[HW3]

I. Continuing with Conditioning

1. Let M_j be the event that Monty Hall opens the j^{th} door. Let S be the event that we get the car by switching. Let C_j be the event that car is behind the j^{th} door.
Assuming that we always choose the 1^{st} door.

(a). $$P(S) = P(S|C_1)P(C_1) + P(S|C_2)P(C_2) + P(S|C_3)P(C_3)$$
$$= 0 \times \frac{1}{3} + 1 \times \frac{1}{3} + 1 \times \frac{1}{3}$$
$$= \frac{2}{3}.$$

(b).

choose		car behind		MH opens	outcome	
door 1	$1/3$	door 1	p	door 2	goat	$\frac{1}{3}p$
			$1-p$	door 3	goat	$\frac{1}{3}(1-p)$
	$1/3$	door 2	1	door 3	car	$\frac{1}{3}$
	$1/3$	door 3	1	door 2	car	$\frac{1}{3}$.

$$P(S|M_2) = \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3}p} = \frac{1}{1+p}.$$

(c). Similar to (b),

$$p(S \mid M_3) = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{3}(1-p)} = \frac{1}{2-p}.$$

2.

(a). False. Let $X=Z$. We then can assume that X and Y are indep., and Z and Y are indep. But X and Z are dependent.

(b). False.
Two fair dices. X is the 1^{st} dice's result; Y is the 2^{nd} dice's result.
$\Rightarrow X, Y$ are independent.
Let $Z = X+Y$. Then,

$$p(X \mid Z) \neq p(X \mid Y, Z)$$

(c). False.
In Problem 5, HW2. (chess opponent)

(d). True.
For all a, we have

$$\sum_z p(X=a \mid Z=z)\, p(Z=z)$$

$$= \sum_z p(Y=a \mid Z=z)\, p(Z=z)$$

$$\Rightarrow p(X=a) = p(Y=a)$$

I. Simpson's Paradox.

1.

(a). $p(A) > p(B)$ is impossible ✓

Pf:

Since,

$$\begin{cases} p(A) = p(A|E)\,p(E) + p(A|E^c)\,p(E^c), \\ p(B) = p(B|E)\,p(E) + p(B|E^c)\,p(E^c) \end{cases}$$

then, if $p(A|E) < p(B|E)$ and
$p(A|E^c) < p(B|E^c)$,

Hence,

$$p(A) < p(B)$$

Q.E.D.

(b). $p(A|B) > p(A|B^c)$ is possible. ✓

Pf:

$$p(A|B) = p(A|E,B)\,p(E|B) + p(A|E^c,B)\,p(E^c|B)$$

$$p(A|B^c) = p(A|E,B^c)\,p(E|B^c) + p(A|E^c,B^c)\,p(E^c|B^c)$$

Given $p(A|B,E) < p(A|B^c,E)$ and
$p(A|B,E^c) < p(A|B^c,E^c)$,
we can't determine the relationship between $p(A|B)$ and $p(A|B^c)$ because we don't know the relationships between $p(E|B)$ and $p(E|B^c)$, and between $p(E^c|B)$ and $p(E^c|B^c)$.

2.

(a). Let D be the event that the man is a dealer. Let H be the event that the man will hurt Stampy. Let E be the event that a man get a lot of ivory.

(b). Lisa states that $P(D|E) > \cancel{P(D|E^c)}$ $P(D)$ and suggest that $P(H|E) > P(H|E^c)$. Homer argues that

$$P(H|E) < P(H|E^c)$$

(c). Homer conclude that $P(H|E) < P(H|E^c)$. But even $P(H|E,D) < P(H|E^c,D)$ and $P(H|E,D^c) < P(H|E^c,D^c)$, it does mean $P(H|E) < P(H|E^c)$ because $P(D|E)$ could be high.

