Homework 7

DATA130021 Financial Econometrics

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Problem 1

Let Y_t be an MA(2) process,

$$Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}.$$

Find formulas for the autocovariance and autocorreletion functions of Y_t

Solution:

Recall the definition of autocovariance and autocorreletion functions, we have

$$\gamma(h) = \mathrm{Cov}(Y_t, Y_{t+h}) = \gamma(0)\rho(h).$$

Find the variance of Y_t, we have

$$\begin{aligned}
\gamma(0) &= \mathrm{Var}(Y_t) \\
&= \mathrm{Var}(\mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}) \\
&= \mathrm{Var}(\epsilon_t) + \theta_1^2 \mathrm{Var}(\epsilon_{t-1}) + \theta_2^2 \mathrm{Var}(\epsilon_{t-2}) \\
&= (1 + \theta_1^2 + \theta_2^2)\sigma^2.
\end{aligned}$$

When $h > 2$, we have

$$\begin{cases} Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} \\ Y_{t+h} = \mu + \epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \theta_2 \epsilon_{t+h-2} \end{cases},$$

then $t-2 < t-1 < t < t+h-2 < t+h-1 < t+h$. Since $\epsilon_t \sim \mathcal{N}(0, \sigma^2)$, then $\mathrm{Cov}(Y_i, Y_j) = 0$ where $i, j \in \{t-2, t-1, t, t+h-2, t+h-1, t+h\}$. Hence,

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}(Y_t, Y_{t+h}) \\
&= \mathrm{Cov}(\mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}, \mu + \epsilon_{t+h} + \theta_1 \epsilon_{t+h-1} + \theta_2 \epsilon_{t+h-2}) \\
&= \mathrm{Cov}(\epsilon_t, \epsilon_{t+h}) + \theta_1 \mathrm{Cov}(\epsilon_t, \epsilon_{t+h-1}) + \theta_2 \mathrm{Cov}(\epsilon_t, \epsilon_{t+h-2}) \\
&\quad + \theta_1 \mathrm{Cov}(\epsilon_{t-1}, \epsilon_{t+h}) + \theta_1^2 \mathrm{Cov}(\epsilon_{t-1}, \epsilon_{t+h-1}) + \theta_1\theta_2 \mathrm{Cov}(\epsilon_{t-1}, \epsilon_{t+h-2}) \\
&\quad + \theta_2 \mathrm{Cov}(\epsilon_{t-2}, \epsilon_{t+h}) + \theta_2\theta_1 \mathrm{Cov}(\epsilon_{t-2}, \epsilon_{t+h-1}) + \theta_2^2 \mathrm{Cov}(\epsilon_{t-2}, \epsilon_{t+h-2}) \\
&= 0,
\end{aligned}$$

and

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = 0.$$

When $h = 2$, we have

$$\begin{cases} Y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} \\ Y_{t+2} = \mu + \epsilon_{t+2} + \theta_1\epsilon_{t+1} + \theta_2\epsilon_t \end{cases},$$

then

$$\begin{aligned} \gamma(2) &= \text{Cov}(Y_t, Y_{t+2}) \\ &= \text{Cov}(\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}, \mu + \epsilon_{t+2} + \theta_1\epsilon_{t+1} + \theta_2\epsilon_t) \\ &= \text{Cov}(\epsilon_t, \epsilon_{t+2}) + \theta_1\text{Cov}(\epsilon_t, \epsilon_{t+1}) + \theta_2\text{Cov}(\epsilon_t, \epsilon_t) \\ &\quad + \theta_1\text{Cov}(\epsilon_{t-1}, \epsilon_{t+2}) + \theta_1^2\text{Cov}(\epsilon_{t-1}, \epsilon_{t+1}) + \theta_1\theta_2\text{Cov}(\epsilon_{t-1}, \epsilon_t) \\ &\quad + \theta_2\text{Cov}(\epsilon_{t-2}, \epsilon_{t+2}) + \theta_2\theta_1\text{Cov}(\epsilon_{t-2}, \epsilon_{t+1}) + \theta_2^2\text{Cov}(\epsilon_{t-2}, \epsilon_t) \\ &= \theta_2\text{Var}(\epsilon_t) \\ &= \theta_2\sigma^2, \end{aligned}$$

and

$$\rho(2) = \frac{\gamma(2)}{\gamma(0)} = \frac{\theta_2\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{\theta_2}{1+\theta_1^2+\theta_2^2}.$$

