# Hackerspace

What my friends think we do


What my mom thinks we do

What society thinks we do

What our neighbors think we do

What I think we do

What we actually do

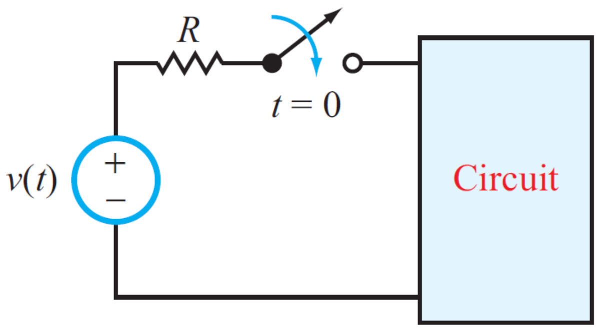
# *EE 40– RL and RC Circuits*

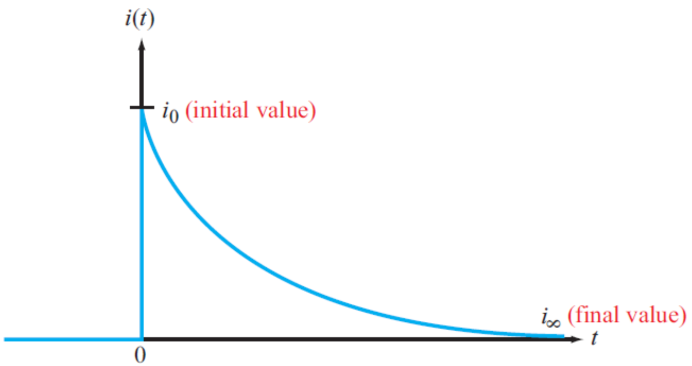
Reading Material:

Chapter 5

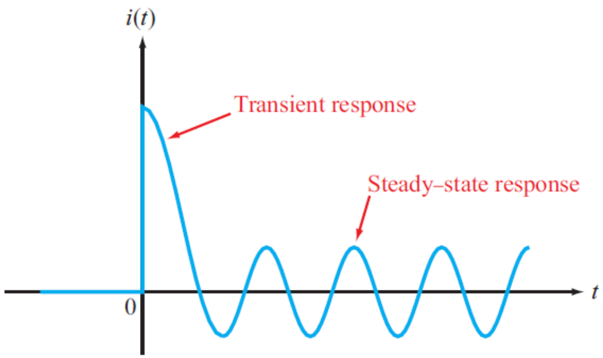
# Transient Response

The transient response represents the initial reaction immediately after a sudden change, such as closing or opening a switch to connect a source to the circuit.



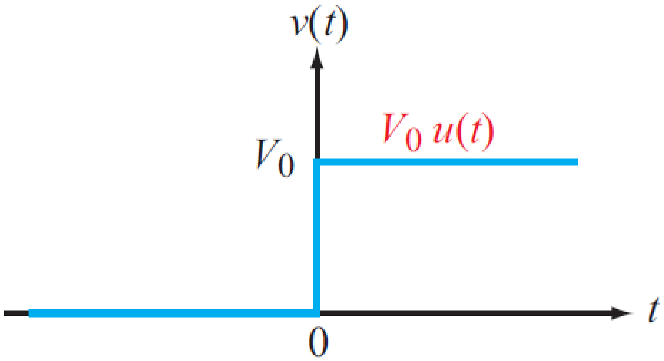

(a) dc transient response



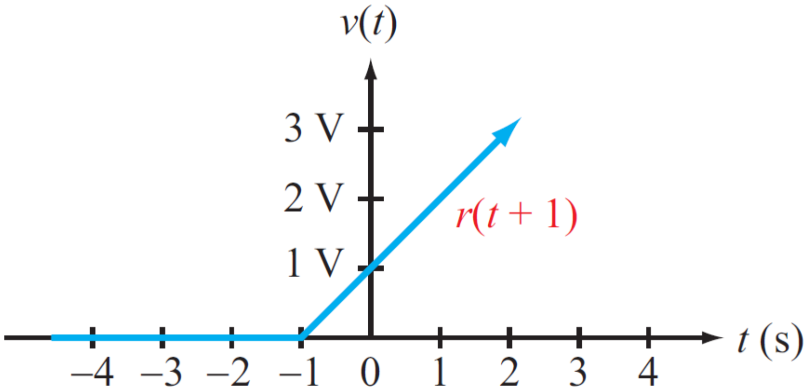

(b) Combined response to ac excitation

# Some Non-Periodic Waveforms

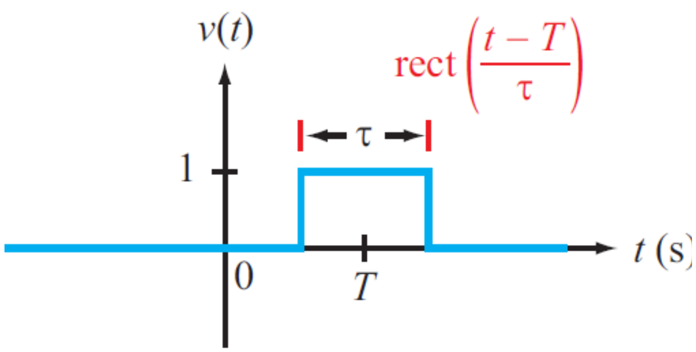
## Step Function



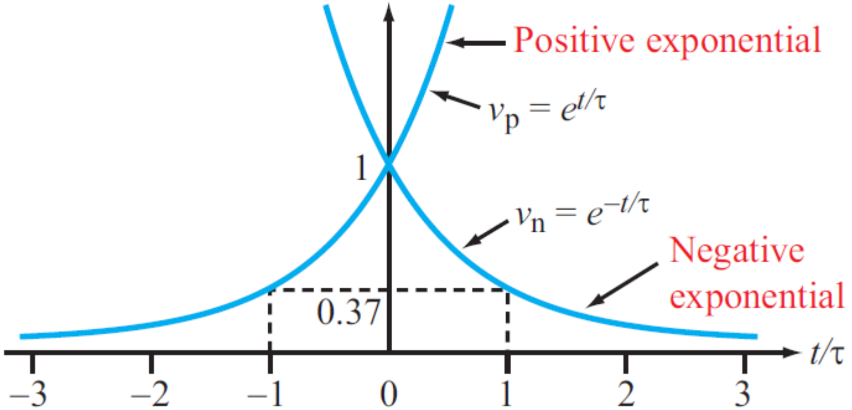

## Ramp Function




## Square Pulse



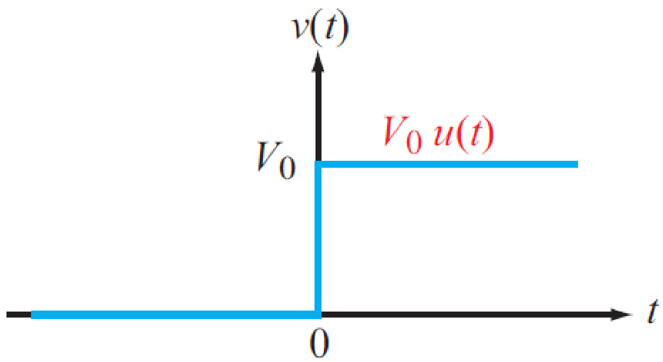

## Exponential



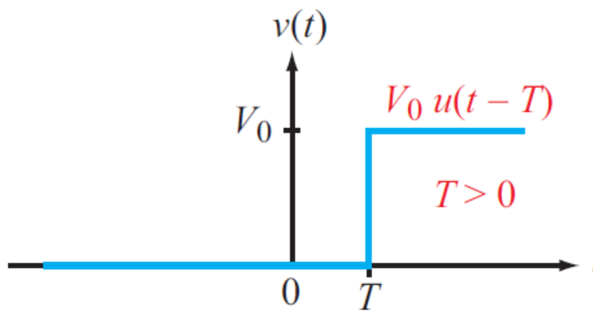
# Non-Periodic Waveforms: Step Function

## Step Functions

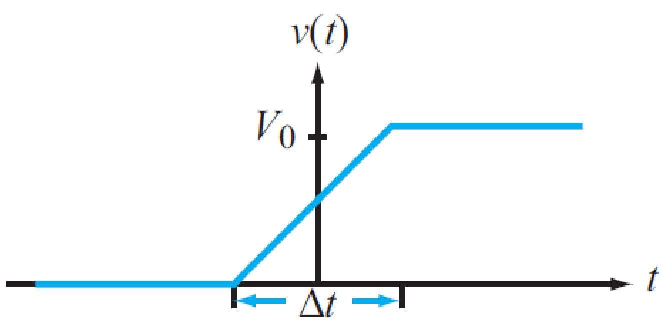
$$v(t) = V_0\, u(t - T) = \begin{cases} 0 & \text{for } t < T, \\ V_0 & \text{for } t > T. \end{cases}$$



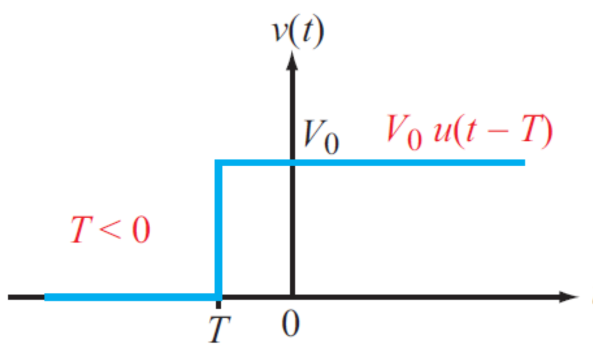

(a) Ideal step function




(c) Time-shifted step with $T > 0$

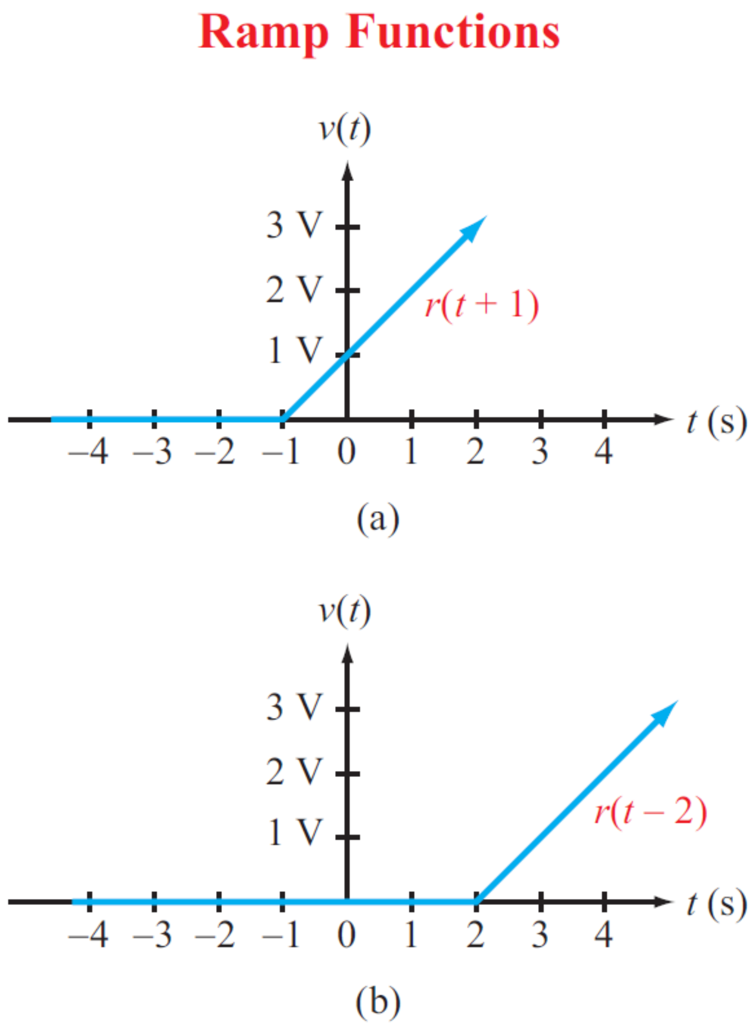



(b) Realistic step function



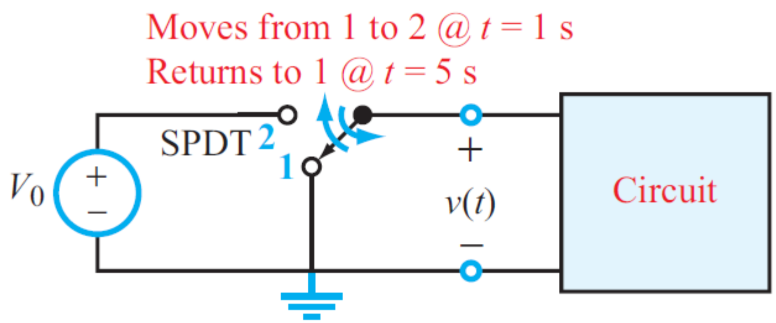

(d) Time-shifted step with $T < 0$

# Non-Periodic Waveforms: Ramp Function

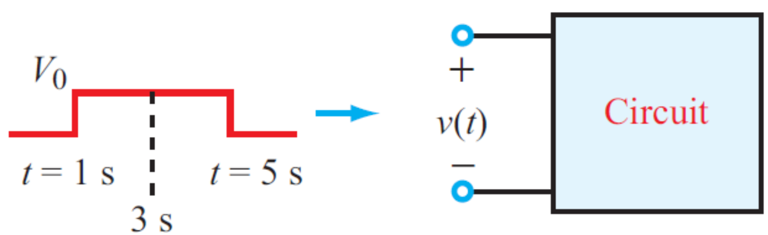



$$r(t-T) = \begin{cases} 0 & \text{for } t \leq T, \\ (t-T) & \text{for } t \geq T. \end{cases}$$

