[SP-5]

I. Poisson Distribution & Poisson Paradigm.

1. The possible distribution is
$\text{Pois}(100t)$.
The reason is that the #raindrops is very large, the probability of a raindrop hits the region is small.

Let $X = (\text{\# raindrops})$. Then,
$X \sim \text{Pois}(5)$
Then,
$$P(X=0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda} = e^{-5}.$$

2.
(a). } I checked the starting paragraph of the solutions and get the hint: de Montmort's matching problems

We label students from 1 to 100 and their first seating as 1 to 100. (i.e., label "1" student picks a seat, then the seat is also labelled as "1".)
Let A_j be the event that student j picks seat j. WTF: $1 - P(A_1 \cup \cdots \cup A_{100})$.

To obtain $P(A_1 \cup \cdots \cup A_{100})$,
$$P(A_j) = 1/100$$
$$P(A_i \cap A_j) = \frac{98!}{100!} = \frac{1}{100 \times 99}.$$
$$P(A_i \cap A_j \cap A_k) = \frac{97!}{100!} = \frac{1}{100 \times 99 \times 98}.$$
$$\vdots$$

By inclusion-exclusion:

$$P(A_1 \cup \cdots \cup A_{100})$$
$$= 100 \cdot \frac{1}{100} - \binom{100}{2}\frac{1}{100\times 99} + \binom{100}{3}\frac{1}{100\times 99\times 98}$$
$$- \cdots + \binom{100}{99}\frac{1}{100\times\cdots\times 2} - \binom{100}{100}\frac{1}{100!}$$

$\Rightarrow$ Pr(no one has same seat)

$$= 1 - P(A_1 \cup \cdots \cup A_{100})$$
$$= \binom{100}{2}\frac{1}{100\times 99} - \binom{100}{3}\frac{1}{100\times 99\times 98} + \cdots$$
$$- \binom{100}{99}\frac{1}{100\times\cdots\times 2} + \binom{100}{100}\frac{1}{100!}$$ ← {can be further simplied}

(b). The problem can be restated as follows:
Let $A_1, \cdots, A_{100}$ be events that student $1, \cdots, 100$ has the same seat with the $Pr(A_j) = \frac{1}{100}$, respectively. $A_1, \cdots, A_{100}$ are weakly dependent. Let X be # students having the same seat. Then,

$$X = \sum_{i=1}^{100} I(A_i),$$

and,

$$E[X] = \sum_{i=1}^{100} E[I(A_i)]$$
$$= \sum_{i=1}^{100} P(A_i) = 1$$

Therefore, we can approximately model
$$X \sim Pois(1).$$

Then,
$$P(X=0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-1}.$$

{comparing with (a), you will find it is e^{-1} taylor expansion at 0.}

(c). $P(X \geq 2) = 1 - P(X=1) - P(X=0)$

$$= 1 - e^{-1}\frac{1^1}{1!} - e^{-1}$$

$$= 1 - 2e^{-1}.$$

3.

(a). $E[e^{-3X}] - \theta$

$$= \left(\sum_{k=0}^{\infty} e^{-3k} e^{-\lambda}\frac{\lambda^k}{k!}\right) - e^{-3\lambda}$$

$$= e^{-\lambda}\left(\sum_{k=0}^{\infty} \frac{(e^{-3}\lambda)^k}{k!}\right) - e^{-3\lambda}$$

$\downarrow$ {Taylor Expan.}

$$= e^{-\lambda} e^{e^{-3}\lambda} - e^{-3\lambda}$$

$$= e^{(e^{-3}-1)\lambda} - e^{-3\lambda} \neq 0$$

$\Rightarrow$ Biased.

(b). Pf:

$E[(-2)^X] - \theta$

$$= \sum_{k=0}^{\infty} -2^k \cdot e^{-\lambda}\frac{\lambda^k}{k!} - e^{-3\lambda}$$

$$= e^{-\lambda} e^{-2\lambda} - e^{-3\lambda} = 0$$

Q.E.D.

(c).

It is silly in the sense that $g(X) = (-2)^X$ can sometimes be negative. However, $e^{-3\lambda} > 0$.

