CS3230 Tutorial 1

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Introduction

This is Tutorial Group 15 for CS3230.

My name is Deng Tianle:

- Year 3 Computer Science and Mathematics (DDP)
- First-time TA :)
- My JC: Raffles Institution (Y5-6)
- I was from Shenzhen, China

Admin

Everyone needs to present 3 times to obtain the 3% tutorial participation marks. (I believe that beyond the presentation, there is no obligation to attend tutorial. But of course, you are encouraged to attend).

$$\begin{aligned} & \text{We have 21 people} \\ \Longrightarrow\ & \text{63 presentations} \\ \Longrightarrow\ & \text{we should have around 6 presentations per class.} \end{aligned}$$

Remark

In later slides I use P to denote presentations, e.g. $P3$ means that the third presenter of the day will present on (possibly a part) of this question.

Agenda

This tutorial is about asymptotic notations: $O, \Omega, \Theta, o, \omega$.

- Analogy with $\leqslant, \geqslant, =, <, >$: Q2
- Computation using limit: Q1 (I rearranged because we can use Q2 here)
- Practical computation, relation between common functions like log, polynomial, exp, factorial: Q3-5
- LeetCode question on removing duplicates (if time permits; I will run the algorithm on the board to show the idea)

Let $\mathbb{N}$ denote the set of positive integers. We consider functions $f : \mathbb{N} \to \mathbb{N}$. Let the class of all such functions be C.

Definition

Let f and g be functions $\mathbb{N} \to \mathbb{N}$. We say that $f(n) \in \Theta(g(n))$ if there exist positive constants c_1, c_2 such that

$$c_1 g(n) \leqslant f(n) \leqslant c_2 g(n)$$

for sufficiently large n. We say that $f(n) \in \omega(g(n))$ if for all positive constants c, we have

$$cg(n) < f(n)$$

for sufficiently large n.

Remark

All the precise definitions are on the tutorial sheet. I only want to caution that $\forall, \exists$ are considered informal shorthands in mathematics (excluding logic and set theory), it is preferred to spell them out in formal writings.

Q2: P1, 2, 3

Q2 says that $O, \Omega, \Theta, o, \omega$ behave very much like $\leqslant, \geqslant, =, <, >$ respectively. We want to make this more precise. Recall that C denotes the set of functions $\mathbb{N} \to \mathbb{N}$. From the reflexivity, transitivity and symmetry parts of Q2, we get:

Theorem

For $f, g \in C$, we define $f \sim g$ iff $f(n) \in \Theta(g(n))$. Then $\sim$ is an equivalence relation. [1]

Now we recall the following result from lecture:

$$\Theta(g) = O(g) \cap \Omega(g).$$

This means that

$$f(n) \in \Theta(g(n)) \iff \left(f(n) \in O(g(n)) \quad \text{and} \quad f(n) \in \Omega(g(n)) \right).$$

[1] Note that in analysis, $\sim$ already has a meaning that is stronger than this; we do not consider that definition.

Theorem

For $[f], [g] \in C/\sim$, we define $[f] \leqslant [g]$ iff $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$. Then $\leqslant$ is a well-defined partial order on $C/\sim$.

Proof.

Well-definedness follows from lecture result and transitivity. Antisymmetry (if $[f] \leqslant [g], [g] \leqslant [f]$ then $[f] = [g]$) follows from lecture result. The rest follow from Q2. □

Theorem

If $f(n) \in o(g(n))$ i.e. $g(n) \in \omega(f(n))$, then $[f] < [g]$ (meaning, $[f] \leqslant [g]$ but $[f] \neq [g]$).

Proof.

If $f(n) \in o(g(n))$ and $f(n) \in \Theta(g(n))$, then there is some $c > 0$ such that

$$cg(n) \leqslant f(n) < cg(n)$$

for sufficiently large n, contradiction. □

Remark

It is not true that $[f] < [g] \implies f(n) \in o(g(n))$ *(and similarly for* ω*)*

However, it turns out that o itself induces naturally a strict partial order $<_o$ on $C/\sim$ which is 'more selective' than $<$ in the sense that $[f] <_o [g] \implies [f] < [g]$ but not vice versa.

