Homework I

Name: Deng Qisheng Student ID: 16307110232

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1. **Solutions:**

First, apply Newton's method to the given function $f(x) = x^5 - 3x^4 + 25$ starting from $x_0 = -2$:

```
% Newton's method starting from -2
k = 1;
m = 100;
x = -2;
xs = -1.5325;
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = x - (x ^ 5 - 3 * x ^ 4 + 25) / (5 * x ^ 4 - 12 * x ^ 3);
    x = xk(k, 1);
    k = k + 1;
end
xk(1 : k - 1, 1)     % iteration results
k - 1     % No. of iterations starting from x0 = -2
```

Since $x_0 = -2$ is sufficiently close to the root $x_* \approx -1.532501$, these iterations converge only in 4 steps. The results are shown below:

```
>> Problem_1_1

xk(1 : k - 1, 1) =

    -1.687500
    -1.555013
    -1.533047
    -1.532501


k - 1 =

    4
```

Then, apply Newton's method to the given function $f(x) = x^5 - 3x^4 + 25$ starting from $x_0 = 0.25$:

```
% Newton's method starting from 0.25
k = 1;
m = 100;
x = 0.25;
xs = -1.5325;
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = x - (x ^ 5 - 3 * x ^ 4 + 25) / (5 * x ^ 4 - 12 * x ^ 3);
    x = xk(k, 1);
    k = k + 1;
end
xk(1 : k - 1, 1)     % iteration results
k - 1     % No. of iterations starting from x0 = 0.25
```

Since $x_0 = 0.25$ is far away from the root $x_* \approx -1.532501$ relatively, these iterations seem to diverge in first few steps. However, it still converges to the root x_* after 58 iterations. The results are shown below:

```
>> Problem_1_2
xk(1 : k - 1, 1) =

    149.023256
    119.340569
    95.594918
    76.599025
    61.403101
    49.247361
    ...
    ...
    2.468099
    1.439995
    2.716267
    2.414779
    -2.487613
    -1.995572
    -1.685115
    -1.554378
    -1.533017
    -1.532501


k - 1 =

    58
```

Hence, the claim that "starting from $x_0 = 0.25$ gives a divergent set of iterations" is **WRONG**. The numerical results in problem_1_2 shows that the iterations will converge in 58 steps, but it seems to diverge in first few steps.

With the help of Sketchpad, it can find that the figure of the given function is rather steep and "sharp". If the iterations start from $x_0 = 0.25$, the derivative at x_0 is nearly zero. Hence, the first step of iterations keeps x_k away from the root x_*, and it seems to diverge at first.

Moreover, the iterations are not monotonically decreasing. The iterations take values near the minimum at $x_1 = 2.4$ oscillately, which can be proved by the results above.

2. **Solution:**

a) *Proof.* Let $x_1 = 1$ and $x_2 = 2$, and it is to prove that $x_* \in (x_1, x_2)$. Given that the function:

$$f(x) = e^x - x - 2$$

Calculate the boundary value $f(x_1)$ and $f(x_2)$, which satisfies that:

$$\begin{aligned} f(x_1) &= e - 3 < 0 \\ f(x_2) &= e^2 - 4 > 0 \\ f(x_1) \cdot f(x_2) &< 0 \end{aligned}$$

From the existence theorem of zeros of continuous function, which is the special case of intermediate value theorem, it claims that there exists at least one root of the given continuous function if it satisfies the condition $f(x_1) \cdot f(x_2) < 0$. That is:

$$f(x) \text{ continuous on } [x_1, x_2] \text{ and } f(x_1) \cdot f(x_2) < 0 \Rightarrow \exists\, \xi \in [x_1, x_2] \text{ s.t. } f(\xi) = 0$$

Hence, there exists $x_* \in (1, 2)$ such that $f(x_*) = 0$. □

b) *Proof.* Consider the first given function $g(x) = e^x - 2$, there exists:

$$x = g(x) = e^x - 2 \Rightarrow e^x - x - 2 = 0$$

which implies $f(x) = e^x - x - 2 = 0$.

Then consider the second given function $g(x) = \ln(x + 2)$, there exists:

$$x = g(x) = \ln(x + 2) \Rightarrow e^x = x + 2 \Rightarrow e^x - x - 2 = 0$$

which also implies $f(x) = e^x - x - 2 = 0$.

