Homework III

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1. **Solutions.**

It is given that $\{\phi_j,\ j=0,\cdots\}$ is a system of orthogonal polynomials on $(0,1)$ with respect to weight $w(x)=1$. To construct a system of orthogonal polynomials on (a,b) with respect to $w(x)=1$ from $\{\phi_j,\ j=0,\cdots\}$, we need to construct a mapping from $(0,1)$ to (a,b), that is:

$$x = (b-a)t + a, \quad 0 \le t \le 1, \quad a \le x \le b.$$

Let $\{\psi_j,\ j=0,\cdots\}$ be the system of orthogonal polynomials on (a,b), we have:

$$\psi_j(x) = \phi_j(t) = \phi_j\left(\frac{x-a}{b-a}\right), \quad a \le x \le b, \quad j = 0,\cdots.$$

Verify that $\{\psi_j,\ j=0,\cdots\}$ is the system of orthogonal polynomials, assume that $i \neq j$, we have:

$$\begin{aligned}
<\psi_i(x),\psi_j(x)> &= <\phi_i\left(\frac{x-a}{b-a}\right),\phi_j\left(\frac{x-a}{b-a}\right)> \\
&= \int_a^b \phi_i\left(\frac{x-a}{b-a}\right)\phi_j\left(\frac{x-a}{b-a}\right)\mathrm{d}x \\
&= (b-a)\int_a^b \phi_i\left(\frac{x-a}{b-a}\right)\phi_j\left(\frac{x-a}{b-a}\right)\mathrm{d}\left(\frac{x-a}{b-a}\right) \\
&= (b-a)\int_0^1 \phi_i(u)\phi_j(u)\,\mathrm{d}u \\
&= (b-a)<\phi_i(x),\phi_j(x)> \\
&= 0.
\end{aligned}$$

And it is obviously that $\psi_j(x)$ is of exact degree j. Hence, $\{\psi_j,\ j=0,\cdots\}$ is a system of orthogonal polynomials on (a,b) with respect to $w(x)=1$. Actually, if we want to deduct $\{\psi_j,\ j=0,\cdots\}$ from Gram-Schmidt orthogonalization, we have a more accurate form of $\{\psi_j,\ j=0,\cdots\}$, that is:

$$\psi_j(x) = (b-a)^j\,\phi_j\left(\frac{x-a}{b-a}\right), \quad a \le x \le b, \quad j = 0, \cdots.$$

This is because Gram-Schmidt orthogonalization has the form as:

$$\phi_n(x) = x^n - \frac{< x^n, \phi_0 >}{< \phi_0, \phi_0 >}\phi_0 - \cdots - \frac{< x^n, \phi_{n-1} >}{< \phi_{n-1}, \phi_{n-1} >}\phi_{n-1},$$

which has coefficient 1 at highest order of $\phi_n(x)$. Hence it is required to multiply a normalized term with degree j on $\phi_n(x)$ when mapping $\phi_n(x)$ to $\psi_n(x)$ on (a,b). But if we only want a system of orthogonal polynomials, the mapping

$$\psi_j(x) = \phi_j\left(\frac{x-a}{b-a}\right), \quad a \le x \le b, \quad j = 0, \cdots.$$

can also satisfy the requirement. For example, we have the orthogonal polynomials over the interval $[0,1]$ and $[-1,1]$ generated by Gram-Schmidt orthogonalization:

$$\phi_0(x) = 1, \quad \phi_1(x) = x - \frac{1}{2}, \quad \phi_2(x) = x^2 - x + \frac{1}{6}, \quad 0 \le x \le 1,$$
$$\psi_0(x) = 1, \quad \psi_1(x) = x, \quad \psi_2(x) = x^2 - \frac{1}{3}, \quad -1 \le x \le 1.$$

which satisfy:

$$\psi_j(x) = 2^j\,\phi_j\left(\frac{x+1}{2}\right), \quad -1 \le x \le 1, \quad j = 0,1,2.$$