# Non-Periodic Waveforms: Pulses



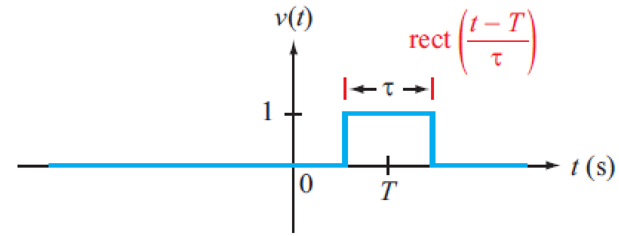

(a) Circuit with input switch



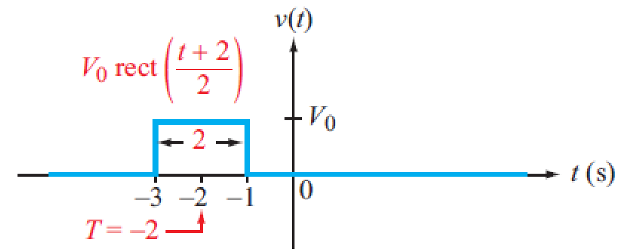

(b) Equivalent input pulse

$$\text{rect}\left(\frac{t-3}{4}\right)$$

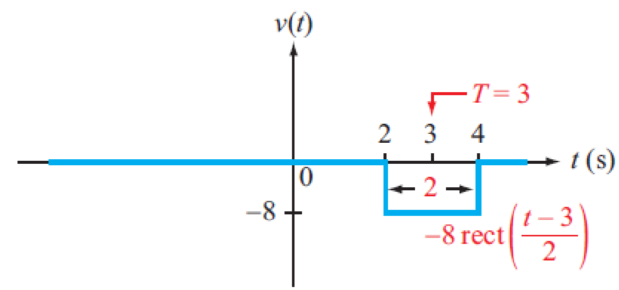
$$\text{rect}\left(\frac{t-T}{\tau}\right) = \begin{cases} 0 & \text{for } t < (T - \tau/2), \\ 1 & \text{for } (T - \tau/2) \le t \le (T + \tau/2), \\ 0 & \text{for } t > (T + \tau/2). \end{cases}$$




(a)

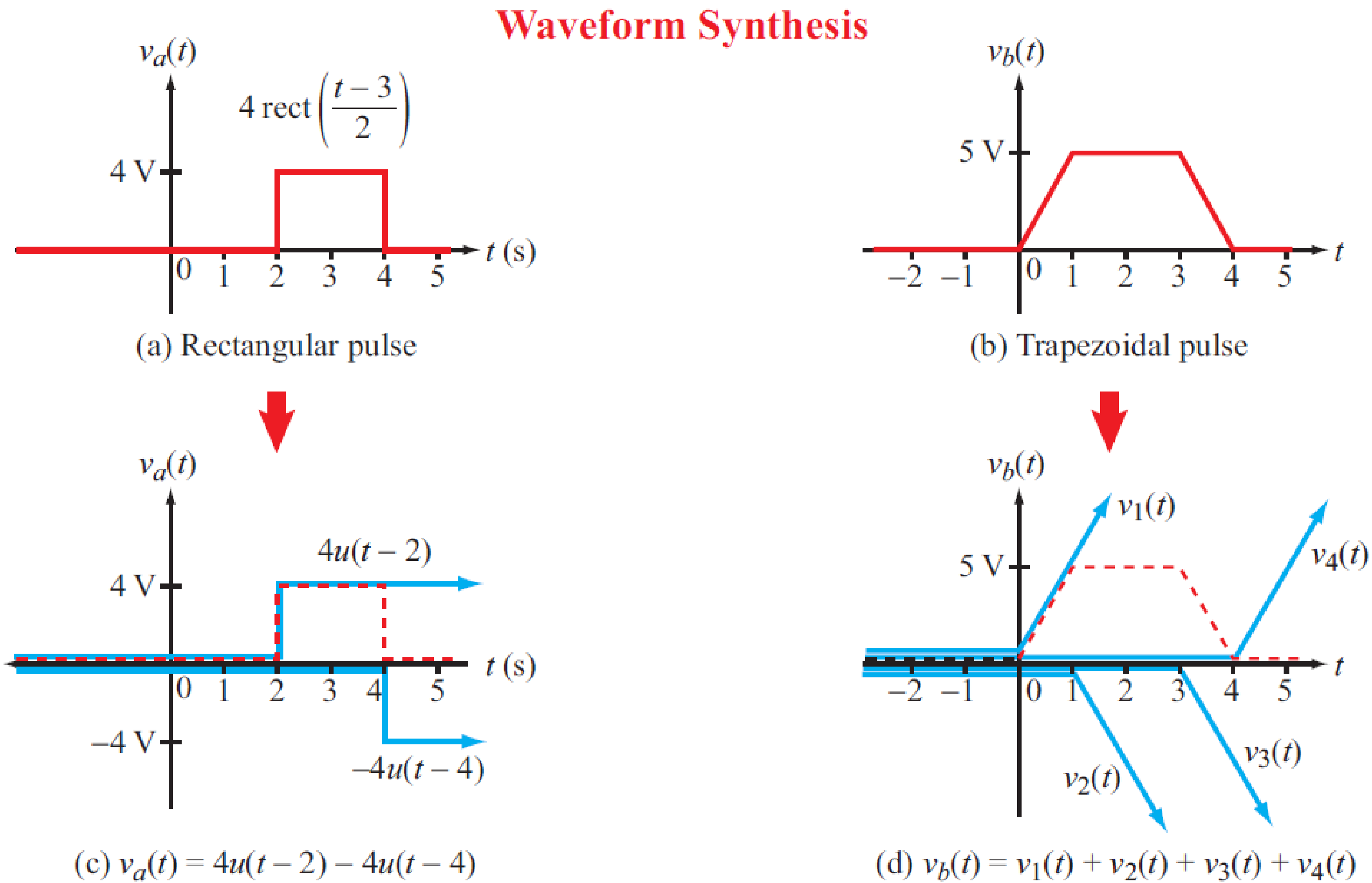



(b)

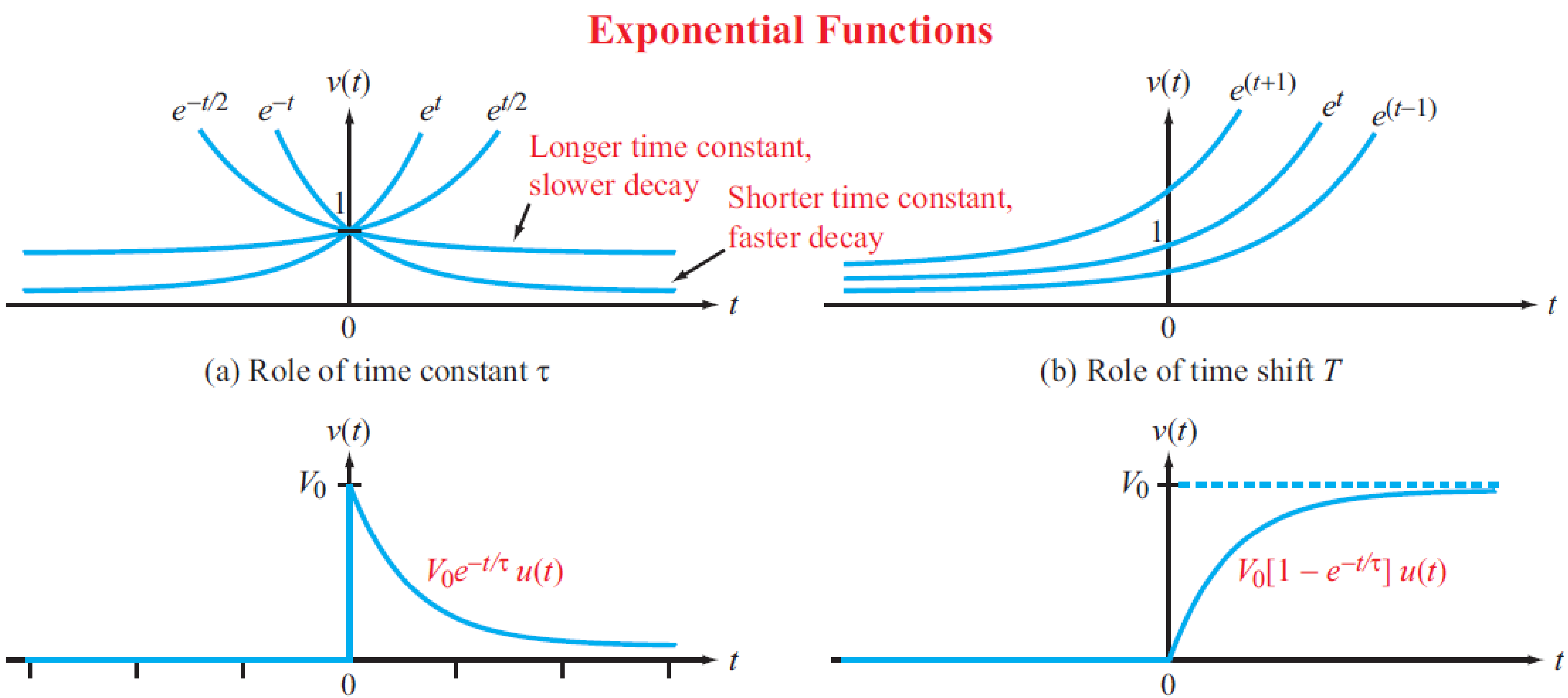



(c)

# Waveform Synthesis



# Non-Periodic Waveforms: Exponentials



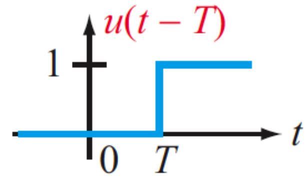
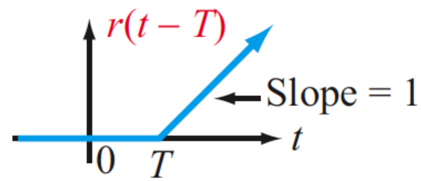
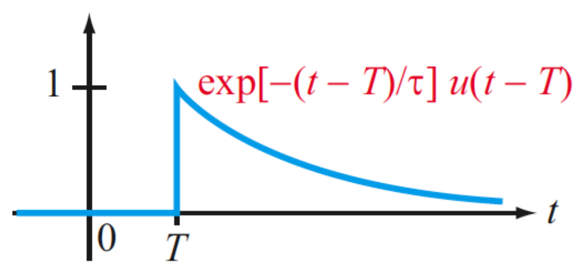

(a) Role of time constant $\tau$

(b) Role of time shift $T$

(c)

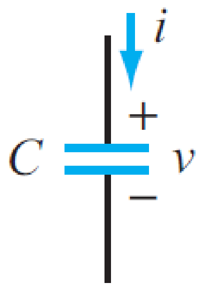
(d)

# Some common non-periodic functions

| Waveform | Expression | General Shape |
|---|---|---|
| **Step** | $u(t-T) = \begin{cases} 0 & \text{for } t < T \\ 1 & \text{for } t > T \end{cases}$ |  |
| **Ramp** | $r(t-T) = (t-T)\, u(t-T)$ |  |
| **Rectangle** | $\text{rect}\left(\frac{t-T}{\tau}\right) = u(t-T_1) - u(t-T_2)$<br>$T_1 = T - \frac{\tau}{2}\,; \quad T_2 = T + \frac{\tau}{2}$ | |
| **Exponential** | $\exp[-(t-T)/\tau]\, u(t-T)$ |  |

