[SP-10]

I. Conditional Expectation & Conditional Variance

1.

$$E[(Y-E[Y|X])^2|X]$$
$$= E[Y^2 - 2YE[Y|X] + (E[Y|X])^2 |X]$$
$$= E[Y^2|X] - 2E[YE[Y|X]|X] + (E[Y|X])^2$$
$$= E[Y^2|X] - 2E[Y|X]\cdot E[Y|X] + (E[Y|X])^2$$
$$= E[Y^2|X] - (E[Y|X])^2.$$

Q.E.D.

2. Prove Eve's Law.

Pf:

$$\mathrm{Var}(E[Y|X]) = E[(E[Y|X])^2] - (E[E[Y|X]])^2$$
$$= E[(E[Y|X])^2] - (E[Y])^2 \quad \cdots (1)$$

$$\mathrm{Var}(Y|X) = E[Y^2|X] - (E[Y|X])^2$$
$$\Rightarrow E[\mathrm{Var}(Y|X)] = E[Y^2] - E[(E[Y|X])^2] \quad \cdots (2)$$

By (1) + (2),

$$E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X])$$
$$= E[Y^2] - (E[Y])^2 = \mathrm{Var}(Y)$$

Q.E.D.

3.

(a). $E[W] = E[Y - E[Y|X]] = 0.$

$$E[W|X] = E[Y - E[Y|X] \mid X]$$
$$= E[Y|X] - E[E[Y|X] \mid X]$$
$$= E[Y|X] - E[Y|X] = 0$$

(b). By Eve's Law,

$$Var(W) = E[Var(W|X)] + Var(E[W|X])$$
$$= E[X^2] + 0$$
$$= 1$$

4.

(a). Write

$$X = X_1 + \cdots + X_N$$

where $N-1 \sim Geom(p)$, $X|N \sim Gamma(N, \lambda)$

Then,

$$E[X] = E[E[X|N]] = E[N/\lambda] = \frac{1}{\lambda p}$$

$$Var(X) = E[Var(X|N)] + Var(E[X|N])$$
$$= E[N/\lambda^2] + Var(N/\lambda)$$
$$= \left(\frac{1}{\lambda p}\right)^2$$

(b). $E[e^{tX}] = E[E[e^{tX}|N]]$

$$= E\left[\left(\frac{\lambda}{\lambda - t}\right)^N\right] \text{ for } t < \lambda.$$

By LOTUS,

$$E\left[\left(\frac{\lambda}{\lambda-t}\right)^N\right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^n P(N=n)$$

$$= \sum_{n=1}^{\infty} \left(\frac{\lambda}{\lambda-t}\right)^n p\, q^{n-1}$$

$$= \frac{p}{q} \sum_{n=1}^{\infty} (qr)^n \quad \{\text{where } r = \frac{\lambda}{\lambda-t}\}$$

$$= \frac{p}{q}\left(\frac{1}{1-qr} - 1\right) \qquad \{\text{Geometric Series}\}$$

$$= \frac{p}{q} \frac{qr}{1-qr} = \frac{p\lambda}{p\lambda - t} \quad \text{for } t < p\lambda$$

$\Rightarrow X \sim \text{Expo}(p\lambda)$ {makes sense}.

5.

Let $Y \sim \text{Bern}(\frac{1}{2})$ s.t.

$Y=1$ if choosing coin with p_1.

