Assignment-2

> 201300069 邓嘉宏
> email: 1739413207@qq.com

Q1

(1)

We construct an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, and $\Delta^{\mathcal{I}} = \{a\}$, for every concept name $A, A^{\mathcal{I}} = \{a\}$, for every role name $r, r^{\mathcal{I}} = \{(a, a)\}$.

We prove by induction on the structure of $\mathcal{EL}$-concept C. The base case is the one where C is a concept name.

- Assume $C = A$, $C^{\mathcal{I}} = A^{\mathcal{I}} = \{a\} \neq \emptyset$
- Assume $C = \top$, $C^{\mathcal{I}} = \top^{\mathcal{I}} = \Delta^{\mathcal{I}} = \{a\} \neq \emptyset$
- Assume $C = D \sqcap E$, $C^{\mathcal{I}} = D^{\mathcal{I}} \cap E^{\mathcal{I}} = \{a\} \cap \{a\} = \{a\} \neq \emptyset$
- Assume $C = \exists r.D$, $C^{\mathcal{I}} = \{a\} \neq \emptyset$

So for this $\mathcal{I}$ and every $\mathcal{EL}$-concept C, $C^{\mathcal{I}} \neq \emptyset$.

(2)

A $\mathcal{EL}$ TBox is a finite set $\mathcal{T}$ of $\mathcal{EL}$ concept inclusions and $\mathcal{EL}$ concept equations. And we can repace a $\mathcal{EL}$ concept equation $C \equiv D$ by two concept inclusions $C \sqsubseteq D$ and $D \sqsubseteq C$, so $\mathcal{T}$ is a finite set of $\mathcal{EL}$ concept inclusions.

We construct an interpretation $\mathcal{I}$ as (1). Then for all $E \sqsubseteq F$ in $\mathcal{T}$, we have $E^{\mathcal{I}} \subseteq F^{\mathcal{I}}$ by $E^{\mathcal{I}} = F^{\mathcal{I}} = \{a\}$, i.e. $\mathcal{I} \models E \sqsubseteq F$, so $\mathcal{I} \models \mathcal{T}$.

Q2

(1)

Consider $\mathcal{T}$:

$$Bird \equiv Vertebrate \sqcap \exists has\_part.Wing$$
$$Reptile \sqsubseteq Vertebrate \sqcap \exists lays.Egg$$

Step 1 gives:

$$Bird \sqsubseteq Vertebrate \sqcap \exists has\_part.Wing$$
$$Vertebrate \sqcap \exists has\_part.Wing \sqsubseteq Bird$$
$$Reptile \sqsubseteq Vertebrate \sqcap \exists lays.Egg$$

Step 2 gives:

$$Bird \sqsubseteq Vertebrate$$
$$Bird \sqsubseteq \exists has\_part.Wing$$
$$Vertebrate \sqcap \exists has\_part.Wing \sqsubseteq Bird$$
$$Reptile \sqsubseteq Vertebrate$$
$$Reptile \sqsubseteq \exists lays.Egg$$

Step 4 gives:

$$Bird \sqsubseteq Vertebrate$$
$$Bird \sqsubseteq \exists has\_part.Wing$$
$$Vertebrate \sqcap X \sqsubseteq Bird$$
$$X \sqsubseteq \exists has\_part.Wing$$
$$\exists has\_part.Wing \sqsubseteq X$$
$$Reptile \sqsubseteq Vertebrate$$
$$Reptile \sqsubseteq \exists lays.Egg$$

This is $\mathcal{T}'$.