# Capacitors

Passive element that stores energy in electric field

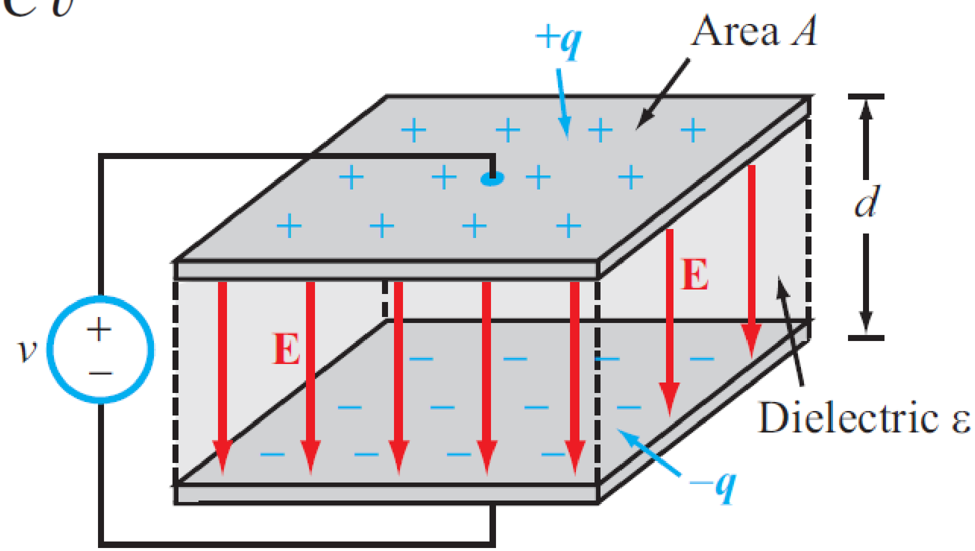



$$i = C\frac{dv}{dt} \qquad q = Cv$$

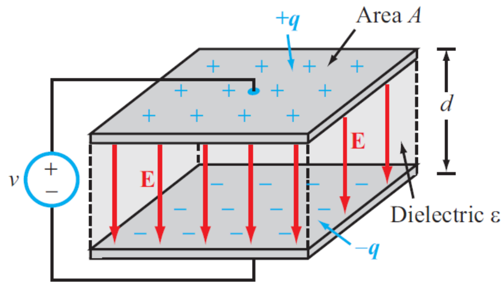
$$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$

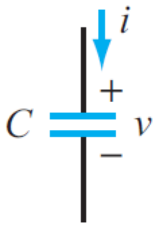
- For DC, capacitor looks like open circuit
- Voltage on capacitor must be continuous (no abrupt change)

Parallel plate capacitor



# Conceptual basis for Q-V relationships

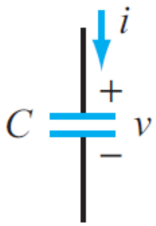



- Since like charges repel, we have to do work to force the charges to accumulate on the conductors. In fact, the smaller the conductor, the more work that we have to do.
- By definition, the potential $v$ across the capacitor represents the amount of work required to move a unit of charge onto the capacitor plates. This is the work done by the current source.
- For linear media, we observe that as we push more charge onto the capacitor with a fixed current, it's voltage increases linearity because it's more and more difficult to do it (like charges repel).

$$V \propto Q$$

# Definition of Capacitance




- The symbol for a capacitor is shown above. Sometimes a $+$ label indicates that a capacitor should only be charged in a given direction. Most capacitors, though, are symmetric and positive or negative charge can be applied to either terminals.
- The proportionality constant between the charge and voltage is defined as the capacitance of the two terminal element

$$q = Cv$$

- The units of capacitance are given charge over voltage, or Farads (in honor of Michael Faraday)

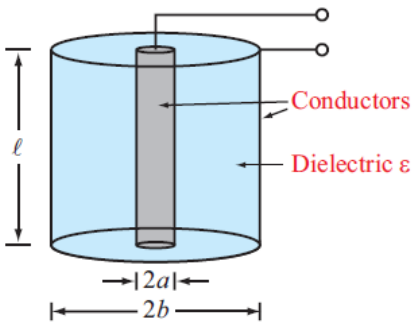
$$[C] = \frac{[q]}{[v]} = \frac{\mathrm{C}}{\mathrm{V}} = \mathrm{F}$$

- Notice that a physically larger conductor has a larger capacitance $C_1 > C_2$, because it has more surface area for the charges to reside. The average distance between the like charges determines how much energy you have to provide to push additional charges onto the capacitor.

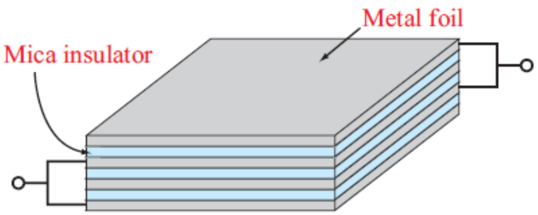
# Capacitor Analogy

- Imagine a tank of water where we pump water into the tank from the bottom. As we initially pump water, there is no water and it takes virtually no work. But as the tank fills up, it takes more and more work since we the liquid obtains gravitational potential energy.
- A smaller tank requires more work (it has less capacity) because the liquid column gets higher and higher.
- Notice that we can always recover the work by emptying the tank.

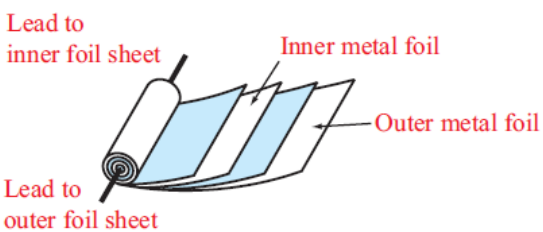
# Various types of capacitors



(a) Coaxial capacitor



(b) Mica capacitor



(c) Plastic foil capacitor

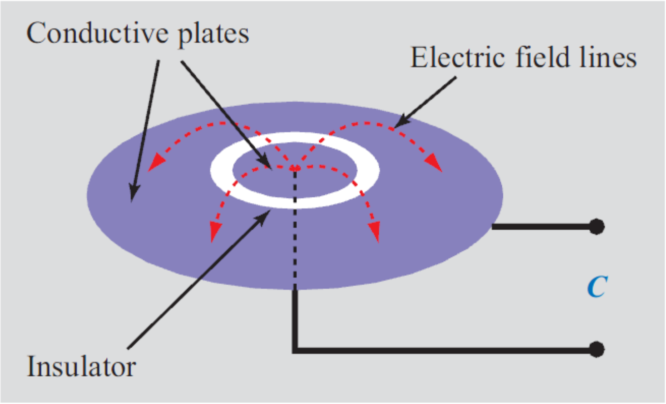



**Figure TF9-4:** Concentric-plate capacitor.

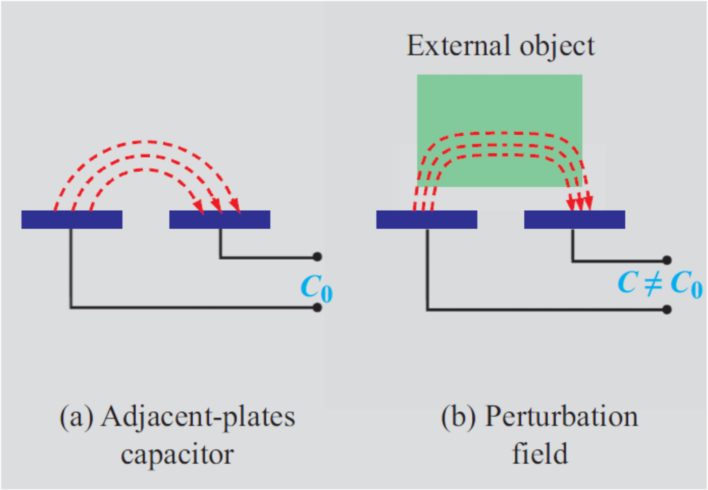



**Figure TF9-5:** (a) Adjacent-plates capacitor; (b) perturbation field.

# *Energy stored in a capacitor*

- The incremental of amount of work done to move a charge $dq$ onto the plates of the capacitor is given by

$$dE = v dq$$

- where $v$ is the potential energy of the capacitor in a given state. Since $q = Cv$, we have $dq = C dv$, or

$$dE = Cv dv$$

- If we now integrate from zero potential (no charge) to some final voltage

$$E = \int_0^{V_0} Cv dv = C \frac{v^2}{2} \Big|_0^{V_0} = \frac{1}{2} C V_0^2$$

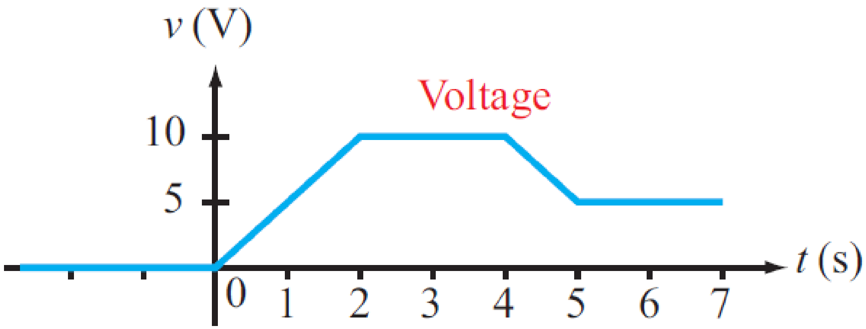
- This is the energy stored in the capacitor. Just like the water tank, it's stored as potential energy that we can later recover.

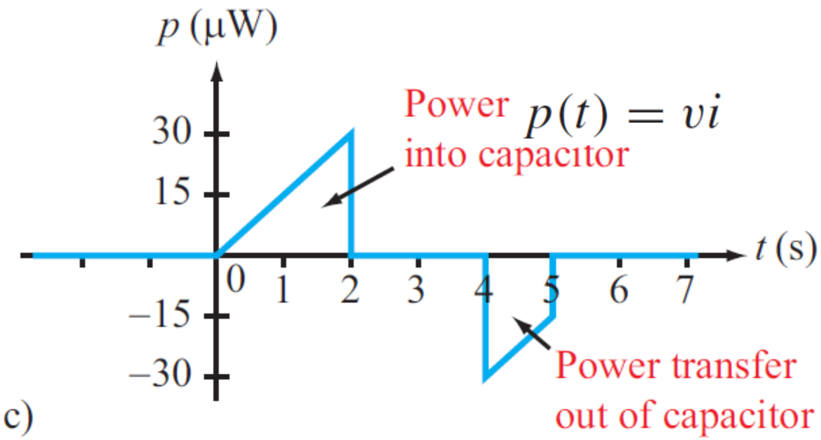
# Capacitor Current

- A steady DC current into a capacitor means that the voltage ramps linearly. In theory it would ramp to infinity but in practice dielectric breakdown (arcing) would short out the capacitor.
- Now consider an AC current. Since a steady positive current increases the voltage linearity, a steady negative current does the opposite. Thus if we pass a time varying current that has positive and negative polarity, the voltage across the capacitor would increase and decrease. For a fixed current, we can infer the voltage.

# Capacitor Response: Given v(t), determine i(t), p(t), and w(t)

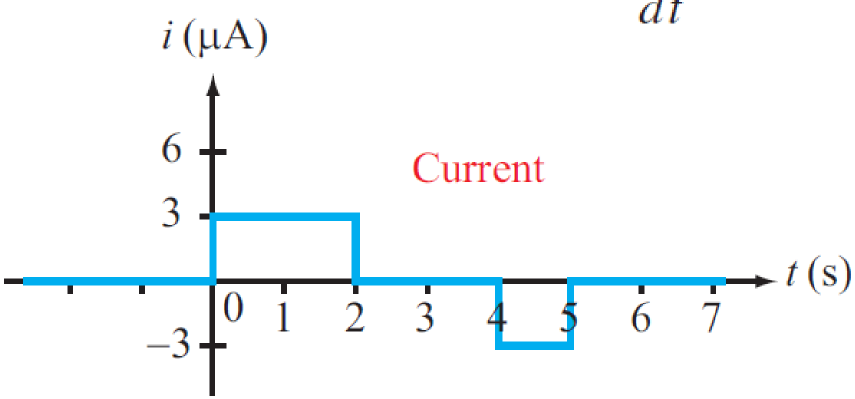
$C = 0.6\text{-}\mu\text{F}$

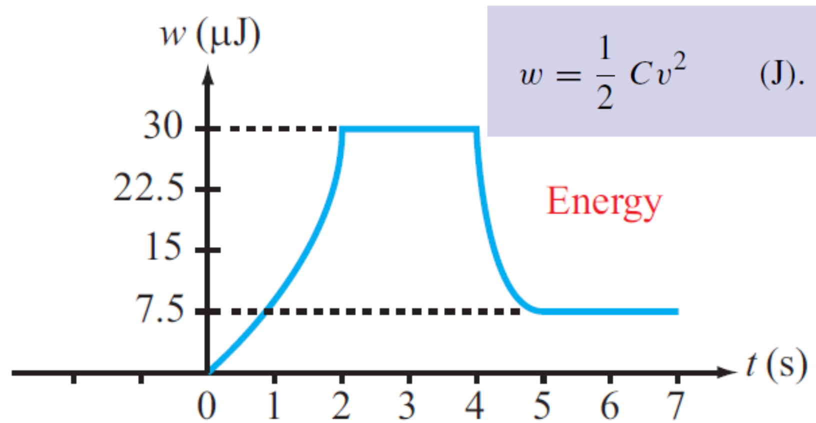


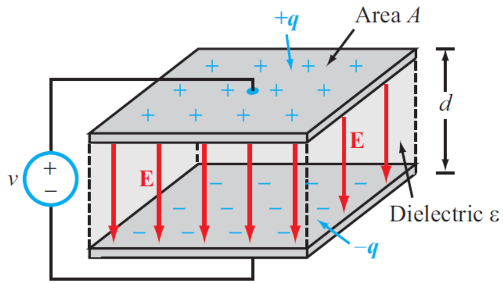

c)

$$i(t) = C\,\frac{dv}{dt}$$



# Parallel plate capacitor




- From basic physics it's easy to show that the capacitance of a parallel plate structure is given by

$$C = \frac{\epsilon A}{d}$$

- where $A$ is the plate area, $\epsilon$ is the permittivity of the dielectric, and $d$ is the gap spacing.
- The permittivity of free space is $eps_0 = 8.854 \times 10^{-12}\,\mathrm{F/m}$. Most materials have a higher permittivity which is captured by the unitless relative permittivity $\epsilon_r = \epsilon/\epsilon_0$, with typical materials $\epsilon \sim 1 - 10$. For instance, air is mostly empty space and so $\epsilon_r \approx 1$.
- Some materials, such as water, have polar molecules that align when an electric field is applied. Thus the dielectric constant is very large.

