1. $p(t|x,\vec{x},\vec{t}) = \int_{-\infty}^{\infty} p(t|x,\vec{w})\, p(\vec{w}|\vec{x},\vec{t})\, dw$

电机乙 0850736 楊馨宇

其中 $\begin{cases} p(t|x,\vec{w}) = N(t|y(x,\vec{w}), \beta^{-1}) = N(t|w^T\phi(x), \beta^{-1}) \\ p(w) = N(w|0, \alpha^{-1}I) \end{cases}$

$\because p(\vec{w}|\vec{x},\vec{t}) \propto p(t|\vec{x},\vec{w}) \times p(\vec{w}) \propto \prod_{n=1}^{N} N(t_n|w^T\phi(x_n), \beta^{-1}) \cdot N(w|0,\alpha^{-1}I)$

$\propto \exp\left[\frac{-\beta}{2}(t_1 - w^T\phi(x_1))^2 + (t_2 - w^T\phi(x_2))^2 + \cdots + (t_N - w^T\phi(x_N))^2\right] \exp\left(\frac{-\alpha}{2} w^T w\right)$

$= \exp\left[\frac{-\beta}{2}\sum_{n=1}^{N}(t_n^2 + w^T\phi(x_n)\phi(x_n)^T w - 2w^T\phi(x_n)t_n) - \frac{\alpha}{2} w^T w\right]$

$\propto \frac{-1}{2} w^T\left(\beta\sum_{n=1}^{N}\phi(x_n)\phi^T(x_n) + \alpha I\right)w - 2\beta w^T \sum_{n=1}^{N}(\phi(x_n)t_n)$

$\Rightarrow S_N^{-1} = \alpha I + \beta\sum_{n=1}^{N}\phi(x_n)\phi^T(x_n)\ ,\ m_N = S\left(\sum_{n=1}^{N}\beta\phi(x_n)t_n\right)$

$\therefore p(\vec{w}|\vec{x},\vec{t}) = N(w|m_N, S_N)$

$p(t|x,\vec{x},\vec{t}) = \int_{-\infty}^{\infty} p(t|x,\vec{w})\, p(\vec{w}|\vec{x},\vec{t})\, dw \propto \int_{-\infty}^{\infty} \exp\left(\frac{-\beta}{2}(t - w^T\phi(x))^2\right) \cdot \exp\left(\frac{-1}{2}(w-m_N)^T S_N^{-1}(w-m_N)\right) dw$

$\propto \int_{-\infty}^{\infty} \exp\left[\frac{-\beta}{2}(t^2 - 2(w^T\phi(x))t + (w^T\phi(x))^2\right] \cdot \exp\left[\frac{-1}{2}(w^T S_N^{-1} w - 2w^T S_N^{-1} m_N + \cancel{m_N^T S_N^{-1} m_N})\right) dw$

$\propto \int_{-\infty}^{\infty} \exp\left[\frac{-1}{2}(\beta t^2 - 2\beta w^T\phi(x)t + \beta w^T\phi(x)\phi^T(x)w + w^T S_N^{-1} w - 2w^T S_N^{-1} m_N)\right] dw$

$= \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2}\left[w^T(\beta\phi(x)\phi^T(x) + S_N^{-1})w - 2w^T(\phi(x)t\beta + S_N^{-1}m_N) + \beta t^2\right]\right\} dw$

compare with : $\frac{-1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu) = \frac{-1}{2}(x^T\Sigma^{-1}x - 2x^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu)$

令 $\boxed{\Sigma^{-1} = \beta\phi(x)\phi^T(x) + S_N^{-1}\ ,\ \mu = \Sigma(\beta\phi(x)t + S_N^{-1}m_N)}$

$= \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2}\left[(w^T\Sigma^{-1}w - 2w^T\Sigma^{-1}\mu + \mu^T\Sigma^{-1}\mu) - \mu^T\Sigma^{-1}\mu + \beta t^2\right]\right\} dw$

$= \int_{-\infty}^{\infty} \underbrace{\cancel{\exp\left\{\frac{-1}{2}\left[(w-\mu)^T\Sigma^{-1}(w-\mu)\right]\right\}}}_{\text{積分為}1} \cdot \exp\left\{\frac{-1}{2}(\beta t^2 - \mu^T\Sigma^{-1}\mu)\right\} dw$

$= \exp\left\{\frac{-1}{2}(\beta t^2 - \mu^T\Sigma^{-1}\mu)\right\} = \exp\left\{\frac{-1}{2}(\beta t^2 - (\Sigma(\beta\phi(x)t + S_N^{-1}m_N))^T \cancel{\Sigma^{-1}}\cancel{\Sigma}(\beta\phi(x)t + S_N^{-1}m_N)\right.$

