Homework 6

DATA130021 Financial Econometrics

Deng Qisheng

2021/05/13

Problem 1

Consider the AR(1) model

$$Y_t = 5 - 0.55Y_{t-1} + \epsilon_t$$

and assume that $\sigma_\epsilon^2 = 1.2$.

(a) Is this process stationary? Why or why not?

Solution:

Rewrite the AR(1) model to standard form, we have

$$Y_t - \mu = \phi\big(Y_{t-1} - \mu\big) + \epsilon_t,$$

where $\mu \approx 3.23$ and $\phi = -0.55$. We find that

$$|\phi| < 1,$$

which means that $\{Y_t\}$ is weakly stationary. Hence, we can conclude that this AR(1) model is **stationary**.

(b) What is the mean of this process?

Solution:

Since this is a stationary process, we know that the mean of this process is

$$\mathbb{E}\big(Y_t\big) = \mu \approx 3.23.$$

(c) What is the variance of this process?

Solution:

From the properties of stationary process, we know that the variance of this process is

$$\mathrm{Var}\big(Y_t\big) = \frac{\sigma_\epsilon^2}{1-\phi^2} = \frac{1.2}{1-0.55^2} \approx 1.72.$$

(d) What is the covariance function of this process?

Solution:

From the definition, the autocovariance function of this process is

$$\gamma\big(h\big) = \mathrm{Var}\big(Y_t\big)\rho\big(h\big) = \frac{\sigma_\epsilon^2}{1-\phi^2}\phi^{|h|} \approx 1.72 \times (-0.55)^{|h|}.$$

Problem 2

Suppose that $Y_1, Y_2, \cdots$ is an AR(1) process with $\mu = 0.5$, $\phi = 0.4$ and $\sigma_\epsilon^2 = 1.2$.

(a) What is the variance of Y_1?

Solution:

From the assumption, we can write down the AR(1) model

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + \epsilon_t,$$

where $\mu = 0.5$, $\phi = 0.4$ and $\sigma_\epsilon^2 = 1.2$. Since

$$|\phi| < 1,$$

we know that this is a stationary process. Hence, the variance of Y_1 is

$$\mathrm{Var}(Y_1) = \mathrm{Var}(Y_t) = \frac{\sigma_\epsilon^2}{1-\phi^2} = \frac{1.2}{1-0.4^2} \approx 1.43.$$

(b) What are the covariances between Y_1 and Y_2 and between Y_1 and Y_3?

Solution:

From the definition of autocovariance function, we know that

$$\gamma(h) = \mathrm{Var}(Y_t)\rho(h) = \frac{\sigma_\epsilon^2}{1-\phi^2}\phi^{|h|} \approx 1.43 \times 0.4^{|h|}.$$

Hence, the covariance between Y_1 and Y_2 is

$$\mathrm{Cov}(Y_1, Y_2) = \gamma(1) = 1.43 \times 0.4 = 0.572,$$

the covariance between Y_1 and Y_3 is

$$\mathrm{Cov}(Y_1, Y_3) = \gamma(2) = 1.43 \times 0.4^2 = 0.229.$$

(c) What is the variance of $(Y_1 + Y_2 + Y_3)/2$?

Solution:

From the definition of variance, we have

$$\begin{aligned}
\mathrm{Var}\left(\frac{1}{2}(Y_1+Y_2+Y_3)\right) &= \frac{1}{4}\Big(\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + \mathrm{Var}(Y_3) + 2\mathrm{Cov}(Y_1,Y_2) + 2\mathrm{Cov}(Y_1,Y_3) + 2\mathrm{Cov}(Y_2,Y_3)\Big) \\
&= \frac{1}{4}\Big(\mathrm{Var}(Y_t) + \mathrm{Var}(Y_t) + \mathrm{Var}(Y_t) + 2\gamma(1) + 2\gamma(2) + 2\gamma(1)\Big) \\
&= \frac{3}{4}\mathrm{Var}(Y_t) + \gamma(1) + \frac{1}{2}\gamma(2) \\
&= \frac{3}{4} \times 1.43 + 0.572 + \frac{1}{2} \times 0.229 \\
&= 1.759.
\end{aligned}$$

Problem 3

In addition to the autocorrelation between Z_t and Z_{t+k}, we may investigate the correlation between Z_t and Z_{t+k} after their mutual linear dependency on the intervening variables $Z_{t+1}, Z_{t+2}, \cdots, Z_{t+k-1}$ has been removed. The **partial autocorrelation** (PACF) in time series analysis is defined as

$$P(k) = \mathrm{Corr}\Big(Z_t - L(Z_t|Z_{t+1}, Z_{t+2}, \cdots, Z_{t+k-1}), Z_{t+k} - L(Z_{t+k}|Z_{t+1}, Z_{t+2}, \cdots, Z_{t+k-1})\Big)$$

where $L(X|Y)$ is the best linear projection of X on Y. In R, the function `pacf` is used to compute and plot the PACF, which is similar to the use of `acf`.

(a) Simulate 300 observations from AR(1) model $Y_t = 0.3 + 0.5Y_{t-1} + \epsilon_t$ and plot both ACF and PACF of Y_t.