(2)

Initialise:

$$
\begin{aligned}
S(Bird) &= \{Bird\} \\
S(Vertebrate) &= \{Vertebrate\} \\
S(Wing) &= \{Wing\} \\
S(X) &= \{X\} \\
S(Reptile) &= \{Reptile\} \\
S(Egg) &= \{Egg\} \\
R(has\_part) &= \emptyset \\
R(lays) &= \emptyset
\end{aligned}
$$

- Application of (simpleR) and axiom 1,6 gives :

$$
\begin{aligned}
S(Bird) &= \{Bird, Vertebrate\} \\
S(Reptile) &= \{Reptile, Vertebrate\}
\end{aligned}
$$

- Application of (rightR) and axiom 2,4,7 gives :

$$
\begin{aligned}
R(has\_part) &= \{(Bird, Wing), (X, Wing)\} \\
R(lays) &= \{(Reptile, Egg)\}
\end{aligned}
$$

- Application of (leftR) and axiom 5 gives :

$$
S(Bird) = \{Bird, Vertebrate, X\}
$$

- No more rules are applicable.

Thus:

$$
\begin{aligned}
S(Bird) &= \{Bird, Vertebrate, X\} \\
S(Reptile) &= \{Reptile, Vertebrate\} \\
R(has\_part) &= \{(Bird, Wing), (X, Wing)\} \\
R(lays) &= \{(Reptile, Egg)\}
\end{aligned}
$$

and no changes for the remaining values.

(3)

Due to $Vertebrate \in S(Reptile)$, but $Bird \notin S(Vertebrate)$, we can obtain $Reptile \sqsubseteq_{\mathcal{T}'} Vertebrate$, but $Vertebrate \sqsubseteq_{\mathcal{T}'} Bird$ is false.

Q3

(1)

We define bisimulation for $\mathcal{ALCN}$:
Let $\mathcal{I}_1 \text{ and } \mathcal{I}_2$ be interpretations. The relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ if:

- (i),(ii),(iii) is defined the same as $\mathcal{ALC}$ bisimulation.
- (iv) $d_1\ \rho\ d_2$ and there are n distinct elements $d_1^i \in \Delta^{\mathcal{I}_1}$, $1 \leq i \leq n$, $(d_1, d_1^i) \in r^{\mathcal{I}_1}$ implies the existence of n distinct elements $d_2^i \in \Delta^{\mathcal{I}_2}$, $(d_2, d_2^i) \in r^{\mathcal{I}_2}$, for all $d_1, d_1^i \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$ and $r \in \mathbf{R}$.
- (v) $d_1\ \rho\ d_2$ and there are n distinct elements $d_2^i \in \Delta^{\mathcal{I}_2}$, $1 \leq i \leq n$, $(d_2, d_2^i) \in r^{\mathcal{I}_2}$ implies the existence of $d_1^i \in \Delta^{\mathcal{I}_1}$, $(d_1, d_1^i) \in r^{\mathcal{I}_1}$, for all $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2, d_2^i \in \Delta^{\mathcal{I}_2}$ and $r \in \mathbf{R}$.

Since we have proved the bisimulation invariance of $\mathcal{ALC}$, we only need add two cases to prove the bisimulation invariance of $\mathcal{ALCN}$:

- Assume that $C = \leq n\ r.\top$. Then $d_1 \in (\leq n\ r.\top)^{\mathcal{I}_1}$ if and only if there exist $m \leq n$ elements $d_1^i \in \Delta^{\mathcal{I}_1} and\ (d_1, d_1^i) \in r^{\mathcal{I}_1}$ (due to semantics,$1 \leq i \leq m$), if and only if there exist $d_2^i \in \Delta^{\mathcal{I}_2}$ such that $(d_2, d_2^i) \in r^{\mathcal{I}_2}$ (due to definition(iv)), if and only if $d_2 \in (\leq n\ r.\top)^{\mathcal{I}_2}$.
- Assume that $C = \geq n\ r.\top$. Then $d_1 \in (\geq n\ r.\top)^{\mathcal{I}_1}$ if and only if there exist $m \geq n$ elements $d_1^i \in \Delta^{\mathcal{I}_1} and\ (d_1, d_1^i) \in r^{\mathcal{I}_1}$ (due to semantics,$1 \leq i \leq m$), if and only if there exist $d_2^i \in \Delta^{\mathcal{I}_2}$ such that $(d_2, d_2^i) \in r^{\mathcal{I}_2}$ (due to definition(iv)), if and only if $d_2 \in (\geq n\ r.\top)^{\mathcal{I}_2}$.