$= \exp\left\{\frac{-1}{2}(\beta t^2 - (\beta\phi(x)t + S_N^{-1}m_N)^T \underbrace{\Sigma^T}_{=\Sigma}(\beta\phi(x)t + S_N^{-1}m_N))\right\}$

$= \exp\left\{\frac{-1}{2}\left[\beta t^2 - ((\beta\phi(x)t)^T\Sigma(\beta\phi(x)t) + 2(S_N^{-1}m_N)^T\Sigma(\beta\phi(x)t) + \text{const}\right]\right\}$

$$\propto \exp\left\{\frac{-1}{2}\left[\beta t^2 - \beta^2 \phi(x)^T \Sigma \phi(x) t^2 + 2(S_N^{-1} m_N)^T \Sigma \beta \phi(x) t\right]\right\}$$

$$= \exp\left\{\frac{-1}{2}\left[\left(\beta - \beta^2 \phi(x)^T \Sigma \phi(x)\right) t^2 - 2(S_N^{-1} m_N)^T \Sigma \beta \phi(x) t\right]\right\}$$

$$\propto \exp\left\{\frac{-1}{2}\left(\beta - \beta^2 \phi(x)^T \Sigma \phi(x)\right)\left(t - \underbrace{\frac{S_N^{-1} m_N^T \Sigma \beta \phi(x) t}{\beta - \beta^2 \phi(x)^T \Sigma \phi(x)}}_{m(x)}\right)^2\right\}$$

$$\Rightarrow S^2(x) = \left(\beta - \beta^2 \phi(x)^T \Sigma \phi(x)\right)^{-1}, \quad \Sigma = (\Sigma^{-1})^{-1} = \left[S_N^{-1} + \beta \phi(x) \phi^T(x)\right]^{-1} = S_N - \frac{S_N \beta \phi(x) \phi(x)^T S_N}{1 + \phi(x)^T S_N (\beta \phi(x))}$$

$$\Rightarrow S^2(x) = \left(\beta - \beta^2 \phi(x)^T \left(S_N - \frac{S_N \beta \phi(x) \phi(x)^T S_N}{1 + \phi(x)^T S_N \beta \phi(x)}\right) \phi(x)\right)^{-1}$$

$$= \left(\beta - \beta^2 \phi(x)^T S_N \left(I - \frac{\beta \phi(x) \phi(x)^T S_N}{1 + \beta \phi(x)^T S_N \phi(x)}\right) \phi(x)\right)^{-1}$$

$$= \left(\beta - \beta^2 \phi(x)^T S_N \frac{\phi(x) + \cancel{\beta \phi(x) \phi(x)^T S_N \phi(x)} - \cancel{\beta \phi(x) \phi(x)^T S_N \phi(x)}}{1 + \beta \phi(x)^T S_N \phi(x)}\right)^{-1}$$

$$= \left(\beta - \beta^2 \frac{\phi(x)^T S_N \phi(x)}{1 + \beta \phi(x)^T S_N \phi(x)}\right)^{-1} = \left[\beta\left(1 - \beta \frac{\phi(x)^T S_N \phi(x)}{1 + \beta \phi(x)^T S_N \phi(x)}\right)\right]^{-1}$$

$$= \left(\beta \cdot \frac{1}{1 + \beta \phi(x)^T S_N \phi(x)}\right)^{-1} = \frac{1 + \beta \phi(x)^T S_N \phi(x)}{\beta}$$

$$\Rightarrow S^2(x) = \beta^{-1} + \phi(x)^T S_N \phi(x), \quad S_N^{-1} = \alpha I + \sum_{n=1}^{N} \phi(x_n) \phi(x_n)^T$$

$$m(x) = y(x, m_N) = \phi(x)^T m_N = \phi(x)^T \left[S_N \left(\sum_{n=1}^{N} \beta \phi(x_n) t_n\right)\right]$$

$$= \beta \phi(x)^T S_N \sum_{n=1}^{N} \phi(x_n) t_n$$

$$\therefore p(t \mid x, \vec{x}, \vec{t}) = N\left(t \mid m(x), S^2(x)\right), \quad \begin{cases} m(x) = \beta \phi(x)^T S_N \sum_{n=1}^{N} \phi(x_n) t_n \\ S^2(x) = \beta^{-1} + \phi(x)^T S_N \phi(x) \\ S_N^{-1} = \alpha I + \sum_{n=1}^{N} \phi(x_n) \phi(x_n)^T \end{cases}$$