Homework 5

DATA130021 Financial Econometrics

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Problem 1

A linear regression model with three predictor variables was fit to a data set with 40 observations. The correlation between Y and $\hat{Y}$ was 0.65. The total sum of squares was 100.

Solution:

From the assumption, we know that $p = 3$, $n = 40$, $\rho\big(Y, \hat{Y}\big) = 0.65$ and $TSS = 100$.

(a) What is the value pf R^2?

Solution:

Recall the definition of correlation and coefficient of determination

$$r_{XY} = \rho\big(X, Y\big) = \frac{\sum_{i=1}^{n}\big(x_i - \bar{x}\big)\big(y_i - \bar{y}\big)}{\sqrt{\sum_{i=1}^{n}\big(x_i - \bar{x}\big)^2}\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}},$$

$$R^2 = \frac{SS}{TSS} = \frac{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2},$$

then we have

$$\begin{aligned}
r_{Y\hat{Y}} = \rho\big(Y, \hat{Y}\big) &= \frac{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)\big(\hat{y}_i - \bar{y}\big)}{\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}\sqrt{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}} \\
&= \frac{\sum_{i=1}^{n}\big(y_i - \hat{y}_i + \hat{y}_i - \bar{y}\big)\big(\hat{y}_i - \bar{y}\big)}{\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}\sqrt{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}} \\
&= \frac{\sum_{i=1}^{n}\big(y_i - \hat{y}_i\big)\big(\hat{y}_i - \bar{y}\big) + \sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}{\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}\sqrt{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}} \\
&= \frac{\sum_{i=1}^{n}\big(y_i - \hat{y}_i\big)\hat{y}_i - \bar{y}\sum_{i=1}^{n}\big(y_i - \hat{y}_i\big) + \sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}{\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}\sqrt{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}} \\
&= \frac{\sum_{i=1}^{n}\big(1 - h_{ii}\big)h_{ii}y_i^2 - 0 + \sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}{\sqrt{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}\sqrt{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}} \\
&= \sqrt{\frac{\sum_{i=1}^{n}\big(\hat{y}_i - \bar{y}\big)^2}{\sum_{i=1}^{n}\big(y_i - \bar{y}\big)^2}} = \sqrt{R^2},
\end{aligned}$$

Hence, the value of R^2 is

$$R^2 = \rho^2\big(Y, \hat{Y}\big) = 0.65^2 = 0.4225.$$

(b) What is the value of the residual sum of square RSS?

Solution:

The value of the residual sum of sqaure is

$$RSS = TSS - SS = TSS - R^2 \times TSS = \big(1 - R^2\big) \times TSS = \big(1 - 0.65^2\big) \times 100 = 57.75.$$

(c) What is the value of the regression sum of square SS?

Solution:

The value of the regression sum of sqaure is

$$SS = R^2 \times TSS = 0.65^2 \times 100 = 42.25.$$

Problem 2

Complete the following ANOVA table for the model $Y_i = \beta_0 + \beta_1 X_{i,1} + \beta_2 X_{i,2} + \epsilon_i$:

Source	df	SS	MS	F	P
Regression	?	?	?	?	0.04
Error	?	5.66	?		
Total	15	?			
R-sq=?					

Solution:

From the given model, we know that $p = 2$. Then

$$\mathrm{df}_R = p = 2, \quad \mathrm{df}_E = \mathrm{df}_T - \mathrm{df}_R = 13.$$

Next, we calculate F value from p-value

$$p = P\big(X > F | X \sim F_{2,13}\big) \Rightarrow F = F_{2,13}^{-1}\big(1 - p\big) = F_{2,13}^{-1}\big(1 - 0.04\big) \approx 4.1655.$$

Then we can get MSE_E, MSE_R, SS, TSS and R^2, and we complete the table as below,

$$MSE_E = \frac{RSS}{\mathrm{df}_E} = \frac{5.66}{13} \approx 0.4354,$$

$$MSE_R = F \times MSE_E = 4.1655 \times 0.4354 \approx 1.8137,$$

$$SS = MSE_R \times \mathrm{df}_R = 1.8137 \times 2 = 3.6274,$$

$$TSS = SS + RSS = 3.6274 + 5.66 = 9.2874,$$

$$R^2 = \frac{SS}{TSS} = \frac{3.6274}{9.2874} \approx 0.3906.$$

Source	df	SS	MS	F	P
Regression	2	3.63	1.81	4.17	0.04
Error	13	5.66	0.44		
Total	15	9.29			
R-sq=0.39					