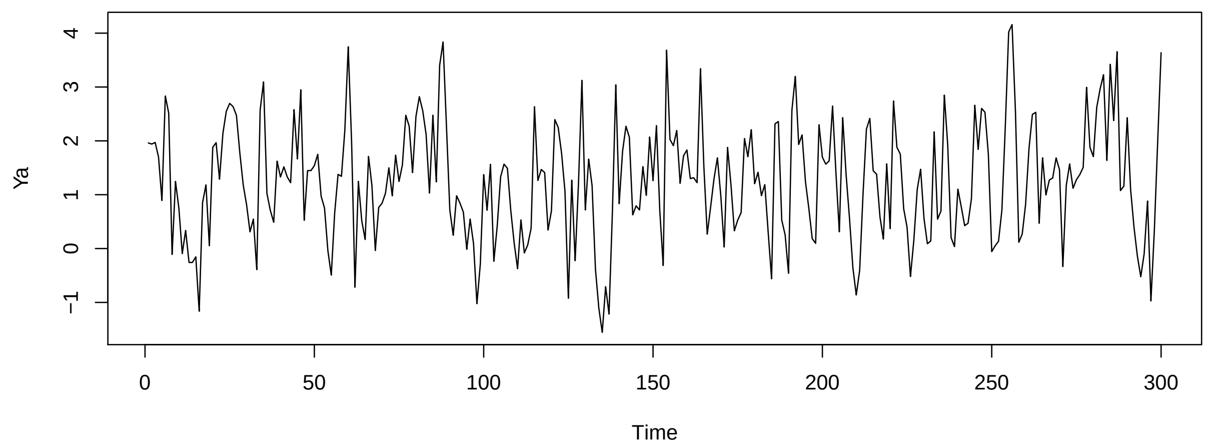
Solution:

Rewrite the AR(1) model, we have

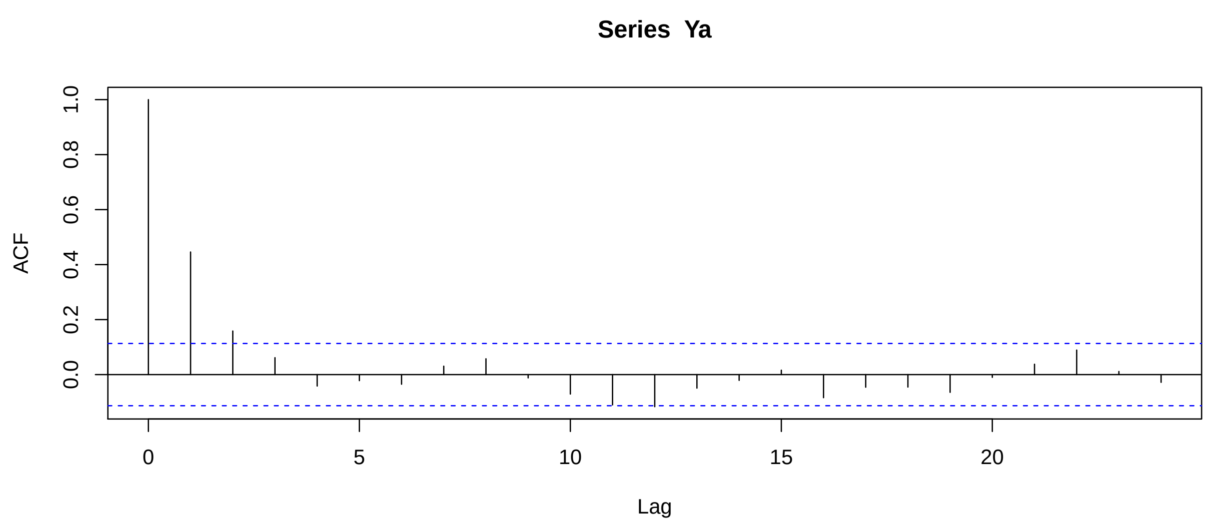
$$Y_t - 0.6 = 0.5\big(Y_{t-1} - 0.6\big) + \epsilon_t,$$

which means $\mu = 0.6$ and $\phi = 0.5$. Simulate 300 observations as follow.

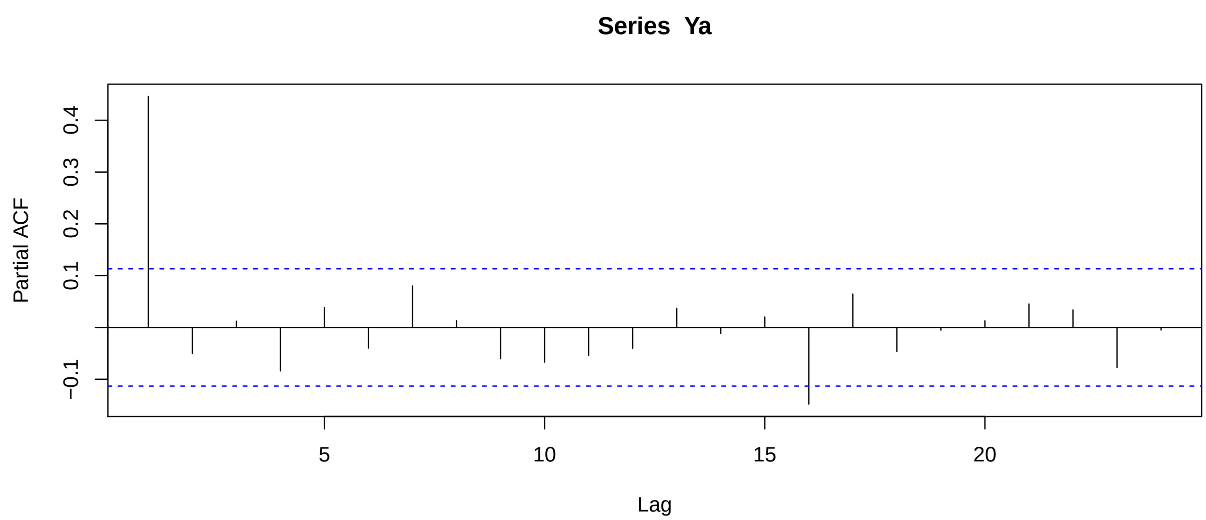
```
set.seed(123)
Ya <- arima.sim(n=300, model=list(order=c(1, 0, 0), ar=0.5), mean=0.6)
plot(Ya, type="l")
```




```
acf(Ya)
```



```r
pacf(Ya)
```



(b) Simulate 300 observations from AR(1) model $Y_t = 0.3 - 0.5Y_{t-1} + \epsilon_t$ and plot both ACF and PACF of Y_t.

