Mechanics of Solids

Course notes to accompany lectures in MEC2025F

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First printing, February 2016

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1. Introduction

The construction of Green Point Stadium, would be impossible without a knowledge of Mechanics.

1.1 Mechanics of Solids

Mechanics of Solids or Mechanics of Materials extends the ideas presented in Statics. The main assumptions of force and moment equilibrium in Statics still apply in Mechanics of Solids. The main difference being that in a Statics course it is assumed that bodies are rigid and cannot change size or shape. In a Mechanics course bodies can deform. In the real world bodies are deformable, and a study of mechanics allows calculation of these deformations which also allows predictions to be made about when things will break.

1.2 Presenting Quantities

There are two issues which will be addressed when representing any engineering quantity: with what unit will it be presented and how many digits or significant figures will be used.

1.2.1 Units

In science and math courses SI[1] derived units (Pa, N,..) and base units (K, m, s...) were the recommended way of representing units. Then any extreme number was represented using exponentiation for example 3.456×10^{-7} or 1.234×10^{5}. In science courses you can deal with objects on the scale of the universe to smaller than atoms, so avoiding prefixes helps to prevent errors in conversion.

In engineering we seldom go to these scales and it is most useful to work with units and prefixes (MPa, mm^2, kN) that are firstly more convenient to individual problems and secondly guide the engineer's intuition to whether a solution calculated is valid. An important restriction is that any unit and prefix should be consistently used from beginning to end in a problem. Changing units can invariably lead to errors.

1.2.2 Significant Figures

In this course it is recommended that all final numerical answers will be presented to four significant figures. In problems with many calculation steps, answers to intermediate steps can be presented with five significant figures to allow for checking and avoiding errors with rounding. For example 1234.5, 1234.0 and 0.012345 are all five significant figures.

[1] from the French Système international d'unités, the modern form of the metric system.

1.3 Review of Statics

To develop any ideas in Mechanics an understanding of Statics is required. The essential ideas of statics are summarised here. An in depth understanding is assumed and good sources of further information are by Beer[1] and Hellaby[2].

1.3.1 Equations of Equilibrium

All bodies in this course are assumed to be in static equilibrium. This means that the sum of the moments and the sum of the forces are equal to zero. This is described in vector form by:

$$\boxed{\begin{aligned}\sum \mathbf{F} &= 0\\ \sum \mathbf{M} &= 0.\end{aligned}} \tag{1.1}$$

It is often more useful to represent these equations in component form in a cartesian coordinate system to help sum these moments and forces and determine any unknown quantities:

$$\boxed{\begin{aligned}\sum F_x = 0, \quad \sum F_y = 0, \quad \sum F_z = 0,\\ \sum M_x = 0, \quad \sum M_y = 0, \quad \sum M_z = 0.\end{aligned}} \tag{1.2}$$

1.3.2 Freebody Diagrams

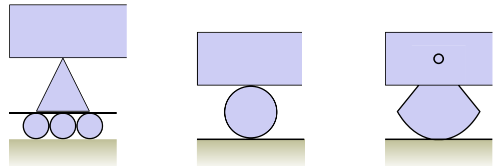
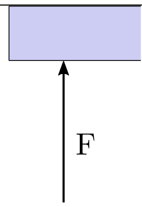
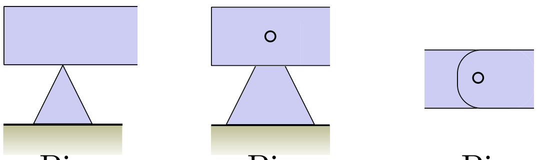
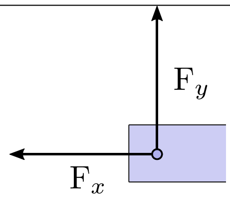
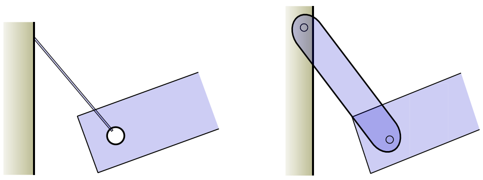
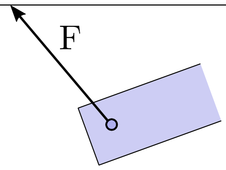
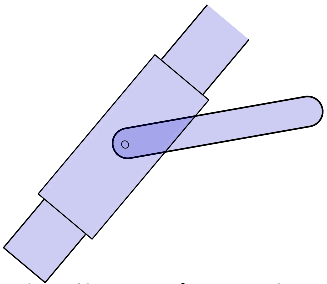
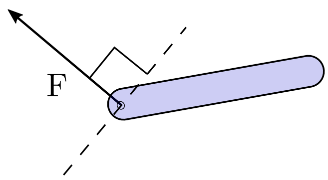
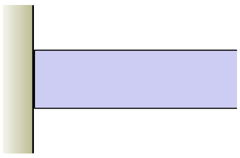
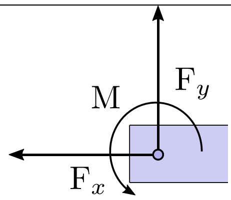
It would be impossible to calculate all unknown quantities from static equilibrium without the ability to draw **free-body diagrams**. Below are a summary of the steps to be followed when constructing such a diagram.

1. Determine the body which is important in analysis and separate it from any bodies attached to it.
2. Indicate all external applied forces and moments at their exact locations on the separated body. **Do not include internal forces.**
3. Show the forces and moments which oppose the displacement and rotation of the separated body. Surface forces that occur between two bodies in contact are called **reactions**. These reactions are shown in Table 1.1. From Newton's third law the force exterted by one body on a second body will be equal and opposite to the force exerted by the second body on the first body. The load may be applied anywhere on the bodies to cause these reactions.
4. Use the **static sign convention** in which forces have the same sign as the coordinate directions in which they are defined. Moment vectors are considered positive if they act in an anticlockwise direction.
5. Include dimensions on figure which will help in determining moments.

1.4 Analysis of Trusses

Trusses are an important class of engineering structure. The truss members are connected by pin jointed connections. Pin joints can resist or transfer forces but not moments between members. This means that the sum of forces at a pin is not zero but the sum of moments about a pin is zero. These members are considered to be slender. That means their length is much greater than their width or breadth.

Table 1.1: Support Reactions for Two Dimensional Bodies

Support or connection	Reaction
 Roller Roller Rocker	 One unknown force F
 Pin Pin Pin	 Two unknown forces F_x and F_y
 Short cable Short link or rod	 One unknown force F
 Pinned collar on frictionless rod	 One unknown force F
 Clamped, fixed or rigid	 One unknown moment M and two unknown forces F_x and F_y

When analysing truss members it is assumed that they can only resist a load along their length known as an **axial load**. They are two-force members. The forces that act on these members are equal, opposite and collinear. They can either be in tension and acted upon by a tensile force as in Figure 1.1(a) or in compression and acted on by a compressive force as in Figure 1.1(b).

Another assumption made about trusses is that the weights of the members are considered negligible i.e. not considered in calculations. This is usually a reasonable assumption since the weight of truss

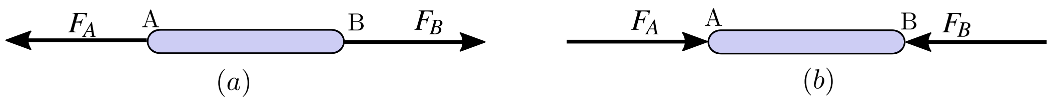


Figure 1.1: Two force members in tension(a) or compression(b)

member is small in comparison to the other loads it should carry.

Trusses where all the forces and members are in the same plane are called **planar trusses** whereas trusses which fill out three dimensions are called **space trusses**.

The basic building block of a truss is a triangle which is a **rigid** structure. It will not collapse under loading and the deformation due to internal strains are considered negligible. It is also called internally stable as seen in Figure 1.2(a). In contrast Figure 1.2(b) is unstable and will collapse under the loading shown.

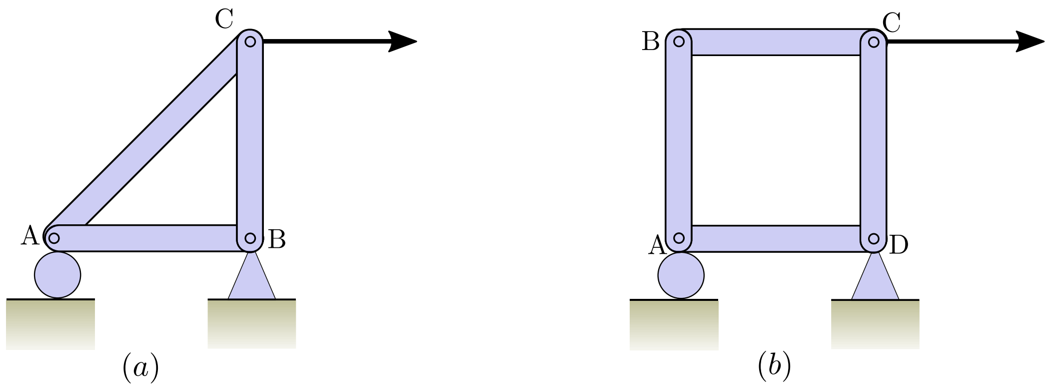


Figure 1.2: Internally stable or rigid structure(a) and unstable structure(b) are shown.

Trusses cannot resist a load perpendicular to their length otherwise known as a **lateral load**. In Figure 1.3 if ACD was a continuous member then the structure would no longer be a truss but a **frame**.

To solve for various forces in trusses there are two methods for calculating the internal forces: Method of Joints and Method of Sections. It should be noted using these methods do not include calculating the external forces which may be required.

1.4.1 Method of Joints

If an entire truss is in equilibrium then every pin in that truss must be in equilibrium. This can be represented by drawing a free body diagram of that pin and writing the equilibrium equations for each pin. Working in 2-dimensions, we have two equilibrium equations to satisfy $\sum F_x = 0$ and $\sum F_y = 0$. This idea is illustrated in the following example.

■ **Example 1.1** Calculate the forces in all the joints in Figure 1.3.

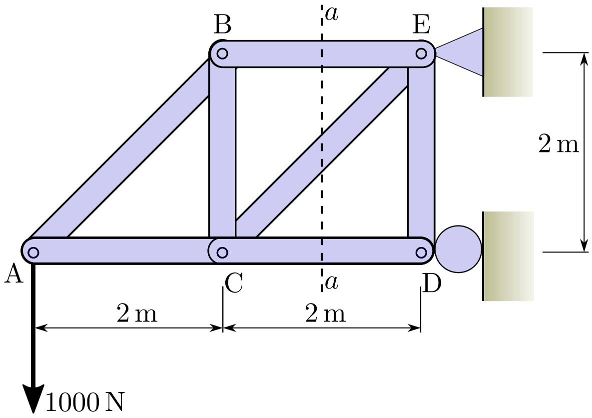


Figure 1.3: Truss structure with a concentrated load at A and simply supported at D and E. Members AC and CD are separate members but collinear. BC and ED are perpendicular to ACD.

Free body diagrams of the pins in Figure 1.3 are shown in Figure 1.4. To start the angle BAC is

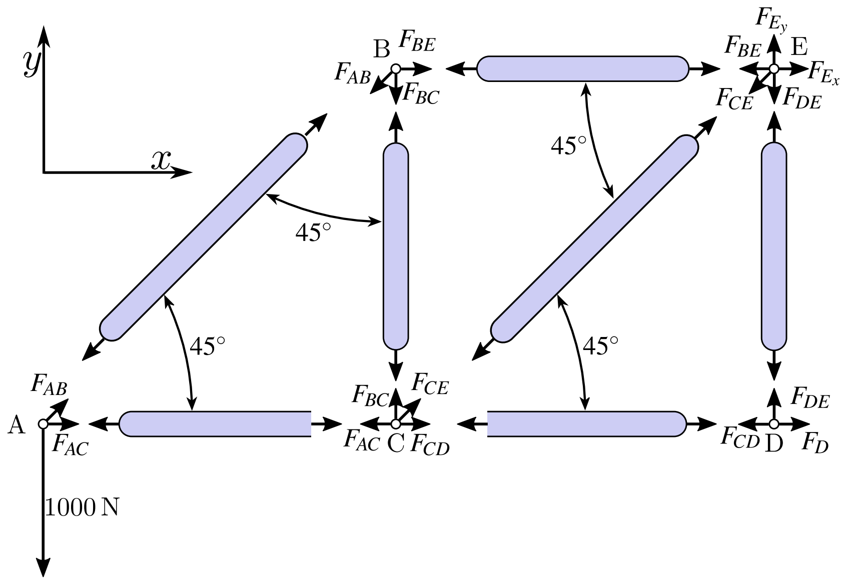


Figure 1.4: Method of joints illustrated for Figure 1.3 with all members assumed to be in tension.

calculated from the beam dimensions, knowing that BC is perpendicular to AC and the lengths of AC and BC are equal. It follows angle BAC = 45°. Similarly angle ECD = 45°.