$X|Y=1 \sim \text{Bin}(n, p_1)$
$X|Y=0 \sim \text{Bin}(n, p_2)$

$$E[X] = E[E[X|Y]]$$
$$= \frac{1}{2}E[X|Y=1] + \frac{1}{2}E[X|Y=0]$$
$$= \frac{1}{2}np_1 + \frac{1}{2}np_2 = \frac{n}{2}(p_1+p_2)$$

$$E[X^2] = E[E[X^2|Y]]$$
$$= \frac{1}{2}E[X^2|Y=1] + \frac{1}{2}E[X^2|Y=0]$$
$$= \frac{1}{2}[np_1(q_1+np_1)] + \frac{1}{2}[np_2(q_2+np_2)]$$

$$\Rightarrow Var(X) = E[X^2] - E[X]$$
$$= \frac{1}{2}(np_1q_1 + np_2q_2) + \frac{1}{2}n^2p_1^2 + \frac{1}{2}n^2p_2^2 - \frac{1}{4}n^2(p_1+p_2)^2$$
$$= \frac{1}{2}(np_1q_1 + np_2q_2) + \frac{1}{4}n^2(p_1-p_2)^2$$

where $q_1 = 1-p_1$, $q_2 = 1-p_2$

{ Note on 4, 5 :

In 4. $X = X_1 + \cdots + X_N$. We shall understand that

$$E[X|N] = E[X_1 + \cdots + X_N] = N\frac{1}{\lambda}$$

instead of

$$E[X] = E[X_1 + \cdots + X_N] = N\frac{1}{\lambda}$$

because

$E[X]$ is a number, $N\frac{1}{\lambda}$ is $\underbrace{\text{a fcn of } N}_{\text{a r.v.}}$.

Similarly, in 5, we can write

$$X = IX_1 + (1-I)X_2$$

where $X_i \sim Bin(n, p_i)$, $I=1$ if having p_1 coin.

{I, X_1, X_2 indep.}

We shall understand that

$$E[X|I] = I \cdot E[X_1] + (1-I)E[X_2]$$
$$= Inp_1 + (1-I)np_2$$

because $X|I$ means that we treat "I" as constant $\Rightarrow$ we can take it out.

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II. Inequality

1.

Given X, Y are i.i.d. positive r.v.s. and $c > 0$

(a). $E[\ln X] \leq \ln E[X]$
because $\ln x$ is concave.

(b). $E[X] \leq \sqrt{E[X^2]}$
because $Var(X) = E[X^2] - (E[X])^2 \geq 0$

(c). Since
$\sin^2 X + \cos^2 X = 1$
$\Rightarrow E[\sin^2 X] + E[\cos^2 X] = 1$

(d). Since $X > 0$,
$E[|X|] = E[X] \leq \sqrt{E[X^2]}$.

(e). By Markov inequality,
$P(X > c) = P(|X^3| > c^3) \leq \frac{E[|X|^3]}{c^3} = \frac{E[X^3]}{c^3}$.
$\Rightarrow P(X > c) \leq \frac{E(X^3)}{c^3}$.

(f). Since X, Y are i.i.d. positive r.v.s.,
by symmetry,
$P(X \leq Y) = P(X \geq Y)$

(g). $E[XY] = |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$
by Cauchy-Schwarz inequality.

(h). We have

$$P(X>5 \text{ or } Y>5) = 1 - P(X\le 5 \text{ and } Y\le 5)$$
$$P(X+Y>10) = 1 - P(X+Y\le 10)$$

Since

$$X\le 5 \text{ and } Y\le 5 \subseteq X+Y\le 10$$
$$\Rightarrow P(X\le 5 \text{ and } Y\le 5) \le P(X+Y\le 10)$$
$$\Rightarrow P(X+Y>10) \le P(X>5 \text{ or } Y>5)$$

(i). We know

$$E[X] = E[Y] \text{ . (i.i.d.)}$$
$$\Rightarrow \min(E[X], E[Y]) = E[X] = E[Y].$$

And

$$\begin{cases} \min(X,Y) \le X \\ \min(X,Y) \le Y \end{cases}$$
$$\Rightarrow E[\min(X,Y)] \le \min(E[X], E[Y])$$

(j). Since X, Y are i.i.d.

$$E[\tfrac{X}{Y}] = E[X]\,E[\tfrac{1}{Y}]$$

Since $g(y) = \frac{1}{y}$ for $y>0$ is convex,
by Jensen's inequality,

$$E[\tfrac{1}{Y}] \geq \frac{1}{E[Y]}$$
$$\Rightarrow E[\tfrac{X}{Y}] \geq \frac{E[X]}{E[Y]}.$$