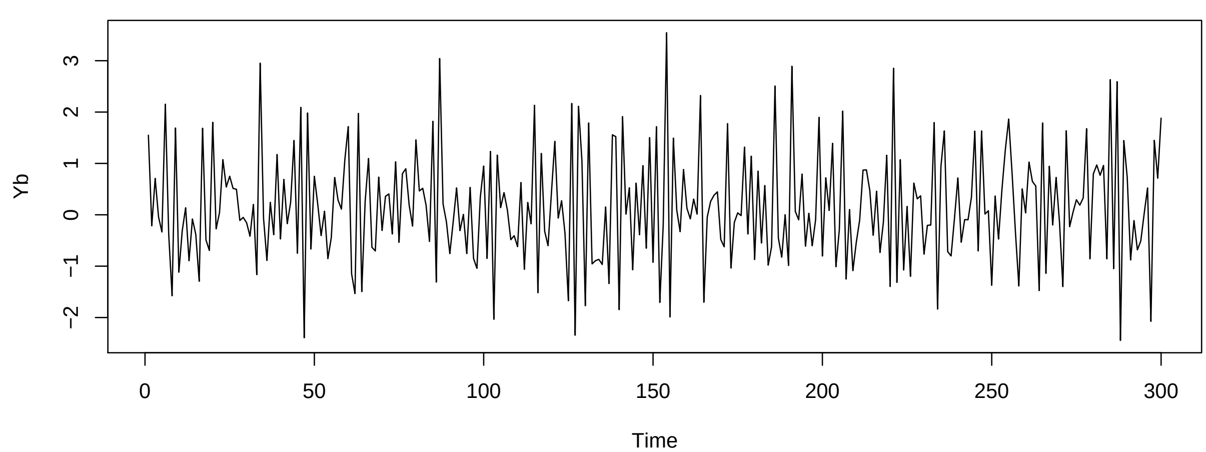
Solution:

Rewrite the AR(1) model, we have

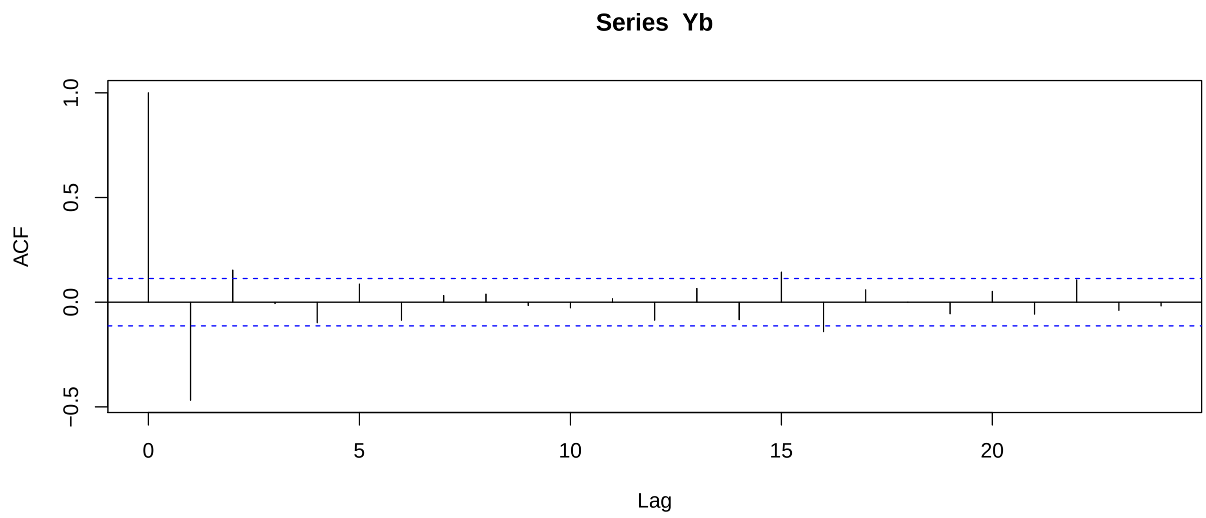
$$Y_t - 0.2 = -0.5\big(Y_{t-1} - 0.2\big) + \epsilon_t,$$

which means $\mu = 0.2$ and $\phi = -0.5$. Simulate 300 observations as follow.

