Stochastic Finance (FIN 519)
Homework Solutions

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HW 1-1 Using martingale property, re-drive that

$$E(\tau) = AB \quad \text{for} \quad \tau = \min\{n : S_n = A \text{ or } S_n = -B\}.$$

Answer You can find the answer in the textbook section 2.3.

HW 1-2. SCFA Exercise 2.4

Answer From the hint, let us define

$$A_{n+1} = A_n + E[(M_{n+1} - M_n)^2 | \mathcal{F}_n]$$

and prove the three required properties:

(i) N_n is a martingale.

$$\begin{aligned} E[N_{n+1} \,|\, \mathcal{F}_n] &= E[M_{n+1}^2 - A_{n+1} \,|\, \mathcal{F}_n] \\ &= E[2M_{n+1}M_n - M_n^2 - A_n \,|\, \mathcal{F}_n] \\ &= 2E[M_{n+1} \,|\, \mathcal{F}_n] \, M_n - M_n^2 - A_n \\ &= M_n^2 - A_n = N_n. \end{aligned}$$

(ii) $A_{n+1} \geq A_n$ is trivial.

(iii) A_n is non-anticipating because it is defined via the expectation under $\mathcal{F}_n$.

HW 2-1 Compute the self-covariance of Brownian bridge, $\mathrm{Cov}(U_s, U_t)$ where $U_t = B_t - tB_1$.

Answer

$$\begin{aligned} \mathrm{Cov}(U_s, U_t) &= E\Big((B_s - sB_1)(B_t - tB_1)\Big) = E\Big(B_sB_t - sB_1B_t - tB_sB_1 + stB_1^2\Big) \\ &= \min(s,t) - s\min(1,t) - t\min(s,t) + st = s(1-t) \end{aligned}$$

HW 2-2 Calculate $\mathrm{Var}(aB_t + bB_s)$ for constants a and b.

Answer

$$\begin{aligned} \mathrm{Var}(aB_t + bB_s) &= E[a^2\, B_t^2 + 2abB_t\, B_s + b^2\, B_s^2] \\ &= a^2\, t + 2ab\, \min(s,t) + b^2\, s \end{aligned}$$

HW 2-3 Derive the price of down-and-out call option with knock-out strike K_D and option strike K. (Obviously, $K_D < F$ and $K_D < K$) See the derivation for up-and-out call option and down-and-out digital option from the previous HW and exams.

Answer Assume $B_T^M = \max_{0\le t\le T} B_t$ and $B_T^m = \min_{0\le t\le T} B_t$. From textbook and class, we know

$$P(\sigma B_T^M < v,\ \sigma B_T < u) = P(B_T^M < v/\sigma,\ B_T < u/\sigma) = \Phi\Big(\frac{u}{\sigma\sqrt{t}}\Big) - \Phi\Big(\frac{u-2v}{\sigma\sqrt{t}}\Big).$$

Using the reflection, $B_t \to -B_t$, we get

$$(-B)_T^m = \min_{0\le t\le T}(-B_t) = -\max_{0\le t\le T} B_t = -B_T^M$$

and

$$P(\sigma B_T^m > v,\ \sigma B_T > u) = \Phi\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - \Phi\Big(\frac{2v-u}{\sigma\sqrt{T}}\Big).$$

As the stock price is given as $S_T = F + \sigma B_T$,

$$P(S_T^m > v,\ S_T > u) = \Phi\Big(\frac{F-u}{\sigma\sqrt{T}}\Big) - \Phi\Big(\frac{2v-u-F}{\sigma\sqrt{T}}\Big).$$

The probability density function on u with the joint condition, $\sigma B_T^m > v$ is obtained from the partial derivative w.r.t. u (with negative sign),

$$f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{F-u}{\sigma\sqrt{T}}\Big) - n\Big(\frac{2v-u-F}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < v \le u.$$

Let $z = (u-F)/\sigma\sqrt{T}$, $d = (F-K)/\sigma\sqrt{T}$ and $d^* = (F-K_D)/\sigma\sqrt{T}$. Then, the down-and-out call option price is given as

$$\begin{aligned} C(K, K_D) =& \int_{u=K}^{\infty} (u-K)\, f(u)du = \int_{z=-d}^{\infty} (F-K+\sigma\sqrt{T}\,z)(n(z) - n(z+2d^*))dz \\ =&(F-K)N(d) + \sigma\sqrt{T}\,n(d) - (F-K-2d^*\sigma\sqrt{T})N(d-2d^*) - \sigma\sqrt{T}\,n(d-2d^*). \end{aligned}$$

The first two terms are exactly the regular call option price, $C(K) = (F-K)N(d) + \sigma\sqrt{T}\,n(d)$. Therefore, the down-and-out option is cheaper than the regular option by $(F - K - 2d^*\sigma\sqrt{T})N(d-2d^*) + \sigma\sqrt{T}\,n(d-2d^*)$.

We can verify two cases:

1. If $K_D \to -\infty$ ($d^* \to \infty$), $C(K, K_D) = C(K)$ because the probably of being knocked out is zero. It is indeed the case because $N(2d^* - d) = n(2d^* - d) = 0$.
2. If $K_D \to F$ from below ($d^* \to 0$) on the other hand, the knock-out probability approaches to 100%, so the price should be zero. This is also the case from the formula.

HW 3-1 SCFA Exercise 8.2 The notation $h \in C^1(\mathbb{R}^+)$ means that the function $h(s)$ is differentiable for $s > 0$.

HW 3-2 For a standard BM B_t, let

$$N_t = B_t^3 - 3t\,B_t.$$

(i) Prove that N_t is a martingale. (Hint: use Proposition 8.1) (ii) By applying Itô's lemma, express N_t as a stochastic integration. (iii) Calculate the variance of N_t.

HW 3-3 SCFA Exercise 8.4 The sub-problem (b) is understood better after **HW 3-2** is solved.

HW 3-4 SCFA Exercise 9.6

HW 3-5 (asser-or-nothing/digital option) The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,

$$S_T \cdot 1_{S_T \geq K} \quad - \quad K \cdot 1_{S_T \geq K}.$$

The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. Under the Black-Scholes model, what are the prices of the asset-or-nothing call option and the binary call option respectively? What is the intuition behind that the asset-or-nothing option price does not include the discounting factor e^{-rT}?