[SP-8]

1. Covariance & Correlation

1.

(a). $Cov(X+Y, X-Y)$

$= Var(X) - Var(Y) - Cov(X,Y)$
$+ Cov(Y,X)$

$= Var(X) - Var(Y)$

$= 0$ by symmetry. ✓

(b). $X+Y$ and $X-Y$ are not independent.

i.e.,

$p(X-Y=0 \mid X+Y=2) = 1$

$\neq p(X-Y=0)$ ✓

2. Given $N \sim Pois(\lambda)$ and
$X|N \sim Bin(N, p)$

<u>WTF</u>: $Corr(N, X)$

$Cov(N, X) = E[NX] - E[N]E[X]$

Since

$E[X] = E[E[X|N]] = E[Np]$
$= pE[N]$
$= \lambda p,$

$E[NX] = E[E[NX|N]]$
$= E[NE[X|N]]$
$= pE[N^2]$
$= p(Var(N) + (E[N])^2)$
$= p(\lambda + \lambda^2)$

$\Rightarrow Cov(N, X) = p(\lambda + \lambda^2) - \lambda(\lambda p) = \lambda p.$

And,

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|N)] + \mathrm{Var}(E[X|N])$$
$$= E[Np(1-p)] + \mathrm{Var}(Np)$$
$$= p(1-p)\lambda + \lambda p^2$$
$$= \lambda p.$$

$\Rightarrow SD(X) = \sqrt{\lambda p}$

$$\Rightarrow \mathrm{Corr}(N,X) = \frac{\mathrm{Cov}(N,X)}{SD(N)\,SD(X)}$$
$$= \frac{\lambda p}{\sqrt{\lambda}\cdot\sqrt{\lambda p}}$$
$$= \sqrt{p}.$$

} make senses! More N tends to cause more X. :)

3. <u>WTF</u>: (a,b,c,d) making Z, W uncorrelated but standardized.

Let $\mathrm{Cov}(Z,W) = 0$.

$\Rightarrow \mathrm{Cov}(aX+bY,\ cX+dY) = 0$

$\Rightarrow ac\,\mathrm{Var}(X) + (ad+bc)\,\mathrm{Cov}(X,Y) + bd\,\mathrm{Var}(Y) = 0$

$\Rightarrow ac + bd + (ad+bc)\rho = 0.$ ... (1).

Since Z, W are standardized,

$$1 = \mathrm{Var}(Z) = \mathrm{Var}(aX+bY)$$
$$= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y)$$

$\Rightarrow 1 = a^2 + 2ab\rho + b^2$ ... (2)

Similarly,

$1 = c^2 + 2cd\rho + d^2.$ ... (3).

} Refering to Ans {

I set $a=1$, $b=0$.
from (1), $c+d\rho=0$
plug into (3), we get

$$d=\frac{1}{\sqrt{1-\rho^2}}, \quad c=-\frac{\rho}{\sqrt{1-\rho^2}}.$$

4. In $\mathrm{Mult}_k(n,(p_1,\cdots,p_k))$,
for $i\neq j$. WTE: $\mathrm{Cov}(X_i,X_j)$. We first find $\mathrm{Cov}(X_1,X_2)$.

As required by the problem, we use indicator r.v.s. to solve it. Let I_i be the i-th object in 1^{st} category, J_j be the j-th object in 2^{nd} category.

$\Rightarrow X_1=\sum_{i=1}^n I_i$, $X_2=\sum_{j=1}^n J_j$.

$$\begin{aligned}\Rightarrow\ &\mathrm{Cov}(X_1,X_2)\\ &=\mathrm{Cov}(\sum I_i,\sum J_j)\\ &=\sum_{i=j}\mathrm{Cov}(I_i,J_j)+\underbrace{\sum_{i\neq j}\mathrm{Cov}(I_i,J_j)}_{=0\ (\text{indep.})}\\ &=\sum_{i=j}\mathrm{Cov}(I_i,J_j)\end{aligned}$$

WLOG, we find $\mathrm{Cov}(I_1,J_1)$.
$E[I_1]=p_1$, $E[J_1]=p_2$.
$E[I_1J_1]=1\times P(I_1=1,J_1=1)$
$\quad +0\times P(\cdots)$
$\quad =0$.

$\Rightarrow \mathrm{Cov}(I_1,J_1)=0-p_1p_2$
$\quad =-p_1p_2$

$=-np_1p_2$.

Therefore,
$$\mathrm{Cov}(X_i, X_j) = -n p_i p_j.$$

5.
It is correct to say
$$\max(X,Y) + \min(X,Y) = X+Y$$
But it is not correct to say
$$\mathrm{Cov}(\max(X,Y), \min(X,Y)) = \mathrm{Cov}(X,Y).$$
} Note {

6. Given $V, W, Z \overset{i.i.d.}{\sim} \mathrm{Pois}(\lambda)$

(a). $\mathrm{Cov}(X,Y)$
$$= \mathrm{Cov}(V+W, V+Z)$$
$$= \mathrm{Var}(V) = \lambda.$$

(b). $\mathrm{Cov}(X,Y) = \lambda \neq 0 \Rightarrow X, Y$ are not indep.

$$P(X=x, Y=y \mid V=v)$$
$$= P(W = x-v, Z = y-v \mid V=v)$$
$$= P(W = x-v, Z = y-v)$$
$$= P(W = x-v)\, P(Z = y-v)$$
$$= P(X = x \mid V=v)\, P(Y=y \mid V=v)$$

$\Rightarrow X, Y$ are conditionally indep. given V.

