Homework IV

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1. **Solution.**

It is required to use forward Euler method to solve the ODE problem given by:

$$\begin{cases} y' = y^2 + g(x) \\ y(0) = 2 \end{cases}$$

where $g(x) = -(x^4 - 6x^3 + 12x^2 - 14x + 9) \,/\, (1 + x)^2$, and the solution $y(x)$ is given by:

$$y(x) = (1 - x)(2 - x) \,/\, (1 + x)$$

The codes to reproduce the figure are as follow:

```
% construct functions
g = @(x) -(x ^ 4 - 6 * x ^ 3 + 12 * x ^ 2 - 14 * x + 9) / (1 + x) ^ 2;
y = @(x) (1 - x) .* (2 - x) ./ (1 + x);

% plot the approximate solution
for h = [0.2 0.1 0.05]
    y0 = 2;
    yk = zeros(2 / h - 1, 1);
    yk(1) = y0;
    for i = 2 : 2 / h - 1
        yk(i) = y0 + h * (y0 ^ 2 + g((i - 2) * h));
        y0 = yk(i);
    end
    plot(linspace(0, 2, 2 / h - 1), yk, '.', 'MarkerSize', 15);
    hold on;
end

% plot the precise solution
x = linspace(0, 2, 99);
plot(x, y(x), 'k-', 'LineWidth', 2);
hold off;
```

The results are shown below, which shows that the situation $h = 0.05$ is closest to the precise solution.

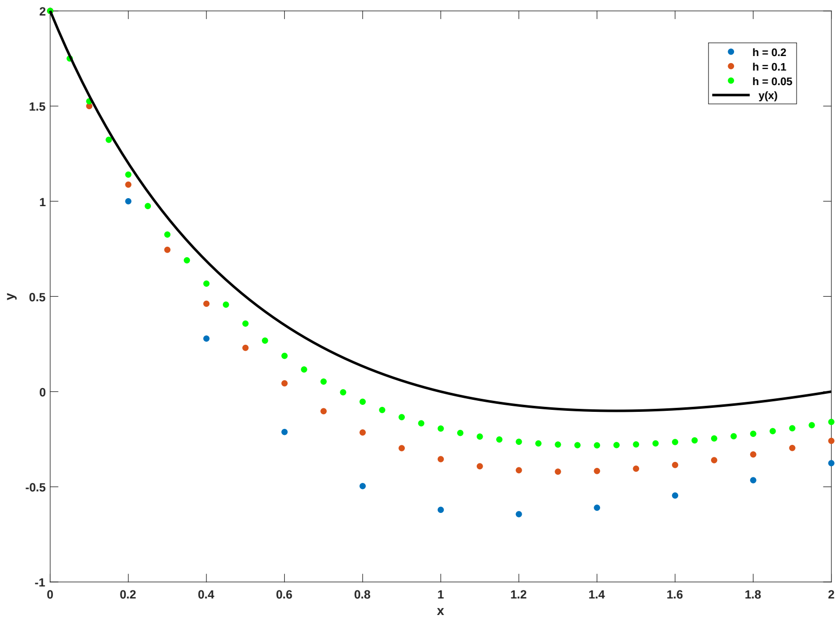

Figure 1: Forward Euler Method

2. **Solution.**

It is required to use Trapezoidal method to solve the ODE problem given by:

$$\begin{cases} y' = y^2 + g(x) \\ y(0) = 2 \end{cases}$$

where $g(x) = -(x^4 - 6x^3 + 12x^2 - 14x + 9) / (1 + x)^2$, and the solution $y(x)$ is given by:

$$y(x) = (1 - x)(2 - x) / (1 + x).$$

To use the Trapezoidal method, it is required to solve a quadratic equation for y_{n+1} given as:

$$y_{n+1} = y_n + \frac{h}{2}(y_n^2 + g(x_n) + y_{n+1}^2 + g(x_{n+1}))$$

Hence, we have the iteration formula as:

$$y_{n+1} = \frac{1 - \sqrt{1 - 2hy_n - h^2\left(y_n^2 + g(x_n) + g(x_{n+1})\right)}}{h}$$