When $h = 1$, we have

$$\begin{cases} Y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} \\ Y_{t+1} = \mu + \epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1} \end{cases},$$

then

$$\begin{aligned} \gamma(1) &= \text{Cov}(Y_t, Y_{t+1}) \\ &= \text{Cov}(\mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2}, \mu + \epsilon_{t+1} + \theta_1\epsilon_t + \theta_2\epsilon_{t-1}) \\ &= \text{Cov}(\epsilon_t, \epsilon_{t+1}) + \theta_1\text{Cov}(\epsilon_t, \epsilon_t) + \theta_2\text{Cov}(\epsilon_t, \epsilon_{t-1}) \\ &\quad + \theta_1\text{Cov}(\epsilon_{t-1}, \epsilon_{t+1}) + \theta_1^2\text{Cov}(\epsilon_{t-1}, \epsilon_t) + \theta_1\theta_2\text{Cov}(\epsilon_{t-1}, \epsilon_{t-1}) \\ &\quad + \theta_2\text{Cov}(\epsilon_{t-2}, \epsilon_{t+1}) + \theta_2\theta_1\text{Cov}(\epsilon_{t-2}, \epsilon_t) + \theta_2^2\text{Cov}(\epsilon_{t-2}, \epsilon_{t-1}) \\ &= \theta_1\text{Var}(\epsilon_t) + \theta_1\theta_2\text{Var}(\epsilon_{t-1}) \\ &= \theta_1(1+\theta_2)\sigma^2, \end{aligned}$$

and

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta_1(1+\theta_2)\sigma^2}{(1+\theta_1^2+\theta_2^2)\sigma^2} = \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}.$$

When $h = 0$, we have

$$\gamma(0) = (1+\theta_1^2+\theta_2^2)\sigma^2,$$

and

$$\rho(0) = 1.$$

Hence, we can conclude that the autocovariance and autocorreletion functions of Y_t are

$$\gamma(h) = \begin{cases} 0, & h > 2 \\ \theta_2\sigma^2, & h = 2 \\ \theta_1(1+\theta_2)\sigma^2, & h = 1 \\ (1+\theta_1^2+\theta_2^2)\sigma^2, & h = 0 \end{cases},$$

and

$$\rho(h) = \begin{cases} 0, & h > 2 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2}, & h = 2 \\ \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}, & h = 1 \\ 1, & h = 0 \end{cases}.$$

Problem 2

The MA(2) model $Y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2}$ was fit to data and the estimates are

Parameter	Estimate
μ	45
θ_1	0.3
θ_2	-0.15

The last two values of the observed time series and residuals are

t	Y_t	$\hat{\epsilon}_t$
$n-1$	39.8	-4.3
n	42.7	1.5

Find the forecasts of Y_{n+1} and Y_{n+2}.

Solution:

From the definition, we know that the forecasts of Y_{n+k} are

$$\hat{Y}_{n+k} = \mathbb{E}\big[Y_{n+k}|Y_n, Y_{n-1}, \cdots, Y_1\big], \quad k > 0.$$

The forecast of Y_{n+1} is

$$\begin{aligned}
\hat{Y}_{n+1} &= \mathbb{E}\big[Y_{n+1}|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mathbb{E}\big[\mu + \epsilon_{n+1} + \theta_1\epsilon_n + \theta_2\epsilon_{n-1}|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mu + \mathbb{E}\big[\epsilon_{n+1}|Y_n, Y_{n-1}, \cdots, Y_1\big] + \theta_1\mathbb{E}\big[\epsilon_n|Y_n, Y_{n-1}, \cdots, Y_1\big] + \theta_2\mathbb{E}\big[\epsilon_{n-1}|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mu + 0 + \theta_1\hat{\epsilon}_n + \theta_2\hat{\epsilon}_{n-1} \\
&= 45 + 0 + 0.3 \times 1.5 + 0.15 \times 4.3 \\
&= 46.095 \approx 46.1.
\end{aligned}$$

The forecast of Y_{n+2} is

$$\begin{aligned}
\hat{Y}_{n+2} &= \mathbb{E}\big[Y_{n+2}|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mathbb{E}\big[\mu + \epsilon_{n+2} + \theta_1\epsilon_{n+1} + \theta_2\epsilon_n|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mu + \mathbb{E}\big[\epsilon_{n+2}|Y_n, Y_{n-1}, \cdots, Y_1\big] + \theta_1\mathbb{E}\big[\epsilon_{n+1}|Y_n, Y_{n-1}, \cdots, Y_1\big] + \theta_2\mathbb{E}\big[\epsilon_n|Y_n, Y_{n-1}, \cdots, Y_1\big] \\
&= \mu + 0 + 0 + \theta_2\hat{\epsilon}_n \\
&= 45 + 0 + 0 - 0.15 \times 1.5 \\
&= 44.775 \approx 44.8.
\end{aligned}$$

Problem 3

Simulate 300 time series observations from an MA(2) model

$$Y_t = \mu + \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2},$$

with

(a) $\theta_1 > 0$, $\theta_2 > 0$.

(b) $\theta_1 < 0$, $\theta_2 > 0$.

(c) $\theta_1 > 0$, $\theta_2 < 0$.

(d) $\theta_1 < 0$, $\theta_2 < 0$.

