Stochastic Finance (2016-17, M3)
Some Problem Solutions for **SCFA**

Exercise 2.1 The roots of the equation qualify for the x

$$E(x_1^X) = \frac{0.52}{x} + 0.45x + 0.03x^2 = 1.$$

Form
$$\frac{0.52}{x} + 0.45x + 0.03x^2 - 1 = \frac{x-1}{x}\big(0.03x^2 + 0.48x - 0.52\big)$$
we have the following three values.

$$x = 1,\ \frac{-0.24 \pm \sqrt{0.24^2 + 0.03 \times 0.24}}{0.03} = 1,\ 1.01850,\ -17.0185.$$

We pick $x = 1.01850$ since the change under high powers are reasonable. If we let τ be the first time Gambler's wealth is either 100, 101 or -100, we have the equation from the Martingale property,

$$1 = E(M_\tau) = x^{100}P(S_\tau = 100) + x^{101}P(S_\tau = 101) + x^{-100}P(S_\tau = -100).$$

Letting $p = P(S_\tau = 100) + P(S_\tau = 101)$ and using the fact that $x > 1$,

$$x^{100}p + x^{-100}(1-p) < 1 < x^{101}p + x^{-100}(1-p)$$
$$\frac{1 - x^{-100}}{x^{101} - x^{-100}} < p < \frac{1 - x^{-100}}{x^{100} - x^{-100}} \quad \Rightarrow \quad 0.13531 < x < 0.13788.$$

Exercise 3.1 (Brownian Bridge) (a) This problem is based on a series representation of Brownian motion (also see p.286 of Steele),

$$B_t = tZ_0 + \sum_{k=0} + \sqrt{2}\sum_{k=1}^{\infty} Z_k \frac{\sin \pi k t}{\pi k}$$

for independent standard normal random variables, $\{Z_{k\geq 0}\}$. (But I think the author somehow dropped this in the current version of the textbook.) So, $\Delta_0(t) = t$ and $\lambda_0 = 1$. Because $B_1 = Z_0$ (from $\Delta_{k\geq 1}(1) = 0$), the first term can be expressed as $\lambda_0 Z_0 \Delta_0(t) = t\,B_1$. Therefore,

$$U_t = B_t - tB_1$$

(b)

$$\begin{aligned}\mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_sB_t - sB_1B_t - tB_sB_1 + stB_1^2\Big)\\ &= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t)\end{aligned}$$

(c) We need to find any set of functions, $g(\cdot)$ and $h(\cdot)$, such that

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)\min(h(s), h(t)) = s(1-t) \quad \text{for} \quad s \leq t.$$

If we narrow down the search by assuming $h(\cdot)$ is monotonically increasing,

$$\mathrm{Cov}(X_s, X_t) = g(s)g(t)h(s) = s(1-t),$$

so we get

$$g(t) = 1 - t, \quad h(s) = \frac{s}{1-s},$$

where $h(s)$ is indeed an increasing function. Therefore we obtained a representation of Brownian bridge,

$$X_t = (1-t)B_{\frac{t}{1-t}}$$

Since U_{1-t} is also a Brownian bridge due to the symmetry,

$$X_{1-t} = tB_{\frac{1-t}{t}}, \quad \Big(g(t) = t,\ h(t) = \frac{1-t}{t}\Big)$$

is also a valid solution.

(d) We use the inequality $s/(1+s) \le t/(1+t)$ if $0 \le s \le t$.

$$\begin{aligned}\mathrm{Cov}(Y_s, Y_t) &= \mathrm{Cov}\Big((1+s)U_{\frac{s}{1+s}}(1+t)U_{\frac{t}{1+t}}\Big) = (1+s)(1+t)\mathrm{Cov}(U_{\frac{s}{1+s}}, U_{\frac{t}{1+t}}) \\ &= (1+s)(1+t)\,\frac{s}{1+s}\Big(1 - \frac{t}{1+t}\Big) = s = \min(s,t).\end{aligned}$$

Exercise 3.2 (Cautionary Tale) Suppose X is a standard normal, consider an independent U such that $P(U=1) = 1/2 = P(U=1)$, and set $Y = UX$. Then, Y is also a standard normal as X and $-X$ are also standard normal.

In order to show X and Y are not independent, we need to show

$$\mathrm{Prob}(I_X \ \&\ J_Y) \neq \mathrm{Prob}(I_X)\mathrm{Prob}(J_Y)$$

for some event I_X and I_Y regarding X and Y respectively.