Joint A is the only joint where there is a known force so we begin there. We start by assuming up and to the right is positive as indicated by the coordinate axes. Let's also assume all the joints are in tension. If any force turns out to be negative then the joint is in compression.

Joint A

$\Sigma F_y = 0 \Rightarrow F_{AB}\sin 45 - 1000 = 0 \Rightarrow F_{AB} = 1414.2\,\mathrm{N}$

$\Sigma F_x = 0 \Rightarrow F_{AB}\cos 45 + F_{AC} = 0 \Rightarrow F_{AC} = -1000.0\,\mathrm{N}$

Negative sign indicates F_{AC} is an opposite direction to that shown in Figure 1.4 so F_{AC} is in compression. We move onto Joint B since we now know a force F_{AB}. We know F_{AC} as well however there are more unknowns at Joint C so we do it later.

Joint B
$\Sigma F_x = 0 \Rightarrow -F_{AB}\sin 45 + F_{BE} = 0 \Rightarrow F_{BE} = 1000.0\,\mathrm{N}$
$\Sigma F_y = 0 \Rightarrow -F_{AB}\cos 45 - F_{BC} = 0 \Rightarrow F_{BC} = -1000.0\,\mathrm{N}$
F_{BC} is negative so member in compression and direction drawn is wrong.
Joint C
$\Sigma F_y = 0 \Rightarrow F_{BC} + F_{CE}\cos 45 = 0 \Rightarrow F_{CE} = 1414.2\,\mathrm{N}$
$\Sigma F_x = 0 \Rightarrow -F_{AC} + F_{CE}\sin 45 + F_{CD} = 0 \Rightarrow F_{CD} = -2000.0\,\mathrm{N}$
Joint D
$\Sigma F_x = 0 \Rightarrow -F_{CD} + F_D = 0 \Rightarrow F_D = -2000.0\,\mathrm{N}$
There is *no force in the vertical direction since Joint D is a roller* so F_{DE} is zero and there is no axial force in member DE. Member DE is therefore called a **zero-force** member.
Joint E
$\Sigma F_y = 0 \Rightarrow F_{E_y} - F_{CE}\cos 45 = 0 \Rightarrow F_{E_y} = 1000.0\,\mathrm{N}$
$\Sigma F_x = 0 \Rightarrow -F_{BE} - F_{CE}\cos 45 + F_{E_x} = 0 \Rightarrow F_{E_x} = -2000.0\,\mathrm{N}$

■

1.4.2 Method of Sections

The method of joints is useful when all the forces in a truss structure have to be determined. If only a few forces have to be determined, the method of sections may be more efficient. A section of the truss is isolated and a free body diagram is drawn for that section. The following example illustrates this idea. The isolated section will be in moment equilibrium.

■ **Example 1.2** Calculate the force in members CD and CE in Figure 1.3.

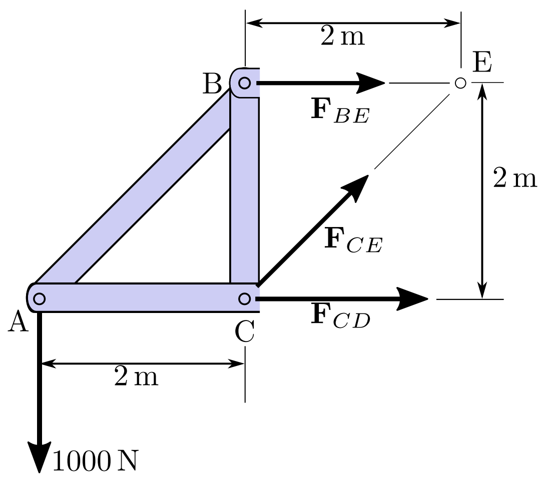


Figure 1.5: Method of sections through a-a in Figure 1.3

The whole section should be in moment equilibrium. Assume counterclockwise moments are positive and F_{CD} is positive as indicated in Figure 1.5.
$\Sigma M_E = 0 \Rightarrow 2 \times F_{CD} + (2+2) \times 1000 = 0 \Rightarrow F_{CD} = -2000.0\,\mathrm{N}$
This indicates F_{CD} is negative and direction is opposite to what is shown in Figure 1.5.
$\Sigma F_y = 0 \Rightarrow -1000 + F_{CE}\cos 45 = 0 \Rightarrow F_{CE} = 1414.2\,\mathrm{N}$ ■

1.5 Distributed Loading

Loads applied to structures are not applied to points but distributed over lines, areas and even volumes. In many instances it is convenient to replace the distributed load with a single resultant force which is **statically equivalent**. By statically equivalent we mean that the sum of forces and sum of moments for the two load cases are the same. The reactions at the same support is the same for statically equivalent systems.

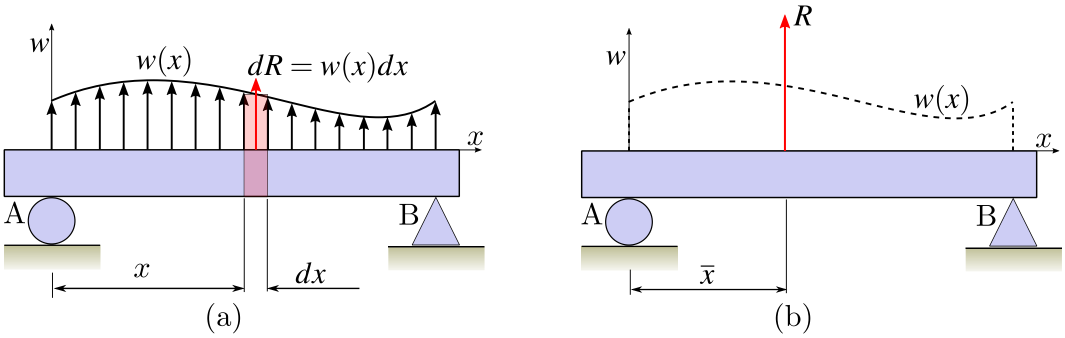


Figure 1.6: Distributed load (a) and statically equivalent force with location (b) shown.

The magnitude of the resultant force, R, is the sum of the differential forces $dR = w(x)dx$:

$$R = \int w(x)dx \tag{1.3}$$

The moment of the forces about a normal axis must be the same for the two cases.

$$R\bar{x} = \int x w(x)dx$$

$$\Rightarrow \bar{x} = \frac{\int x w(x)dx}{\int w(x)dx} \tag{1.4}$$

The resultant is located at the **centroid** of the area under the distributed load. These ideas are illustrated with the following example.

■ **Example 1.3** Determine the magnitude and location of the equivalent resultant force acting on the shaft in Figure 1.7.

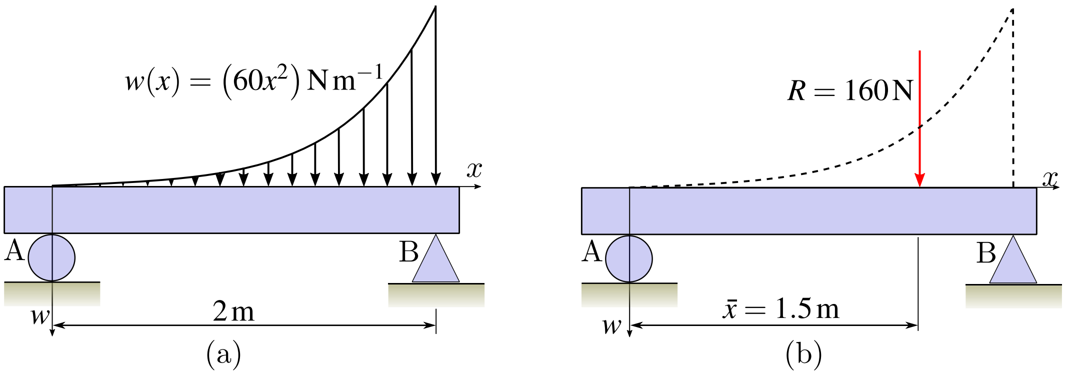


Figure 1.7: Distributed load(a) and statically equivalent force with location(b) shown.

The problem can be solved by integration since $w(x)$ is given.

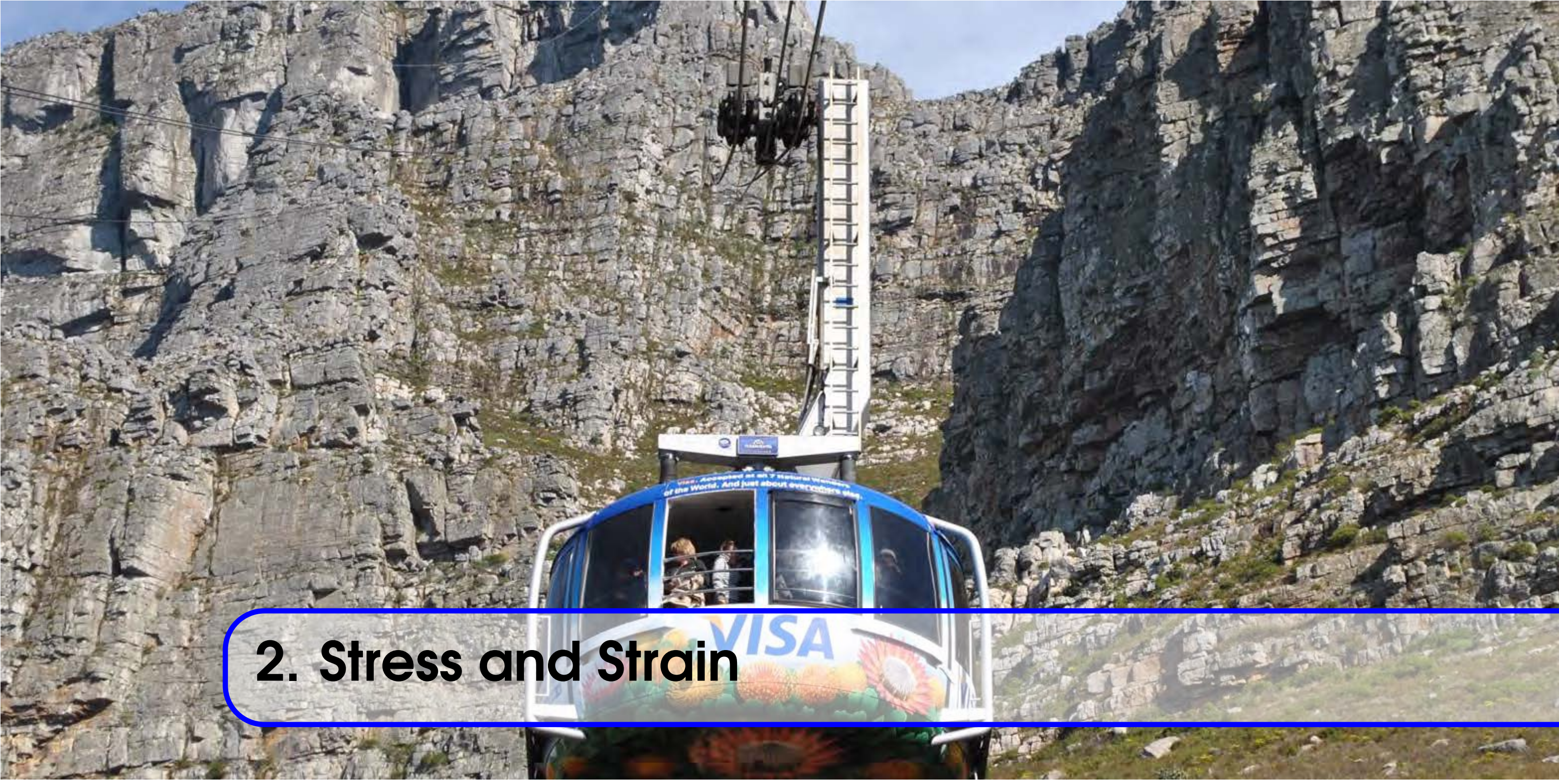
$$R = \int w(x)dx = \int_0^2 60x^2 dx = 160.0\,\mathrm{N}$$

The location of $\bar{x}$ is measured from A

$$\bar{x} = \frac{\int x w(x) dx}{\int w(x) dx} = \frac{\int_0^2 x 60x^2 dx}{\int_0^2 60x^2 dx} = 1.500\,\mathrm{m}$$

■

2. Stress and Strain



View of the cable car from lower cable station on Table Mountain. An understanding of stresses and strains in the design of the cables and other structural members is essential in their design.