Hence, for each $g(x)$ above, $x = g(x)$ implies $f(x) = 0$. □

c) Apply fixed point iteration to the function $g(x) = e^x - 2$ and let the initial value x_0 be 2, which satisfies $x_0 > x_* \approx 1.146193$.

```matlab
% g(x) = e^x - 2
k = 1;
m = 100;
x = 2;      % x_0 > x_*
xs = 1.14619322;     % x_*
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = exp(x) - 2;
    x = xk(k, 1);
    k = k + 1;
    if x == inf
        k = inf;
        break;
    end
end
xk(1 : k - 1, 1)     % iteration results
k - 1    % No. of iterations using g(x) = e^x - 2
```

The results show that the iteration diverges, if choosing $g(x) = e^x - 2$ and $x_0 = 2 > x_*$.

```
>> Problem_2_3_1

xk(1 : k - 1, 1) =

    5.3891e+00
    2.1700e+02
    1.7395e+94
    Inf


k - 1 =

    Inf
```

Then apply fixed point iteration to function $g(x) = \ln(x+2)$ and let the initial value x_0 be 2.

```
% g(x) = ln(x + 2)
k = 1;
m = 100;
x = 2;     % x_0 > x_*
xs = 1.14619322;     % x_*
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = log(x + 2);
    x = xk(k, 1);
    k = k + 1;
    if x == inf
        k = inf;
        break;
    end
end
xk(1 : k - 1, 1)     % iteration results
k - 1     % No. of iterations using g(x) = ln(x + 2)
```

The results show that the iteration has converges, if choosing $g(x) = \ln(x+2)$ and $x_0 = 2 > x_*$.

```
>> Problem_2_3_2

xk(1 : k - 1, 1) =

    1.386294
    1.219736
    1.169299
    1.153511
    1.148516
    1.146931
    1.146428
    1.146268
    1.146217
    1.146201
    1.146196
    1.146194


k - 1 =

    12
```

Above all, considering these two given functions above, after applying fixed point iteration to $x = g(x)$ with $x_0 > x_*$, the result **diverges** if choosing $g(x) = e^x - 2$, while it **converges** to the root $x_* \approx 1.146193$ if choosing $g(x) = \ln(x+2)$.

As a matter of fact, if choosing $g(x) = e^x - 2$ and $x_0 = 1 < x_*$, the result will converge to another root $x_{**} \approx -1.841406$. If choosing $g(x) = \ln(x+2)$ and $x_0 = 1 < x_*$, the result will still converge to the root $x_* \approx 1.146193$, which is in $[1,2]$.

3. **Solutions:**

Construct a function $f(x) = 1/x - a$ $(a \neq 0)$ for computing $1/a$ to avoid calculating division in each iteration. Apply Newton's method to the function $f(x)$, there exists the Newton's formula:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \Rightarrow x_{k+1} = x_k - \frac{\frac{1}{x_k} - a}{-\frac{1}{x_k^2}} = 2x_k - ax_k^2$$

First consider positive a. With the help of Sketchpad, it can find that the value of x can **NOT** be negative for $a > 0$, that is, $x_0 > 0$. Considering the figure of $f(x) = 1/x - a$, it means that the intercept of the tangent lines of $f(x)$ at $(x_0, \frac{1}{x_0} - a)$ must satisfies:

$$y = f'(x_0)x + b \Rightarrow \frac{1}{x_0} - a = -\frac{1}{x_0^2}x_0 + b \Rightarrow x_0 = \frac{2}{a+b},\ b = \frac{2}{x_0} - a > 0$$

Hence, the range of x_0 is $(0, \frac{2}{a})$, which means that the result will converge if and only if $x_0 \in (0, \frac{2}{a})$ $(a > 0)$. For the same reason, for negative a, it converge if and only if $x_0 \in (\frac{2}{a}, 0)$ $(a < 0)$.

Following is the algorithm for computing $1/a$ to test different inputs of a and initial value x_0:

```
% the algorithm of Newton's method for computing 1/a
function reciprocal(a, x)

sgn = a / abs(a);
k = 1;
m = 100;
x0 = x;
xs = 1 / a;
xk = zeros(m, 1);

while abs(x - xs) > 1e-6
    xk(k, 1) = 2 * x - a * x ^ 2;     % Newton's formula
    x = xk(k, 1);
    k = k + 1;
    if x == inf | k > 1000
        fprintf('NUMBER: a = %f\n', a);
        if sgn == 1
            fprintf('CONVERGENT INTERVAL: (0, %f)\n', 2 / a);
        else
            fprintf('CONVERGENT INTERVAL: (%f, 0)\n', 2 / a);
        end
        fprintf('PRECISE VALUE: x_* = %f\nINITIAL VALUE: x_0 = ...
            %f\nRESULT: DIVERGE\nNO. OF ITERATIONS: Inf\n', xs, x0)
        return;
    end
end

```

```
fprintf('NUMBER: a = %f\n', a);
if sgn == 1
    fprintf('CONVERGENT INTERVAL: (0, %f)\n', 2 / a);
else
    fprintf('CONVERGENT INTERVAL: (%f, 0)\n', 2 / a);
end
fprintf('PRECISE VALUE: x_* = %f\nINITIAL VALUE: x_0 = %f\nRESULT: ...
    CONVERGE\nNO. OF ITERATIONS: %d\n', xs, x0, k - 1);
```