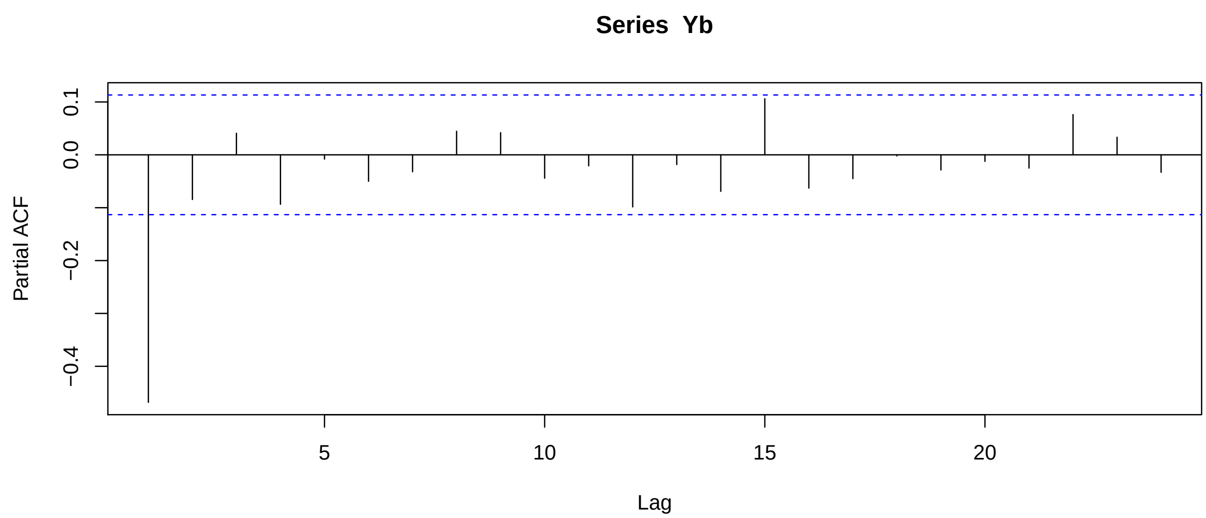
```r
set.seed(123)
Yb <- arima.sim(n=300, model=list(order=c(1, 0, 0), ar=-0.5), mean=0.2)
plot(Yb, type="l")
```



```r
acf(Yb)
```




```r
pacf(Yb)
```



Problem 4

Simulate 300 time series observations from an AR(2) model $Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \phi_2(Y_{t-2} - \mu) + \epsilon_t$ with

(a) $\phi_1 > 0$, $\phi_2 > 0$.

(b) $\phi_1 > 0$, $\phi_2 < 0$.

(c) $\phi_1 < 0$, $\phi_2 > 0$.

(d) $\phi_1 < 0$, $\phi_2 < 0$.

Plot both ACF and PACF for each model. What do you observe?

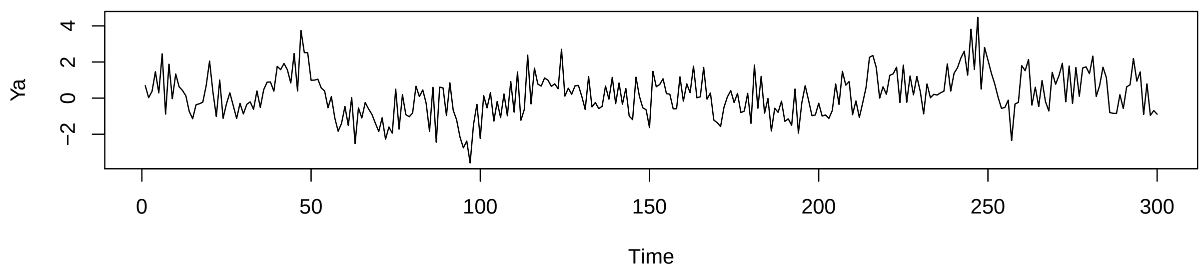
Solution:

For simplicity, we set $\mu = 0$.

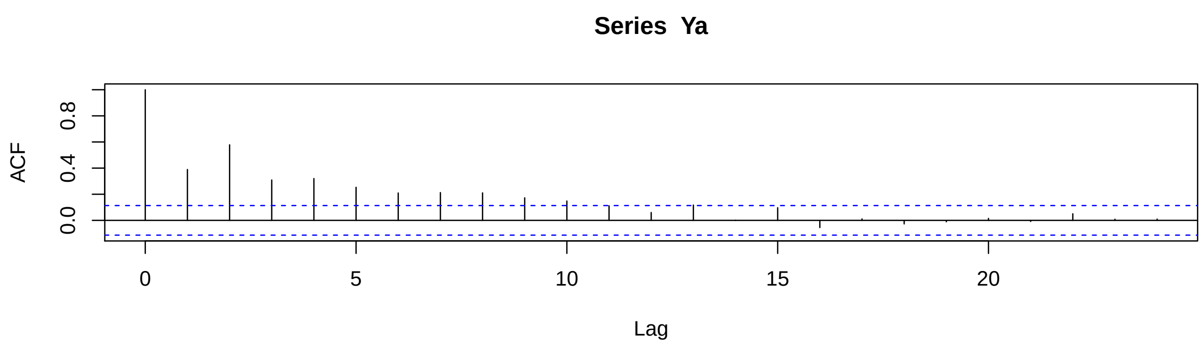
(a) $\phi_1 = 0.2$, $\phi_2 = 0.6$

Simulate 300 time series observations from $Y_t = 0.2Y_{t-1} + 0.6Y_{t-2} + \epsilon_t$ as follow.