2.1 Introduction

In this course we make certain assumptions about the materials that we analyse, they are **isotropic** and **homogeneous**. Isotropic materials have the same properties in all directions. These properties can be mechanical like strength or physical like thermal conductivity. Therefore if a mechanical load is applied in any direction the response of an isotropic material to that load will be the same. A material that is not isotropic is anisotropic like a fiber reinforced composite. This composite is stronger parallel to the fibres than perpendicular to the fibres. Homogeneous materials have a uniform composition with no second phase[1]. A material that is not homogeneous is heterogeneous like concrete which is mixture of sand, stone, gravel with a binder cement. A well mixed concrete is also isotropic. An example of a homogeneous material which is anisotropic is rolled steel which is stronger in the direction of rolling.

2.2 Normal Stress

One of the most important concepts in mechanics is that of stress, it is represented by the Greek letter σ (sigma). It is the force per unit area or the intensity of the internal force at a point.

[1]A phase should not be confused with a state of matter which can be solid, liquid or gas. The same state can have different phases such as oil mixed with water which is in the liquid state but has two phases, the oil and the water. A single phase however can only be a single state.

Consider a rectangular bar with uniform cross section in figure 2.1(a) which is subject to an axial force F. This is a **tension force** which elongates the bar. The **normal stress** is the stress experienced by the cross section A and is defined as follows:

$$\textbf{Normal Stress} = \frac{\text{Force normal to an area}}{\text{Area over which the force acts}}$$

Let's consider a small area ΔA and a very small force ΔF which is the resultant internal force acting normal to this area in figure 2.1(b). As ΔA tends to zero this corresponding force ΔF also approaches zero. The stress on this cross section is defined as:

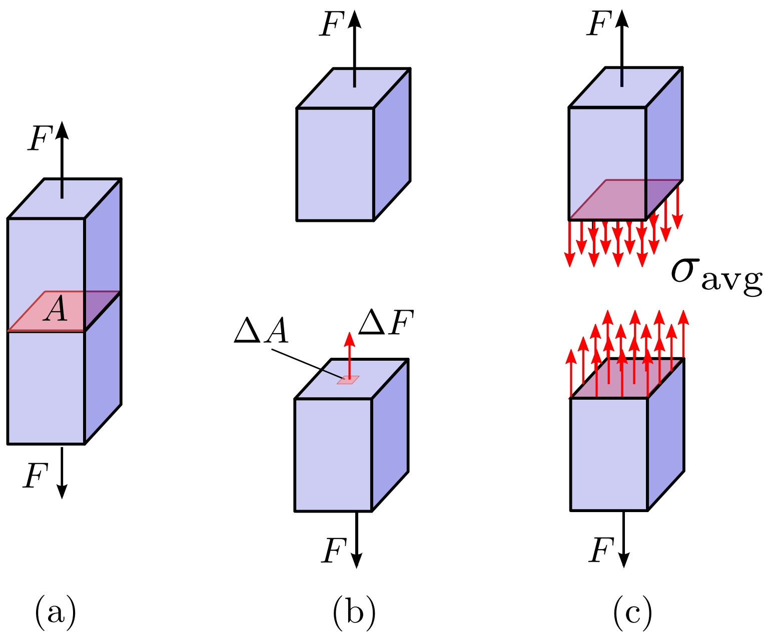


Figure 2.1: Illustration of axial forces and stresses

$$\boxed{\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}} \tag{2.1}$$

The average force intensity over the whole cross section as shown in figure 2.1(c) is:

$$\boxed{\sigma_{avg} = \frac{F}{A}} \tag{2.2}$$

Here F is the load in Newtons (N) and A is the area in square meters (m^2). The unit of stress is the Pascal (Pa). The Pascal is a very small quantity. Usually stress in engineering applications is expressed in megapascals (MPa) where 1 MPa = 1 000 000 Pa. A useful method of calculating stress is to express force in Newtons and area in square millimetres to get:

$$\boxed{1\,\text{MPa} = \frac{1\,\text{N}}{1\,\text{mm}^2}} \tag{2.3}$$

The following sign convention is followed when a tensile force acts on a body as seen in figure 1.1 the resulting **tensile normal stress** is positive and when a compressive force acts on a body **compressive normal stress** is negative.

■ **Example 2.1** Determine the average normal stress in each of the 20 mm diameter bars of the truss shown in figure 2.2(a).

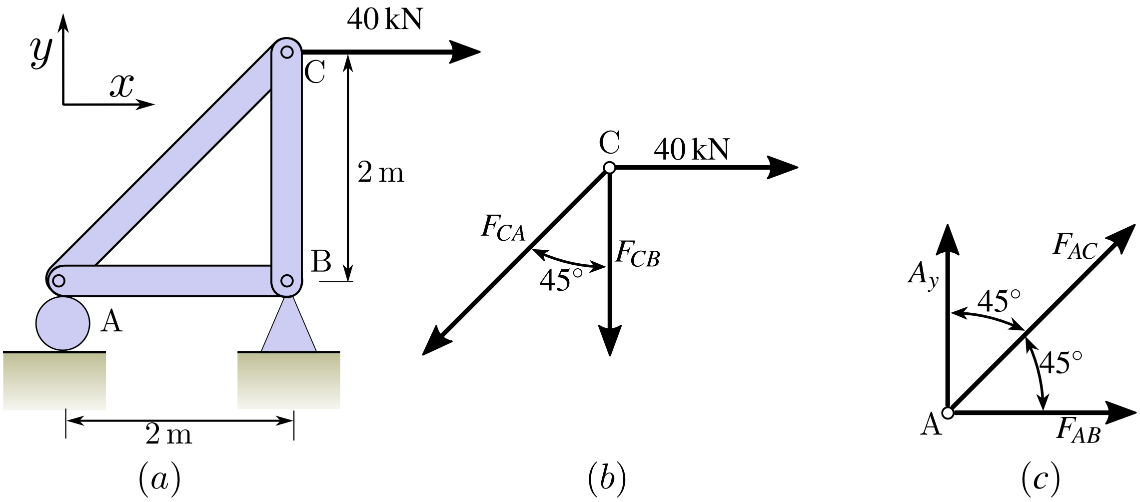


Figure 2.2: Truss with (b) joint C and (c) joint A isolated

Solution
This example is calculated using in units of N, mm and MPa.
In a similar way to example 1.1 the angles BAC = 45° and ACB = 45° because the length of AB = BC.
To start the angle BAC is calculated from the beam dimensions, knowing that BC is perpendicular to AB and the lengths of AB and BC are equal. It follows BAC = 45°. Similarly angle ACB = 45°.

Joint C is the only joint where there is a known force so we begin there using the method of joints.
Joint C
$\Sigma F_x = 0 \Rightarrow 40\,000\,\mathrm{N} - F_{CA}\sin 45 = 0 \Rightarrow F_{CA} = 56\,568\,\mathrm{N}$
$\Sigma F_y = 0 \Rightarrow -F_{CB} - F_{CA}\cos 45 = 0 \Rightarrow F_{CB} = -40\,000\,\mathrm{N}$ compression

Joint A
$\Sigma F_x = 0 \Rightarrow F_{AB} + F_{AC}\cos 45 = 0 \Rightarrow F_{AB} = -40\,000\,\mathrm{N}$ compression

The area of each cross section A = $\pi\left(\frac{20}{2}\right)^2 = 314.16\,\mathrm{mm}^2$

$$\sigma_{AC} = \frac{F_{AC}}{A} = \frac{56568}{314.16} = 180.1\,\mathrm{MPa}$$

$$\sigma_{CB} = \sigma_{AB} = \frac{F_{CB}}{A} = \frac{F_{CB}}{A} = \frac{-40000}{314.16} = -127.3\,\mathrm{MPa} \text{ compression}$$

■

2.3 Normal Strain

When a solid body is subjected to some loading (mechanical or thermal), it undergoes **deformation**. This means it changes size and/or shape which means the lengths and/or the angles in the body can change.

Strain, represented by the Greek letter ε (epsilon), is the intensity of deformation or deformation per unit length just as stress is the force intensity.

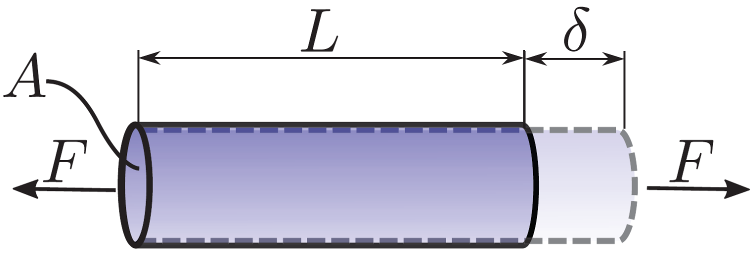


Figure 2.3: Normal strain in a cylinder

The average normal strain is

$$\boxed{\varepsilon_{avg} = \frac{\delta}{L}} \tag{2.4}$$

where δ (delta) is the Greek letter representing the change in length in meters and L is the original length also in meters. A positive value of ε indicates extension of the body and a negative value indicates contraction.

Strain is a dimensionless quantity but is often expressed in terms of ratios of lengths for example m/m. In most engineering application it is convenient to use units of microstrain $\mu\varepsilon$ where $1\ \mu\varepsilon = 1 \times 10^{-6}$ m/m. It is also useful to represent strains in terms of percent. For most metal objects the strain will seldom be bigger than 0.2% which is 0.002 m/m.

2.4 Stress-Strain Diagrams

To design or analyse structures an understanding of the properties of the material being used is needed. The most effective way of determining these properties is in a **tensile test**. We have either a round or flat sample (see insets in figure 2.4) pulled with a controlled displacement which is related to the strain. The force is measured with a load cell and the strain calculated. The plot of the stress against strain output is also shown in the figure 2.4.

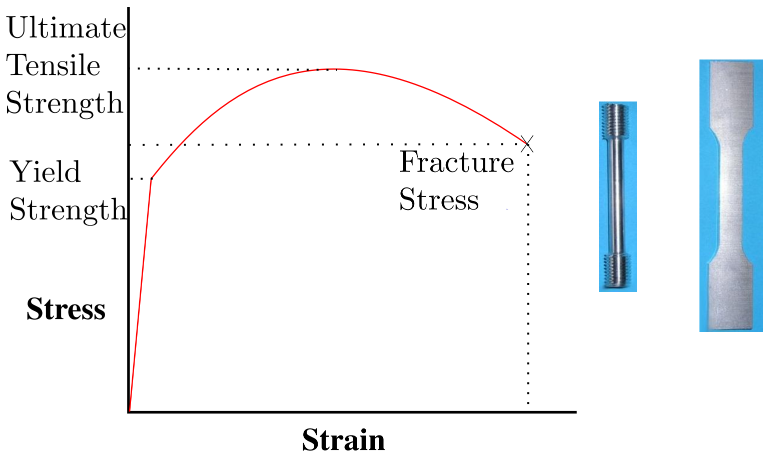


Figure 2.4: Stress versus strain for a typical metal

The **yield strength** can be considered to be the elastic limit in many cases. Any stress applied above the yield strength will permanently deform the material. This is called **plastic deformation**. The

ultimate tensile strength represents the maximum stress the material can tolerate. The fracture stress represents the stress when the material will break completely[2].