Plot both ACF and PACF for each model. Considering the result in Problem 1, what do you observe?

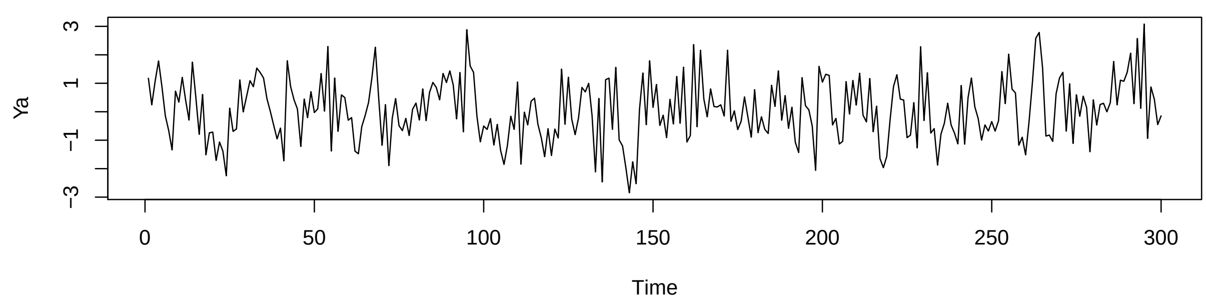
Solution:

For simplicity, we set $\mu = 0$.

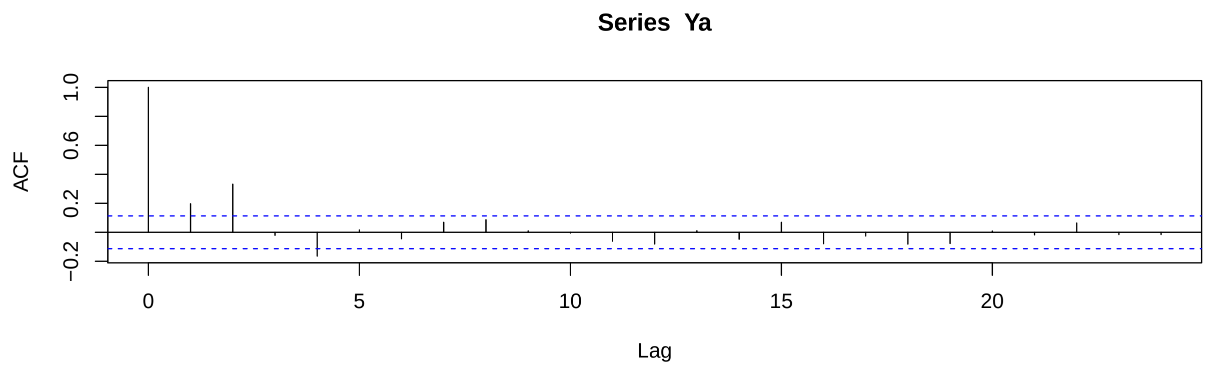
(a) $\theta_1 = 0.2$, $\theta_2 = 0.6$

Simulate 300 time series observations from $Y_t = \epsilon_t + 0.2\epsilon_{t-1} + 0.6\epsilon_{t-2}$ as follow.

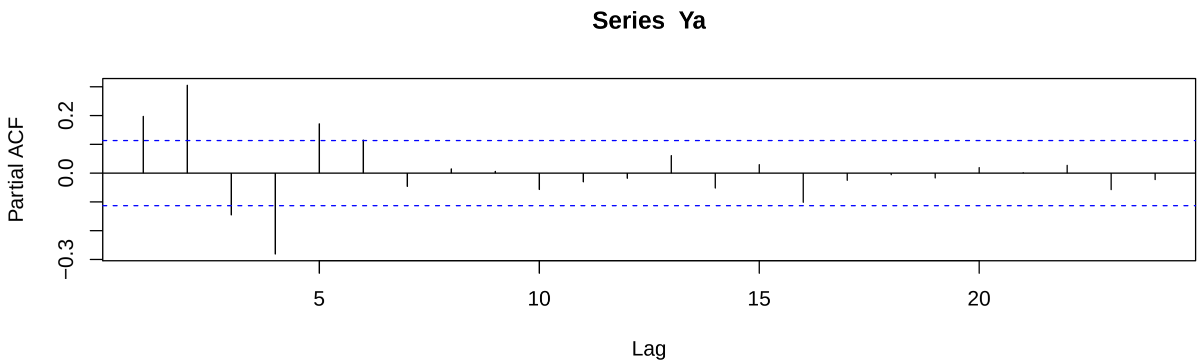
```
set.seed(123)
Ya <- arima.sim(n=300, model=list(order=c(0, 0, 2), ma=c(0.2, 0.6)))
plot(Ya, type="l")
```




```
acf(Ya)
```



