[sp-2]

I. Inclusive - Exclusive.

1. Let A_j be the event that j^{th} season does not occur among these 7 people. We want to find $1-P(A_1 \cup A_2 \cup A_3 \cup A_4)$.

$$P(A_1) = \frac{3^7}{4^7} = \left(\frac{3}{4}\right)^7$$

$$P(A_1 \cap A_2) = \frac{2^7}{4^7} = \left(\frac{1}{2}\right)^7$$

$$P(A_1 \cap A_2 \cap A_3) = \left(\frac{1}{4}\right)^7$$

$$P(A_1 \cap \cdots \cap A_4) = 0$$

By inclusion-exclusion and symmetry,

$$\begin{aligned} P(A_1 \cup A_2 \cup A_3 \cup A_4) &= \binom{4}{1} P(A_1) - \binom{4}{2} P(A_1 \cap A_2) \\ &\quad + \binom{4}{3} P(A_1 \cap A_2 \cap A_3) \\ &\quad - \binom{4}{4} P(A_1 \cap \cdots \cap A_4) \\ &= 4\left(\frac{3}{4}\right)^7 - 6\left(\frac{1}{2}\right)^7 \\ &\quad + 4\left(\frac{1}{4}\right)^7 \end{aligned}$$

Then, we can compute

$$1 - P(A_1 \cup \cdots \cup A_4).$$

2.

Doing this by inclusion-exclusion, we define A_j be the event that there is no class on j^{th} day. We want to find $1 - P(A_1 \cup \cdots \cup A_5)$.

But I want to solve this problem by counting directly.

$$\# \text{favorable outcomes} = \underbrace{\binom{5}{1}\binom{6}{3}6^4}_{\text{1 day with 3 classes}} + \underbrace{\binom{5}{2}\binom{6}{2}^2 6^3}_{\text{2 days with 2 classes}}$$

$$\# \text{total outcomes} = \binom{30}{7}.$$

$\Rightarrow Pr = 0.3024.$ ✓

II. Independence.

1. Not possible. ✗

2. Always true.

<u>Pf</u>: Given $P(A \cap B) = P(A)P(B)$,

<u>WTS</u>: $P(A^c \cap B^c) = P(A^c)P(B^c)$

$$\begin{aligned} P(A^c \cap B^c) &= 1 - P(A \cup B) \\ &= 1 - [P(A) + P(B) - P(A)P(B)] \\ &= 1 - P(B) - P(A)(1 - P(B)) \\ &= (1 - P(B))(1 - P(A)) \\ &= P(A^c)P(B^c) \end{aligned}$$

Q.E.D. ✓

3. We have two fair dices.

Let A be the event that 1^{st} dice is 1

Let B be the event that 2^{nd} dice is 6

Let C be the event that the sum of 1^{st} & 2^{nd} is 7.

We have,

$$P(A) = \frac{1}{6}, \quad P(B) = \frac{1}{6}, \quad P(C) = \frac{1}{6}.$$

And,

$$P(C|A) = P(C|B) = P(C)$$
$$P(B|A) = P(B)$$

$\Rightarrow$ A, B, C are pairwise independent.

But $P(C|A,B) = 1 \neq P(C)$

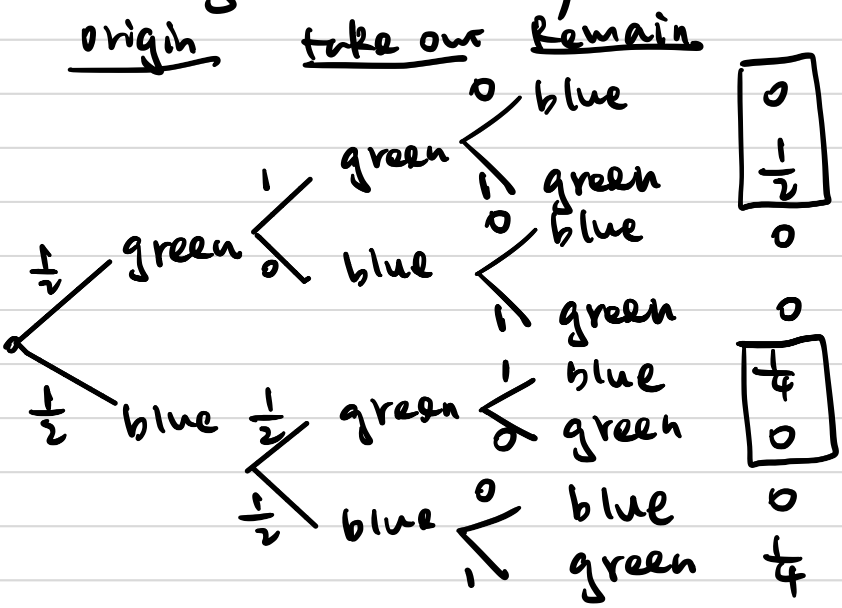
$\Rightarrow$ A, B, C are not independent. ✓