In design we would use the yield strength as our maximum stress, however because there are many uncertainties in our loads, our material properties and even our calculations, a factor of safety is used:

$$\boxed{\text{Factor of Safety} = \frac{\text{Material Strength}}{\text{Required Strength}}} \tag{2.5}$$

The material strength is typically the yield strength and the required strength is usually calculated from some mechanics equation based on the required loading.

Factors of safety depend on the application and can range from just over 1 all the way to more than 10. Issues such as the weight, how critical part the part is and uncertainty in the calculations all influence the safety factor used.

2.5 Hooke's Law

Below the yield strength, a material typically exhibits linear elastic behaviour. The relationship between the stress and strain is linear. Elastic behaviour means it will return to the same state after a load is removed. The strain measured is proportional to the applied stress and described by the equation:

$$\boxed{\sigma = \varepsilon E} \tag{2.6}$$

where E is called the **Young's Modulus** or the **Modulus of Elasticity**. The unit for E is Pa which is the same as for stress. For typical engineering materials a more convenient unit is 1 GPa = 1×10^9 Pa. Steel, which is one of the most widely used engineering materials, has a modulus of 200 GPa.

2.6 Poisson's Ratio

When a *tensile force is applied to a material in the longitudinal direction* then there will typically be a contraction of the material in the lateral direction [3]. The ratio of the strain in the lateral direction to strain in the longitudinal direction is called Poisson's Ratio. It is represented by the Greek symbol ν (pronounced nu) and is defined as follows:

$$\boxed{\nu = -\frac{\varepsilon_{lateral}}{\varepsilon_{longitudinal}}} \tag{2.7}$$

Take note of the negative sign. It shows that for positive Poisson's Ratios if there is an extension in the longitudinal direction, there will be a contraction in the lateral direction. Alternatively if there is a contraction in the longitudinal direction there will be extension in the lateral direction.

Most engineering materials have Poisson's Ratio between 0.3 and 0.4. The maximum Poisson's Ratio is 0.5 and indicates the volume will stay constant.

[2] See **MecMovies M3.1** at http://web.mst.edu/ mecmovie/ for a complete explanation of the tensile test

[3] There are some exceptions to this and materials that expand in the lateral direction when pulled in the longitudinal direction are called Auxetic Materials

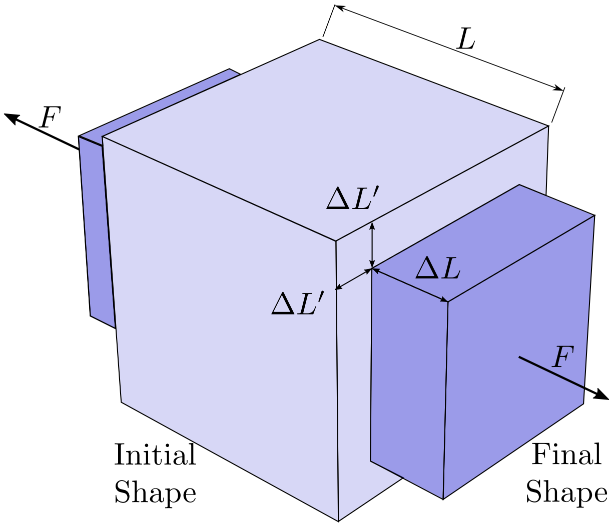


Figure 2.5: Cube is pulled with tensile force F in longitudinal direction

2.7 Shear Stress

Stress discussed in section 2.2 has the load perpendicular to area A over which it acts. If the load acts tangential to the area then this is called a **shear force** V and the stress is called the **shear stress** [4] . The shear stress is denoted by the Greek letter τ (tau) and defined as follows similar to normal stress:

$$\boxed{\tau = \lim_{\Delta A \to 0} \frac{\Delta V}{\Delta A}} \tag{2.8}$$

The average shear stress τ_{avg} is then:

$$\boxed{\tau_{avg} = \frac{V}{A}} \tag{2.9}$$

The stress that is experienced directly below or above the offset block in Figure 2.6 is the **bearing stress** or **compressive stress**.

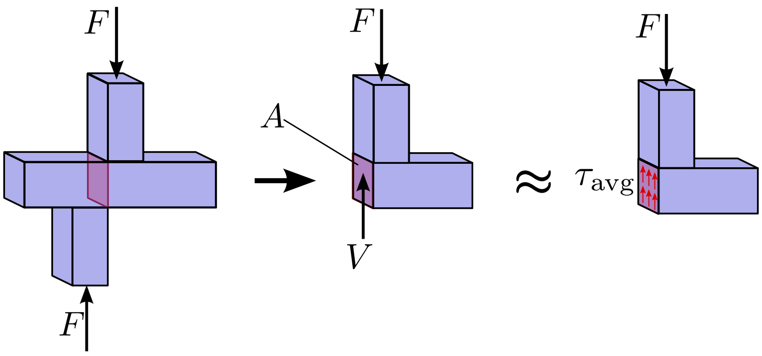


Figure 2.6: Horizontal bar loaded with offset blocks showing the shear stress

Shear stresses are often found in bolts, rivets and pins used to connect various objects.

[4] See **MecMovies M1.3** at http://web.mst.edu/∼mecmovie/

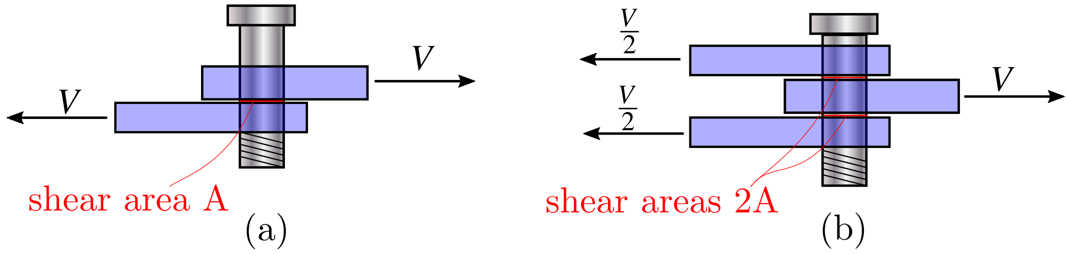


Figure 2.7: Bolts shown in (a) single shear and (b) double shear.

For the **double shear** shown in Figure 2.7 the area is double that for **single shear**. The shear stress in double shear is:

$$\boxed{\tau_{avg} = \frac{V}{2A}} \tag{2.10}$$

■ **Example 2.2** Determine the average shear stress developed in pin B of the truss of figure 2.2. Each pin has a diameter of 25 mm and is in double shear.

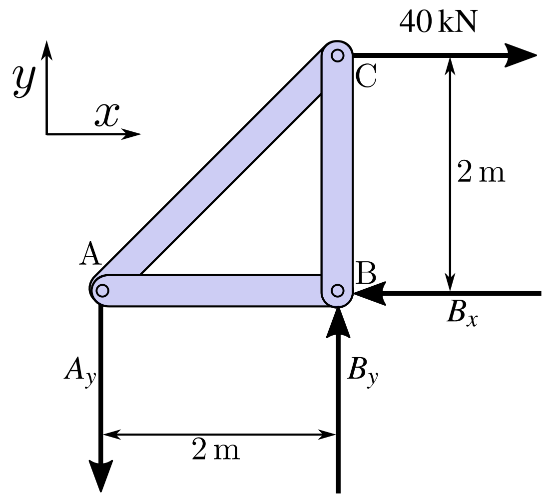


Figure 2.8: Free body diagram for figure 2.2

The shear area $A = \pi \left(\frac{25}{2}\right)^2 = 490.87\,\text{mm}^2$

Next the total force on the joint is calculated.

$\Sigma M_A = 0 \Rightarrow B_y \times 2 - 40000 \times 2 = 0 \Rightarrow B_y = 40\,000\,\text{N}$

$\Sigma F_x = 0 \Rightarrow 40000 - B_x = 0 \Rightarrow B_x = 40\,000\,\text{N}$

Total force on joint is $\sqrt{40000^2 + 40000^2} = 56\,568\,\text{N}$

The average shear stress $\tau_{avg} = \dfrac{V}{2A} = \dfrac{56568}{2 \times 490.87} = 57.62\,\text{MPa}$

■

2.8 Shear Strain

Deformation involving a change of shape is called **shear strain**. This is defined in the figure 2.9 with the Greek letter γ (gamma) and $\tan\gamma = \frac{\delta_s}{L}$. Since the shear strain angle is small, $\tan\gamma = \gamma$. Remember that the angle measurement is in radians. Strain can then be defined this as:

$$\boxed{\gamma = \frac{\delta_s}{L}} \tag{2.11}$$

It is the change in angle between two faces of an element of material.

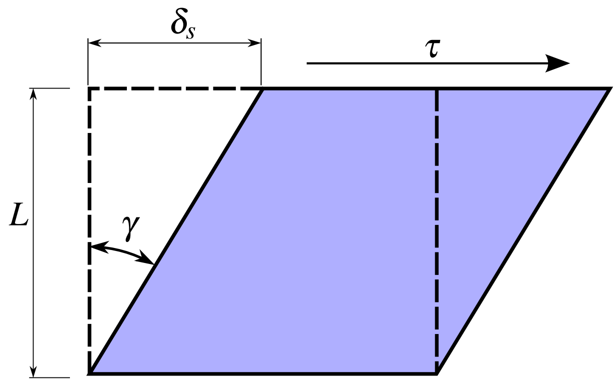


Figure 2.9: Shear strain shown for a block of height L under a shear stress of τ.

The shear strain is related to the shear stress by the equation:

$$\boxed{\tau = \gamma G} \tag{2.12}$$

where G is the **Shear Modulus** or **Modulus of Rigidity**.

For isotropic homogeneous materials we can relate the Young's modulus E, Poisson's ratio ν and Modulus of Rigidity G with:

$$\boxed{G = \frac{E}{2(1+\nu)}} \tag{2.13}$$

3. Stress Transformations

Colour variations seen in plastic cutlery is due to the photoelastic effect. The manufacturing process causes stresses in the cutlery. Polarised light interacts with the stresses and causes the different colours.

3.1 Introduction

In previous sections normal and shear stresses were calculated in various situations. These stresses interact with each other in a complex fashion. Occasionally stresses need to be calculated on planes at some angle to the applied forces. This may be the case when you need to calculate the stress on a crack orientated at an angle to the applied load. You may need to know the stress normal to the crack to know whether the crack will open or close.

If you know how to calculate the stress in any arbitrary orientation you may need to know the orientation and magnitude of the maximum stress. This will help in knowing when a structure will fail. Calculation of these maximum stresses and their directions is unlike calculating the forces and their directions. Forces are vectors and stresses are higher rank **tensors**[1].
In this section a more general definition of normal stress is required which is related to a coordinate axis:

$$\boxed{\sigma_x = \sigma_{xx} = \lim_{\Delta A \to 0} \frac{\Delta F_x}{\Delta A_x}} \tag{3.1}$$

In σ_{xx}, the first x subscript indicates the normal of the area over which force acts and the second x subscript indicates the force direction. The repeated subscript, in this case x, is often dropped for convenience.

For normal stresses the area normal and the force direction are parallel. The sign convention for normal stress was established previously with tensile stresses being positive and compressive stresses being negative.

[1] A description of tensors is beyond the scope of this course. They are geometric objects which are a generalisation of scalars and vectors which are tensors of rank 0 and 1, respectively. Stress is a tensor of rank 2.

