# Homework 9

DATA130021 Financial Econometrics

Deng Qisheng

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## Problem 1

Suppose the risk measure $\mathcal{R}$ is $\mathrm{VaR}(\alpha)$ for some $\alpha$. Let $P_1$ and $P_2$ be two portfolios whose returns $R_1$ and $R_2$ have a joint normal distribution with means $\mu_1$ and $\mu_2$, standard deviations $\sigma_1$ and $\sigma_2$, correlation $\rho$. Suppose the initial investments are $S_1$ and $S_2$. Show that $\mathcal{R}(R_1 + R_2) \leq \mathcal{R}(R_1) + \mathcal{R}(R_2)$ under joint normality.

**Solution:**

For any portfolio with initial investment $S$ whose return $R$ has a normal distribution with mean $\mu$ and standard deviation $\sigma$, we know that the value-at-risk is

$$\mathcal{R}(R) = \mathrm{VaR}_R(\alpha) = -S\big(\mu + \sigma\Phi^{-1}(\alpha)\big).$$

Consider the portfolio $R_1 + R_2$, we have:

$$\begin{aligned}
&P\big(\mathcal{L}_{R_1+R_2} > \mathrm{VaR}_{R_1+R_2}(\alpha)\big) = \alpha \\
&\Leftrightarrow P\big(-(S_1R_1 + S_2R_2) > \mathrm{VaR}_{R_1+R_2}(\alpha)\big) = \alpha \\
&\Leftrightarrow P\left(\frac{S_1R_1 + S_2R_2 - \mu_P}{\sigma_P} < \frac{-\mathrm{VaR}_{R_1+R_2}(\alpha) - \mu_P}{\sigma_P}\right) = \alpha \\
&\Leftrightarrow \Phi^{-1}(\alpha) = \frac{-\mathrm{VaR}_{R_1+R_2}(\alpha) - \mu_P}{\sigma_P} \\
&\Leftrightarrow \mathcal{R}(R_1 + R_2) = \mathrm{VaR}_{R_1+R_2}(\alpha) = -\big(\mu_P + \sigma_P\Phi^{-1}(\alpha)\big).
\end{aligned}$$

where

$$\mu_P = S_1\mu_1 + S_2\mu_2,\ \ \sigma_P = \sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\rho\sigma_1\sigma_2}.$$

Hence, we have:

$$\begin{aligned}
\mathcal{R}(R_1 + R_2) - \mathcal{R}(R_1) - \mathcal{R}(R_2) &= -\big(\mu_P + \sigma_P\Phi^{-1}(\alpha)\big) + S_1\big(\mu_1 + \sigma_1\Phi^{-1}(\alpha)\big) + S_2\big(\mu_2 + \sigma_2\Phi^{-1}(\alpha)\big) \\
&= (S_1\mu_1 + S_2\mu_2 - \mu_P) + (S_1\sigma_1 + S_2\sigma_2 - \sigma_P)\Phi^{-1}(\alpha) \\
&= \left(\sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\sigma_1\sigma_2} - \sqrt{S_1^2\sigma_1^2 + S_2^2\sigma_2^2 + 2S_1S_2\rho\sigma_1\sigma_2}\right)\Phi^{-1}(\alpha).
\end{aligned}$$

Since $-1 \leq \rho \leq 1$ and $\alpha > 0.5$, we can get that

$$\mathcal{R}(R_1 + R_2) \leq \mathcal{R}(R_1) + \mathcal{R}(R_2)$$

under joint normality. ∎

## Problem 2

Use negative daily stock return data of IBM from 2010/01 to 2015/12 to calculate the monthly value-at-risk and expected shortfall under risk level 95% and 99%. Show your results of 12 months in a table and sketch them in a time series plot for every year. Make a conclusion about your observation.

**Solution:**

```
library(lubridate)
library(knitr)
library(ggplot2)

raw.ibm <- read.csv("./IBM.csv")[, c(1, 6)]
ibm <- data.frame(
    "Year"=year(raw.ibm$Date), "Month"=month(raw.ibm$Date),
    "Return"=c(0, diff(raw.ibm$Adj.Close)) / raw.ibm$Adj.Close)

measure <- function (Y, M, S, A) {
    ibm.monthly <- ibm[which((ibm$Year==Y) & (ibm$Month==M)), ]$Return
    VaR <- - S * as.numeric(quantile(ibm.monthly, 0.05))
    IEVaR <- (ibm.monthly < as.numeric(quantile(ibm.monthly, 0.01)))
    ES <- - S * sum(ibm.monthly * IEVaR) / sum(IEVaR)
    return(c(VaR, ES))
}

Year <- 2010:2015
Month <- 1:12
S <- 20000

result <- array(0, dim=c(12, 2, 6), dimnames=list(Month, c("VaR", "ES"), Year))
for (y in 1:length(Year)) {
    for (m in 1:length(Month)){
        result[m, , y] <- measure(Year[y], Month[m], S)
    }
}
```

