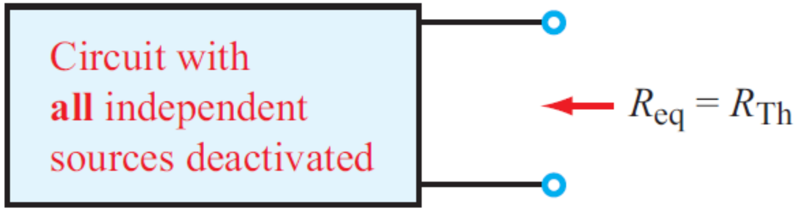
v_{oc} or i_{sc}



(a) $v_{Th} = v_{oc}$

2. Deactivate all independent sources by replacing voltage sources with short circuits and current sources with open circuits.
3. Simplify circuit to find equivalent resistance

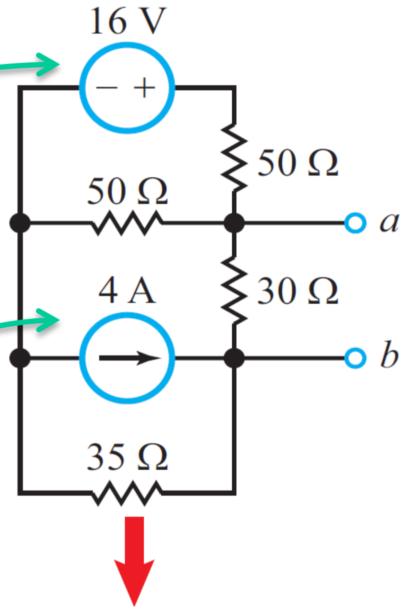
Equivalent-Resistance Method



Note: This method does not apply to circuits that contain dependent sources.

Example: R_{Th}

Replace with SC



Replace with OC

(Circuit has no dependent sources)

How Do We Find Thévenin/Norton Equivalent Circuits?

Method 3:

External-Source Method

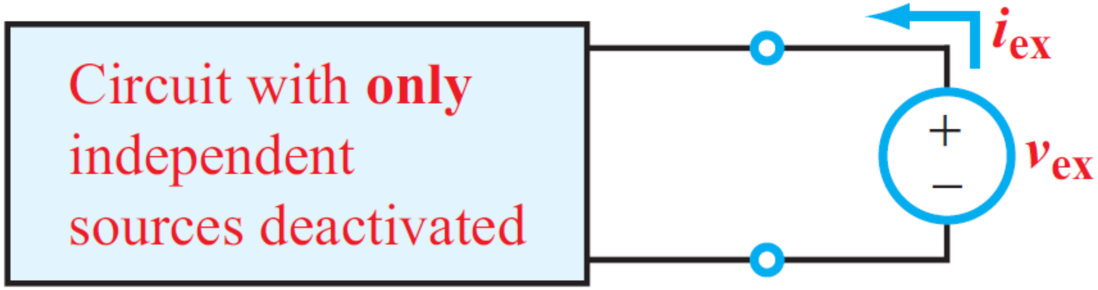


Figure 3-22: If a circuit contains both dependent and independent sources, R_{Th} can be determined by (a) deactivating independent sources (only), (b) adding an external source v_{ex}, and then (c) solving the circuit to determine i_{ex}. The solution is $R_{Th} = v_{ex}/i_{ex}$.

Example

Find the Thévenin equivalent circuit at terminals (a, b) for the circuit in Fig. 3-23(a) by applying the combination of open-circuit-voltage and external-source methods.

Solution:
The equations for mesh currents I_1 and I_2 in Fig. 3-23(a) are given by

$$-68 + 6I_1 + 2(I_1 - I_2) + 4I_x = 0$$

and

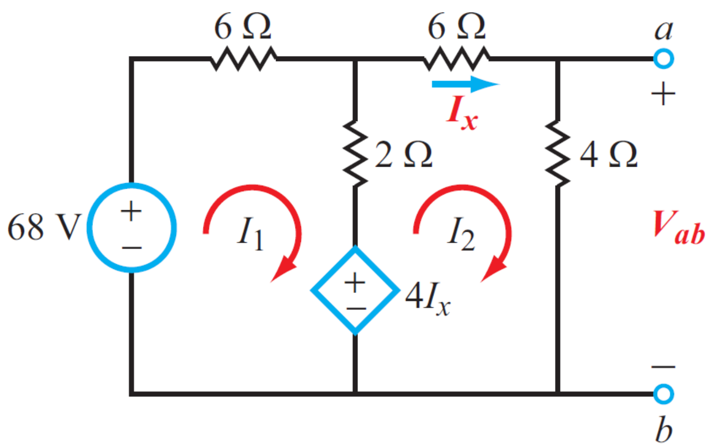
$$-4I_x + 2(I_2 - I_1) + 6I_2 + 4I_2 = 0.$$

