[SP-6]

I. Exponential Distribution and Memoryless

1.

(a). Let X be an r.v. representing offers. Given the problem,

$$X \sim \text{Expo}(\lambda)$$

Since

$$E[X] = \frac{1}{\lambda} = 10000$$

$\Rightarrow \lambda = 1/10000$

$\Rightarrow X \sim \text{Expo}\left(\frac{1}{10000}\right)$

Let Y be an r.v. representing #offers Fred will have. Then, by problem,

$$Y \sim \text{Geom}(p)$$

$$p = P(X \geq 15000)$$

$$= 1 - P(X < 15000)$$

$$= 1 - \left(1 - e^{-\frac{15000}{10000}}\right) = 0.223$$

$\Rightarrow Y \sim \text{Geom}(0.223)$

$\Rightarrow E[Y] = (1 - 0.223)/0.223 = 3.48.$

$\Rightarrow$ Expected #offers $= 3.48 + 1 = 4.48.$

(b). Find the expected amount of money that Fred gets for the car.

We know that Fred accept the offer only if the offer is Larger than \$15000.

WTF: $E[X \mid X \geq 15000]$

By memoryless property,

$$E[X \mid X \geq 15000] = E[X - 15000 + 15000 \mid X \geq 15000]$$
$$= 15000 + E[X - 15000 \mid X \geq 15000]$$
$$= 15000 + E[X] = 25000.$$

2. Find $E[X^3]$ for $X \sim Expo(\lambda)$

By LOTUS

$$E[X^3] = \int_0^\infty x^3 \lambda e^{-\lambda x}\, dx$$

$$= x^3(-e^{-\lambda x})\Big|_0^\infty - \int_0^\infty 3x^2(-e^{-\lambda x})\, dx$$

$$= 0 + \frac{3}{\lambda}\int_0^\infty x^2 \lambda e^{-\lambda x}\, dx$$

$$= \frac{3}{\lambda} E[X^2]$$

$$= \frac{3}{\lambda}\{ Var(X) + (E[X])^2 \}$$

$$= \frac{3}{\lambda}\left[\frac{1}{\lambda^2} + \left(\frac{1}{\lambda}\right)^2\right]$$

$$= \frac{6}{\lambda^3}.$$

✓

3. We find the CDF of M.

$$P(M \le m) = P(\min(X_1, \cdots, X_n) \le m)$$

$$= 1 - P(\min(X_1, \cdots, X_n) > m).$$

Then,

$$P(\min(X_1, \cdots, X_n) > m)$$

$$= P(X_1 > m, \cdots, X_n > m)$$

$$= P(X_1 > m) \cdots P(X_n > m) \quad \text{by independence.}$$

$$= e^{-\lambda_1 m} e^{-\lambda_2 m} \cdots e^{-\lambda_n m}$$

$$= e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)m}$$

Therefore,

$$P(M \le m) = 1 - e^{-(\lambda_1 + \lambda_2 + \cdots + \lambda_n)m}$$

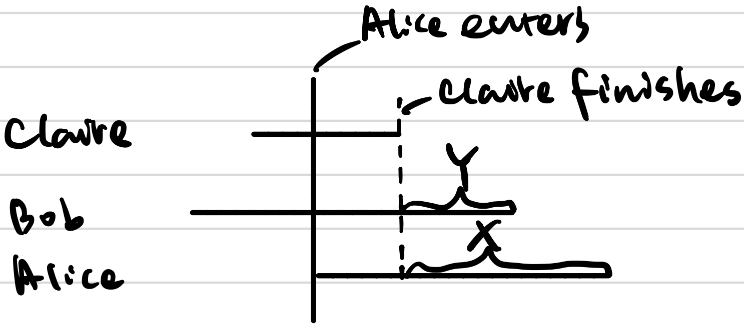
$$\Rightarrow M \sim Expo(\lambda_1 + \lambda_2 + \cdots + \lambda_n).$$

✓

Intuition: r, } Check the end of the file { .

4.

(a).



