EE 40 – RLC Circuits

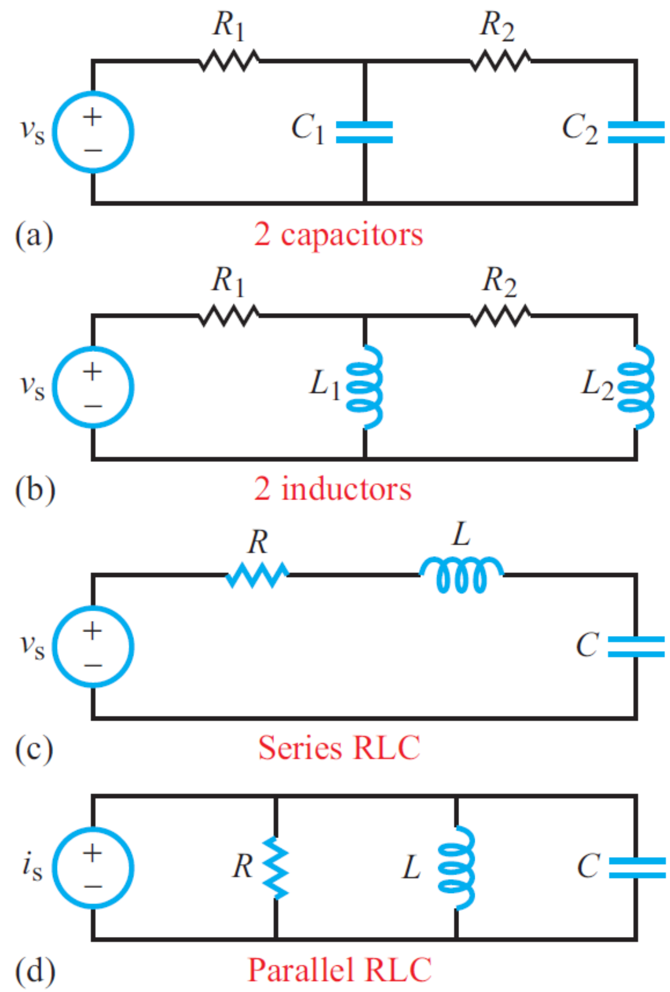
Reading Material:

Chapter 6

Second Order Circuits

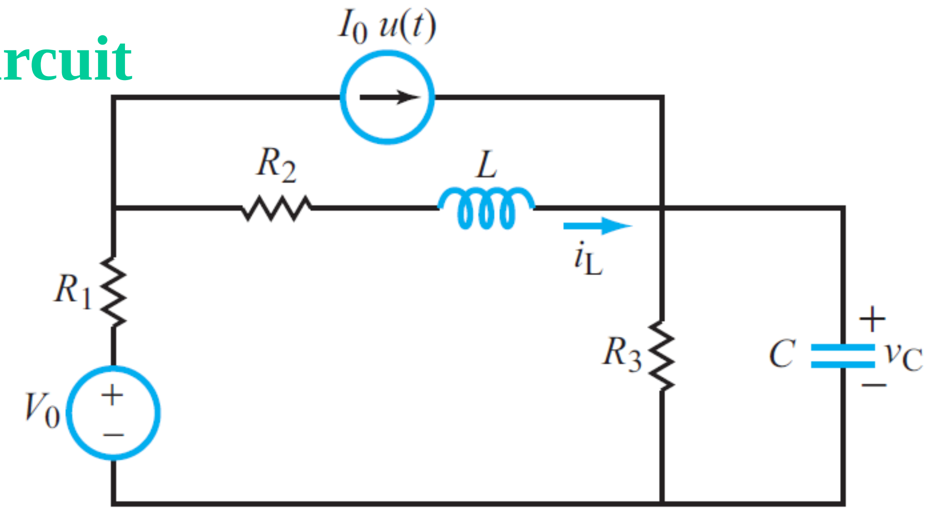
A second order circuit is characterized by a second order differential equation

- Resistors and two energy storage elements
- Determine voltage/current as a function of time
- Initial/final values of voltage/current, *and their derivatives are* needed



