Hidden Markov Models and Gaussian Mixture Models

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Automatic Speech Recognition— ASR Lecture 2
16 January 2020

Overview

HMMs and GMMs

- Introduction to HMMs: Hidden Markov models
- Univariate and multivariate Gaussians
- Gaussian mixture models
- Introduction to the EM algorithm

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- Univariate and multivariate Gaussians
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Warning: the maths starts here!

Fundamental Equation of Statistical Speech Recognition

If $\mathbf{X}$ is the sequence of acoustic feature vectors (observations) and $\mathbf{W}$ denotes a word sequence, the most likely word sequence $\mathbf{W}^*$ is given by

$$\mathbf{W}^* = \arg\max_{\mathbf{W}} P(\mathbf{W} \mid \mathbf{X})$$

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Applying Bayes' Theorem:

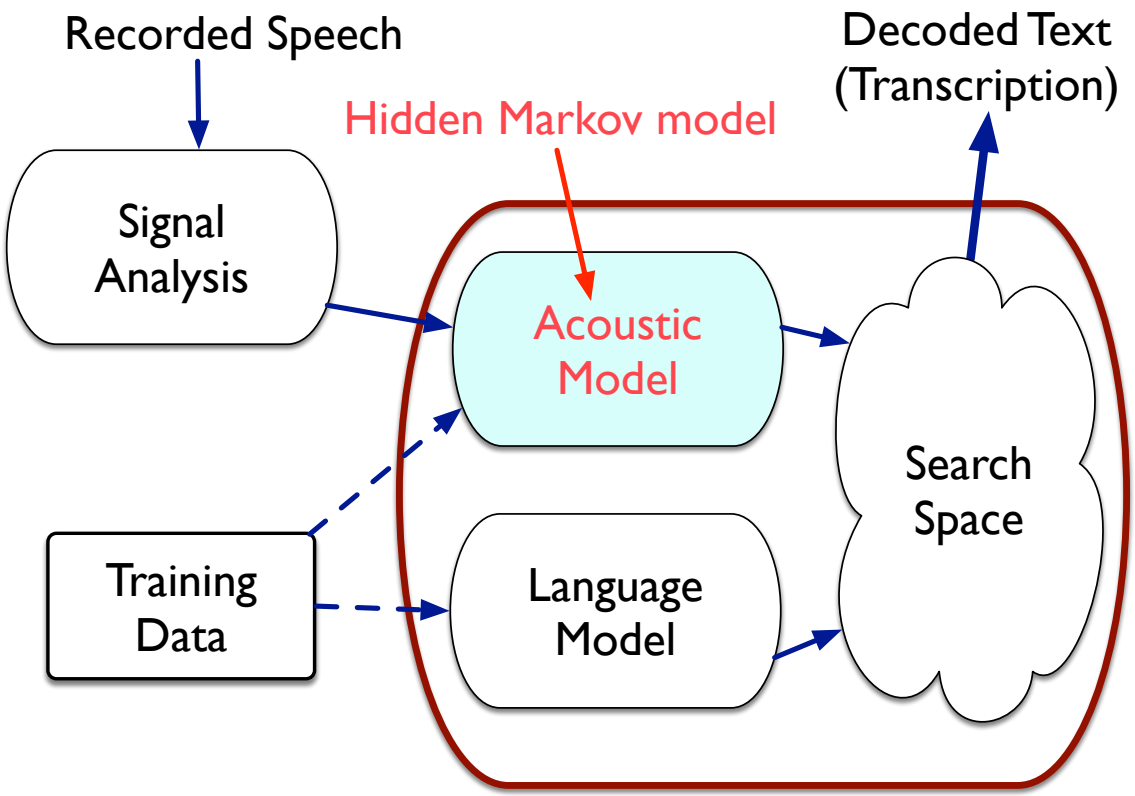
$$P(\mathbf{W}|\mathbf{X}) = \frac{p(\mathbf{X}|\mathbf{W})\,P(\mathbf{W})}{p(\mathbf{X})}$$

$$\propto p(\mathbf{X}|\mathbf{W})\,P(\mathbf{W})$$

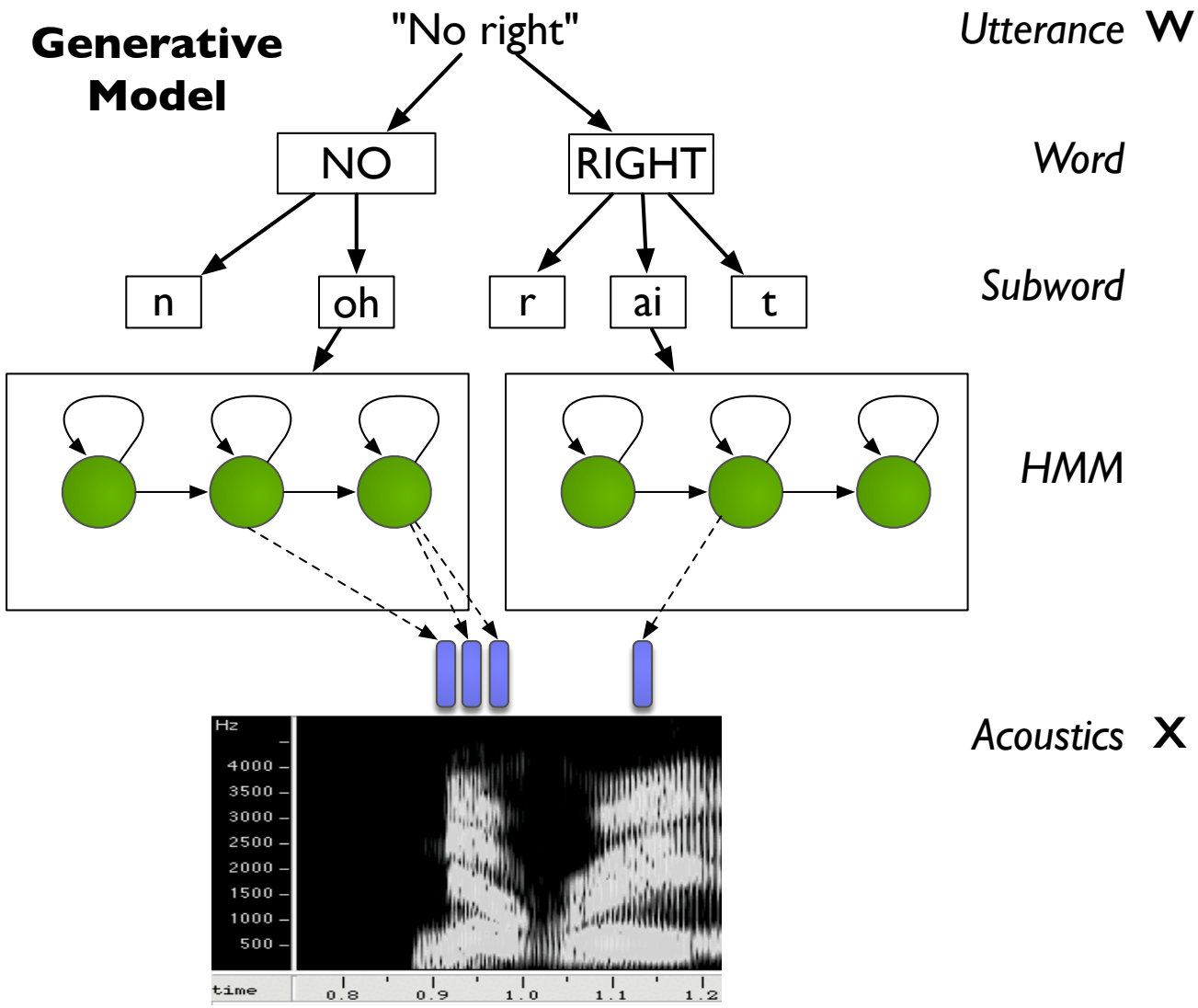
$$\mathbf{W}^* = \arg\max_{\mathbf{W}} \underbrace{p(\mathbf{X}|\mathbf{W})}_{\text{Acoustic model}} \underbrace{P(\mathbf{W})}_{\text{Language model}}$$

NB: $\mathbf{X}$ is used hereafter to denote the output feature vectors from the signal analysis module rather than DFT spectrum.

Acoustic Modelling



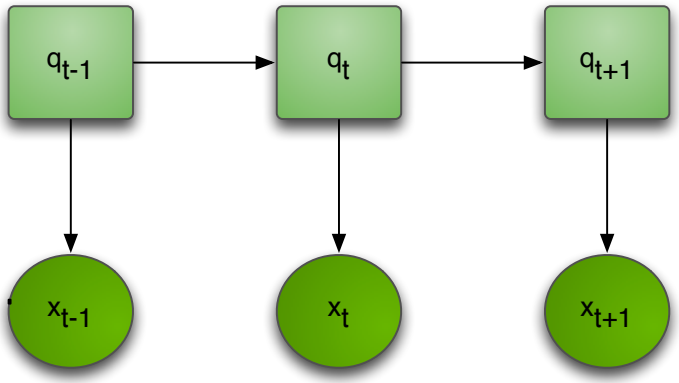
Hierarchical modelling of speech



The Hidden Markov model

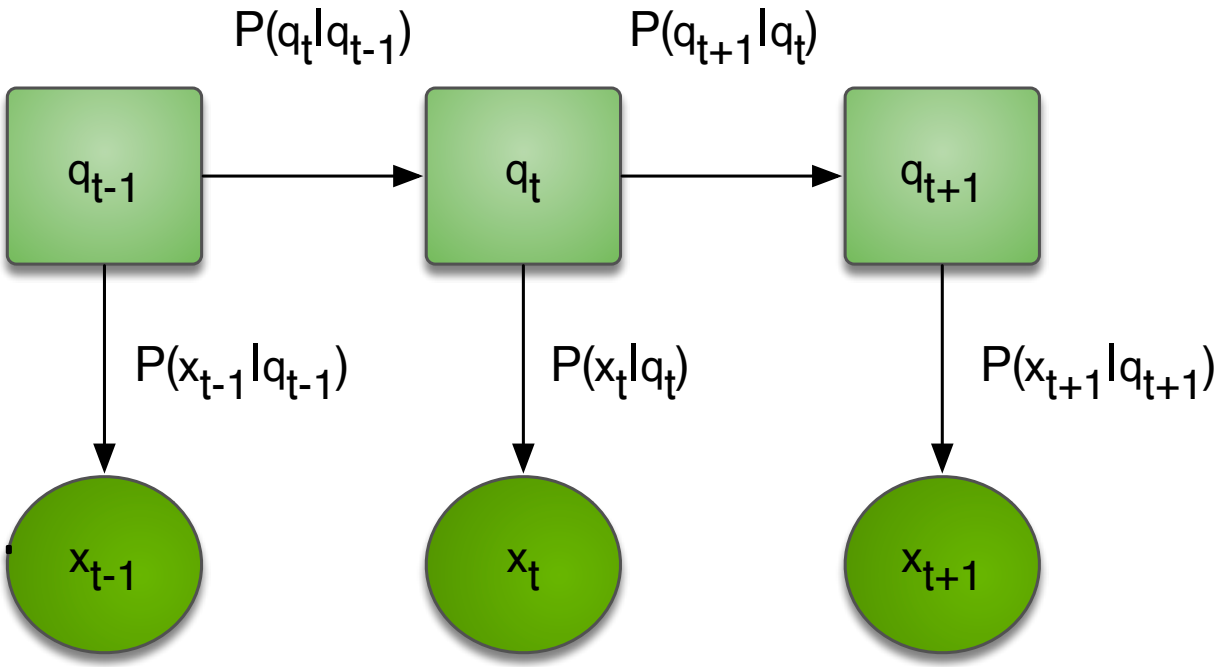
- A statistical model for time series data with a set of **discrete** states $\{1, \ldots, J\}$ (we index them by j or k)
- At each time step t:
  - the model is in a fixed state q_t.
  - the model generates an observation, $\mathbf{x}_t$, according to a probability distribution that is specific to the state
- We don't actually observe which state the model is in at each time step – hence **"hidden"**.
- Observations can be either continous or discrete (usually the former)

HMM probabilities



- Imagine we know the state at a given time step t, $q_t = k$
- Then the probability of being in a new state, j at the next time step, is dependent only on q_t. This is the **Markov** assumption.
- Alternatively: q_{t+1} is *conditionally independent* of $q_1, \ldots, q_{t-1}$, given q_t.
- This means we can parametrise the model with parameters λ:
  - Transition probabilities $a_{kj} = P(q_{t+1} = j | q_t = k)$
  - Observation probabilities $b_j(\mathbf{x}) = P(\mathbf{x} | q = j)$

HMM assumptions



Note that **observation independence** is an assumption that naturally arises from the model: the probability of $\mathbf{x_t}$ depends only on the state that generated it, q_t.

