24352: Homework #4

Daniel Deng

March 5, 2021

Question 1

State space representation from differential equations

a)

$2\ddot{x} + 4\dot{x} + 4x = 3u$, where the state vector is $\begin{bmatrix} x \\ \dot{x} \end{bmatrix}$. The out put is x.

solution

$\ddot{x} + 2\dot{x} + 2x = 1.5u \Rightarrow \begin{cases} \dot{x} = 0x + \dot{x} \\ \ddot{x} = -2x - 2\dot{x} + 1.5u \end{cases}$, thus

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} u \tag{1}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u \tag{2}$$

We get:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix} \tag{3}$$

b)

$2\ddot{x} + 4\dot{x} + 4x = 3u$, where the state vector is $\begin{bmatrix} x + \dot{x} \\ \dot{x} \end{bmatrix}$. The out put is x.

solution

$\begin{cases} \dot{x} + \ddot{x} = \dot{x} + (-2x - 2\dot{x} + 1.5u) = -2(x + \dot{x}) + \dot{x} + 1.5u \\ \ddot{x} = -2x - 2\dot{x} + 1.5u = -2(x + \dot{x}) + 0\dot{x} + 1.5u \\ x = (x + \dot{x}) + (-1)\dot{x} \end{cases}$, thus

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} + \ddot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x + \dot{x} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix} u \tag{4}$$

$$\dot{\mathbf{y}} = \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} x + \dot{x} \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u \tag{5}$$

We get:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 \end{bmatrix} \tag{6}$$

c)

Using the A,B,C,D matrices found in parts (a) and (b), use Matlab's step command to simulate the step responses to both systems. How do these step responses compare and does that comparison make sense?

solution

Matlab code:

```
t = 0:0.01:10;

A = [0, 1; -2 -2];
B = [0; 1.5];
C = [1 0];
D = [0];
sys1 = ss(A,B,C,D);

A = [-2, 1; -2 0];
B = [1.5; 1.5];
C = [1 -1];
D = [0];
sys2 = ss(A,B,C,D);

[y1,t] = step(sys1,t);
[y2,t] = step(sys2,t);
plot(t,y1,'r',t,y2,'b--','linewidth',2)
grid minor
title('Response to step')
xlabel('Time (s)')
ylabel('Amplitude')
legend('y1', 'y2')
```

Result:
We can tell from Fig.1 that the step responses are exactly the same. This make sense because we are representing the same system, and the output is the same for the two representations.

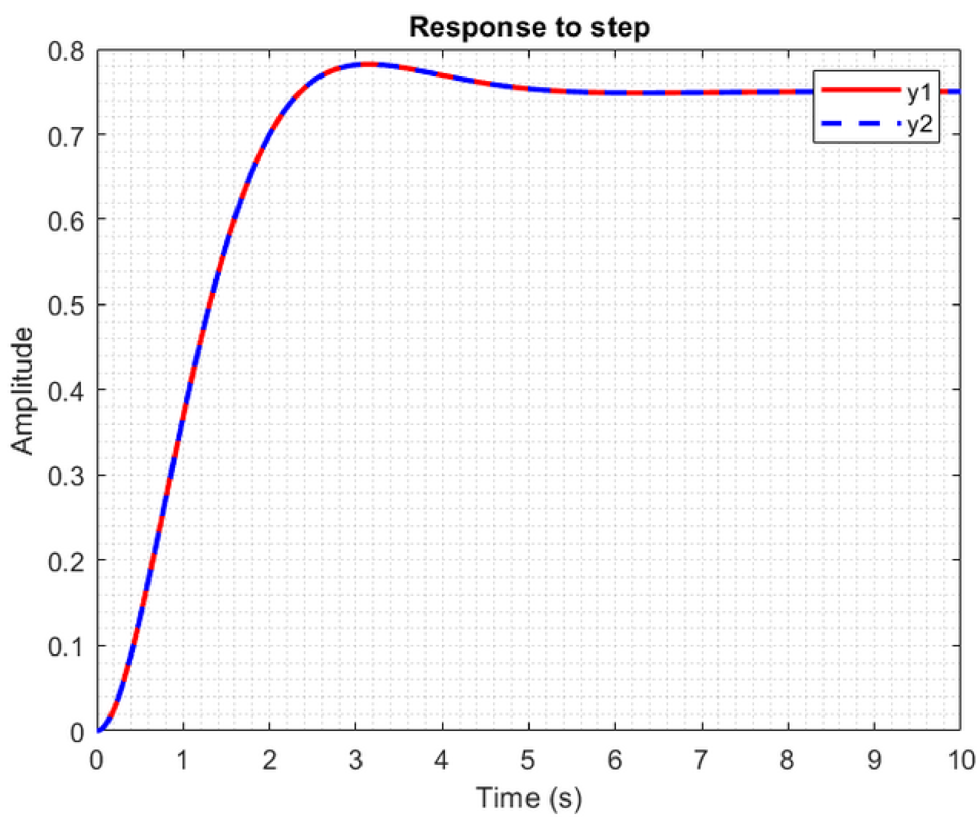
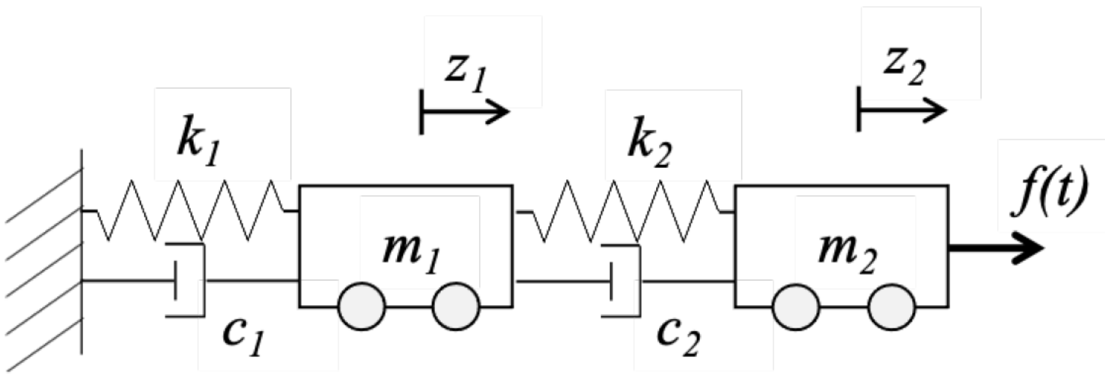