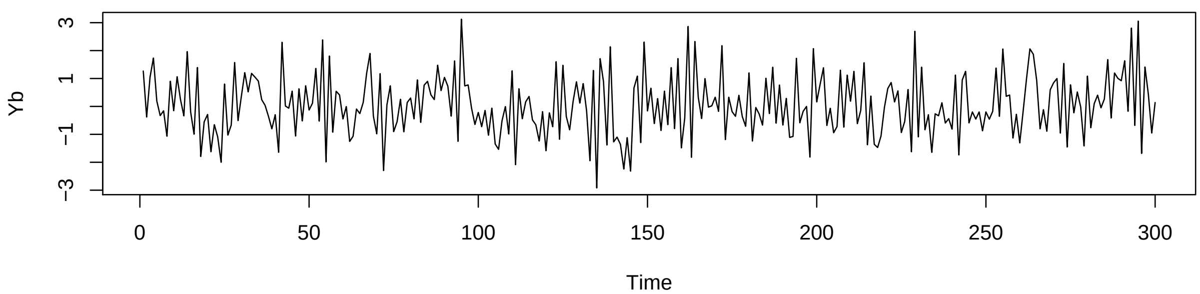
```r
pacf(Ya)
```



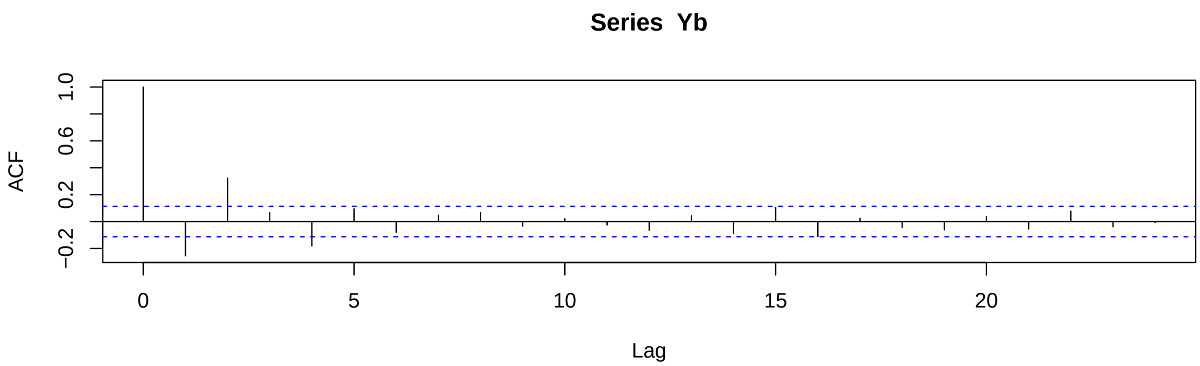
(b) $\theta_1 = -0.2$, $\theta_2 = 0.6$

Simulate 300 time series observations from $Y_t = \epsilon_t - 0.2\epsilon_{t-1} + 0.6\epsilon_{t-2}$ as follow.

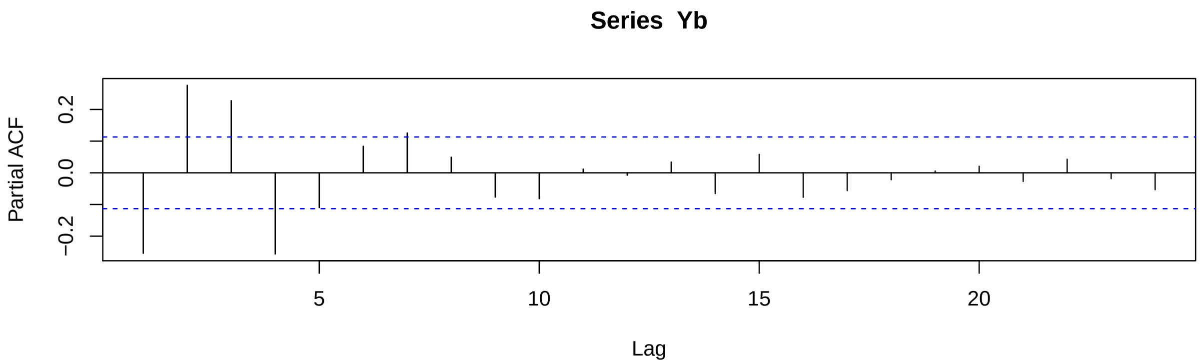
```r
set.seed(123)
Yb <- arima.sim(n=300, model=list(order=c(0, 0, 2), ma=c(-0.2, 0.6)))
plot(Yb, type="l")
```




```r
acf(Yb)
```