HMM topologies

- The HMM topology determines the set of allowed transitions between states

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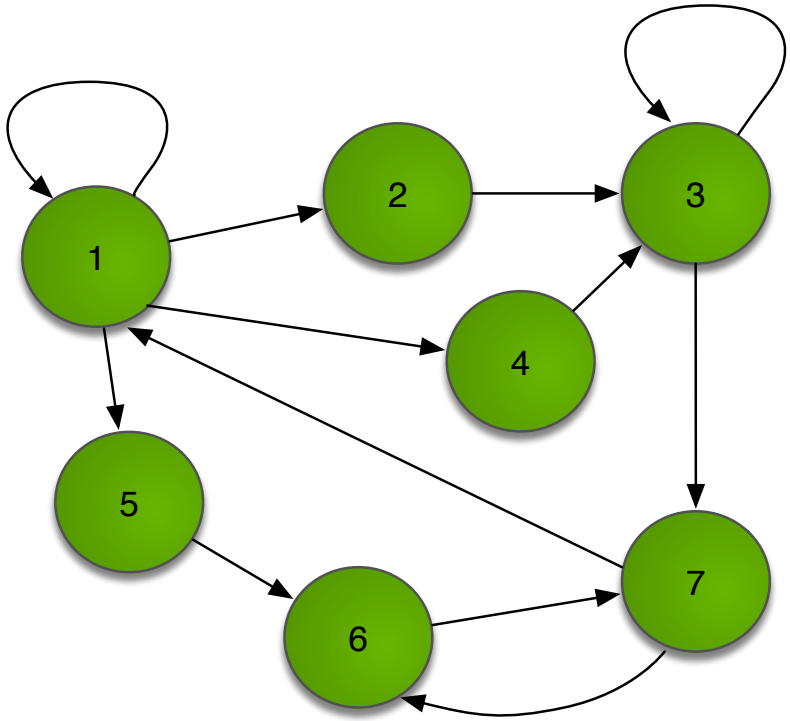

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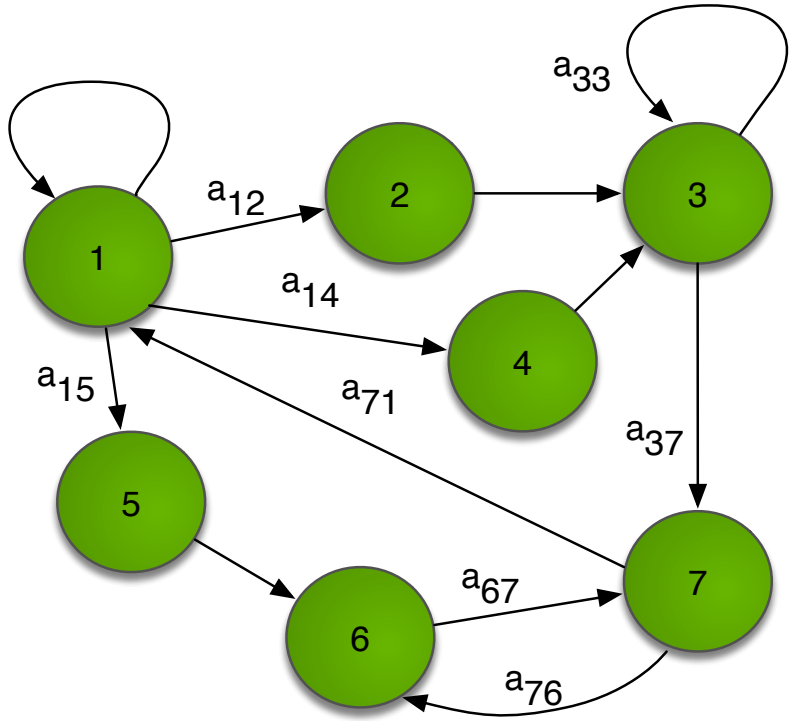
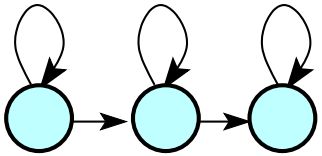
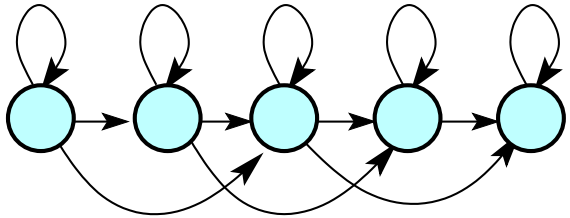


Not all transition probabilities are shown

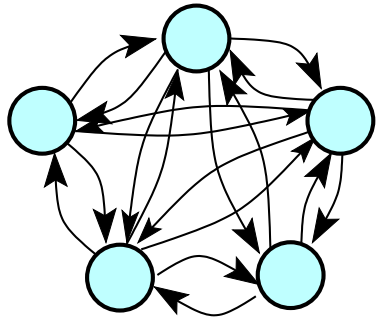
Example topologies



left–to–right model



parallel path left–to–right model



ergodic model

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

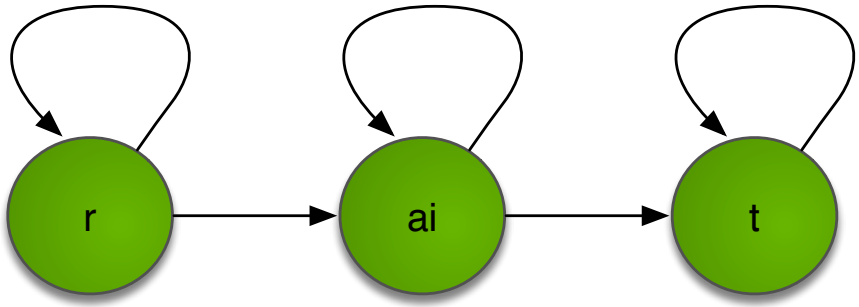
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{pmatrix}$$

Speech recognition: left-to-right HMM with $3 \sim 5$ states
Speaker recognition: ergodic HMM

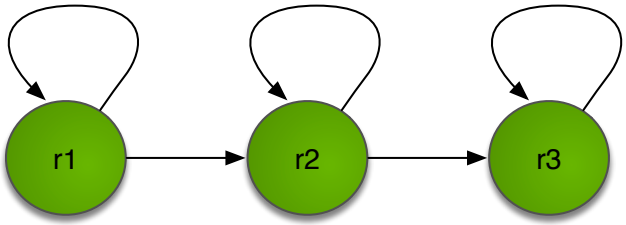
HMMs for ASR

We generally model words or phones with a left-to-right topology with self loops.



HMMs for ASR

Traditional HMMs for ASR tend to model each phone with three distinct states (this also enforces a minimum phone duration of three frames of observations)



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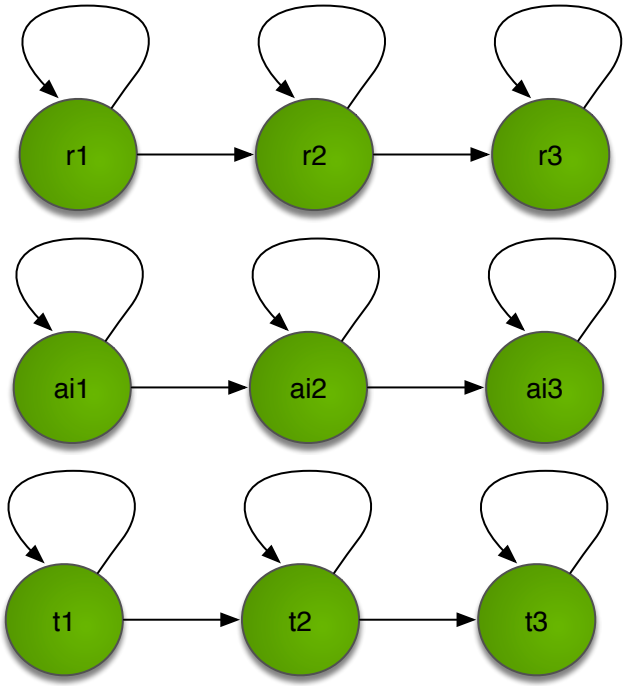

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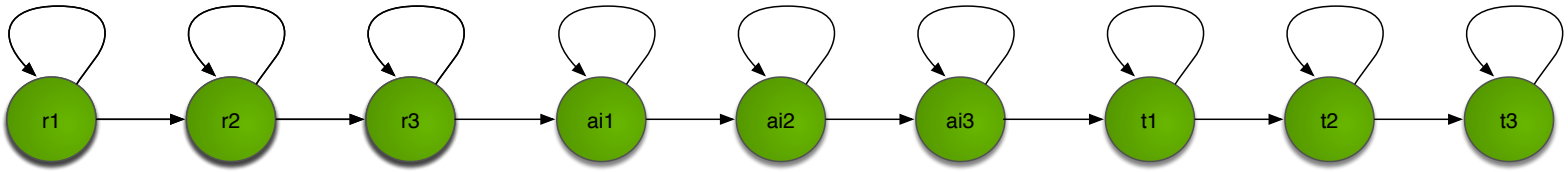
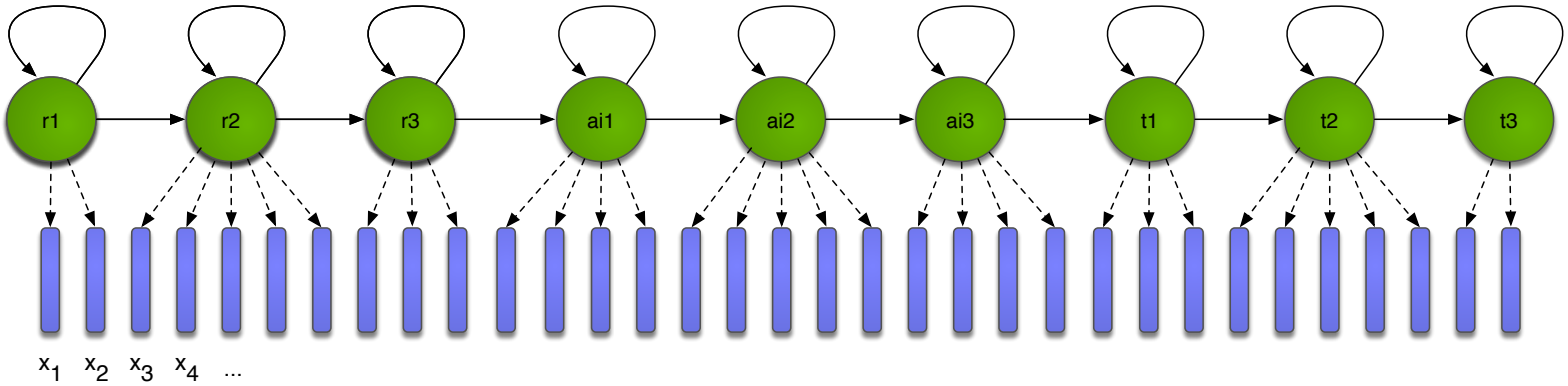


The phone model topologies can be concatenated to form a HMM for the whole word

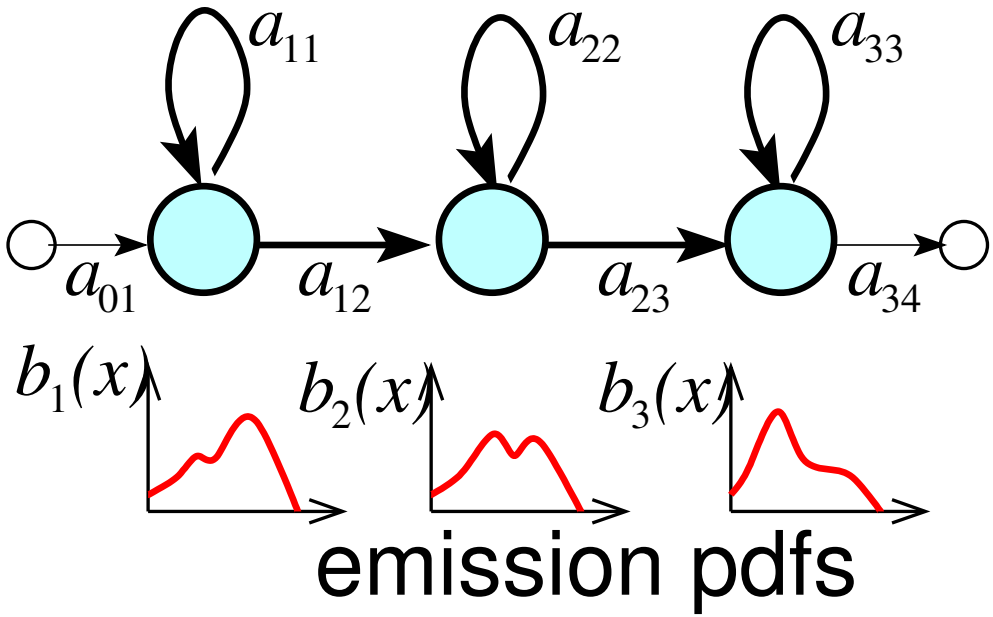
HMMs for ASR

Traditional HMMs for ASR tend to model each phone with three distinct states (this also enforces a minimum phone duration of three frames of observations)



This model naturally generates an alignment between states and observations (and hence words/phones).

A note on HMM observation probabilities



	Observation prob.	
Continuous (density) HMM	continuous	GMM, DNN
Discrete (probability) HMM	discrete	Vector quantisation
Semi-continuous HMM (tied-mixture HMM)	continuous	tied mixture

Computing likelihoods with the HMM

Suppose we have a sequence of observations of length T, $X = (\mathbf{x}_1, \ldots, x_T)$, and Q is a known state sequence, $(q_1, \ldots, q_T)$. Then we can use the HMM to compute the joint likelihood of X and Q:

$$P(X, Q; \lambda) = P(q_1)P(\mathbf{x}_1|q_1)P(q_2|q_1)P(\mathbf{x}_2|q_2)\ldots \quad (1)$$

$$= P(q_1)P(\mathbf{x}_1|q_1)\prod_{t=2}^{T} P(q_t|q_{t-1})P(\mathbf{x}_t|q_t) \quad (2)$$

$P(q_1)$ denotes the initial occupancy probability of each state

Background: cdf

Consider a real valued random variable X

- Cumulative distribution function (cdf) $F(x)$ for X:

$$F(x) = P(X \leq x)$$

- To obtain the probability of falling in an interval we can do the following:

$$\begin{aligned} P(a < X \leq b) &= P(X \leq b) - P(X \leq a) \\ &= F(b) - F(a) \end{aligned}$$

Background: pdf

- The rate of change of the cdf gives us the *probability density function* (pdf), $p(x)$:

$$p(x) = \frac{d}{dx}F(x) = F'(x)$$

$$F(x) = \int_{-\infty}^{x} p(x)\mathrm{d}x$$

- $p(x)$ is **not** the probability that X has value x. But the pdf is proportional to the probability that X lies in a small interval centred on x.
- Notation: p for pdf, P for probability

The Gaussian distribution (univariate)

