Stochastic Finance (FIN 519)
Midterm Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**.

1. (2 points) **(Recurrence relation)** A startup company in Shenzhen either fails or succeeds every year with probability of 25% and 75% respectively. If a company is successful, employees spin off a new startup at the end of the year (and it will face with the same fail/success probability afterwards). There is no correlation between companies. Assume that Shenzhen sets off with one startup company when the city was established as a special economic zone in 1980. What is the probability that all startups in Shenzhen eventually fail.

Solution: Let p be the probability and ranch on the whether the first company fails or succeeds. If the first company succeeds and there are two companies in Shenzhen next year, the probability that both fail is p^2 due to the independence assumption. Therefore,

$$p = \frac{1}{4}\cdot 1 + \frac{3}{4}\cdot p^2 \quad \Rightarrow \quad 3p^2 - 4p + 1 = (3p-1)(p-1) = 0$$

Among the two solutions, $p = 1/3$ is the right solution.

2. (4 points) **(Box-Muller algorithm for generating normal random number)** The probability and cumulative density functions (PDF and CDF) of exponential RV, Z, are given respectively as

$$f(z) = \lambda e^{-\lambda z}, \quad P(z) = 1 - e^{-\lambda z} \quad \text{for} \quad \lambda > 0, z \geq 0.$$

   (a) If U is a uniform RV, how can you generate the RNs of Z?

   (b) Let X and Y be two independent standard normal RVs. Show that the squared radius, $Z = X^2 + Y^2$, follows an exponential distribution by computing $P(X^2 + Y^2 < z)$. What is λ?

   (c) How can you generate the RNs of X and Y from uniform RNs? Hint: introduce another uniform RV, V, and consider the random angle $2\pi V$.

Solution:

(a) The RN can be generated from the inverse CDF:

$$Z = P^{-1}(U) = -\frac{1}{\lambda}\log(1-U) \quad \text{or} \quad Z = -\frac{1}{\lambda}\log U,$$

where we use that $1-U$ is also a uniform random RV.

(b) With the change of variable $r^2 = x^2 + y^2$ and radial symmetry,

$$P(X^2+Y^2 < z) = \frac{1}{2\pi}\int_{x^2+y^2<z} e^{-(x^2+y^2)/2} dxdy = \frac{1}{2\pi}\int_{r=0}^{\sqrt{z}} e^{-r^2/2} 2\pi r\, dr = 1 - e^{z/2}.$$

Therefore Z follows an exponential distribution with $\lambda = 1/2$.

(c) The RVs X and Y can be thought as x- and y-components of $\sqrt{Z}$ with random angle $2\pi V$. Also, from the results of (a) and (b), the pair (X, Y) is generated as

$$(X, Y) = \sqrt{Z}(\cos(2\pi V), \sin(2\pi V)) = \sqrt{-2\log U}(\cos(2\pi V), \sin(2\pi V))$$

3. (4 points) If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Provide a brief reason for your answer.

(a) $\frac{1}{2}B_{4t}$

(b) $\frac{1}{2}(B_{1+2t} - B_1)$

(c) $\frac{3}{5}B_t + \frac{4}{5}W_t$, where W_t is another BM independent from B_t.

Solution:

(a) **Yes**. $\text{Var}(\frac{1}{2}B_{4t}) = \frac{1}{4}\cdot 4t = t$.

(b) **No**. It is equivalent to $\frac{1}{2}B_{2t}$, however $\frac{1}{2}B_{2t}$ is not a standard BM.

(c) **Yes**. $aB_t + bW_t$ is a standard BM when $a^2 + b^2 = 1$.

4. (3 points) **(Exponentially decaying volatility)** Assume that a stock follows BM with time-varying volatility:

$$dS_t = f(t)dB_t \quad \text{for} \quad f(t) = a + be^{-\lambda t} \quad (a, b > 0)$$

What is the ATM call option price at expiry T? What is the equivalent normal model volatility? In other words, what value of σ gives the same option price when the stock price follows $dS_t = \sigma\, dW_t$ at $t = T$.