Recognizing that $I_x = I_2$, solution of these two simultaneous equations leads to

$$I_1 = 8 \text{ A},$$

and

$$I_2 = 2 \text{ A}.$$



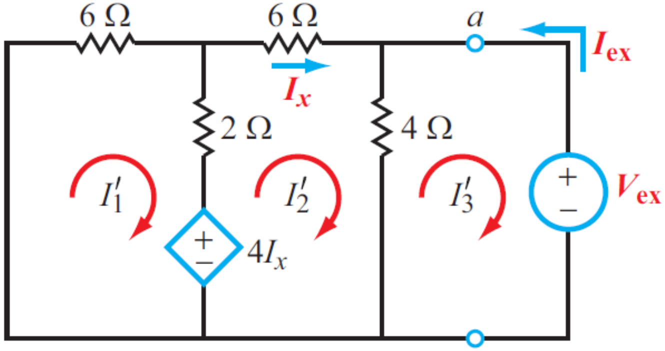
(a) Solving for V_{Th}

The Thévenin voltage is V_{ab}. Hence,

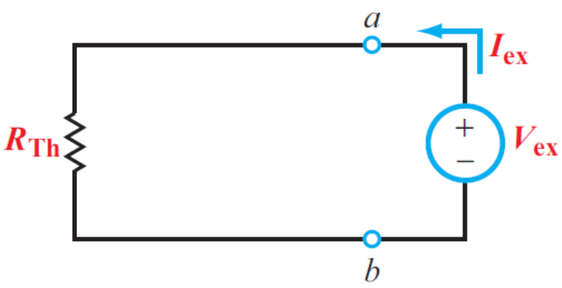
$$\begin{aligned} V_{\text{Th}} &= V_{ab} \\ &= 4I_2 \\ &= 8 \text{ V}. \end{aligned}$$

Example

To find R_{Th}

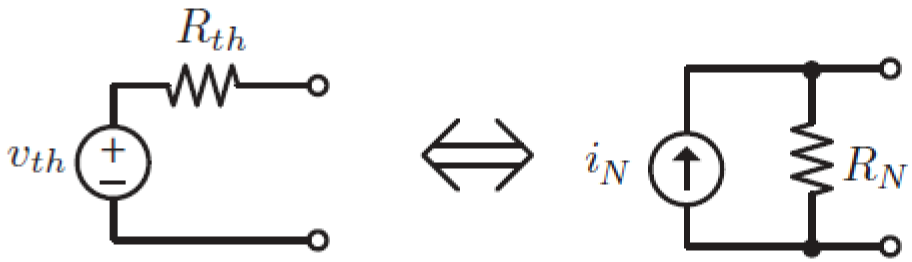


(b) Solving for I_{ex}



(c) Equivalent circuit for calculating R_{Th}

Source Transformations

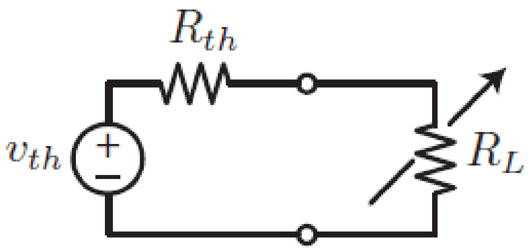


- A trivial application of Norton or Thevenin's Theorem shows us that we can transform from one representation to the other. For instance, starting from the Thevenin, let's find the Norton. Short circuit the Thevenin to find

$$i_n = i_{sc} = \frac{V_{th}}{R_{th}}$$

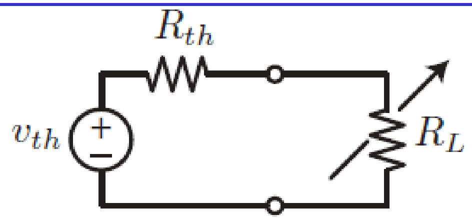
and it's trivial to see that $R_n = R_{th}$.