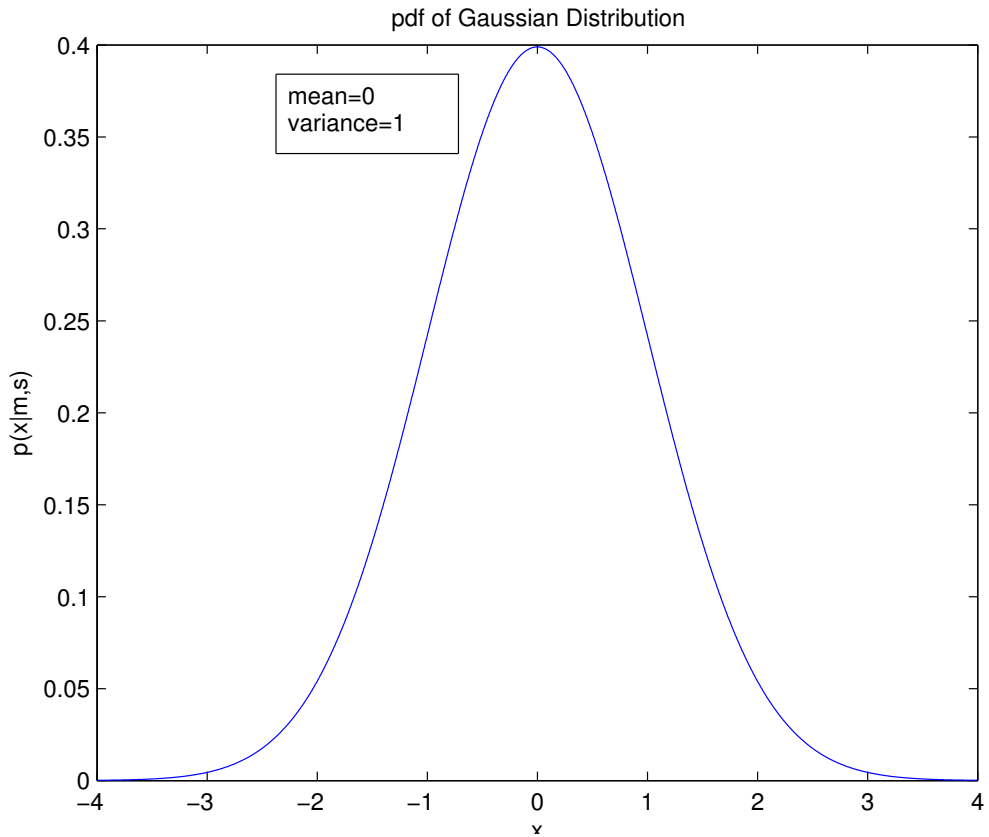
- The Gaussian (or Normal) distribution is the most common (and easily analysed) continuous distribution
- It is also a reasonable model in many situations (the famous "bell curve")
- If a (scalar) variable has a Gaussian distribution, then it has a probability density function with this form:

$$p(x \,|\, \mu, \sigma^2) = \mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$

- The Gaussian is described by two parameters:
  - the mean μ (location)
  - the variance σ^2 (dispersion)

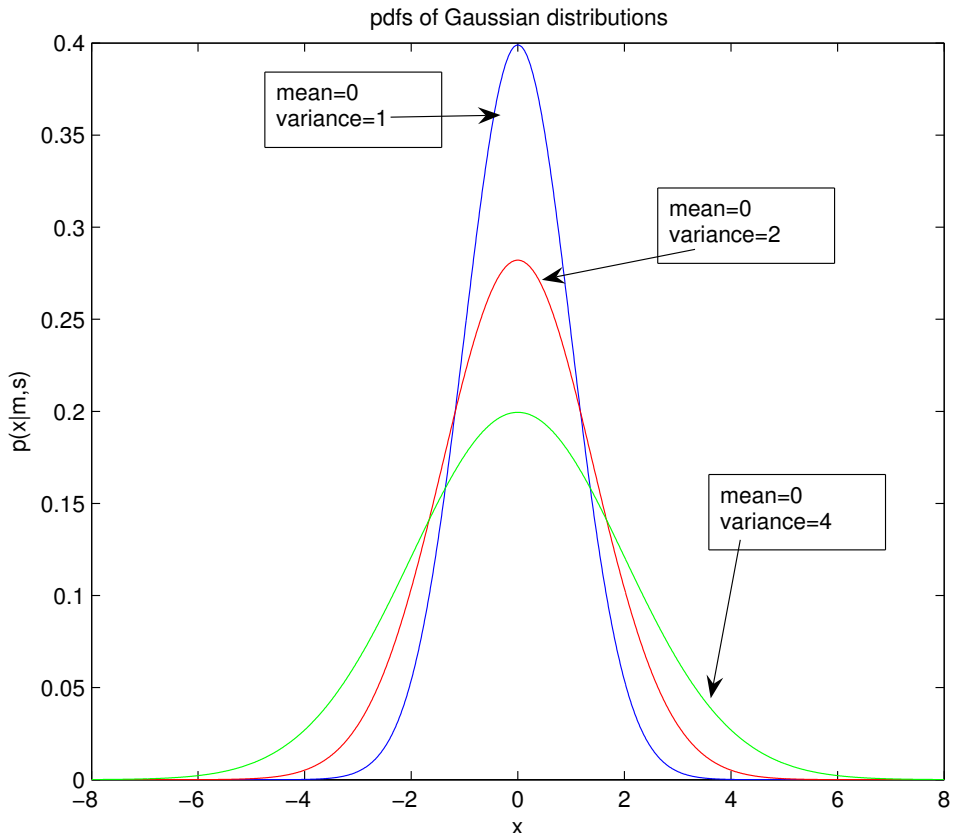
Plot of Gaussian distribution

- Gaussians have the same shape, with the location controlled by the mean, and the spread controlled by the variance
- One-dimensional Gaussian with zero mean and unit variance ($\mu = 0$, $\sigma^2 = 1$):



Properties of the Gaussian distribution

$$\mathcal{N}(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)$$



Parameter estimation

- Estimate mean and variance parameters of a Gaussian from data $x_1, x_2, \ldots, x_T$
- Use the following as the estimates:

$$\hat{\mu} = \frac{1}{T} \sum_{t=1}^{T} x_t \qquad \text{(mean)}$$

$$\hat{\sigma}^2 = \frac{1}{T} \sum_{t=1}^{T} (x_t - \hat{\mu})^2 \qquad \text{(variance)}$$

Exercise — maximum likelihood estimation (MLE)

Consider the log likelihood of a set of T training data points $\{x_1, \ldots, x_T\}$ being generated by a Gaussian with mean μ and variance σ^2:

$$L = \ln p(\{x_1, \ldots, x_T\} \,|\, \mu, \sigma^2) = -\frac{1}{2} \sum_{t=1}^{T} \left(\frac{(x_t - \mu)^2}{\sigma^2} - \ln \sigma^2 - \ln(2\pi) \right)$$

$$= -\frac{1}{2\sigma^2} \sum_{t=1}^{T} (x_t - \mu)^2 - \frac{T}{2} \ln \sigma^2 - \frac{T}{2} \ln(2\pi)$$

By maximising the the log likelihood function with respect to μ show that the maximum likelihood estimate for the mean is indeed the sample mean:

$$\mu_{ML} = \frac{1}{T} \sum_{t=1}^{T} x_t.$$

The multivariate Gaussian distribution

- The D-dimensional vector $\mathbf{x} = (x_1, \ldots, x_D)^T$ follows a multivariate Gaussian (or normal) distribution if it has a probability density function of the following form:

$$p(\mathbf{x}|\boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

The pdf is parameterised by the mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_D)^T$ and the covariance matrix $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \ldots & \sigma_{1D} \\ \vdots & \ddots & \vdots \\ \sigma_{D1} & \ldots & \sigma_{DD} \end{pmatrix}$.

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- The 1-dimensional Gaussian is a special case of this pdf

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- The 1-dimensional Gaussian is a special case of this pdf
- The argument to the exponential $0.5(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})$ is referred to as a *quadratic form*.

Covariance matrix

- The mean vector $\boldsymbol{\mu}$ is the expectation of $\mathbf{x}$:

$$\boldsymbol{\mu} = E[\mathbf{x}]$$

- The covariance matrix $\boldsymbol{\Sigma}$ is the expectation of the deviation of $\mathbf{x}$ from the mean:

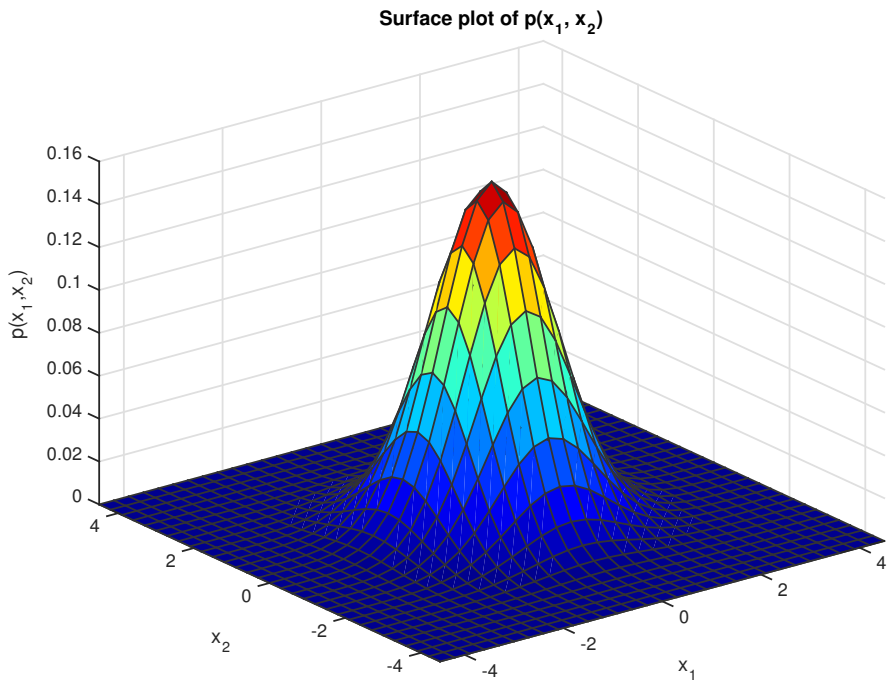
$$\boldsymbol{\Sigma} = E[(\mathbf{x} - \boldsymbol{\mu})(\mathbf{x} - \boldsymbol{\mu})^T]$$

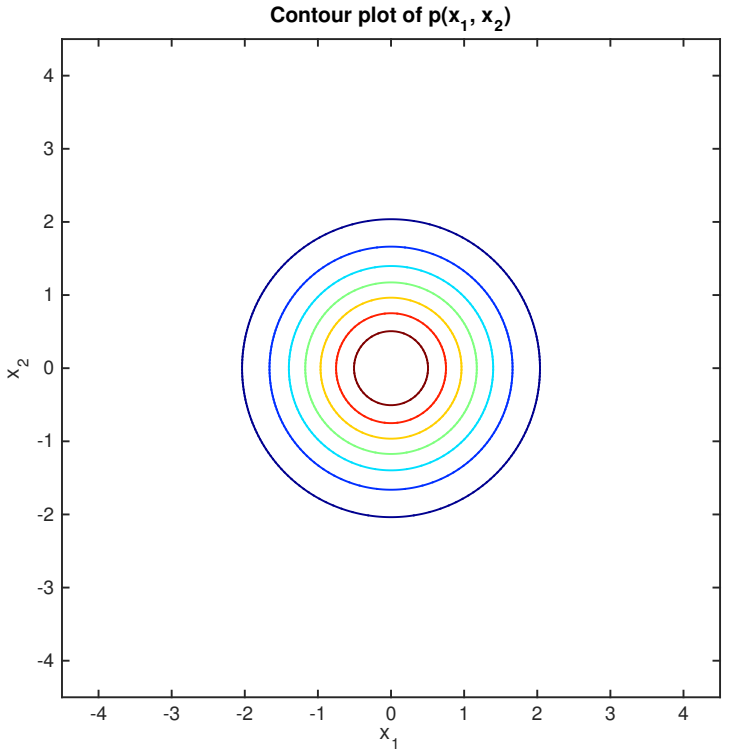
- $\boldsymbol{\Sigma}$ is a $D \times D$ symmetric matrix:

$$\sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)] = E[(x_j - \mu_j)(x_i - \mu_i)] = \sigma_{ji}$$

- The sign of the covariance helps to determine the relationship between two components:
  - If x_j is large when x_i is large, then $(x_i - \mu_i)(x_j - \mu_j)$ will tend to be positive;
  - If x_j is small when x_i is large, then $(x_i - \mu_i)(x_j - \mu_j)$ will tend to be negative.