Example: Determine Initial/Final Conditions

Circuit



$V_0 = 24$ V

$I_0 = 4$ A

$R_1 = 2\ \Omega$

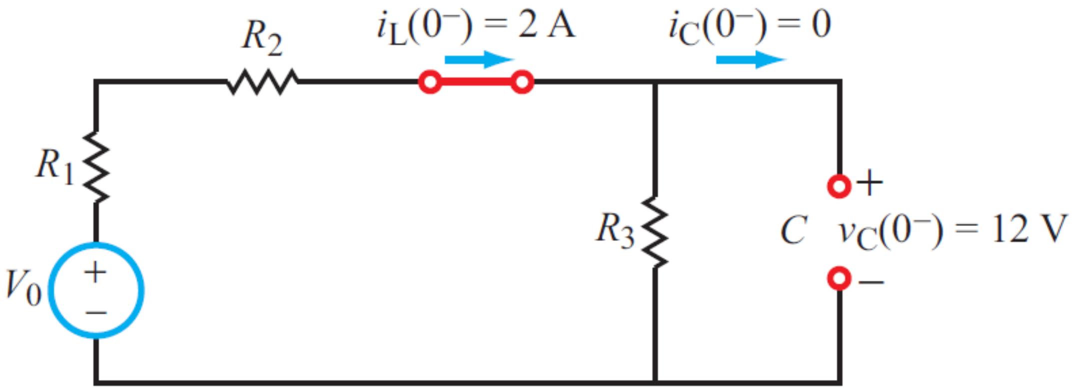
$R_2 = 4\ \Omega$

$R_3 = 6\ \Omega$

$L = 0.2$ H

$C = 8$ mF

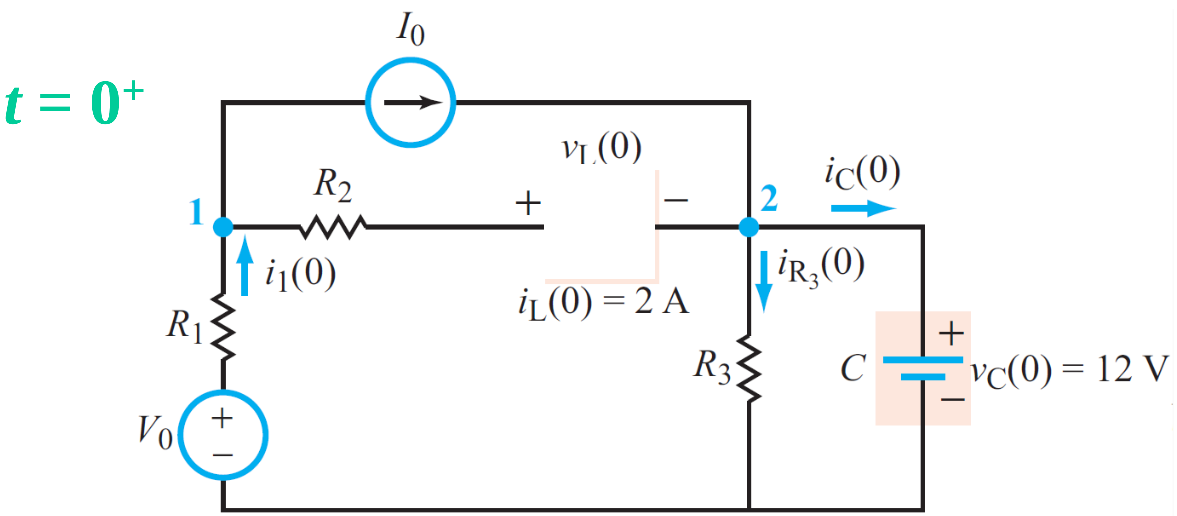
$t = 0^-$



$$i_L(0^-) = \frac{V_0}{R_1 + R_2 + R_3} = 2\text{ A},$$

$$v_C(0^-) = i_L(0^-)\ R_3 = 12\text{ V}.$$

Example: Initial/Final Conditions (cont.)