Maximum Power Transfer



- An important question arises in many electrical circuits when we wish to interface one component to another while maximizing the power transfer to the second component.
- A good example is a battery with internal resistance and a motor. What’s the best “load" resistance to choose in order to maximize the power transfer?
- Interestingly, if we maximize the current or voltage transfer, the power transfer is exactly zero.

Maximum Power Transfer



- No matter how complicated the black box source, we can represent it as a Thevenin equivalent circuit, v_{th} and R_{th}.
- The general procedure is to find the power through the load and then to find the optimal load value. We can take the derivative of the load power with respect to the load resistance and set it equal to zero (occurs at only a maximum or minimum). A second derivative test confirms that it's a peak.

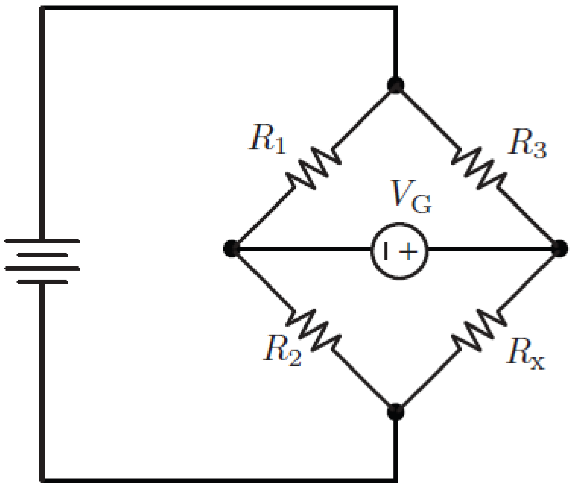
$$P_L = I_L^2 R_L = \left(\frac{v_{th}}{R_L + R_{th}}\right)^2 R_L$$

$$\frac{dP_L}{dR_L} = \left(\frac{v_{th}}{R_L + R_{th}}\right)^2 - 2R_L v_{th}^2 \left(\frac{1}{R_L + R_{th}}\right)^3 = 0$$

$$(R_L + R_{th}) = 2R_L \rightarrow R_L = R_{th}$$

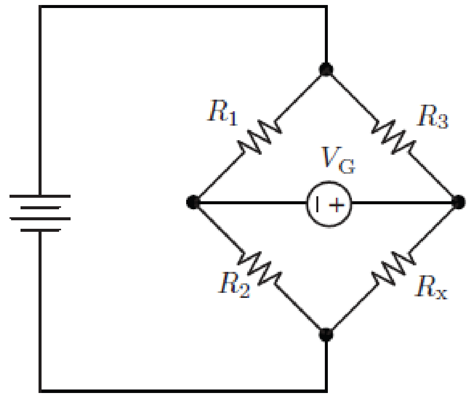
The optimum load resistance is equal to the Thevenin Equivalent Value, or it's *Matched*.

The Wheatstone Bridge



- The Wheatstone Bridge (originally invted by Samuel Hunter Christie in 1833 and then popularized by Sir Charles Wheatstone in 1843) is used to measure an unknown resistance. It is highly accurate and only requires an adjustable resistor (or set of well known calibrated resistors) and a method of measuring zero current, such as a galvanometer.
- Since we only need to measure if the current is zero, we can do this very precisely with a galvanometer.
- The Wheatstone bridge is often used with strain gauges, thermocouples, and other transducers.