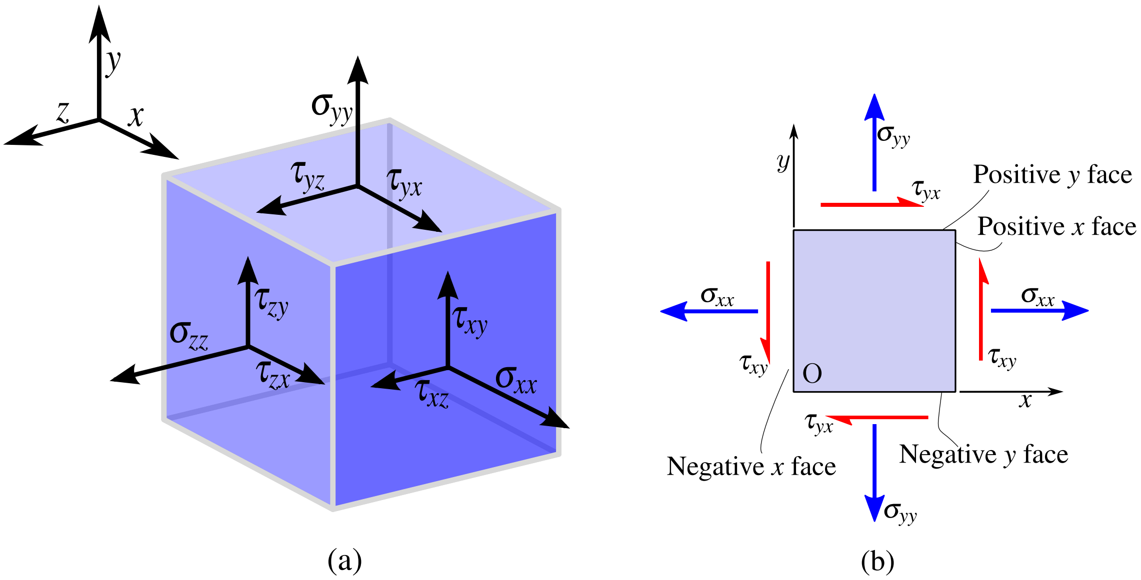


Figure 3.1: Stress depicted in (a) three dimensions and on (b) two dimensional stress element.

In a similar fashion to normal stress, shear stress can be represented as follows:

$$\boxed{\tau_{xy} = \sigma_{xy} = \lim_{\Delta A \to 0} \frac{\Delta V_y}{\Delta A_x}} \tag{3.2}$$

Here the shear force ΔV_y and the area normal are perpendicular to each other. This is unlike normal stress where these vector quantities are parallel to each other. The sign conventions for the shear stress is as follows:

Shear Stress is Positive

- acts in the positive coordinate direction on a positive face of the stress element, or
- acts in the negative coordinate direction on a negative face of the stress element.

Shear Stress is Negative

- acts in the positive coordinate direction on a negative face of the stress element, or
- acts in the negative coordinate direction on a positive face of the stress element.

For example all shear stresses are all positive in figure 3.1(b). The **stress element** in figure 3.1(b) is a two-dimensional graphical representation of a three-dimensional object.

3.1.1 Shear Stress Equilibrium

If an object is in static equilibrium then any part isolated from it must be under static equilibrium no matter the size or shape. Let's consider a small volume element depicted in figure 3.2 with a shear stress τ_{xy} that is known. The positive and negative z-normal faces are stress free. The shear stresses τ_{yx}, τ'_{xy} and τ'_{yx} are unknown.

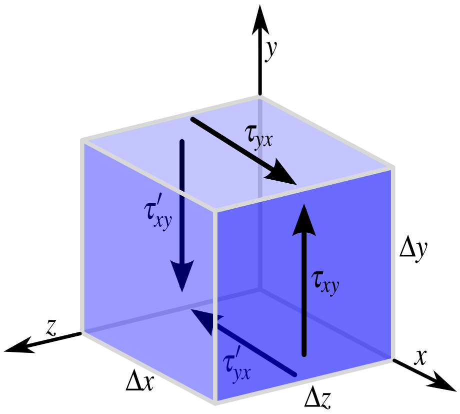


Figure 3.2: Shear stress acting on a small volume element.

Since the volume element is in equilibrium the sum of forces and moments about any point is zero. Summing forces in the x direction:

$$\begin{aligned}\Sigma F_x &= \tau_{yx}\Delta x\Delta z - \tau'_{yx}\Delta x\Delta z = 0\\ &\Rightarrow \tau_{yx} = \tau'_{yx}.\end{aligned} \tag{3.3}$$

Summing forces in the y direction:

$$\begin{aligned}\Sigma F_y &= \tau_{xy}\Delta y\Delta z - \tau'_{xy}\Delta y\Delta z = 0\\ &\Rightarrow \tau_{xy} = \tau'_{xy}.\end{aligned} \tag{3.4}$$

Taking moments about the z-axis:

$$\begin{aligned}\Sigma M_z &= (\tau_{xy}\Delta y\Delta z)\,\Delta x - (\tau_{yx}\Delta x\Delta z)\,\Delta y = 0\\ &\Rightarrow \tau_{xy} = \tau_{yx}.\end{aligned} \tag{3.5}$$

Combining equations (3.3), (3.4) and (3.5) gives:

$$\tau_{xy} = \tau_{yx} = \tau'_{xy} = \tau'_{yx}. \tag{3.6}$$

So the shear stress on any of the four faces would be the same. This property is called the *complementary* property of shear.

Stresses occur in three dimensions, represented with three axes, x, y and z. For simplicity out of plane stresses are assumed to be zero in a condition called **plane stress**:

$$\boxed{\sigma_{zz} = \tau_{zx} = \tau_{zy} = 0} \tag{3.7}$$

From equation (3.6) we see then that $\tau_{xz} = \tau_{yz} = 0$ for plane stress.

3.2 Plane Stress Transformations using Equilibrium Equations

A state of stress can be completely described by three components in two-dimensions , σ_{xx}, σ_{yy} and τ_{xy} acting on a plane with x and y coordinate axes. Often we want to represent it on a different plane represented by normal n and tangential t coordinate axes giving stress components σ_n, σ_t and τ_{nt}.[2] Changing from one set of axes to another is termed a **stress transformation**.

■ **Example 3.1** Consider an element with a shear stress of 110 MPa and a tensile stress of 185 MPa in the horizontal plane. Determine the normal and shear stresses at a slope 60° to the horizontal.

Solution

Sketch a free-body diagram of a wedge shaped portion of the stress element, shown in figure 3.3(a). The wedge shape is formed with a plane cutting the element at the 30° to the vertical. Let the area of the inclined surface be dA. Remember we are looking at a two dimensional representation of what is a three-dimensional stress state. The angle 30° the new plane makes to the vertical is also the angle the normal rotates from x to n. The area of the vertical surface will be $dA\cos 30$ and the area of the horizontal surface will be $dA\sin 30$.

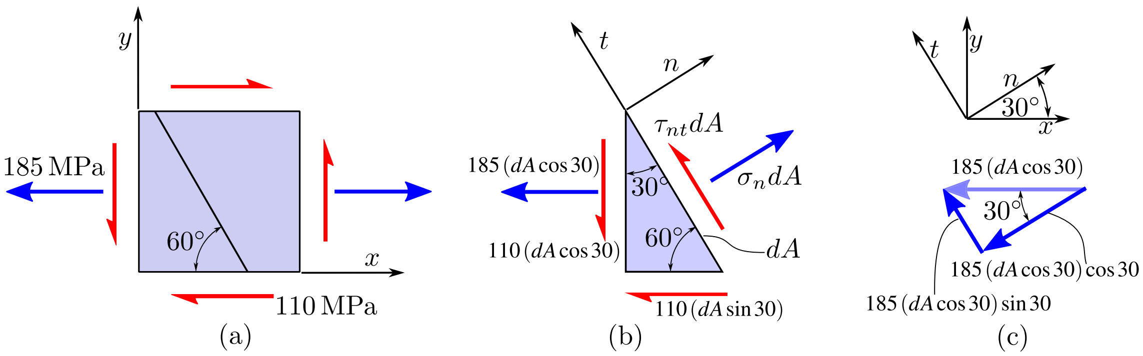


Figure 3.3: Illustration of (a) plane stress element and (b) determination of stress on an arbitrarily chosen plane at 60° clockwise to the horizontal. The resolution of the x direction force is shown in (c).

The forces on these areas are the product of the areas and the stresses. For example the force in the x direction is $185(dA\cos 30)$. The horizontal force is resolved into components in the n and t axes, see figure 3.3(c). The shear force is also resolved in a similar way.

Summing forces parallel to the inclined plane in the t-direction.

$$\begin{aligned}\Sigma F_t =& \tau_{nt} dA - 110\,(dA\cos 30)\cos 30 + 110\,(dA\sin 30)\sin 30\\ &+ 185\,(dA\cos 30)\sin 30 = 0\end{aligned}$$

All the dA factors disappear which gives: $\tau_{nt} = -25.1$ MPa. The shear is negative so it is in a opposite direction to what is shown in figure 3.3(b).

Summing forces in the normal to the inclined surface.

$$\begin{aligned}\Sigma F_n =& \sigma_n dA - 110\,(dA\cos 30)\sin 30 - 110\,(dA\sin 30)\cos 30\\ &- 185\,(dA\cos 30)\cos 30 = 0\end{aligned}$$

The normal stress $\sigma_n = 234$ MPa. This is positive and therefore in tension. It is indicated in the correct direction in figure 3.3(b). ■

[2] Many authors use the notation $\sigma_{x'}$, $\sigma_{y'}$ and $\tau_{x'y'}$ to represent σ_n, σ_t and τ_{nt}, respectively.

3.3 General Equations of Plane Stress Transformation

The method of transforming the normal and shear stresses from x-y coordinates to n-t coordinates can be developed in a general fashion and expressed as a set of stress transformation equations. [3]

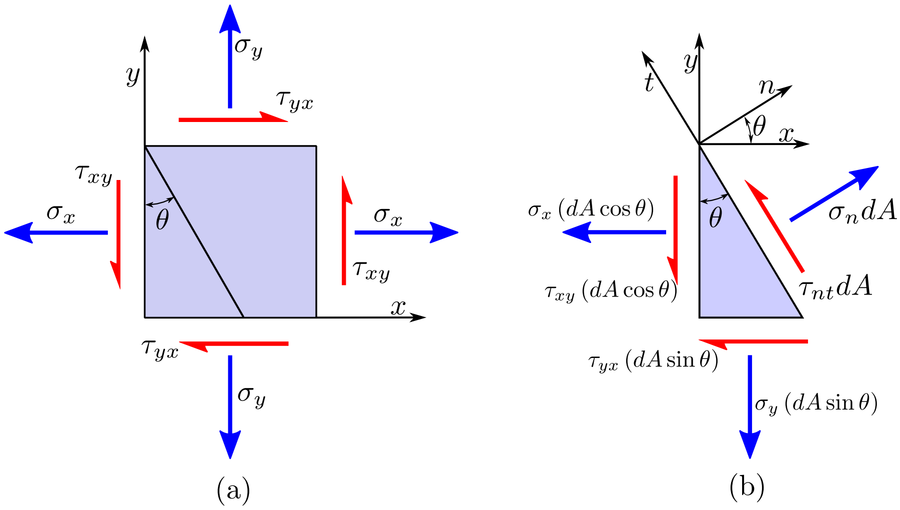


Figure 3.4: Drawing of a (a) Stress element with an arbitrary plane at an angle θ to vertical and (b) free-body diagram of wedge shaped element.

