pca

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1 Homework 5 Machine Learning

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First, we try to load the data into python

```
In [1]: import numpy as np
        import pandas as pd
        import matplotlib.pyplot as plt
        from numpy import linalg as la

In [2]: data = pd.read_csv("data/mfeat-pix.txt",sep='\s+', header=None)

In [3]: darr = data.values.astype(float)

In [4]: def img_cat(darr):
            """
            reshape the image and show the image
            """
            img_mat = darr.reshape(16, 15) # reshape the d array
            plt.imshow(img_mat, cmap='gray')
            plt.show()
        def imgs_cat(darr):
            for rows in darr:
                img_cat(rows)

In [5]: img_threes = darr[600:800] #select class 1
```

2 PCA algorithms

2.1 Steps in PCA algorithms

- centering the data: mapping each vector $x_i \to \bar{x}_i$, where $\bar{x}_i = x_i - \bar{x}$
- calculating the $\mu_1, \mu_2, ...\mu_m$, where m is the number of principle components μ_i is calculated via SVD algorithm.

1. We calculate the correlation matrix R.

2. Calculate the SVD via

$$[U, \Sigma, V^*] = SVD(R)$$

, where

$$R = U\Sigma V^*$$

3. Extract the principle components $\mu_1, \mu_2, ...\mu_m$, where \$ m < n\$, by taking the first m columns of U.

- Compression: we take each $\mu_1, \mu_2, \mu_3 \dots$ and dot product the already centered $\bar{x}_i$ to obtain a vector $v \in \mathbb{R}^m$

$$v = \begin{bmatrix} \mu_1 \cdot \bar{x}_i \\ \mu_2 \cdot \bar{x}_i \\ \mu_3 \cdot \bar{x}_i \\ \mu_4 \cdot \bar{x}_i \\ ... \\ \mu_m \cdot \bar{x}_i \end{bmatrix}$$

- Decompression: we simply calculate

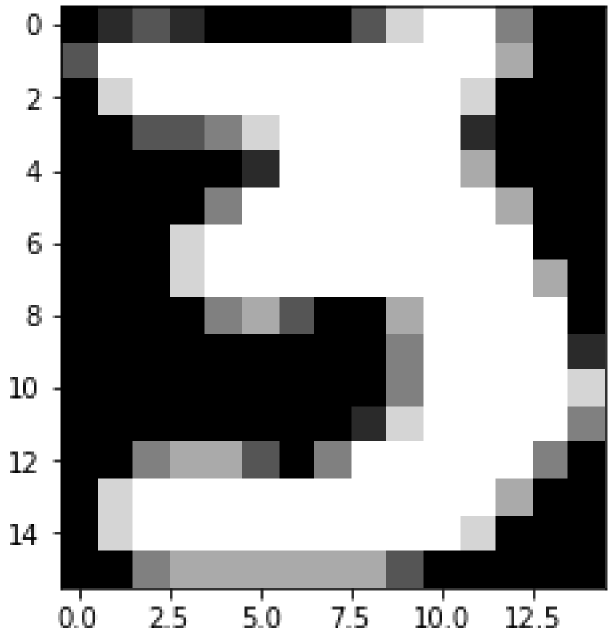
$$recovered = \sum_{i=1}^{m} v_i \cdot \mu_i$$

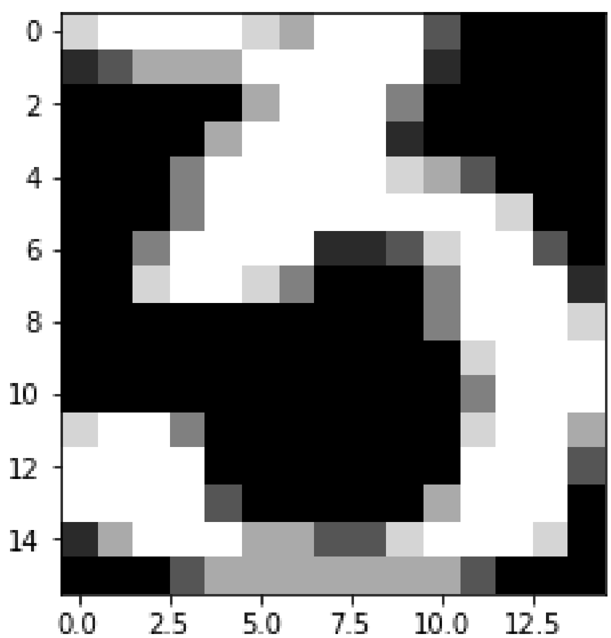
2.1.1 Concrete implementation of PCA algorithm

```
In [6]: def distance(p1, p2):
            diff = p1 - p2
            s = np.sum(np.power(diff, 2))
            return np.sqrt(s)
```

Let's first take a look at what is in the picture.

```
In [7]: imgs_cat(img_threes[0:2]) # let's take a look into the picture
```