Given:
$V_0 = 24$ V
$I_0 = 4$ A
$R_1 = 2\ \Omega$
$R_2 = 4\ \Omega$
$R_3 = 6\ \Omega$
$L = 0.2$ H
$C = 8$ mF

$$v_{R_3}(0) = v_C(0) = 12 \text{ V},$$

it follows that

$$i_{R_3}(0) = \frac{12}{6} = 2 \text{ A}.$$

Application of KCL at node 2 leads to

$$\begin{aligned} i_C(0) &= I_0 + i_L(0) - i_{R_3}(0) \\ &= 4 + 2 - 2 \\ &= 4 \text{ A}. \end{aligned}$$

Next, we need to determine $v_L(0)$. At node 1,

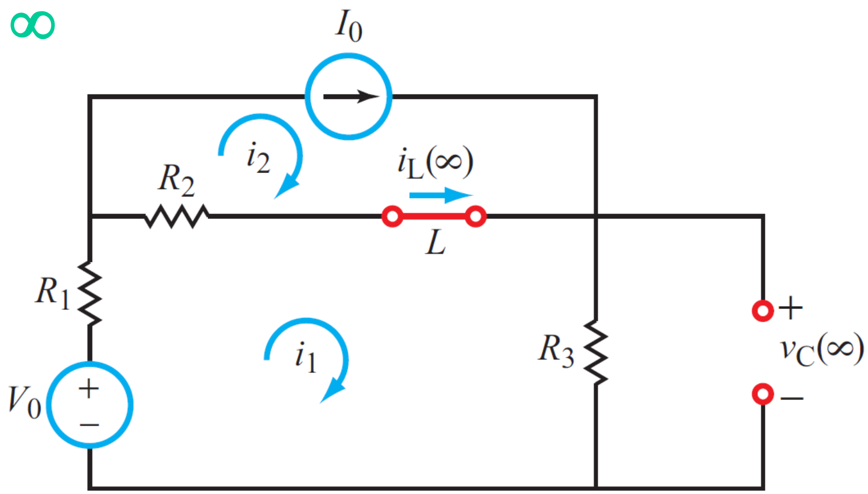
$$i_1(0) = I_0 + i_L(0) = 4 + 2 = 6 \text{ A}.$$

By applying KVL around the lower left loop, we find that

$$v_L(0) = -8 \text{ V}.$$

Example: Initial/Final Conditions (cont.)

$t \Rightarrow \infty$



$V_0 = 24$ V

$I_0 = 4$ A

$R_1 = 2\ \Omega$

$R_2 = 4\ \Omega$

$R_3 = 6\ \Omega$

$L = 0.2$ H

$C = 8$ mF

$$-V_0 + R_1 i_1 + R_2(i_1 - i_2) + R_3 i_1 = 0,$$

and for loop 2,

$$i_2 = I_0 = 4 \text{ A}.$$

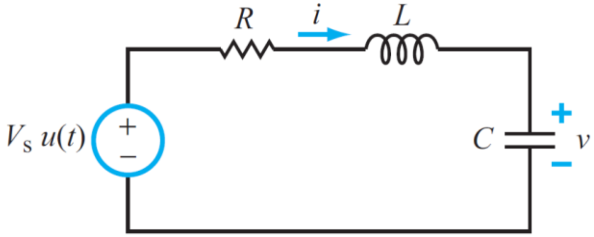
Solving for i_1 gives

$$i_1 = 3.33 \text{ A},$$

$$i_L(\infty) = i_1 - I_0 = 3.33 - 4 = -0.67 \text{ A}$$

$$v_C(\infty) = i_1 R_3 = 3.33 \times 6 = 20 \text{ V}.$$

Series RLC Circuit : General Solution



$$Ri + L\,\frac{di}{dt} + v = V_s \qquad \text{(for } t \geq 0\text{)}. \tag{6.3}$$

By incorporating the relation

$$i = C\,\frac{dv}{dt} \tag{6.4}$$

and rearranging terms, Eq. (6.3) becomes

$$\frac{d^2v}{dt^2} + \frac{R}{L}\,\frac{dv}{dt} + \frac{1}{LC}\,v = \frac{V_s}{LC}. \tag{6.5}$$

For convenience, we rewrite Eq. (6.5) in the abbreviated form as

$$\boxed{v'' + av' + bv = c,} \tag{6.6}$$

where

$$a = \frac{R}{L}, \qquad b = \frac{1}{LC}, \qquad \text{and} \qquad c = \frac{V_s}{LC}. \tag{6.7}$$

But wait…

- If you’ve been coming to lecture, you know I paused here to look at AC analysis and phasors first (i.e. jump to Lecture 7, then we’ll come back and pickup here).