(c).

$$\begin{aligned}
&p(X=x, Y=y) \\
&= \sum_{v=0}^{\infty} p(X=x, Y=y \mid V=v)\, p(V=v) \\
&= \sum_{v=0}^{\infty} p(W=x-v)\, p(Z=y-v)\, p(V=v) \\
&= \sum_{v=0}^{\infty} \frac{\lambda^{x-v} e^{-\lambda}}{(x-v)!} \frac{\lambda^{y-v} e^{-\lambda}}{(y-v)!} \frac{\lambda^{v} e^{-\lambda}}{v!} \\
&= e^{-3\lambda} \lambda^{x+y} \sum_{v=0}^{\cancel{\infty}\ \min(x,y)} \frac{\lambda^{-v}}{(x-v)!\,(y-v)!\,v!}
\end{aligned}$$

using $\min(x,y)$ because $V \le X$, $V \le Y$.

7. Let $X \sim \text{HGeo}(w, b, n)$

(a). WTF: $E\left[\binom{X}{2}\right]$.

$\binom{X}{2}$ = (# pairs of sampled balls s.t. both balls are white)

Let I_i be the indicator r.v. s.t. both balls in the i-th pair are white.

$\Rightarrow E\left[\binom{X}{2}\right] = E\left[I_1 + I_2 + \cdots + I_{\binom{n}{2}}\right]$.

By symmetry,

$$\begin{aligned}
E\left[\binom{X}{2}\right] &= \binom{n}{2} E[I_1] \\
&= \binom{n}{2} \left(\frac{w}{w+b}\right) \left(\frac{w-1}{w+b-1}\right)
\end{aligned}$$

(b).

$$E\left[\binom{X}{2}\right] = E\left[\frac{X(X-1)}{2}\right]$$

$$= \frac{1}{2}\left\{E[X^2] - E[X]\right\} = \binom{n}{2}\left(\frac{w}{w+b}\right)\left(\frac{w-1}{w+b-1}\right)$$

Since

$$E[X] = E[X_1 + \cdots + X_n]$$

where X_i is the indicator r.v. such that i-th ball is white. By symmetry,

$$\Rightarrow E[X] = n\,E[X_1] = \frac{nw}{w+b} = np.$$

Then, based on the above equation, we can solve

$$E[X^2] = E[X] + n(n-1)\left(\frac{w}{w+b}\right)\left(\frac{w-1}{w+b-1}\right)$$

$$= np + n(n-1)p\,\frac{w-1}{N-1}$$

$$\Rightarrow Var(X) = E[X^2] - (E[X])^2$$

$$= np + n(n-1)p\,\frac{w-1}{N-1} - n^2p^2$$

$$= np\left(1 + \frac{(n-1)(w-1)}{N-1} - np\right)$$

$$= np\left(\frac{nw-n-w+N}{N-1} - np\right)$$

$$= np\,\frac{(N-n)(N-w)}{N(N-1)}$$

$$= \frac{N-n}{N-1}\,npq.$$

II. Transformations

1. Given $X \sim Unif(0,1)$.
WTF: PDF of X^2 and $\sqrt{X}$.

We know $f_X(x) = 1$ for $0 \leq x \leq 1$.
Let $Y = X^2$, $Z = \sqrt{X}$.
$\Rightarrow \frac{dy}{dx} = 2x$, $\frac{dz}{dx} = \frac{1}{2} x^{-\frac{1}{2}}$.

Since $f(x) = x^2$ and $g(x) = \sqrt{x}$ are both strictly increasing continuous fcns,

$$\Rightarrow f_Y(y) = f_X(x) \left(\frac{dy}{dx}\right)^{-1} = \frac{1}{2x} = \frac{1}{2\sqrt{y}}$$ ✓

$$f_Z(z) = f_X(x) \left(\frac{dz}{dx}\right)^{-1} = 2\sqrt{x} = 2z$$ ✓

2. We have

$$f_{U,T}(u,t) = f_U(u) \cdot f_T(t) = \frac{1}{2\pi} e^{-t}.$$

Given

$$\begin{cases} X = \sqrt{2T} \cos U, \\ Y = \sqrt{2T} \sin U. \end{cases}$$

$$\frac{\partial(x,y)}{\partial(u,t)} = \begin{bmatrix} -\sqrt{2t} \sin u & (2t)^{-\frac{1}{2}} \cos u \\ \sqrt{2t} \cos u & (2t)^{-\frac{1}{2}} \sin u \end{bmatrix}$$

$$\Rightarrow f_{U,T}(u,t) = f_{X,Y}(x,y) \left| \left| \frac{\partial(x,y)}{\partial(u,t)} \right| \right|$$

$$= f_{X,Y}(x,y) \, |-1|$$

$$\Rightarrow f_{X,Y}(x,y) = f_{U,T}(u,t)$$
$$= \frac{1}{2\pi} e^{-t}$$
$$= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$
$$= \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right)$$

X, Y are independent and $X, Y \sim N(0,1)$.