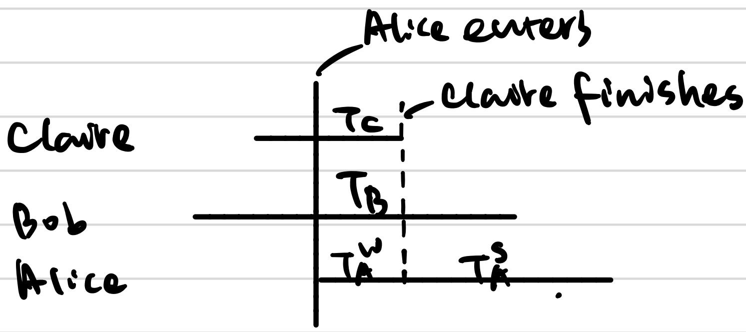
Without loss of generality, we assume Claire finishes first. By memoryless property, $X, Y \sim Expo(\lambda)$. By symmetry,

$$P(X>Y) = P(Y>X).$$

Since $P(X=Y)=0$,

$$\Rightarrow P(X>Y) = \frac{1}{2}.$$

(b).



Let T_A be an r.v. representing the total time Alice spends. T_A consists of waiting time T_A^w and serving time T_A^s:

$$T_A = T_A^w + T_A^s.$$

Based on Question 3,

$$T_A^w = \min(T_B, T_C) \sim Expo(2\lambda)$$

where $T_B, T_C \sim Expo(\lambda)$ and are independent.

And,

$$T_A^s \sim Expo(\lambda)$$

$$E[T_A] = E[T_A^w] + E[T_A^s] = \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{3}{2\lambda}.$$

II. Moment Generating Functions (MGFs).

1. Find $E[X^3]$ for $X \sim Expo(\lambda)$ by using MGF.
In Lecture, we know

$$E[X^3] = \frac{3!}{\lambda^3} = \frac{6}{\lambda^3}.$$

{comparing to I.2.}

2. Suppose X has MGF $M(t)$.
The MGF of $-X$ is

$$M_{-X}(t) = E[e^{t(-X)}] = E[e^{-tX}] = M(-t)$$

The MGF of $Y = aX + b$ is → copied wrongly.

$$M_Y(t) = E[e^{t(aX+b)}] = E[e^{atX + bt}] = e^{bt} E[e^{atX}] = e^{bt} M(at)$$

3. Since $U_1, U_2, \cdots, U_{60}$ be i.i.d. Unif(0,1),

$$M_X(t) = M_{U_1}(t) \cdots M_{U_{60}}(t).$$

Let $U \sim Unif(0,1)$. Then,

$$M_U(t) = E[e^{tU}] = \int_0^1 e^{tu}\,du = \frac{1}{t} e^{tu}\Big|_0^1 = \frac{1}{t}(e^t - 1)$$

for $t \neq 0$

Therefore,

$$M_X(t) = \frac{(e^t - 1)^{60}}{t^{60}}$$

for $t = 0$, $M_X(t) = 1$.

4. In lecture, we have MGF of $X \sim \text{Pois}(\lambda)$

$$M(t) = e^{\lambda(e^t-1)}$$

Then,

$$\begin{aligned} g(t) &= \ln M(t) \\ &= \lambda(e^t-1) \\ &= \lambda \sum_{j=1}^{\infty} \frac{t^j}{j!} \\ &= \sum_{j=1}^{\infty} \frac{\lambda}{j!} t^j . \end{aligned}$$

$\Rightarrow$ j^{th} cumulant of X is λ for all $j \geq 1$. ✓

2.3.

Intuition: $M = \min(X_1, \cdots, X_n)$, it makes sense that M should have a continuous, memoryless distribution ($\Rightarrow$ Exponential).

If we intepréte λ_j as rate, it makes sense that M has a combined rate $\lambda_1 + \lambda_2 + \cdots + \lambda_n$. X_1: wait time for a green car to pass by, X_2: ... blue car, ... M: wait time for a car with any of these colors.