Series RLC Circuit: Natural Response

Find response when $V_S=0$ after t=0

Solution of Diff. Equation

$$v'' + av' + bv = 0,$$

Assume: $v(t) = Ae^{st},$

where A and s are constants to be determined later.

It follows that: $s^2Ae^{st} + asAe^{st} + bAe^{st} = 0,$

which simplifies to

$$s^2 + as + b = 0.$$

Considering roots of quadratic

Hence, we should generalize the form of our solution

to

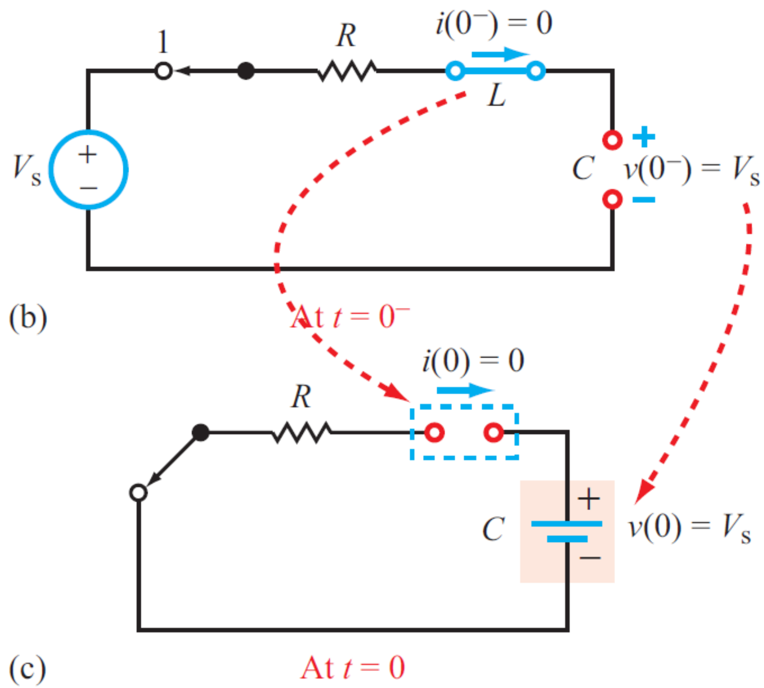
$$v(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad (\text{for } t \geq 0),$$

where A_1 and A_2 are to be determined next.

Solution of Diff. Equation (cont.)

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (\text{for } t \geq 0),}$$

Invoke Initial Conditions to determine A1 and A2



$$v(0) = v(0^-) = V_s$$

and

$$i(0) = i(0^-) = 0.$$

But, remember: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

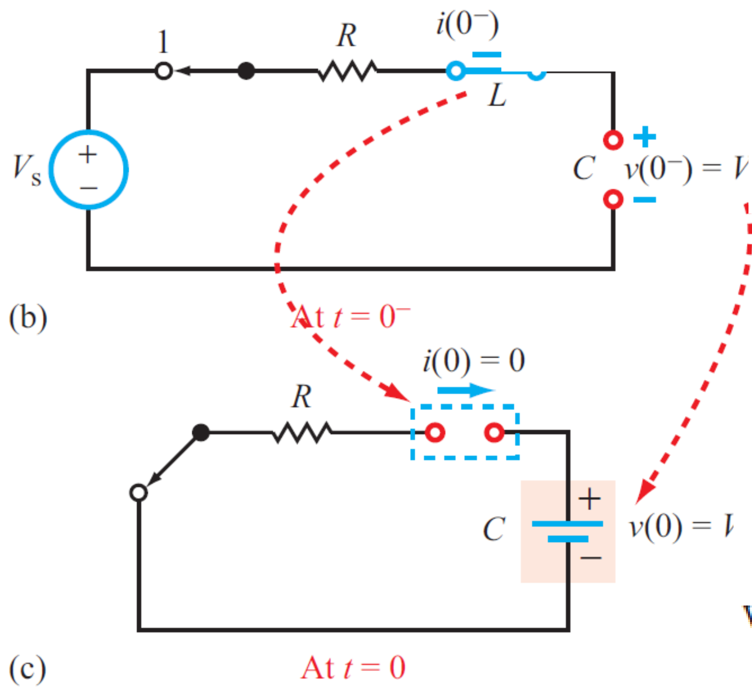
Therefore:

$$v(0) = A_1 + A_2 = V_s.$$

Solution of Diff. Equation (cont.)