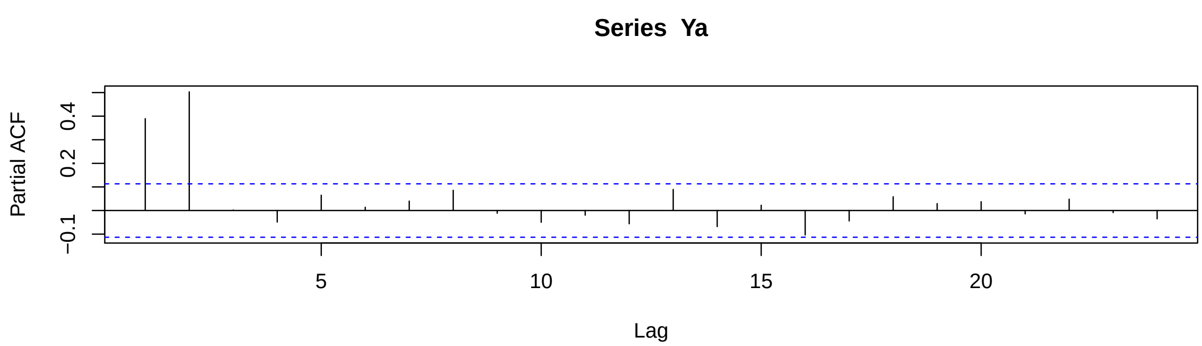
```
set.seed(123)
Ya <- arima.sim(n=300, model=list(order=c(2, 0, 0), ar=c(0.2, 0.6)))
plot(Ya, type="l")
```




```
acf(Ya)
```



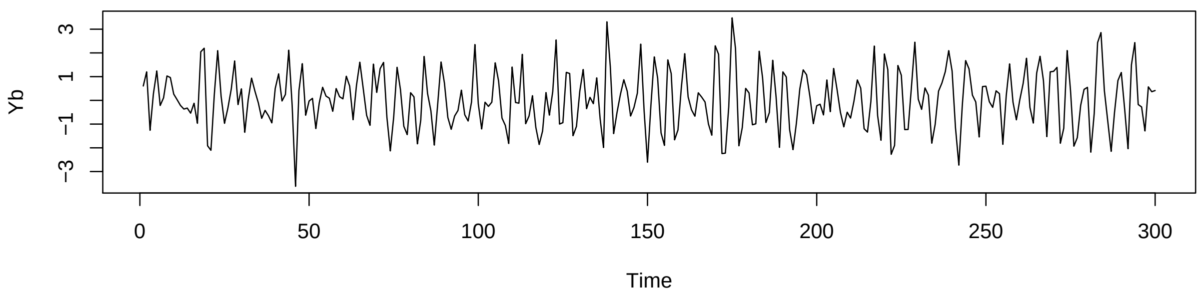

```
pacf(Ya)
```



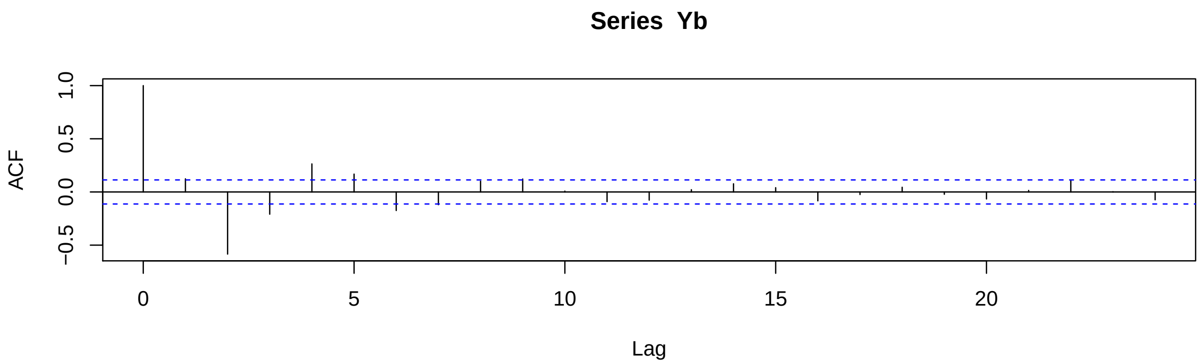
(b) $\phi_1 = 0.2$, $\phi_2 = -0.6$

Simulate 300 time series observations from $Y_t = 0.2Y_{t-1} - 0.6Y_{t-2} + \epsilon_t$ as follow.

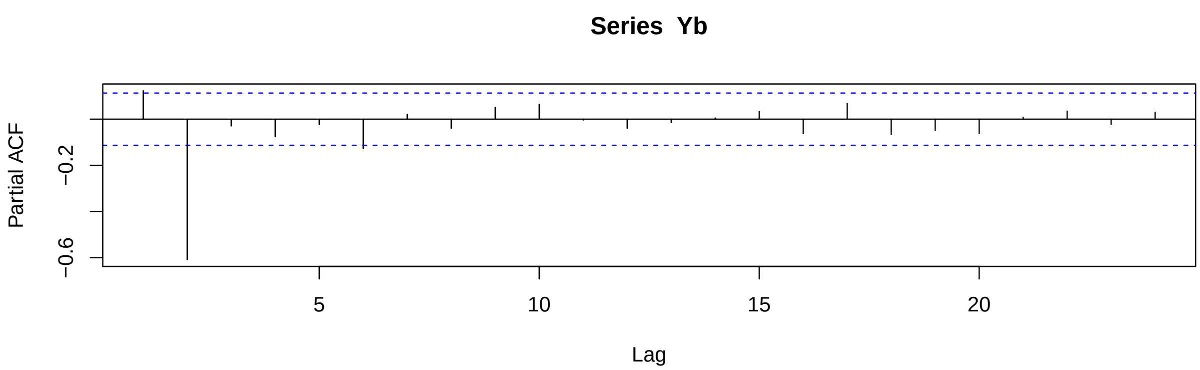
```
set.seed(123)
Yb <- arima.sim(n=300, model=list(order=c(2, 0, 0), ar=c(0.2, -0.6)))
plot(Yb, type="l")
```




```
acf(Yb)
```