The codes to reproduce the figure and the results are shown below, which shows that the Trapezoidal method is more precise than the forward Euler method.

```matlab
% construct functions
g = @(x) -(x ^ 4 - 6 * x ^ 3 + 12 * x ^ 2 - 14 * x + 9) / (1 + x) ^ 2;
y = @(x) (1 - x) .* (2 - x) ./ (1 + x);
h = 0.2;

% forward Euler method
y0 = 2;
yk = zeros(2 / h + 1, 1);
yk(1) = y0;
for i = 2 : 2 / h + 1
    yk(i) = y0 + h * (y0 ^ 2 + g((i - 2) * h));
    y0 = yk(i);
end
plot(linspace(0, 2, 2 / h + 1), yk, 'b.', 'MarkerSize', 15);
hold on;

% Trapezoidal Method
y0 = 2;
yk = zeros(2 / h + 1, 1);
yk(1) = y0;
for i = 2 : 2 / h + 1
    yk(i) = (1 - sqrt(1 - 2 * h * (y0 + 0.5 * h * (y0 ^ 2 + g((i - 2) * ...
        h) + g((i - 1) * h))))) / h;
    y0 = yk(i);
end
plot(linspace(0, 2, 2 / h + 1), yk, 'r.', 'MarkerSize', 15);
hold on;

% plot the precise solution
x = linspace(0, 2, 99);
plot(x, y(x), 'k-', 'LineWidth', 2);
hold off;
```

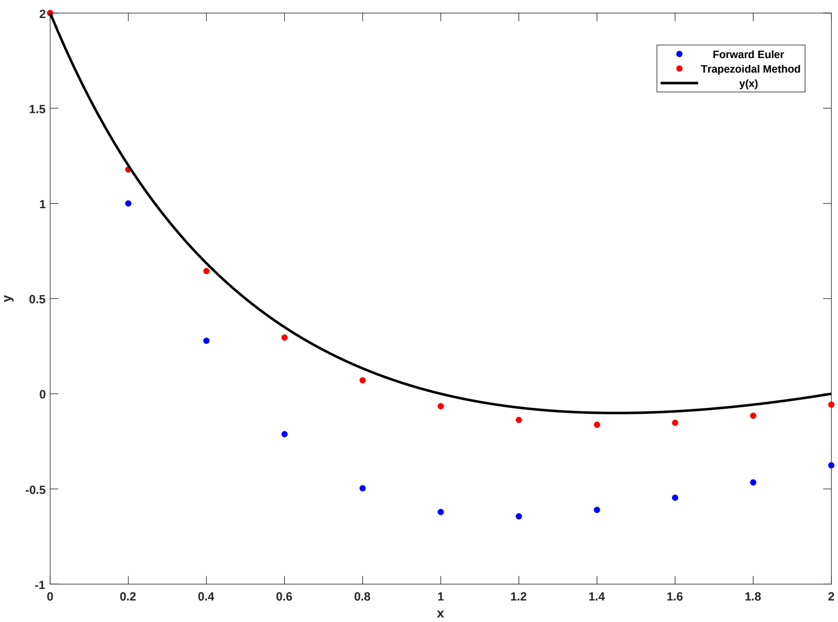

Figure 2: Trapezoidal Method

3. **Solution.**

Proof. We have the formula of Trapezoidal method as:

$$y_{n+1} = y_n + \frac{h}{2}\big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\big)$$

for $n = 0, 1, 2, \cdots$. Note that there exists:

$$\begin{aligned} y(t_{n+1}) &= y(t_n) + \int_{t_n}^{t_{n+1}} f(t, y(t))\,\mathrm{d}t \\ &\approx y(t_n) + \frac{h}{2}\big(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\big) \end{aligned}$$

And we hold that the following formula is correct:

$$y_{n+1} = y(t_n) + \frac{h}{2}\big(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\big)$$

Since we have the equation $y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1}))$, we apply the Taylor's expansion to the derivative $y'(t_{n+1})$:

$$\begin{aligned} y'(t_{n+1}) &= y'(t_n) + y''(t_n)(t_{n+1} - t_n) + \frac{1}{2}y'''(\xi)(t_{n+1} - t_n)^2 \\ &= y'(t_n) + y''(t_n)h + O(h^2) \end{aligned}$$

where $\xi \in (t_n,\ t_{n+1})$. Hence, we have:

$$\begin{aligned} y_{n+1} &= y(t_n) + \frac{h}{2}\big(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\big) \\ &= y(t_n) + \frac{h}{2}\big(y'(t_n) + y'(t_{n+1})\big) \\ &= y(t_n) + \frac{h}{2}\big(y'(t_n) + y'(t_n) + y''(t_n)h + O(h^2)\big) \\ &= y(t_n) + y'(t_n)h + \frac{1}{2}y''(t_n)h^2 + O(h^3) \end{aligned}$$

Then apply Taylor's expansion to $y(t_{n+1})$:

$$y(t_{n+1}) = y(t_n) + y'(t_n)h + \frac{1}{2}y''(t_n)h^2 + O(h^3)$$

Hence, the local truncation error is calculated by:

$$\begin{aligned} \text{LTE} &= \int_{t_n}^{t_{n+1}} f(t, y(t))\,\mathrm{d}t - \frac{h}{2}\big(f(t_n, y(t_n)) + f(t_{n+1}, y(t_{n+1}))\big) \\ &= y(t_{n+1}) - y_{n+1} \\ &= O(h^3) \end{aligned}$$

Hence, we have proved that the local truncation error of the Trapezoidal method for the ODE initial value problem is $O(h^3)$.