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (\text{for } t \geq 0),}$$

Invoke Initial Conditions to determine A_1 and A_2



Moreover, because $i = C\, dv/dt$,

$$v'(0) = \left.\frac{dv}{dt}\right|_{t=0} = \frac{1}{C}\, i(0) = 0.$$

But, remember: $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

Therefore:

$$v'(0) = \left.(s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t})\right|_{t=0} = 0,$$

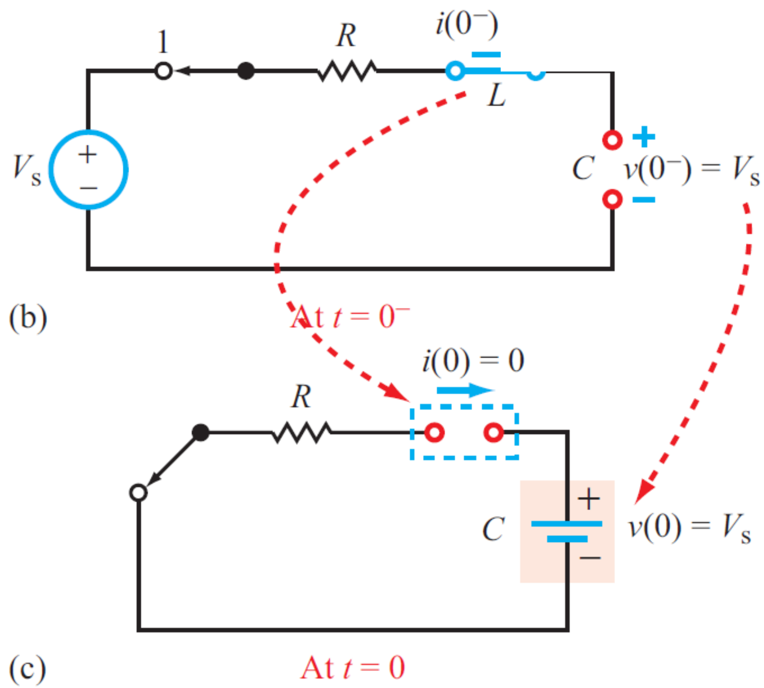
which simplifies to

$$s_1 A_1 + s_2 A_2 = 0.$$

Solution of Diff. Equation (cont.)

$$\boxed{v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad (\text{for } t \geq 0),}$$

Invoke Initial Conditions to determine A1 and A2



$$s_1 A_1 + s_2 A_2 = 0.$$

$$v(0) = A_1 + A_2 = V_s.$$

The solution is:

$$A_1 = \left(\frac{s_2}{s_2 - s_1}\right) V_s$$

$$A_2 = -\left(\frac{s_1}{s_2 - s_1}\right) V_s.$$

Circuit Response: Damping Conditions

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(for } t \geq 0\text{)},$$

$$s_1 = -\frac{a}{2} + \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

$$= -\alpha + \sqrt{\alpha^2 - \omega_0^2},$$

$$s_2 = -\frac{a}{2} - \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

$$= -\alpha - \sqrt{\alpha^2 - \omega_0^2}.$$

Damping coefficient

$$\alpha = \frac{a}{2} = \frac{R}{2L},$$

Resonant frequency

$$\omega_0 = \sqrt{b} = \frac{1}{\sqrt{LC}}.$$

Overdamped Response ($\alpha > \omega_0$)

s_1 and s_2 are real

Critically Damped Response ($\alpha = \omega_0$)