For any $h > 0$,

$$\begin{aligned}\mathrm{Prob}(X > h \ \&\ Y > h) &= \frac{1}{2}\mathrm{Prob}(X > h \ \&\ X > h \mid U = 1) \\ &\quad + \frac{1}{2}\mathrm{Prob}(X > h \ \&\ -X > h \mid U = -1) \\ &= \frac{1}{2}(1 - \Phi(h)) + 0.\end{aligned}$$

However,

$$\mathrm{Prob}(X > h)\mathrm{Prob}(Y > h) = (1 - \Phi(h))(1 - \Phi(h))$$

is not same as the previous value. Therefore

Exercise 3.3 (Multivariage Gaussians)

(a) Let us work on each components of the vectors and matrices; $V = (v_i)$, $\mu = (\mu_i)$, $A = (a_{ij})$ and $\Sigma = (\sigma_{ij})$.

$$E\Big((AV)_i\Big) = E\Big(\sum_j a_{ij}V_j\Big) = \sum_j a_{ij}E\Big(V_j\Big) = \sum_j a_{ij}\mu_j = (A\mu)_i$$

$$E\Big((AV)\Big) = A\mu$$

$$
\begin{aligned}
\mathrm{Cov}\Big((AV)_i,(AV)_j\Big) &= \mathrm{Cov}\Big(\sum_l a_{il}V_l, \sum_m a_{jm}V_m\Big) = \sum_{l,m} a_{il}\mathrm{Cov}(V_l,V_m)a_{jm} \\
&= \sum_{l,m} a_{il}\sigma_{lm}a_{jm} = (A\Sigma A^T)_{ij}
\end{aligned}
$$

Therefore,

$$\mathrm{Cov}\Big(AV,AV\Big) = A\Sigma A^T.$$

(b)

$$E(X \pm Y) = E(X) \pm E(Y) = 0 \pm 0 = 0$$

$$\mathrm{Var}(X \pm Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \pm 2\,\mathrm{Covar}(X,Y) = 1 + 1 + 0 = 2$$

$$\mathrm{Cov}(X+Y, X-Y) = \mathrm{Var}(X) - \mathrm{Var}(Y) = 1 - 1 = 0$$

(c) When $\mathrm{Cov}(X,Y) = 0$, the covariance matrix Σ is given as

$$\Sigma = \begin{pmatrix} \sigma_{11} & 0 \\ 0 & \sigma 22 \end{pmatrix}, \quad \Sigma^{-1} = \begin{pmatrix} 1/\sigma_{11} & 0 \\ 0 & 1/\sigma 22 \end{pmatrix}, \quad \det\Sigma = \sigma_{11}\,\sigma_{22}$$

The joint density function can be factored to the product of the single variable density function,

$$
\begin{aligned}
f(x,y) &= \frac{1}{2\pi\sqrt{\sigma_{11}\,\sigma_{22}}}\exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}} - \frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) \\
&= \frac{1}{2\pi\sqrt{\sigma_{11}}}\exp\Big(-\frac{(x-\mu_X)^2}{2\sigma_{11}}\Big)\,\frac{1}{2\pi\sqrt{\sigma_{22}}}\exp\Big(-\frac{(y-\mu_Y)^2}{2\sigma_{22}}\Big) = f(x)f(y).
\end{aligned}
$$

Therefore X and Y are independent.

(d) We first find the matrix A such that, for the independent standard normal variables W and Z,

$$\begin{pmatrix} X-\mu_X \\ Y-\mu_Y \end{pmatrix} = A\begin{pmatrix} W \\ Z \end{pmatrix}$$

has the given covariance matrix

$$\begin{pmatrix} \sigma_{XX} & \sigma_{XY} \\ \sigma_{XY} & \sigma_{YY} \end{pmatrix} = A\,I\,A^T = A\,A^T.$$

One of the solution from Cholesky decomposition is

$$A = \begin{pmatrix} \sqrt{\sigma_{XX}} & 0 \\ \sigma_{XY}/\sqrt{\sigma_{XX}} & \sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}} \end{pmatrix}.$$

Conditional on that $X = x$,

$$Y = \mu_Y + \frac{\sigma_{XY}}{\sqrt{\sigma_{XX}}}\frac{x-\mu_X}{\sqrt{\sigma_{XX}}} + Z\sqrt{\sigma_{YY} - \sigma_{XY}^2/\sigma_{XX}}.$$