Theorem

For $[f], [g] \in C/\sim$*, we define* $[f] <_o [g]$ *iff* $f(n) \in o(g(n))$ *iff* $g(n) \in \omega(f(n))$*. Then* $<_o$ *is a well-defined strict partial order on* $C/\sim$*.*

Remark

They are not total orders on $C/\sim$*.*

I move the proof to the appendix. Exercise: give counterexamples for the two remarks above. (Hint: consider g that 'oscillates')

Q1: P4

The conclusions here are very important tools for computations (as we will see in Q3-5). We recall the definition of limit:

Definition

We say that $\lim_{n\to\infty} \phi(n) = l$ if for all $\epsilon > 0$, there exists N such that

$$n \geqslant N \implies |\phi(n) - l| < \epsilon.$$

We say that $\lim_{n\to\infty} \phi(n) = \infty$ if for all M, there exists N such that

$$n \geqslant N \implies \phi(n) > M.$$

We would have someone presenting on the one for O. Then observe that by Q2, Ω follows from O, Θ follows from O and Ω, ω follows from o (make sure you understand the proof in lecture!)

Q3-5: P5, 6

We know from calculus/analysis that (c, d are positive constants)

$$[\log n] <_o [n^d] <_o [(1+c)^n] <_o [n!].$$

Some useful facts that follow immediately from definition:

- $[cf(n)] = [f(n)]$ for constant $c > 0$
- If $f(n)$ is a (finite) sum, then if a term of the sum has some other term $\geqslant$ it, it can be ignored.

Most if not all of Q3-5 can be done using limits, order properties and such basic facts if you do not want to directly use definition.

I am asked to discuss this 'hidden' variant of Q4: consider $2^{\log_4 n}$, is it in $O(n)$? $\Omega(n)$? $\Theta(\sqrt{n})$? $\omega(n)$?

Appendix

Theorem

For $[f], [g] \in C/\sim$, we define $[f] <_o [g]$ iff $f(n) \in o(g(n))$ iff $g(n) \in \omega(f(n))$. Then $<_o$ is a well-defined strict partial order on $C/\sim$.

Proof.

Well-definedness: if $[f_1] = [f_2]$ and $[f_2] <_o [g]$, then there exists $c_0 > 0$ such that for all $c > 0$, we have

$$f_1(n) \leqslant c_0 f_2(n) < c_0 \frac{c}{c_0} g(n) = cg(n)$$

for sufficiently large n, showing that $[f_1] <_o [g]$. Analogously, if $[g_1] = [g_2]$ and $[f] <_o [g_2]$, then $[f] <_o [g_1]$.
Irreflexivity: we have shown that if $[f] <_o [f]$ then $[f] \neq [f]$, contradiction.
Assymmetry: trivial exercise in CLRS
Transitivity: from Q2. $\square$

Q3

- True: $3^{n+1} = 3 \cdot 3^n \in \Theta(3^n)$ because we are just multiplying by constant. In particular, it is in $O(3^n)$. (If you want to show by definition, take $c = 3$ and it goes through for all n).
- False. I will explain this in two ways (there were some questions relating to this part, please feel free to clarify with me further):
  - Use the theorems on analogy with orders that we have established (this allows you to see immediately whether it is true, and is good for MCQs): by limit, $[2^n] <_o [4^n]$, so $[2^n] < [4^n]$ and so it is not true that $[2^n] \geqslant [4^n]$ i.e. $4^n \in O(2^n)$.
  - Alternatively, you can use definition (this is good if you know it is false already and is asked to prove it in exam): to prove negation of $4^n \in O(2^n)$, you want to show that for all $c > 0$ and for all n_0, there is some $n \geqslant n_0$ such that

    $$2^n \cdot 2^n = 4^n > c \cdot 2^n.$$

    This is indeed true because for large enough n we always have $2^n > c$.

