Homework 2

DATA130021 Financial Econometrics

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Problem 1

The unit of yearly log returns is *year*. We have the definition as

$$r_t = \log\big(1 + R_t\big) = \log P_t - \log P_{t-1} \overset{\text{i.i.d.}}{\sim} \mathcal{N}\big(\mu, \sigma^2\big),$$

where $\mu = 0.08$ and $\sigma = 0.15$. From the assumption, the price P_0 is \$80 today. Then we have

$$\begin{aligned}
\mathrm{P}\big(P_1 \geq 90\big) &= \mathrm{P}\left(\log\left(\frac{P_1}{P_0}\right) \geq \log\left(\frac{90}{80}\right)\right) \\
&= \mathrm{P}\big(r_1 \geq 0.11778\big) \\
&= \mathrm{P}\left(\frac{r_1 - 0.08}{0.15} \geq \frac{0.11778 - 0.08}{0.15}\right) \\
&= 1 - \Phi\big(0.2519\big) \\
&\approx 0.4006.
\end{aligned}$$

Hence, the probability that the price one year from now is no less than \$90 is approximately 40.06%.

Problem 2

Definitions of simple returns and log returns are

$$R_t = \frac{P_t + D_t}{P_{t-1}} - 1,$$

$$r_t = \log\big(1 + R_t\big) = \log\left(\frac{P_t + D_t}{P_{t-1}}\right),$$

and multiple-period gross returns are products of single-period gross returns so that

$$1 + R_t(k) = \prod_{i=t-k+1}^{t} \big(1 + R_i\big) = \prod_{i=t-k+1}^{t} \frac{P_i + D_i}{P_{i-1}}.$$

Then we have the k-period returns and log returns as

$$R_t(k) = -1 + \prod_{i=t-k+1}^{t} \big(1 + R_i\big) = -1 + \prod_{i=t-k+1}^{t} \frac{P_i + D_i}{P_{i-1}},$$

$$r_t(k) = \log\big(1 + R_t(k)\big) = \sum_{i=t-k+1}^{t} \log\left(\frac{P_i + D_i}{P_{i-1}}\right) = \sum_{i=t-k+1}^{t} r_i.$$

a. $R_3(2)$

$$\begin{aligned}R_3(2) &= \big(1+R_2\big)\big(1+R_3\big)-1\\ &= \frac{P_2+D_2}{P_1}\cdot\frac{P_3+D_3}{P_2}-1\\ &= \frac{85+0.1}{82}\cdot\frac{83+0.1}{85}-1\\ &\approx 0.0146.\end{aligned}$$

b. $r_4(3)$

$$\begin{aligned}r_4(3) &= r_2+r_3+r_4\\ &= \log\big(1+R_2\big)+\log\big(1+R_3\big)+\log\big(1+R_4\big)\\ &= \log\left(\frac{P_2+D_2}{P_1}\right)+\log\left(\frac{P_3+D_3}{P_2}\right)+\log\left(\frac{P_4+D_4}{P_3}\right)\\ &= \log\left(\frac{85+0.1}{82}\right)+\log\left(\frac{83+0.1}{85}\right)+\log\left(\frac{87+0.125}{83}\right)\\ &\approx 0.0630.\end{aligned}$$

Problem 3

a. From the assumption, variables $r_1, r_2, \cdots \overset{\text{i.i.d}}{\sim} \mathcal{N}\big(\mu,\sigma^2\big)$. Then

$$\log\left(\frac{X_k}{X_0}\right)=\log X_k-\log X_0=\sum_{i=1}^{k} r_i\sim\mathcal{N}\big(k\mu,k\sigma^2\big)$$

Hence, we have

$$\begin{aligned}\mathrm{P}\big(X_2>1.3X_0\big) &= \mathrm{P}\left(\log\left(\frac{X_2}{X_0}\right)>\log(1.3)\right)\\ &= \mathrm{P}\big(r_1+r_2>0.26236\big)\\ &= \mathrm{P}\left(\frac{r_1+r_2-2\mu}{\sqrt{2}\sigma}>\frac{0.26236-2\mu}{\sqrt{2}\sigma}\right)\\ &= 1-\Phi\left(\frac{0.26236-2\mu}{\sqrt{2}\sigma}\right).\end{aligned}$$

b. The relationship between variables X_1 and r_1 is

$$X_1=X_0\mathrm{e}^{r_1}\ \Leftrightarrow\ h(X_1)=r_1=\log X_1-\log X_0\sim\mathcal{N}\big(\mu,\sigma^2\big),$$

and the density of r_1 is

$$f_{r_1}(x)=\frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{1}{2\sigma^2}(x-\mu)^2},\ x\in(-\infty,+\infty).$$