So we have proved the bisimulation invariance of $\mathcal{ALCN}$.

(2)

We construct a $\mathcal{ALCQ}$ concept $C = \leq 1 r.A$, and two interpretations $\mathcal{I}_1, \mathcal{I}_2$, The relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$\begin{aligned}
\Delta^{\mathcal{I}_1} &= \{a, b, c, d\} \\
A^{\mathcal{I}_1} &= \{b\} \\
B^{\mathcal{I}_1} &= \{c, d\} \\
r^{\mathcal{I}_1} &= \{(a, b), (a, c), (a, d)\} \\
\Delta^{\mathcal{I}_2} &= \{e, f, g, h\} \\
A^{\mathcal{I}_2} &= \{f, g\} \\
B^{\mathcal{I}_2} &= \{h\} \\
r^{\mathcal{I}_2} &= \{(e, f), (e, g), (e, h)\} \\
\rho &= \{(a, e), (b, f), (b, g), (c, h), (d, h)\}
\end{aligned}$$

For $\mathcal{ALCQ}$ concept $C = \leq 1 r.A$, we have $a \in \leq 1 r.A, e \notin \leq 1 r.A$. But since $(\mathcal{I}_1, a) \sim (\mathcal{I}_2, e)$, due to the bisimulation invariance of $\mathcal{ALCN}$, we can know $\mathcal{ALCN}$ do not support this concept. So, $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

Q4

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $\mathcal{J} = \biguplus_{v \in \Omega} \mathcal{I}_v$, we extend the notion of disjoint union:

$$\begin{aligned}
\Delta^{\mathcal{J}} &= \{(d, v) \mid v \in \Omega \; and \; d \in \Delta^{\mathcal{I}_v}\} \\
A^{\mathcal{J}} &= \{(d, v) \mid v \in \Omega \; and \; d \in A^{\mathcal{I}_v}\} \text{ for all } A \in \mathbf{C} \\
r^{\mathcal{J}} &= \{((d, v), (e, v)) \mid v \in \Omega \; and \; (d, e) \in r^{\mathcal{I}_v}\} \text{ for all } r \in \mathbf{R} \\
a^{\mathcal{J}} &= (d, v_0), \; d = a^{\mathcal{I}_{v0}} \in \Delta^{\mathcal{I}_{v0}}, v_0 \in \Omega
\end{aligned}$$

From $\textbf{Lemma 3.7.}$ we know:

$$\begin{gathered}
d \in C^{\mathcal{I}_v} \text{ if and noly if } (d, v) \in C^{\mathcal{J}} \\
(d, e) \in r^{\mathcal{I}_v} \text{ if and only if } ((d, v), (e, v)) \in C^{\mathcal{J}}
\end{gathered}$$

We prove $\mathcal{J}$ is a model of $\mathcal{K}$:
Assume $\mathcal{J}$ is not a model of $\mathcal{K}$, since $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, we have these cases:\

- if $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there is a GCI $C \sqsubseteq D$ in $\mathcal{T}$, and an element $(d, v) \in C^{\mathcal{J}}, (d, v) \notin D^{\mathcal{J}}$, in this case, from $\textbf{Lemma 3.7.}$, we know $d \in C^{\mathcal{I}_v}, d \notin D^{\mathcal{I}_v}$, this contradicts to $\mathcal{I}_v$ is a model of $\mathcal{K}$.
- if $\mathcal{J}$ is not a model of $\mathcal{A}$.
  If there exist a concept assertion $a : C$ and an element $(a^{\mathcal{I}_v}, v) \notin C^{\mathcal{J}}$, from $\textbf{Lemma 3.7.}$, we know $a^{\mathcal{I}_v} \notin C^{I_v}$, this contradicts to $\mathcal{I}_v$ is a model of $\mathcal{K}$.