We will try to normalize the data to the center by calculating the mean value and minus every vectors in the training set by the mean value.

```
In [8]: def normalize(data):
            # normalize the data set to perform the algorithm
```

```
        mean = np.sum(data, axis = 0)
        N, _ = data.shape
        mean = mean / float(N)
        mean_mat = [mean] * N
        print(mean_mat)
        normalized = data - mean_mat
        return normalized, mean
```

```
In [9]: def denormalize(normalized, mean):
            # recover from the normalization
            N, _ = normalized.shape
            mean_mat = [mean] * N
            data = normalized + mean_mat
            return data
```

```
In [10]: %%capture
         normalized_threes, mean_threes = normalize(img_threes) # first normalize the data

         # calculate the correlation matrix
         R = np.corrcoef(normalized_threes.T)

         # compute the SVD algorithm
         U, s, Vt = la.svd(R)
```

We have two functions that compress and decompress the data respectively.

```
In [11]: def compress(normalized, pc, k):
             # compress the normalized data using pc, the princle component
             pc = pc[:, :k]
             return np.matmul(normalized,pc)

         def decompress(compressed, pc, k):
             # recover from compression
             pc = pc[:, :k]
             return np.matmul(compressed,pc.T)
```

```
In [12]: def pca_test(normalized, pc, k):
             return decompress(compress(normalized, pc, k), pc, k)
```

The variance error can be calculated using Σ returned by the SVD algorithm. To be specific, the error term

$$E_m = \frac{\sum_{j=m+1} n\sigma_j^2}{\sum_{j=1}^{n} \sigma_j^2}$$

```
In [13]: def variance_error(sigma,k):
             # for a given sigma vector and k component, calculate the error term
             return np.sum(sigma[k:]) / np.sum(sigma)
```

The expected m, or k mentioned in the question can be quickly determined if given a preserved variance p.

$$m_{expected} = \min_{k \leq m}\{k \mid (1 - E_k) \geq q\}$$

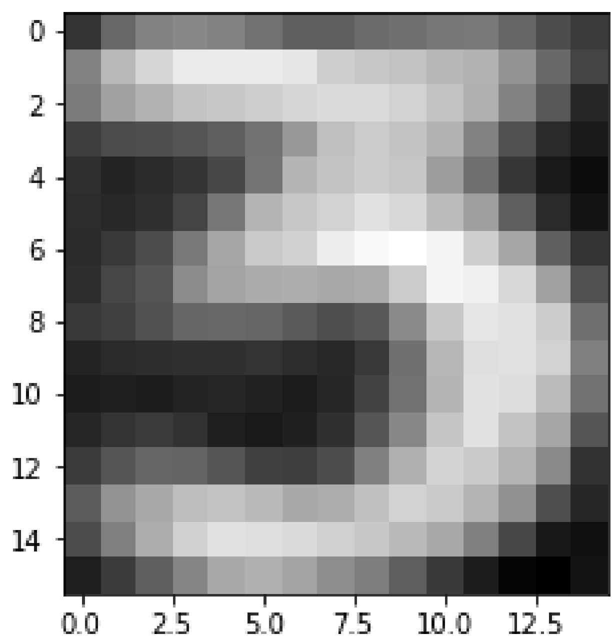
```
In [14]: def pick_k(sigma, percentage):
             for k in range(1, 240):
                 if variance_error(sigma, k) < (1.0 - percentage):
                     print('k = {} obtains a variance of {}'.format(k, 1.0 - variance_error(si
                     return k
```

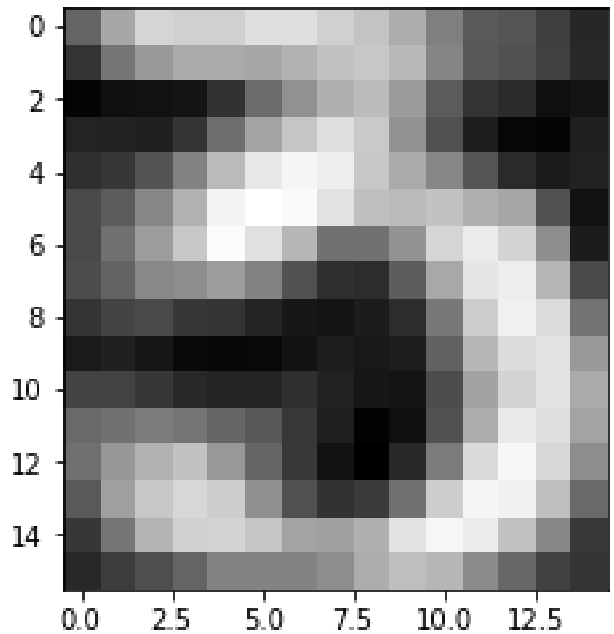
2.1.2 With preseverd variance of 50%

Using the technique above, $m_{expected} = 8$. As you can see below, the image is pretty blurry.

```
In [15]: imgs_cat(denormalize(pca_test(normalized_threes, U, pick_k(s, 0.5)), mean_threes)[:5]
```

```
k = 8 obtains a variance of 0.5293816034413219
```