Problem 3

When we were finding the best linear predictor of Y given X, we derived the equations

$$0 = -\mathbb{E}(Y) + \beta_0 + \beta_1 \mathbb{E}(X)$$

$$0 = -\mathbb{E}(XY) + \beta_0 \mathbb{E}(X) + \beta_1 \mathbb{E}(X^2)$$

Show that their solution is

$$\beta_1 = \frac{\sigma_{XY}}{\sigma_X^2}$$

and

$$\beta_0 = \mathbb{E}(Y) - \beta_1 \mathbb{E}(X) = \mathbb{E}(Y) - \frac{\sigma_{XY}}{\sigma_X^2} \mathbb{E}(X).$$

Proof:

From the first equation, we have

$$0 = -\mathbb{E}(X)\mathbb{E}(Y) + \beta_0 \mathbb{E}(X) + \beta_1 \mathbb{E}^2(X).$$

Subtract this eqaution from the second equation

$$\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = \beta_1 \Big(\mathbb{E}(X^2) - \mathbb{E}^2(X)\Big).$$

Then we get

$$\beta_1 = \frac{\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)}{\mathbb{E}(X^2) - \mathbb{E}^2(X)} = \frac{\sigma_{XY}}{\sigma_X^2},$$

and

$$\beta_0 = \mathbb{E}(Y) - \beta_1 \mathbb{E}(X) = \mathbb{E}(Y) - \frac{\sigma_{XY}}{\sigma_X^2} \mathbb{E}(X).$$

$\blacksquare$

Problem 4*

Recall the general setting of linear regression model $\boldsymbol{Y} = \boldsymbol{X}\beta + \epsilon$ where n is the number of observations (X_i, Y_i), p is the number of parameters in β (including the intercept), $\mathbb{E}(\epsilon) = 0$ and $\mathrm{Var}(\epsilon) = \sigma_\epsilon^2 I_n$. Let $\hat\beta$ be the LSE of β and $RSS = (\boldsymbol{Y} - \boldsymbol{X}\hat\beta)^\top (\boldsymbol{Y} - \boldsymbol{X}\hat\beta)$ be the residual sum of square.

(a) Show that $(\boldsymbol{Y} - \boldsymbol{X}\beta)^\top (\boldsymbol{Y} - \boldsymbol{X}\beta) = RSS + (\hat\beta - \beta)^\top \boldsymbol{X}^\top \boldsymbol{X} (\hat\beta - \beta)$.

Proof:

From the definition, we have

$$\begin{aligned}
(\boldsymbol{Y} - \boldsymbol{X}\beta)^\top (\boldsymbol{Y} - \boldsymbol{X}\beta) &= \|\boldsymbol{Y} - \boldsymbol{X}\beta\|^2 = \|\boldsymbol{Y} - \boldsymbol{X}\hat\beta + \boldsymbol{X}\hat\beta - \boldsymbol{X}\beta\|^2 \\
&= \|\boldsymbol{Y} - \boldsymbol{X}\hat\beta\|^2 + \|\boldsymbol{X}\hat\beta - \boldsymbol{X}\beta\|^2 + 2(\boldsymbol{Y} - \boldsymbol{X}\hat\beta)^\top (\boldsymbol{X}\hat\beta - \boldsymbol{X}\beta) \\
&= RSS + (\hat\beta - \beta)^\top \boldsymbol{X}^\top \boldsymbol{X} (\hat\beta - \beta) + 2(\boldsymbol{Y}^\top \boldsymbol{X}\hat\beta - \hat\beta^\top \boldsymbol{X}^\top \boldsymbol{X}\hat\beta - \boldsymbol{Y}^\top \boldsymbol{X}\beta + \hat\beta^\top \boldsymbol{X}^\top \boldsymbol{X}\beta).
\end{aligned}$$

Since $\hat\beta$ is the LSE of β, then $\hat\beta = (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y}$. Plug in and we have

$$\boldsymbol{Y}^\top \boldsymbol{X}\hat\beta = \boldsymbol{Y}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y},$$

$$\hat\beta^\top \boldsymbol{X}^\top \boldsymbol{X}\hat\beta = \boldsymbol{Y}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y} = \boldsymbol{Y}^\top \boldsymbol{X} (\boldsymbol{X}^\top \boldsymbol{X})^{-1} \boldsymbol{X}^\top \boldsymbol{Y} = \boldsymbol{Y}^\top \boldsymbol{X}\hat\beta,$$

$$\hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\beta = \boldsymbol{Y}^\top \boldsymbol{X}\left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top \boldsymbol{X}\beta = \boldsymbol{Y}^\top \boldsymbol{X}\beta.$$

Hence

$$\boldsymbol{Y}^\top \boldsymbol{X}\hat{\beta} - \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\hat{\beta} - \boldsymbol{Y}^\top \boldsymbol{X}\beta + \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\beta = 0,$$

and

$$\left(\boldsymbol{Y} - \boldsymbol{X}\beta\right)^\top \left(\boldsymbol{Y} - \boldsymbol{X}\beta\right) = RSS + \left(\hat{\beta} - \beta\right)^\top \boldsymbol{X}^\top \boldsymbol{X}\left(\hat{\beta} - \beta\right)$$

■

(b) Show that $RSS = \boldsymbol{Y}^\top \boldsymbol{Y} - \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\hat{\beta}$.