□

4. **Solution.**

It is required to derive the Adams-Moulton formula for $k = 3$ given by:

$$y_{n+1} = y_n + \frac{h}{12}\left(5f_{n+1} + 8f_n - f_{n-1}\right)$$

Consider the points at (t_{n-1}, f_{n-1}), (t_n, f_n) and (t_{n+1}, f_{n+1}), note that:

$$t_{n+1} - t_n = h, \ t_n - t_{n-1} = h$$

We use these points to construct Lagrange polynomials to substitute $f(t, y(t))$. We have:

$$\begin{aligned}
y_{n+1} &= y_n + \int_{t_n}^{t_{n+1}} f(t, y(t))\,\mathrm{d}t \\
&= y_n + \int_{t_n}^{t_{n+1}} P_2(t)\,\mathrm{d}t \\
&= y_n + f_{n-1}\int_{t_n}^{t_{n+1}} L_0(t)\,\mathrm{d}t + f_n\int_{t_n}^{t_{n+1}} L_1(t)\,\mathrm{d}t + f_{n+1}\int_{t_n}^{t_{n+1}} L_2(t)\,\mathrm{d}t
\end{aligned}$$

where

$$\begin{aligned}
L_0(t) &= \frac{1}{2h^2}(t - t_n)(t - t_{n+1}), \\
L_1(t) &= -\frac{1}{h^2}(t - t_{n-1})(t - t_{n+1}), \\
L_2(t) &= \frac{1}{2h^2}(t - t_{n-1})(t - t_n)
\end{aligned}$$

Plug into the formula above and we have:

$$\begin{aligned}
\int_{t_n}^{t_{n+1}} L_0(t)\,\mathrm{d}t &= \int_{t_n}^{t_{n+1}} \frac{1}{2h^2}(t - t_n)(t - t_{n+1})\,\mathrm{d}t = -\frac{1}{12}h, \\
\int_{t_n}^{t_{n+1}} L_1(t)\,\mathrm{d}t &= \int_{t_n}^{t_{n+1}} -\frac{1}{h^2}(t - t_{n-1})(t - t_{n+1})\,\mathrm{d}t = \frac{2}{3}h, \\
\int_{t_n}^{t_{n+1}} L_2(t)\,\mathrm{d}t &= \int_{t_n}^{t_{n+1}} \frac{1}{2h^2}(t - t_{n-1})(t - t_n)\,\mathrm{d}t = \frac{5}{12}h
\end{aligned}$$

Hence, we have the Adams-Moulton formula for $k = 3$ as:

$$\begin{aligned}
y_{n+1} &= y_n + f_{n-1}\int_{t_n}^{t_{n+1}} L_0(t)\,\mathrm{d}t + f_n\int_{t_n}^{t_{n+1}} L_1(t)\,\mathrm{d}t + f_{n+1}\int_{t_n}^{t_{n+1}} L_2(t)\,\mathrm{d}t \\
&= y_n - \frac{1}{12}hf_{n-1} + \frac{2}{3}hf_n + \frac{5}{12}hf_{n+1} \\
&= y_n + \frac{h}{12}\left(5f_{n+1} + 8f_n - f_{n-1}\right)
\end{aligned}$$

which completes the deduction.

5. **Solution.**

Proof. Since $g(t) = h(t)f(t)$, from the properties of Fourier transform and convolution, we have:

$$
\begin{aligned}
\hat{g}(\omega) &= \int_{-\infty}^{+\infty} g(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} h(t)f(t)e^{-i\omega t}\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{h}(\omega_1)e^{i\omega_1 t}\,\mathrm{d}\omega_1\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(\omega_2)e^{i\omega_2 t}\,\mathrm{d}\omega_2\right)e^{-i\omega t}\,\mathrm{d}t \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \hat{h}(\omega_1)\left(\int_{-\infty}^{+\infty} \hat{f}(\omega_2)e^{i(\omega_1+\omega_2)t}\,\mathrm{d}\omega_2\right)\mathrm{d}\omega_1\right)e^{-i\omega t}\,\mathrm{d}t \quad (\text{Let } u = \omega_1 + \omega_2) \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \hat{h}(\omega_1)\left(\int_{-\infty}^{+\infty} \hat{f}(u-\omega_1)e^{iut}\,\mathrm{d}u\right)\mathrm{d}\omega_1\right)e^{-i\omega t}\,\mathrm{d}t \\
&= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \hat{h}(\omega_1)\hat{f}(u-\omega_1)\,\mathrm{d}\omega_1\right)e^{iut}\,\mathrm{d}u\right)e^{-i\omega t}\,\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(\int_{-\infty}^{+\infty} \hat{h}(\omega_1)\hat{f}(u-\omega_1)\,\mathrm{d}\omega_1\right)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i(u-\omega)t}\,\mathrm{d}t\right)\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\hat{h}\star\hat{f})(u)\,\delta(u-\omega)\,\mathrm{d}u \quad (\text{Let } u' = \omega - u) \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\hat{h}\star\hat{f})(\omega-u')\,\delta(-u')\,\mathrm{d}u' \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (\hat{h}\star\hat{f})(\omega-u')\,\delta(u')\,\mathrm{d}u' \\
&= \frac{1}{2\pi}(\hat{h}\star\hat{f})(\omega)
\end{aligned}
$$

