Gen IMS II Lecture Note

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1 ODEs

1.1 Spring-Peudulem

Given the ODE:

$$Mx'' + Kx = 0$$

We have a ansaz:

$$x(t) = Acos(\omega t) + Bsin(\omega t)$$

Plugin:

$$M(-A\omega^2 cos(\omega t) - B\omega^2 sin(\omega t)) + K(Acos(\omega t) + Bsin(\omega t)) = 0$$

Solving and grouping:

$$\begin{aligned} -MA\omega^2 + kA &= 0 \\ -MB\omega^2 + kB &= 0 \\ \omega^2 &= \frac{K}{M} \end{aligned}$$

Other way to solve for ODE by using **Laplace transform**

$$\begin{aligned} \mathcal{L}(x'') &= s^2 X(s) - sx_0 - x_0' \\ \mathcal{L}(x') &= sX(s) - x_0 \end{aligned}$$

Using the above formula, we have

$$\begin{aligned} Ms^2 X(s) - Msx_0 - Mx_0' + KX(s) = 0 \\ X(s) = \frac{sx_0 + x_0'}{s^2 + K/M} \end{aligned}$$

However, we need to consider the Laplace transform of Trignometric function: $\mathcal{L}(cos(\omega t)) =?$ and $\mathcal{L}(sin(\omega t)) =?$
Using some trick:

$$\begin{aligned} \mathcal{L}(cos''(\omega t)) &= s^2 \mathcal{L}(cos(\omega t)) - s \\ \mathcal{L}(cos''(\omega t)) &= -\omega^2 \mathcal{L}(cos(\omega t)) \end{aligned}$$

Solve for Laplace term

$$\begin{aligned} \mathcal{L}(cos(\omega t)) &= \frac{s}{s^2 + \omega^2} \\ \mathcal{L}(sin(\omega t)) &= \frac{\omega}{s^2 + \omega^2} \end{aligned}$$

Using the Laplace transform, we can solve the ODE easily.

$$x(t) = x_0 cos(\omega t) + \frac{x_0'}{\omega} sin(\omega t)$$

1.2 Complex Number

We write a complex number as $z = a + ib$
Linear rule: $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$
Product rule: $z_1 \cdot z_2 = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$
Euler's representation: $e^x = cos(x) + isin(x)$
Using Power Series, we can express the following

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$cos(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + ...$$
$$sin(x) = x - \frac{1}{6}x^3 + ...$$

Euler's formula: $z = re^{i\theta}$

2 Transfer Function

Having a system equation:

$$a_2\ddot{x} + a_1\dot{x} + a_0x = b_2\ddot{r} + b_1\dot{r} + b_0$$

Solving using Laplace:

$$(a_2s^2 + a_1s + a_0)X(s) = (s_2s^2 + b_1s + b_0)R(s) + \text{ initial value term}$$

We ignore the "initial value term", we have

$$X(s) = \frac{b_2s^2 + b_1s + b_0}{a_2s^2 + a_1s + a_0} \cdot R(s) = T(s) \cdot R(s)$$

If you have two transfer function, thus we have

$$Y = T_2(s) \cdot X = T_2(s) \cdot T_1(s) \cdot R(s)$$

2.1 Model in Frequency domain

Using a model with one resistor and one capacitor.
Mode current:

$$V_S(s)$$

Voltage:

$$V_S(s) = R \cdot I(s)$$
$$V_C(s) = \frac{1}{Cs} \cdot I(s)$$

Remember:

$$I_C(t) = \dot{Q} = C \cdot \dot{V}_C$$
$$I_C(s) = C \cdot s \cdot V_C(s)$$
$$-V_S + V_R + V_C = 0$$
$$V_S(s) = (R + \frac{1}{CS})I(s)$$

With two of such system, we have

$$(R + \frac{1}{Cs})I_1(s) - \frac{1}{Cs}I_2(s) = Vs$$
$$-\frac{1}{Cs}I_1(s) + (R + \frac{2}{Cs}I_2(s)) = 0$$

Write all the term in matrices and vectors

$$M \cdot \vec{I} = \vec{V} \implies$$
$$\vec{I} = M^-1 \cdot \vec{V}$$

2.2 Op-Amps

Use such device to amptify the voltage. Using a feedback, we have the relationship like:

$$V_{in} = \frac{V_{in} - V_-}{R_1} = \frac{V_- - V_{out}}{R_f}$$

$$V_{out} = (1 + \frac{R_f}{R_1})V_- - \frac{R_f}{R_1}V_{in}$$

$$V_- = V_+ + \epsilon$$

if we have feedback, using virtual ground, the effective gain is.

$$V_{out} = -\frac{R_f}{R_1}V_{in}$$