- If there exist a role assertion $(a, b) : r$ and $((a^{\mathcal{I}_v}, v), (b^{\mathcal{I}_v}, v)) \notin r^{\mathcal{J}}$, from $\mathbf{Lemma\ 3.7.}$, we know $(a^{\mathcal{I}_v}, b^{\mathcal{I}_v}) \notin r^{I_v}$, this contradicts to $\mathcal{I}_v$ is a model of $\mathcal{K}$.

So we have proved $\mathcal{J}$ is a model of $\mathcal{K}$.

Q5

$\Longrightarrow$:
if $C \sqsubseteq_{\mathcal{K}} D$,

$\Longleftarrow$:
if $C \sqsubseteq_{\mathcal{T}} D$, then for every model $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds. As we know every model $\mathcal{I}$ of $\mathcal{K}$ is a model of $\mathcal{T}$, so if every model $\mathcal{I}$ of $\mathcal{T}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds , we have for every model $\mathcal{I}$ of $\mathcal{K}$, $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds, that is said $C \sqsubseteq_{\mathcal{K}} D$.

Q6

(1)

True.

As C is a $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, by finite model property, we know there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $|C^{\mathcal{I}}| \geq 1$.

Let $\mathcal{I}_m = \biguplus_{1 \leq v \leq m} \mathcal{I}_v$ be m-fold disjoint union of $\mathcal{I}$ with itself, then by $\mathbf{Lemma\ 3.7.}$ and $\mathbf{Theorem\ 3.8.}$, we know $\mathcal{I}_m$ is a model of $\mathcal{T}$ and $|C^{\mathcal{I}_m}| = m|C^{\mathcal{I}}| \geq m$.

(2)

False.

Construct $\mathcal{T} = \{A \sqsubseteq \exists r.\neg A, \neg A \sqsubseteq \exists r.A\}$, $C = \top$, and C is satisfiable w.r.t $\mathcal{T}$.

Let $m = 1$, assume there is a finite model $\mathcal{I}$ of $\mathcal{T}$ such that $|C^{\mathcal{I}}| = |\Delta^{\mathcal{I}}| = m = 1$. That is said the domain of $\mathcal{T}$ has only one element, so there must have $A = \emptyset\ or\ \neg A = \emptyset$. But:

- If $A = \emptyset$, then $\exists r.A = \emptyset$. As $\mathcal{I}$ is a model of $\mathcal{T}$, so $\mathcal{T} \models \neg A \sqsubseteq \exists r.A$, so $\neg A = \emptyset$, contradiction occur.
- If $\neg A = \emptyset$, then $\exists r.\neg A = \emptyset$. As $\mathcal{I}$ is a model of $\mathcal{T}$, so $\mathcal{T} \models A \sqsubseteq \exists r.\neg A$, so $A = \emptyset$, contradiction occur.

Q7

False.

We construct $\mathcal{T} = \{A \sqsubseteq \exists r.\top\}$, $C = \top$, as shown in lecture. Then $sub(C) \cup sub(\mathcal{T}) = \{\top, A, \exists r.\top\}$, we construct $\mathcal{I}$ as follow:

$$\begin{aligned} \Delta^{\mathcal{I}} &= \{d_1, d_1', d_2, d_2'\} \\ A^{\mathcal{I}} &= \{d_1, d_1', d_2\} \\ r^{\mathcal{I}} &= \{(d_1, d_2), (d_1', d_2')\} \end{aligned}$$

and its filtration $\mathcal{J}$ w.r.t. $sub(C) \cup sub(\mathcal{T})$ are as follow:

$$\begin{aligned} \Delta^{\mathcal{J}} &= \{[d_1], [d_1'], [d_2], [d_2']\} \\ A^{\mathcal{J}} &= \{[d_1], [d_1'], [d_2]\} \\ r^{\mathcal{J}} &= \{([d_1], [d_2]), ([d_1'], [d_2]), ([d_1], [d_2']), ([d_1'], [d_2'])\} \end{aligned}$$