(k). Comparing $E[X^2(X^2+1)]$ and $E[X^2(Y^2+1)]$
is equivalent to comparing
$E[X^4]$ and $E[X^2Y^2]$.
Since X, Y are i.i.d.,

$$E[X^2Y^2] = E[X^2]\,E[Y^2] = (E[X^2])^2.$$

Since $\mathrm{Var}(X^2) = E[X^4] - (E[X^2])^2 \geq 0$

$$\Rightarrow E[X^4] \geq (E[X^2])^2 \Rightarrow E[X^2(X^2+1)] \geq E[X^2(Y^2+1)]$$

(e). $E\left[\frac{X^3}{X^3+Y^3}\right] = E\left[\frac{Y^3}{X^3+Y^3}\right]$ by symmetry.

2.

(a). Since $g(x) = \frac{1}{x}$ for $x>0$ is convex,

$\Rightarrow E[1/X] > 1/E[X]$

for any positive nonconstant r.v..

(b). Given X, Y are two positive nonconstant r.v.'s.

By Cauchy-Schwarz inequality,

$$\sqrt{E\left[\frac{X}{Y}\right] E\left[\frac{Y}{X}\right]} \geq E\left[\sqrt{\frac{X}{Y}} \cdot \sqrt{\frac{Y}{X}}\right] = 1$$

Let $W = \frac{X}{Y}$, $\Rightarrow \frac{1}{W} = \frac{Y}{X}$.

$$\Rightarrow E\left[\frac{X}{Y}\right] E\left[\frac{Y}{X}\right] = E[W] \, E\left[\frac{1}{W}\right]$$

$$> E[W] \cdot \frac{1}{E[W]} = 1$$

by Jensen's inequality

3.

Let sample mean be

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

We want to find an "n" s.t.

$$P(|\bar{X}_n - \mu| > 2\sigma) \leq 0.01$$

$$Var(\bar{X}_n) = \frac{1}{n^2} \cdot n\, Var(X_i) = \frac{\sigma^2}{n}.$$

By Chebyshev inequality,

$$P(|\bar{X}_n - \mu| > 2\sigma) \le \frac{Var(\bar{X}_n)}{4\sigma^2} = \frac{\sigma^2/n}{4\sigma^2} = \frac{1}{4n}.$$

$$\Rightarrow \frac{1}{4n} \le 0.01 \Rightarrow n \geq 25.$$

4. Suppose $g(x) = \log x$ for $x > 0$.
$g''(x) = -x^{-2} < 0 \Rightarrow g(x)$ is concave.

$\Rightarrow E[\log(X)] \le \log(E[X])$ by Jensen's inequality

Assume a_j's are distinct. Let X be a random variable which takes values from $a_1, \cdots, a_n$ with equal prob.

Then, for left hand side, by LOTUS,

$$E[\log X] = \frac{1}{n}\sum_{j=1}^{n} \log a_j = \log\left(\prod_{j=1}^{n} a_j\right)^{\frac{1}{n}}.$$

For right hand side,

$$\log E[X] = \log \frac{1}{n}\sum_{j=1}^{n} a_j$$

$$\Rightarrow \log\left(\prod_{j=1}^{n} a_j\right)^{1/n} \le \log \frac{1}{n}\sum_{j=1}^{n} a_j$$

$$\Rightarrow \frac{1}{n}\sum_{j=1}^{n} a_j \geq \left(\prod_{j=1}^{n} a_j\right)^{1/n}.$$

If a_j have some repeated values,

let $P(X=a_j) = \frac{m_j}{n}$, where m_j is # a_j's

$\Rightarrow E[X] = \sum_x x \cdot P(X=x)$

$= \frac{1}{n} \cdot (\text{sum of all } a_j\text{'s})$