Q3

- True: We have

$$\frac{1}{2}n \leqslant 2^{\log n - 1} \leqslant 2^{\lfloor \log n \rfloor} \leqslant 2^{\log n} = n$$

  Hence $2^{\lfloor \log n \rfloor} \in O(n)$ (take $c = 1$ if you want to show using definition) and $2^{\lfloor \log n \rfloor} \in \Omega(n)$ (take $c = 1/2$)
- True: From binomial expansion of $(n + a)^i$, the dominant term is n^i (all lower n powers can be ignored), so this is in $\Theta(n^i)$.

Q4

I will only discuss the hidden question. We have $2^{\log_4 n} = \sqrt{n}$ (there are many ways to see this depending on your high school background, e.g. use $\log_{2^2} \sqrt{n}^2 = \log \sqrt{n}$, or change of base $log_4 n = \frac{\log n}{\log 4} = \frac{\log n}{2}$)

Then for the same options shown, it is in $\Theta(\sqrt{n})$ and $O(n)$ but not in $\Omega(n)$ and hence not in $\omega(n)$ (all can be seen from $\sqrt{n} <_o n$)

Q5

As explained in class, $\log(n^2) = 2\log(n)$ so $[\log(n^2)] = [\log(n)]$. The rest follow from the facts shown in the slides before Appendix.

Final answer:

$$[f_1] = [f_5] <_o [f_4] <_o [f_3] <_o [f_2].$$

For $[n!] >_o a^n$, one way to see this is that

$$n! \geqslant n(n-1)\ldots(n/2) \geqslant (n/2)^{n/2}$$

(no need to be too careful but odd/even case of n since we can multiply $n!$ by constant anyway) and $\sqrt{n/2}$ exceeds a for large n.

Q5

$$T(n) = 4T(n/2) + \sqrt{n}$$

Step 1: Guess the answer (without using Master theorem, because it may not always apply). Need some experience/intuition/luck, not 100% methodological

Possibility 1: try substitution: $T(n) = cn$, RHS is $2cn + \sqrt{n}$, it seems that LHS might be too small. Next reasonable guess is n^2 and you happily observe that n^2 satisfies $f(n) = 4f(n/2)$.

Possibility 2: try dropping the $\sqrt{n}$ term, then observe that

$$f(n) = 4f(n/2) = \cdots = 4^{\log n} f(1) = f(1)n^2.$$

In both cases it is then easy to show that $T(n) \geqslant cn^2$ and hence $T(n) \in \Omega(n^2)$.

Q5

$$T(n) = 4T(n/2) + \sqrt{n}$$

Step 2: prove bounds (in which case, the hard case is the upper bound).

Most reasonable to try substituting $T(n) = An^2 + B\sqrt{n}$. Now you can reverse engineer to make induction work: $T(1) = A + B$,

$$\begin{aligned} T(n) &= 4T(n/2) + \sqrt{n} \\ &\leqslant 4\left(An^2/4 + B\sqrt{n/2}\right) + \sqrt{n} \\ &= An^2 + (4B/\sqrt{2} + 1)\sqrt{n} \\ &\leqslant An^2 + B\sqrt{n} \end{aligned}$$

where we want the last inequality to hold. We can solve for $B = -\sqrt{2}/(4 - \sqrt{2})$ and take A accordingly. Hence $T(n) \in O(n^2)$ and we are done.

Q5

Question: why is it that lower bound is cn^2 and upper bound is $An^2 + B\sqrt{n}$ for **negative** B?

The 'paradox' resolves when you realise that for $f(n) = 4f(n/2)$ such that $c := f(1) = T(1)$ we have

$$f(n) = cn^2$$

and

$$T(n) = \left(c + \frac{\sqrt{2}}{4-\sqrt{2}}\right)n^2 - \frac{\sqrt{2}}{4-\sqrt{2}}\sqrt{n}$$

(basically same induction but we actually have equalities).

Q6

Alternative method:

$$T(k,n) = 2T(k/2,n) + \Theta(nk)$$

$$\frac{T(k,n)}{k} = \frac{T(k/2,n)}{k/2} + \frac{\Theta(nk)}{k}$$

by telescoping or pushing $\log k$ times we get

$$\frac{T(k,n)}{k} = T(1,n) + \frac{\log k\Theta(nk)}{k}.$$

Same answer, $T(k,n) \in \Theta(nk \log k)$.

$\nVdash$