Consider the free body diagram of figure 3.4(b). Using the equations of equilibrium by summing forces in the n and t axes directions. Note we drop the repeated subscript in σ_{xx} and make it σ_x

$$\begin{aligned}\Sigma F_n =&\sigma_\mathrm{n} \mathrm{d}A - \tau_{xy}\left(\mathrm{d}A\cos\theta\right)\sin\theta - \tau_{yx}\left(\mathrm{d}A\sin\theta\right)\cos\theta \\ &- \sigma_x\left(\mathrm{d}A\cos\theta\right)\cos\theta - \sigma_y\left(\mathrm{d}A\sin\theta\right)\sin\theta = 0 \\ \Rightarrow&\sigma_\mathrm{n} = \sigma_x\cos^2\theta + \sigma_y\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta\end{aligned}$$

and

$$\begin{aligned}\Sigma F_t =&\tau_{nt} \mathrm{d}A - \tau_{xy}\left(\mathrm{d}A\cos\theta\right)\cos\theta + \tau_{yx}\left(\mathrm{d}A\sin\theta\right)\sin\theta \\ &+ \sigma_x\left(\mathrm{d}A\cos\theta\right)\sin\theta - \sigma_y\left(\mathrm{d}A\sin\theta\right)\cos\theta = 0 \\ \Rightarrow&\tau_\mathrm{nt} = -\left(\sigma_x - \sigma_y\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)\end{aligned}$$

Using the following trigonometric identities:

$$\sin(2\theta) = 2\sin\theta\cos\theta; \qquad \sin^2\theta = \frac{1-\cos(2\theta)}{2}; \qquad \cos^2\theta = \frac{1+\cos(2\theta)}{2}$$

We obtain the following formulae called the **plane stress transformation equations**:

$$\boxed{\sigma_\mathrm{n} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta} \tag{3.8}$$

and

$$\boxed{\tau_\mathrm{nt} = -\left(\frac{\sigma_x-\sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta} \tag{3.9}$$

[3] see UCT MecMovies 3.8 for an animated derivation of the plane stress transformations

■ **Example 3.2** Consider an element on the surface of an aircraft. What would the stress be on an element inclined at 60° clockwise to the vertical?

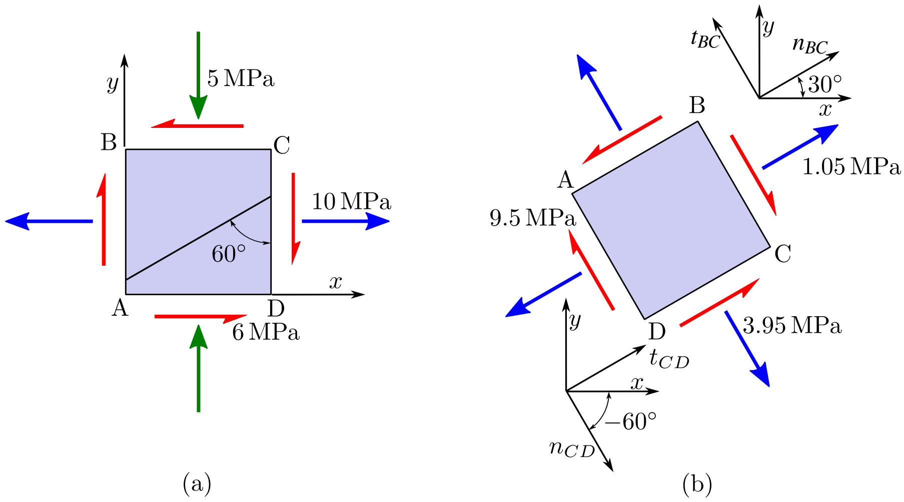


Figure 3.5: (a) Element on the surface of an aircraft and (b) the element transformed .

Solution
First consider the stresses according to the established sign conventions.

$$\sigma_x = 10\,\text{MPa}; \qquad \sigma_y = -5\,\text{MPa}; \qquad \tau_{xy} = -6\,\text{MPa}$$

The transformed element has two perpendicular faces. Consider the transformation to each of these faces in planes CD and BC separately shown in figure 3.5(b).

Plane BC
Using equation (3.8) to calculate the normal stress and use the angle that plane BC is transformed in figure 3.5(b)

$$\sigma_{\text{n}} = \frac{10-5}{2} + \frac{10+5}{2}\cos(2(30)) - 6\sin(2(30)) = 1.05\,\text{MPa}$$

Using equation (3.9) to calculate the shear stress

$$\tau_{\text{nt}} = -\frac{10+5}{2}\sin(2(30)) - 6\cos(2(30)) = -9.5\,\text{MPa}$$

Considering the shear sign convention the shear on face BC is pointing in the negative t_{BC} direction on the positive n_{BC} face.

Plane CD
Again using equation (3.8) to calculate the normal stress with the angle −60° instead of 30°.

$$\sigma_{\text{n}} = \frac{10-5}{2} + \frac{10+5}{2}\cos(2(-60)) - 6\sin(2(-60)) = 3.95\,\text{MPa}$$

Check the previous shear stress with equation (3.9).

$$\tau_{\text{nt}} = -\frac{10+5}{2}\sin(2(-60)) - 6\cos(2(-60)) = 9.5\,\text{MPa}$$

Consider our shear sign convention again, the shear on face AD is pointing in the positive t_{CD} direction on the positive n_{CD} face. Notice the shear stress changed signs according to the axes which were considered. ■

3.4 Principal Stresses

As seen in the previous section stresses at a point vary according to the plane direction in which we consider the stresses. In engineering it is very important to determine the largest normal and shear stresses in a component to determine when it will fail. In this section we will determine the magnitude of these stresses and the direction in which they will act. To get the maximum, the normal stress equation (3.8) is differentiated with respect to the angle θ and see where it is zero. This gives:

$$\frac{d\sigma_n}{d\theta} = -\frac{\sigma_x - \sigma_y}{2} 2\sin 2\theta + \tau_{xy} 2\cos 2\theta = 0 \tag{3.10}$$

Solving this equation and with the solution $\theta = \theta_p$ the following is obtained:

$$\boxed{\tan 2\theta_p = \frac{\tau_{xy}}{(\sigma_x - \sigma_y)/2}} \tag{3.11}$$

The tan function recurs every 180° so $2\theta_{p1}$ and $2\theta_{p2}$ are 180° apart and therefore the two roots θ_{p1} and θ_{p2} will be 90° apart, i.e. $\theta_{p1} = \theta_{p2} \pm 90°$. The two roots are called the **principal angles**. Equation (3.11) is represented geometrically in figure 3.6.

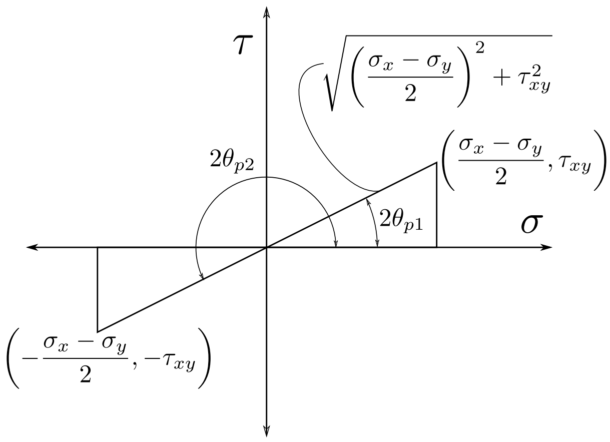


Figure 3.6: Illustration of principal angles in equation (3.11)

The values of θ_{p1} and θ_{p2} must be substituted into equation (3.8) to give the maximum and minimum normal stresses called the **principal stresses**, σ_{p1} and σ_{p2}.

$$\boxed{\sigma_{p1,p2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \tag{3.12}$$

Here σ_{p1} is the major principal stress which is greater than minor principal stress σ_{p2}. The plane on which these stresses act are the **principal stress planes**. Looking at the right hand side of equation (3.9) for shear stress ($-\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$), it is exactly half the right hand side of equation (3.10) for the normal stress derivative. Then the normal stress derivative and the shear stress will be zero at the same time therefore **the shear stress will be zero on the principal stress plane**.

3.5 Maximum In-Plane Shear Stress

As with obtaining the principal stress we differentiate the appropriate stress equation and set to zero to get the maximum in-plane shear stress τ_{max}. Differentiate equation (3.9) to give

$$\frac{d\tau_{\text{nt}}}{d\theta} = -(\sigma_x - \sigma_y)\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0 \tag{3.13}$$

The solution gives the orientation $\theta = \theta_s$ of the plane where the shear is either maximum or minimum.

$$\boxed{\tan 2\theta_s = -\frac{(\sigma_x - \sigma_y)/2}{\tau_{xy}}} \tag{3.14}$$

As with the principal angles the θ_s angles are 90° apart. Notice that equation (3.11) and equation (3.14) are negative reciprocals of each other therefore the angles $2\theta_s$ and $2\theta_p$ are 90° apart from each other. The angles θ_s and θ_p are 45° apart, i.e. $\theta_s = \theta_p \pm 45°$. The maximum shear plane is therefore 45° to the principal plane.

Substitute θ_s into equation (3.9) gives the magnitude of the in-plane shear stress τ_{max}.

$$\boxed{\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \tag{3.15}$$

The maximum shear stress is ambiguous in sign. Unlike normal stress where tension and compression has different effects on material behaviour, a different sign in shear stress has no difference in fact the same shear stress will have a different sign but same magnitude if taken on an orthogonal plane.

Unlike the principal plane where no shear stress acts normal stress acts on the plane of maximum shear stress and equals:

$$\boxed{\sigma_{avg} = \frac{\sigma_x + \sigma_y}{2}} \tag{3.16}$$

The σ_{avg} will be the same on both orthogonal $\tau_{\max}$ planes.

Another relation which relates principal and maximum shear stress is given from equation (3.12) and equation (3.15):

$$\boxed{\tau_{\max} = \frac{\sigma_{p1} - \sigma_{p2}}{2}} \tag{3.17}$$

■ **Example 3.3** A stress element has a tensile stress of 50 MPa, a compressive stress of 10 MPa and a shear stress of 40 MPa as shown in figure 3.7.

Determine the principal stresses and at which angles they occur, the maximum shear stress and the angle at which it occurs. Also determine the normal stress on the plane of maximum shear stress.

Solution
The stresses are according to the sign conventions: $\sigma_x = 50\,\text{MPa}$; $\sigma_y = -10\,\text{MPa}$; $\tau_{xy} = 40\,\text{MPa}$

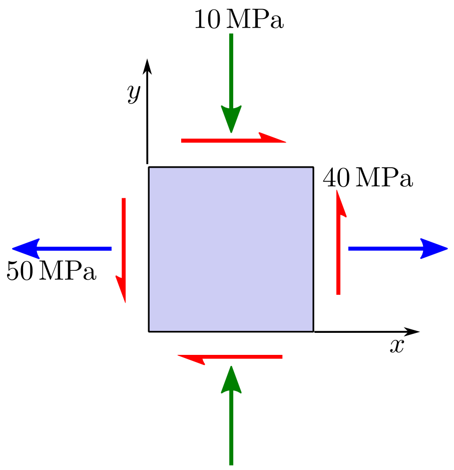


Figure 3.7: Element and stresses shown

Principal Angles
Substitute stress values into equation (3.11) to give:

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{4}{3}$$

$2\theta_p = 53.1^\circ$ and $180^\circ + 53.1^\circ = 233.1^\circ$
$\theta_p = 26.6^\circ$ and 116.6° which gives the two principal angles.

Principal Stresses
Substitute stress values and principal angles into equation (3.8) to give:

$$\sigma_p = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta_p + \tau_{xy}\sin 2\theta_p$$

$\sigma_{p1} = 70\,\text{MPa}$ corresponds to $\theta_{p1} = 26.6^\circ$ and $\sigma_{p2} = -30\,\text{MPa}$ corresponds to $\theta_{p1} = 116.6^\circ$. We know that $\sigma_{p1} = 70\,\text{MPa}$ corresponds to the major principal stress since it is larger than all the normal stresses. Similarly $\sigma_{p2} = -30\,\text{MPa}$ since it is smaller than all the normal stresses.

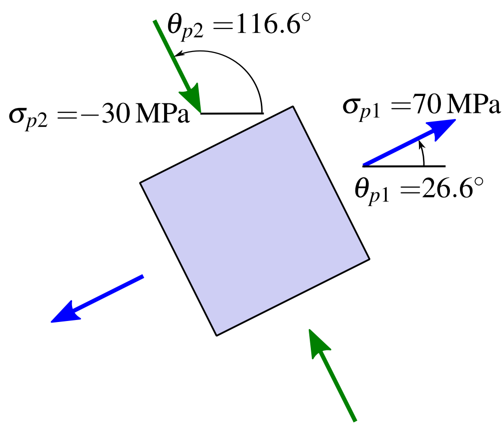


Figure 3.8: Principal stress shown for figure 3.7.