Assume $\rho = \{(d, [d]) \mid d \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$. So $(d_1, [d_1]) \in \rho$, but $[d_1]$ has an r-successor in $\mathcal{J}$ that does not belong to the extension of A, whereas d_1 does not have such an r-successor in $\mathcal{I}$, contradiction occur， so this ρ is not a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

Q8

(1)

(i) $d \approx_{\mathcal{I}} e$ implies there is a bisimulation ρ on $\mathcal{I}$ such that $d \ \rho \ e$, i.e. ,$d \in A^{\mathcal{I}}$ if and only if $e \in A^{\mathcal{I}}$, for all $d, e \in \Delta^{\mathcal{I}}$, and $A \in \mathbf{C}$;

(ii)$d \approx_{\mathcal{I}} e$ and $(d, d') \in r^{\mathcal{I}}$ implies $d \ \rho \ e$ and $(d, d') \in r^{\mathcal{I}}$, implies the existence of $e' \in \Delta^{\mathcal{I}}$ such that $d' \ \rho \ e'$ and $(e, e') \in r^{\mathcal{I}}$, i.e.,$d' \approx_{\mathcal{I}} e'$ and $(e, e') \in r^{\mathcal{I}}$, for all $d, d', e \in \Delta^{\mathcal{I}}$ and $r \in \mathbf{R}$.

(iii) $d \approx_{\mathcal{I}} e$ and $(e, e') \in r^{\mathcal{I}}$ implies $d \ \rho \ e$ and $(e, e') \in r^{\mathcal{I}}$, implies the existence of $d' \in \Delta^{\mathcal{I}}$ such that $d' \ \rho \ e'$ and $(d, d') \in r^{\mathcal{I}}$, i.e.,$d' \approx_{\mathcal{I}} e'$ and $(d, d') \in r^{\mathcal{I}}$, for all $d, e, e' \in \Delta^{\mathcal{I}}$ and $r \in \mathbf{R}$.

So $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

(2)

We modify the definition about filtration:

$$
\begin{aligned}
[d]_{\approx_\mathcal{I}} &= \{e \in \Delta^\mathcal{I} \mid d \approx_\mathcal{I} e\};\\
\Delta^\mathcal{J} &= \{[d]_{\approx_\mathcal{I}} \mid d \in \Delta^\mathcal{I}\};\\
A^\mathcal{J} &= \{[d]_{\approx_\mathcal{I}} \mid \text{there is } d' \in [d]_{\approx_\mathcal{I}} \text{ with } d' \in A^\mathcal{I}\} \text{ for all } A \in \mathbf{C};\\
r^\mathcal{J} &= \{([d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}}) \mid \text{there are } d' \in [d]_{\approx_\mathcal{I}}, e' \in [e]_{\approx_\mathcal{I}} \text{ with } (d', e') \in r^\mathcal{I}\} \text{ for all } r \in \mathbf{R}.
\end{aligned}
$$

(i) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ implies:
$d \in A^\mathcal{I}$ if and only if $e \in A^\mathcal{I}$,(if $d \approx_\mathcal{I} e, \text{for all } e$) if and only if $[d]_{\approx_\mathcal{I}} \in A^\mathcal{J}$,(due to all e and d, by definition $d, e \in [d]_{\approx_\mathcal{I}}$), for all $d \in \Delta^\mathcal{I}, [d]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}, \text{and } A \in \mathbf{C}$;

(ii) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ and $(d, d') \in r^\mathcal{I}$ implies:
the existence of $[d']_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$ such that $d \in [d]_{\approx_\mathcal{I}}, d' \in [d']_{\approx_\mathcal{I}}$ with $(d, d') \in r^\mathcal{I}$, so $([d]_{\approx_\mathcal{I}}, [d']_{\approx_\mathcal{I}}) \in r^\mathcal{J}$ and $(d', [d']_{\approx_\mathcal{I}}) \in \rho$, for all $d, d' \in \Delta^\mathcal{I}, [d]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}, \text{and } r \in \mathbf{R}$;