$s_1 = s_2$

Underdamped Response ($\alpha < \omega_0$)

s_1 and s_2 are complex

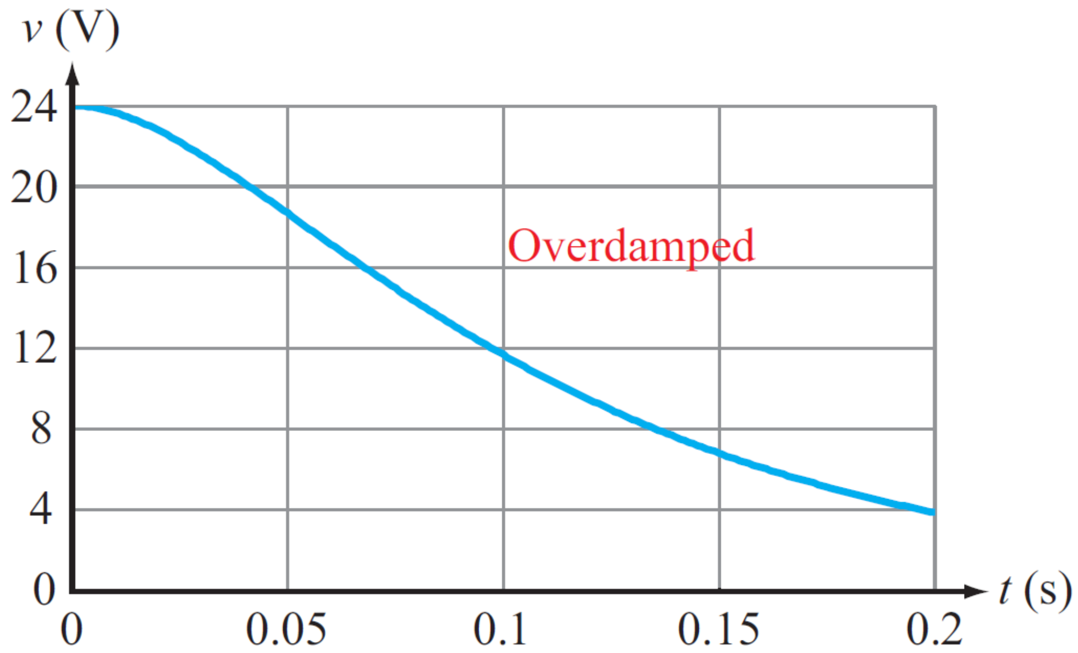
Overdamped Response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \text{damping factor}$$

$$\omega_0 = \text{resonant frequency}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$



Overdamped, $\alpha > \omega_0$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Underdamped Response

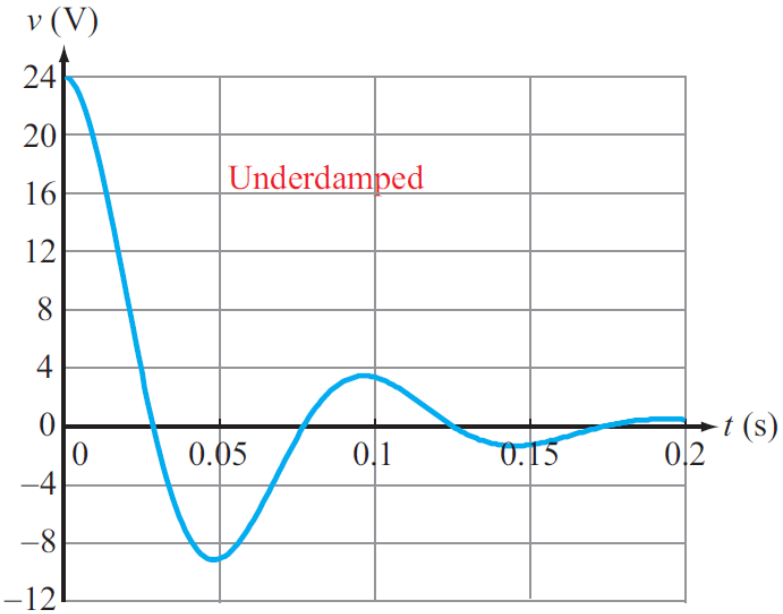
Damping: loss of stored energy

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \text{damping factor}$$

$$\omega_0 = \text{resonant frequency}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$



Underdamped $\alpha < \omega_0$

$$v(t) = e^{-\alpha t}\left(D_1 \cos \omega_d t + D_2 \sin \omega_d t\right)$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$ Damped natural frequency