The principal stresses can be checked using equation (3.12). If equation (3.12) was used directly the principal angle corresponding to each principal stress would not be known.

Maximum Shear Stress
This is calculated from equation (3.15)

$$\tau_{\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm 50\,\text{MPa}.$$

The maximum shear angle can be obtained knowing that it is 45° from the principal angle so $\theta_S = \theta_p - 45^\circ = -18.4^\circ$.

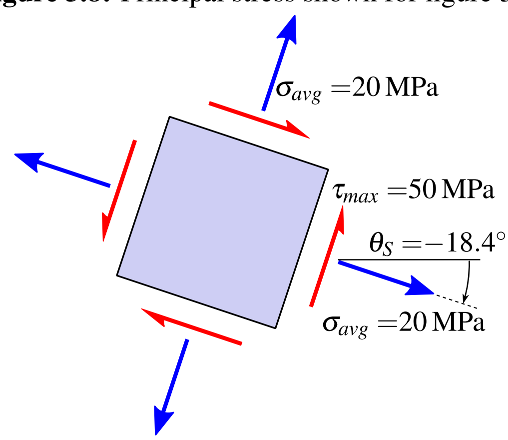


Figure 3.9: Maximum shear stress shown

The shear sign can be checked by substituting -18.4° into equation (3.9) to see its positive. The normal stress is obtained from equation (3.16) to give 20 MPa. ■

3.6 Mohr's Circle for Plane Stress

The basic equations of stress transformation derived previously can be interpreted graphically in a way that is easy to remember and convenient to use. It also aids in rapid understanding of the physical significance of the transformation equations.

To derive the circle equation write equation (3.8) and equation (3.9) in the form with the trigonometric functions on one side [4]:

$$\sigma_n - \frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

and

$$\tau_{nt} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

Both equations above are squared, added and simplified to give an equation with no θ:

$$\boxed{\left(\sigma_n - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{nt}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{3.18}$$

This is the equation for a circle in terms of variables σ_n and τ_{nt}. If we let:

$$\boxed{C = \frac{\sigma_x + \sigma_y}{2}} \tag{3.19}$$

then C is centre of the circle located on the horizontal σ axis. Also let

$$\boxed{R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}} \tag{3.20}$$

which is the square root of the right hand side of equation (3.18). This is the radius of the circle. Equation (3.18) is then rewritten as

$$\boxed{(\sigma_n - C)^2 + \tau_{nt}^2 = R^2} \tag{3.21}$$

This equation represents a circle with radius R and center C. This is called **Mohr's Circle** after Otto Mohr[5].

[4] See MecMovies 12.15 for a step by step guide of the derivation of Mohr's circle stress transformation equations

[5] Otto Christian Mohr was a German structural engineer who developed this circle approach in 1882

3.6.1 Construction of Mohr's Circle

The following steps can be used to plot Mohr's Circle. We assume that we have been given σ_x, σ_y and τ_{xy} of an element in plane stress.

The sign convention for plotting stresses are as follows:

1. Normal stresses are positive in tension and negative in compression as in previous sections
2. Shear stress that turn the element face clockwise are plotted above the σ-axis and shear stresses that turn the face anti-clockwise are plotted below the σ-axis, see figure 3.10.

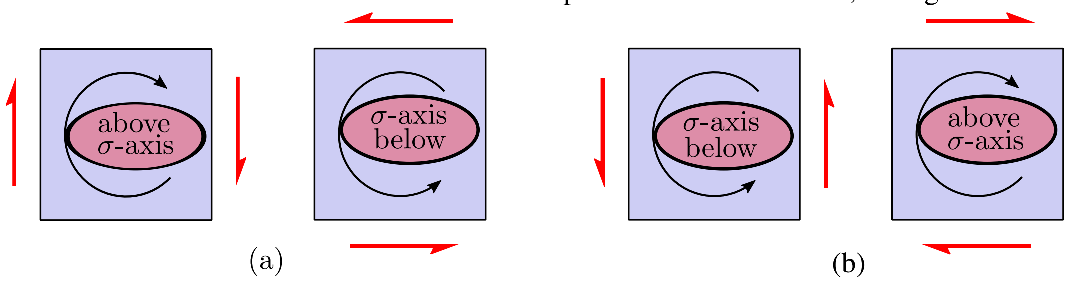


Figure 3.10: Sign convention for shear stresses used in constructing Mohr's Circle: (a) is negative shear and (b) is positive shear.

1. Draw the coordinate axes with σ as the horizontal axis or abscissa and with τ as the vertical axis or ordinate [6].
2. Locate the centre of the circle C with coordinate $\left(\sigma_{\text{avg}}, 0\right)$ where $\sigma_{\text{avg}} = \frac{\sigma_x+\sigma_y}{2}$.
3. Plot a point A, (σ_x, τ_{xy}) which may relate to two opposite faces on the stress element; however it is easier to specify the stresses on the positive x face. This represents $\theta = 0°$. Importantly **σ is positive to the right and τ which turns a face anti-clockwise is below the σ-axis**.
4. Plot a point B, (σ_y, τ_{yx}) which would be on the opposite side of the σ-axis to τ_{xy}. The plane going through this point represents $\theta = 90°$.
5. Draw line AB through C which represents the diameter of the circle
6. Draw Mohr's circle through points A and B with centre C. The circle has radius R.

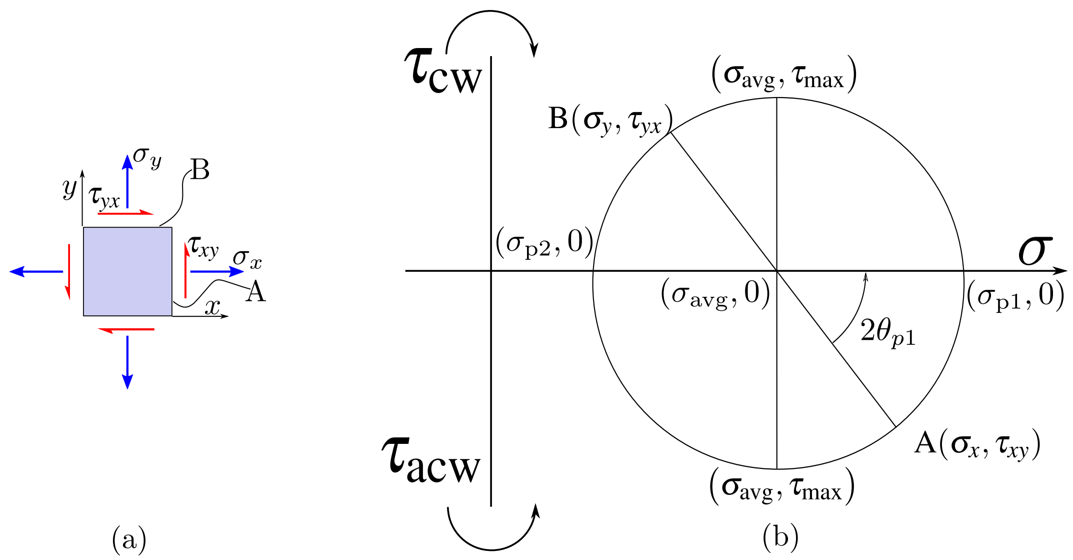


Figure 3.11: Stress element (a) and Mohr's circle of stress (b) shown.

The angle $2\theta_p$ in figure 3.11 is twice the principal angle. Any angle in Mohr's Circle space represents twice that angle in real space.

[6] MecMovies 12.16 presents a step by step guide for plotting Mohr's Circle.

■ **Example 3.4** An element in plane stress is subjected to stresses $\sigma_x = 100\,\text{MPa}$, $\sigma_y = 34\,\text{MPa}$ and $\tau_{xy} = 28\,\text{MPa}$, as shown in figure 3.12. Using Mohr's circle, determine the following quantities: (a) the principal stresses and angles, (b) the maximum shear stress and angle, and (c) the stresses acting on an element inclined at an angle $\theta = 40°$ anticlockwise. (Consider only the in-plane stresses, and show (c) on a sketch of a properly oriented element.)

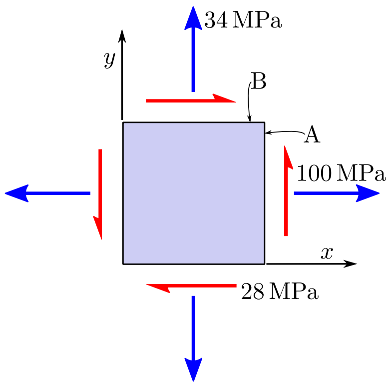


Figure 3.12: Plane stress element with various loadings.

Solution

1. The normal stress and shear axes are seen in figure 3.13
2. Centre of Mohr's circle from equation (3.19): $C = \dfrac{100+34}{2} = 67\,\text{MPa}$
3. Point A is at $(100, 28)$ which is below the σ axis since the shear on the x-face turns the element anti-clockwise
   Point B is at $(34, 28)$ which is above the is above the σ axis since the shear on the x-face turns the element anti-clockwise
4. Line AB is drawn which goes through point C in figure 3.13
5. The radius of the circle is determined using equation (3.20):

$$R = \sqrt{\left(\frac{100-34}{2}\right)^2 + 28^2} = 43.2\,\text{MPa}$$

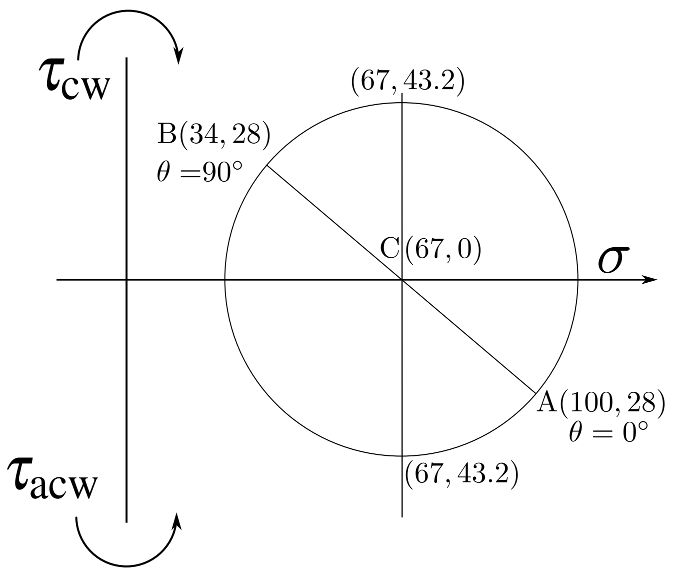


Figure 3.13: Mohr's Circle for stress element shown in figure 3.12

(a) The principal stresses are:

$$\sigma_{\text{p1}} = 67 + 43.2 = 110.2\,\text{MPa}$$

$$\sigma_{\text{p2}} = 67 - 43.2 = 23.7\,\text{MPa}$$

Principal angle (maximum) from circle geometry

$$\theta_{\text{p1}} = \frac{1}{2}\tan^{-1}\frac{100-67}{28} = 20.15^\circ$$

Principal angle (minimum)

$$\theta_{\text{p2}} = \theta_{\text{p1}} + 90 = 110.2^\circ$$

(b) Maximum shear stress from the ordinate of circle below centre:

$$\tau_{\text{max}} = 43.2\,\text{MPa}$$

Shear angle of maximum stress is $20.15^\circ - 45^\circ = -24.85^\circ$

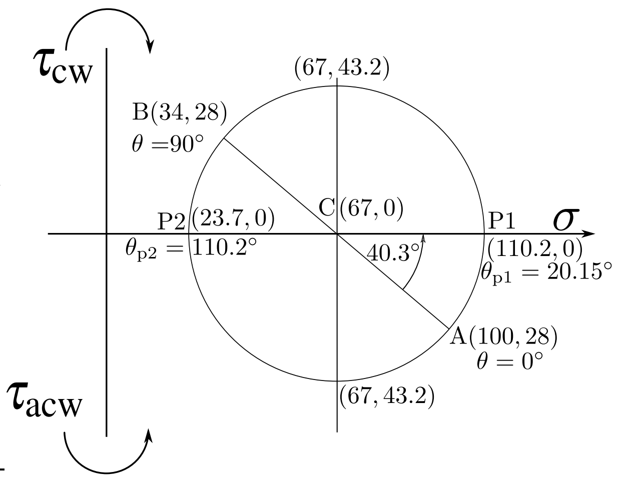


Figure 3.14: Principal stresses shown

(c) If an element is rotated 40° then we rotate 80° on Mohr's Circle from point A to D.