Hence, the Fourier transform of $g(t) = h(t)f(t)$ is given by $\hat{g}(\omega) = \frac{1}{2\pi}(\hat{h}\star\hat{f})(\omega)$. □

6. **Solution.**

Proof. We have a vector of length n given as $f = [f_0, f_1, \cdots, f_{n-1}]^\top$, and the discrete Fourier transform of f is given as $\hat{f} = [\hat{f}_0, \hat{f}_1, \cdots, \hat{f}_{n-1}]^\top$. From the definition, we have:

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j\, e^{-\frac{2\pi ijk}{n}} = \sqrt{n} < f, e_k >$$

where $k = 0, 1, \cdots, n-1$. By the unitary property of $\{e_k\}_{k=0}^{n-1}$, we have the following inverse DFT:

$$f = \sum_{k=0}^{n-1} f_k e_k = \sum_{k=0}^{n-1} < f, e_k > e_k = \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} \hat{f}_k e_k$$

Hence, we have the 2-norm of the vector f:

$$\|f\|_2^2 = \sum_{k=0}^{n-1} \left| < f, e_k > \right|^2 = \sum_{k=0}^{n-1} \left| \frac{1}{\sqrt{n}}\hat{f}_k \right|^2 = \frac{1}{n}\sum_{k=0}^{n-1} |\hat{f}_k|^2 = \frac{1}{n}\|\hat{f}\|_2^2$$

which gives the proof. □

7. **Solution.**

Proof. It is required to prove the odd frequencies given as:

$$\hat{f}_{2k+1} = \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} \big(f_j - f_{j+\frac{n}{2}}\big) e^{-\frac{2\pi i j k}{\frac{n}{2}}}$$

for $k = 0, 1, \cdots, n-1$, which appears in the derivation of FFT. Since we have the DFT formula as:

$$\hat{f}_k = \sum_{j=0}^{n-1} f_j\, e^{-\frac{2\pi i j k}{n}}$$

where $k = 0, 1, \cdots, n-1$. Consider the values of k from 0 to $\frac{n}{2} - 1$, we have:

$$\begin{aligned}
\hat{f}_{2k+1} &= \sum_{j=0}^{n-1} f_j\, e^{-\frac{2\pi i j(2k+1)}{n}} \\
&= \sum_{j=0}^{n-1} f_j\, e^{-\frac{2\pi i j}{n}}\, e^{-\frac{2\pi i j k}{\frac{n}{2}}} \\
&= \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} f_j e^{-\frac{2\pi i j k}{\frac{n}{2}}} + \sum_{j=\frac{n}{2}}^{n-1} e^{-\frac{2\pi i j}{n}} f_j e^{-\frac{2\pi i j k}{\frac{n}{2}}} \quad \Big(\text{Let } j = j' + \frac{n}{2},\ j' = 0, 1, \cdots, \frac{n}{2} - 1\Big) \\
&= \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} f_j e^{-\frac{2\pi i j k}{\frac{n}{2}}} + \sum_{j'=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i (j'+\frac{n}{2})}{n}} f_{j'+\frac{n}{2}} e^{-\frac{2\pi i (j'+\frac{n}{2})k}{\frac{n}{2}}} \\
&= \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} f_j e^{-\frac{2\pi i j k}{\frac{n}{2}}} + \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} e^{-i\pi} f_{j+\frac{n}{2}} e^{-\frac{2\pi i j k}{\frac{n}{2}}} e^{-2\pi i k} \\
&= \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} f_j e^{-\frac{2\pi i j k}{\frac{n}{2}}} - \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} f_{j+\frac{n}{2}} e^{-\frac{2\pi i j k}{\frac{n}{2}}} \\
&= \sum_{j=0}^{\frac{n}{2}-1} e^{-\frac{2\pi i j}{n}} \big(f_j - f_{j+\frac{n}{2}}\big) e^{-\frac{2\pi i j k}{\frac{n}{2}}}
\end{aligned}$$

which gives the proof. □