FUDAN UNIVERSITY, SCHOOL OF DATA SCIENCE

Homework II

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Lecture Date: March 20, 2019

1. **Solutions.**

   (a) It is required to do **Lagrange interpolation** for $f(x) = \frac{1}{1+25x^2}$ over $[-1, 1]$ with $n = 10$ using equally spaced points. Following is the codes for Lagrange interpolation.

```
n = 10;    % degree of polynomial
f = @(x) 1 ./ (1 + 25 * x .^ 2);    % generating function
xScatter = -1 : 2 / n : 1;    % scatter of x0, x1, x2, ..., xn
xCurve = -1 : 0.01 : 1;    % continuous interval of x in [-1, 1]

row = length(xScatter);    % number of interpolation points
col = length(xCurve);    % number of points for generating the curve
L = ones(row, col);    % Lagrange Polynomial of generating points
for k = 1 : row
    for j = 1 : col
        for i = 1 : row
            if i ≠ k
                L(k, j) = L(k, j) * (xCurve(j) - xScatter(i)) / ...
                    (xScatter(k) - xScatter(i));
            end
        end
    end
end

P = zeros(col, 1);    % calculate the value of the interpolation ...
    polynomial for generating points
for j = 1 : col
    for k = 1 : row
        P(j) = P(j) + f(xScatter(k)) * L(k, j);
    end
end

```

```
plot(xScatter, f(xScatter), 'bo', 'MarkerSize', 10)     % scatter of ...
    (x1, y1), ..., (xn, yn)
hold on
plot(xCurve, f(xCurve), 'b-', xCurve, P, 'r-', 'LineWidth', 2)    % ...
    figure of function and interpolation
hold off
xlabel('\bf\fontsize{14}x')
ylabel('\bf\fontsize{14}y')
leg = legend('\bf(x,y)', '\bf f(x)', '\bf p(x)', 'Location', 'Best');
set(leg, 'FontName', 'Consolas', 'FontSize', 10, 'FontWeight', ...
    'normal');
```

Then plot $f(x)$ and interpolation polynomials in the same figure, we have:

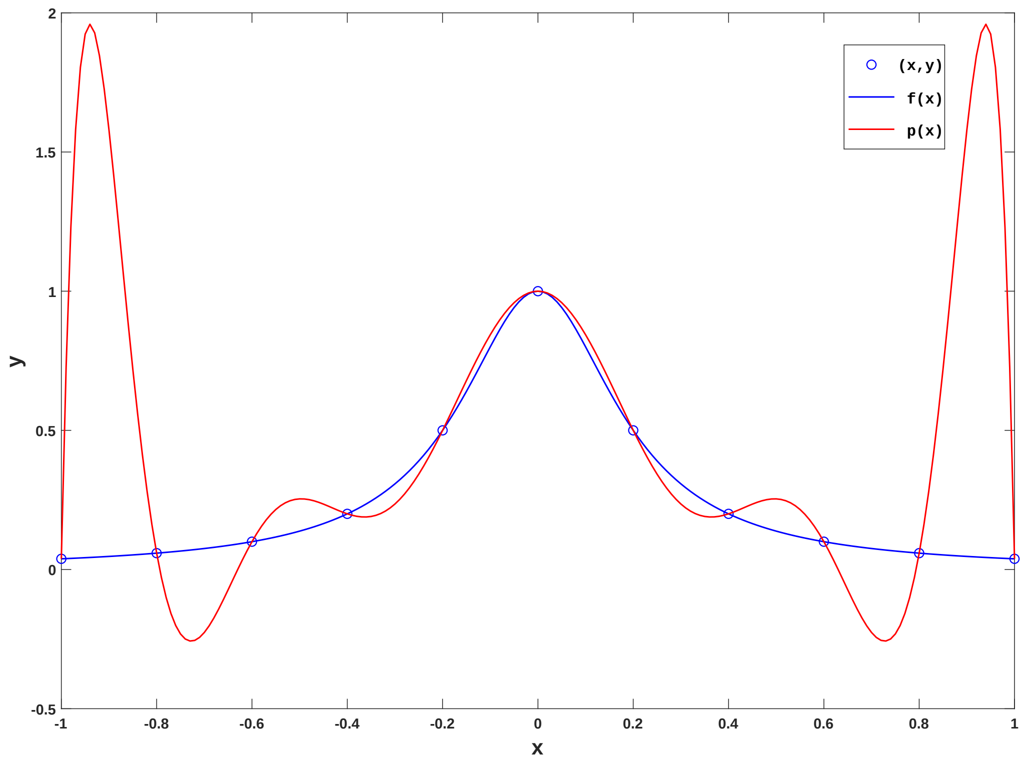

Figure 1: Interpolation at equally spaced points

(b) Let $T_n(x) = \cos(n \arccos(x)) = 0$ over $[-1, 1]$, we have:

$$\begin{aligned}
\cos(n \arccos(x)) = 0 \Rightarrow n \arccos(x) &= \frac{2k+1}{2}\pi \\
\Rightarrow \arccos(x) &= \frac{2k+1}{2n}\pi \\
\Rightarrow x = \cos\frac{(2k+1)\pi}{2n} &\in [-1, 1]
\end{aligned}$$

where $k = 0, 1, \ldots n-1$. Hence, the n **distinct roots** of $T_n(x)$ is $x_k = \cos\frac{2k+1}{2n}\pi$.

(c) Using the same method in (a), we have the following codes for Lagrange interpolation for $f(x)$ with $n = 10$ using the roots of the Chebshev polynomial $T_{n+1}(x)$.

```
n = 10;    % degree of polynomial
f = @(x) 1 ./ (1 + 25 * x .^ 2);    % generating function
x = @(k) cos((2 * k + 1) * pi / (2 * (n + 1)));    % roots of the ...
    Chebshev polynomial T_(n + 1)(x)
xRoot = x(0 : n);    % interpolation points x0, x1, ... , xn
xCurve = -1 : 0.01 : 1;    % continuous interval of x in [-1, 1]

row = length(xRoot);    % number of interpolation points
col = length(xCurve);    % number of points for generating the curve
L = ones(row, col);    % Lagrange Polynomial of generating points
for k = 1 : row
    for j = 1 : col
        for i = 1 : row
            if i ≠ k
                L(k, j) = L(k, j) * (xCurve(j) - xRoot(i)) / ...
                    (xRoot(k) - xRoot(i));
            end
        end
    end
end

P = zeros(col, 1);    % calculate the value of the interpolation ...
    polynomial for generating points
for j = 1 : col
    for k = 1 : row
        P(j) = P(j) + f(xRoot(k)) * L(k, j);
    end
end

plot(xRoot, f(xRoot), 'bo', 'MarkerSize', 10)    % interpolation ...
    points of (x1, y1), ..., (xn, yn)
hold on
plot(xCurve, f(xCurve), 'b-', xCurve, P, 'r-', 'LineWidth', 2)    % ...
    figure of function and interpolation
hold off
axis([-1 1 -0.5 2]);
xlabel('\bf\fontsize{14}x')
ylabel('\bf\fontsize{14}y')
leg = legend('\bf(x,y)', '\bf f(x)', '\bf p(x)', 'Location', 'Best');
set(leg, 'FontName', 'Consolas', 'FontSize', 10, 'FontWeight', ...
    'normal');
```

Then plot $f(x)$ and interpolation polynomials in the same figure, we have the following figure on the next page. Comparing the figures in (a) and (c), it can find that the Lagrange interpolation at **equally spaced points DIVERGES** in $[-1, 1]$ while the Lagrange interpolation at **Chebshev points CONVERGES** in $[-1, 1]$. This is because there exists

$$M_{n+1} = \max_{x \in [-1,1]} \left| f^{(n+1)}(x) \right| \sim 10^{16},$$

but we also have

$$\left| \pi_{n+1}(x) \right| \leq \frac{1}{2^n}$$

in **Chebshev polynomial**. Hence, the interpolation diverges in (a) but converges in (c).