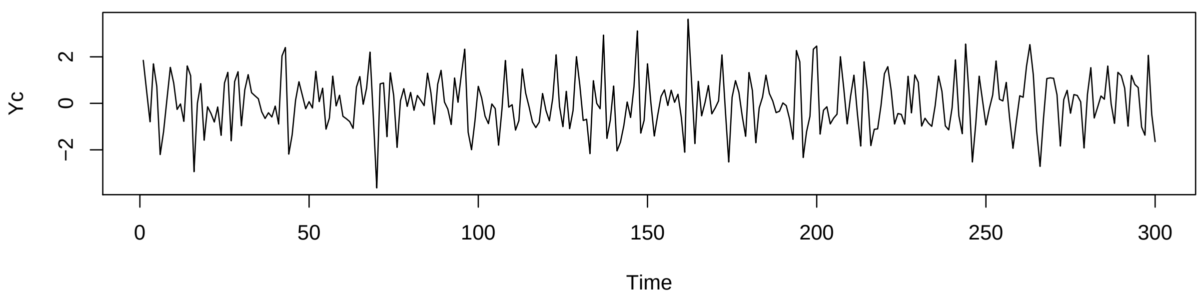
```r
pacf(Yb)
```



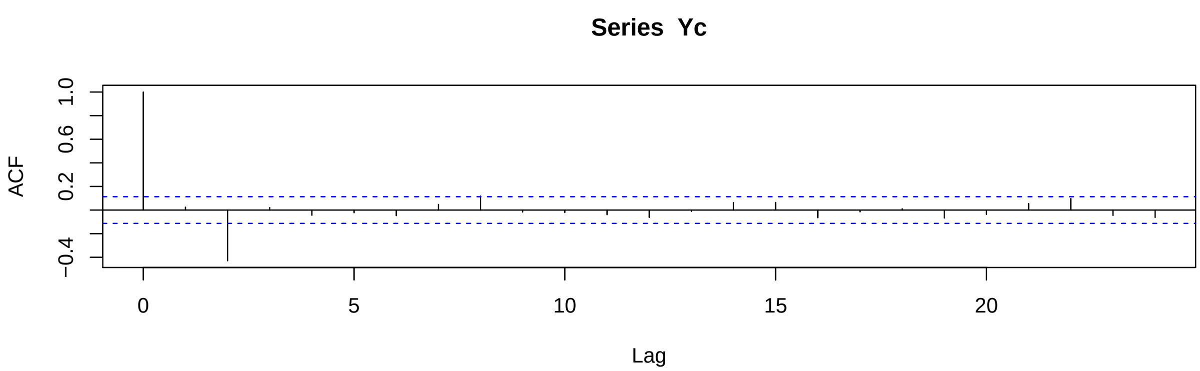
(c) $\theta_1 = 0.2$, $\theta_2 = -0.6$

Simulate 300 time series observations from $Y_t = \epsilon_t + 0.2\epsilon_{t-1} - 0.6\epsilon_{t-2}$ as follow.

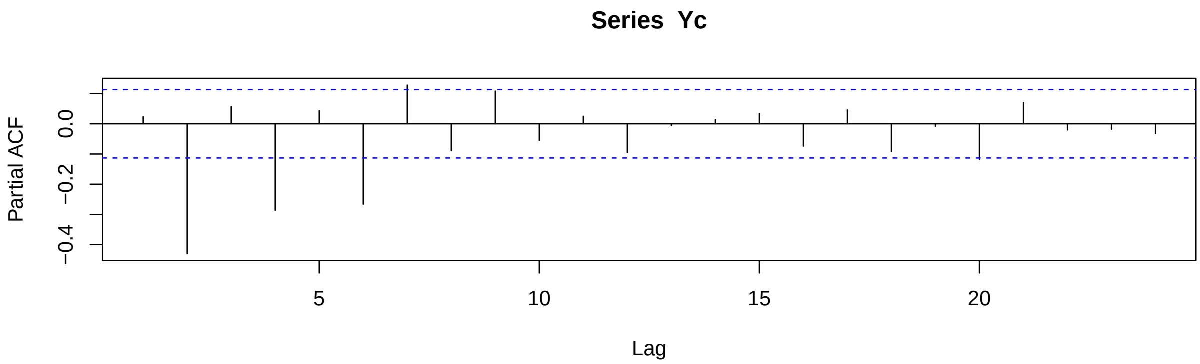
```r
set.seed(123)
Yc <- arima.sim(n=300, model=list(order=c(0, 0, 2), ma=c(0.2, -0.6)))
plot(Yc, type="l")
```




```r
acf(Yc)
```