Thus, let

$$h(X) = \begin{cases} 0, & \text{if } g(X) \le 0 \\ g(X), & \text{otherwise.} \end{cases}$$

II. Seeking Sublime Symmetry.

1. Let $T = SZ$.

$$\begin{aligned}
F_T(t) &= P(T \le t) \\
&= P(SZ \le t) \\
&= P(SZ \le t \mid S=1)P(S=1) \\
&\quad + P(SZ \le t \mid S=-1)P(S=-1) \\
&= P(Z \le t)P(S=1) + P(Z \ge -t)P(S=-1) \\
&= \tfrac{1}{2}\Phi(t) + \tfrac{1}{2}(1-\Phi(-t)) \\
&= \tfrac{1}{2}\Phi(t) + \tfrac{1}{2}\Phi(t) \\
&= \Phi(t)
\end{aligned}$$

{S, Z are indep.} $\Downarrow$

Hence,

$T \sim N(0,1)$. Q.E.D.

2.

1). Why $p(X<Y) = p(Y<X)$ if X and Y are i.i.d.?

The reason is that in such case, we can interchange X and Y. i.e., both $P(X<Y)$ and $P(Y<X)$ represent the Pr. that one draw is less than another independent draw, from the same distribution.

2). Is $P(X<Y) = \frac{1}{2}$?

In discrete case, $p(X=Y) \neq 0$

$\Rightarrow P(X<Y) < 1/2$.

In continuous case, $p(X=Y) = 0$

$\Rightarrow P(X<Y) = 1/2$.

3). If X and Y are dependent, {same dist.} it may not be true that $p(X<Y) = P(Y<X)$.

For example. {sp-4, 1, 2}. X is a random day of a week. $Y = X+1$.

(Monday = 1, Tuesday = 2, ...)

3. If $X \sim Bin(n, p)$, then $n-X$ is a random variable representing # failures in n independent trials with Pr. $1-p$.

4. Based on Hint, we label seats as $1, 2, \cdots, 100$, such that seat j is assigned to j^{th} passenger.

The last passenger (100^{th}) only has two possibilities:

1). seat 1

2). seat 100 (assigned one)

1). Any of 1^{st} to 99^{th} passenger taking seat 100, the last passenger will get seat 1.

2). On the other hand, Any of 1^{st} to 99^{th} passenger taking seat 1, the last passenger will get assigned seat.

At each step among 1^{st} to 99^{th} choice, only taking seat 1 or seat 100 can decide the fate of the last passenger. If j^{th} passenger's seat is occupied, he can choose either seat 1 or seat 100 with equal Pr. if the fate of last passenger is settled.

I saw an interesting solution from web:

Say there are n passengers in line, and $n \geq 2$. Let $P(n)$ be the Pr. that the last passenger gets the assigned seat (seat n).

If $n=2$, $P(n) = \frac{1}{2}$.

If $n>2$,

$$p(n)=\frac{1}{n}\times 1+\frac{1}{n}\times 0+\frac{1}{n}\sum_{j=2}^{n-1}p(j). \qquad (1)$$

"$\frac{1}{n}\times 1$": 1^{st} passenger chooses Seat 1.

"$\frac{1}{n}\times 0$": 1^{st} passenger chooses Seat n.

"$\frac{1}{n}\sum_{j=2}^{n-1}p(j)$": 1^{st} passenger chooses Seat j, $j\in\{2,\dots,n-1\}$.
Then, everything starts over at j^{th} passenger, with $n-j+1$ passengers.
$\Rightarrow p(n-j+1)$.
$\Rightarrow p(n-1), p(n-2), \dots, p(2)$.

Rearrange (1), we have

$$n\,p(n)=1+\sum_{j=2}^{n-1}p(j). \qquad (2).$$

And $n:=n+1$, we get

$$(n+1)\,p(n+1)=1+\sum_{j=2}^{n}p(j) \qquad (3).$$

(3)−(2), we get

$$(n+1)\,p(n+1)-n\,p(n)=p(n)$$

$$\Rightarrow \quad p(n+1)=p(n).$$

And, $p(3)=\frac{1}{3}\times 1+\frac{1}{3}\times 0+\frac{1}{3}p(2)$

$$\Rightarrow p(3)=\frac{1}{2}.$$

Thus, $p(n)=p(3)=\frac{1}{2}$ for $n>2$.

Hence, $p(n)=1/2$ for $n\geq 2$.