# Capacitor I-V Relationship

- We can re-write the capacitor "I-V" relationship as

$$\frac{dv}{dt} = \frac{i}{C}$$

- If we integrate the above relation, we have

$$\int_{t_0}^{t} \frac{dv}{d\tau} d\tau = \int_{t_0}^{t} \frac{i}{C} d\tau$$

$$v(t) = v(t_0) + \frac{1}{C} \int_{t_0}^{t} i(\tau) d\tau$$

- The integral of current over time is nothing but the *net charge* flow into the capacitor

$$Q = \int_{t_0}^{t} i(\tau) d\tau$$

- Let's break this down term-by-term

$$\underbrace{v(t)}_{\text{voltage now}} = \underbrace{v(t_0)}_{\text{voltage then}} + \underbrace{Q/C}_{\text{net charge flow over capacitance}}$$

# Driving with AC

- If we drive the capacitor with an AC sinusoidal voltage, we can calculate the current

$$v(t) = V_0 \cos \omega_0 t$$

$$i(t) = C\frac{dv}{dt} = -CV_0\omega_0 \sin \omega_0 t$$

- Current is out of phase with the voltage. It is exactly 90° out of phase, which is very important! The power flow onto the capacitor is given by

$$p(t) = v(t)i(t) = -CV_0^2\omega_0 \cos \omega_0 t \sin \omega_0 t$$

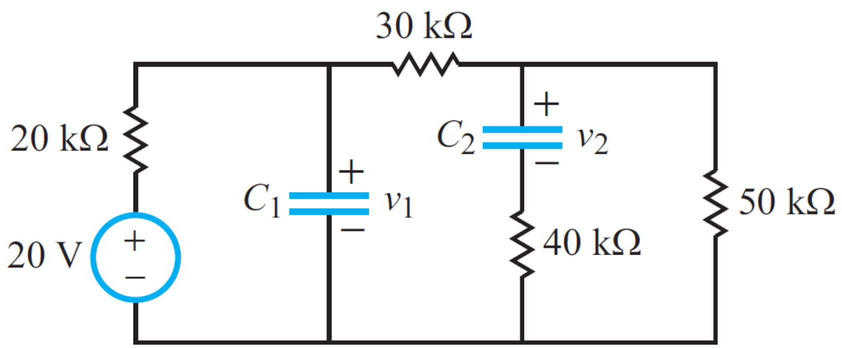
$$= -\frac{CV_0^2}{2}\omega_0 \sin 2\omega_0 t$$

- The instantaneous power flow switches signs twice per cycle, as in each cycle energy is first delivered onto the capacitor, but then the energy is returned back to the source. In other words, there is no net energy flow into the capacitor

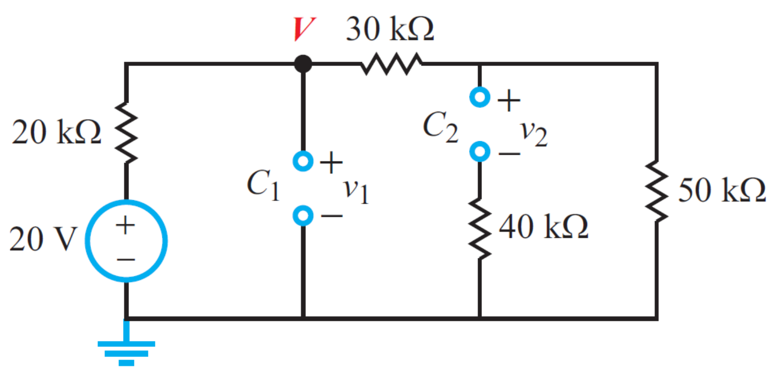
$$\int_0^T p(\tau)d\tau = -\frac{CV_0^2}{2}\omega_0 \int_0^T \sin 2\omega_0 \tau d\tau \equiv 0$$

# RC Circuits at dc

- At dc no currents flow through capacitors: open circuits




$$\frac{V-20}{20\times 10^3}+\frac{V}{(30+50)\times 10^3}=0,$$



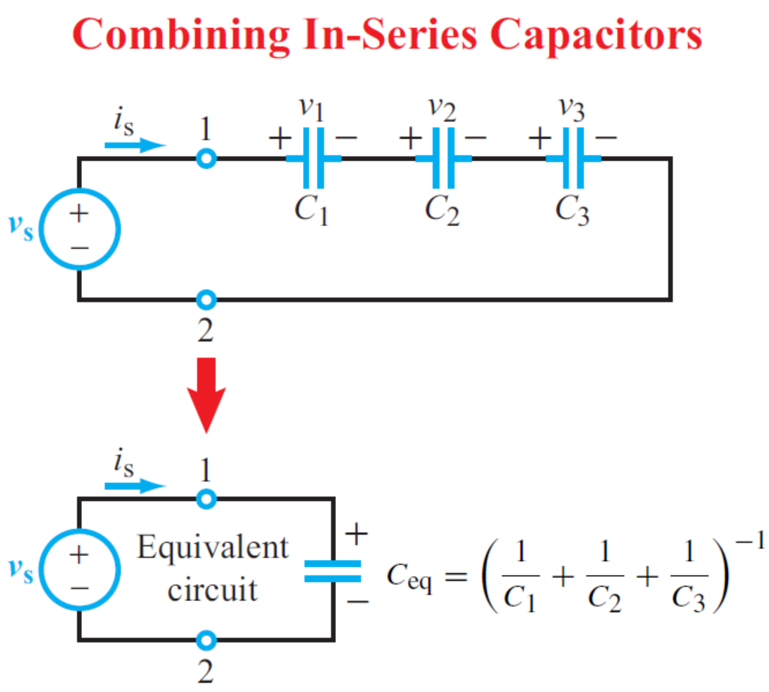

which gives $V=16$ V. Hence,

$$v_1=V=16\text{ V}.$$

Through voltage division, $v_2$ across the 50-kΩ resistor is given by

$$v_2=\frac{V\times 50\text{k}}{(30+50)\text{k}}=\frac{16\times 50}{80}=10\text{ V}.$$

# Capacitors in Series




Use KVL, current same through each capacitor

$$i_s = C_1 \frac{dv_1}{dt} = C_2 \frac{dv_2}{dt} = C_3 \frac{dv_3}{dt}.$$

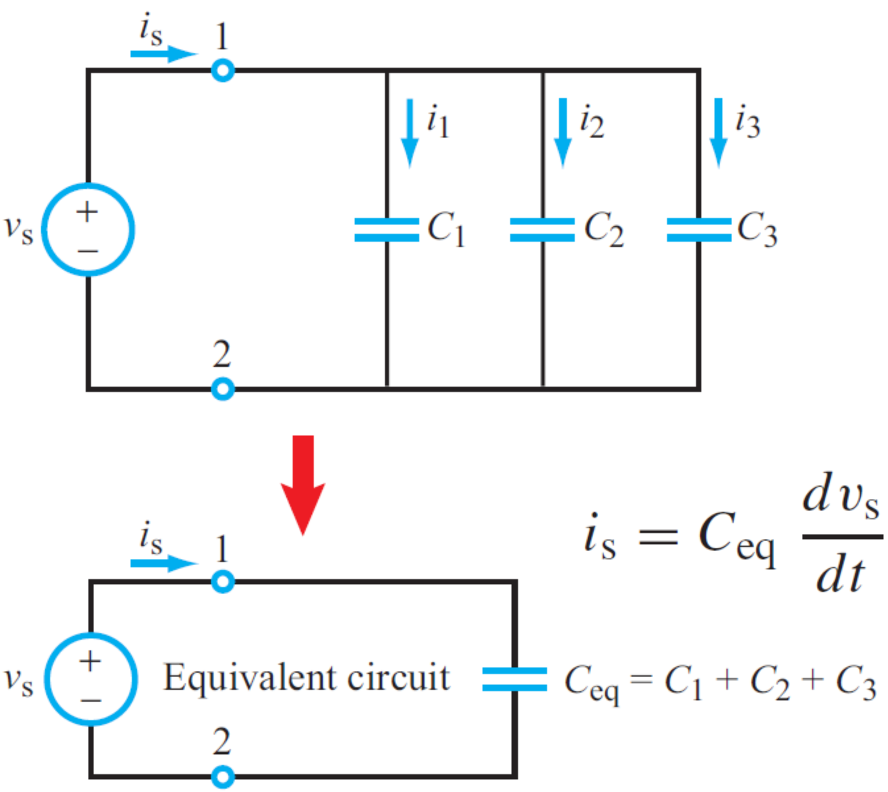
Also,

$$\begin{aligned} i_s &= C_{eq} \frac{dv_s}{dt} \\ &= C_{eq} \left( \frac{dv_1}{dt} + \frac{dv_2}{dt} + \frac{dv_3}{dt} \right) \\ &= C_{eq} \left( \frac{i_s}{C_1} + \frac{i_s}{C_2} + \frac{i_s}{C_3} \right), \end{aligned}$$

which leads to

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

# Capacitors in Parallel




$$i_s = C_{eq} \frac{dv_s}{dt}$$

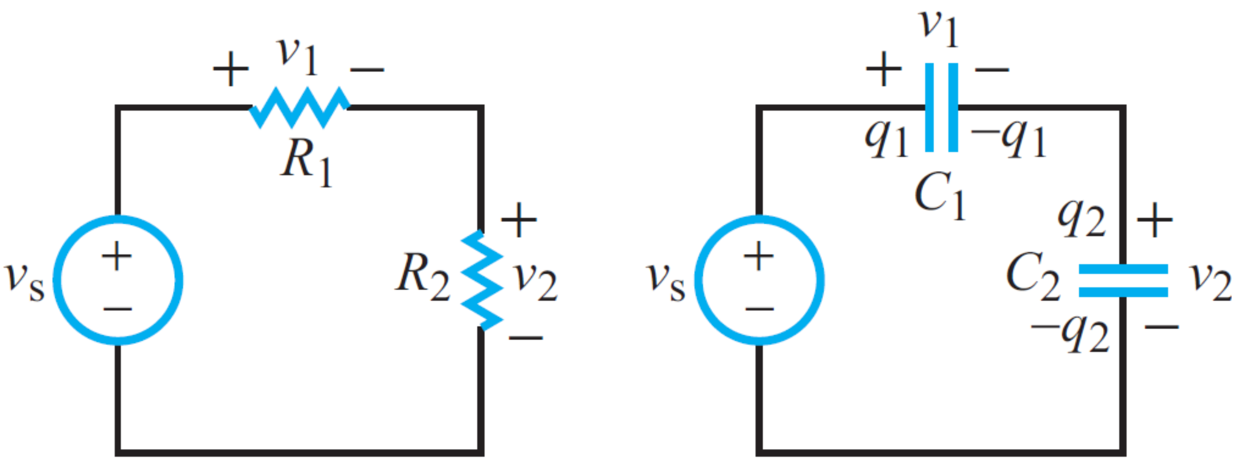
Use KCL, voltage same across each capacitor

$$i_s = i_1 + i_2 + i_3$$

$$= C_1 \frac{dv_s}{dt} + C_2 \frac{dv_s}{dt} + C_3 \frac{dv_s}{dt}$$

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N}$$

# *Voltage Division*




(a) $v_1 = \left(\dfrac{R_1}{R_1 + R_2}\right) v_s$

$v_2 = \left(\dfrac{R_2}{R_1 + R_2}\right) v_s$

(b) $v_1 = \left(\dfrac{C_2}{C_1 + C_2}\right) v_s$

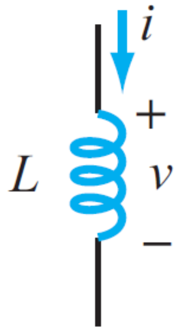
$v_2 = \left(\dfrac{C_1}{C_1 + C_2}\right) v_s$

# Inductance

- Inductance is a fundamental property of circuits, much like capacitance. While capacitors relate charges and potential, inductance relates currents and magnetic flux.
- In fact there is an important duality between inductance and capacitance that you'll quickly recognize. You can simply interchange the role of current/voltage in the circuit equations and you'll go from capacitance to inductance! This duality is only broken due to the absence of magnetic charge.
- Inductance is related to the magnetic field and flux generated around any current carrying conductor, including wires. The energy of the voltage source goes into building a magnetic field, and the energy stays in the field.
- The magnetic flux is proportional to the current (for linear media)

