

# CS3230 Tutorial 1

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# Introduction

This is Tutorial Group 15 for CS3230.

My name is Deng Tianle:

- ▶ Year 3 Computer Science and Mathematics (DDP)
- ▶ First-time TA :)
- ▶ My JC: Raffles Institution (Y5-6)
- ▶ I was from Shenzhen, China

# Admin

Everyone needs to present 3 times to obtain the 3% tutorial participation marks. (I believe that beyond the presentation, there is no obligation to attend tutorial. But of course, you are encouraged to attend).

We have 21 people

⇒ 63 presentations

⇒ we should have around 6 presentations per class.

## Remark

*In later slides I use  $P$  to denote presentations, e.g.  $P3$  means that the third presenter of the day will present on (possibly a part) of this question.*

# Agenda

This tutorial is about asymptotic notations:  $O, \Omega, \Theta, o, \omega$ .

- ▶ Analogy with  $\leq, \geq, =, <, >$ : Q2
- ▶ Computation using limit: Q1 (I rearranged because we can use Q2 here)
- ▶ Practical computation, relation between common functions like log, polynomial, exp, factorial: Q3-5
- ▶ LeetCode question on removing duplicates (if time permits; I will run the algorithm on the board to show the idea)

Let  $\mathbb{N}$  denote the set of positive integers. We consider functions  $f : \mathbb{N} \rightarrow \mathbb{N}$ . Let the class of all such functions be  $\mathcal{C}$ .

### Definition

Let  $f$  and  $g$  be functions  $\mathbb{N} \rightarrow \mathbb{N}$ . We say that  $f(n) \in \Theta(g(n))$  if there exist positive constants  $c_1, c_2$  such that

$$c_1 g(n) \leq f(n) \leq c_2 g(n)$$

for sufficiently large  $n$ . We say that  $f(n) \in \omega(g(n))$  if for all positive constants  $c$ , we have

$$c g(n) < f(n)$$

for sufficiently large  $n$ .

### Remark

*All the precise definitions are on the tutorial sheet. I only want to caution that  $\forall, \exists$  are considered informal shorthands in mathematics (excluding logic and set theory), it is preferred to spell them out in formal writings.*

## Q2: P1, 2, 3

Q2 says that  $O, \Omega, \Theta, o, \omega$  behave very much like  $\leq, \geq, =, <, >$  respectively. We want to make this more precise. Recall that  $C$  denotes the set of functions  $\mathbb{N} \rightarrow \mathbb{N}$ . From the reflexivity, transitivity and symmetry parts of Q2, we get:

### Theorem

*For  $f, g \in C$ , we define  $f \sim g$  iff  $f(n) \in \Theta(g(n))$ . Then  $\sim$  is an equivalence relation.*<sup>1</sup>

Now we recall the following result from lecture:

$$\Theta(g) = O(g) \cap \Omega(g).$$

This means that

$$f(n) \in \Theta(g(n)) \iff (f(n) \in O(g(n)) \text{ and } f(n) \in \Omega(g(n))).$$

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<sup>1</sup>Note that in analysis,  $\sim$  already has a meaning that is stronger than this; we do not consider that definition.

## Theorem

For  $[f], [g] \in C/\sim$ , we define  $[f] \leq [g]$  iff  $f(n) \in O(g(n))$  iff  $g(n) \in \Omega(f(n))$ . Then  $\leq$  is a well-defined partial order on  $C/\sim$ .

## Proof.

Well-definedness follows from lecture result and transitivity.

Antisymmetry (if  $[f] \leq [g], [g] \leq [f]$  then  $[f] = [g]$ ) follows from lecture result. The rest follow from Q2. □

## Theorem

If  $f(n) \in o(g(n))$  i.e.  $g(n) \in \omega(f(n))$ , then  $[f] < [g]$  (meaning,  $[f] \leq [g]$  but  $[f] \neq [g]$ ).

## Proof.

If  $f(n) \in o(g(n))$  and  $f(n) \in \Theta(g(n))$ , then there is some  $c > 0$  such that

$$cg(n) \leq f(n) < cg(n)$$

for sufficiently large  $n$ , contradiction. □

## Remark

*It is not true that  $[f] < [g] \implies f(n) \in o(g(n))$  (and similarly for  $\omega$ )*

However, it turns out that  $o$  itself induces naturally a strict partial order  $<_o$  on  $C/\sim$  which is 'more selective' than  $<$  in the sense that  $[f] <_o [g] \implies [f] < [g]$  but not vice versa.

## Theorem

*For  $[f], [g] \in C/\sim$ , we define  $[f] <_o [g]$  iff  $f(n) \in o(g(n))$  iff  $g(n) \in \omega(f(n))$ . Then  $<_o$  is a well-defined strict partial order on  $C/\sim$ .*

## Remark

*They are not total orders on  $C/\sim$ .*

I move the proof to the appendix. Exercise: give counterexamples for the two remarks above. (Hint: consider  $g$  that 'oscillates')



## Q1: P4

The conclusions here are very important tools for computations (as we will see in Q3-5). We recall the definition of limit:

### Definition

We say that  $\lim_{n \rightarrow \infty} \phi(n) = l$  if for all  $\epsilon > 0$ , there exists  $N$  such that

$$n \geq N \implies |\phi(n) - l| < \epsilon.$$

We say that  $\lim_{n \rightarrow \infty} \phi(n) = \infty$  if for all  $M$ , there exists  $N$  such that

$$n \geq N \implies \phi(n) > M.$$

We would have someone presenting on the one for  $O$ . Then observe that by Q2,  $\Omega$  follows from  $O$ ,  $\Theta$  follows from  $O$  and  $\Omega$ ,  $\omega$  follows from  $o$  (make sure you understand the proof in lecture!)