The Wheatstone Bridge

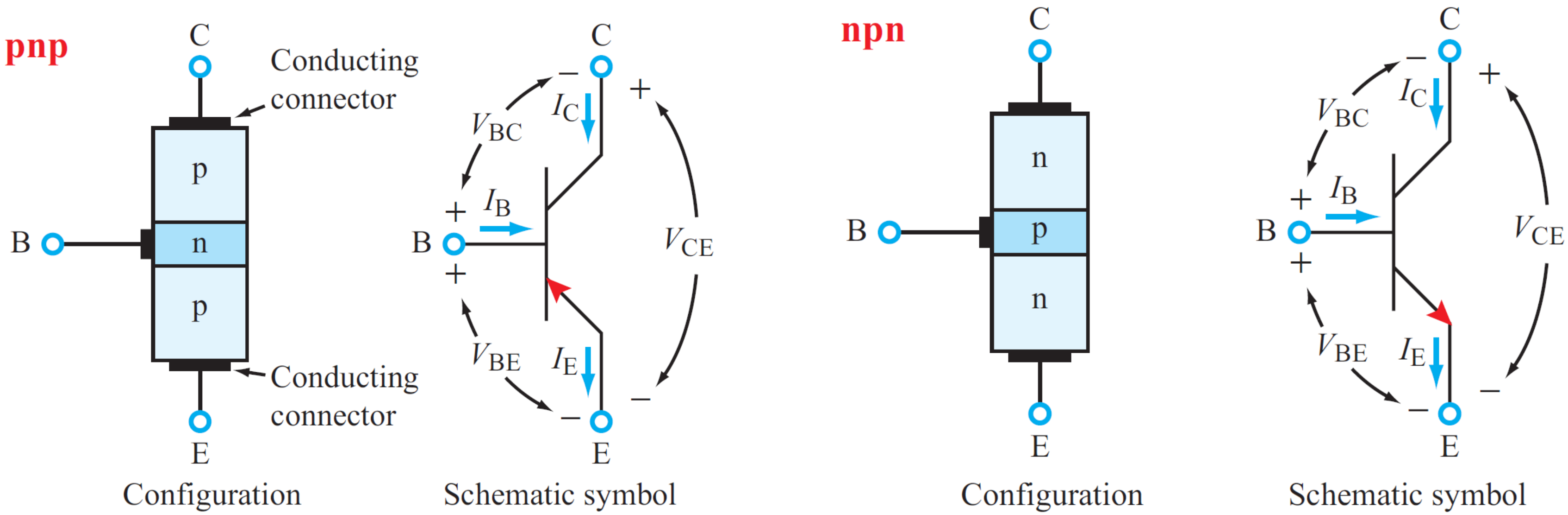


- The operation of the Wheatstone Bridge is as follows. One leg of the bridge contains an unknown resistance which we would like to find. The other leg contains an adjustable resistor R_2 (of known value). The goal is to adjust the resistor R_2 until the circuit is "balanced", in other words until no current flows through the galvanometer.
- Under the balanced condition, there is no current I_g, so the current in R_1 and R_2 is the same, say I_1, and the current through R_3 and R_x is also the same, I_3. By KVL, under the balanced condition $V_g = 0$, we have

$$I_3 R_3 = I_1 R_1$$

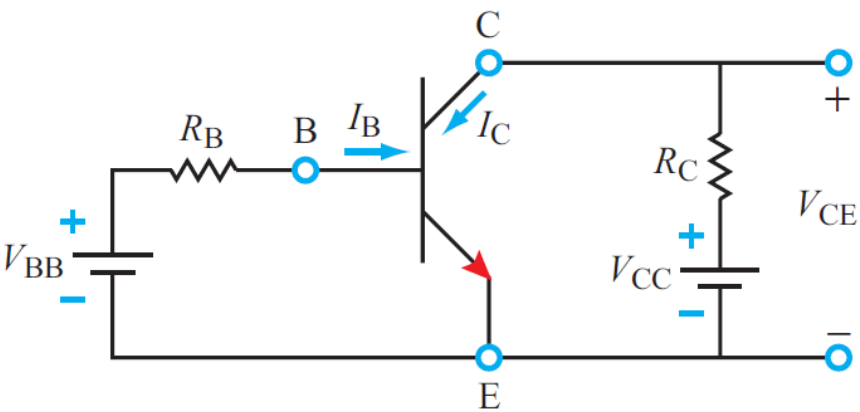
$$I_3 R_x = I_1 R_2$$

BJT: Our First 3 Terminal Device!

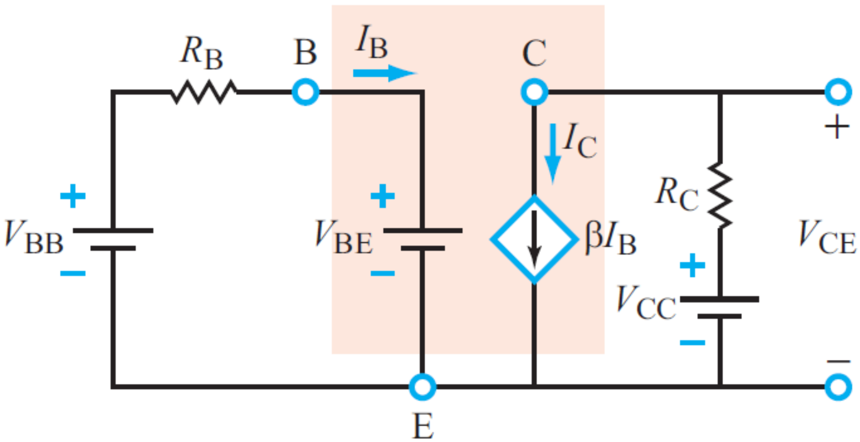


- Active device with dc sources
- Allows for input/output, gain/amplification, etc