# Inductors

Passive element that stores energy in magnetic field



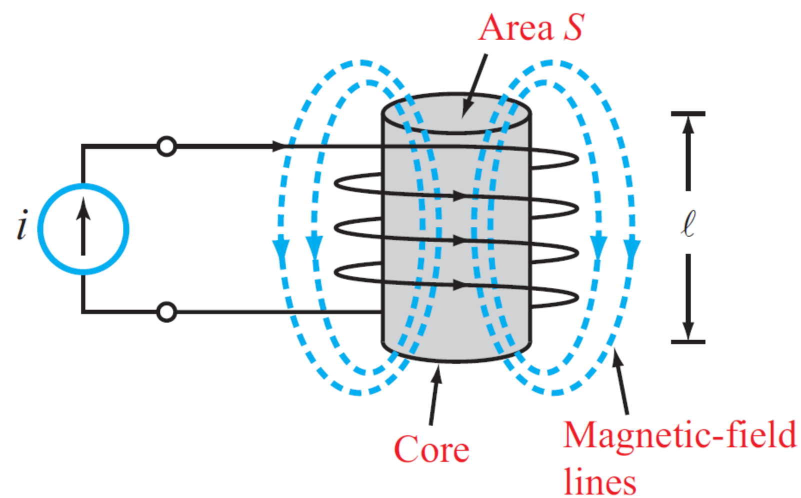

$$v = L\frac{di}{dt}$$

$$i = \frac{1}{L}\int_{t_o}^{t} v(t)\,dt + i(t_0)$$

- At dc, inductor looks like a short circuit
- Current through inductor must be continuous (no abrupt change)

$$w = \frac{1}{2}Li^2 \qquad \text{(J)},$$

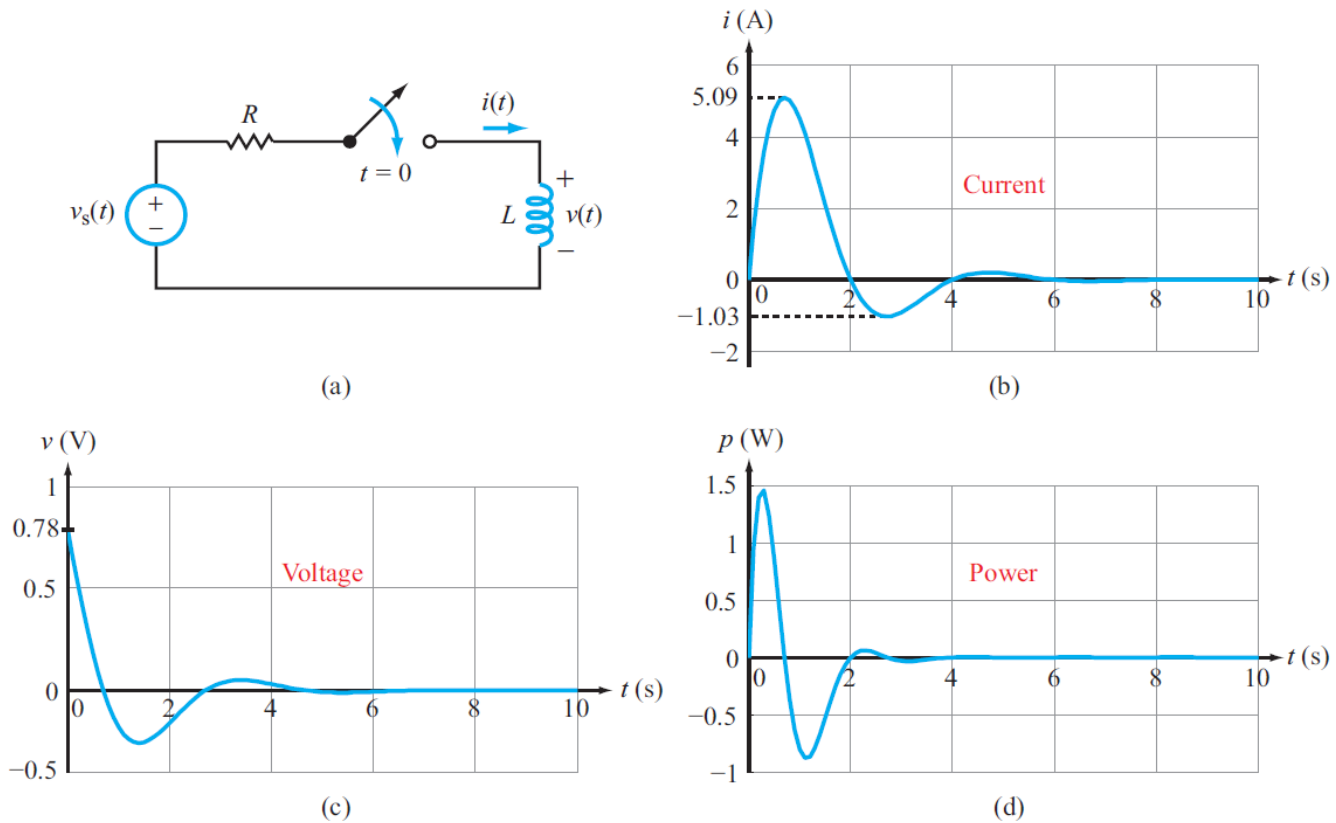
Solenoid Wound Inductor




$$L = \frac{N^2 \mu A}{l}$$

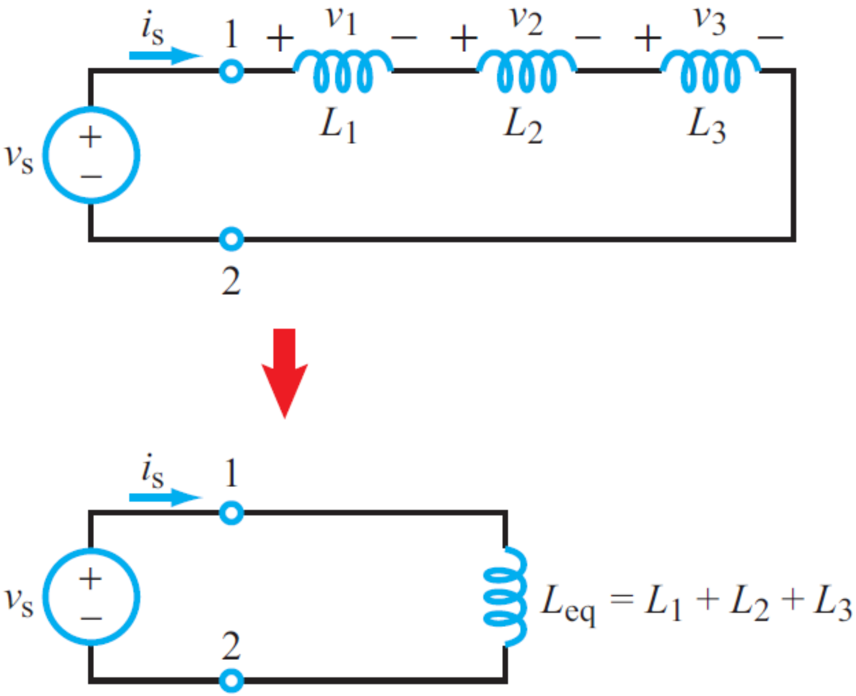
# Inductor Response

$$i(t) = 10e^{-0.8t}\sin(\pi t/2)\text{ A}, \qquad (\text{for } t \geq 0)$$




$$\begin{aligned}
v(t) &= L\,\frac{di}{dt} \\
&= L\,\frac{d}{dt}[10e^{-0.8t}\sin(\pi t/2)] \\
&= 50 \times 10^{-3} \cdot [-8e^{-0.8t}\sin(\pi t/2) + 5\pi e^{-0.8t}\cos(\pi t/2)] \\
&= [-0.4\sin(\pi t/2) + 0.25\pi\cos(\pi t/2)]e^{-0.8t}\text{ V.}
\end{aligned}$$

# Inductors in Series

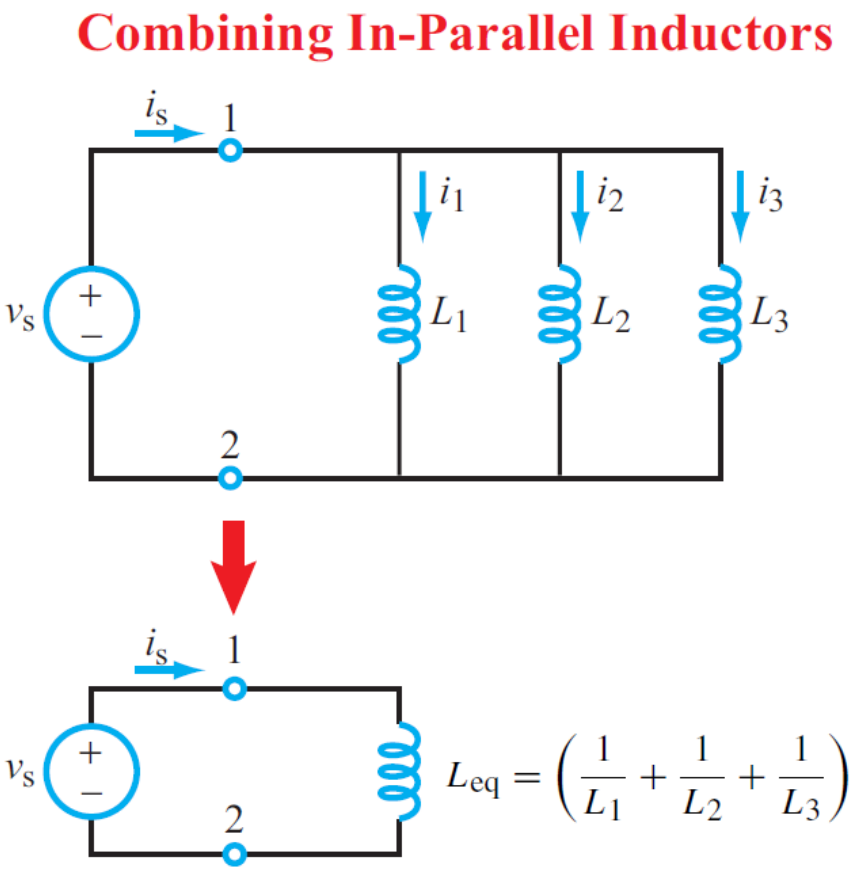



Use KVL, current is same through all inductors

$$v_s = v_1 + v_2 + v_3$$
$$= L_1 \frac{di_s}{dt} + L_2 \frac{di_s}{dt} + L_3 \frac{di_s}{dt}$$
$$= (L_1 + L_2 + L_3) \frac{di_s}{dt},$$
$$v_s = L_{eq} \frac{di_s}{dt}$$

$$\rightarrow \quad L_{eq} = L_1 + L_2 + L_3$$

# Inductors in Parallel



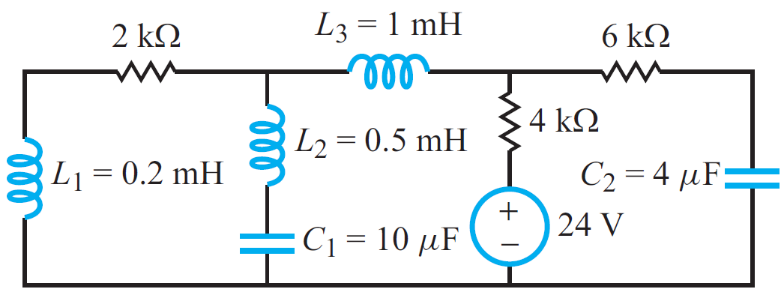

$$L_{eq} = \left(\frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}\right)^{-1}$$

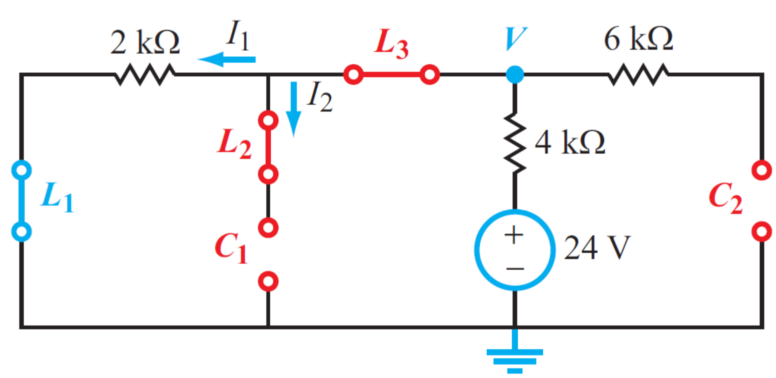
Voltage is same across all inductors

Inductors add together in the same way resistors do

# RL Circuits at dc

- At dc no voltage across inductors: short circuit




$$I_1 = \frac{24}{(2+4)\text{k}} = 4 \text{ mA},$$

and node voltage $V$ is

$$V = 24 - (4 \times 10^{-3} \times 4 \times 10^{3}) = 8 \text{ V}.$$

Hence, the amounts of energy stored in $C_1$, $C_2$, $L_1$, $L_2$, and $L_3$ are

$$C_1: \quad W = \frac{1}{2}C_1V^2 = \frac{1}{2} \times 10^{-5} \times 64 = 0.32 \text{ mJ},$$

$$C_2: \quad W = \frac{1}{2}C_2V^2 = \frac{1}{2} \times 4 \times 10^{-6} \times 64 = 0.128 \text{ mJ},$$

