[HW 2]

I. Inclusive - Exclusive.

1. Let A_j be the event that j^{th} season does not occur among these 7 people. We want to find $1 - P(A_1 \cup A_2 \cup A_3 \cup A_4)$.

$$P(A_1) = \frac{3^7}{4^7} = \left(\frac{3}{4}\right)^7$$

$$P(A_1 \cap A_2) = \frac{2^7}{4^7} = \left(\frac{1}{2}\right)^7$$

$$P(A_1 \cap A_2 \cap A_3) = \left(\frac{1}{4}\right)^7$$

$$P(A_1 \cap \cdots \cap A_4) = 0$$

By inclusion-exclusion and symmetry,

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = \binom{4}{1} P(A_i) - \binom{4}{2} P(A_1 \cap A_2) + \binom{4}{3} P(A_1 \cap A_2 \cap A_3) - \binom{4}{4} P(A_1 \cap \cdots \cap A_4)$$

$$= 4\left(\frac{3}{4}\right)^7 - 6\left(\frac{1}{2}\right)^7 + 4\left(\frac{1}{4}\right)^7$$ ✓

Then, we can compute

$$1 - P(A_1 \cup \cdots \cup A_4).$$

2.

Doing this by inclusion-exclusion, we define A_j be the event that there is no class on j^{th} day. We want to find $1-P(A_1 \cup \cdots \cup A_5)$.

But I want to solve this problem by counting directly.

favorable outcomes

$$= \underbrace{\binom{5}{1}\binom{6}{3}6^4}_{\text{1 day with 3 classes}} + \underbrace{\binom{5}{2}\binom{6}{2}^2 6^3}_{\text{2 days with 2 classes}}$$

total outcomes

$$= \binom{30}{7}.$$

$\Rightarrow Pr = 0.3024.$ ✓

II. Independence

1. Not possible. ✗

2. Always true.

Pf: Given $P(A \cap B) = P(A)P(B)$,

WTS: $P(A^c \cap B^c) = P(A^c)P(B^c)$

$$\begin{aligned}
P(A^c \cap B^c) &= 1 - P(A \cup B) \\
&= 1 - [P(A) + P(B) - P(A)P(B)] \\
&= 1 - P(B) - P(A)(1-P(B)) \\
&= (1-P(B))(1-P(A)) \\
&= P(A^c)P(B^c) \quad \text{Q.E.D.}
\end{aligned}$$

✓

3. We have two fair dices.
Let A be the event that 1^{st} dice is 1
Let B be the event that 2^{nd} dice is 6
Let C be the event that the sum of 1^{st} & 2^{nd} is 7.
We have,
$$P(A) = \frac{1}{6}, \quad P(B) = \frac{1}{6}, \quad P(C) = \frac{1}{6}.$$
And,
$$P(C|A) = P(C|B) = P(C)$$
$$P(B|A) = P(B)$$
$\Rightarrow$ A, B, C are pairwise independent.

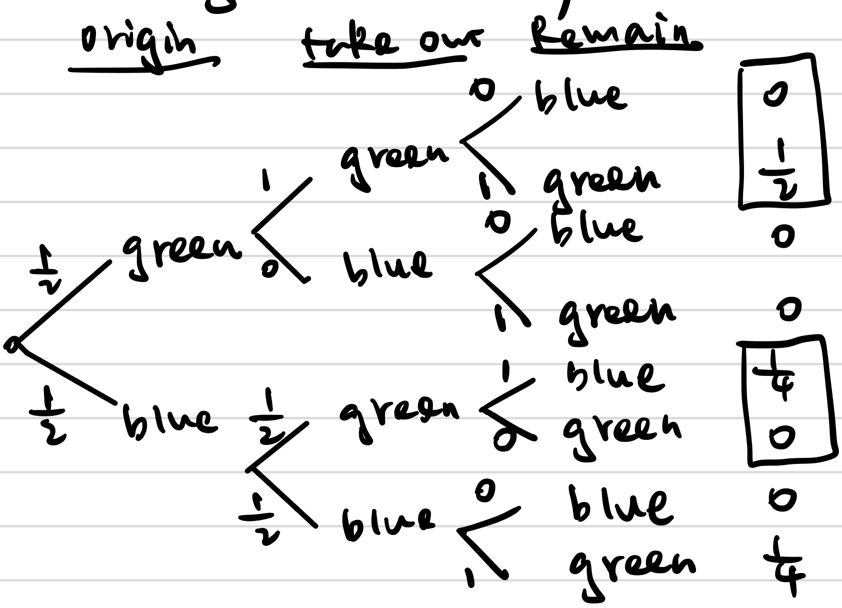
But $P(C|A,B) = 1 \neq P(C)$

$\Rightarrow$ A, B, C are not independent. ✓

4. We can let $P(A) = 0$, B and C are dependent.
$$P(A \cap B \cap C) = 0 = P(A)P(B)P(C).$$ ✓

III. Thinking Conditionally.

1.