Spherical Gaussian

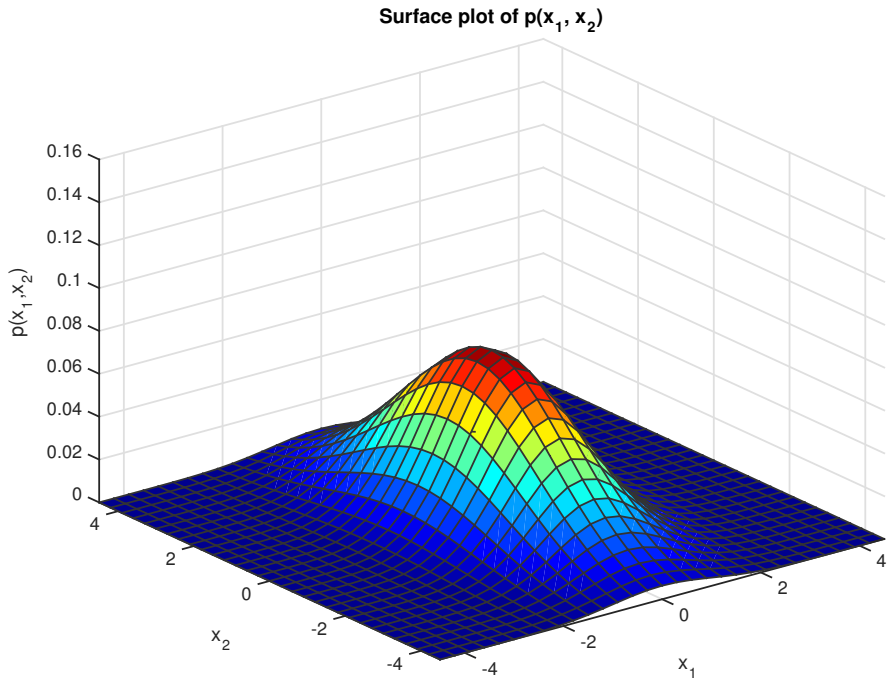


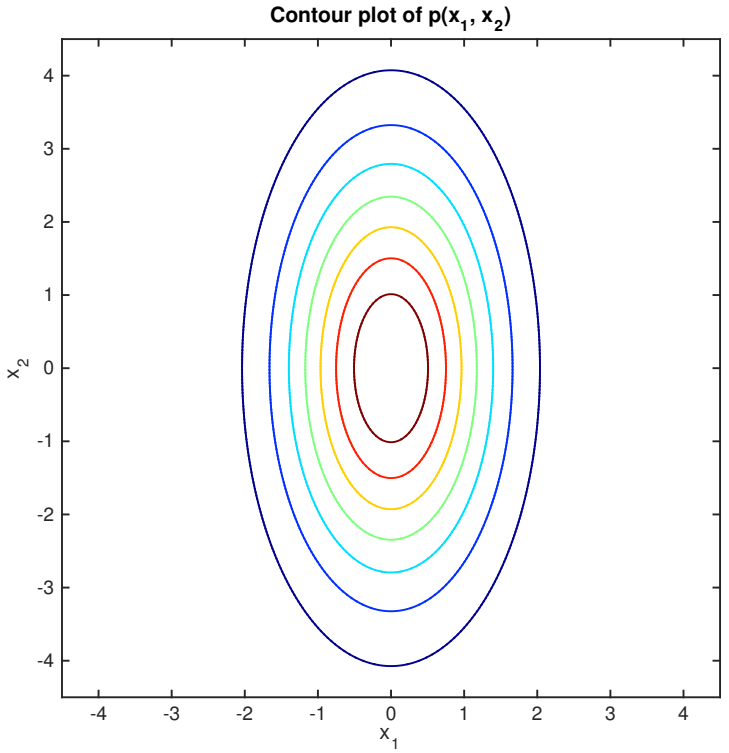


$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho_{12} = 0$$

NB: Correlation coefficient $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \qquad (-1 \leq \rho_{ij} \leq 1)$

Diagonal Covariance Gaussian

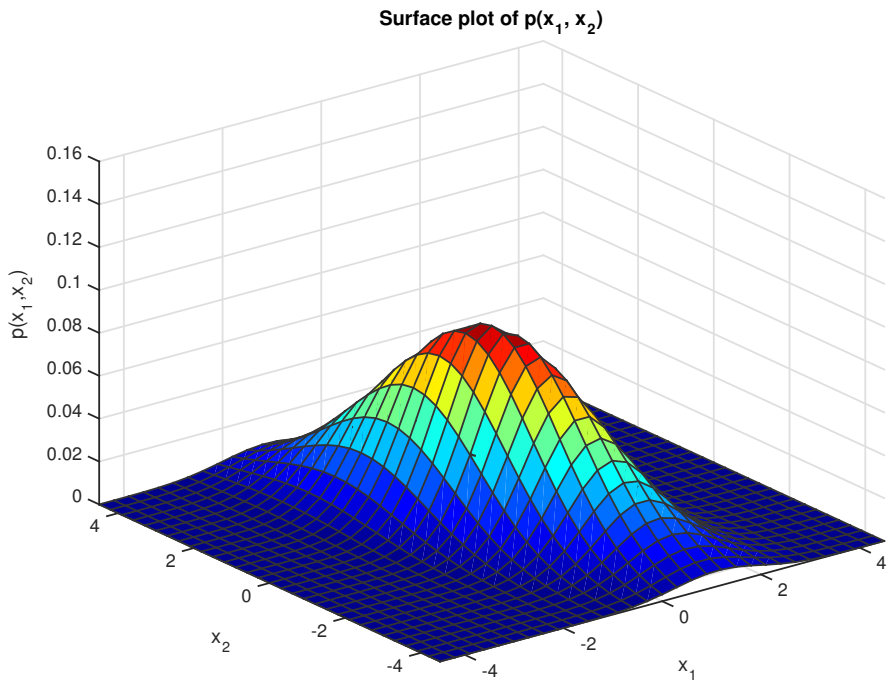


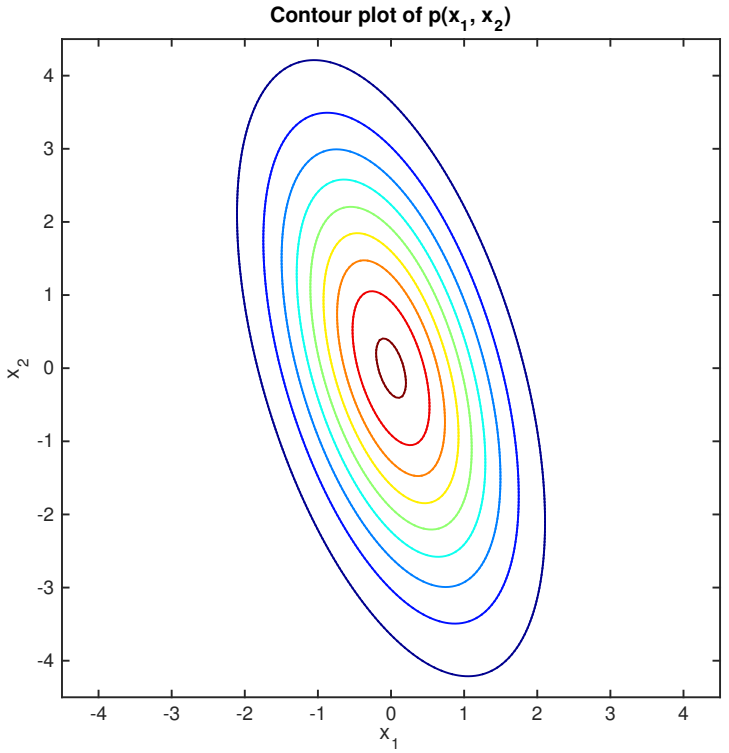


$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \qquad \rho_{12} = 0$$

NB: Correlation coefficient $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \qquad (-1 \leq \rho_{ij} \leq 1)$

Full covariance Gaussian





$$\boldsymbol{\mu} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix} \qquad \rho_{12} = -0.5$$

NB: Correlation coefficient $\rho_{ij} = \dfrac{\sigma_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}} \qquad (-1 \leq \rho_{ij} \leq 1)$

Parameter estimation of a multivariate Gaussian distribution

- It is possible to show that the mean vector $\hat{\boldsymbol{\mu}}$ and covariance matrix $\hat{\boldsymbol{\Sigma}}$ that maximise the likelihood of the training data are given by:

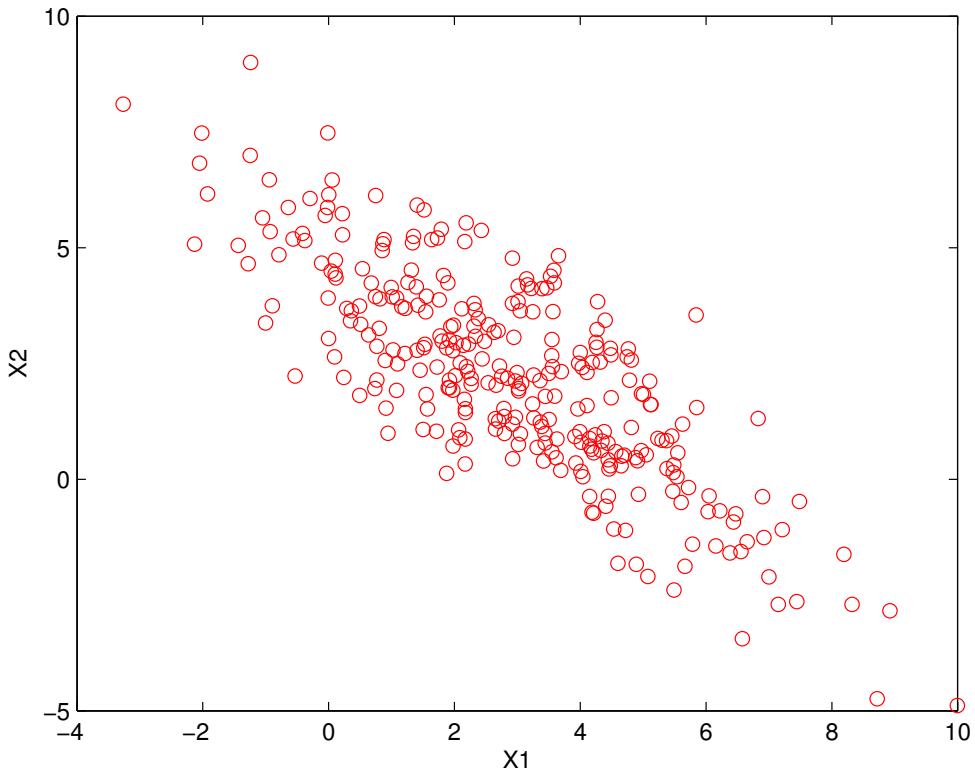
$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{x}_t$$

$$\hat{\boldsymbol{\Sigma}} = \frac{1}{T}\sum_{t=1}^{T} (\boldsymbol{x}_t - \hat{\boldsymbol{\mu}})(\boldsymbol{x}_t - \hat{\boldsymbol{\mu}})^T$$

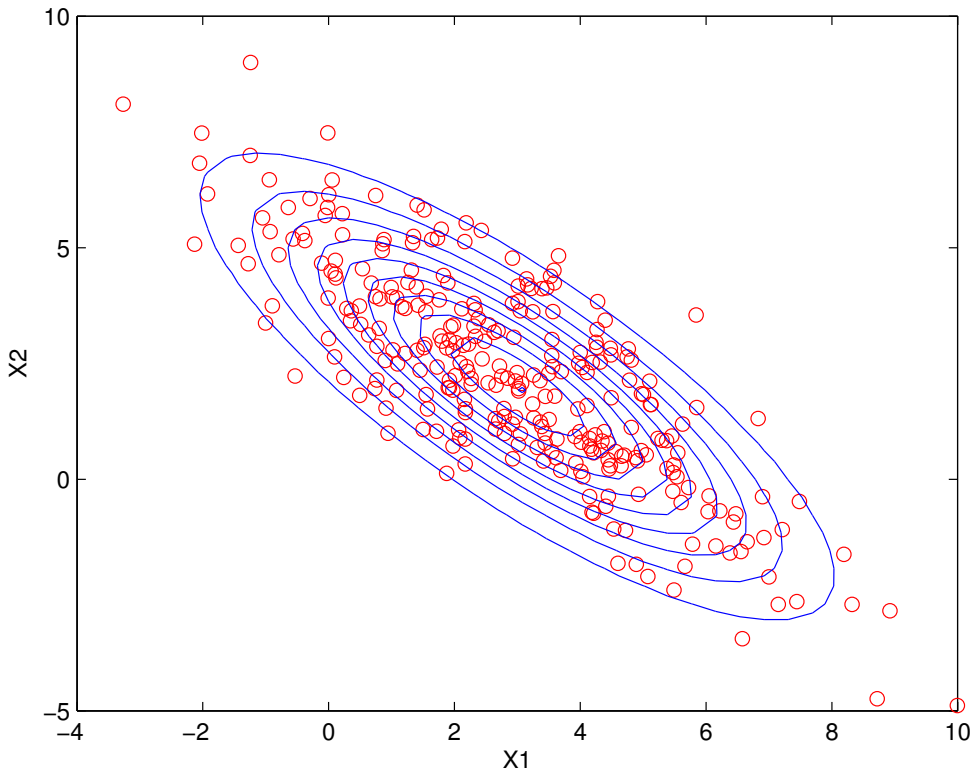
where $\boldsymbol{x}_t = (x_{t1}, \ldots, x_{tD})^T$.

NB: T denotes either the number of samples or vector transpose depending on context.