Table 1: VaR and ES in 2010

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 561.7 | 204.2 | 141.9 | 296.5 | 484.0 | 421.4 | 420.2 | 306.7 | 199.8 | 75.23 | 265.5 | 132.7 |
| ES | 597.3 | 432.5 | 180.5 | 391.7 | 817.4 | 622.0 | 512.0 | 405.6 | 258.8 | 695.49 | 308.7 | 152.0 |

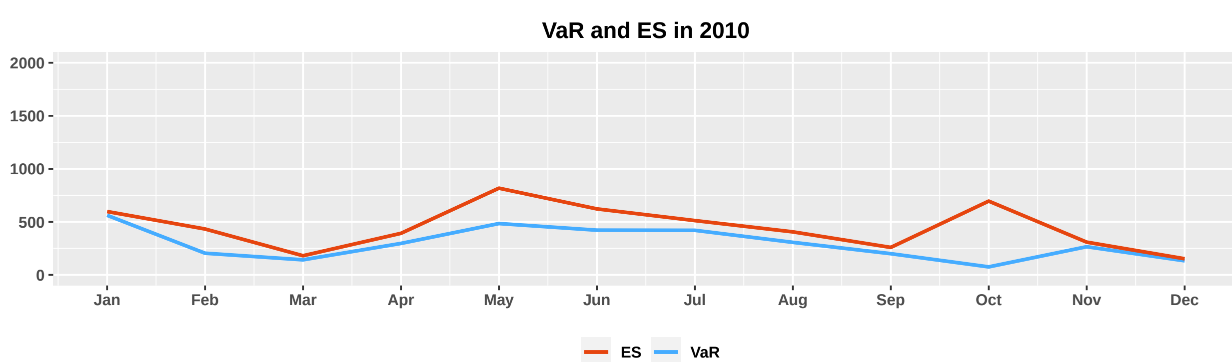


Table 2: VaR and ES in 2011

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 105.4 | 234.6 | 456.4 | 79.25 | 263.7 | 219.7 | 171.6 | 926.2 | 491.8 | 422.3 | 414.4 | 426.0 |
| ES | 233.7 | 356.9 | 786.9 | 85.75 | 273.1 | 284.6 | 174.2 | 993.0 | 521.9 | 859.7 | 549.8 | 635.9 |

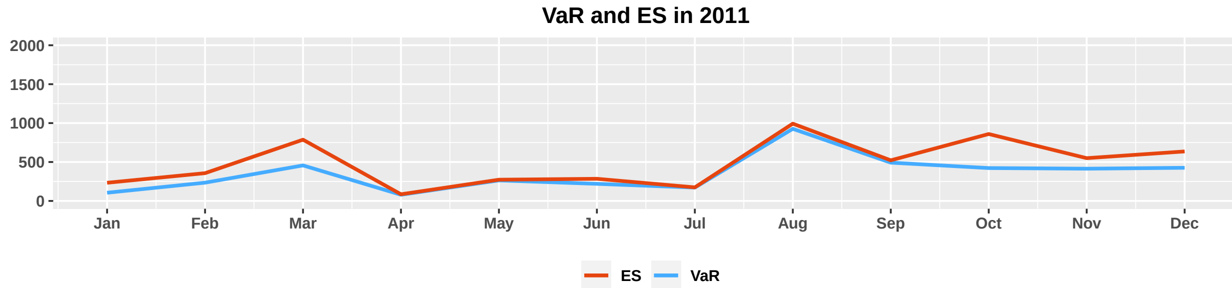



Table 3: VaR and ES in 2012

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 197.9 | 114.5 | 139.6 | 354.7 | 204.9 | 404.1 | 366.1 | 185.3 | 70.11 | 581.7 | 320.1 | 147.8 |
| ES | 232.3 | 127.1 | 344.7 | 731.5 | 219.5 | 557.4 | 405.4 | 212.6 | 96.08 | 1033.7 | 384.6 | 303.4 |

