[SP-4]

1. Distribution and Expected Values for Discrete R.V.S.

1. Say I toss a fair coin $2n+1$ times $(n=0,1,\cdots)$,
$X = (\#\text{Heads})$, $Y = (\#\text{Tails})$. Then,
both X and Y follows $\text{Bin}(2n+1, 1/2)$,
but the event $X=Y$ never occurs.

2. $P(Y=k) = P(X+1=k) = P(X=k-1) = \frac{1}{7}$.
$\Rightarrow$ Y and X has the same distribution.
$P(X<Y) = P(1 \le X < 7) = \frac{6}{7}$.
{Note that X, Y are not independent.}

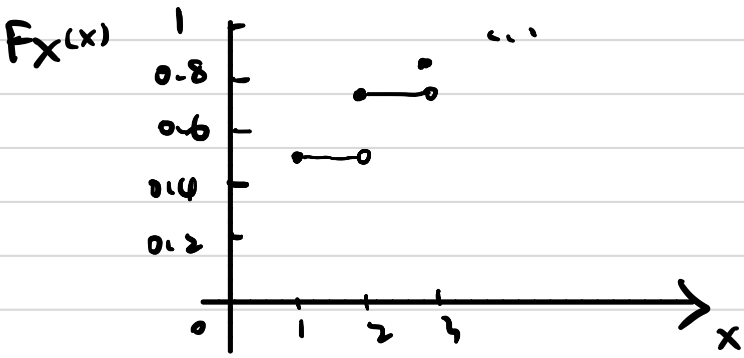
3. Given the problem statement,
$$Y = X-1 \sim \text{Geom}(p)$$
Then, PDF of Y is
$$P(Y=k) = (1-p)^k p. \quad k=0,1,\cdots$$
Thus,
$$P(X=k) = P(Y=k-1) = (1-p)^{k-1} p. \quad k=1,2,\cdots$$
And CDF of X is
$$\begin{aligned} P(X \le x) &= \textstyle\sum_{j=1}^{x} P(X=j) \\ &= \textstyle\sum_{j=1}^{x} (1-p)^{j-1} p. \\ &= p \textstyle\sum_{j=1}^{x} (1-p)^{j-1} \\ &= p \frac{1-(1-p)^x}{1-(1-p)} \\ &= 1-(1-p)^x. \quad \text{where } x \ge 0 \ \& \ x \in \mathbb{Z} \end{aligned}$$

More generally, if x is not an integer, then,

$$F_X(x) = \begin{cases} 1-(1-p)^{\lfloor x \rfloor}, & \text{if } x \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

When $p = 1/2$,

$$F_X(x) = \begin{cases} 1-(1/2)^{\lfloor x \rfloor}, & \text{if } x \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$



4. It is possible. Say let PMF of X is

$$P(X=0) = \frac{99}{100}, \quad P(X=10^6) = \frac{1}{100}$$

$\Rightarrow E[X] = 10000$

Let $Y = 99 \Rightarrow E[Y] = 99$

5.

$$E[X] = \sum_{n=1}^{\infty} n \cdot P(X=n)$$

$$= P(X=1) + 2P(X=2) + 3P(X=3) + \ldots$$

$$= [P(X=1) + P(X=2) + \ldots] + [P(X=2) + P(X=3) + \ldots] + \ldots$$

$$= P(X>0) + P(X>1) + \ldots$$

$$= (1-F(0)) + (1-F(1)) + \ldots$$

$$= \sum_{n=0}^{\infty} (1-F(n))$$

6. {Note the hint: ... to the previous problem.}

For $n \in \{0, 1\}$, $P(X>n) = 1$ {1st one is highest.}

For $n \geq 2$,

$$P(X>n) = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

Based on the problem 5,

$$E[X] = \sum_{n=1}^{\infty} n \cdot P(X=n)$$

$$= \sum_{n=0}^{\infty} P(X>n)$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$= 1 + \{\text{harmonic series}\}$$

$$= \infty$$

↳ diverges slowly.

II. Indicator R.V.s & Linearity of Expectation.

1.

