[SP-7]

1. Joint, Conditional, and Marginal Distribution.

1.

(a). The volume of ball B is

$$V = \frac{4}{3}\pi 1^3 = \frac{4}{3}\pi.$$

$$\Rightarrow f(x,y,z) = \begin{cases} \frac{3}{4\pi}, & x^2+y^2+z^2 \le 1. \\ 0, & \text{otherwise} \end{cases}$$

(b).

$$f_{X,Y}(x,y) = \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\,dz$$

$$= \frac{3}{2\pi}\sqrt{1-x^2-y^2}. \qquad x^2+y^2 \le 1$$

(c).

$$f_X(x) = \int_y \int_z f(x,y,z)\,dz\,dy$$

$$= \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{3}{2\pi}\sqrt{1-x^2-y^2}\,dy$$

2. <u>WTF</u>: $E[|X-Y|]$ and $SD(|X-Y|)$

By LOTUS:

$$E[|X-Y|] = \iint |x-y|\,f(x,y)\,dx\,dy$$

$$= \iint |x-y| \cdot 1 \cdot dx\,dy$$

$$= \iint_{x \ge y} (x-y)\,dx\,dy + \iint_{x<y} (y-x)\,dx\,dy$$

(two disjoint triangles)

$x \ge y$

{Changing x, y makes no difference}.

By symmetry, we have

$$= 2\iint_{x\geq y} (x-y)\,dx\,dy$$

$$= 2\int_0^1\int_y^1 (x-y)\,dx\,dy$$

$$= 2\int_0^1 \tfrac{1}{2}y^2 - y + \tfrac{1}{2}\,dy$$

$$= \frac{1}{3}.$$

To find $SD(|X-Y|)$, we first find $Var(|X-Y|) = E[|X-Y|^2] - (E[|X-Y|])^2$

By LOTUS,

$$E[|X-Y|^2] = \iint (x-y)^2 f(x,y)\,dx\,dy$$

$$= \int_0^1\int_0^1 (x^2 - 2xy + y^2)\,dx\,dy$$

$$= \frac{1}{6}$$

$$\Rightarrow Var(|X-Y|) = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}.$$

$$\Rightarrow SD(|X-Y|) = \sqrt{\frac{1}{18}} = \frac{\sqrt{2}}{6}.$$

3.

(a). WTF: CDF & PDF of M.

$$P(M\leq t) = P(U_1\leq t, U_2\leq t, U_3\leq t)$$

$$= P(U_1\leq t)\,P(U_2\leq t)\,P(U_3\leq t)$$

$$= t^3 \quad 0\leq t\leq 1$$

$$\Rightarrow F_M(t) = \begin{cases} 0, & t<0 \\ t^3, & 0\leq t\leq 1 \\ 1, & t>1 \end{cases}$$

Therefore, PDF of M is

$$f_M(t) = \begin{cases} 3t^2, & 0 \le t \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

WTF: joint CDF & PDF of L and M.

$\{ F(l,m) = P(L \le l, M \le m) \}$

We first consider $P(L \ge l, M \le m)$.

$$P(L \ge l, M \le m) = P(l \le U_1 \le m,\ l \le U_2 \le m,\ l \le U_3 \le m)$$
$$= [P(l \le U_1 \le m)]^3$$
$$= (m-l)^3 \quad \text{for } l \le m,\ 0 \le l, m \le 1$$

Since

$$P(M \le m) = P(L \le l, M \le m) + P(L > l, M \le m)$$

$\Rightarrow$

$$F(l,m) = P(M \le m, L \le l)$$
$$= P(M \le m) - P(L > l, M \le m)$$
$$= m^3 - (m-l)^3$$
$$\text{for } 0 \le l \le m \le 1$$

$$\Rightarrow f(l,m) = \frac{\partial^2}{\partial l \partial m} F(l,m)$$
$$= 6(m-l) \quad \text{for } 0 \le l \le m \le 1.$$

(b). WTF: $f_{M|L}(m|l)$.