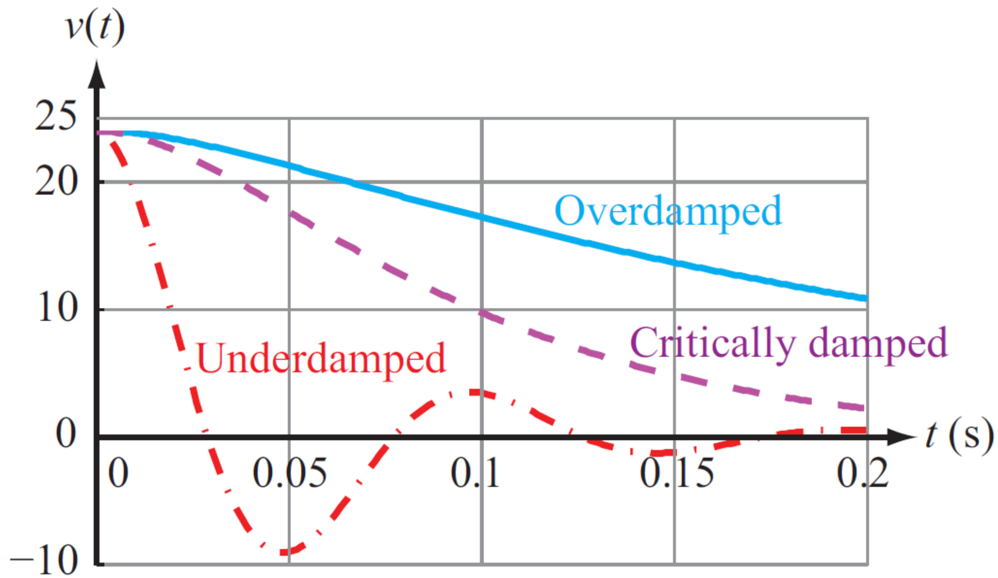
Critically Damped Response

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\,2}}$$

$\alpha =$ damping factor

$\omega_0 =$ resonant frequency

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$



Critically damped $\alpha = \omega_0$

$$v(t) = (B_1 + B_2 t)e^{-\alpha t}$$

Total Response of Series RLC Circuit

Need to add Forced/Steady State Solution

$$v(t) = v_{ss} + v_t(t)$$

Natural solution represents transient response, decays to 0 as $t \Rightarrow \infty$.
$v(\infty)$ represents forced/steady state solution.

Overdamped ($\alpha > \omega_0$)

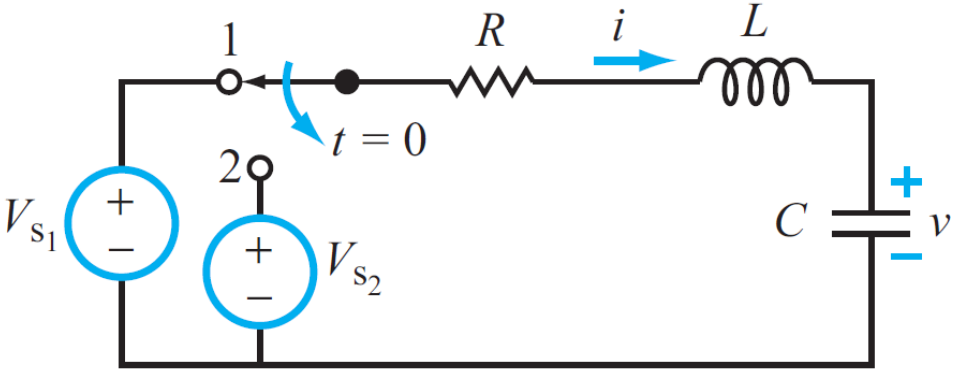
$$v(t) = v(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Critically Damped ($\alpha = \omega_0$)

$$v(t) = v(\infty) + (B_1 + B_2 t)e^{-\alpha t}$$

Underdamped ($\alpha < \omega_0$)