Proof:

From the definition, we have

$$\begin{aligned}
RSS &= \left(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta}\right)^\top \left(\boldsymbol{Y} - \boldsymbol{X}\hat{\beta}\right) \\
&= \boldsymbol{Y}^\top \boldsymbol{Y} - \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \boldsymbol{X}\hat{\beta} + \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\hat{\beta} \\
&= \boldsymbol{Y}^\top \boldsymbol{Y} - 2\boldsymbol{Y}^\top \boldsymbol{H}\boldsymbol{Y} + \boldsymbol{Y}^\top \boldsymbol{H}^\top \boldsymbol{H}\boldsymbol{Y} \\
&= \boldsymbol{Y}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \boldsymbol{H}\boldsymbol{Y} \\
&= \boldsymbol{Y}^\top \boldsymbol{Y} - \hat{\beta}^\top \boldsymbol{X}^\top \boldsymbol{X}\hat{\beta},
\end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{X}\left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top$ and $\boldsymbol{H}$ is a symmetric idempotent matrix, that is, $\boldsymbol{H}^\top = \boldsymbol{H}$ and $\boldsymbol{H}^2 = \boldsymbol{H}$. ■

(c) Show that $RSS/(n-p)$ is an unbiased estimator of σ_ϵ^2.

Proof:

Consider the expectation of $RSS/(n-p)$

$$\begin{aligned}
\mathbb{E}\left[\frac{RSS}{n-p}\right] &= \frac{1}{n-p}\mathbb{E}\left[RSS\right] \\
&= \frac{1}{n-p}\mathbb{E}\left[\boldsymbol{Y}^\top \boldsymbol{Y} - \boldsymbol{Y}^\top \boldsymbol{H}\boldsymbol{Y}\right] = \frac{1}{n-p}\mathbb{E}\left[\boldsymbol{Y}^\top \left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{Y}\right] \\
&= \frac{1}{n-p}\mathbb{E}\left[\mathrm{tr}\left(\boldsymbol{Y}^\top \left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{Y}\right)\right] = \frac{1}{n-p}\mathbb{E}\left[\mathrm{tr}\left(\left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{Y}\boldsymbol{Y}^\top\right)\right] \\
&= \frac{1}{n-p}\mathrm{tr}\left(\mathbb{E}\left[\left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{Y}\boldsymbol{Y}^\top\right]\right) = \frac{1}{n-p}\mathrm{tr}\left(\left(\boldsymbol{I} - \boldsymbol{H}\right)\mathbb{E}\left[\boldsymbol{Y}\boldsymbol{Y}^\top\right]\right) \\
&= \frac{1}{n-p}\mathrm{tr}\left(\left(\boldsymbol{I} - \boldsymbol{H}\right)\left(\mathrm{Cov}\left[\boldsymbol{Y}\right] + \mathbb{E}\left[\boldsymbol{Y}\right]\mathbb{E}\left[\boldsymbol{Y}\right]^\top\right)\right) \\
&= \frac{1}{n-p}\mathrm{tr}\left(\left(\boldsymbol{I} - \boldsymbol{H}\right)\left(\sigma_\epsilon^2 \boldsymbol{I}_n + \boldsymbol{X}\beta\beta^\top \boldsymbol{X}^\top\right)\right) \\
&= \frac{1}{n-p}\left(\sigma_\epsilon^2 \mathrm{tr}\left(\boldsymbol{I} - \boldsymbol{H}\right) + \mathrm{tr}\left(\left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{X}\beta\beta^\top \boldsymbol{X}^\top\right)\right) \\
&= \frac{1}{n-p}\left(\sigma_\epsilon^2\left(n-p\right) + 0\right) \\
&= \sigma_\epsilon^2,
\end{aligned}$$

where $\boldsymbol{H} = \boldsymbol{X}\left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1}\boldsymbol{X}^\top$ and $\boldsymbol{H}$ satisfies $\left(\boldsymbol{I} - \boldsymbol{H}\right)\boldsymbol{X} = \boldsymbol{0}$.

Hence, $RSS/(n-p)$ is an unbiased estimator of σ_ϵ^2. ■