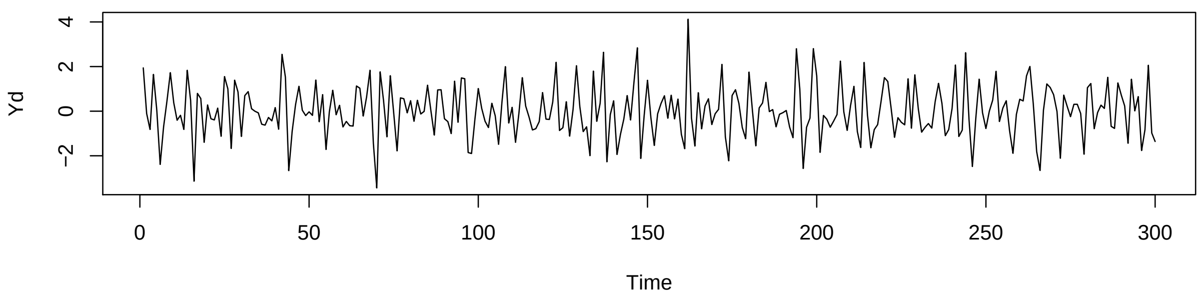
```r
pacf(Yc)
```



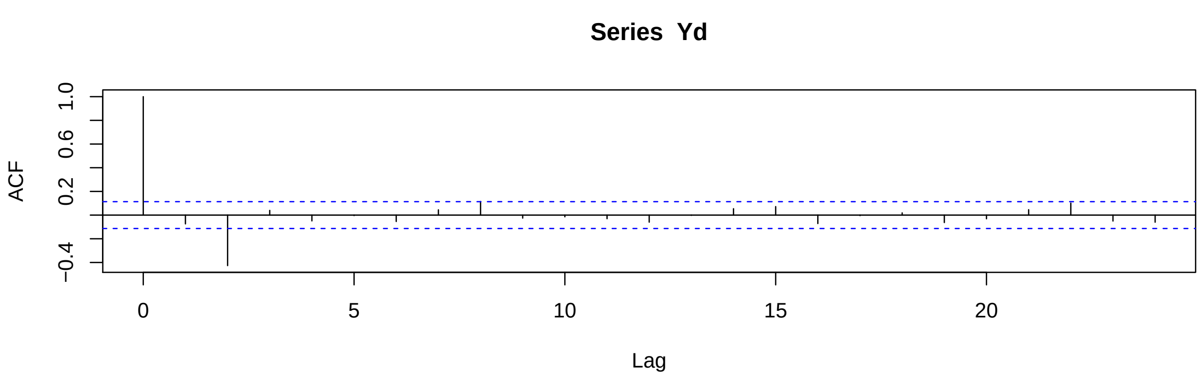
(d) $\theta_1 = -0.2$, $\theta_2 = -0.6$

Simulate 300 time series observations from $Y_t = \epsilon_t - 0.2\epsilon_{t-1} - 0.6\epsilon_{t-2}$ as follow.

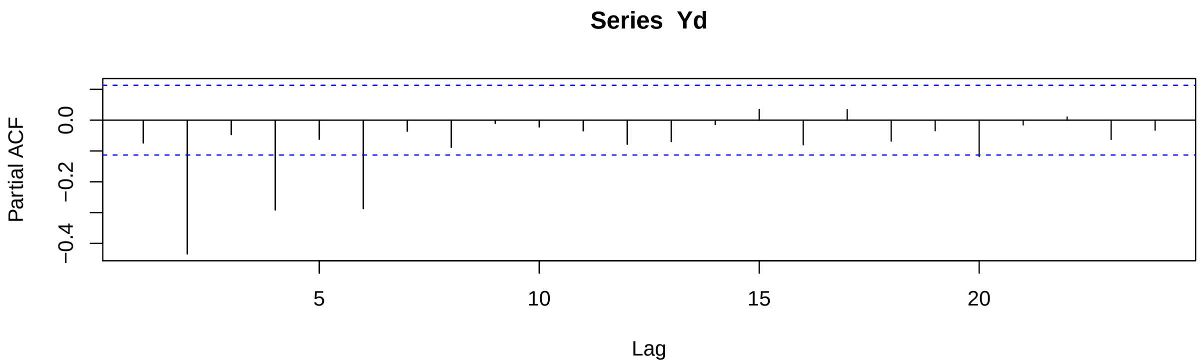
```r
set.seed(123)
Yd <- arima.sim(n=300, model=list(order=c(0, 0, 2), ma=c(-0.2, -0.6)))
plot(Yd, type="l")
```




```r
acf(Yd)
```



```r
pacf(Yd)
```



Observation:

- ACF shows the signs and significances of θ_1 and θ_2 for these MA(2) models.
- If $\theta_1 > 0$, then there is a positive peak at lag 1 in ACF; If $\theta_1 < 0$, then there is a negative peak at lag 1 in ACF.
- If $\theta_2 > 0$, then there is a positive peak at lag 2 in ACF; If $\theta_2 < 0$, then there is a negative peak at lag 2 in ACF.
- In ACF, all lags larger than 2 are not significant for MA(2) model. Hence, we can use ACF to observe the significant peaks at lags smaller than p to get p for any MA(p) model.
- Comparing to the results of Problem 1, we find that all peaks at lags larger than 2 in MA(2) model are close to 0, which are consistent with the theoretical results.