# Summary of properties

| Property | $R$ | $L$ | $C$ |
|---|---|---|---|
| $i$–$v$ relation | $i = \frac{v}{R}$ | $i = \frac{1}{L}\int_{t_0}^{t} v\, dt + i(t_0)$ | $i = C\frac{dv}{dt}$ |
| $v$-$i$ relation | $v = iR$ | $v = L\frac{di}{dt}$ | $v = \frac{1}{C}\int_{t_0}^{t} i\, dt + v(t_0)$ |
| $p$ (power transfer in) | $p = i^2 R$ | $p = Li\frac{di}{dt}$ | $p = Cv\frac{dv}{dt}$ |
| $w$ (stored energy) | 0 | $w = \frac{1}{2}Li^2$ | $w = \frac{1}{2}Cv^2$ |
| Series combination | $R_{eq} = R_1 + R_2$ | $L_{eq} = L_1 + L_2$ | $C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$ |
| Parallel combination | $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$ | $L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$ | $C_{eq} = C_1 + C_2$ |
| dc behavior | no change | short circuit | open circuit |
| Can $v$ change instantaneously? | yes | yes | no |
| Can $i$ change instantaneously? | yes | no | yes |

# Response Terminology

## Source dependence

Natural response – response in absence of sources

Forced response – response due to external source

Complete response = Natural + Forced

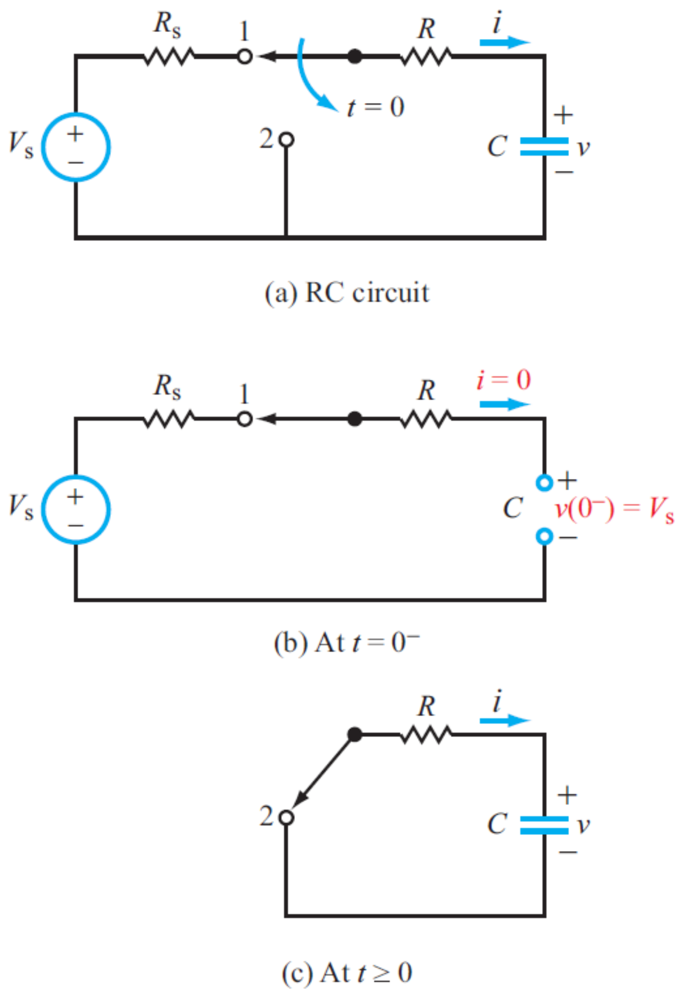
## Time dependence

Transient response – time-varying response (temporary)

Steady state response – time-independent or periodic (permanent)

Complete response = Transient + Steady State

# Natural Response of Charged Capacitor




(a) $t = 0^-$ is the instant just before the switch is moved from terminal 1 to terminal 2

(b) $t = 0$ is the instant just after it was moved; $t = 0$ is synonymous with $t = 0^+$

since the voltage across the capacitor cannot change instantaneously, it follows that

$$v(0) = v(0^-) = V_s.$$

For $t \geq 0$, application of KVL to the loop in Fig. 5-28(c) gives

$$Ri + v = 0 \qquad (\text{for } t \geq 0), \tag{5.68}$$

where $i$ is the current through and $v$ is the voltage across the capacitor. Since $i = C\,dv/dt$,

$$RC\,\frac{dv}{dt} + v = 0. \tag{5.69}$$

Upon dividing both terms by RC, Eq. (5.69) takes the form

$$\boxed{\frac{dv}{dt} + av = 0 \qquad (\text{source-free}),} \tag{5.70}$$

where

$$a = \frac{1}{RC}. \tag{5.71}$$

# Solution to the differential equation

The standard procedure for solving Eq. (5.70) starts by multiplying both sides by $e^{at}$,

$$\frac{dv}{dt}\,e^{at} + ave^{at} = 0. \tag{5.72}$$

Next, we recognize that the sum of the two terms on the left-hand side is equal to the expansion of the differential of $(ve^{at})$,

$$\frac{d}{dt}(ve^{at}) = \frac{dv}{dt}\,e^{at} + ave^{at}. \tag{5.73}$$

Hence, Eq. (5.72) becomes

$$\frac{d}{dt}(ve^{at}) = 0. \tag{5.74}$$

Integrating both sides, we have

$$\int_0^t \frac{d}{dt}(ve^{at})\,dt = 0, \tag{5.75}$$

# Solution to the differential equation

Performing the integration gives

$$v e^{at}\Big|_0^t = 0$$

or

$$v(t)\, e^{at} - v(0) = 0. \qquad (5.76)$$

Solving for $v(t)$, we have

$$\begin{aligned} v(t) &= v(0)\, e^{-at}, \\ &= v(0)\, e^{-t/RC} \qquad (\text{for } t \geq 0), \end{aligned} \qquad (5.77)$$

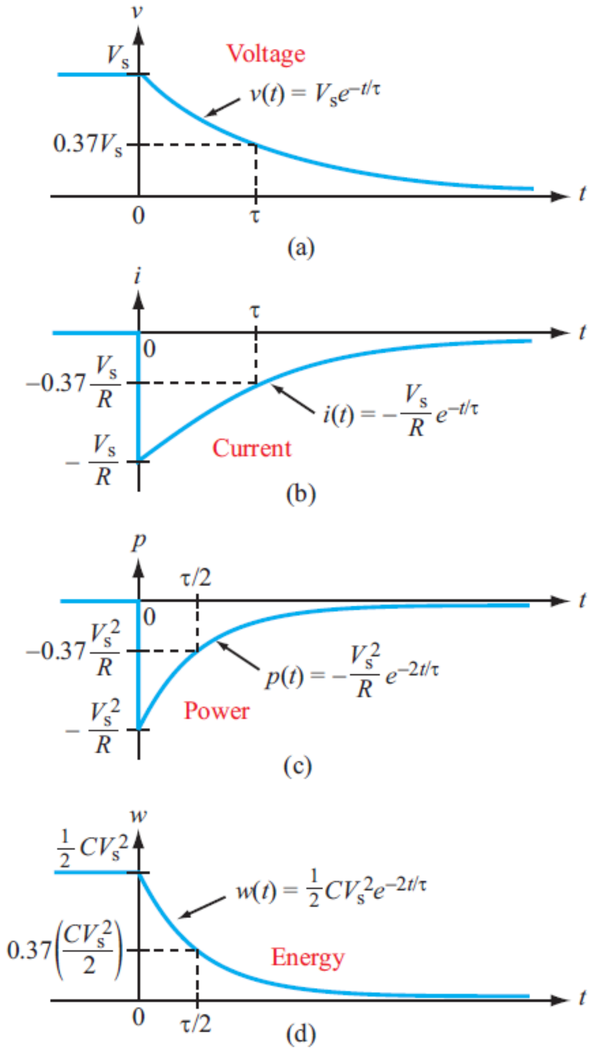
$$v(t) = v(0)\, e^{-t/\tau} \qquad (\text{natural response}),$$

with

$$\tau = RC \qquad (\text{s}),$$

*$\tau$ is called the* ***time constant of the circuit.***

# Natural Response of Charged Capacitor




$$i(t) = C\,\frac{dv}{dt} = C\,\frac{d}{dt}(V_s e^{-t/\tau})$$
$$= -C\,\frac{V_s}{\tau}\,e^{-t/\tau} \qquad (\text{for } t \geq 0),$$

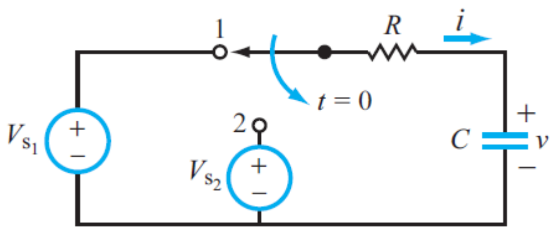
which simplifies to

$$i(t) = -\frac{V_s}{R}\,e^{-t/\tau}\,u(t) \qquad (\text{for } t \geq 0)$$

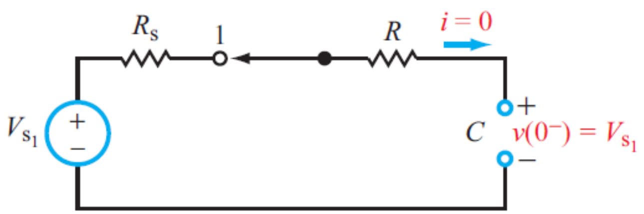
(natural response).

$$p(t) = iv = -\frac{V_s}{R}\,e^{-t/\tau} \times V_s e^{-t/\tau}$$
$$= -\frac{V_s^2}{R}\,e^{-2t/\tau} \qquad (\text{for } t \geq 0).$$

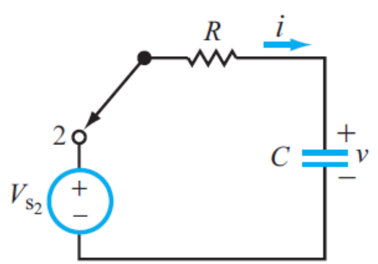
# General Response of RC Circuit




(a) RC circuit




(b) At $t = 0^-$




(c) At $t \geq 0$

$$v(0) = v(0^-) = V_{s_1}. \qquad (5.86)$$

For $t \geq 0$, the voltage equation for the loop in Fig. 5-30(c) is

$$-V_{s_2} + iR + v = 0. \qquad (5.87)$$

Upon using $i = C\, dv/dt$ and rearranging its terms, Eq. (5.87) can be written in the differential-equation form

$$\boxed{\frac{dv}{dt} + av = b,} \qquad (5.88)$$

where

$$a = \frac{1}{RC} \qquad \text{and} \qquad b = \frac{V_{s_2}}{RC}. \qquad (5.89)$$

# Solution of $\frac{dv}{dt} + av = b,$

$$\frac{d}{dt}(ve^{at}) = be^{at}.$$

Integrating both sides,

$$\int_0^t \frac{d}{dt}(ve^{at})\, dt = \int_0^t be^{at}$$

gives

$$ve^{at}\big|_0^t = \frac{b}{a}\, e^{at}\Big|_0^t .$$

Upon evaluating the functions at the two limits, we have $$v(t)\, e^{at} - v(0) = \frac{b}{a}\, e^{at} - \frac{b}{a},$$

and then solving for $v(t)$, we have

$$v(t) = v(0)\, e^{-at} + \frac{b}{a}\,(1 - e^{-at}).$$

As $t \to \infty$, $v(t)$ reduces to

$$v(\infty) = \frac{b}{a} = V_{s_2}.$$

# Solution of $\frac{dv}{dt} + av = b,$

We have:

$$v(t) = v(0)\, e^{-at} + \frac{b}{a}\,(1 - e^{-at}). \qquad v(\infty) = \frac{b}{a} = V_{s_2}.$$

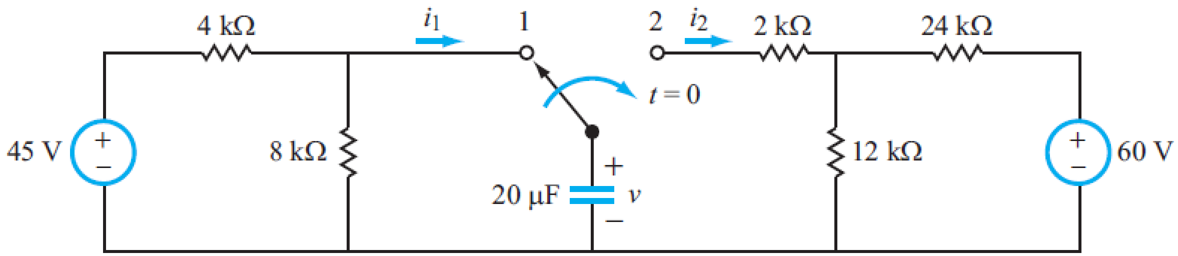
By reintroducing the time constant $\tau = RC = 1/a$ and replacing $b/a$ with $v(\infty)$, we can rewrite Eq. (5.94) in the general form:

$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau} \qquad (\text{for } t \geq 0)$$
$$(\text{switch action at } t = 0).$$

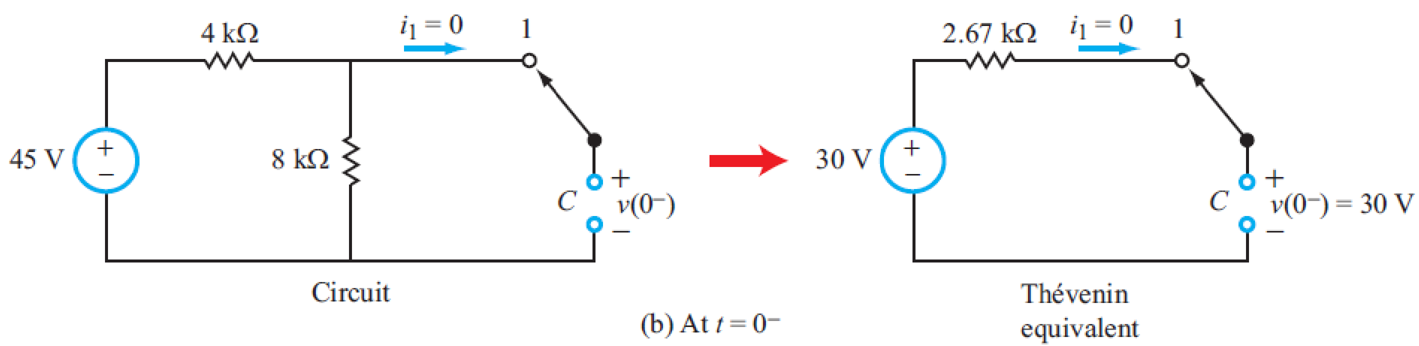
If the switch action causing the change in voltage across the capacitor occurs at time $T_0$ instead of at $t = 0$, Eq. (5.96) assumes the form

$$v(t) = v(\infty) + [v(T_0) - v(\infty)]e^{-(t-T_0)/\tau} \qquad (\text{for } t \geq T_0)$$
$$(\text{switch action at } t = T_0),$$

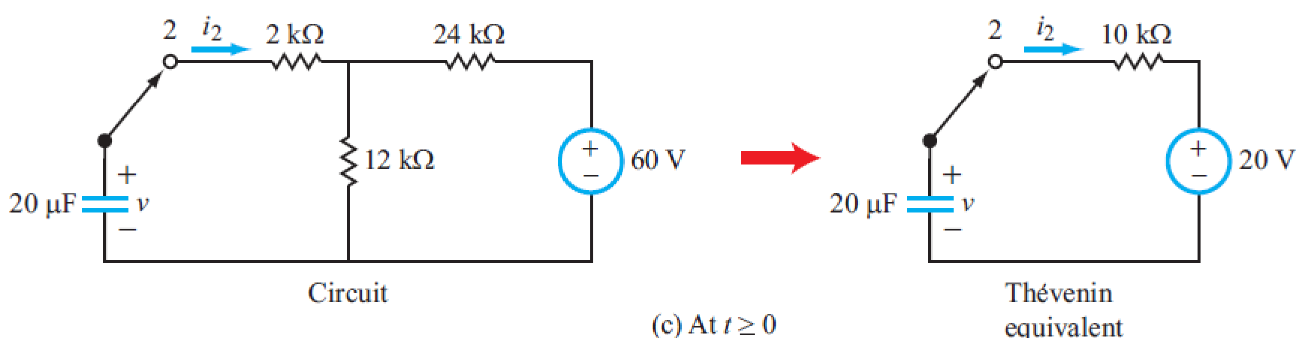
# Example: Determine Capacitor Voltage




(a) Original circuit

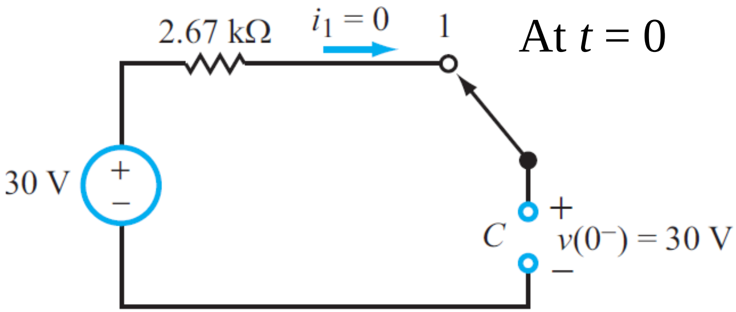



(b) At $t = 0^-$




(c) At $t \geq 0$

# Solution




At $t = 0$

Thévenin equivalent

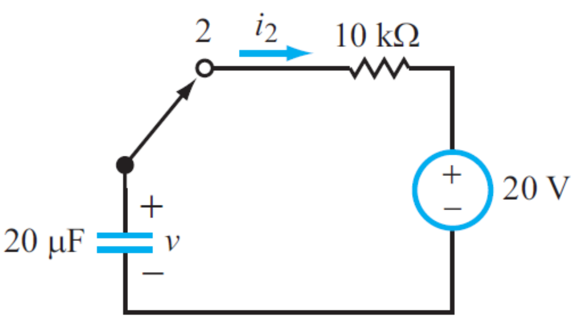
$$v(t) = v(\infty) + [v(0) - v(\infty)]e^{-t/\tau}$$

$$v(0) = v(0^-) = 30 \text{ V}$$

$$v(\infty) = \left(\frac{12\text{k}}{12\text{k} + 24\text{k}}\right) \times 60 = 20 \text{ V}$$

$$R = R_{\text{Th}} = 2 \text{ k}\Omega + 12 \text{ k}\Omega \parallel 24 \text{ k}\Omega$$

$$= 2 \text{ k}\Omega + \frac{12\text{k} \times 24\text{k}}{12\text{k} + 24\text{k}} = 10 \text{ k}\Omega.$$




Thévenin equivalent

At $t > 0$

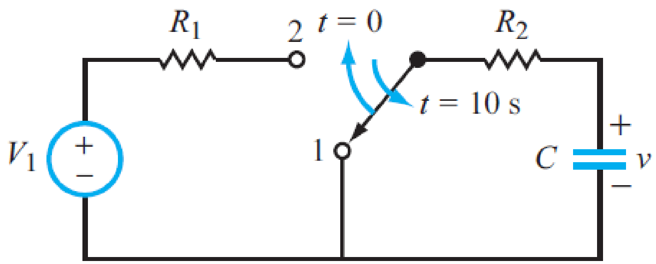
Hence,

$$\tau = RC = 10 \times 10^3 \times 20 \times 10^{-6} = 0.2 \text{ s}.$$

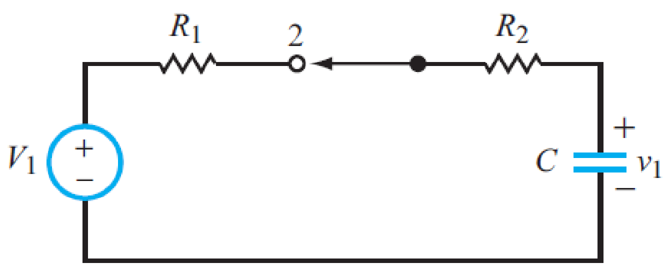
Substituting the values we obtained for $v(0)$, $v(\infty)$, and $\tau$ in Eq. (5.99) leads to

$$v(t) = (20 + 10e^{-5t}) \text{ V} \qquad (\text{for } t \geq 0).$$

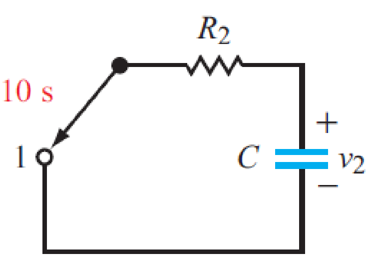
# Example: Charge/Discharge Action




(a) Actual circuit




(b) Circuit during $0 \le t \le 10$ s




(c) Circuit after $t = 10$ s

Given that the switch in Fig. 5-32 was moved to position 2 at $t = 0$ (after it had been in position 1 for a long time) and then returned to position 1 at $t = 10$ s, determine the voltage response $v(t)$ for $t \geq 0$ and evaluate it for $V_1 = 20$ V, $R_1 = 80$ kΩ, $R_2 = 20$ kΩ, and $C = 0.25$ mF.

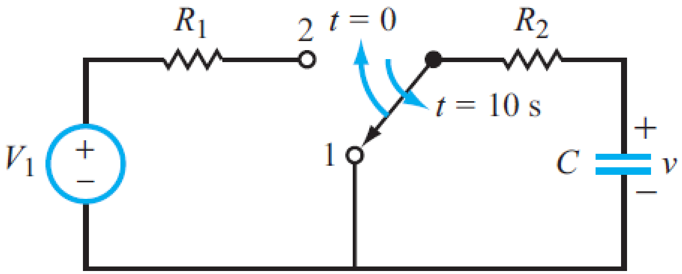
**Time Segment 1:** $0 \leq t \leq 10$ s

When the switch is in position 2 (Fig. 5-32(b)), the resistance of the circuit is $R = R_1 + R_2$. Hence, the time constant during this first time segment is

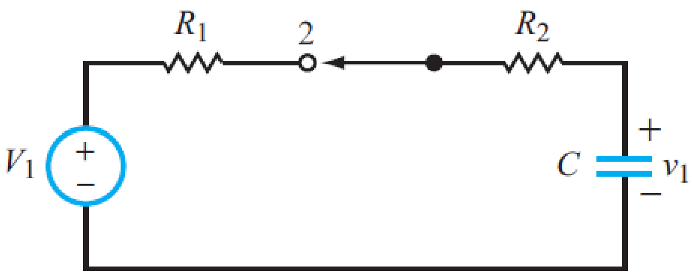
$$\tau_1 = (R_1 + R_2)C$$
$$= (80 + 20) \times 10^3 \times 0.25 \times 10^{-3} = 25 \text{ s}.$$

$$v_1(t) = v_1(\infty) + [v_1(0) - v_1(\infty)]e^{-t/\tau_1}$$
$$= 20(1 - e^{-0.04t}) \quad \text{V} \quad (\text{for } 0 \leq t \leq 10 \text{ s}).$$

# Example (cont.)

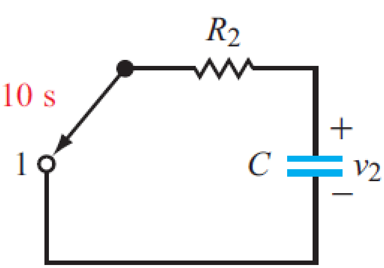



(a) Actual circuit




(b) Circuit during $0 \le t \le 10$ s

Circuit after $t = 10$ s




(c)

**Time Segment 2:** $t \geq 10$ s

Voltage $\upsilon_2(t)$, corresponding to the second time segment (Fig. 5-32(c)), is given by Eq. (5.98) with a new time constant $\tau_2$ as

$$\upsilon_2(t) = \upsilon_2(\infty) + [\upsilon_2(10) - \upsilon_2(\infty)]e^{-(t-10)/\tau_2}.$$

The new time constant is associated with the capacitor circuit remaining after returning the switch to position 1,

$$\tau_2 = R_2 C$$
$$= 20 \times 10^3 \times 0.25 \times 10^{-3} = 5 \text{ s}.$$

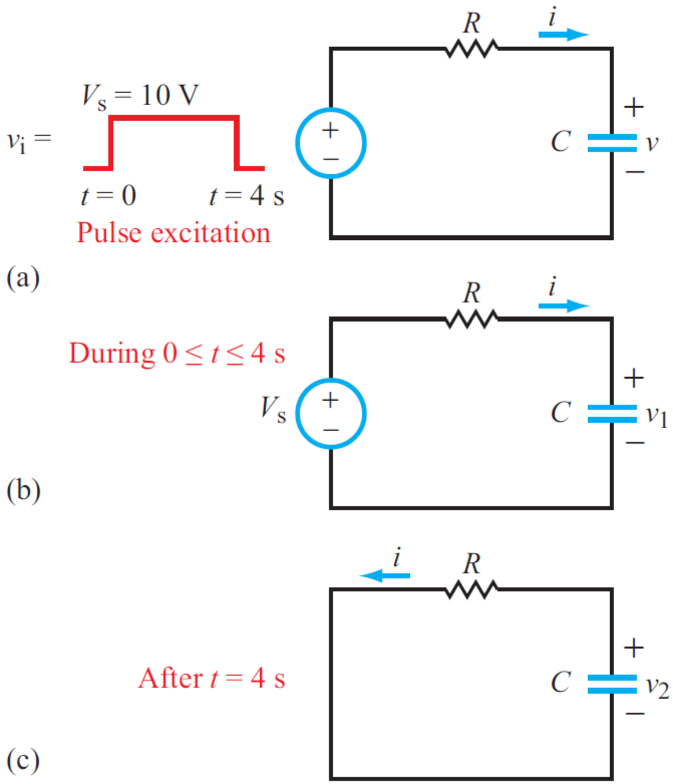
The initial voltage $\upsilon_2(10)$ is equal to the capacitor voltage $\upsilon_1$ at the end of time segment 1, namely

$$\upsilon_2(10) = \upsilon_1(10) = 20(1 - e^{-0.04\times 10})$$
$$= 6.59 \text{ V}.$$