Therefore

$$
\begin{aligned}
E(Y|X=x) &= \frac{\sigma_{XY}}{\sigma_{XX}}(x-\mu_X) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}(x-\mu_X) \\
\mathrm{Var}(Y|X=x) &= \sigma_{YY} - \frac{\sigma_{XY}^2}{\sigma_{XX}} = \mathrm{Var}(Y) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(X)}
\end{aligned}
$$

Exercise 3.4 (Auxilary Functions and Moments)

$$E(e^{tz}) = \int_{-\infty}^{\infty} e^{tz}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz = \int_{-\infty}^{\infty} e^{t^2/2}\frac{1}{\sqrt{2\pi}}e^{-(z-t)^2/2}dz = e^{t^2/2}$$

If M_n is the $n-$th moment,

$$E(e^{tz}) = \sum_{k=0}^{\infty} M_k \frac{t^k}{k!} = 1 + \frac{t^2}{2} + \frac{1}{2!}\left(\frac{t^2}{2}\right)^2 + \frac{1}{3!}\left(\frac{t^2}{2}\right)^3 + \cdots$$

By matching the coefficients, we get

$$\begin{aligned}
&M_0 = 1\\
&M_1 = M_3 \ (= M_{2k-1}) = 0\\
&M_2 = 1\\
&M_4 = 4!/(2!\ 2^2) = 3\\
&M_6 = 6!/(3!\ 2^3) = 15.
\end{aligned}$$

For $t > 0$,

$$E(e^{tz^4}) = \int_{-\infty}^{\infty} e^{tz^4}\frac{1}{\sqrt{2\pi}}e^{-z^2/2}dz \to \infty,$$

so the MGF of Z^4 does not exist.

Exercise 4.6 The stopped process is a martingale. By the symmetry, $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -A) = 0.5$.

$$1 = E(X_\tau) = \frac{1}{2}e^{\alpha A}E(e^{-\alpha^2\tau/2}) + \frac{1}{2}e^{-\alpha A}E(e^{-\alpha^2\tau/2}).$$

Therefore, we have

$$E\left(e^{-\alpha^2\tau/2}\right) = \frac{1}{\cosh(\alpha A)}$$

or

$$\phi(\lambda) = E(e^{-\lambda\tau}) = \frac{1}{\cosh(A\sqrt{2\lambda})}.$$

In order to calculate $E(\tau^2)$, we need to obtain the x^4 term in the expansion of $1/\cosh(x)$ given that $\sqrt{\lambda}$ appears in the expression. From the expansion, $\cosh x \sim 1 + x^2/2! + x^4/4! + \cdots$,

$$\frac{1}{\cosh x} \sim \frac{1}{1 + (x^2/2! + x^4/4! + \cdots)} = 1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \left(\frac{x^2}{2!} + \cdots\right)^2 = 1 - \frac{x^2}{2} + \frac{5}{24}x^4 + \cdots .$$

Finally we get

$$E(\tau^2) = 2\frac{5}{24}(A\sqrt{2\lambda})^4|_{\lambda=1} = \frac{5}{3}A^4$$

For the non-symmetric case ($A \neq B$), we can not use $\mathrm{P}(B_\tau = A) = \mathrm{P}(B_\tau = -B) = 0.5$ anymore.

Extra Problem on Ch. 5 Derive the probability results on the running maximum and the first hitting time to the BM with the volatility σ. Using the scaling, $\sigma B_t = B_{\sigma^2 t}$, you are going to replace t with $\sigma^2 t$.

For the CDF (on both time and space) and the PDF on space, the simple replacement works.

$$P(\sigma B_t^* < x) = P(\tau_x > t) = 2\Phi(x/\sigma\sqrt{t}) - 1.$$

$$f_{\sigma B_t^*}(x) = \frac{2}{\sigma\sqrt{t}}\phi\left(\frac{x}{\sigma\sqrt{t}}\right)$$

For the PDF on time, however, we need to consider the normalization because the time is scaled. The original PDF $f_{\tau_x}(t)$ satisfies $\int_0^\infty f_{\tau_x}(t)dt = 1$. After the scaling,

$$\int_0^\infty f_{\tau_x}(\sigma^2 t)dt = \frac{1}{\sigma^2},$$

so the new PDF should be

$$\sigma^2 f_{\tau_x}(\sigma^2 t) = \frac{x}{\sigma t^{3/2}}\,\phi\left(\frac{x}{\sigma\sqrt{t}}\right).$$

Exercise 6.1 The mean of the two expressions are zero.