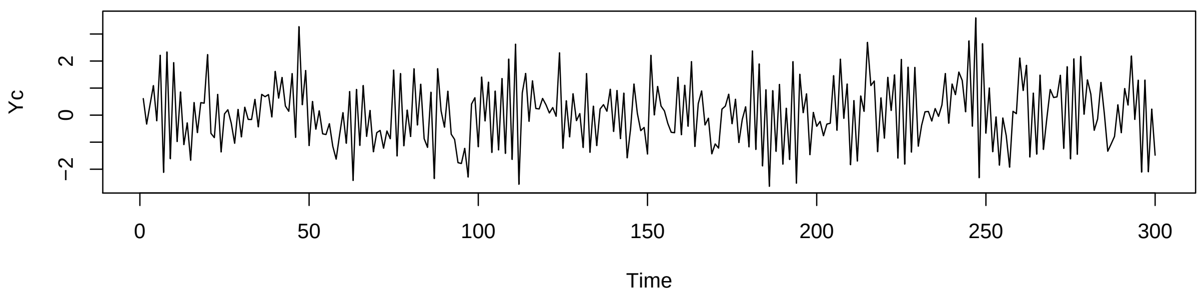

```
pacf(Yb)
```



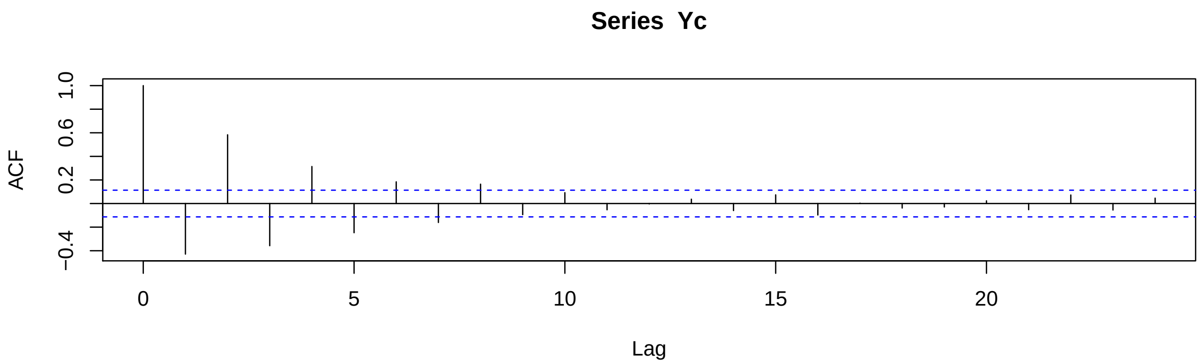
(c) $\phi_1 = -0.2$, $\phi_2 = 0.6$

Simulate 300 time series observations from $Y_t = -0.2Y_{t-1} + 0.6Y_{t-2} + \epsilon_t$ as follow.

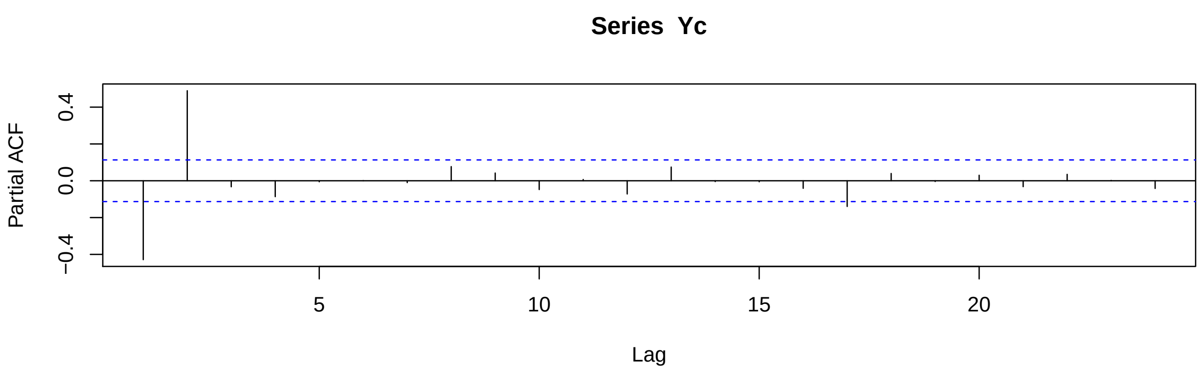
```
set.seed(123)
Yc <- arima.sim(n=300, model=list(order=c(2, 0, 0), ar=c(-0.2, 0.6)))
plot(Yc, type="l")
```




```
acf(Yc)
```