Problem 4*

For an ARMA(1, 1) model

$$Y_t - \mu = \phi_1(Y_{t-1} - \mu) + \epsilon_t + \theta_1 \epsilon_{t-1}.$$

(a) Write it into a pure MA process. (Determine the coefficients of MA by using ϕ_1, θ_1)

Solution:

From the assumption, we have

$$\begin{aligned}
Y_t - \mu &= \phi_1(Y_{t-1} - \mu) + \epsilon_t + \theta_1 \epsilon_{t-1} \\
&= \phi_1\Big(\phi_1(Y_{t-2} - \mu) + \epsilon_{t-1} + \theta_1 \epsilon_{t-2}\Big) + \epsilon_t + \theta_1 \epsilon_{t-1} \\
&= \phi_1^2(Y_{t-2} - \mu) + \epsilon_t + (\phi_1 + \theta_1)\epsilon_{t-1} + \phi_1\theta_1\epsilon_{t-2} \\
&= \phi_1^2\Big(\phi_1(Y_{t-3} - \mu) + \epsilon_{t-2} + \theta_1 \epsilon_{t-3}\Big) + \epsilon_t + (\phi_1 + \theta_1)\epsilon_{t-1} + \phi_1\theta_1\epsilon_{t-2} \\
&= \phi_1^3(Y_{t-3} - \mu) + \epsilon_t + (\phi_1 + \theta_1)\epsilon_{t-1} + \phi_1(\phi_1 + \theta_1)\epsilon_{t-2} + \phi_1^2\theta_1\epsilon_{t-3} \\
&= \lim_{k \to +\infty} \phi_1^k(Y_{t-k} - \mu) + \epsilon_t + (\phi_1 + \theta_1)\epsilon_{t-1} + \phi_1(\phi_1 + \theta_1)\epsilon_{t-2} + \cdots + \phi_1^{k-1}(\phi_1 + \theta_1)\epsilon_{t-k} + \cdots \\
&= \epsilon_t + \sum_{i=1}^{+\infty} \phi_1^{i-1}(\phi_1 + \theta_1)\epsilon_{t-i},
\end{aligned}$$

and this implies that

$$Y_t = \mu + \epsilon_t + \sum_{i=1}^{+\infty} \phi_1^{i-1}\big(\phi_1 + \theta_1\big)\epsilon_{t-i} = \mu + \epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i},$$

where $\tilde{\theta}_i = \phi_1^{i-1}\big(\phi_1 + \theta_1\big)$, which means that $\{Y_t\}$ is an MA(∞) process.

(b) Find the ACF of it.

Solution:

First calculate the variance of Y_t

$$\begin{aligned}
\gamma(0) = \mathrm{Var}(Y_t) &= \mathrm{Var}\left(\mu + \epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}\right) \\
&= \mathrm{Var}(\epsilon_t) + \sum_{i=1}^{+\infty} \mathrm{Var}\big(\tilde{\theta}_i \epsilon_{t-i}\big) \\
&= \sigma^2 + \sum_{i=1}^{+\infty} \tilde{\theta}_i^2 \mathrm{Var}(\epsilon_{t-i}) = \sigma^2\left(1 + \sum_{i=1}^{+\infty} \tilde{\theta}_i^2\right) \\
&= \sigma^2\left(1 + \sum_{i=1}^{+\infty} \phi_1^{2i-2}\big(\phi_1 + \theta_1\big)^2\right) \\
&= \sigma^2\left(1 + \big(\phi_1 + \theta_1\big)^2\left(\sum_{i=0}^{+\infty} \big(\phi_1^2\big)^i\right)\right) \\
&= \sigma^2\left(1 + \frac{(\phi_1 + \theta_1)^2}{1 - \phi_1^2}\right) = \frac{1 + 2\theta_1\phi_1 + \theta_1^2}{1 - \phi_1^2}\sigma^2.
\end{aligned}$$