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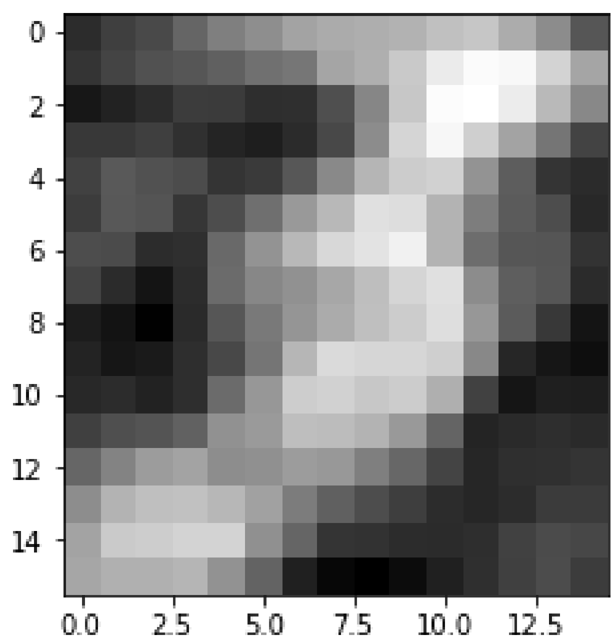
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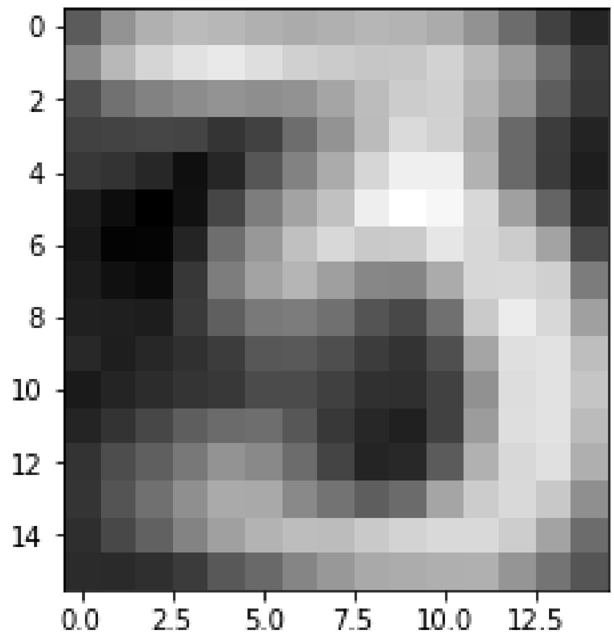
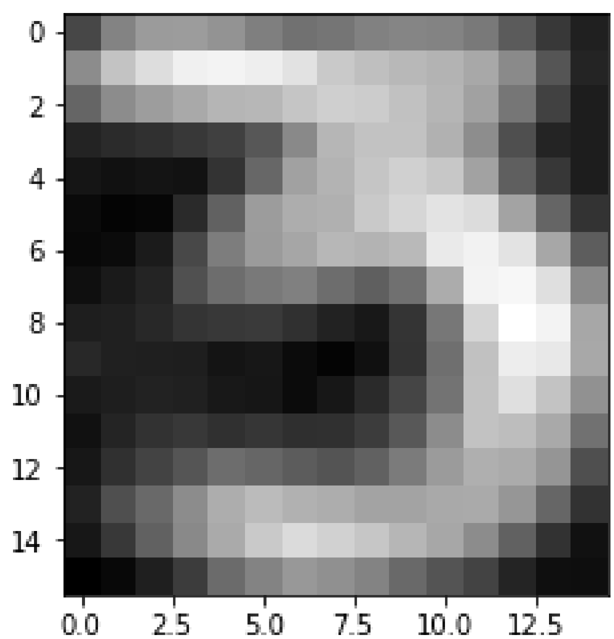