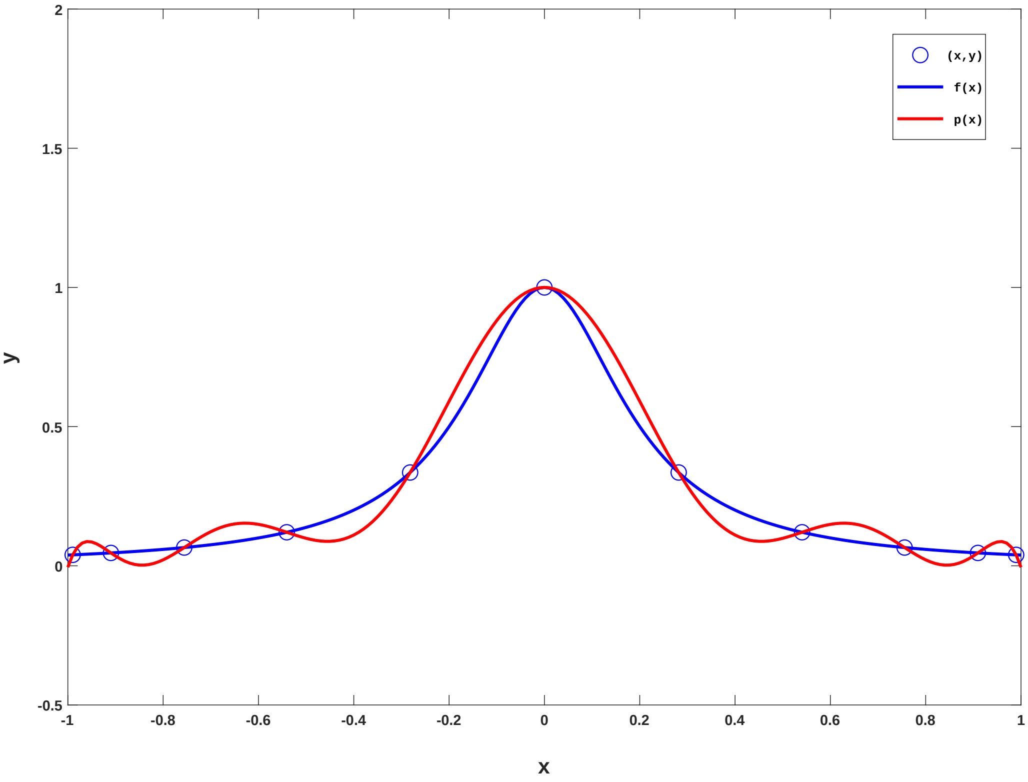

Figure 2: Interpolation at Chebshev points

2. **Solution.**

Proof. In the Hermite interpolation, we have the form of Hermite polynomials $H_k(x)$ and $K_k(x)$ as follow:

$$H_k(x) = [L_k(x)]^2(1 - 2L'_k(x_k)(x - x_k))$$
$$K_k(x) = [L_k(x)]^2(x - x_k),$$

where

$$L_k(x) = \prod_{i=0;i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

is the Lagrange polynomial. From the properties of the Lagrange polynomial, we have:

$$L_k(x_k) = \prod_{i=0;i\neq k}^{n} \frac{x_k - x_i}{x_k - x_i} = 1, \; L_k(x_i) = \prod_{i=0;i\neq k}^{n} \frac{x_i - x_i}{x_k - x_i} = 0,$$

that is:

$$L_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

Consider $H_k(x)$ first, when $i = k$, we have:

$$H_k(x_i) = [L_k(x_k)]^2(1 - 2L'_k(x_k)(x_k - x_k)) = [L_k(x_k)]^2 = 1.$$

When $i \neq k$, we have:

$$H_k(x_i) = [L_k(x_i)]^2(1 - 2L_k'(x_k)(x_i - x_k)) = 0.$$

Then consider the derivative of $H_k(x)$, we have:

$$H_k'(x) = 2L_k(x)L_k'(x)[1 - 2L_k'(x_k)(x - x_k)] + [L_k(x)]^2(-2L_k'(x_k)).$$

When $i = k$, we have:

$$\begin{aligned} H_k'(x_i) &= 2L_k(x_k)L_k'(x_k)[1 - 2L_k'(x_k)(x_k - x_k)] + [L_k(x_k)]^2(-2L_k'(x_k)) \\ &= 2L_k'(x_k) - 2L_k'(x_k) \\ &= 0, \end{aligned}$$

and when $i \neq k$, we have:

$$H_k'(x_i) = 2L_k(x_i)L_k'(x_i)[1 - 2L_k'(x_k)(x_i - x_k)] + [L_k(x_i)]^2(-2L_k'(x_k)) = 0.$$

Hence, there exists:

$$H_k(x_i) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}, \ H_k'(x_i) = 0.$$

Then consider $K_k(x)$, when $i = k$, we have:

$$K_k(x_i) = [L_k(x_i)]^2(x_i - x_k) = 0,$$

and when $i \neq k$, we have:

$$K_k(x_k) = [L_k(x_k)]^2(x_k - x_k) = 0.$$

Finally, consider the derivative of $K_k(x)$, we have:

$$K_k(x) = 2L_k(x)L_k'(x)(x - x_k) + [L_k(x)]^2$$

When $i = k$, we have:

$$K_k(x_i) = 2L_k(x_k)L_k'(x_k)(x_k - x_k) + [L_k(x_k)]^2 = 1,$$

and when $i \neq k$, we have:

$$K_k(x_i) = 2L_k(x_i)L_k'(x_i)(x_i - x_k) + [L_k(x_i)]^2 = 0.$$

Hence, there exists:

$$K_k(x) = 0, \ K_k'(x) = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

which gives the proof. $\square$

3. **Solution.**

Since it is required to construct a system of orthogonal polynomials $\{\phi_0, \phi_1, \phi_2, \phi_3\}$ on the interval $(-1, 1)$ with respect to the weight function $w(x) = 1$ starting from $\phi_0 = 1$, the highest degree of these polynomials is 3. For an orthogonal polynomial with degree n, it has the basis of $\{1, x, ..., x^n\}$. Hence, we have the following formula. When $n = 0$:

$$\phi_0 = 1.$$

When $n = 1$:

$$< \phi_0, \phi_0 > = \int_{-1}^{1} \phi_0^2 \, dx = \int_{-1}^{1} dx = 2,$$

$$< x^1, \phi_0 > = \int_{-1}^{1} x\,dx = 0,$$

$$\boldsymbol{\phi_1} = x^1 - \frac{< x^1, \phi_0 >}{< \phi_0, \phi_0 >}\phi_0 = \boldsymbol{x}.$$

When $n = 2$:

$$< \phi_1, \phi_1 > = \int_{-1}^{1} \phi_1^2\,dx = \int_{-1}^{1} x^2\,dx = \frac{2}{3},$$

$$< x^2, \phi_0 > = \int_{-1}^{1} x^2\,dx = \frac{2}{3},$$

$$< x^2, \phi_1 > = \int_{-1}^{1} x^3\,dx = 0,$$

$$\boldsymbol{\phi_2} = x^2 - \frac{< x^2, \phi_0 >}{< \phi_0, \phi_0 >}\phi_0 - \frac{< x^2, \phi_1 >}{< \phi_1, \phi_1 >}\phi_1 = \boldsymbol{x^2 - \frac{1}{3}}.$$