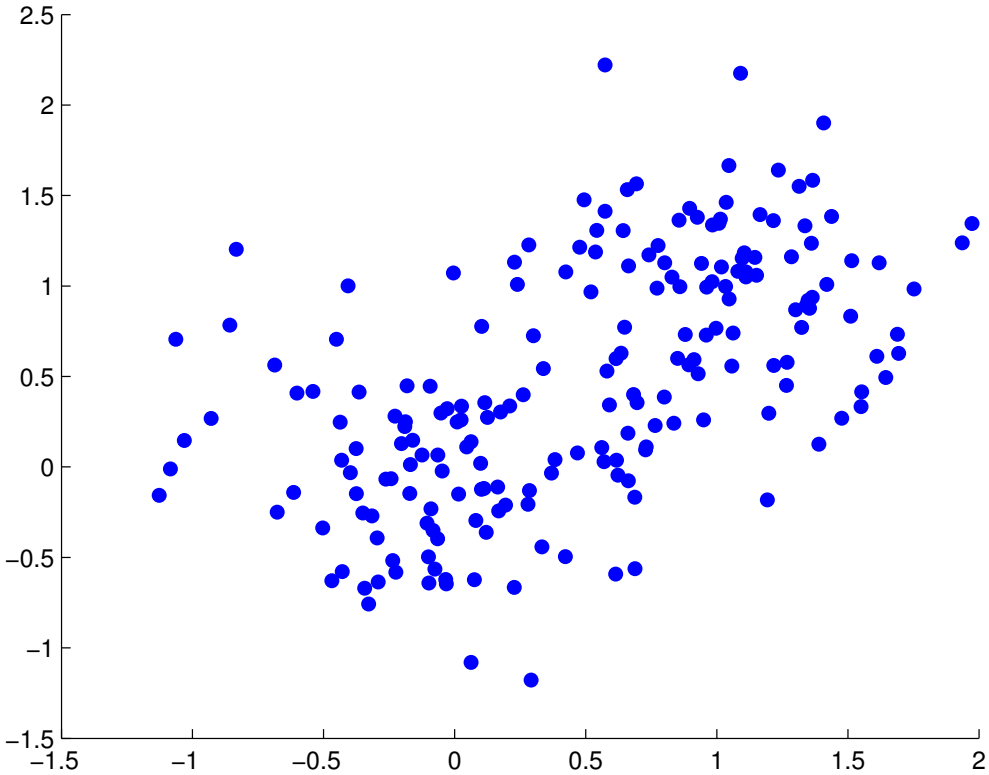
Example data



Maximum likelihood fit to a Gaussian

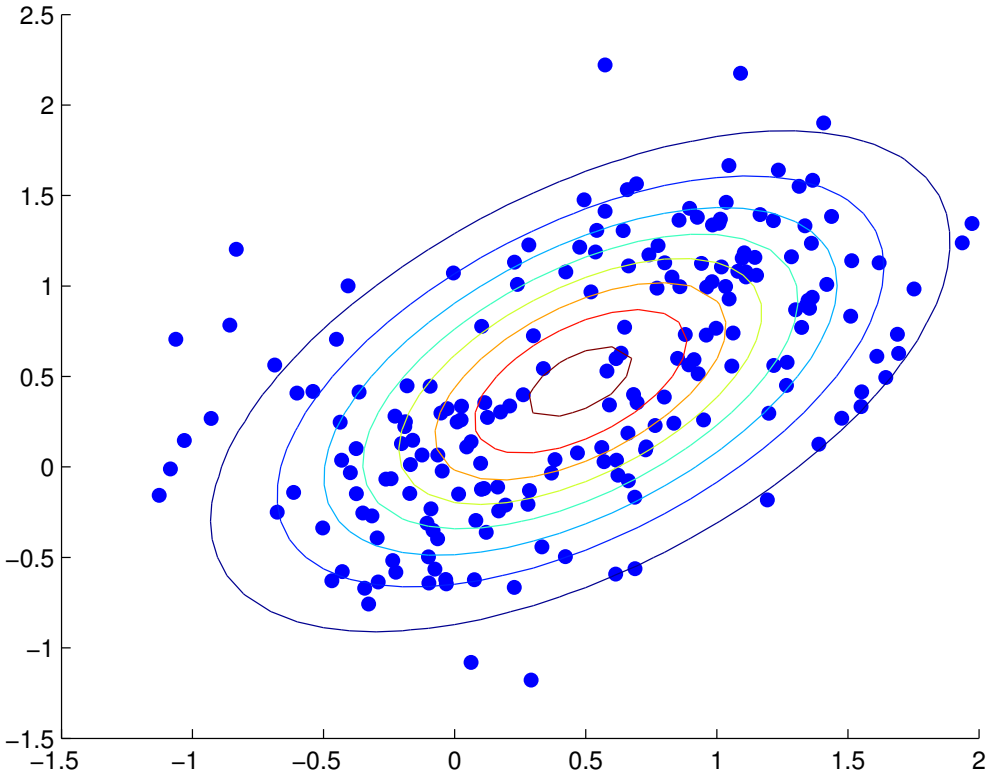


Data in clusters (example 1)



$$\boldsymbol{\mu}_1 = (0,0)^T \qquad \boldsymbol{\mu}_2 = (1,1)^T \qquad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = 0.2\,\mathbf{I}$$

Example 1 fit by a Gaussian

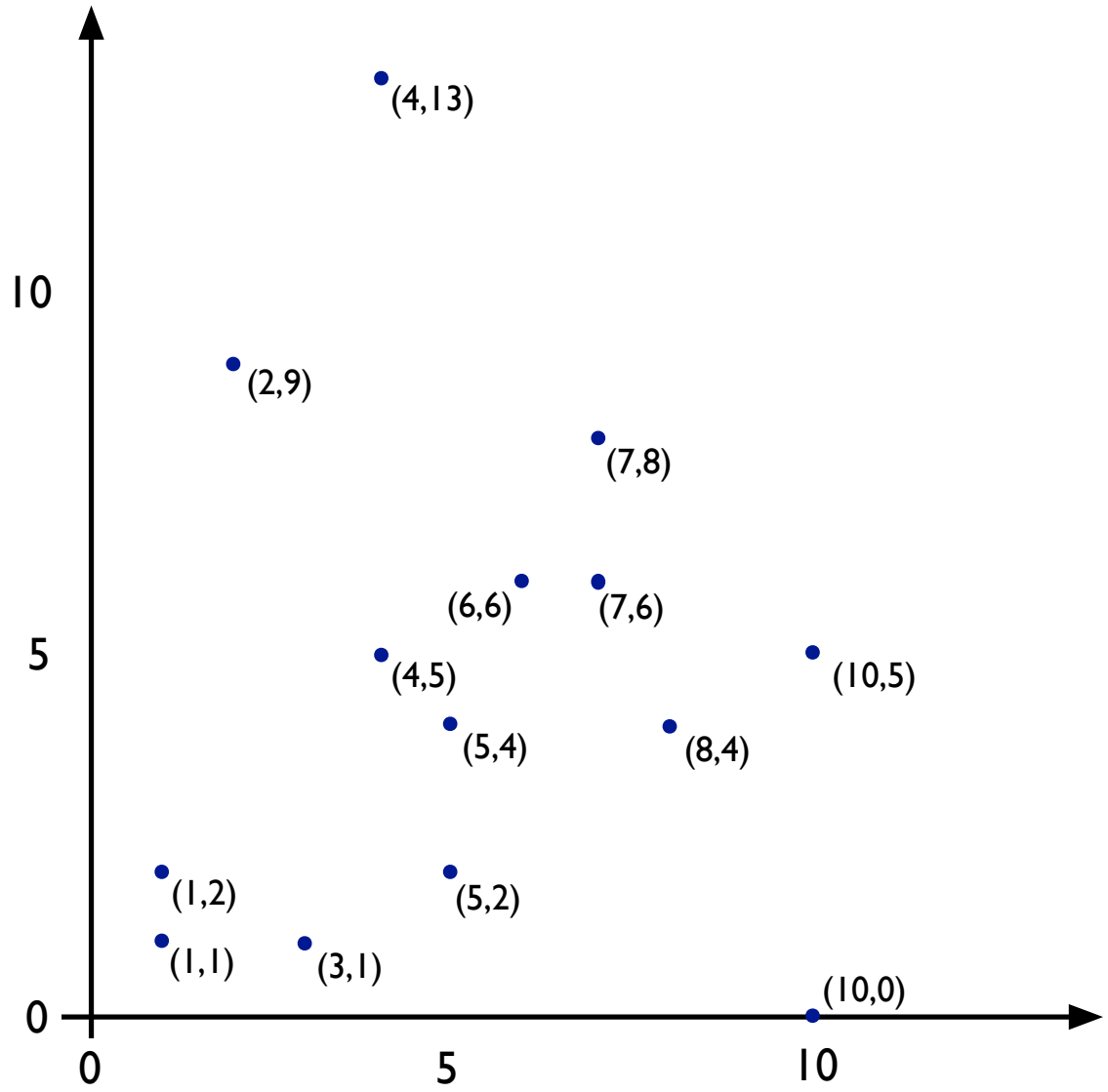


$$\boldsymbol{\mu}_1 = (0,0)^T \qquad \boldsymbol{\mu}_2 = (1,1)^T \qquad \boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = 0.2\,\mathbf{I}$$

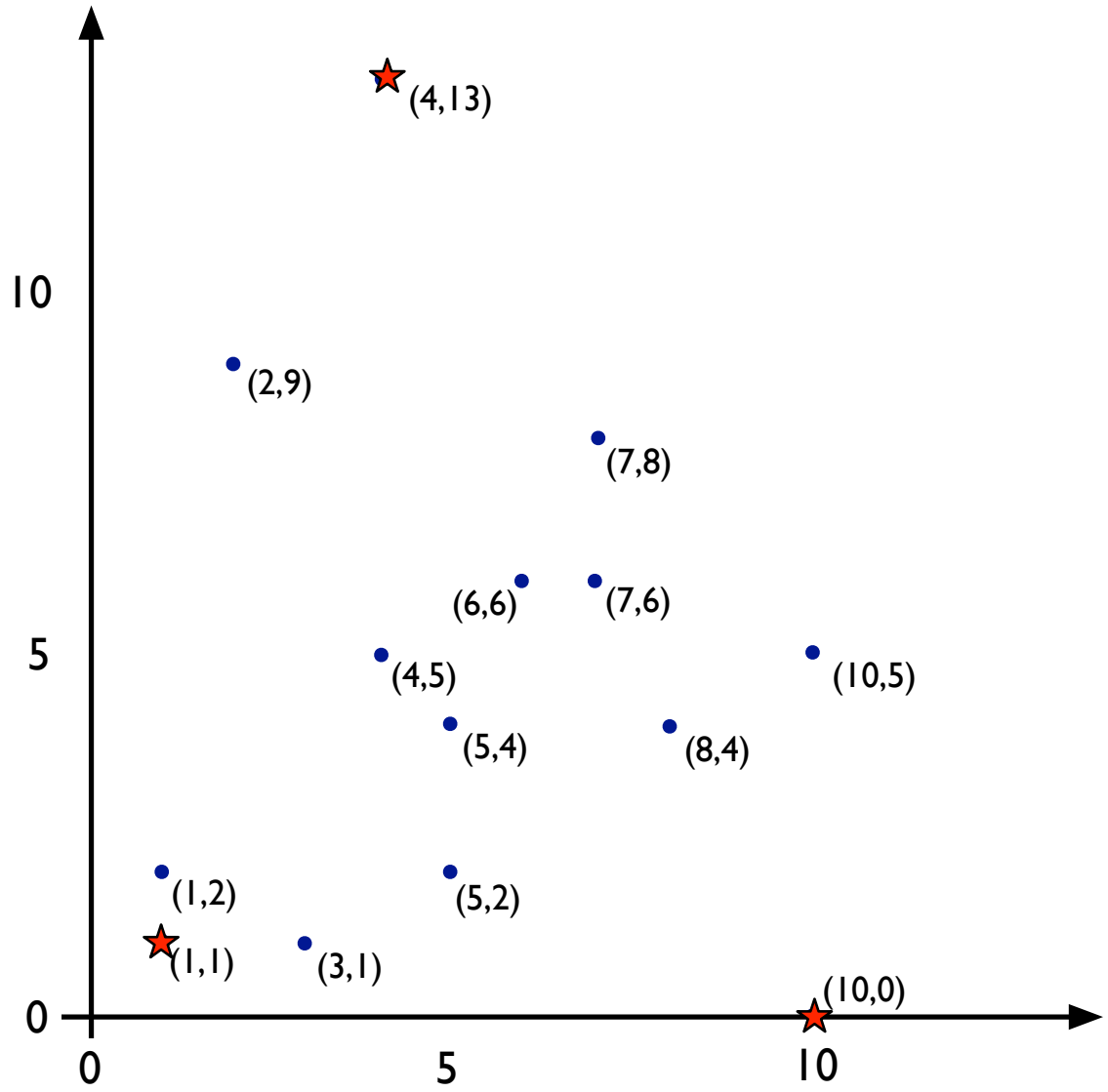
k-means clustering

- k-means is an automatic procedure for clustering unlabelled data
- Requires a prespecified number of clusters
- Clustering algorithm chooses a set of clusters with the minimum within-cluster variance
- Guaranteed to converge (eventually)
- Clustering solution is dependent on the initialisation