Solution: The variance of S_t computed as

$$\begin{aligned}\mathrm{Var}(S_t) &= \int_0^t f^2(s)ds = \int_0^t (a^2 + 2ab\, e^{-\lambda s} + b^2\, e^{-2\lambda s})ds \\ &= a^2 t + \frac{2ab}{\lambda}(1 - e^{-\lambda t}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda t}).\end{aligned}$$

The option price is given as

$$C = 0.4\sqrt{a^2 T + \frac{2ab}{\lambda}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda}(1 - e^{-2\lambda T})}.$$

From $\sigma^2 T = \mathrm{Var}(S_T)$,

$$\sigma = \sqrt{a^2 + \frac{2ab}{\lambda T}(1 - e^{-\lambda T}) + \frac{b^2}{2\lambda T}(1 - e^{-2\lambda T})}$$

5. (6 points) **(Stochastic integral)** Assume that you follow a trading strategy based on *momentum*, where you long more stock if the stock price is up and short more if the stock price is down. If S_t is the process for the stock price, the amount you long/short is $S_t - S_0$.
   (a) If X_T is the profit and loss from this strategy at time $t = T$, express X_T using stochastic integral.
   (b) When the stock follows a BM, $S_t = S_0 + \sigma B_t$, what is X_T? You may directly use the result from the class.
   (c) Imagine a scenario where the stock price goes up a lot but loses later recovering the original price, $S_T = S_0$. How much do you profit or lose at $t = T$? Intuitively explain why you profit or lose? (The opposite is called *mean-reversion* strategy, where you long/short by $S_0 - S_t$.)
   (d) Calculate the variance of X_T. You can either use the result from (b) or use Itô's isometry.

Solution:

(a)

$$X_T = \int_0^T (S_t - S_0)dS_t$$

(b)

$$X_T = \sigma^2 \int_0^T B_t\, dB_t = \frac{\sigma^2}{2}(B_T^2 - T)$$

(c) Since $B_T = B_0 = 0$, the momentum strategy loses by $\sigma^2 T/2$ (where as the mean-reversion strategy profits the same amount). The strategy profits when the stock price goes up but loses when it goes down. Intuitively, loss is bigger than profit because you start from zero long position when the price is up where as you already have some short position when the price is down.

(d) From Itô's isometry,

$$\mathrm{Var}(X_t) = \sigma^4 \int_0^T E(B_t^2)dt = \sigma^4 \int_0^T t\,dt = \frac{\sigma^4}{2}T^2$$

From (b),

$$\mathrm{Var}(X_t) = \frac{\sigma^4}{4}\mathrm{Var}(B_T^2 - T) = \frac{\sigma^4}{4}(B_T^4 - 2T\,B_T^2 + T^2) = \frac{\sigma^4}{4}E(3T^2 - 2T\cdot T + T^2) = \frac{\sigma^4}{2}T^2$$

6. (6 points) **(Basket/Spread option under normal model)** Let W_t and Z_t are standard **BM**s with correlation ρ, i.e., $E(W_t\,Z_t) = \rho\,t$. For (b) and (c), assume that two stock prices follow

$$S_{1t} = 100 + 20W_t, \quad S_{2t} = 80 + 10Z_t.$$

and you may use the simple ATM option price formula, $C = P = 0.4\sigma\sqrt{T}$.

(a) Calculate $\mathrm{Var}(a\,W_T + b\,Z_T)$ for some constants a and b.

(b) Consider a spread option, whose payout is on *the difference of the prices*, $\max\big(S_{1T} - S_{2T} - K, 0\big)$. What is the price of the ATM option ($K = 20$) as a function of ρ? What is the min and max price and when do you get the those values?

(c) Consider a basket option, whose payout is on *the weighted average of the prices*, $\max\big((S_{1T} + S_{2T})/2 - K, 0\big)$. What is the price of the ATM option ($K = 90$) as a function of ρ? What is the min and max price and when do you get the those values?

Solution:

(a)

$$\mathrm{Var}(a\,W_T + b\,Z_T) = (a^2 + 2\rho ab + b^2)T$$

(b) $\mathrm{Var}(S_{1T} - S_{2T}) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 20$ and $b = -10$. Therefore,

$$C = 0.4\sqrt{(a^2 + 2\rho ab + b^2)T} = 4\sqrt{(5 - 4\rho)T},$$

and

$$4\sqrt{T}\;\;(\rho = 1, \text{ i.e., correlated}) \le C \le 12\sqrt{T}\;\;(\rho = -1, \text{ i.e., anti-correlated}).$$

(c) $\mathrm{Var}((S_{1T} + S_{2T})/2) = \mathrm{Var}(a\,W_T + b\,Z_T)$ with $a = 10$ and $b = 5$. Therefore,

$$C = 0.4\sqrt{(a^2 + 2\rho ab + b^2)T} = 2\sqrt{(5 + 4\rho)T},$$

and

$$2\sqrt{T}\;\;(\rho = -1, \text{ i.e., anti-correlated}) \le C \le 6\sqrt{T}\;\;(\rho = 1, \text{ i.e., correlated}).$$