The results are shown below, which implies the convergent interval is:

$$\begin{cases} (0, \dfrac{2}{a}) \ , \ a > 0 \\ (\dfrac{2}{a}, 0) \ , \ a < 0 \end{cases}$$

```
>> reciprocal(3, -1)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = -1.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(3, 0)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = 0.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(3, 0.1)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = 0.100000
    RESULT: CONVERGE
    NO. OF ITERATIONS: 6


>> reciprocal(3, 0.5)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = 0.500000
    RESULT: CONVERGE
    NO. OF ITERATIONS: 5
```

```
>> reciprocal(3, 0.7)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = 0.700000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(3, 1)

    NUMBER: a = 3.000000
    CONVERGENT INTERVAL: (0, 0.666667)
    PRECISE VALUE: x_* = 0.333333
    INITIAL VALUE: x_0 = 1.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(-4, 1)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = 1.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(-4, 0)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = 0.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(-4, -0.1)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = -0.100000
    RESULT: CONVERGE
    NO. OF ITERATIONS: 5


>> reciprocal(-4, -0.3)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = -0.300000
    RESULT: CONVERGE
    NO. OF ITERATIONS: 3
```

```
>> reciprocal(-4, -0.5)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = -0.500000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf


>> reciprocal(-4, -1)

    NUMBER: a = -4.000000
    CONVERGENT INTERVAL: (-0.500000, 0)
    PRECISE VALUE: x_* = -0.250000
    INITIAL VALUE: x_0 = -1.000000
    RESULT: DIVERGE
    NO. OF ITERATIONS: Inf
```

Hence, the Newton's iteration will **NOT** converge starting from **ANY** initial value x_0. According to the analysis above, the convergent interval is:

$$\begin{cases} (0, \dfrac{2}{a}) \ , \ a > 0 \\ (\dfrac{2}{a}, 0) \ , \ a < 0 \end{cases}$$

It shows that the **convergent range** of initial value x_0 is rather **small**, which guarantees x_0 is **sufficiently** close to the precise root x_*.

According to the local convergence theorem for Newton's method, the derivative of $f(x)$ satisfies $f'(x_*) = -a^2 \neq 0$, which means that the Newton's method exhibits quadratic convergence locally depending on a sufficiently closed initial value. The value out of $(0, \frac{2}{a})$ or $(\frac{2}{a}, 0)$ is too far away from $x_* = \frac{1}{a}$, which causes divergence. Therefore, the Newton's iteration will not converge starting from any initial value.

4. **Solutions:**

Apply secant method to the given function $f(x) = x^4 - 2x^2 - 4$ starting from $x_{-1} = 2$ and $x_0 = 3$:

```
% secant method
k = 1;
m = 20;
x0 = 2;     % represent x_{-1}
x = 3;     % represent x_{0}
xs = 1.79890744;
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = x - (x ^ 4 - 2 * x ^ 2 - 4) * (x - x0) / ...
    ((x ^ 4 - 2 * x ^ 2 - 4) - (x0 ^ 4 - 2 * x0 ^ 2 - 4));
    x0 = x;
    x = xk(k, 1);
    k = k + 1;
end
xk(1 : k - 1, 1)     % iteration results
k - 1     % No. of iterations using secant method
```

The iterations converge in 6 steps and the results are shown below:

```
>> Problem_4_1

xk(1 : k - 1, 1) =

    1.927273
    1.882421
    1.809063
    1.799771
    1.798917
    1.798907


k - 1 =

    6
```

The secant method shows the super-linear local convergence of order $q \in (1, 2)$.

Apply Newton's method to the given function $f(x) = x^4 - 2x^2 - 4$ starting from $x_0 = 3$:

```
% Newton's method
k = 1;
m = 20;
x = 3;
xs = 1.79890744;
xk = zeros(m, 1);
while abs(x - xs) > 1e-6
    xk(k, 1) = x - (x ^ 4 - 2 * x ^ 2 - 4) / (4 * x ^ 3 - 4 * x);
    x = xk(k, 1);
    k = k + 1;
end
xk(1 : k - 1, 1)    % iteration results
k - 1    % No. of iterations using Newton's method
```

The iterations also converge in 6 steps and the results are shown below:

```
>> Problem_4_2

xk(1 : k - 1, 1) =

    2.385417
    2.005592
    1.835058
    1.800257
    1.798909
    1.798907


k - 1 =

    6
```

The Newton's method shows the quadratic local convergence.

Sum up the data above, the numerical results can be written in the following table:

k	0	1	2	3	4	5	6
x_k (Secant)	3	1.927273	1.882421	1.809063	1.799771	1.798917	1.798907
x_k (Newton)	3	2.385417	2.005592	1.835058	1.800257	1.798909	1.798907

Table 1: Root finding applying secant method and Newton's method