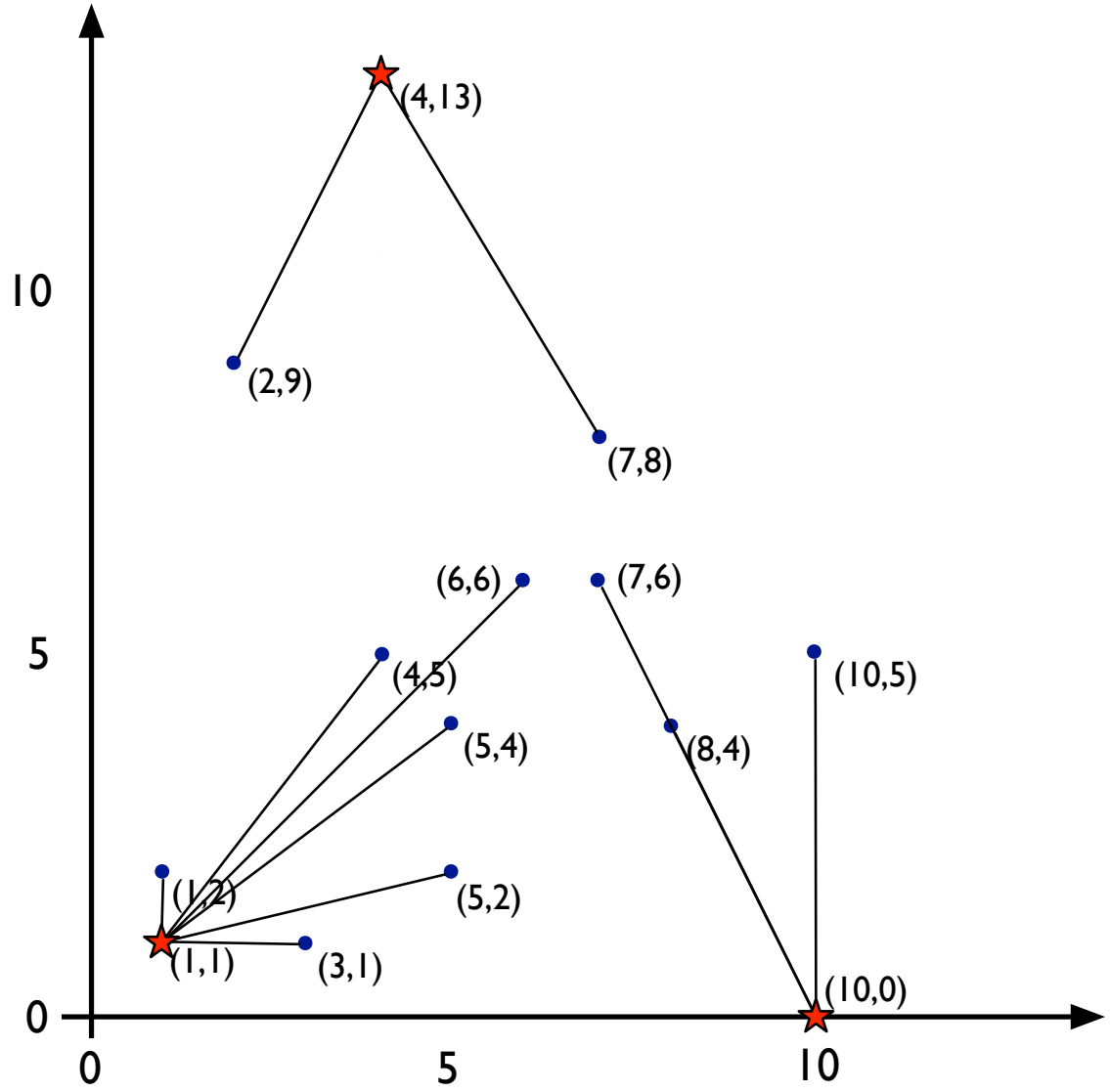
k-means example: data set



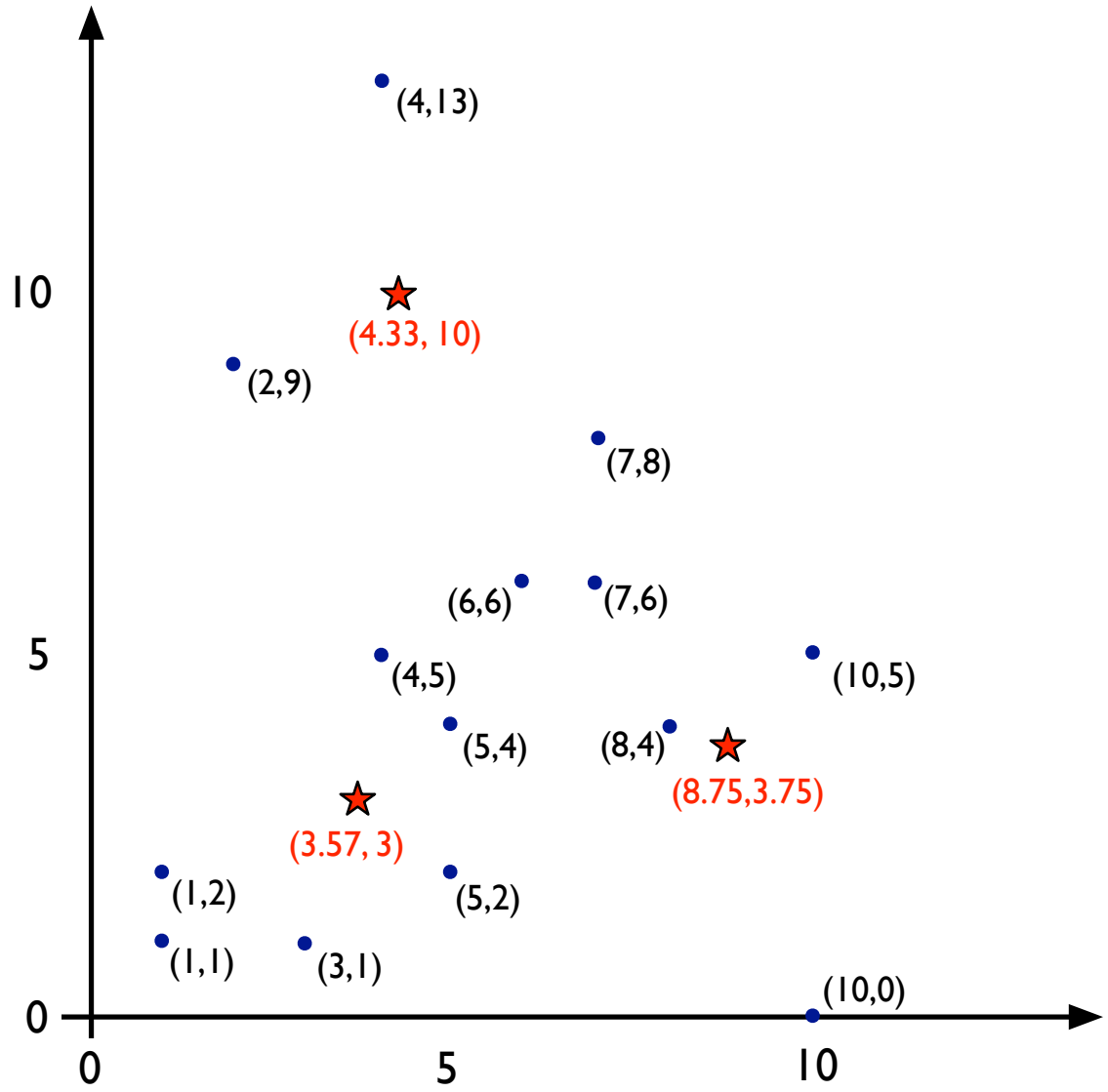
k-means example: initialisation



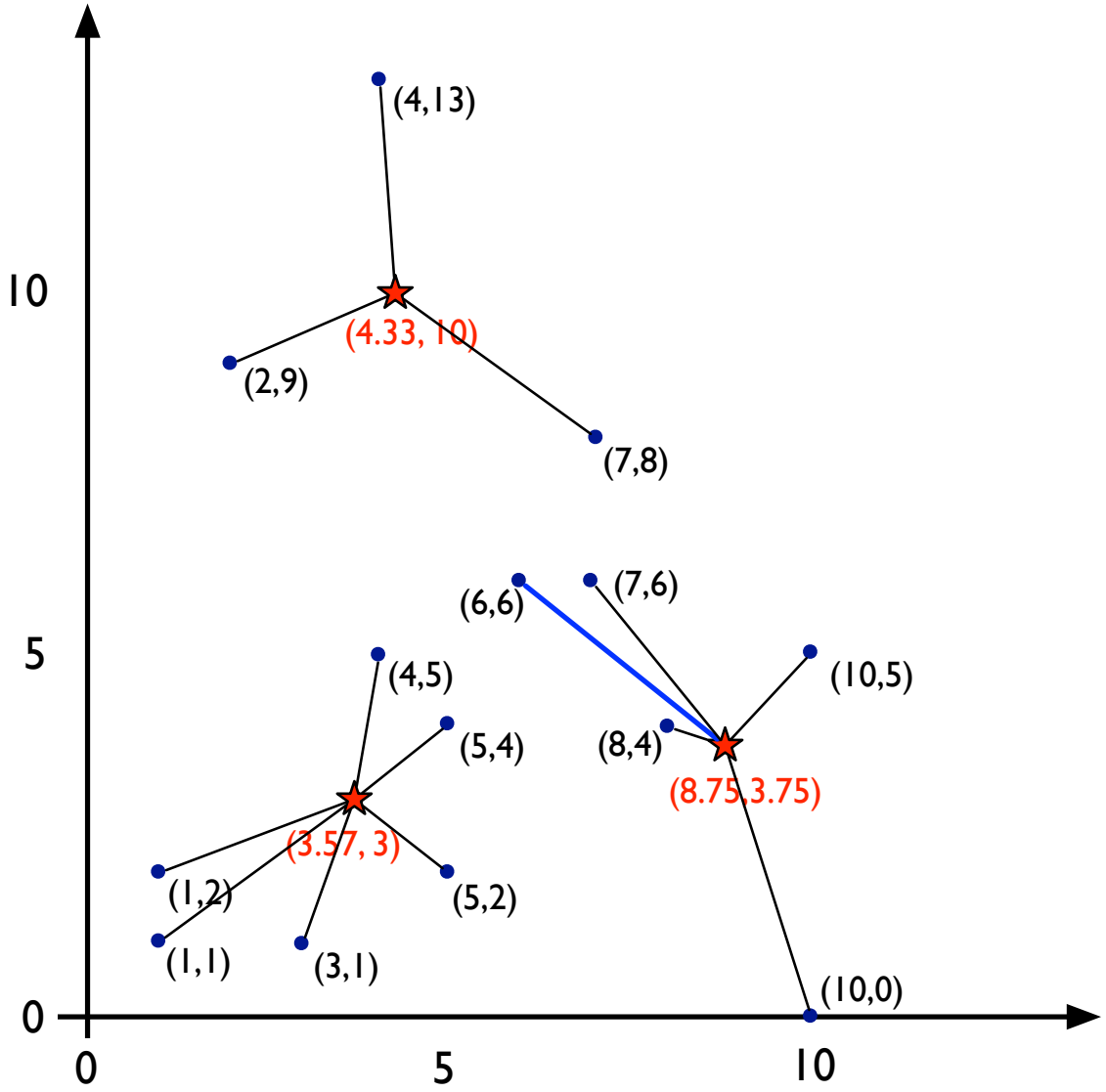
k-means example: iteration 1 (assign points to clusters)



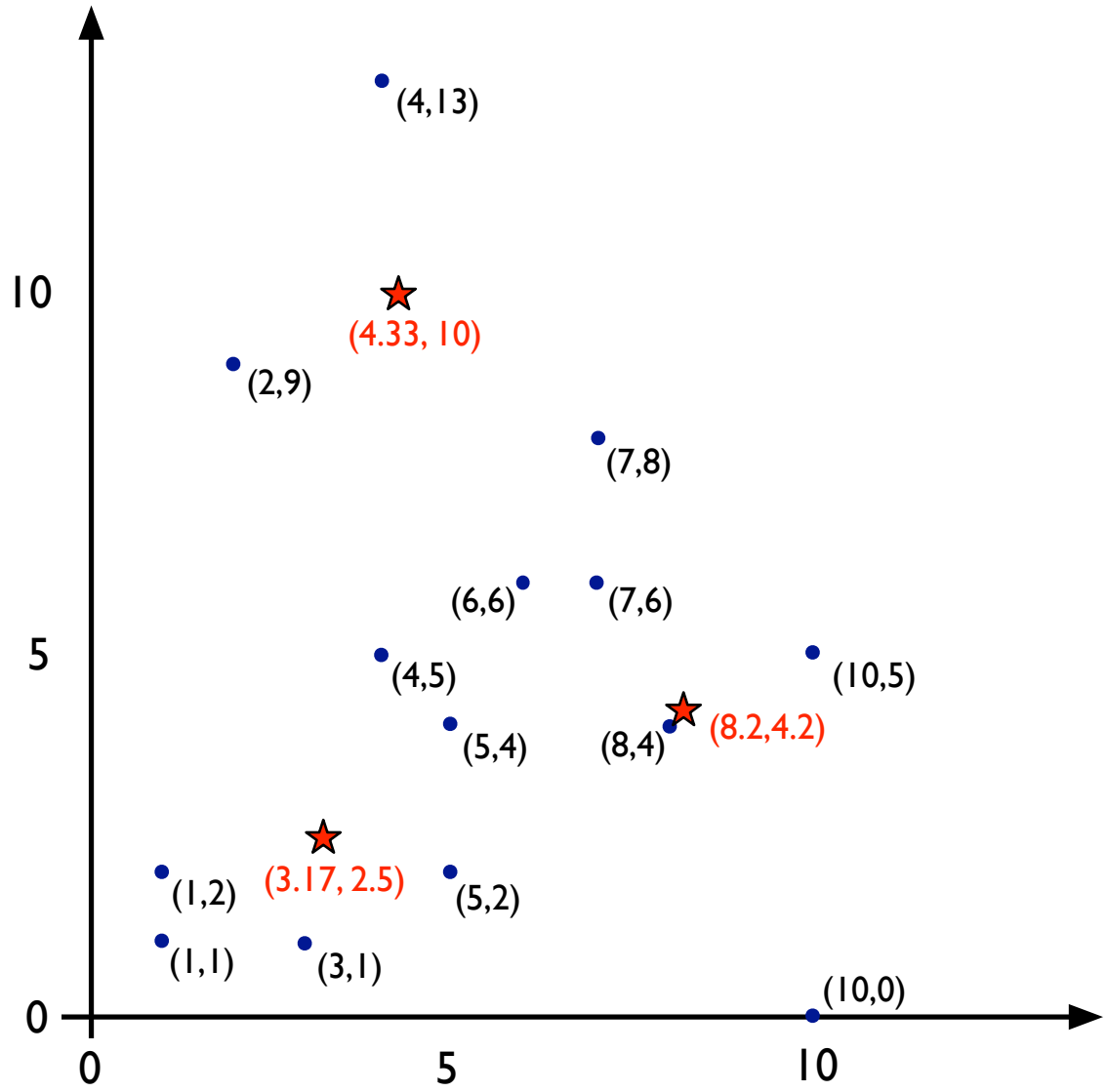
k-means example: iteration 1 (recompute centres)



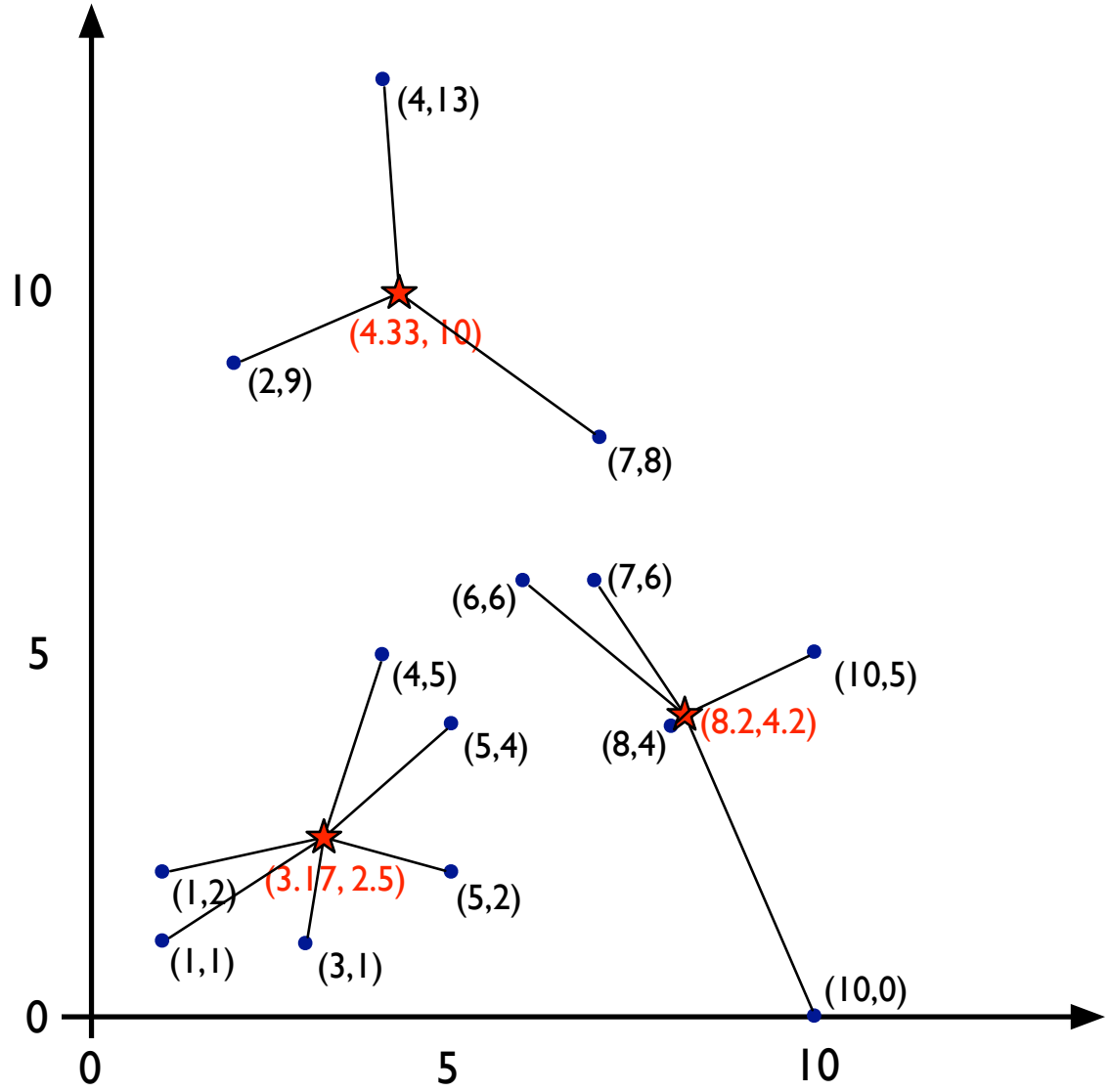
k-means example: iteration 2 (assign points to clusters)



k-means example: iteration 2 (recompute centres)



k-means example: iteration 3 (assign points to clusters)