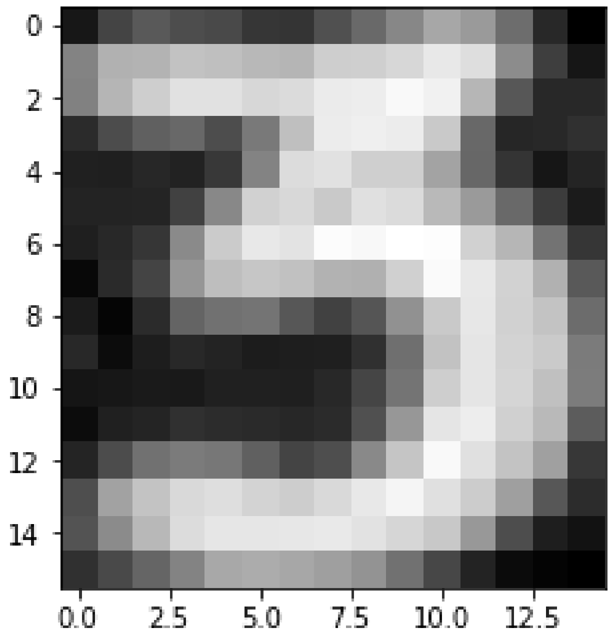


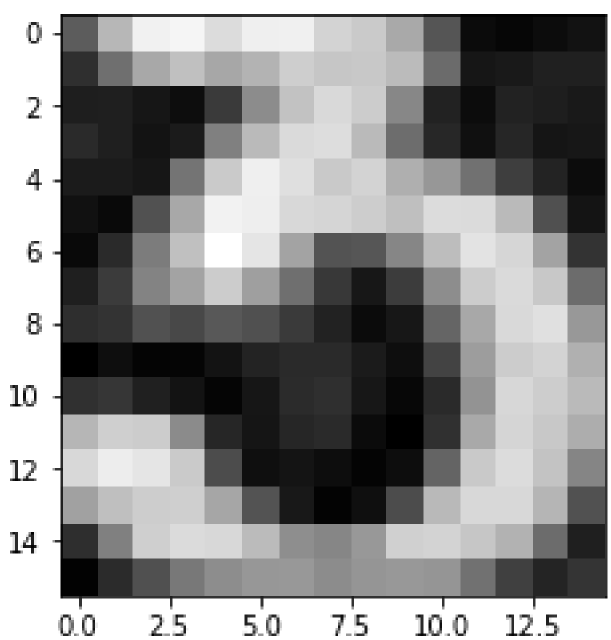
2.1.3 With preseverd variance of 80%

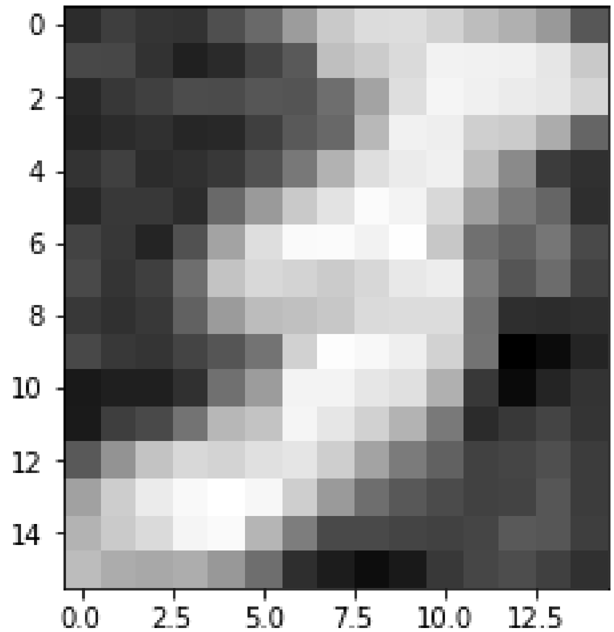
Using the technique above, $m_{expected} = 27$. As you can see below, the image is less blurry.

```
In [16]: imgs_cat(denormalize(pca_test(normalized_threes, U, pick_k(s, 0.8)), mean_threes)[:5]
```

```
k = 27 obtains a variance of 0.8063078690150481
```



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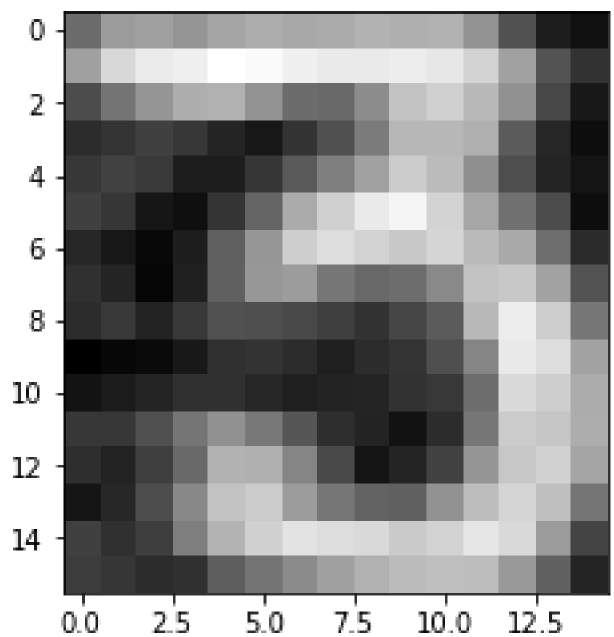
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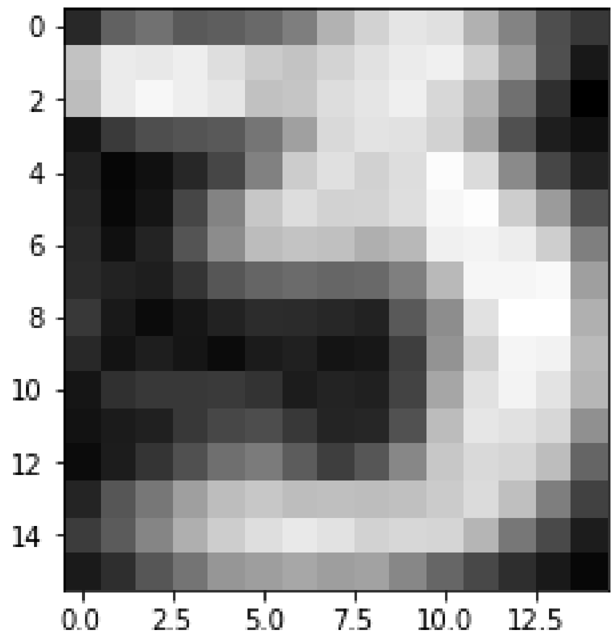
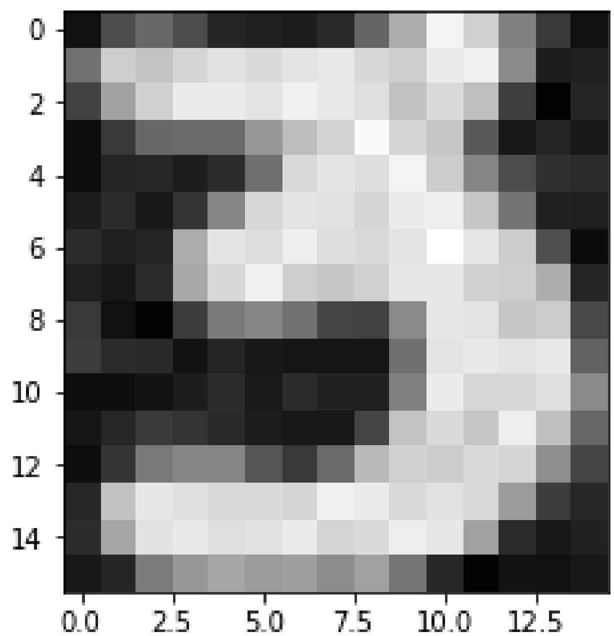