When $n = 3$:

$$< \phi_2, \phi_2 > = \int_{-1}^{1} \phi_2^2\,dx = \int_{-1}^{1} (x^2 - \frac{1}{3})^2\,dx = \frac{8}{45},$$

$$< x^3, \phi_0 > = \int_{-1}^{1} x^3\,dx = 0,$$

$$< x^3, \phi_1 > = \int_{-1}^{1} x^4\,dx = \frac{2}{5},$$

$$< x^3, \phi_2 > = \int_{-1}^{1} x^3(x^2 - \frac{1}{3})\,dx = 0,$$

$$\boldsymbol{\phi_3} = x^3 - \frac{< x^3, \phi_0 >}{< \phi_0, \phi_0 >}\phi_0 - \frac{< x^3, \phi_1 >}{< \phi_1, \phi_1 >}\phi_1 - \frac{< x^3, \phi_2 >}{< \phi_2, \phi_2 >}\phi_2 = \boldsymbol{x^3 - \frac{3}{5}x}.$$

Plot the polynomials $\{\phi_0 = 1,\ \phi_1 = x,\ \phi_2 = x^2 - \frac{1}{3},\ \phi_3 = x^3 - \frac{3}{5}x\}$ on the interval $(-1, 1)$ as follow:

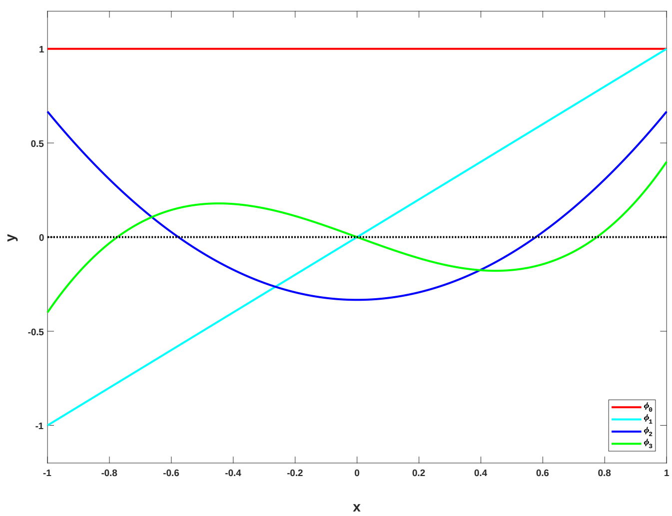

Figure 3: Orthogonal polynomials

From the figure above, we can find that in the interval $(-1, 1)$:

$$\phi_0 \text{ has } \mathbf{0} \text{ root,}$$
$$\phi_1 \text{ has } \mathbf{1} \text{ root,}$$
$$\phi_2 \text{ has } \mathbf{2} \text{ roots,}$$
$$\phi_3 \text{ has } \mathbf{3} \text{ roots.}$$

Hence, it is verified that the roots of each polynomial are distinct and lie in the interval $(-1, 1)$, and for $k = 0, 1, 2, \ldots$, the polynomial ϕ_k has $\boldsymbol{k}$ root(s) in the interval $(-1, 1)$.

4. **Solution.**

Following is the algorithm codes for natural cubic spline interpolation through points