No changes, so converged

Mixture model

- A more flexible form of density estimation is made up of a linear combination of component densities:

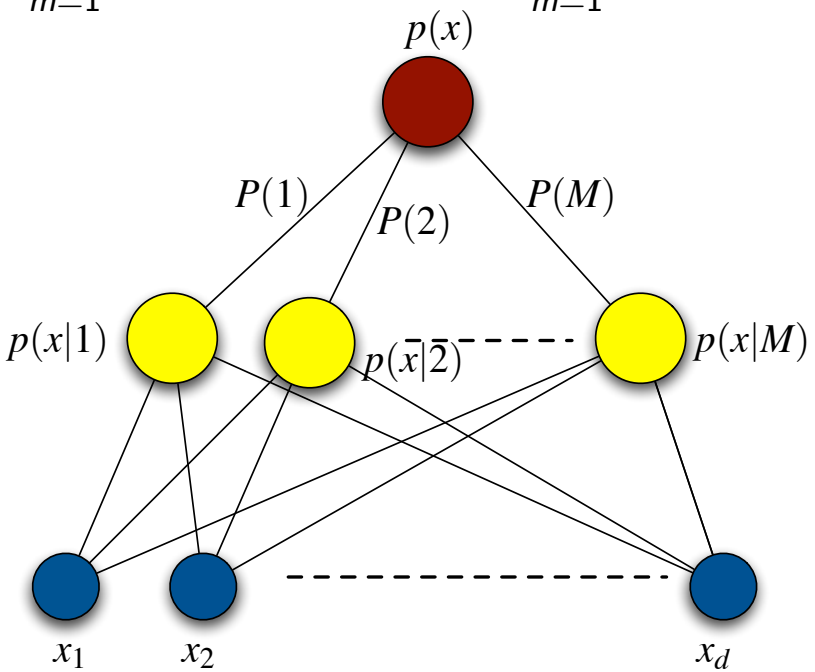
$$p(\mathbf{x}) = \sum_{m=1}^{M} P(m)p(\mathbf{x}\,|\,m)$$

- This is called a *mixture model* or a *mixture density*
- $p(\mathbf{x}\,|\,m)$: component densities
- $P(m)$: mixing parameters
- Generative model:
  1. Choose a mixture component based on $P(m)$
  2. Generate a data point $\mathbf{x}$ from the chosen component using $p(\mathbf{x}\,|\,m)$

Gaussian mixture model

- The most important mixture model is the *Gaussian Mixture Model* (GMM), where the component densities are Gaussians
- Consider a GMM, where each component Gaussian $\mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_m, \boldsymbol{\Sigma}_m)$ has mean $\boldsymbol{\mu}_m$ and a spherical covariance $\boldsymbol{\Sigma}_m = \sigma_m^2 \mathbf{I}$

$$p(\mathbf{x}) = \sum_{m=1}^{M} P(m)\, p(\mathbf{x} \mid m) = \sum_{m=1}^{M} P(m)\, \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_m, \sigma_m^2 \mathbf{I})$$



GMM Parameter estimation when we know which component generated the data

- Define the indicator variable $z_{mt} = 1$ if component m generated data point $\boldsymbol{x}_t$ (and 0 otherwise)
- If z_{mt} wasn't hidden then we could count the number of observed data points generated by m:

$$N_m = \sum_{t=1}^{T} z_{mt}$$

- And estimate the mean, variance and mixing parameters as:

$$\hat{\boldsymbol{\mu}}_m = \frac{\sum_t z_{mt} \boldsymbol{x}_t}{N_m}$$

$$\hat{\sigma}_m^2 = \frac{\sum_t z_{mt} \|\boldsymbol{x}_t - \hat{\boldsymbol{\mu}}_m\|^2}{N_m}$$

$$\hat{P}(m) = \frac{1}{T} \sum_t z_{mt} = \frac{N_m}{T}$$

GMM Parameter estimation when we don't know which component generated the data

- *Problem:* we don't know z_{mt} - which mixture component a data point comes from...

GMM Parameter estimation when we don't know which component generated the data

- *Problem:* we don't know z_{mt} - which mixture component a data point comes from...
- Idea: use the posterior probability $P(m|\mathbf{x})$, which gives the probability that component m was responsible for generating data point $\mathbf{x}$.

$$P(m|\mathbf{x}) = \frac{p(\mathbf{x}|m)\,P(m)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|m)\,P(m)}{\sum_{m'=1}^{M} p(\mathbf{x}|m')P(m')}$$

- The $P(m|\mathbf{x})$s are called the *component occupation probabilities* (or sometimes called the *responsibilities*)
- Since they are posterior probabilities:

$$\sum_{m=1}^{M} P(m|\mathbf{x}) = 1$$

Soft assignment

- Estimate "*soft counts*" based on the component occupation probabilities $P(m|\boldsymbol{x}_t)$:

$$N_m^* = \sum_{t=1}^{T} P(m|\boldsymbol{x}_t)$$

- We can imagine assigning data points to component m weighted by the component occupation probability $P(m|\boldsymbol{x}_t)$
- So we could imagine estimating the mean, variance and prior probabilities as:

$$\hat{\boldsymbol{\mu}}_m = \frac{\sum_t P(m|\boldsymbol{x}_t)\boldsymbol{x}_t}{\sum_t P(m|\boldsymbol{x}_t)} = \frac{\sum_t P(m|\boldsymbol{x}_t)\boldsymbol{x}_t}{N_m^*}$$

$$\hat{\sigma}_m^2 = \frac{\sum_t P(m|\boldsymbol{x}_t)\,\|\boldsymbol{x}_t - \hat{\boldsymbol{\mu}}_m\|^2}{\sum_t P(m|\boldsymbol{x}_t)} = \frac{\sum_t P(m|\boldsymbol{x}_t)\,\|\boldsymbol{x}_t - \hat{\boldsymbol{\mu}}_m\|^2}{N_m^*}$$

$$\hat{P}(m) = \frac{1}{T}\sum_t P(m|\boldsymbol{x}_t) = \frac{N_m^*}{T}$$

EM algorithm

- *Problem!* Recall that:

$$P(m|\mathbf{x}) = \frac{p(\mathbf{x}|m)P(m)}{p(\mathbf{x})} = \frac{p(\mathbf{x}|m)P(m)}{\sum_{m'=1}^{M} p(\mathbf{x}|m')P(m')}$$

  We need to know $p(\mathbf{x}|m)$ and $P(m)$ to estimate the parameters of $P(m|\mathbf{x})$, and to estimate $P(m)$....
- Solution: an iterative algorithm where each iteration has two parts:
  - Compute the component occupation probabilities $P(m|\mathbf{x})$ using the current estimates of the GMM parameters (means, variances, mixing parameters) (E-step)
  - Computer the GMM parameters using the current estimates of the component occupation probabilities (M-step)
- Starting from some initialisation (e.g. using k-means for the means) these steps are alternated until convergence
- This is called the *EM Algorithm* and can be shown to maximise the likelihood. (NB: local maximum rather than global)

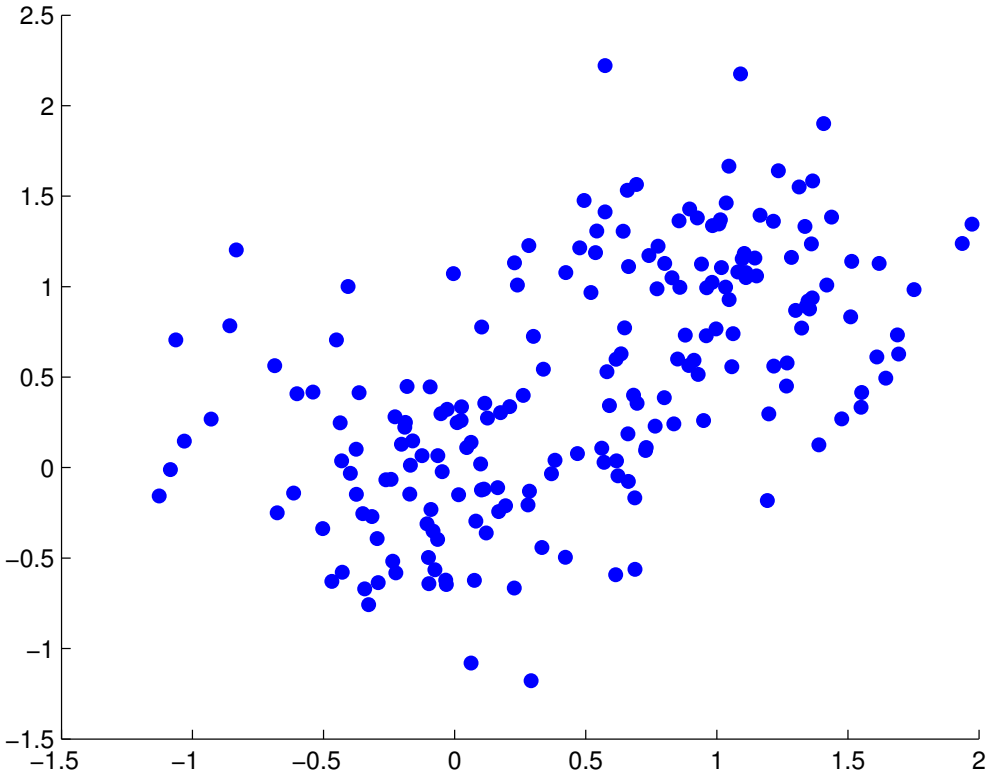
Maximum likelihood parameter estimation

- The likelihood of a data set $\mathbf{X} = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_T\}$ is given by:

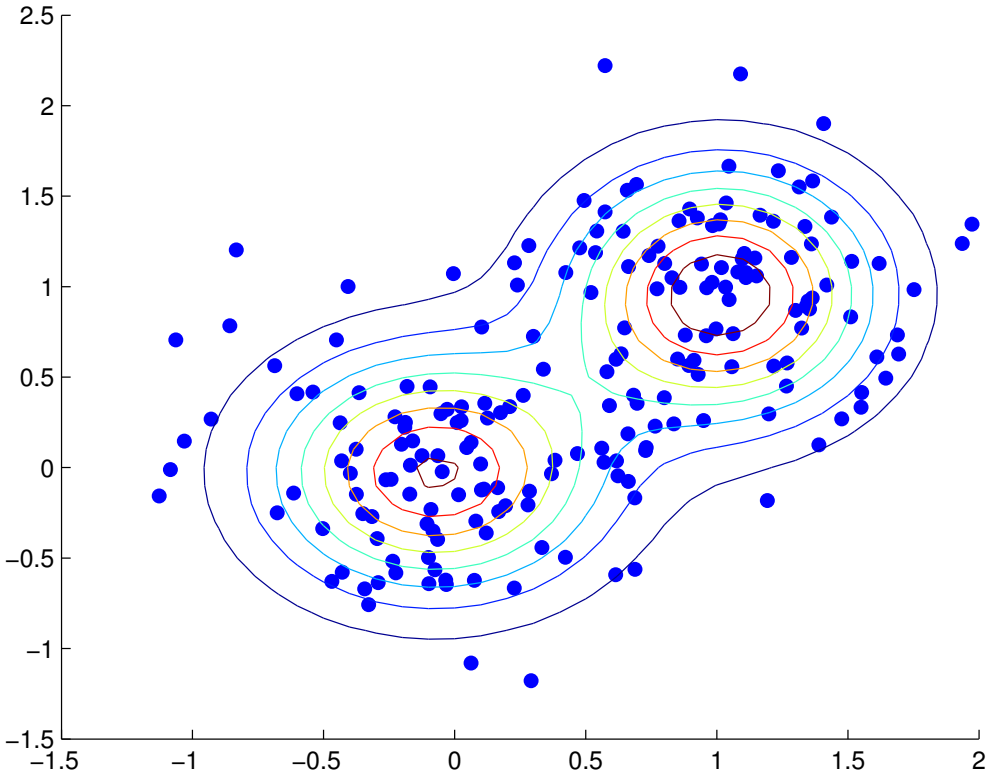
$$\mathcal{L} = \prod_{t=1}^{T} p(\boldsymbol{x}_t) = \prod_{t=1}^{T} \sum_{m=1}^{M} p(\boldsymbol{x}_t \,|\, m)\, P(m)$$