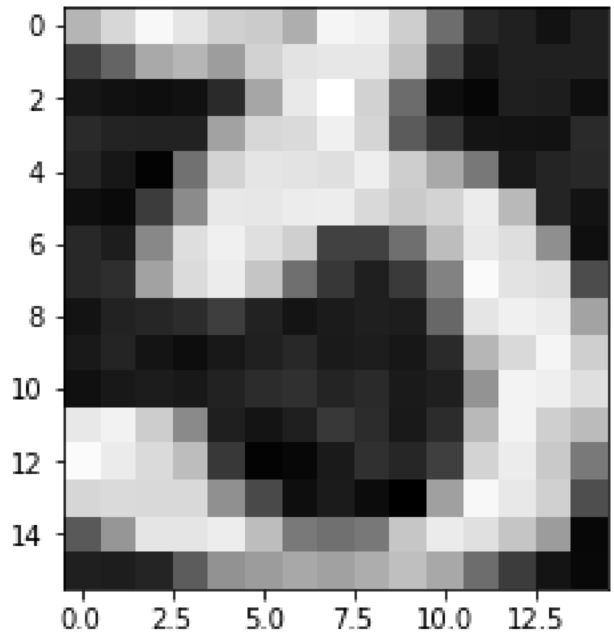
2.1.4 With preseverd variance of 95%

Using the technique above, $m_{expected} = 71$. As you can see below, the image is closer to the original.

```
In [17]: imgs_cat(denormalize(pca_test(normalized_threes, U, pick_k(s, 0.95)), mean_threes)[:5
```

```
k = 71 obtains a variance of 0.9509743872148163
```

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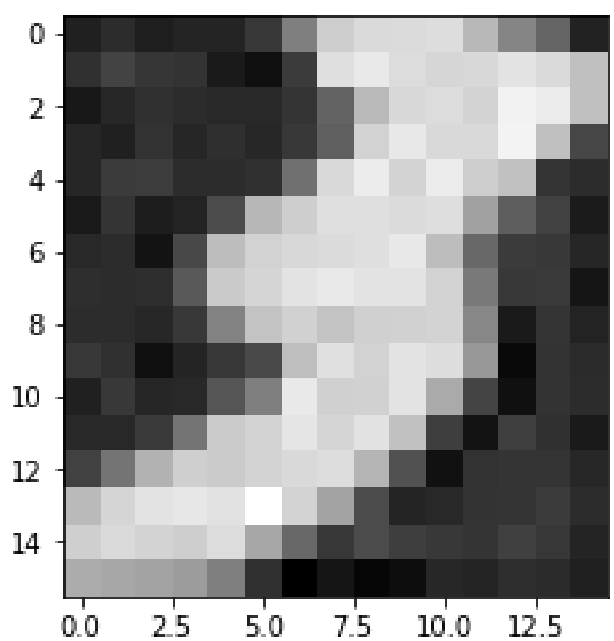
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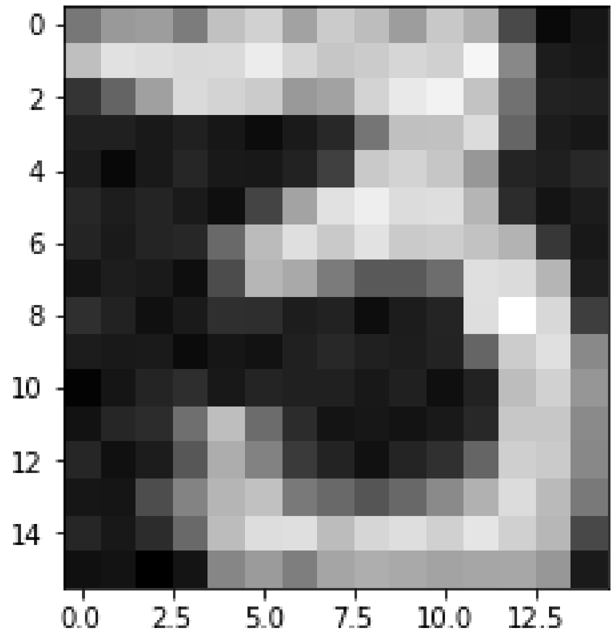
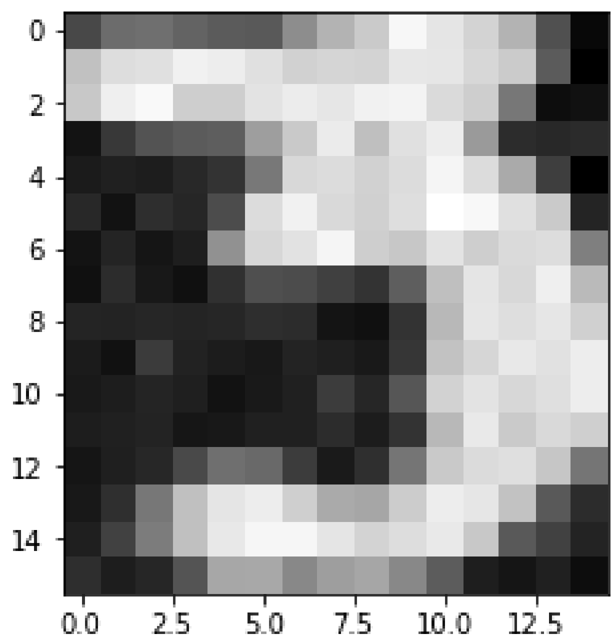