$$(1,16),\ (2,18),\ (3,21),\ (4,17),\ (5,15),\ (6,12)$$

which are equally spaced.

```
xi = [1, 2, 3, 4, 5, 6];
yi = [16, 18, 21, 17, 15, 12];

h = diff(xi);
H = [2 * (h(1) + h(2)), h(2), 0, 0;
     h(2), 2 * (h(2) + h(3)), h(3), 0;
     0, h(3), 2 * (h(3) + h(4)), h(4);
     0, 0, h(4), 2 * (h(4) + h(5))];
b = [6 * ((yi(3) - yi(2)) / h(2) - (yi(2) - yi(1)) / h(1));
     6 * ((yi(4) - yi(3)) / h(3) - (yi(3) - yi(2)) / h(2));
     6 * ((yi(5) - yi(4)) / h(4) - (yi(4) - yi(3)) / h(3));
     6 * ((yi(6) - yi(5)) / h(5) - (yi(5) - yi(4)) / h(4))];
sigma = [0; inv(H) * b; 0];

CSI = @(i, x) (xi(i) - x) .^ 3 * sigma(i - 1) / (6 * h(i - 1))  ...
              + (x - xi(i - 1)) .^ 3 * sigma(i) / (6 * h(i - 1)) ...
              + (yi(i) - sigma(i) * h(i - 1) / 6) * (x - xi(i - 1)) ...
              + (yi(i - 1) - sigma(i - 1) * h(i - 1) / 6) * (xi(i) - x);

i = [linspace(xi(1), xi(2), 1000);
     linspace(xi(2), xi(3), 1000);
     linspace(xi(3), xi(4), 1000);
     linspace(xi(4), xi(5), 1000);
     linspace(xi(5), xi(6), 1000)];

plot(i(1, :), CSI(2, i(1, :)), 'r-', i(2, :), CSI(3, i(2, :)), 'r-', ...
     i(3, :), CSI(4, i(3, :)), 'r-', i(4, :), CSI(5, i(4, :)), 'r-', ...
     i(5, :), CSI(6, i(5, :)), 'r-', 'LineWidth', 2);
hold on;
plot(xi, yi, 'bo', 'MarkerSize', 6);
hold off;

axis([0 7 10 22]);
xlabel('\bf\fontsize{14}x');
ylabel('\bf\fontsize{14}y');
```

The result is shown on the next page.

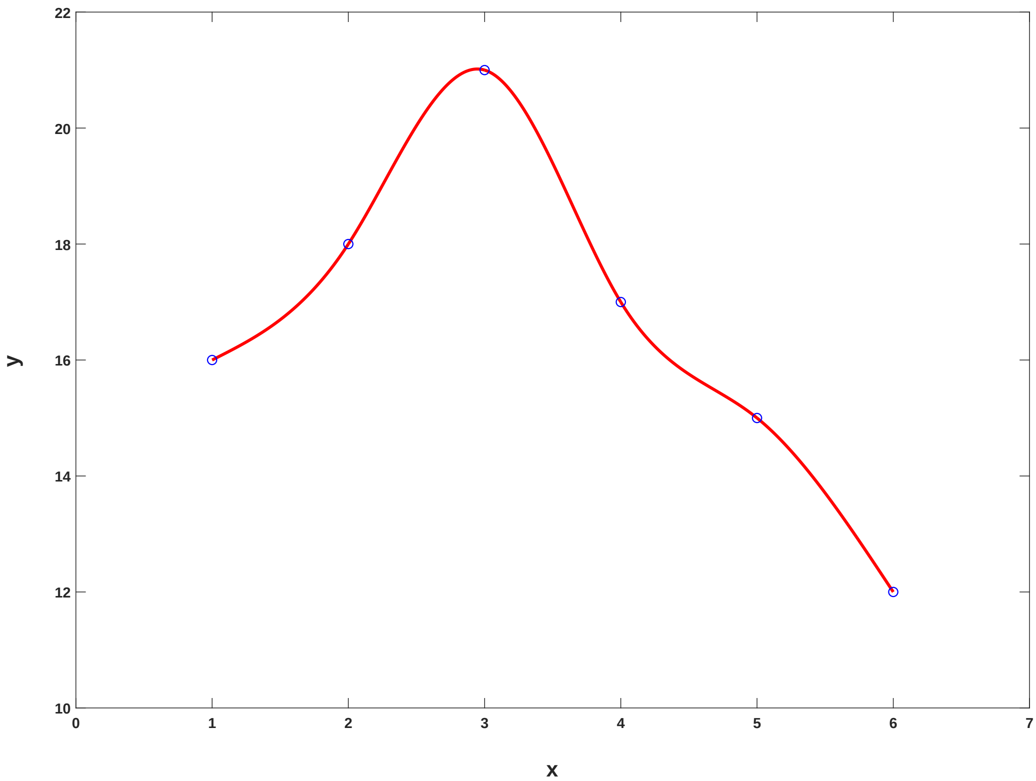

Figure 4: Piecewise cubic interpolation