Figure 1: Step Response

Problem 2

State space representation of a 2-mass system.
For the following mechanical system, there is one input, f, and two outputs: the position of the first mass, z_1, and the distance between the masses, $z_1 - z_2$.



a)

Write the state equation, $\dot{x} = Ax + Bu$, where x is a state vector that you choose, A is the system matrix, B is the input matrix, and u is the input.

solution

First we identify all energy storage elements:

Energy Storage Elements	State variable
m_1 (Kinetic)	$\dot{z_1}$
m_2 (Kinetic)	$\dot{z_2}$
k_1 (Potential)	z_1
k_2 (Potential)	$z_2 - z_1$

From all energy storage elements we define the state vector: $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 - z_1 \\ \dot{z_1} \\ \dot{z_2} \end{bmatrix}$, and input $u = f$

Draw FBD to analyze system dynamics:

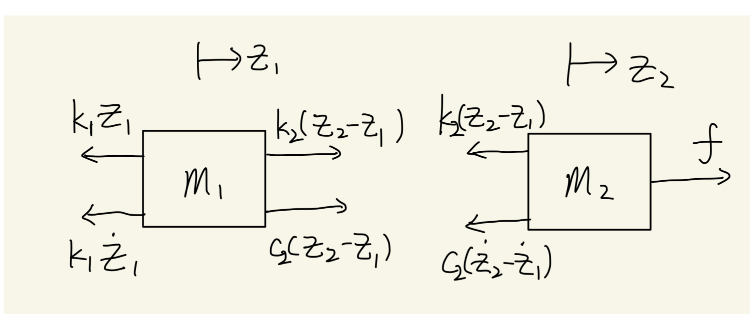


Figure 2: Free body diagram

We get the following dynamic equations:

$$\begin{cases} m_1\ddot{z_1} = k_2(z_2 - z_1) + c_2(\dot{z_2} - \dot{z_1}) - k_1 z_1 - c_1\dot{z_1} \\ m_2\ddot{z_2} = f - k_2(z_2 - z_1) - c_2(\dot{z_2} - \dot{z_1}) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x_1} = \dot{z_1} \\ \dot{x_2} = \dot{z_2} - \dot{z_1} \\ \dot{x_3} = \ddot{z_1} = \frac{k_2}{m_1}(z_2 - z_1) + \frac{c_2}{m_1}(\dot{z_2} - \dot{z_1}) - \frac{k_1}{m_1}z_1 - \frac{c_1}{m_1}\dot{z_1} \\ \dot{x_4} = \ddot{z_2} = \frac{f}{m_2} - \frac{k_2}{m_2}(z_2 - z_1) - \frac{c_2}{m_2}(\dot{z_2} - \dot{z_1}) \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x_1} = 0x_1 + 0x_2 + x_3 + 0x_4 + 0u \\ \dot{x_2} = 0x_1 + 0x_2 + (-1)x_3 + x_4 + 0u \\ \dot{x_3} = -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}x_2 - \frac{c_1+c_2}{m_1}x_3 + \frac{c_2}{m_1}x_4 + 0u \\ \dot{x_4} = 0x_1 - \frac{k_2}{m_2}x_2 + \frac{c_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 + \frac{1}{m_2}u \end{cases}$$

We can write the state space equation as:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -\frac{k_1}{m_1} & \frac{k_2}{m_1} & -\frac{c_1+c_2}{m_1} & \frac{c_2}{m_1} \\ 0 & -\frac{k_2}{m_2} & \frac{c_2}{m_2} & -\frac{c_2}{m_2} \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2} \end{bmatrix} u \tag{7}$$

b)

Write the output equation, $y = Cx + Du$.

solution

The output vector: $\mathbf{y} = \begin{bmatrix} z_1 \\ z_1 - z_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$, thus:

$$\dot{\mathbf{y}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{8}$$