II. Gambler's Ruin.

1. Let W be the event that the gambler is ahead of \$2. Let A be the event that the gambler wins the first game.

Let

$$p_i = P(W \mid \text{start at } \$i).$$

Conditioning on the first step,

$$\begin{aligned} p_i &= P(W \mid \text{start at } \$i, A)\, P(A) \\ &\quad + P(W \mid \text{start at } \$i, A^c)\, P(A^c) \\ &= p_{i+1} \cdot \tfrac{1}{3} + p_{i-1} \cdot \tfrac{2}{3} \end{aligned}$$

where $i \in \mathbb{Z}$ and $i \geq 1$.

Following the same procedure in the textbook, we have

$$\begin{aligned} p_i &= a \cdot 1^i + b\left(\frac{2/3}{1/3}\right)^i \\ &= a \cdot 1^i + b \cdot 2^i. \end{aligned}$$

The boundary conditions are

$$\begin{cases} P_0 = 0 \\ P_{i+2} = 1 \end{cases}$$

Then,

$$\begin{cases} 0 = a + b \\ 1 = a + b \cdot 2^{i+2}. \end{cases}$$

$$\Rightarrow \quad b = -a = \frac{1}{2^{i+2}-1} > 0$$

We want to find P_N where $N = 10^6$.

$$\begin{aligned} P_N &= a \cdot 1^N + b \cdot 2^N \\ &= b(2^N - 1) \\ &= \frac{2^N - 1}{2^{N+2}-1} \\ &= \frac{1}{4}\left(\frac{2^{N+2}-1-3}{2^{N+2}-1}\right) < \frac{1}{4}. \end{aligned}$$

Q.E.D.

IV. Bernoulli and Binomial

1.

(a). The possible # total games are 4, 5, 6, and 7. Then, the Pr. that team A wins the series is

$$p^4 + \binom{4}{3} p^3 (1-p) p + \binom{5}{3} p^3 (1-p)^2 p + \binom{6}{3} p^3 (1-p)^3 p$$

[Last game's winner must be Team A.]

{I misunderstand the problem.}

(b). ... teams stop playing more games as soon as one team has won 4 games.

2.

Let X = #success. Then $X \sim Bin(n,p)$.
Let A_i be the i^{th} possible list of outcomes
Then,

$$P(A_i \mid X=k) = \frac{P(A_i, X=k)}{P(X=k)}$$

$\{A_i \subseteq X=k\}$

$$= \frac{P(A_i)}{P(X=k)}$$

$$= \frac{p^k q^{n-k}}{\binom{n}{k} p^k q^{n-k}} = \frac{1}{\binom{n}{k}}$$

for any i, k. Q.E.D.

3.

(a). story proof:
$n+m$ independent trials with Pr. of success p. $X+Y$ = (#success)

(b). Let $Z = X-Y$. Then.

$$P(Z=k) = \sum_{j=0}^{n} P(Z=k \mid X=j)\, P(X=j)$$

$$= \sum_{j=0}^{n} P(Y=j-k)\, P(X=j)$$

Assuming $Z \sim Bin$, so $k>0$. This will definitely cause $Y=j-k<0$ when $j=0$, which is not allowed.

(c).

$$P(X=k \mid X+Y=j)$$

$$= \frac{P(Y=j-k)\,P(X=k)}{P(X+Y=j)}$$

$$= \frac{\binom{m}{j-k} p^{j-k} q^{m-j+k} \binom{n}{k} p^k q^{n-k}}{\binom{n+m}{j} p^j q^{n+m-j}}$$

$$= \frac{\binom{m}{j-k}\binom{n}{k}}{\binom{n+m}{j}}$$

✓

$$\Rightarrow X \mid X+Y=j \sim \mathrm{HGeom}(n, m, j)$$

IV. 1. (b).

Ans to (a) does not depend on whether the teams play all seven games. Imaging telling players to continue playing even after the match is decided, THIS WON'T INFLUENCE THE RESULT, i.e., A wins at 5, no matter A wins or loses for the remaining 2 games, A still wins the series.

We then let $X = \#(A \text{ wins}) \sim \mathrm{Bin}(7, p)$

$$P(A \text{ wins}) = P(X \geq 4)$$

$$= \sum_{k=4}^{7} P(X=k)$$