{Box-Muller Method for generating Normal r.v.s.}.

3. <u>WTF</u>: Joint PDF of (T, X).

$$\begin{cases} T = X+Y \\ W = X \end{cases} \Rightarrow \begin{cases} X = W \\ Y = T-W \end{cases}$$

$$\Rightarrow \frac{\partial(x,y)}{\partial(t,w)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(x,t)}\right| = -1$$

$$\Rightarrow f_{T,W}(t,w) = f_{X,Y}(x,y) \cdot |-1|$$
$$= f_{X,Y}(x,y)$$
$$= f_X(x) f_Y(y)$$
$$= f_X(w) f_Y(t-w)$$

$$\Rightarrow f_T(t) = \int_{-\infty}^{\infty} f_X(w) f_Y(t-w)\, dw$$

Q.E.D.

IV. Existence

2.

Randomly choose two students, let X be the number of questions collectively correct.
Let $I_1, \cdots, I_8$ be indicators s.t.

$$I_j = \begin{cases} 1, & \text{one of two students got } j\text{-th question correct,} \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X] &= E[I_1 + \cdots + I_8] \\ &= 8 \cdot E[I_1] \\ &= 8 \cdot P(I_1 = 1) \\ &= 8 \cdot (1 - P(I_1 = 0)) \end{aligned}$$

Since

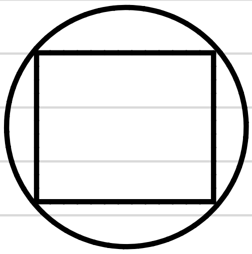
$$P(I_1 = 0) \leq \frac{35}{100} \times \frac{34}{99} = 0.1202$$

then

$$E[X] \geq 8(1 - 0.1202) = 7.04$$

$\Rightarrow$ There exists a pair of students with collectively eight right answers.

3.



Let X be the number of corners touching red ink for an inscribed square.
Let $I_1, \cdots, I_4$ be indicator r.v.s. s.t. $I_j = 1$ if j-th corner touches red ink.

$$\begin{aligned} \Rightarrow E[X] &= E[I_1 + \cdots + I_4] \\ &= 4 E[I_1] \\ &= 4 \cdot \frac{2}{3} = \frac{8}{3} \end{aligned}$$

$\Rightarrow$ There exists an inscribed square with 3 corners touching red ink.

4. Let $I_1, \cdots, I_{10}$ be indicator r.v.s.
s.t. $I_j = 1$ if j-th point is in the circle.

$\Rightarrow E[\#\text{points in circle}] = E[I_1 + \cdots + I_{10}]$
$= 10 \cdot E[I_1]$
$> 10 \times 0.9 = 9$

$\Rightarrow \ldots$

1. $\left\{ \begin{array}{l} 1 \le i_1 < \cdots < i_k \le n \\ i_1, i_2, \cdots, i_k \text{ don't have to be contiguous} \end{array} \right\}$

Let S be the set consisting of randomly chosen m strings of length n.
<u>WTS</u>: $P(S \text{ is } k\text{-complete}) > 0$.

Let A be the event that S is k-complete.
Let
$$N = 2^k \binom{n}{k}$$
and $A_1, \cdots, A_N$ are events s.t.
A_j means S has a string containing a particular $b_1 \cdots b_k$ (from 2^k) at a particular position $i_1, \cdots, i_k$ (from $\binom{n}{k}$).

$$P(A) = P(A_1 \cap \cdots \cap A_N)$$

$$\Rightarrow P(A^c) = P(A_1^c \cup \cdots \cup A_N^c)$$
$$\le \sum_{j=1}^{N} P(A_j^c)$$
$$= N \cdot \left(1 - \frac{1}{2^k}\right)^m < 1$$

$$\Rightarrow P(A) = 1 - P(A^c) > 0.$$

Why $p(A_j^c) = (1-\frac{1}{2^k})^m$?

A_j^c is the event that none of m strings in S contains a particular $b_1 \cdots b_k$ at a particular position $i_1, \cdots, i_k$.

Say one of m string, its bits on $i_1, \cdots i_k$ are randomly generated $\Rightarrow 2^k$ possibilities.

$\Rightarrow p(\text{It is } b_1 \cdots b_k) = \frac{1}{2^k}$,

$\Rightarrow p(\text{It does not contain } b_1 \cdots b_k$
$\text{at } i_1, \cdots, i_k) = 1 - 2^{-k}$.

$\Rightarrow$ m strings are generated independently.

$\Rightarrow \cdots$