(iii) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ and $([d]_{\approx_\mathcal{I}}, [d']_{\approx_\mathcal{I}}) \in r^\mathcal{J}$ implies:
$d_1 \in [d]_{\approx_\mathcal{I}}, d'_1 \in [d']_{\approx_\mathcal{I}}$ with $(d_1, d'_1) \in r^\mathcal{I}$, because $d \approx_\mathcal{I} d_1$,so there exist $d' \in \Delta^\mathcal{I}$ such that $d' \approx_\mathcal{I} d'_1$ and $(d, d') \in r^\mathcal{I}$, $(d', [d']_{\approx_\mathcal{I}}) \in \rho$, for all $d \in \Delta^\mathcal{I}, [d]_{\approx_\mathcal{I}}, [d']_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$ and $r \in \mathbf{R}$.

So $\rho = \{(d, [d]_{\approx_\mathcal{I}}) \mid d \in \Delta^\mathcal{I}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

(3)

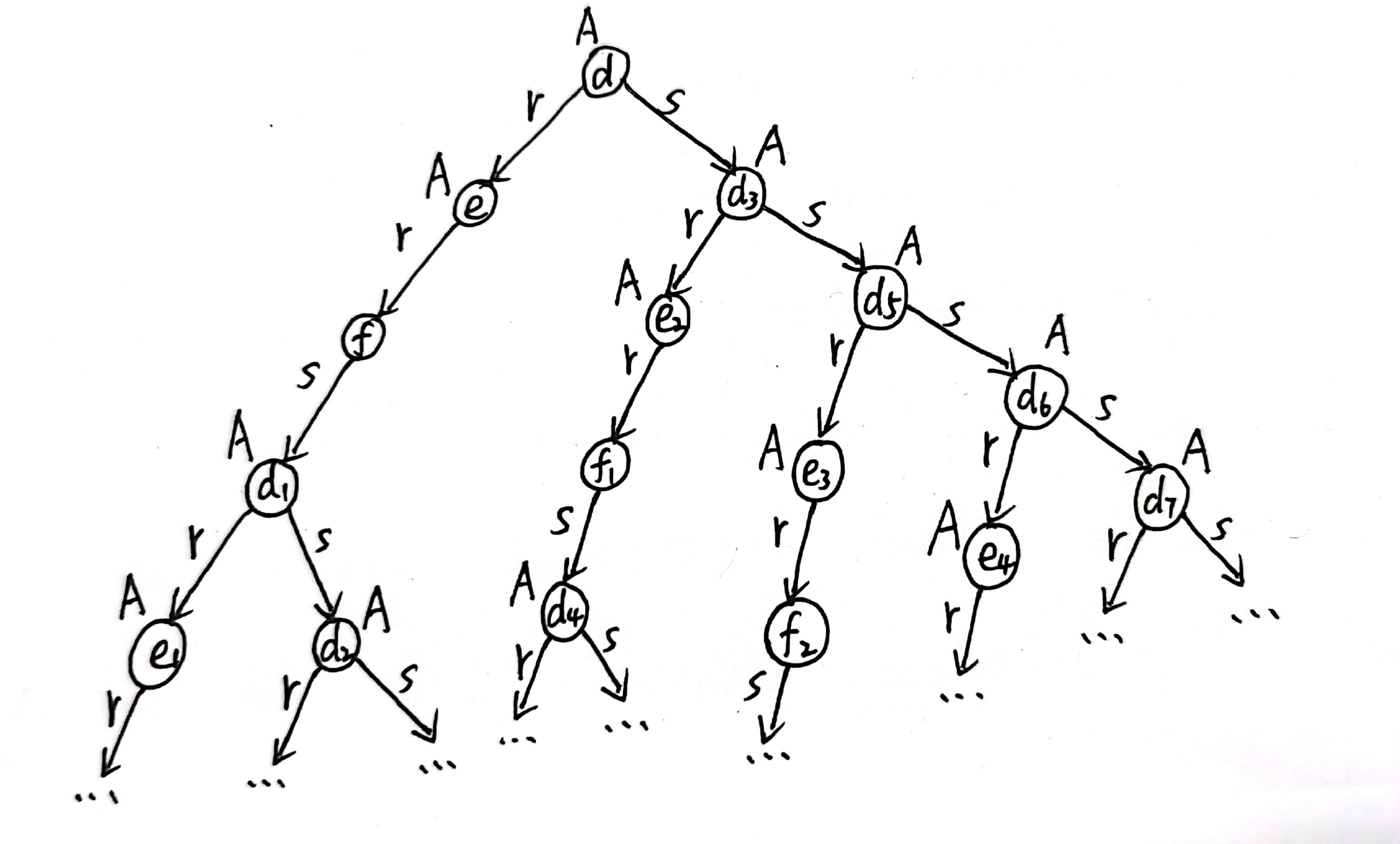
If $\mathcal{I}$ is a model of an $\mathcal{ALC}$- concept C w.r.t. an $\mathcal{ALC}$-Tbox $\mathcal{T}$, then:

- For every GCI $D \sqsubseteq E$ in $\mathcal{T}$, we have if $a \in D^\mathcal{I}, then\ a \in E^\mathcal{I}$, by bisimulation invariance of $\mathcal{ALC}$, we have if $[a]_{\approx_\mathcal{I}} \in D^\mathcal{J}, then\ [a]_{\approx_\mathcal{I}} \in E^\mathcal{J}$.
- If $a \in C^\mathcal{I}$, by bisimulation invariance of $\mathcal{ALC}$, we have $[a]_{\approx_\mathcal{I}} \in C^\mathcal{J}$.

So $\mathcal{J}$ is also a model of the $\mathcal{ALC}$-concept C w.r.t. the $\mathcal{ALC}$-Tbox $\mathcal{T}$.

Q9

The unravelling of the following interpretation $\mathcal{I}$ at d up to depth 5 :



Q10

False.
We construct a $\mathcal{K}$, where $\mathcal{T} = \emptyset$, $\mathcal{A} = \{a : A, (a, a) : r\}$. And an $\mathcal{ALC}$-concept $C = \top$.
In this case, for every model $\mathcal{I}$ of $\mathcal{K}$, we know $a^{\mathcal{I}} \in A^{\mathcal{I}}, (a^{\mathcal{I}}, a^{\mathcal{I}}) \in r^{\mathcal{I}}$, and the graph of $\mathcal{I}$ have a cycle "$a^{\mathcal{I}} \rightarrow_r a^{\mathcal{I}}$", this is not a tree, so every model $\mathcal{I}$ of $\mathcal{K}$ is not a tree model.

Q11

$\mathcal{A}_0 = \{(b, a) : r, (a, b) : r, (a, c) : s, (c, b) : s, a : \exists s.A, b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B)), c : \forall s.(B \sqcap (\forall s.\bot))\}$

Apply rule $\rightarrow_\exists$ gives:

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{(a, d) : s, d : A\}$$

Apply rule $\rightarrow_\forall$ gives:

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{a : (\forall s.\neg A) \sqcup (\exists r.B)\}$$

Apply rule $\rightarrow_\forall$ gives:

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{b : B \sqcap (\forall s.\bot)\}$$

Apply rule $\rightarrow_\sqcap$ gives:

$$\mathcal{A}_4 = \mathcal{A}_3 \cup \{b : B, b : \forall s.\bot\}$$

Apply rule $\rightarrow_\sqcup$. We have two possibilities:

- Firstly we can try :

  $$\mathcal{A}_5 = \mathcal{A}_4 \cup \{a : \forall s.\neg A\}$$

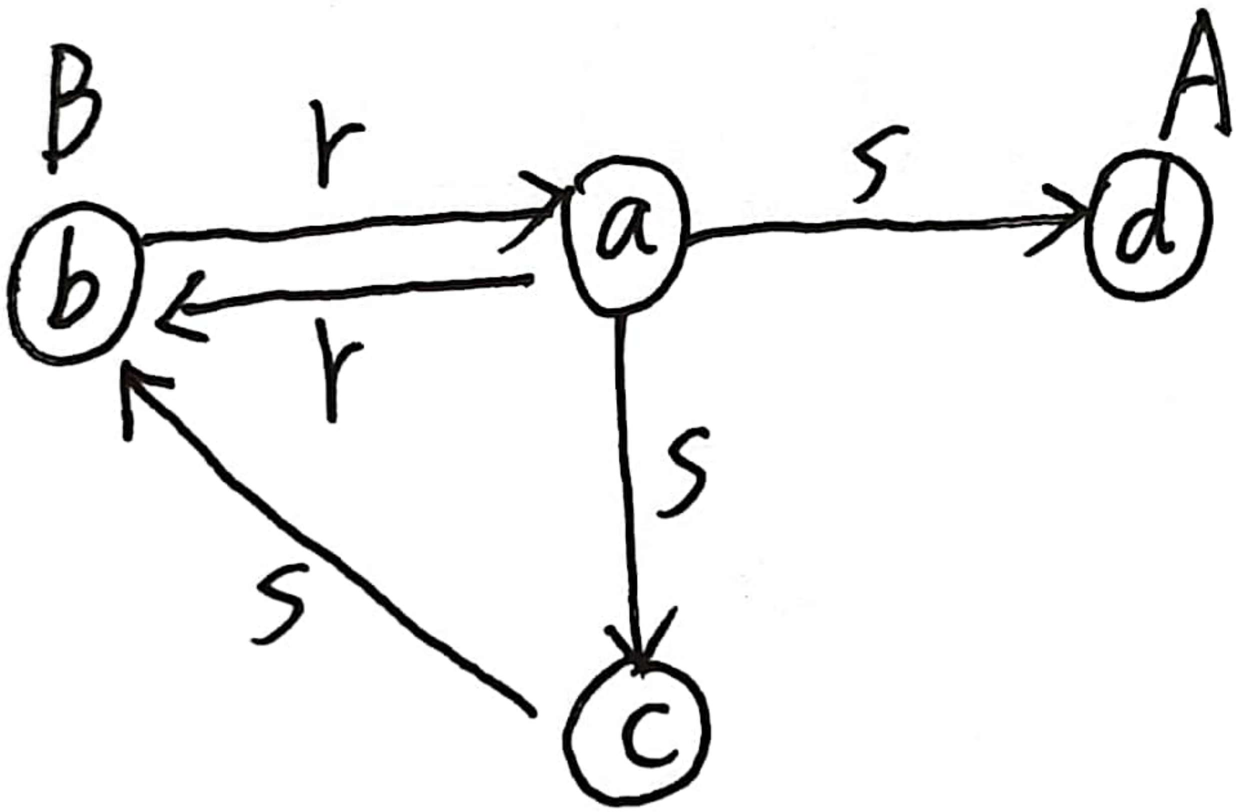
  Then apply rule $\rightarrow_\forall$ gives:

  $$\mathcal{A}_6 = \mathcal{A}_5 \cup \{c : \neg A, d : \neg A\}$$

  Due to $d : A, d : \neg A$, we have obtained a clash, thus this choice was unsuccessful.
- Secondly, we can try:

  $$\mathcal{A}_5^* = \mathcal{A}_4 \cup \{a : \exists r.B\}$$

No rule is applicable to $\mathcal{A}_5^*$ and it does not contain a clash.
So $\mathcal{A}$ is consistent:



Q12