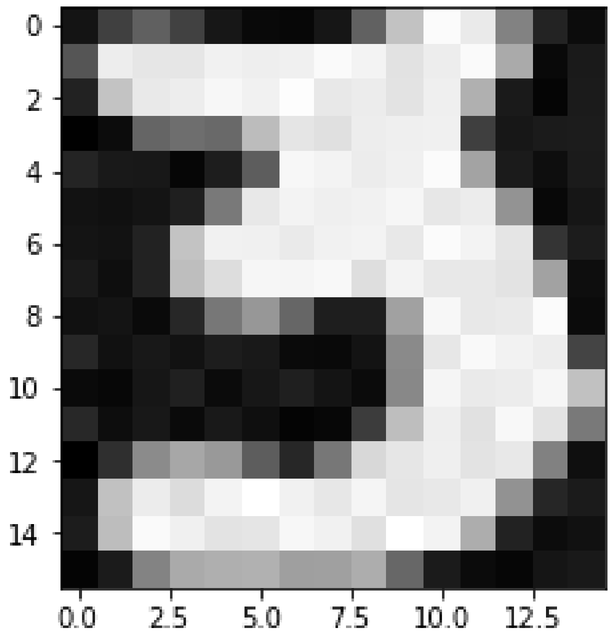


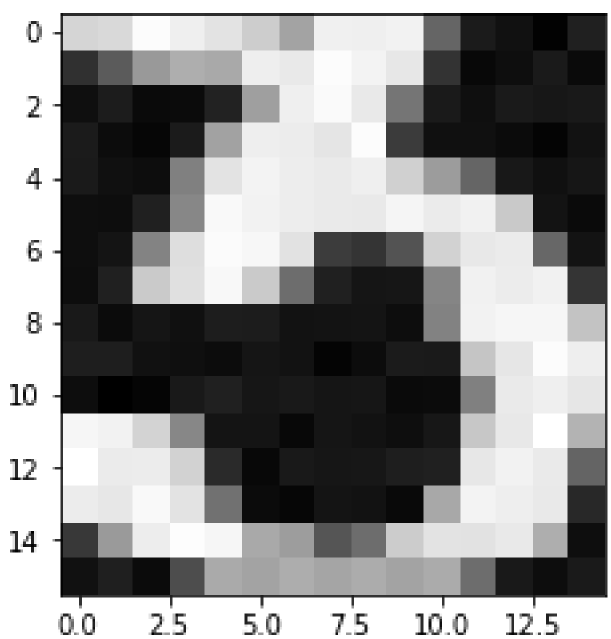
2.1.5 With preseverd variance of 99%

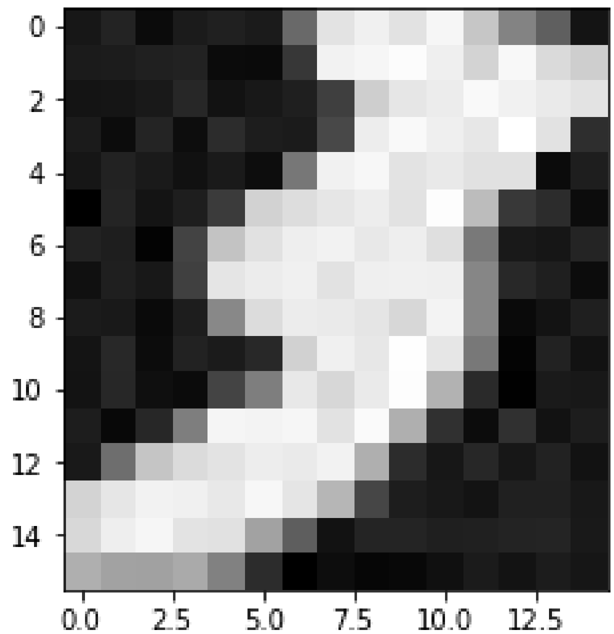
Using the technique above, $m_{expected} = 122$. As you can see below, the image is almost identical to the original.

```
In [18]: imgs_cat(denormalize(pca_test(normalized_threes, U, pick_k(s, 0.99)), mean_threes)[:5]
```

```
k = 122 obtains a variance of 0.9902471965368619
```