2. **Solution.**

No. The Simpson's rule is **NOT** always more accurate than the Trapezium rule in numerical integration. Here gives a counter example, for the integral as follow:

$$A(x) = \int_0^1 \left(x^5 - \frac{11}{10}x^4\right)\mathrm{d}x,$$

the Trapezium rule is more precise than Simpson's rule. We have the precise value as:

$$A(x) = \int_0^1 \left(x^5 - \frac{11}{10}x^4\right)\mathrm{d}x = -\frac{4}{75} = -0.053333.$$

For Trapezium rule, we have:

$$A(x) \approx \frac{1-0}{2}\left(\left(0^5 - \frac{11}{10}\times 0^4\right) + \left(1^5 - \frac{11}{10}\times 1^4\right)\right) = -0.05.$$

For Simpson's rule, we have:

$$A(x) \approx \frac{1-0}{6}\left(\left(0^5 - \frac{11}{10}\times 0^4\right) + 4\times\left((\frac{0+1}{2})^5 - \frac{11}{10}\times(\frac{0+1}{2})^4\right) + \left(1^5 - \frac{11}{10}\times 1^4\right)\right) = -0.041667.$$

Hence, the result using the Trapezium rule is closer to the precise value than the Simpson's rule, which indicates that the Simpson's rule is not always more accurate than the Trapezium rule in numerical integration for all situations.

3. **Solution.**

It is required to evaluate result for $\int_0^1 \frac{e^{-2x}}{1+4x}\,\mathrm{d}x \approx 0.220458$. Using the Richardson extrapolation, we have the form of polynomial $T(m)$ which is deduced by composite Trapezium rule. There exists the follow formula:

$$\int_a^b f(x)\,\mathrm{d}x \approx h\Big[\frac{1}{2}f(x_0) + f(x_1) + \cdots f(x_{m-1}) + \frac{1}{2}f(x_m)\Big] = T(m).$$

From delicate error analysis for Trapezium rule, we have the expression of the Euler-Maclaurin expansion as:

$$\int_a^b f(x)\,\mathrm{d}x = T(m) + Ch^2 + O(h^4).$$

Using the Richardson extrapolation with high order terms, we have the expression of $T_1(m)$:

$$T_1(m) = \frac{4T(2m) - T(m)}{3},$$

which has the expansion as:

$$\int_a^b f(x)\,\mathrm{d}x = T_1(m) + O(h^4).$$

Here attaches the MATLAB codes for substituting parameter m into the expression of $T(m)$:

```
% evaluate the integral of f(x) over (0, 1)
f = @(x) exp(- 2 * x) / (1 + 4 * x);
a = 0;
b = 1;

% storage of T(m) and T1(m)
T = zeros(5, 1);
T1 = zeros(4, 1);

% substitute m into T(m)
for i = 1 : 5
    m = 2 ^ (i + 1);     % m = [4 8 16 32 64]
    h = (b - a) / m;

    % calculate f(x_1) + ... + f(x_m-1)
    sum = 0;
    for j = 1 : m - 1
        sum = sum + f(a + h * j);
    end

    % calculate T(m)
    T(i) = h * (0.5 * f(a) + sum + 0.5 * f(b));
end

    % calculate T1(m)
for k = 1 : 4
    T1(k) = (4 * T(k + 1) - T(k)) / 3;
end

% output the result
T
T1
```

and we have the results of $T(m)$ and $T_1(m)$:

```
>> hw3_3

T =

    0.248801969651892
    0.227978745877260
    0.222373671948229
    0.220939503026484
    0.220578695605905


T1 =

    0.221037671285716
    0.220505313971885
    0.220461446719236
    0.220458426465712
```

Hence, the evaluation is shown as follow:

m	$T(m)$	$T_1(m)$
4	0.248802	0.221038
8	0.227979	0.220505
16	0.222374	0.220461
32	0.220940	0.220458
64	0.220579	

It shows that $T_1(m)$ is more precise than $T(m)$ for evaluating the integral results.