## Q3-5: P5, 6

We know from calculus/analysis that ( $c, d$  are positive constants)

$$[\log n] <_o [n^d] <_o [(1+c)^n] <_o [n!].$$

Some useful facts that follow immediately from definition:

- ▶  $[cf(n)] = [f(n)]$  for constant  $c > 0$
- ▶ If  $f(n)$  is a (finite) sum, then if a term of the sum has some other term  $\geq$  it, it can be ignored.

Most if not all of Q3-5 can be done using limits, order properties and such basic facts if you do not want to directly use definition.

I am asked to discuss this 'hidden' variant of Q4: consider  $2^{\log_4 n}$ , is it in  $O(n)$ ?  $\Omega(n)$ ?  $\Theta(\sqrt{n})$ ?  $\omega(n)$ ?

# Appendix

## Theorem

For  $[f], [g] \in C/\sim$ , we define  $[f] <_o [g]$  iff  $f(n) \in o(g(n))$  iff  $g(n) \in \omega(f(n))$ . Then  $<_o$  is a well-defined strict partial order on  $C/\sim$ .

## Proof.

Well-definedness: if  $[f_1] = [f_2]$  and  $[f_2] <_o [g]$ , then there exists  $c_0 > 0$  such that for all  $c > 0$ , we have

$$f_1(n) \leq c_0 f_2(n) < c_0 \frac{c}{c_0} g(n) = cg(n)$$

for sufficiently large  $n$ , showing that  $[f_1] <_o [g]$ . Analogously, if  $[g_1] = [g_2]$  and  $[f] <_o [g_2]$ , then  $[f] <_o [g_1]$ .

Irreflexivity: we have shown that if  $[f] <_o [f]$  then  $[f] \neq [f]$ , contradiction.

Assymetry: trivial exercise in CLRS

Transitivity: from Q2.



### Q3

- ▶ True:  $3^{n+1} = 3 \cdot 3^n \in \Theta(3^n)$  because we are just multiplying by constant. In particular, it is in  $O(3^n)$ . (If you want to show by definition, take  $c = 3$  and it goes through for all  $n$ ).
- ▶ False. I will explain this in two ways (there were some questions relating to this part, please feel free to clarify with me further):
  - ▶ Use the theorems on analogy with orders that we have established (this allows you to see immediately whether it is true, and is good for MCQs): by limit,  $[2^n] <_o [4^n]$ , so  $[2^n] < [4^n]$  and so it is not true that  $[2^n] \geq [4^n]$  i.e.  $4^n \in O(2^n)$ .
  - ▶ Alternatively, you can use definition (this is good if you know it is false already and is asked to prove it in exam): to prove negation of  $4^n \in O(2^n)$ , you want to show that for all  $c > 0$  and for all  $n_0$ , there is some  $n \geq n_0$  such that

$$2^n \cdot 2^n = 4^n > c \cdot 2^n.$$

This is indeed true because for large enough  $n$  we always have  $2^n > c$ .

## Q3

- True: We have

$$\frac{1}{2}n \leq 2^{\log n - 1} \leq 2^{\lfloor \log n \rfloor} \leq 2^{\log n} = n$$

Hence  $2^{\lfloor \log n \rfloor} \in O(n)$  (take  $c = 1$  if you want to show using definition) and  $2^{\lfloor \log n \rfloor} \in \Omega(n)$  (take  $c = 1/2$ )

- True: From binomial expansion of  $(n+a)^i$ , the dominant term is  $n^i$  (all lower  $n$  powers can be ignored), so this is in  $\Theta(n^i)$ .

## Q4

I will only discuss the hidden question. We have  $2^{\log_4 n} = \sqrt{n}$  (there are many ways to see this depending on your high school background, e.g. use  $\log_{2^2} \sqrt{n^2} = \log \sqrt{n}$ , or change of base  $\log_4 n = \frac{\log n}{\log 4} = \frac{\log n}{2}$ )

Then for the same options shown, it is in  $\Theta(\sqrt{n})$  and  $O(n)$  but not in  $\Omega(n)$  and hence not in  $\omega(n)$  (all can be seen from  $\sqrt{n} <_o n$ )

## Q5

As explained in class,  $\log(n^2) = 2 \log(n)$  so  $\lceil \log(n^2) \rceil = \lceil \log(n) \rceil$ .  
The rest follow from the facts shown in the slides before Appendix.

Final answer:

$$\lceil f_1 \rceil = \lceil f_5 \rceil <_o \lceil f_4 \rceil <_o \lceil f_3 \rceil <_o \lceil f_2 \rceil.$$

For  $\lceil n! \rceil >_o a^n$ , one way to see this is that

$$n! \geq n(n-1) \dots (n/2) \geq (n/2)^{n/2}$$

(no need to be too careful but odd/even case of  $n$  since we can multiply  $n!$  by constant anyway) and  $\sqrt{n/2}$  exceeds  $a$  for large  $n$ .