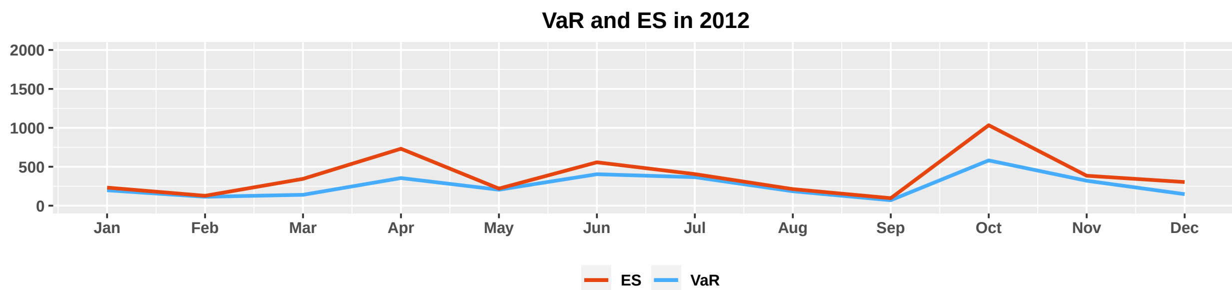



Table 4: VaR and ES in 2013

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 132.0 | 172.9 | 165.6 | 242.7 | 196.2 | 465.7 | 371.3 | 217.3 | 353.2 | 356.4 | 274.1 | 237.8 |
| ES | 190.0 | 362.5 | 263.8 | 1805.3 | 291.5 | 475.1 | 459.9 | 472.3 | 354.7 | 1361.3 | 312.2 | 247.9 |

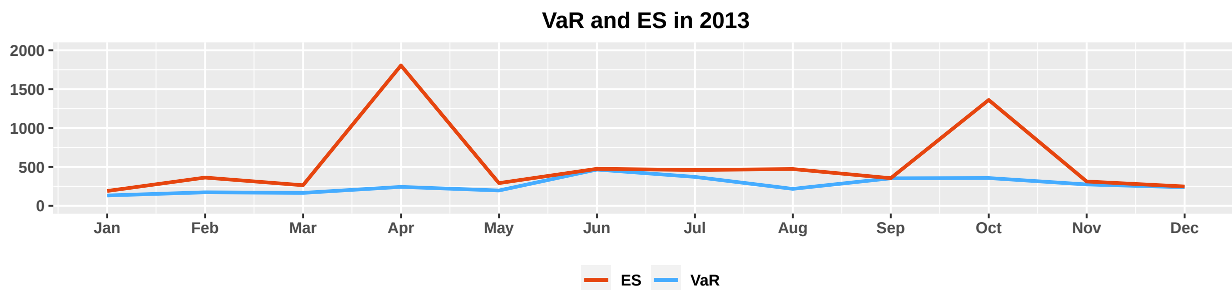


Table 5: VaR and ES in 2014

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 344.0 | 188.5 | 252.3 | 158.8 | 303.8 | 209.5 | 171.5 | 266.5 | 155.5 | 683.2 | 217.3 | 308.7 |
| ES | 678.2 | 437.2 | 293.9 | 672.6 | 367.7 | 223.9 | 243.1 | 271.5 | 349.2 | 1531.6 | 280.4 | 732.4 |

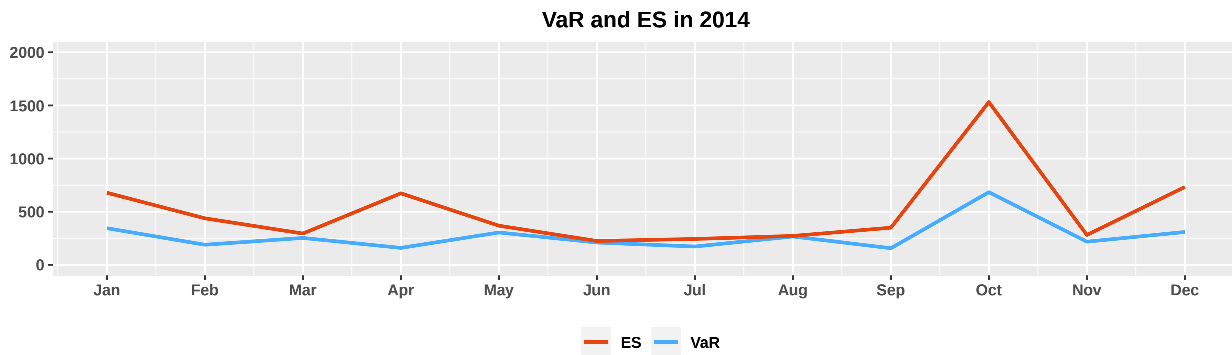
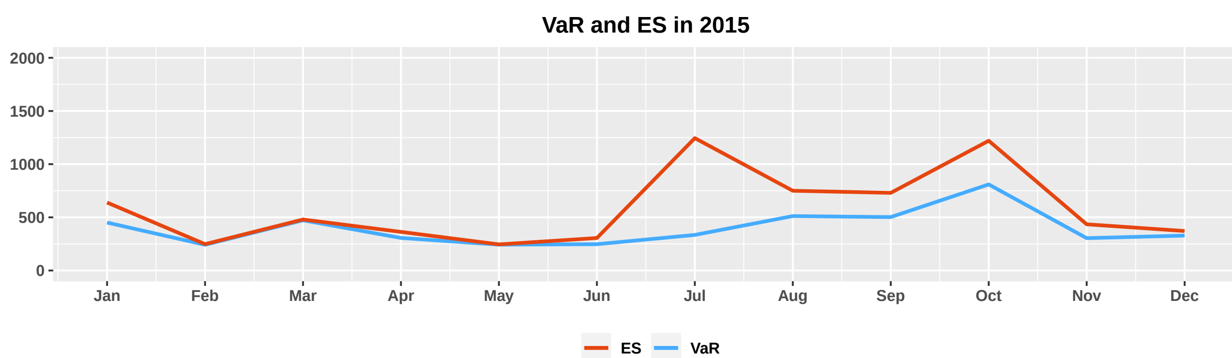