With no voltage source present in the $R_2C$ circuit, the charged capacitor will dissipate its energy into $R_2$, exhibiting a ***natural response*** with a final voltage of $\upsilon_2(\infty) = 0$. Consequently,

$$\upsilon_2(t) = \upsilon_2(10)\ e^{-(t-10)/\tau_2}$$
$$= 6.59e^{-0.2(t-10)} \text{ V} \quad (\text{for } t \geq 10 \text{ s}).$$

# Example:  Rectangular Pulse




$$v_i(t) = V_s\, u(t) - V_s\, u(t-4).$$

Since the circuit is linear, we can apply the superposition theorem to determine the capacitor response $v(t)$. Thus,

$$v(t) = v_1(t) + v_2(t),$$

$$v_1(t) = v_1(\infty) + [v_1(0) - v_1(\infty)]e^{-t/\tau}$$
$$= V_s(1 - e^{-t/\tau}) \qquad (\text{for } t \geq 0).$$

For $V_s = 10$ V and $\tau = RC = 25 \times 10^3 \times 0.2 \times 10^{-3} = 5$ s,

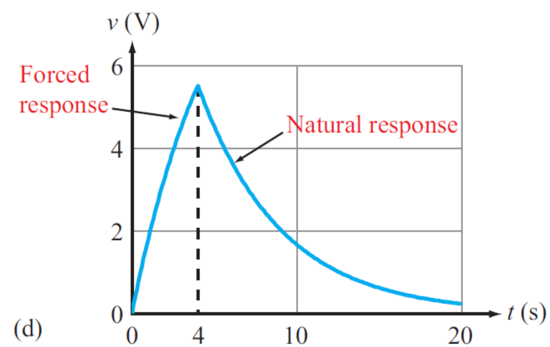
$$v_1(t) = 10(1 - e^{-0.2t}) \text{ V} \qquad (\text{for } t \geq 0).$$

The second step function has an amplitude of $-V_s$ and is delayed in time by 4 s. Upon reversing the polarity of $V_s$ and replacing $t$ with $(t-4)$, we have

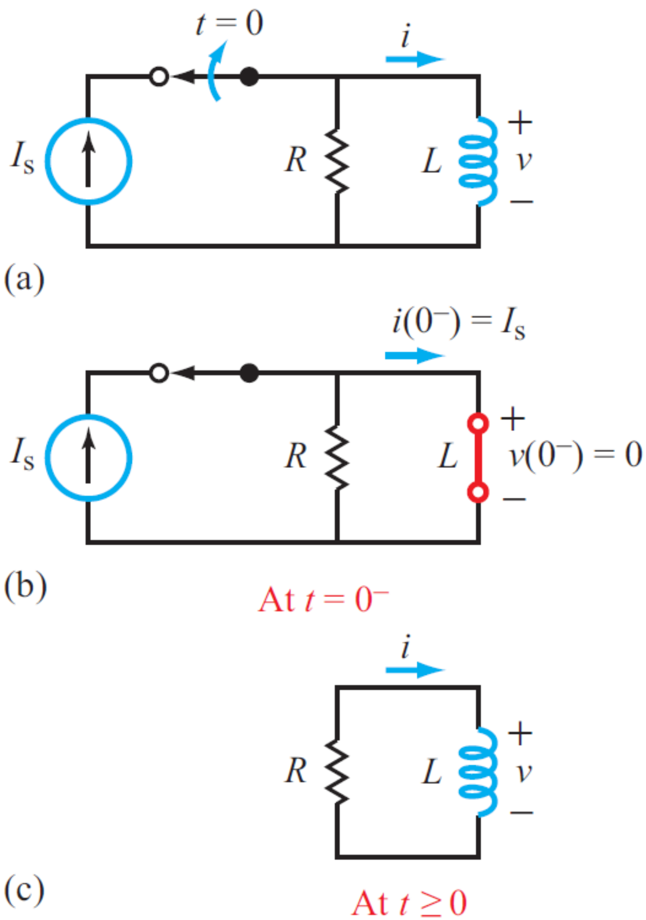
$$v_2(t) = -10[1 - e^{-0.2(t-4)}] \text{ V} \qquad (\text{for } t \geq 4 \text{ s}).$$

The total response for $t \geq 0$ therefore is given by

$$v(t) = v_1(t) + v_2(t)$$
$$= 10[1 - e^{-0.2t}] - 10[1 - e^{-0.2(t-4)}]\, u(t-4) \text{ V}$$



# Natural Response of the RL Circuit




$$i(0) = i(0^-) = I_s$$

$$Ri + L\frac{di}{dt} = 0,$$

which can be cast in the form

$$\frac{di}{dt} + ai = 0,$$

where $a$ is a temporary constant given by

$$a = \frac{R}{L}.$$

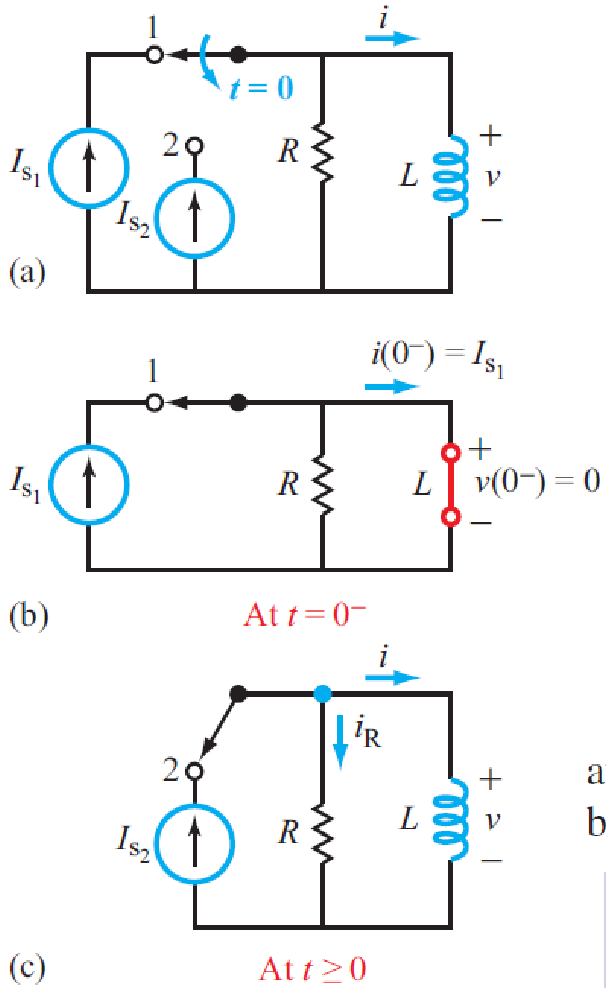
$$i(t) = i(0)\, e^{-t/\tau} \qquad (\text{for } t \geq 0)$$

(natural response),

where for the RL circuit, the ***time constant*** is given by

$$\tau = \frac{1}{a} = \frac{L}{R}.$$

# General Response of the RL Circuit




$$-I_{s_2} + i_R + i = 0.$$

Since $v$ is common to $R$ and $L$, $i_R = v/R$, and by applying $v = L\, di/dt$, the KCL equation becomes

$$\frac{di}{dt} + ai = b, \qquad (5.105)$$

where $a$ is as given previously by Eq. (5.102) and

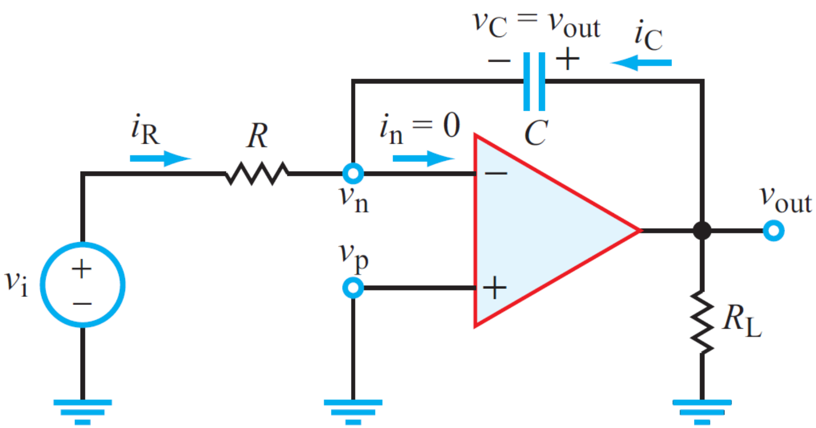
$$b = aI_{s_2} = \frac{R}{L} I_{s_2}. \qquad (5.106)$$

$$i(t) = i(\infty) + [i(0) - i(\infty)]e^{-t/\tau} \qquad (\text{for } t \geq 0)$$
$$(\text{switch action at } t = 0),$$

If the sudden change in the circuit configuration happens at $t = T_0$ instead of at $t = 0$, the general expression for $i(t)$ becomes

$$i(t) = i(\infty) + [i(T_0) - i(\infty)]e^{-(t-T_0)/\tau} \qquad (\text{for } t \geq T_0)$$
$$(\text{switch action at } t = T_0),$$

# RC Op-Amp Circuits: Ideal Integrator




The output voltage $v_{out}$ of such an integrator circuit is directly proportional to the time integral of the input signal $v_i$.

$$i_R = \frac{v_i}{R}. \tag{5.123}$$

Given that $v_n = 0$, the voltage $v_C$ across $C$ is simply $v_{out}$, and the current flowing through it is

$$i_C = C\,\frac{dv_{out}}{dt}. \tag{5.124}$$

At node $v_n$,

$$i_R + i_C - i_n = 0. \tag{5.125}$$

In view of the second op-amp constraint, namely $i_n = i_p = 0$, it follows that

$$i_C = -i_R \tag{5.126}$$

or

$$\frac{dv_{out}}{dt} = -\frac{1}{RC}\,v_i. \tag{5.127}$$

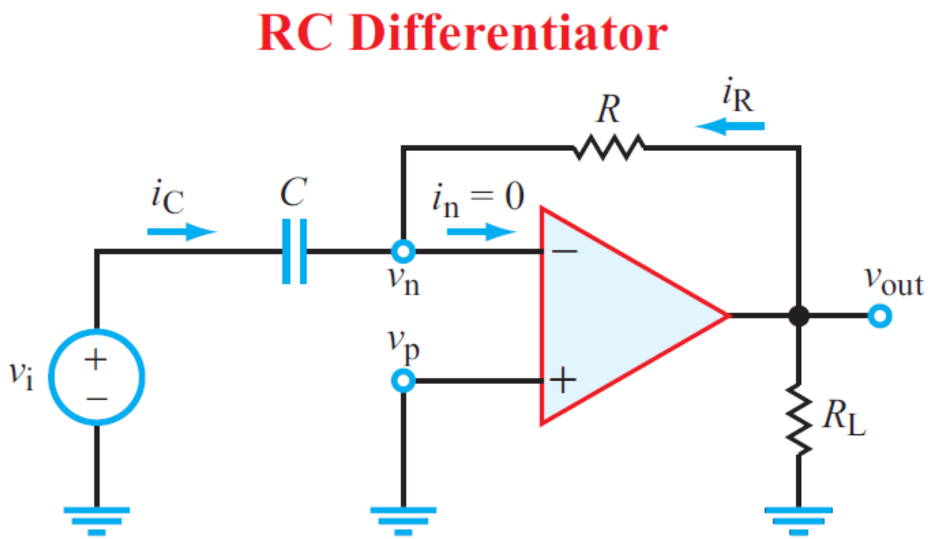
Upon integrating both sides of Eq. (5.127) from a reference time $t_0$ to time $t$, we have

$$\int_{t_0}^{t} \left(\frac{dv_{out}}{dt}\right) dt = -\frac{1}{RC}\int_{t_0}^{t} v_i\,dt, \tag{5.128}$$

which leads to

$$\boxed{v_{out}(t) = -\frac{1}{RC}\int_{t_0}^{t} v_i\,dt + v_{out}(t_0).} \tag{5.129}$$

# RC Op-Amp Circuits: Ideal Differentiator




$$i_C = C\,\frac{dv_i}{dt},$$

$$i_R = \frac{v_{out}}{R},$$

$$i_C = -i_R.$$

$$v_{out} = -RC\,\frac{dv_i}{dt},$$