- We can regard the *negative log likelihood* as an error function:
- Considering the derivatives of E with respect to the parameters, gives expressions like the previous slide

Example 1 fit using a GMM

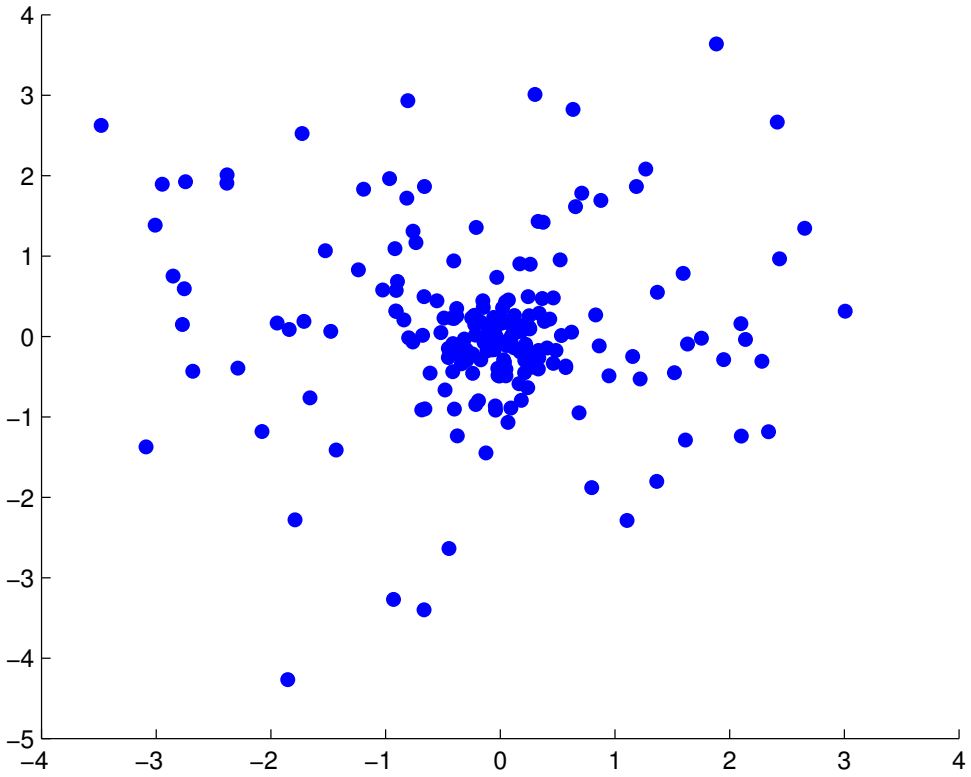


Example 1 fit using a GMM



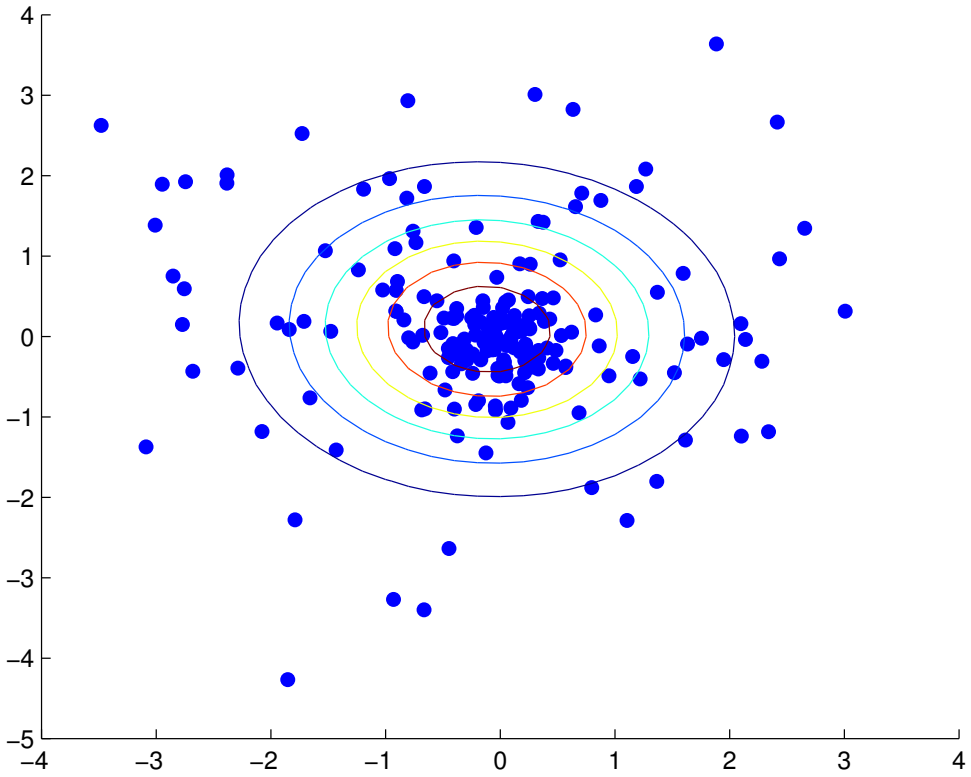
Fitted with a two component GMM using EM

Peakily distributed data (Example 2)



$$\boldsymbol{\mu}_1 = \boldsymbol{\mu}_2 = [0 \quad 0]^T \qquad \boldsymbol{\Sigma}_1 = 0.1\mathbf{I} \qquad \boldsymbol{\Sigma}_2 = 2\mathbf{I}$$

Example 2 fit by a Gaussian



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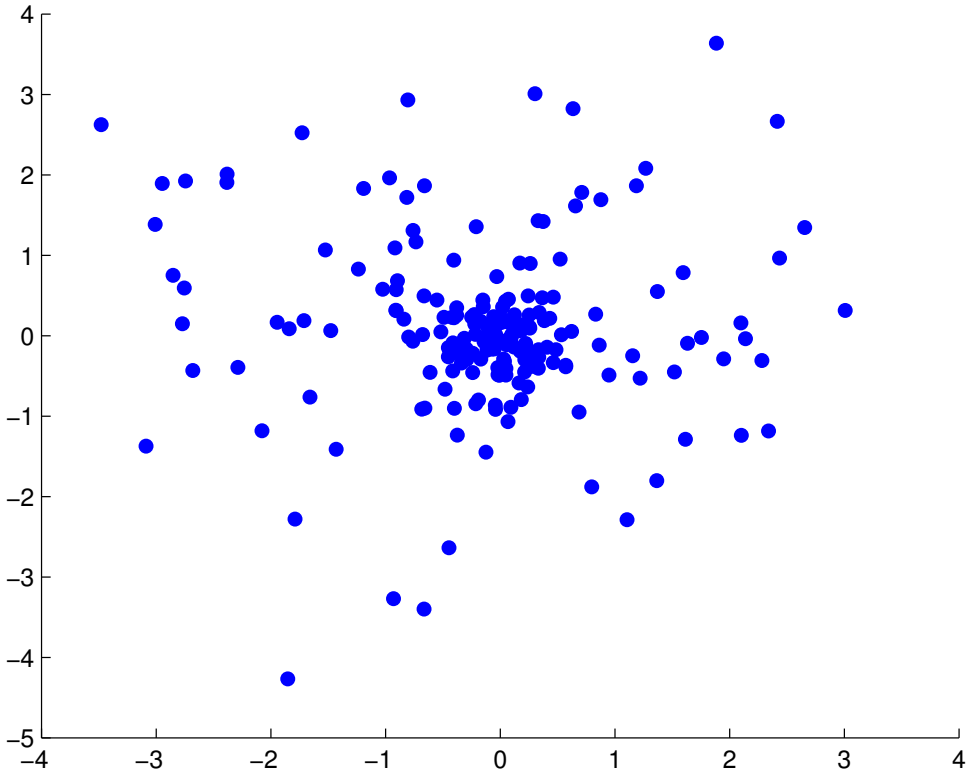
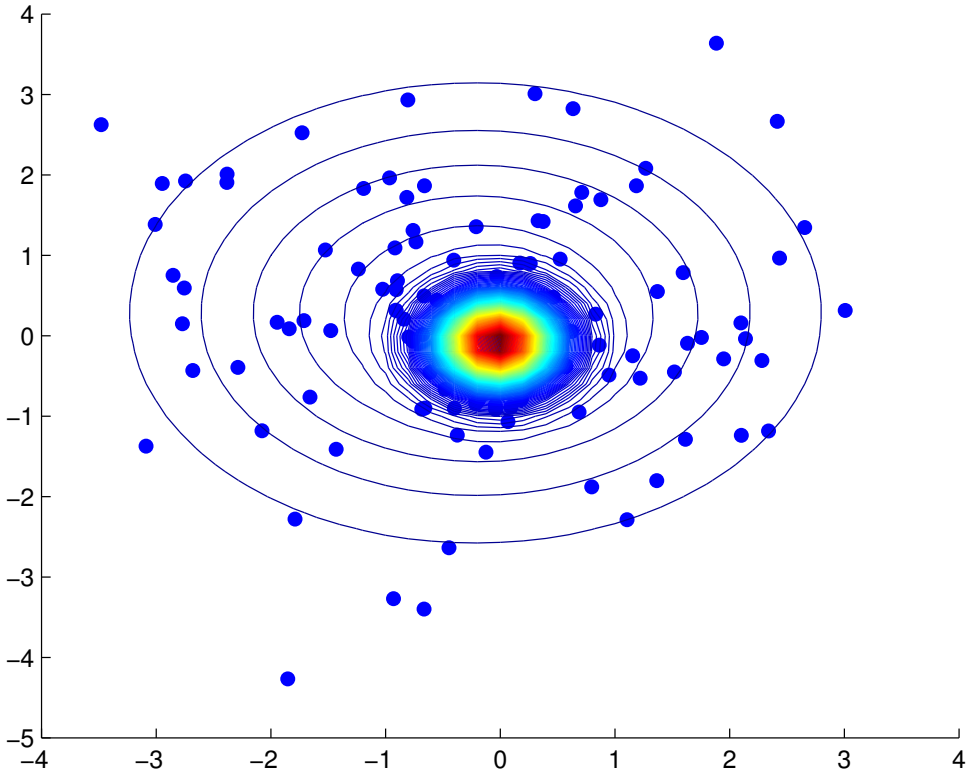
Example 2 fit by a GMM

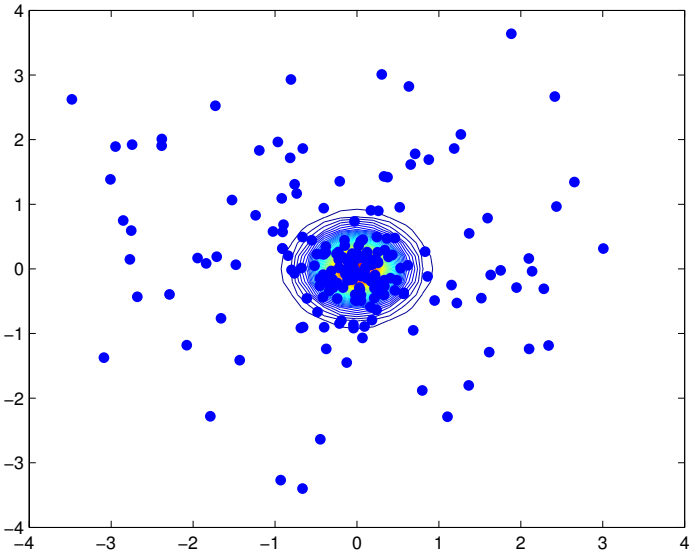


Example 2 fit by a GMM

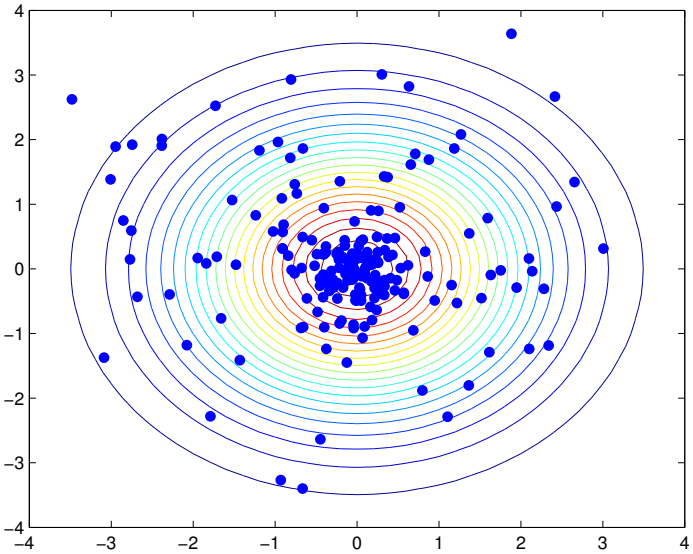


Fitted with a two component GMM using EM

Example 2: component Gaussians



$P(\mathbf{x} \mid m=1)$



$P(\mathbf{x} \mid m=2)$

Comments on GMMs

- GMMs trained using the EM algorithm are able to self organise to fit a data set
- Individual components take responsibility for parts of the data set (probabilistically)
- Soft assignment to components not hard assignment — "soft clustering"
- GMMs scale very well, e.g.: large speech recognition systems can have 30,000 GMMs, each with 32 components: sometimes 1 million Gaussian components!! And the parameters all estimated from (a lot of) data by EM

Next

Back to HMMs:

- Likelihood computation with the Forward algorithm
- Finding the most likely path with the Viterbi algorithm
- Parameter estimation with the Forward-Backward algorithm

References: HMMs

- Gales and Young (2007). "The Application of Hidden Markov Models in Speech Recognition", *Foundations and Trends in Signal Processing*, **1** (3), 195–304: section 2.2.
- Jurafsky and Martin (2008). *Speech and Language Processing* (2nd ed.): sections 6.1–6.5; 9.2; 9.4. (Errata at http://www.cs.colorado.edu/~martin/SLP/Errata/SLP2-PIEV-Errata.html)
- Rabiner and Juang (1989). "An introduction to hidden Markov models", *IEEE ASSP Magazine*, **3** (1), 4–16.
- Renals and Hain (2010). "Speech Recognition", *Computational Linguistics and Natural Language Processing Handbook*, Clark, Fox and Lappin (eds.), Blackwells.