Apply formula (A.4) and we get that the density of X_1 is

$$\begin{aligned}f_{X_1}(y) &= f_{r_1}\big(h(y)\big)\big|h'(y)\big|\\ &= \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-\frac{1}{2\sigma^2}\big(h(y)-\mu\big)^2}\cdot\frac{1}{y}\\ &= \frac{1}{\sqrt{2\pi}\sigma y}\mathrm{e}^{-\frac{1}{2\sigma^2}\big(\log y-\log X_0-\mu\big)^2},\ y\in(X_0,+\infty).\end{aligned}$$

c. From problem 3.a, we know that

$$\log\left(\frac{X_k}{X_0}\right) = \log X_k - \log X_0 = \sum_{i=1}^{k} r_i \sim \mathcal{N}\big(k\mu, k\sigma^2\big).$$

Then

$$h(X_k) = Z = \frac{\log\left(\frac{X_k}{X_0}\right) - k\mu}{\sqrt{k}\sigma} \sim \mathcal{N}\big(0,1\big),$$

Since $h(\cdot)$ is strictly increasing, we can get its inverse function $g(\cdot)$

$$g(Z) = X_k = X_0 \mathrm{e}^{\sqrt{k}\sigma Z + k\mu}.$$

Then we have

$$\begin{aligned}
F_{X_k}^{-1}(p) &= g\big(F_Z^{-1}(p)\big) \\
&= g\big(\Phi^{-1}(p)\big) \\
&= X_0 \mathrm{e}^{\sqrt{k}\sigma\Phi^{-1}(p) + k\mu}.
\end{aligned}$$

For the 0.9 quantile of X_k for all k, we have the following formula

$$\begin{aligned}
F_{X_k}^{-1}(0.9) &= X_0 \mathrm{e}^{\sqrt{k}\sigma\Phi^{-1}(0.9) + k\mu} \\
&= X_0 \mathrm{e}^{1.28155\sqrt{k}\sigma + k\mu}.
\end{aligned}$$

d. From problem 3.c, we know that

$$X_k = X_0 \mathrm{e}^{\sqrt{k}\sigma Z + k\mu},$$

where $Z \sim \mathcal{N}(0,1)$. Recall the moment-generating function of normal distribution

$$M_Z(t) = \mathbb{E}\big(\mathrm{e}^{tZ}\big) = \mathrm{e}^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

Then the expected value of X_K^2 for any k is

$$\begin{aligned}
\mathbb{E}\big(X_k^2\big) &= \mathbb{E}\Big(X_0^2 \mathrm{e}^{2\sqrt{k}\sigma Z + 2k\mu}\Big) \\
&= X_0^2 \mathrm{e}^{2k\mu} \mathbb{E}\Big(\mathrm{e}^{2\sqrt{k}\sigma Z}\Big) \\
&= X_0^2 \mathrm{e}^{2k\mu} M_Z\big(2\sqrt{k}\sigma\big) \\
&= X_0^2 \mathrm{e}^{2k\mu} \cdot \mathrm{e}^{2\sqrt{k}\mu\sigma + 2k\sigma^4} \\
&= X_0^2 \mathrm{e}^{2k\mu + 2\sqrt{k}\mu\sigma + 2k\sigma^4}.
\end{aligned}$$

e. From problem 3.d, we know that

$$\mathbb{E}\big(X_k^2\big) = X_0^2 \mathrm{e}^{2k\mu + 2\sqrt{k}\mu\sigma + 2k\sigma^4}.$$

Similarly, we can get that

$$\begin{aligned}
\mathbb{E}\big(X_k\big) &= \mathbb{E}\Big(X_0 \mathrm{e}^{\sqrt{k}\sigma Z + k\mu}\Big) \\
&= X_0 \mathrm{e}^{k\mu} \mathbb{E}\Big(\mathrm{e}^{\sqrt{k}\sigma Z}\Big) \\
&= X_0 \mathrm{e}^{k\mu} M_Z\big(\sqrt{k}\sigma\big) \\
&= X_0 \mathrm{e}^{k\mu} \cdot \mathrm{e}^{\sqrt{k}\mu\sigma + \frac{1}{2}k\sigma^4} \\
&= X_0 \mathrm{e}^{k\mu + \sqrt{k}\mu\sigma + \frac{1}{2}k\sigma^4}.
\end{aligned}$$