$$f_{M|L}(m|l) = \frac{f(m,l)}{f_L(l)} \quad 0 \le l \le m \le 1 .$$

We first find $f_L(l)$. The CDF of L is

$$\begin{aligned} F_L(l) &= P(L \le l) \\ &= 1 - P(L > l) \\ &= 1 - P(U_1 > l, U_2 > l, U_3 > l) \\ &= 1 - P(U_1 > l)P(U_2 > l)P(U_3 > l) \\ &= 1 - (1-l)^3 \quad \text{for } 0 \le l \le 1 \end{aligned}$$

$$\Rightarrow f_L(l) = \frac{\partial}{\partial l} F_L(l) = 3(1-l)^2$$

Hence,

$$f_{M|L}(m|l) = \frac{6(m-l)}{3(1-l)^2} = \frac{2(m-l)}{(1-l)^2} \quad \text{for } 0 \le l \le m \le 1$$

4.

(a). From the problem, we know $(X, Y, Z) \sim \text{Mult}_3(n, (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}))$

Therefore, the joint PMF of X, Y, Z is

$$P(X=x, Y=y, Z=z)$$

$$= \frac{n!}{x!y!z!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^z$$

$$= \frac{n!}{x!y!z!} \left(\frac{1}{3}\right)^n \quad \text{where } x+y+z=n.$$

(b). The game is decisive is equivalent to the situation that ONLY one of {#Rock, #Paper, #Scissors} is zero.

$$P(\text{decisive}) = \sum_{x=1}^{n-1} P(X=x, Y=n-x, Z=0)$$
$$+ \sum_{x=1}^{n-1} P(X=x, Y=0, Z=n-x)$$
$$+ \sum_{y=1}^{n-1} P(X=0, Y=y, Z=n-y)$$

By symmetry, the three terms on the right are equal. We focus on the first term.

$$\sum_{x=1}^{n-1} P(X=x, Y=n-x, Z=0)$$
$$= \sum_{x=1}^{n-1} \frac{n!}{x!(n-x)!0!}\left(\frac{1}{3}\right)^n$$
$$= \left(\frac{1}{3}\right)^n \sum_{x=1}^{n-1} \binom{n}{x}$$
$$= \left(\frac{1}{3}\right)^n \left(\sum_{x=0}^{n} \binom{n}{x} - \binom{n}{0} - \binom{n}{n}\right)$$
$$= \left(\frac{1}{3}\right)^n (2^n - 2)$$

$$\Rightarrow P(\text{decisive}) = 3 \cdot \left(\frac{1}{3}\right)^n (2^n - 2)$$
$$= \frac{2^n - 2}{3^{n-1}}.$$

(c). When $n=5$,

$$P(\text{decisive}) = \frac{2^5-2}{3^{5-1}} = \frac{10}{27}$$

when $n \to \infty$,

$$P(\text{decisive}) \to 0.$$

This makes sense because intuitively, when total number of participants is very large, it is highly likely that there is at least one Rock, Paper and Scissors.

5. From problem, we have

$$X \mid N \sim Bin(N, s) \quad \# \text{ survived}$$

$$Y \mid N \sim Bin(N, 1-s) \quad \# \text{ died.}$$

We first find $P(X=x, Y=y)$.

$$P(X=x, Y=y)$$

$$= \sum_{k=0}^{n} P(X=x, Y=y \mid N=k) P(N=k)$$

$$= P(X=x \mid N=x+y) P(N=x+y)$$

$$= \binom{x+y}{x} s^x (1-s)^y \binom{n}{x+y} p^{x+y} (1-p)^{n-x-y}$$

$$= \frac{n!}{x!\,y!\,(n-x-y)!} (sp)^x ((1-s)p)^y (1-p)^{n-x-y}$$

$$\Rightarrow (X, Y, Z) \sim Mult(n, (sp, (1-s)p, 1-p)).$$

where $X+Y+Z=n$ and Z is # non-hatched.

Intuitively, each egg independently fall into 3 categories: hatched-and-survived, hatched-and-died, don't hatched.

Then, the marginal PMF of X is

$$P(X=x) = \binom{n}{x} (sp)^x (1-sp)^{n-x}$$

$\{ X \sim Bin(n, sp) \}$

$\{$ We can also get it via a story proof $\}$.