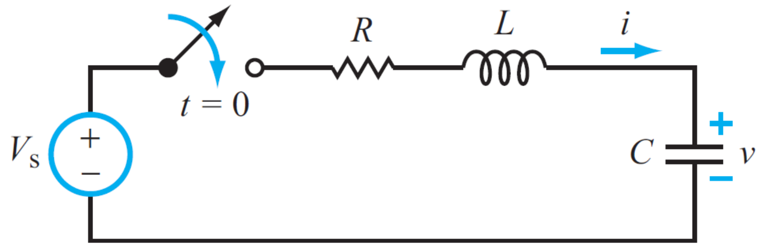
$$v(t) = v(\infty) + e^{-\alpha t}(D_1 \cos \omega_d t + D_2 \sin \omega_d t)$$



Now find unknown constants from initial conditions $v(0^+)$ and dv/dt at $t = 0^+$

Example: Overdamped RLC Circuit

Given that in the circuit of Fig. 6-12(a) $V_s = 16$ V, $R = 64\ \Omega$, $L = 0.8$ H, and $C = 2$ mF, determine $v(t)$ and $i(t)$ for $t \geq 0$. The capacitor had no charge prior to $t = 0$.



$$\alpha = \frac{R}{2L} = \frac{64}{2 \times 0.8} = 40 \text{ Np/s}$$

and

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.8 \times 2 \times 10^{-3}}} = 25 \text{ rad/s.}$$

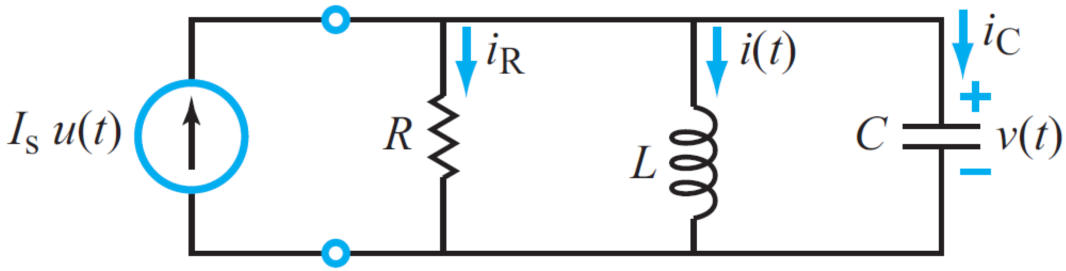
Therefore, the circuit is overdamped, so we should use the overdamped solutions

Parallel RLC Circuit

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s$$

$$v = L\frac{di}{dt}$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$



Same form of diff. equation as series RLC

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^{\,2}}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Overdamped ($\alpha > \omega_0$)

$$i(t) = i(\infty) + A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Critically Damped ($\alpha = \omega_0$)

$$i(t) = i(\infty) + (B_1 + B_2 t)e^{-\alpha t}$$

Underdamped ($\alpha < \omega_0$)

$$i(t) = i(\infty) + e^{-\alpha t}\left(D_1 \cos \omega_d t + D_2 \sin \omega_d t\right)$$

Oscillators

If R=0 in a series or parallel RLC circuit, the circuit becomes an oscillator

Exercise 6-14: Develop an expression for $i_C(t)$ in the circuit of Fig. E6.14 for $t \geq 0$.

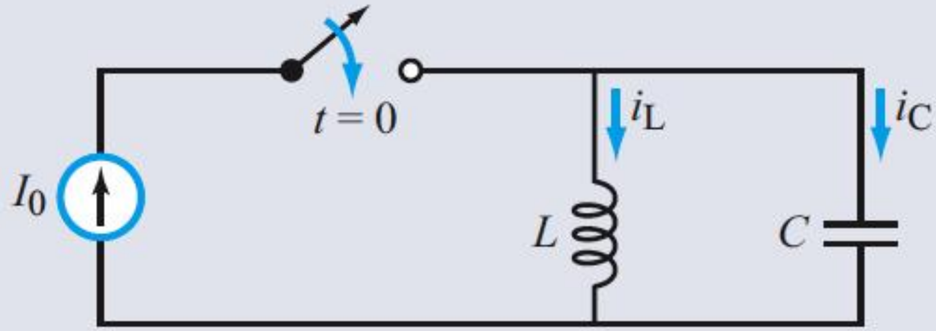


Figure E6.14

Answer: $i_C(t) = I_0 \cos \omega_0 t$ with $\omega_0 = 1/\sqrt{LC}$. This is an LC ***oscillator*** circuit in which dc energy provided by the current source is converted into ac energy in the LC circuit. (See)