$$\mathrm{Var}\left(\int_0^t |B_s|^{\frac{1}{2}} dB_s\right) = E\left(\int_0^t |B_s| ds\right) = \int_0^t E(|B_s|)ds = \int_0^t \sqrt{\frac{2s}{\pi}}ds = \frac{2}{3}\sqrt{\frac{2}{\pi}}\,t^{\frac{3}{2}}$$

$$\begin{aligned}\mathrm{Var}\left(\int_0^t (B_s+s)^2 dB_s\right) &= E\left(\int_0^t (B_s+s)^4 ds\right) = \int_0^t E(B_s^4 + 4sB_s^3 + 6s^2B_s^2 + 4s^3B_s + s^4)ds \\ &= \int_0^t (3s^2 + 0 + 6s^2\cdot s + 0 + s^4)ds = \frac{1}{5}t^5 + \frac{3}{2}t^4 + t^3\end{aligned}$$

Exercise 6.2 For I_1,

$$E(I_1) = \int_0^t E(B_s)ds = \int_0^t 0\,ds = 0.$$

Using Itô's lemma applied to sB_s, $d(sB_s) = sdB_s + B_s ds$, we can express I_1 as

$$I_1 = tB_t - \int_0^t s dB_s = t\int_0^t dB_s - \int_0^t s dB_s = \int_0^t (t-s)dB_s,$$

where we used a trick of $B_t = \int_0^t dB_s$ in order to make the expression suitable for Ito's isometry. We get

$$\mathrm{Var}(I_1) = \int_0^t (t-s)^2 ds = \frac{1}{3}t^3$$

For I_2,

$$E(I_2) = \int_0^t E(B_s^2)ds = \int_0^t s\,ds = \frac{t^2}{2}.$$

Using Itô's lemma applied to sB_s^2, $d(sB_s^2) = B_s^2 ds + 2sB_s dB_s + sds$, we can express I_2 as

$$I_2 = tB_t^2 - 2\int_0^t sB_s dB_s - \frac{t^2}{2}.$$

We apply a similar trick, $d(B_s^2) = 2B_s dB_s + ds$, to replace B_t^2 with a more suitable expression for Itô's isometry,

$$I_2 = t\left(2\int_0^t B_s dB_s + t\right) - 2\int_0^t sB_s dB_s - \frac{t^2}{2} = 2\int_0^t (t-s)B_s dB_s + \frac{t^2}{2},$$

where we can reconfirm that $E(I_2) = t^2/2$. Finally,

$$\mathrm{Var}(I_2) = E\left(\left(I_2 - \frac{t^2}{2}\right)^2\right) = 4\int_0^t E\Big((t-s)^2 B_s^2\Big)ds = 4\int_0^t (t-s)^2 s\,ds = 4\cdot\frac{t^4}{12} = \frac{t^4}{3}$$

Exercise 6.3 At any time s, X_s and B_s has the same distribution, normal distribution with mean 0 and variance s, so $E(f(B_s)) = E(f(X_s))$ and

$$E(U_t) = \int_0^t E(f(B_s))ds = \int_0^t E(f(X_s))ds = E(V_t)$$

For variance, simply let $f(x) = x$. Using that $V_t = \int_0^t \sqrt{s}\, Z ds = \frac{2}{3}t^{\frac{3}{2}}\, Z$,

$$\mathrm{Var}(V_t) = \frac{4}{9}t^3.$$

According to **Exercise 6.2**, however,

$$\mathrm{Var}(U_t) = \frac{1}{3}t^3 \neq \mathrm{Var}(V_t).$$

Exercise 7.1

$$\tau_t = \mathrm{Var}(Y_t) = \mathrm{Var}(X_t) = \int_0^t e^{2s}ds = \frac{1}{2}(e^{2t} - 1)$$

$$E(X_t^2) = \int_0^t e^{2s}ds = \frac{1}{2}(e^{2t} - 1), \quad E(Y_t^2) = \tau_t = \frac{1}{2}(e^{2t} - 1)$$

$$E(X_t^4) = E(Y_t^2 = B_{\tau_t}^4) = 3\tau_t^2 = \frac{3}{4}(e^{2t} - 1)^2$$

Note the difference between this problem and **Corollary 7.1** (Brownian motion time change). Given B_t is a standard BM,

$$X_t = \int_0^t f(s)dB_s, \quad \text{and} \quad \tau(t) = v = \mathrm{Var}(X_t) = \int_0^t f^2(s)ds,$$

this exercise problem is effectively stating that X_t and $B_{\tau(t)}$ are same processes. Whereas, the Corollary 7.1 states that $X_{\tau^{-1}(v)}$ and B_v are same processes where $\tau^{-1}(\cdot)$ is the inverse function of $\tau(\cdot)$, i.e., $t = \tau^{-1}(v)$. Although they look different in forms, the intuitions behind them are same in that the variance of X_t can be used as a new *time scale* of a standard BM.