4. **Solution.**

First deduce the orthogonal polynomials over the interval $(-1,1)$ of degree $n+1=2$ to choose x_0 and x_1 to be the zeros, which makes $v_k = 0$. We have:

$$
\begin{aligned}
\phi_0 &= 1,\\
\phi_1 &= x - \frac{\int_{-1}^{1} x\,\mathrm{d}x}{\int_{-1}^{1} \mathrm{d}x} = x,\\
\phi_2 &= x^2 - \frac{\int_{-1}^{1} x^3\,\mathrm{d}x}{\int_{-1}^{1} x^2\,\mathrm{d}x}x - \frac{\int_{-1}^{1} x^2\,\mathrm{d}x}{\int_{-1}^{1} \mathrm{d}x} = x^2 - \frac{1}{3}.
\end{aligned}
$$

Let $\phi_2 = x^2 - \frac{1}{3} = 0$, we have the roots of the polynomial $\pi_2(x)$ of degree 2 as follow:

$$x_0 = \frac{\sqrt{3}}{3}, \quad x_1 = -\frac{\sqrt{3}}{3}.$$

Then calculate the coefficients of the interpolation polynomial from Lagrange interpolation, we have:

$$
\begin{aligned}
w_0 &= \int_{-1}^{1} L_0\,\mathrm{d}x = \int_{-1}^{1} \frac{x - x_1}{x_0 - x_1}\,\mathrm{d}x = \int_{-1}^{1} \frac{x + \frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{3}}\,\mathrm{d}x = 1,\\
w_1 &= \int_{-1}^{1} L_1\,\mathrm{d}x = \int_{-1}^{1} \frac{x - x_0}{x_1 - x_0}\,\mathrm{d}x = \int_{-1}^{1} \frac{x - \frac{\sqrt{3}}{3}}{-\frac{\sqrt{3}}{3} - \frac{\sqrt{3}}{3}}\,\mathrm{d}x = 1.
\end{aligned}
$$

Hence, we have the Gauss quadrature formula:

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx f\Big(\frac{\sqrt{3}}{3}\Big) + f\Big(-\frac{\sqrt{3}}{3}\Big),$$

which is the Gauss quadrature rule over $[-1,1]$ for $n=1$. First calculate the exact value, we have:

$$
\begin{aligned}
\int_{-1}^{1} (x^3 + 3x^2 - 5)\,\mathrm{d}x &= -8,\\
\int_{-1}^{1} e^x\,\mathrm{d}x &= 2.350402.
\end{aligned}
$$

Then use the rule to evaluate the given functions, we have:

$$
\begin{aligned}
\int_{-1}^{1} (x^3 + 3x^2 - 5)\,\mathrm{d}x &\approx \Big(\big(\frac{\sqrt{3}}{3}\big)^3 + 3\big(\frac{\sqrt{3}}{3}\big)^2 - 5\Big) + \Big(\big(-\frac{\sqrt{3}}{3}\big)^3 + 3\big(-\frac{\sqrt{3}}{3}\big)^2 - 5\Big) = -8,\\
\int_{-1}^{1} e^x\,\mathrm{d}x &\approx e^{\frac{\sqrt{3}}{3}} + e^{-\frac{\sqrt{3}}{3}} = 2.342696.
\end{aligned}
$$

We can find that the evaluation using Gauss quadrature rule is quite precise about the exact value.

5. **Solution.**

It is required to evaluate $f'(1)$ for $f(x) = \sin(x^2)$ with $h = 0.1$ using forward difference, backward difference, centered difference and Richardson extrapolation based on forward difference. We have the following MATLAB codes for the evaluation:

```
% evaluate f'(1) given f(x) = sin(x^2) and h = 0.1
f = @(x) sin(x ^ 2);
h = 0.1;

% FD (forward difference)
FD = (f(1 + h) - f(1)) / h

% BD (backward difference)
BD = (f(1) - f(1 - h)) / h

% CD (centered difference)
CD = (f(1 + h) - f(1 - h)) / (2 * h)

% FD1 (Richardson extrapolation based on forward difference)
FD1 = 2 * (f(1 + h / 2) - f(1)) / (h / 2) - (f(1 + h) - f(1)) / h
```

and we have the results:

```
>> hw3_5

FD =

    0.941450167454895


BD =

    1.171838104377539


CD =

    1.056644135916217


FD1 =

    1.093253006717650
```

Hence, we have the numerical results of given problem as follow, which the exact value of $f'(1)$ is approximately equal to 1.080605:

FD	0.941450
BD	1.171838
CD	1.056644
FD_1	1.093253

It can find that the result using the Richardson extrapolation based on forward difference is the most precise about the exact value.