BJT Equivalent Circuit



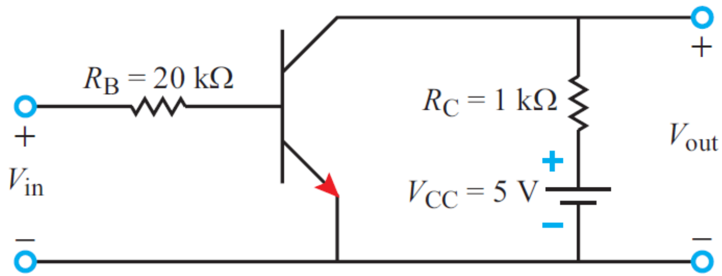
(a) Transistor circuit



(b) Equivalent circuit

ooks like a current
mplifier
with gain β

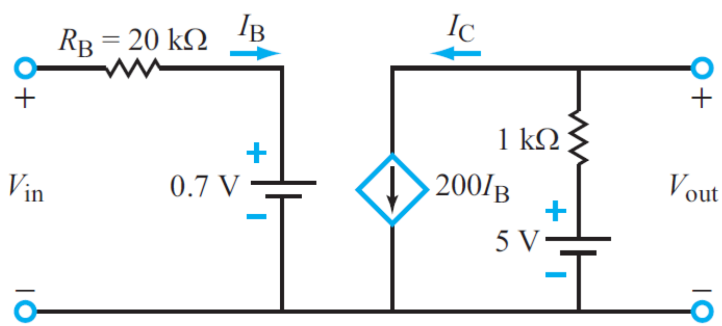
Digital Inverter With BJTs



(a) Inverter circuit

BJT Rules:
Vout cannot exceed Vcc=5V
Vin cannot be negative

In	Out
0	1
1	0



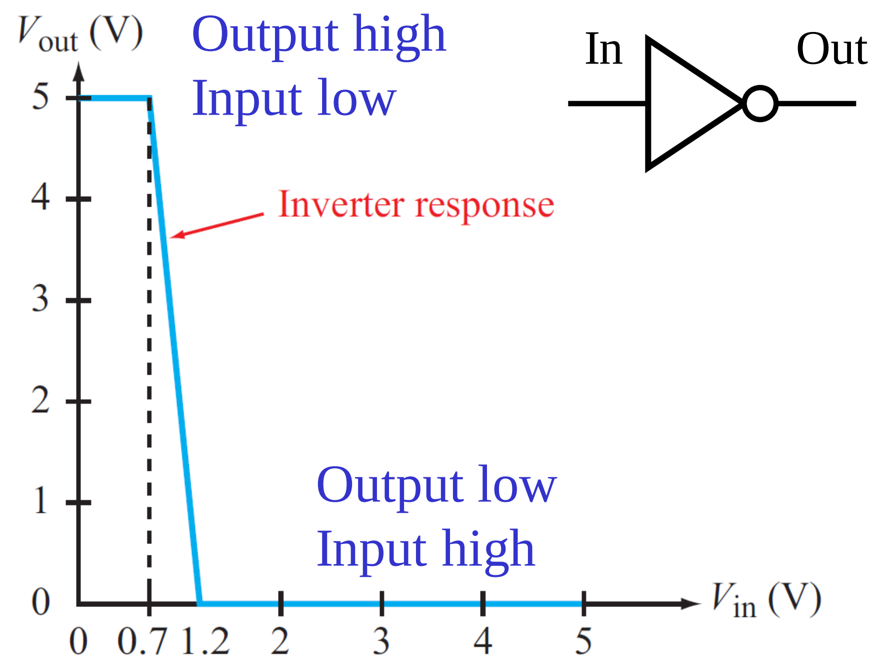
(b) Equivalent circuit

$$I_B = \frac{V_{in} - 0.7}{20k},$$

$$I_C = \beta I_B = 200 I_B,$$

$$V_{out} = V_{CC} - I_C R_C.$$

$$V_{out} = V_{CC} - \frac{\beta R_C}{R_B}(V_{in} - 0.7) = 12 - 10 V_{in} \quad (V).$$



(c) V_{out} versus V_{in}