Given the taken-out ball is green, the "new universe" is the boxed four possibilities. Therefore, the Pr. is

$$\left(\frac{1}{2}\right) / \left(\frac{1}{2} + \frac{1}{4}\right) = \frac{2}{3}$$

2. Let S be the event that the email is spam. Let F be the event that the email contains "free money".
<u>WTF</u>: $P(S|F)$.

By Bayes' Rule,

$$P(S|F) = \frac{P(F|S)P(S)}{P(F)}$$

$$= \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|S^c)P(S^c)}$$

$$= \frac{0.1 \times 0.8}{0.1 \times 0.8 + 0.01 \times 0.2}$$

$$= 0.9756.$$

3.
(a). Yes, it is possible. { I can't show it mathematically }
For example, a murder happened during 8 p.m. to 9 p.m. E_1 is the event that the suspect were in a nearby bar from 8 p.m. to 8:30 p.m. E_2 is the event that the suspect were in a nearby bar from 8:30 p.m to 9 p.m.

(b).

$$P_{new}(G \mid E_2) = \frac{P_{new}(G, E_2)}{P_{new}(E_2)}$$

$$= \frac{P(G, E_2 \mid E_1)}{P(E_2 \mid E_1)}$$

multiply by $P(E_1)$ on both numerator & denominator

$$= \frac{P(G, E_1, E_2)}{P(E_1, E_2)}$$

$$= P(G \mid E_1, E_2)$$

4.

(a). Let A, B be events that A, B commit the crime, respectively. Let M_A, M_B be the events that A, B match the blood type, respectively.

$$P(A \mid M_A) = \frac{P(M_A \mid A) P(A)}{P(M_A)}$$

$$= \frac{P(M_A \mid A) P(A)}{P(M_A \mid A) P(A) + P(M_A \mid B) P(B)}$$

$$= \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{10} \times \frac{1}{2}} = \frac{10}{11}.$$

(b). WTF: $P(M_B \mid M_A)$

$$P(M_B \mid M_A) = P(M_B \mid M_A, B)\,P(B \mid M_A) + P(M_B \mid M_A, A)\,P(A \mid M_A)$$

$$= P(M_B \mid M_A, B)\,P(B \mid M_A) + P(M_B \mid A)\,P(A \mid M_A)$$

$$= 1 \times \left(1 - \frac{10}{11}\right) + \frac{1}{10} \times \frac{10}{11}$$

$$= \frac{2}{11}$$ ✓

5.

Let L_1, L_2, L_3 be the events that my opponent is beginner, intermediate, and master, respectively.

(a). Let A be the event of winning the 1st game.

$$P(A) = \sum_{i=1}^{3} P(A \mid L_i)\,P(L_i)$$

$$= \frac{9}{10} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + \frac{3}{10} \times \frac{1}{3}$$

$$= \frac{17}{30}$$ ✓

(b). Let B be the event of winning the 2nd game. WTF: $P(B \mid A)$

$$
\begin{aligned}
p(B|A) &= \sum_{i=1}^{3} p(B|A, L_i)\, p(L_i|A) \\
&= \sum_{i=1}^{3} p(B|L_i)\, p(L_i|A) \quad \Big\downarrow \{\text{conditional indep.}\} \\
&= \sum_{i=1}^{3} p(B|L_i)\, \frac{p(A|L_i)\,p(L_i)}{p(A)} \\
&= \frac{30}{17}\Big(\frac{9}{10}\times\frac{9}{10}\times\frac{1}{3} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{3} \\
&\qquad + \frac{3}{10}\times\frac{3}{10}\times\frac{1}{3}\Big) \\
&= \frac{23}{34}.
\end{aligned}
$$

(c). Assuming Conditional indep. is more reasonable.

II. 1.

Assuming an event A is independence s.t.
$p(A \cap A) = p(A) \cdot p(A)$.
Since $p(A) = p(A \cap A)$, then
$p(A) = p(A) \cdot p(A)$
$\Rightarrow p(A) = 0$ or $p(A) = 1$.