4. We can let $P(A) = 0$, B and C are dependent.

$$P(A \cap B \cap C) = 0 = P(A)P(B)P(C).$$ ✓

III. Thinking Conditionally.

1.



Given the taken-out ball is green, the "new universe" is the boxed four possibilities. Therefore, the Pr. is

$$\left(\frac{1}{2}\right) / \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{2}{3}$$

2. Let S be the event that the email is spam. Let F be the event that the email contains "free money".
<u>WTF</u>: $P(S|F)$.

By Bayes' Rule,

$$P(S|F) = \frac{P(F|S)P(S)}{P(F)}$$

$$= \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|S^c)P(S^c)}$$

$$= \frac{0.1 \times 0.8}{0.1 \times 0.8 + 0.01 \times 0.2}$$

$$= 0.9756.$$

3.
(a). Yes, it is possible. ~~{I can't show it mathematically}~~
For example, a murder happened during 8 p.m. to 9 p.m. E_1 is the event that the suspect were in a nearby bar from 8 p.m. to 8:30 p.m. E_2 is the event that the suspect were in a nearby bar from 8:30 p.m to 9 p.m.

(b).

$$P_{new}(G \mid E_2) = \frac{P_{new}(G, E_2)}{P_{new}(E_2)}$$

$$= \frac{P(G, E_2 \mid E_1)}{P(E_2 \mid E_1)}$$

multiply by $P(E_1)$ on both numerator & denominator

$$= \frac{P(G, E_1, E_2)}{P(E_1, E_2)}$$

$$= P(G \mid E_1, E_2)$$

4.

(a). Let A, B be events that A, B commit the crime, respectively. Let M_A, M_B be the events that A, B match the blood type, respectively.

$$P(A \mid M_A) = \frac{P(M_A \mid A) P(A)}{P(M_A)}$$

$$= \frac{P(M_A \mid A) P(A)}{P(M_A \mid A) P(A) + P(M_A \mid B) P(B)}$$

$$= \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2}} = \frac{10}{11}.$$

(b). WTF: $P(M_B | M_A)$

$$P(M_B|M_A) = P(M_B|M_A, B)\,P(B|M_A) + P(M_B|M_A, A)\,P(A|M_A)$$
$$= P(M_B|M_A, B)\,P(B|M_A) + P(M_B|A)\,P(A|M_A)$$
$$= 1 \times \left(1 - \frac{10}{11}\right) + \frac{1}{10} \times \frac{10}{11}$$
$$= \frac{2}{11}$$

5.

Let L_1, L_2, L_3 be the events that my opponent is beginner, intermediate, and master, respectively.

(a). Let A be the event of winning the 1^{st} game.

$$P(A) = \sum_{i=1}^{3} P(A|L_i)\,P(L_i)$$
$$= \frac{9}{10} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{3}{10} \times \frac{1}{3}$$
$$= \frac{17}{30}$$

(b). Let B be the event of winning the 2^{nd} game. WTF: $P(B|A)$

$$P(B|A) = \sum_{i=1}^{3} P(B|A, L_i)\, P(L_i|A)$$

$$= \sum_{i=1}^{3} P(B|L_i)\, P(L_i|A) \quad \downarrow \{\text{conditional indep.}\}$$

$$= \sum_{i=1}^{3} P(B|L_i) \frac{P(A|L_i) P(L_i)}{P(A)}$$

$$= \frac{30}{17}\left(\frac{9}{10} \times \frac{9}{10} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} + \frac{3}{10} \times \frac{3}{10} \times \frac{1}{3}\right)$$

$$= \frac{23}{34}.$$

(c). Assuming Conditional indep. is more reasonable.

II. 1.

Assuming an event A is independence s.t.
$P(A \cap A) = P(A) \cdot P(A)$.

Since $P(A) = P(A \cap A)$, then
$P(A) = P(A) \cdot P(A)$

$\Rightarrow P(A) = 0$ or $P(A) = 1$.