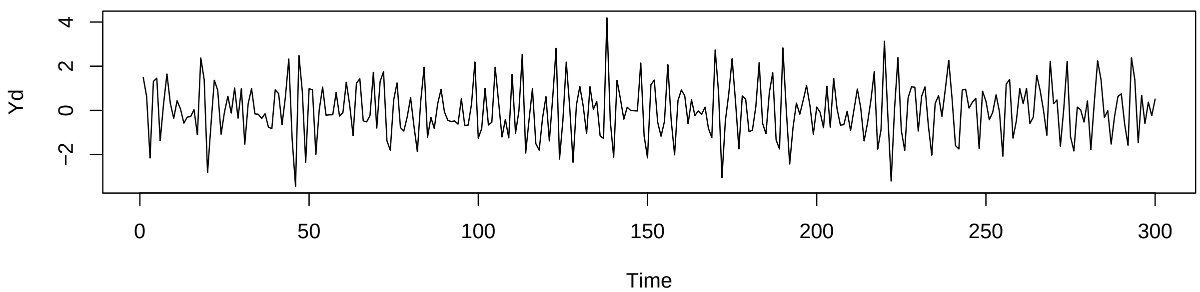

```
pacf(Yc)
```



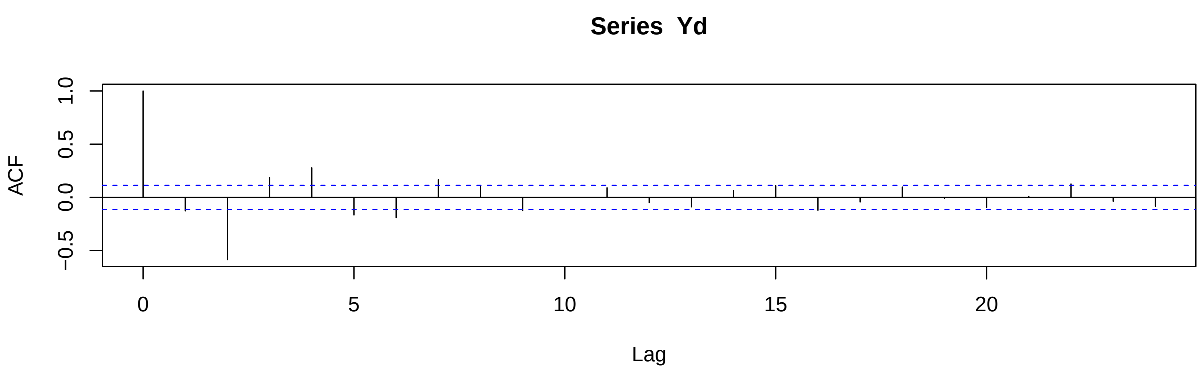
(d) $\phi_1 = -0.2$, $\phi_2 = -0.6$

Simulate 300 time series observations from $Y_t = -0.2Y_{t-1} - 0.6Y_{t-2} + \epsilon_t$ as follow.

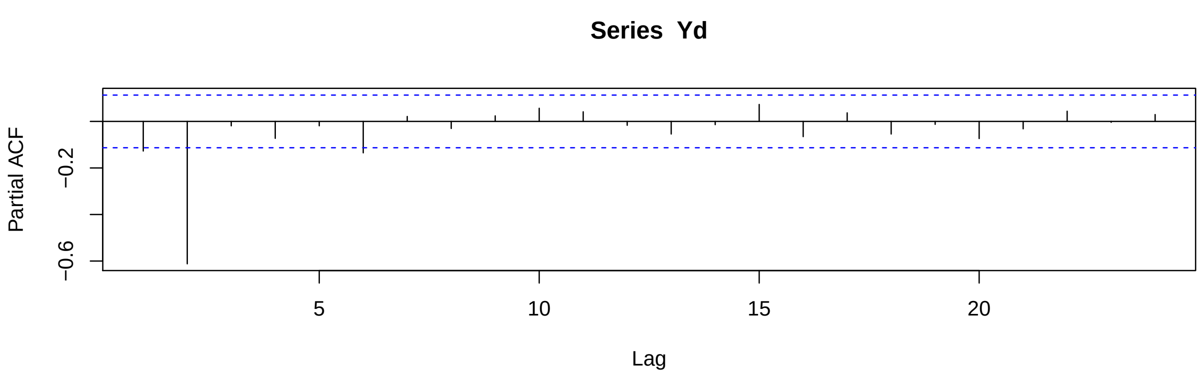
```
set.seed(123)
Yd <- arima.sim(n=300, model=list(order=c(2, 0, 0), ar=c(-0.2, -0.6)))
plot(Yd, type="l")
```




```
acf(Yd)
```




```
pacf(Yd)
```



Observation:

- PACF shows the signs and significances of ϕ_1 and ϕ_2 for these AR(2) models.
- If $\phi_1 > 0$, then there is a positive peak at lag 1 in PACF; If $\phi_1 < 0$, then there is a negative peak at lag 1 in PACF.
- If $\phi_2 > 0$, then there is a positive peak at lag 2 in PACF; If $\phi_2 < 0$, then there is a negative peak at lag 2 in PACF.
- In PACF, all lags larger than 2 are not significant for AR(2) model and peaks at lag 2 are highest. Hence, we can use PACF to observe the highest peak at lag p to get p for any AR(p) model.

Problem 5*

Find the stationarity condition for an AR(2) model.

Solution:

For any AR(2) model, we have the standard form

$$Y_t - \mu = \phi_1\big(Y_{t-1} - \mu\big) + \phi_2\big(Y_{t-2} - \mu\big) + \epsilon_t.$$

If this process is stationary, we need the condition that all roots z of $\Phi(z)$ lie outside the unit circle, where the the characteristic polynomial $\Phi(z)$ for AR(2) model is defined as $\Phi(z) = 1 - \phi_1 z - \phi_2 z^2$.

Solve $\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 = 0$ for z and we get

$$z_{1,2} = \frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}.$$

Then we need to find ϕ_1 and ϕ_2 to satisfy $|z_1| > 1$ and $|z_2| > 1$. Since $\Phi(z)$ is a quadratic function, then we have the discriminant $\Delta = \phi_1^2 + 4\phi_2$.