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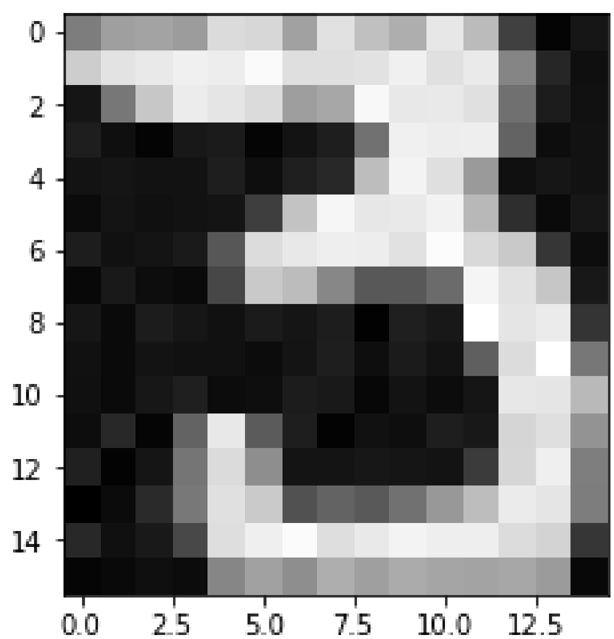
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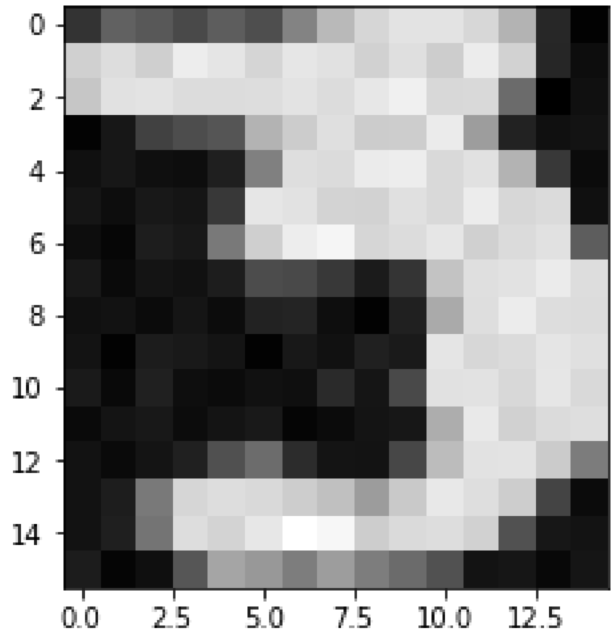


2.2 With preseverd variance of 100%

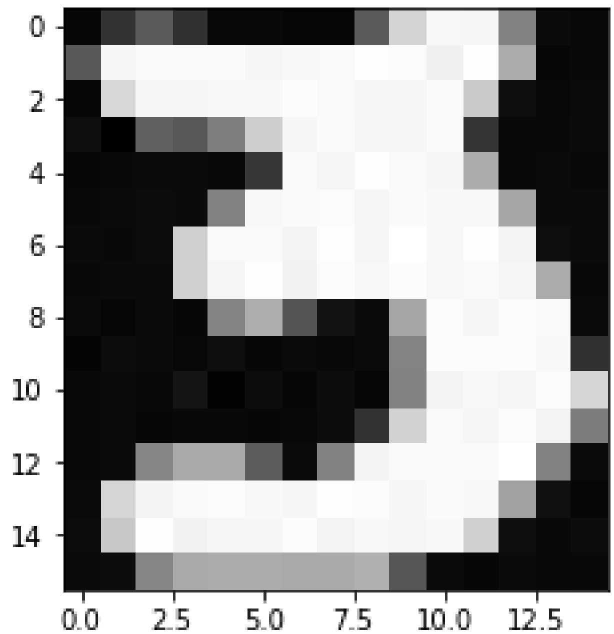
We manually tries out values of k, or m such that it satisfies such a condition. It turns out to be $m = 199$. As you can see below, the error is close to zero(not actually zero due to numerical error).