III. Continuous Distributions

1. Let $y > 0$. Then,

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(e^X \le y) \\ &= P(X \le \ln y) \\ &= P\left(\frac{X-\mu}{\sigma} \le \frac{\ln y - \mu}{\sigma}\right) \\ &= \Phi\left(\frac{\ln y - \mu}{\sigma}\right) \end{aligned}$$

Therefore,

$$F_Y(y) = \begin{cases} 0, & \text{if } y \le 0 \\ \Phi\left(\frac{\ln y - \mu}{\sigma}\right), & \text{if } y > 0. \end{cases}$$

And, PDF of Y is

$$\begin{aligned} f_Y(y) &= F_Y'(y) \\ &= \varphi\left(\frac{\ln y - \mu}{\sigma}\right) \frac{1}{\sigma} \cdot \frac{1}{y} \\ &= \frac{1}{y\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln y - \mu)^2}{2\sigma^2}\right\} \end{aligned}$$

where $y > 0$.

2. Given PDF of X is $f(x) = \lambda e^{-\lambda x}$ for $x > 0$.
The CDF of X is $F(x) = 1 - e^{-\lambda x}$ for $x > 0$.
Then,

$$F^{-1}(u) = -\lambda^{-1} \ln(1-u).$$

Based on Universality of Uniform,

$$X = F^{-1}(U) \text{ where } U \sim \text{Unif}(0,1).$$

3. Find $E[\Phi(Z)]$.

Based on Universality of Uniform,

$U = \Phi(Z) \sim \text{Unif}(0,1)$.

Then,

$$E[\Phi(Z)] = E[U]$$
$$= \int_0^1 u \cdot 1 \, du$$
$$= \frac{1}{2}.$$

4. Let the Length of stick be ℓ and Let X be the r.v. representing the Length of the part on the left side of break point. We have

$$X \sim \text{Unif}(0, \ell).$$
$$\ell - X \sim \text{Unif}(0, \ell).$$

Let

$$Y = \max(X, \ell - X).$$

Then, CDF of Y is

$$F_Y(y) = P(Y \le y)$$
$$= P(Y \le y, X \le \tfrac{\ell}{2}) + P(Y \le y, X > \tfrac{\ell}{2})$$
$$= P(\ell - X \le y, X \le \tfrac{\ell}{2}) + P(X \le y, X > \tfrac{\ell}{2})$$
$$= (\tfrac{y}{\ell} - \tfrac{1}{2}) + (\tfrac{y}{\ell} - \tfrac{1}{2})$$
$$= 2\tfrac{y}{\ell} - 1$$

{In sol., it sets $\ell = 1$}

$\Rightarrow F_Y(y) = 2y - 1$, where $\frac{1}{2} \le y \le 1$

Then, PDF of Y is $f_Y(y) = 2$

$\Rightarrow Y \sim \text{Unif}(\frac{1}{2}, 1)$ {Verify: $(1 - \frac{1}{2}) \cdot 2 = 1$.}

$\Rightarrow E[Y] = (\frac{1}{2} + 1)/2 = \frac{3}{4}$

IV. LOTUS.

1. $X \sim \text{Pois}(\lambda)$

$$E[X!] = \sum_{k=0}^{\infty} k!\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} k! \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=0}^{\infty} \lambda^k$$

} Assuming $0 < \lambda < 1$. or it diverges {

$$= e^{-\lambda} \lim_{k \to \infty} (1 + \lambda + \lambda^2 + \cdots + \lambda^k)$$

$$= \frac{e^{-\lambda}}{1-\lambda}.$$

2. $Z \sim N(0,1)$. Find $E[|Z|]$

$$E[|Z|] = \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= 2\int_0^{\infty} z \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \sqrt{\frac{2}{\pi}} \int_0^{\infty} z\, e^{-z^2/2}\, dz$$

$$= \sqrt{\frac{2}{\pi}} \left(-e^{-z^2/2}\right)\Big|_0^{\infty}$$

$$= \sqrt{\frac{2}{\pi}} \times 1 = \sqrt{\frac{2}{\pi}}.$$

3. Given $X \sim Geom(p)$. Find $E[e^{tX}]$ where t is a constant.

$$E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} (1-p)^k p$$

$$= p \sum_{k=0}^{\infty} [e^{t}(1-p)]^k$$

{Assuming $e^t(1-p) < 1$ $\Rightarrow t < -\ln(1-p)$}

$$= p \frac{1}{1-(1-p)e^t}.$$