- **Situation 1:** $\Delta \geq 0$

If $\Delta \geq 0$, then z_1 and z_2 are real. From the geometric properties of quadratic function, we have the conditions

$$\begin{cases} \Phi(-1) \cdot \Phi(1) > 0, \\ |z_1| \cdot |z_2| = |z_1 z_2| = \left|\dfrac{1}{\phi_2}\right| > 1. \end{cases}$$

From the condition above, we have

$$\begin{cases} \big(\phi_2 - \phi_1 - 1\big)\big(\phi_2 + \phi_1 - 1\big) > 0, \\ |\phi_2| < 1. \end{cases}$$

Then we have the condition as

$$\begin{cases} \phi_1^2 + 4\phi_2 \geq 0 \\ -1 < \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ \phi_2 + \phi_1 < 1 \end{cases}. \tag{1}$$

- **Situation 2:** $\Delta < 0$

If $\Delta < 0$, then z_1 and z_2 are complex and conjugate.

$$z_1 = -\frac{\phi_1}{2\phi_2} + \frac{\sqrt{-\Delta}}{2\phi_2} i, \quad z_2 = -\frac{\phi_1}{2\phi_2} - \frac{\sqrt{-\Delta}}{2\phi_2} i.$$

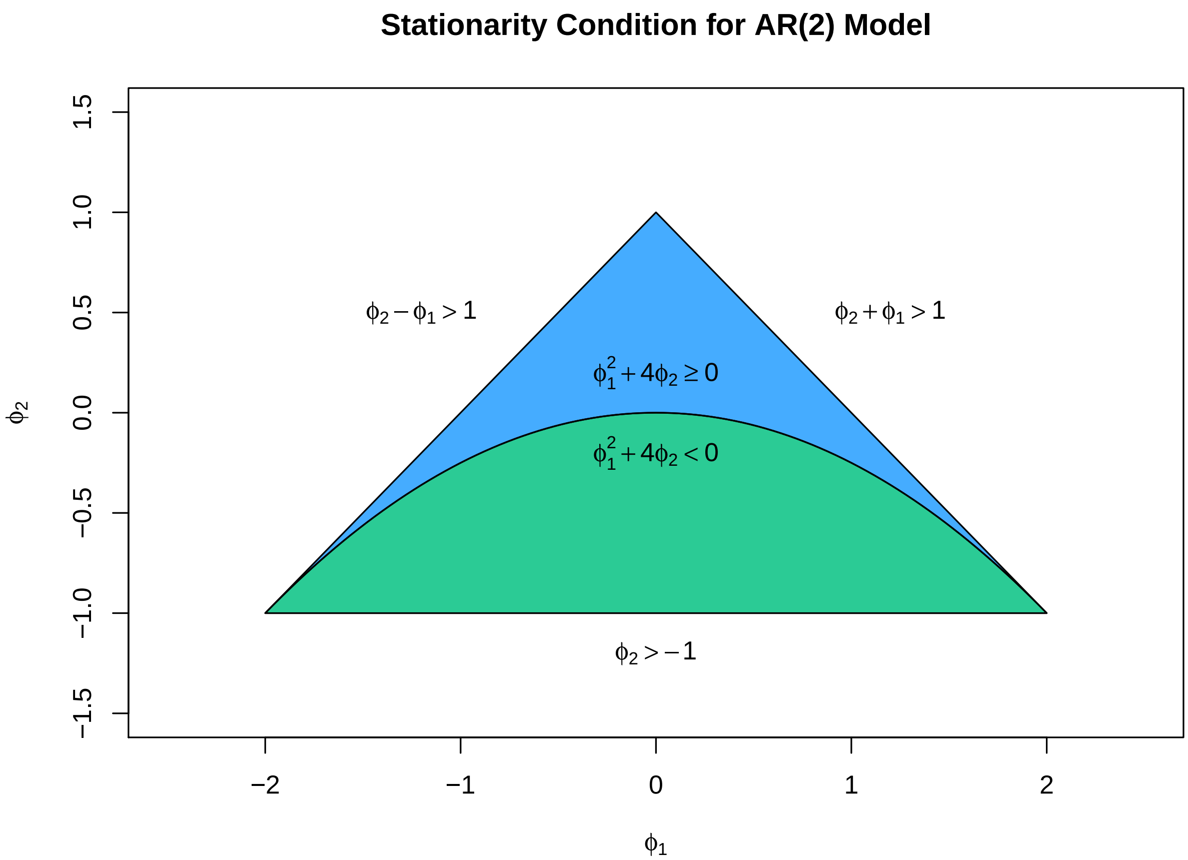
Plug z_1 and z_2 into $|z_1| > 1$ and $|z_2| > 1$, we have

$$|z_1| = |z_2| = \frac{\phi_1^2}{4\phi_2^2} + \frac{-\phi_1^2 - 4\phi_2}{4\phi_2^2} = -\frac{1}{\phi_2} > 1.$$

Then we have the condition as

$$\begin{cases} \phi_1^2 + 4\phi_2 < 0 \\ -1 < \phi_2 < 0 \end{cases}. \tag{2}$$

Show condition (1) and (2) in the following figure.



Hence, we can conclude that the stationarity condition for an AR(2) model is

$$\begin{cases} -1 < \phi_2 < 1 \\ \phi_2 - \phi_1 < 1 \\ \phi_2 + \phi_1 < 1 \end{cases}.$$