Table 6: VaR and ES in 2015

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| VaR | 450.7 | 241.9 | 472.3 | 306.2 | 243.0 | 247.4 | 334.7 | 511.9 | 502.4 | 809.5 | 304.5 | 328.5 |
| ES | 639.1 | 248.1 | 479.6 | 363.1 | 245.7 | 305.6 | 1244.9 | 750.0 | 730.3 | 1220.1 | 434.6 | 371.5 |




**Observations:**

- We find that ESs under risk level 99% are always larger than VaRs under risk level 95%.
- For all years, we find that there are peaks of ES in October, which may mean that something bad happened in trades in every Octobers.

## Problem 3

Suppose $X_i$ are i.i.d. Cauchy$(0, 1)$.

(a) What is the tail index of Cauchy$(0, 1)$?

**Solution:**

We know that Cauchy$(0, 1)$ is $t(1)$ distribution and the tail index of $t(\nu)$ distribution is $\nu$. Hence, the tail index of Cauchy$(0, 1)$ is **1**.

(b) Show that $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \sim$ Cauchy$(0, 1)$ as well.

**Proof:**

Recall the characteristic function of Cauchy$(0, 1)$ distribution:

$$\varphi_X(t) = e^{-|t|}.$$

We have:

$$\varphi_{\bar{X}_n}(t) = \varphi_X^n\left(\frac{t}{n}\right) = e^{-n|\frac{t}{n}|} = e^{-|t|},$$

which means $\bar{X}_n \sim$ Cauchy$(0, 1)$. ■

(c) Use simulation study to compute both the 95% Value-at-Risks of $X_1$ and $\bar{X}_{1000}$. What does it imply for the portfolio diversification in Value-at-Risk?

**Solution:**

For $X_1$, we simulate 10000 historical returns and use nonparametric estimation to find the VaR. For $\bar{X}_{1000}$, we simulate 10000 historical returns for 1000 times and use parametric estimation to find the VaR:

$$\mathrm{VaR}(\bar{X}_{1000}) = -\Big(\hat{\mu}_P + \tan\left[\pi(\alpha - \frac{1}{2})\right]\hat{\sigma}_P\Big),$$

where

$$\hat{\mu}_P = \frac{1}{1000}\sum_{i=1}^{1000}\mu_i, \ \hat{\sigma}_P^2 = \frac{1}{1000^2}\sum_{i=1}^{1000}\sigma_i^2.$$

```
sim <- rcauchy(10000)
VaR.1 <- -as.numeric(quantile(sim, 0.05))
sim.avg <- replicate(1000, rcauchy(10000))
mu_P <- mean(apply(sim.avg, 2, mean))
sigma_P <- sqrt(mean(apply(sim.avg, 2, var)) / 1000)
VaR.1000 <- -(mu_P + tan(pi * (0.05 - 0.5)) * sigma_P)
```

Table 7: 95% Value-at-Risk

| | $X_1$ | $\bar{X}_{1000}$ |
|---|---|---|
| VaR | 6.0998 | 580.8216 |

From the results, we find that

$$\mathrm{VaR}(\bar{X}_{1000}) > \mathrm{VaR}(X_1) = \frac{1}{1000}\sum_{i=1}^{1000}\mathrm{VaR}(X_i),$$

which shows that VaR is not subadditive. In common sense, portfolio should have a lower risk than one asset. But we can not show this result by VaR measure.