Many of people will expect $m = 240$ instead of $m = 199$ for preserved variance of 100%. This is due to the overfitting problem. In the calculation process, the rank of X is 200. However, after the centering(or normalizing), we have reduced the rank by 1. Thus, the correlation matrix R has at most a rank of 199. Hence, only 199 principle components are usable.

```
In [19]: variance_error(s, 199) # variance error for k = 199, it is ignorable
```

```
Out[19]: 4.48395556079047e-16
```

```
In [20]: imgs_cat(denormalize(pca_test(normalized_threes, U, 199), mean_threes)[:5]) # show so
```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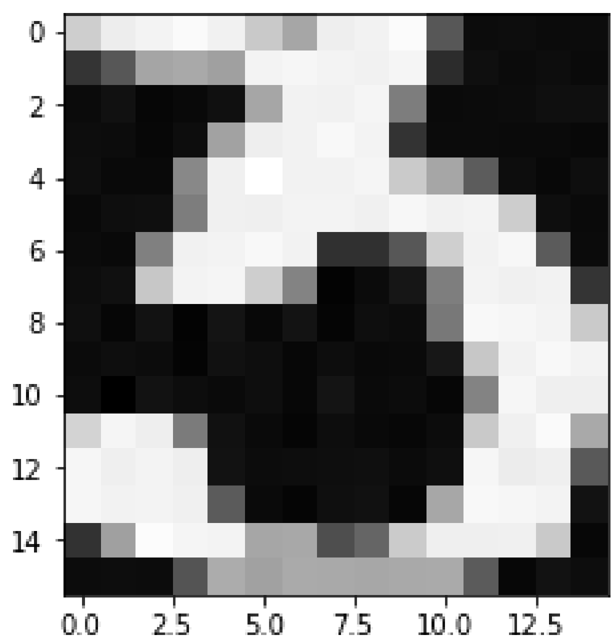
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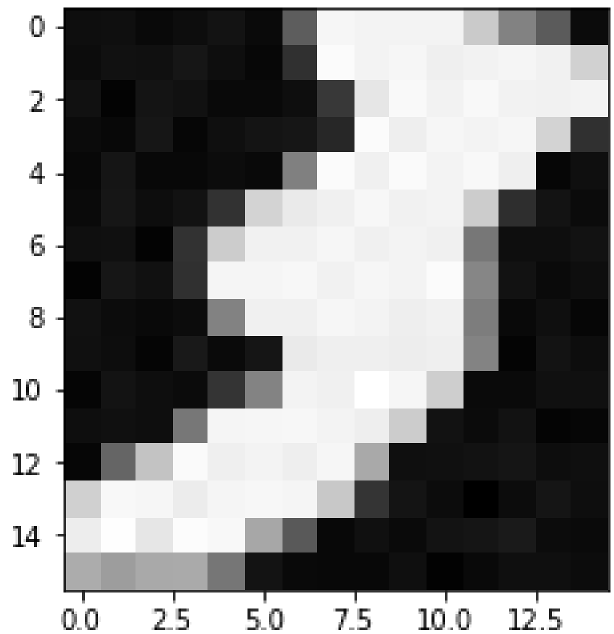
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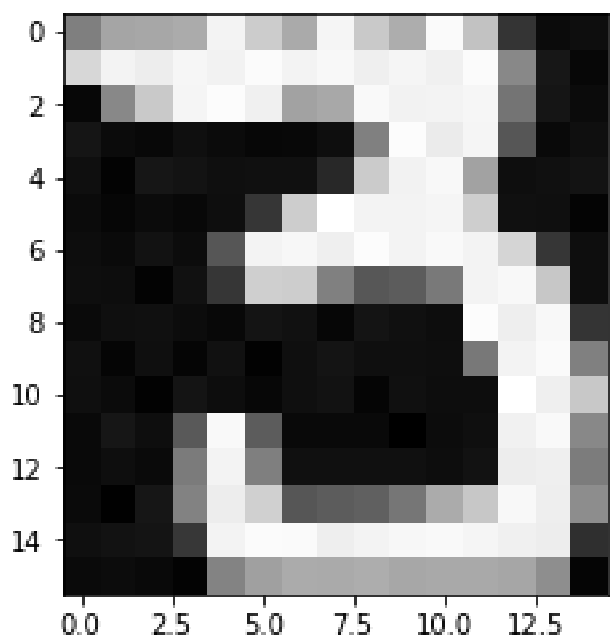
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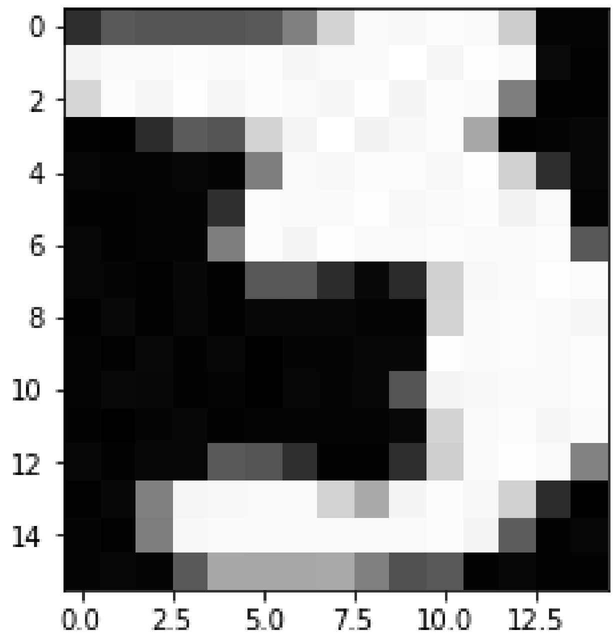
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