Since $\epsilon \sim \mathcal{N}(0, \sigma^2)$, when $h = 1$, we have

$$\begin{aligned}
\gamma(1) = \mathrm{Cov}(Y_t, Y_{t+1}) &= \mathrm{Cov}\left(\mu + \epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \mu + \epsilon_{t+1} + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i+1}\right) \\
&= \mathrm{Cov}\left(\epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \epsilon_{t+1} + \tilde{\theta}_1 \epsilon_t + \sum_{i=2}^{+\infty} \tilde{\theta}_i \epsilon_{t-i+1}\right) \\
&= \mathrm{Cov}\big(\epsilon_t, \tilde{\theta}_1 \epsilon_t\big) + \mathrm{Cov}\left(\sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \sum_{i=1}^{+\infty} \tilde{\theta}_{i+1} \epsilon_{t-i}\right) \\
&= \tilde{\theta}_1 \mathrm{Var}(\epsilon_t) + \sum_{i=1}^{+\infty} \tilde{\theta}_i \tilde{\theta}_{i+1} \mathrm{Var}(\epsilon_{t-i}) = \sigma^2\left(\tilde{\theta}_1 + \sum_{i=1}^{+\infty} \tilde{\theta}_i \tilde{\theta}_{i+1}\right) \\
&= \sigma^2\left(\phi_1 + \theta_1 + \sum_{i=1}^{+\infty} \phi_1^{i-1}\big(\phi_1 + \theta_1\big)\phi_1^i\big(\phi_1 + \theta_1\big)\right) \\
&= \sigma^2\big(\phi_1 + \theta_1\big)\left(1 + \phi_1\big(\phi_1 + \theta_1\big)\sum_{i=0}^{+\infty} \big(\phi_1^2\big)^i\right) \\
&= \sigma^2\big(\phi_1 + \theta_1\big)\left(1 + \frac{\phi_1(\phi_1 + \theta_1)}{1 - \phi_1^2}\right) \\
&= \frac{(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 - \phi_1^2}\sigma^2.
\end{aligned}$$

When $h > 1$, we can derive that

$$\begin{aligned}
\gamma(h) &= \mathrm{Cov}\big(Y_t, Y_{t+h}\big) \\
&= \mathrm{Cov}\left(\mu + \epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \mu + \epsilon_{t+h} + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t+h-i}\right) \\
&= \mathrm{Cov}\left(\epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \epsilon_{t+h} + \sum_{i=1}^{h-1} \tilde{\theta}_i \epsilon_{t+h-i} + \sum_{i=h}^{+\infty} \tilde{\theta}_i \epsilon_{t+h-i}\right) \\
&= \mathrm{Cov}\left(\epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \epsilon_{t+h} + \sum_{i=1}^{h-1} \tilde{\theta}_i \epsilon_{t+h-i}\right) + \mathrm{Cov}\left(\epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \sum_{i=h}^{+\infty} \tilde{\theta}_i \epsilon_{t+h-i}\right) \\
&= 0 + \mathrm{Cov}\left(\epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_i \epsilon_{t-i}, \tilde{\theta}_h \epsilon_t + \sum_{i=1}^{+\infty} \tilde{\theta}_{i+h} \epsilon_{t-i}\right) \\
&= \mathrm{Cov}\big(\epsilon_t, \tilde{\theta}_h \epsilon_t\big) + \sum_{i=1}^{+\infty} \mathrm{Cov}\big(\tilde{\theta}_i \epsilon_{t-i}, \tilde{\theta}_{i+h} \epsilon_{t-i}\big) \\
&= \tilde{\theta_h} \mathrm{Var}\big(\epsilon_t\big) + \sum_{i=1}^{+\infty} \tilde{\theta}_i \tilde{\theta}_{i+h} \mathrm{Var}\big(\epsilon_{t-i}\big) \\
&= \sigma^2 \left(\tilde{\theta_h} + \sum_{i=1}^{+\infty} \tilde{\theta}_i \tilde{\theta}_{i+h}\right) \\
&= \sigma^2 \left(\phi_1^{h-1}\big(\phi_1 + \theta_1\big) + \sum_{i=1}^{+\infty} \phi_1^{i-1}\big(\phi_1 + \theta_1\big)\phi_1^{i+h-1}\big(\phi_1 + \theta_1\big)\right) \\
&= \sigma^2 \phi_1^{h-1}\big(\phi_1 + \theta_1\big)\left(1 + \phi_1\big(\phi_1 + \theta_1\big)\sum_{i=0}^{+\infty} \big(\phi_1^2\big)^i\right) \\
&= \sigma^2 \phi_1^{h-1}\big(\phi_1 + \theta_1\big)\left(1 + \frac{\phi_1(\phi_1 + \theta_1)}{1 - \phi_1^2}\right) = \sigma^2 \phi_1^{h-1}\big(\phi_1 + \theta_1\big)\frac{1 + \phi_1\theta_1}{1 - \phi_1^2} \\
&= \frac{\phi_1^{h-1}(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 - \phi_1^2}\sigma^2.
\end{aligned}$$

Hence, we get the autocovariance function as

$$\gamma(h) = \begin{cases} \dfrac{1 + 2\theta_1\phi_1 + \theta_1^2}{1 - \phi_1^2}\sigma^2, & h = 0 \\[2ex] \dfrac{\phi_1^{h-1}(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 - \phi_1^2}\sigma^2, & h \geq 1 \end{cases}$$

and the ACF of ARMA(1, 1) as

$$\rho(h) = \begin{cases} 1, & h = 0 \\[2ex] \dfrac{\phi_1^{h-1}(\phi_1 + \theta_1)(1 + \phi_1\theta_1)}{1 + 2\theta_1\phi_1 + \theta_1^2}, & h \geq 1 \end{cases}$$