Let $m = (\# \text{pairs}) = \binom{50}{2}$. Let I_j be the indicator r.v. such that

$$I_j = \begin{cases} 1, & \text{same birthday} \\ 0, & \text{otherwise} \end{cases}.$$

Let $X = (\#$ pairs with the same birthday)
Then,

$$E[X] = E[I_1 + I_2 + \cdots + I_m]$$
$$= m \cdot E[I_1]$$
$$= m \cdot P(I_1 = 1) = \binom{50}{2} / 365$$

✓

Let $Y = (\#$ days in the year on which at least two of these people were born)
Let A_j be the indicator r.v. such that

$$A_j = \begin{cases} 1, & \text{at least two are born on } j^{th} \text{ day.} \\ 0, & \text{otherwise} \end{cases}$$

$$P(A_j = 1) = 1 - P(A_j = 0)$$
$$= 1 - \binom{50}{1} \frac{1}{365} \left(\frac{364}{365}\right)^{49} - \binom{50}{0} \left(\frac{364}{365}\right)^{50}.$$

Then,

$$E[Y] = E[A_1 + \cdots + A_{365}]$$
$$= 365 \cdot E[A_1]$$
$$= 365 \cdot P(A_j = 1)$$
$$= 365 \cdot \left(1 - 50 \frac{1}{365} \left(\frac{364}{365}\right)^{49} - \left(\frac{364}{365}\right)^{50}\right)$$

✓

2.

Let X = (# bags distributed to first three students)

Then,

$$X \sim Bin(20, \frac{3}{20})$$

Thus,

$$E[X] = 20 \times \frac{3}{20} = 3.$$

Let I_j be the indicator r.v. such that

$$I_j = \begin{cases} 1, & j^{th} \text{ student get at least one bag.} \\ 0, & \text{otherwise.} \end{cases}$$

Then,

$$E[I_1 + \cdots + I_{20}]$$
$$= 20\, E[I_1]$$
$$= 20\, P(I_1 = 1)$$
$$= 20\,(1 - P(I_1 = 0))$$
$$= 20\,(1 - (\frac{19}{20})^{20})$$

3. Each step leads to decrease of ends by 2. Thus,

$$\#\text{steps} = \frac{200}{2} = 100.$$

Let I_j be an indicator r.v. such that

$$I_j = \begin{cases} 1, & j^{th} \text{ step results in a loop} \\ 0, & \text{otherwise.} \end{cases}$$

$$E[I_j] = P(I_j = 1) = \frac{\binom{100-(j-1)}{1}}{\binom{200-2(j-1)}{2}} = \frac{1}{201-2j}$$

{ official sol. is $n / \binom{2n}{2} = \frac{1}{2n-1}$, n is # unlooped shoelaces at current step. }.

Then,

$E[I_1 + \cdots + I_{100}]$

$= \frac{1}{199} + \frac{1}{197} + \cdots + \frac{1}{3} + 1$ ✓

4. Based on the problem statement, h is true random. That is, k people will be assigned to one of n locations equally likely.

Let I_j be the indicator r.v. such that

$$I_j = \begin{cases} 1, & \text{no phone number in } j^{th} \text{ location} \\ 0, & \text{otherwise.} \end{cases}$$

$$P(I_j = 1) = \left(\frac{n-1}{n}\right)^k = \left(1 - \frac{1}{n}\right)^k$$

$\Rightarrow$ Expected # locations without phone number

$= E[I_1 + \cdots + I_n]$

$= n\left(1 - \frac{1}{n}\right)^k$ ✓

Let M_j be the indicator r.v. such that

$$M_j = \begin{cases} 1, & \text{exact one phone number in } j^{th} \text{ location} \\ 0, & \text{otherwise.} \end{cases}$$

$$P(M_j = 1) = \frac{\binom{k}{1}(n-1)^{k-1}}{n^k} = \frac{k}{n}\left(1 - \frac{1}{n}\right)^{k-1}$$ ✓

$\Rightarrow$ Expected # locations with exact one phone number

$= E[M_1 + \cdots + M_n]$

$= n \cdot \frac{k}{n}\left(1 - \frac{1}{n}\right)^{k-1}$

$= k\left(1 - \frac{1}{n}\right)^{k-1}$

Let N_j be the indicator r.v. such that

$$N_j = \begin{cases} 1, & \text{more than one phone number in } j^{th} \text{ location} \\ 0, & \text{otherwise.} \end{cases}$$

$P(N_j=1) = 1 - P(I_j=1) + P(M_j=1)$.

$\Rightarrow E[N_1 + \cdots + N_n]$

$= n E[N_j]$

$= n(1 - E[I_j] - E[M_j])$

$= n - \{\text{expected \# locations without phone number}\}$

$- \{\text{expected \# locations with exactly one phone number}\}$

$= n - n(1-\frac{1}{n})^k - k(1-\frac{1}{n})^{k-1}$ ✓

$\Rightarrow$ The sum of expected values must be n. ✓