Then the variance of X_k for any k is

$$\begin{aligned}\mathrm{Var}\big(X_k\big) &= \mathbb{E}\big(X_k^2\big) - \mathbb{E}^2\big(X_k\big)\\ &= X_0^2 \mathrm{e}^{2k\mu+2\sqrt{k}\mu\sigma+2k\sigma^4} - \Big(X_0 \mathrm{e}^{k\mu+\sqrt{k}\mu\sigma+\frac{1}{2}k\sigma^4}\Big)^2\\ &= X_0^2 \mathrm{e}^{2k\mu+2\sqrt{k}\mu\sigma+2k\sigma^4} - X_0^2 \mathrm{e}^{2k\mu+2\sqrt{k}\mu\sigma+k\sigma^4}\\ &= X_0^2\big(\mathrm{e}^{k\sigma^4}-1\big)\mathrm{e}^{2k\mu+2\sqrt{k}\mu\sigma+k\sigma^4}.\end{aligned}$$

Problem 4

From the assumption, daily log returns are normally distributed, that is, $r_1, r_2, \cdots \overset{\text{i.i.d.}}{\sim} \mathcal{N}\big(\mu, \sigma^2\big)$, where $\mu = 0.0005$ and $\sigma^2 = 0.012$. From the definition, we have

$$r_t(t) = \log\left(\frac{P_t}{P_0}\right) \sim \mathcal{N}\big(t\mu, t\sigma^2\big).$$

Then

$$\begin{aligned}\mathrm{P}\left(\frac{P_t}{P_0} \geq 2\right) \geq 0.9 &\Leftrightarrow \mathrm{P}\left(\log\left(\frac{P_t}{P_0}\right) \geq \log(2)\right) \geq 0.9\\ &\Leftrightarrow \mathrm{P}\Big(r_t(t) \geq \log(2)\Big) \geq 0.9\\ &\Leftrightarrow \mathrm{P}\left(\frac{r_t(t)-t\mu}{\sqrt{t}\sigma} \geq \frac{\log(2)-t\mu}{\sqrt{t}\sigma}\right) \geq 0.9\\ &\Leftrightarrow 1-\Phi\left(\frac{\log(2)-t\mu}{\sqrt{t}\sigma}\right) \geq 0.9\\ &\Leftrightarrow \Phi\left(\frac{\log(2)-t\mu}{\sqrt{t}\sigma}\right) \leq 0.1\\ &\Leftrightarrow \mu t + \sigma\Phi^{-1}(0.1)\sqrt{t} - \log(2) \geq 0\\ &\Leftrightarrow \mu^2 t^2 - \Big[\sigma^2\Phi^{-1}(0.1)^2 + 2\log(2)\mu\Big]t + \log^2(2) \geq 0 \quad (*).\end{aligned}$$

Solve the quadratic inequality $(*)$ and we get

$$t \geq 81584 \text{ days}.$$

In conclusion, the probability that the price has doubled is at least 90% after 81584 days (about 223.5 years), which means it is almost impossible in reality.