Exercise 9.1 This is slightly modified from the OU process with the extra βdt term. We use the same initial guess, $e^{\alpha t}X_t$, for the OU process.

$$\begin{aligned} d\big(e^{\alpha t}X_t\big) &= \alpha e^{\alpha t}X_t dt + e^{\alpha t}dX_t + \frac{1}{2}0(dX_t)^2 \\ &= e^{\alpha t}\big(\alpha X_t dt - \alpha X_t dt + \beta dt\big) + \sigma e^{\alpha t}dB_t = -\beta e^{\alpha t}dt + \sigma e^{\alpha t}dB_t. \end{aligned}$$

Therefore, we get

$$e^{\alpha t}X_t - X_0 = \beta\big(e^{\alpha t} - 1\big) + \sigma \int_0^t e^{\alpha s}dB_s$$

$$X_t = e^{-\alpha t}X_0 + \beta\big(1 - e^{-\alpha t}\big) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dB_s.$$

Exercise 9.2 We first guess from the traditional calculus. We let $x = X_t$ and solve

$$e^{-t^2/2}dx = txe^{-t^2/2} \quad \Rightarrow \quad \frac{dx}{x} = -t\,dt$$

Luckily we can solve this to have $x = Ce^{t^2/2}$ or $e^{-t^2/2}x = C$ for some constant C, so we stochastically differentiate $e^{-t^2/2}X_t$ to get

$$d\big(e^{-t^2/2}X_t\big) = -te^{-t^2/2}X_t dt + e^{-t^2/2}\big(tX_t\,dt + e^{t^2/2}dB_t\big) = dB_t.$$

We can solve the SDE as

$$e^{-t^2/2}X_t - X_0 = B_t \quad \Rightarrow \quad X_t = e^{t^2/2}(B_t + X_0).$$

Exercise 9.3 The guess from the traditional calculus is

$$\frac{dx}{x} = \frac{-2}{1-t}\,dt \quad \Rightarrow \quad x = C(1-t)^2.$$

Therefore we start by differentiating $(1-t)^{-2}X_t$:

$$d\left(\frac{X_t}{(1-t)^2}\right) = 2\frac{X_t}{(1-t)^3}dt + \frac{1}{(1-t)^2}\Big(-2\frac{X_t}{1-t}dt + \sqrt{2t(1-t)}\,dB_t\Big) = \frac{\sqrt{2t}}{(1-t)^{3/2}}dB_t$$

and finally solve the SDE as

$$X_t = (1-t)^2 \int_0^t \frac{\sqrt{2u}}{(1-u)^{3/2}}dB_u.$$

Since the integrand $\sqrt{2u}(1-u)^{-3/2}$ depends only on the time variable u, X_t is a Gaussian process with the variance

$$\begin{aligned}
\mathrm{Var}(X_t) &= (1-t)^4 \int_0^t \frac{2u}{(1-u)^3}du = (1-t)^4 \int_{1-t}^1 \frac{2(1-u')}{u'^3}du \quad (u' = 1-u) \\
&= (1-t)^4\left(1 - \frac{2}{1-t} + \frac{1}{(1-t)^2}\right) = (1-t)^4\frac{t^2}{(1-t)^2} = t^2(1-t)^2
\end{aligned}$$

The covariance can be obtained similarly. Assuming that $s < t$,

$$\begin{aligned}
\mathrm{Cov}(X_s, X_t) = E(X_sX_t) &= (1-s)^2(1-t)^2\, E\left[\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}}dB_u \int_0^t \frac{\sqrt{2v}}{(1-v)^{3/2}}dB_v\right] \\
&= (1-s)^2(1-t)^2\, E\left[\left(\int_0^s \frac{\sqrt{2u}}{(1-u)^{3/2}}dB_u\right)^2\right] \\
&= (1-s)^2(1-t)^2 \cdot \frac{s^2}{(1-s)^2} = s^2(1-t)^2.
\end{aligned}$$

The covariance is square of that of the Brownian bridge $\mathrm{Cov}(X_s, X_t) = s(1-t)$.