CS3230 Tutorial 1

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Introduction

This is Tutorial Group 15 for CS3230.

My name is Deng Tianle:

- Year 3 Computer Science and Mathematics (DDP)
- First-time TA :)
- My JC: Raffles Institution (Y5-6)
- I was from Shenzhen, China

Admin

Everyone needs to present 3 times to obtain the 3% tutorial participation marks. (I believe that beyond the presentation, there is no obligation to attend tutorial. But of course, you are encouraged to attend).

$$\begin{aligned} &\text{We have 21 people} \\ \implies &\text{63 presentations} \\ \implies &\text{we should have around 6 presentations per class.} \end{aligned}$$

Remark

In later slides I use P to denote presentations, e.g. $P3$ means that the third presenter of the day will present on (possibly a part) of this question.

Agenda

This tutorial is about asymptotic notations: $O, \Omega, \Theta, o, \omega$.

- Analogy with $\leqslant, \geqslant, =, <, >$: Q2
- Computation using limit: Q1 (I rearranged because we can use Q2 here)
- Practical computation, relation between common functions like log, polynomial, exp, factorial: Q3-5
- LeetCode question on removing duplicates (if time permits; I will run the algorithm on the board to show the idea)

Let $\mathbb{N}$ denote the set of positive integers. We consider functions $f : \mathbb{N} \to \mathbb{N}$. Let the class of all such functions be C.

Definition

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say that $f(n) \in \Theta(g(n))$ if there exist positive constants c_1, c_2 such that

$$c_1 g(n) \leqslant f(n) \leqslant c_2 g(n)$$

for sufficiently large n. We say that $f(n) \in \omega(g(n))$ if for all positive constants c, we have

$$cg(n) < f(n)$$

for sufficiently large n.

Remark

All the precise definitions are on the tutorial sheet. I only want to caution that $\forall, \exists$ are considered informal shorthands in mathematics (excluding logic and set theory), it is preferred to spell them out in formal writings.

Q2: P1, 2, 3

Q2 says that $O, \Omega, \Theta, o, \omega$ behave very much like $\leqslant, \geqslant, =, <, >$ respectively. We want to make this more precise. Recall that C denotes the set of functions $\mathbb{N} \to \mathbb{N}$. From the reflexivity, transitivity and symmetry parts of Q2, we get:

Theorem

For $f, g \in C$, we define $f \sim g$ iff $f(n) \in \Theta(g(n))$. Then $\sim$ is an equivalence relation. [1]

Now we recall the following result from lecture:

$$\Theta(g) = O(g) \cap \Omega(g).$$

This means that

$$f(n) \in \Theta(g(n)) \iff (\, f(n) \in O(g(n)) \quad \text{and} \quad f(n) \in \Omega(g(n))\,)\,.$$

[1] Note that in analysis, $\sim$ already has a meaning that is stronger than this; we do not consider that definition.

Theorem

For $[f], [g] \in C/\sim$, we define $[f] \leqslant [g]$ iff $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$. Then $\leqslant$ is a well-defined partial order on $C/\sim$.

Proof.

Well-definedness follows from lecture result and transitivity. Antisymmetry (if $[f] \leqslant [g], [g] \leqslant [f]$ then $[f] = [g]$) follows from lecture result. The rest follow from Q2. □

Theorem

If $f(n) \in o(g(n))$ i.e. $g(n) \in \omega(f(n))$, then $[f] < [g]$ (meaning, $[f] \leqslant [g]$ but $[f] \neq [g]$).

Proof.

If $f(n) \in o(g(n))$ and $f(n) \in \Theta(g(n))$, then there is some $c > 0$ such that

$$cg(n) \leqslant f(n) < cg(n)$$

for sufficiently large n, contradiction. □

Remark

It is not true that $[f] < [g] \implies f(n) \in o(g(n))$ *(and similarly for* ω*)*

However, it turns out that o itself induces naturally a strict partial order $<_o$ on $C/\sim$ which is 'more selective' than $<$ in the sense that $[f] <_o [g] \implies [f] < [g]$ but not vice versa.

Theorem

For $[f], [g] \in C/\sim$*, we define* $[f] <_o [g]$ *iff* $f(n) \in o(g(n))$ *iff* $g(n) \in \omega(f(n))$*. Then* $<_o$ *is a well-defined strict partial order on* $C/\sim$*.*

Remark

They are not total orders on $C/\sim$*.*

I move the proof to the appendix. Exercise: give counterexamples for the two remarks above. (Hint: consider g that 'oscillates')

Q1: P4

The conclusions here are very important tools for computations (as we will see in Q3-5). We recall the definition of limit:

Definition

We say that $\lim_{n\to\infty}\phi(n) = l$ if for all $\epsilon > 0$, there exists N such that

$$n \geqslant N \implies |\phi(n) - l| < \epsilon.$$

We say that $\lim_{n\to\infty}\phi(n) = \infty$ if for all M, there exists N such that

$$n \geqslant N \implies \phi(n) > M.$$

We would have someone presenting on the one for O. Then observe that by Q2, Ω follows from O, Θ follows from O and Ω, ω follows from o (make sure you understand the proof in lecture!)

Q3-5: P5, 6

We know from calculus/analysis that (c, d are positive constants)

$$[\log n] <_o [n^d] <_o [(1+c)^n] <_o [n!].$$

Some useful facts that follow immediately from definition:

- $[cf(n)] = [f(n)]$ for constant $c > 0$
- If $f(n)$ is a (finite) sum, then if a term of the sum has some other term $\geqslant$ it, it can be ignored.

Most if not all of Q3-5 can be done using limits, order properties and such basic facts if you do not want to directly use definition.

I am asked to discuss this 'hidden' variant of Q4: consider $2^{\log_4 n}$, is it in $O(n)$? $\Omega(n)$? $\Theta(\sqrt{n})$? $\omega(n)$?

Appendix

Theorem

For $[f], [g] \in C/\sim$, we define $[f] <_o [g]$ iff $f(n) \in o(g(n))$ iff $g(n) \in \omega(f(n))$. Then $<_o$ is a well-defined strict partial order on $C/\sim$.

Proof.

Well-definedness: if $[f_1] = [f_2]$ and $[f_2] <_o [g]$, then there exists $c_0 > 0$ such that for all $c > 0$, we have

$$f_1(n) \leqslant c_0 f_2(n) < c_0 \frac{c}{c_0} g(n) = cg(n)$$

for sufficiently large n, showing that $[f_1] <_o [g]$. Analogously, if $[g_1] = [g_2]$ and $[f] <_o [g_2]$, then $[f] <_o [g_1]$.
Irreflexivity: we have shown that if $[f] <_o [f]$ then $[f] \neq [f]$, contradiction.
Assymmetry: trivial exercise in CLRS
Transitivity: from Q2. □