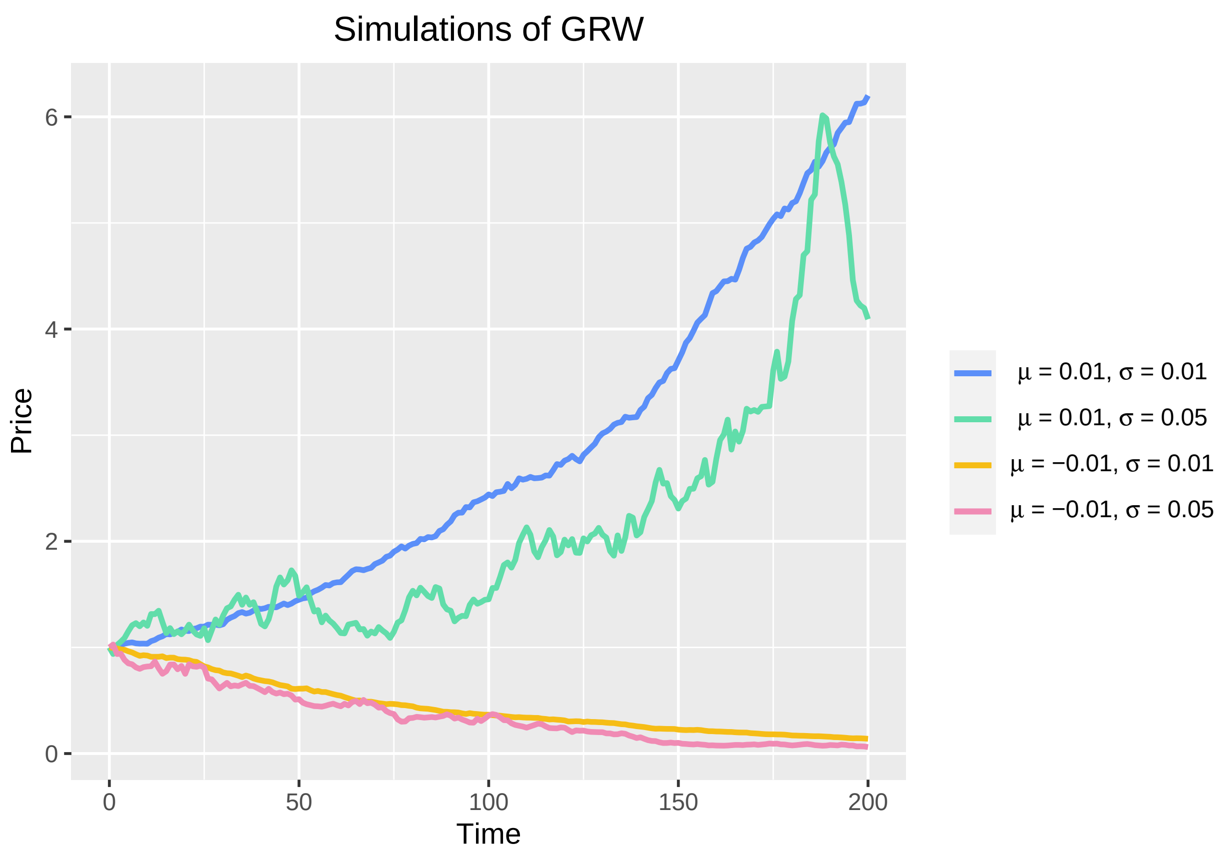
Problem 5

Codes for simulations of log-normal geometric random walk models with $P_0 = 1, \mu = \pm 0.01$ and $\sigma = 0.01, 0.05$:

```r
library(ggplot2)
set.seed(999)  # Set random seed

Time <- c(0:200)  # Simulates 200 times
Price_1 <- c(1, exp(cumsum(rnorm(200, 0.01, 0.01))))
Price_2 <- c(1, exp(cumsum(rnorm(200, 0.01, 0.05))))
Price_3 <- c(1, exp(cumsum(rnorm(200, -0.01, 0.01))))
Price_4 <- c(1, exp(cumsum(rnorm(200, -0.01, 0.05))))
df <- data.frame(Time, Price_1, Price_2, Price_3, Price_4)
```

```r
ggplot() +
    geom_line(data=df, aes(x=Time, y=Price_1, colour="Price_1"), size=1) +
    geom_line(data=df, aes(x=Time, y=Price_2, colour="Price_2"), size=1) +
    geom_line(data=df, aes(x=Time, y=Price_3, colour="Price_3"), size=1) +
    geom_line(data=df, aes(x=Time, y=Price_4, colour="Price_4"), size=1) +
    labs(title="Simulations of GRW", x="Time", y="Price") +
    theme(plot.title=element_text(hjust=0.5)) +
    scale_colour_manual(
        name="",
        values=c(
            "Price_1"="#5B8FF9",
            "Price_2"="#61DDAA",
            "Price_3"="#F6BD16",
            "Price_4"="#F08BB4"
        ),
        labels=c(
            expression(paste(mu, " = 0.01, ", sigma, " = 0.01 ")),
            expression(paste(mu, " = 0.01, ", sigma, " = 0.05 ")),
            expression(paste(mu, " = -0.01, ", sigma, " = 0.01")),
            expression(paste(mu, " = -0.01, ", sigma, " = 0.05"))
        )
    )
```



From the figure above, we find that the chains with $\sigma = 0.05$ have a larger volatility than the ones with $\sigma = 0.01$. Chains with positive drift $\mu = 0.01$ show the increasing trend and chains with negative drift $\mu = -0.01$ seem to converge to zero. These simulation results are consistent with common sense.