The angle ACP1 is $\tan^{-1}\frac{100-67}{28} = 40.3^\circ$

so angle DCP1$= 80^\circ - 40.3^\circ = 39.7^\circ$

(Point D)

$$\sigma_n = 67 + (43)(\cos 39.7^\circ) = 100\,\text{MPa}$$

$$\tau_{nt} = (43)(\sin 39.7^\circ) = 27.5\,\text{MPa}$$

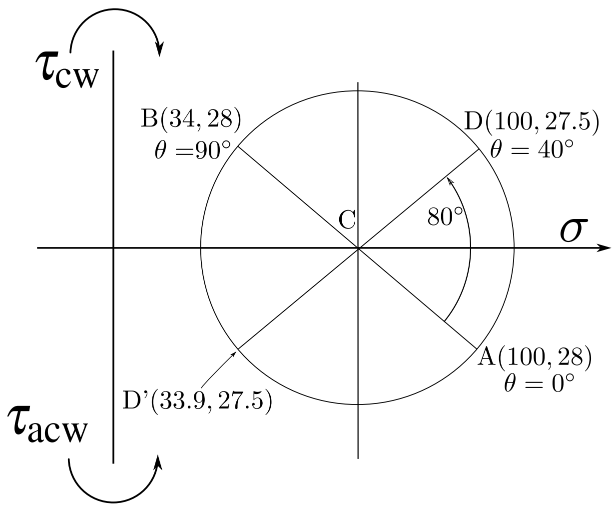


Figure 3.15: Transformation of stresses on Mohr's Circle

Stresses represented by point D' correspond to an angle $\theta = 130^\circ$ (or $2\theta = 260^\circ$):

(Point D')

$$\sigma_t = 67 + (43)(\cos 130^\circ) = 33.9\,\text{MPa}$$

$$\tau_{tn} = (43)(\sin 130^\circ) = 27.5\,\text{MPa}$$

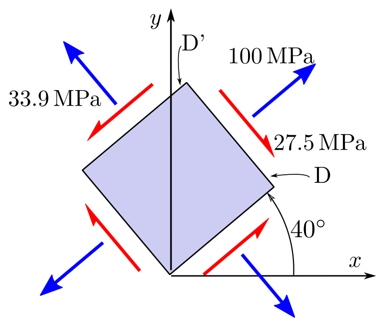


Figure 3.16: Transformed element

4. Axial Loading

The portico at Jameson Hall has a overhang supported by columns under axial compression.

4.1 Introduction

Axial loading occurs along the longitudinal axis of a load bearing member. Examples include trusses, ropes, cables and columns. Axial stress differs from normal stress in that all axial stresses are normal stresses however not all normal stresses are axial stresses. You can get normal stress from bending and this will be discussed in a later chapter.

4.2 Saint-Venant's Principle

Consider a rectangular bar subject to a concentrated load P applied to the ends of the bar, see figure 4.1. Due to the loading there is an increased local stress concentration close to the force P. This stress becomes constant across the section as we move away from the load application point toward the midsection. Consider the largest dimension of the cross-section which is the width of the bar not the thickness. At a distance greater than the width from load point can the stress be considered uniform. See Figure 4.1 for the stress distributions at various cross-sections. This effect is independent of the distribution of the forces as long as the forces are statically equivalent. This is known as **Saint-Venant's Principle** [1].

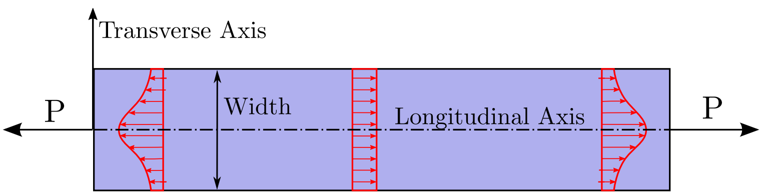


Figure 4.1: Rectangular bar subjected to concentrated force showing stress distributions at various cross sections.

Any loading can be replaced with another statically equivalent one for the purpose of simplifying the loading. This idea is also extended to fillets, grooves, holes or any feature that can cause a stress concentration. At a distance greater than the width from these features can the stress be considered uniform. The stress close to the load application is complex and is dependant on

[1] This is named after the French mathematician and engineer Adhémar Barré de Saint-Venant (1797 - 1886).

the material properties, shape of the member and the loading details. Further away we can use equation 2.2. Since the effects of the stress concentrations are so localised general formulas for strain and elongation give results in good agreement with reality.

4.3 Changes in Lengths of Members under Axial Loads

When a member is under axial loading with an internal force $\mathscr{F}$ a few assumptions are made to calculate the deflection:

1. Saint-Venant's Principle holds and furthermore the localised stress effects are ignored. In other words the stress is assumed to be constant across the member.
2. The material is assumed to isotropic, homogeneous and linear elastic.
3. The member is prismatic which means all cross-sections are the same.

These assumptions give us that the axial strain will be constant across the section. Equations 2.2, 2.4 and 2.6 can then be combined to give:

$$\boxed{\delta = \frac{\mathscr{F}L}{AE}.} \tag{4.1}$$

If a member is subjected to axial loads at intermediate points (points other than at the ends) or the member consists of segment with different, areas and/or stiffnesses. The overall deflection can be determined as follows:

$$\boxed{\delta = \sum \frac{\mathscr{F}_i L_i}{A_i E_i}.} \tag{4.2}$$

Here $\mathscr{F}_i$, L_i, A_i and E_i are the internal force, the length, the cross sectional area, and the Young's Modulus of the i^{th} segment respectively.

■ **Example 4.1** Consider the stepped shaft shown in Figure 4.2. The shaft is manufactured from steel with a Young's Modulus of 200 GPa. The segment areas are $A_1 = 100\,\text{mm}^2$, $A_2 = 200\,\text{mm}^2$ and $A_3 = 150\,\text{mm}^2$. The applied forces are $P_1 = 10\,\text{kN}$, $P_2 = 15\,\text{kN}$ and $P_3 = 5\,\text{kN}$. The lengths are $L_1 = 240\,\text{mm}$, $L_2 = 200\,\text{mm}$ and $L_3 = 180\,\text{mm}$. Calculate the overall change in length.

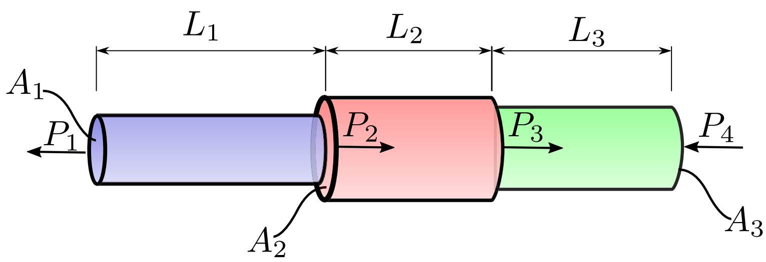


Figure 4.2: Elongation of an stepped shaft with 3 segments of varying diameters.

Solution

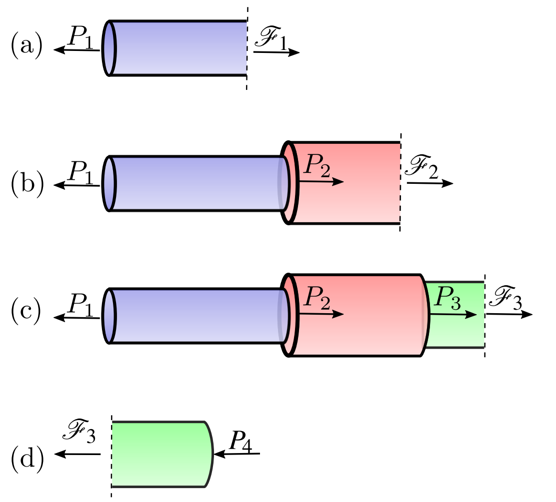


Figure 4.3: Construction of free body diagrams

To begin we draw free body diagrams for each section. Positive defined to the right and P_1 points to the left.

Figure 4.3(a)
$\sum F_x = 0 \Rightarrow \mathscr{F}_1 - P_1 = 0 \Rightarrow \mathscr{F}_1 = 10\,\text{kN}$

Figure 4.3(b)
$\sum F_x = 0 \Rightarrow \mathscr{F}_2 - P_1 + P_2 = 0$
$\Rightarrow \mathscr{F}_2 = 10 - 15 = -5\,\text{kN}$ compression

Figure 4.3(c)
$\sum F_x = 0 \Rightarrow \mathscr{F}_2 - P_1 + P_2 + P_3 = 0$
$\Rightarrow \mathscr{F}_3 = 10 - 15 - 5 = -10\,\text{kN}$ compression

To check Figure 4.3(d) First find P_4
$\sum F_x = 0 \Rightarrow -P_1 + P_2 + P_3 - P_4 = 0$
$\Rightarrow P_4 = -10 + 15 + 5 = 10\,\text{kN}$ right to left
$\sum F_x = 0 \Rightarrow -\mathscr{F}_3 - P_4 = 0$
$\Rightarrow \mathscr{F}_3 = -10\,\text{kN}$ compression

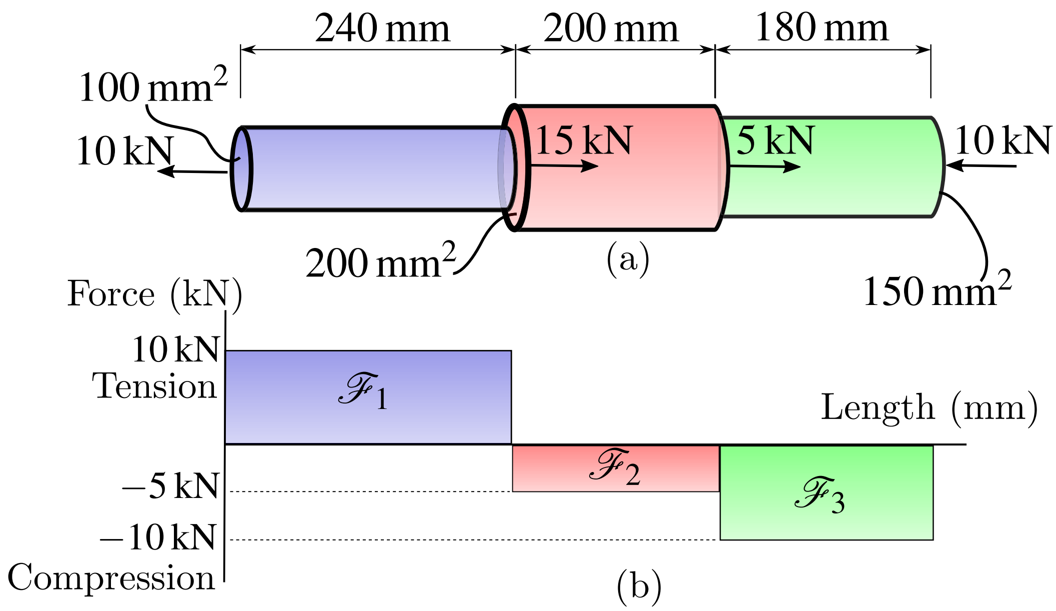


Figure 4.4: Illustration of (a) Compound bar with axial loading and (b) axial force diagram where P_i represents applied forces and F_i represents internal forces.

Now Calculate overall change in length
Units in kN, GPa and mm

$$\delta = \sum \frac{\mathscr{F}_i L_i}{A_i E_i} = \frac{(10)(240)}{(100)(200)} + \frac{(-5)(200)}{(200)(200)} + \frac{(-10)(180)}{(180)(200)} = 0.045\,\text{mm extension} \quad \blacksquare$$

4.4 Statically Indeterminate Axially Loaded Structures

In many structures all the reactions and loads can be determined by drawing free-body diagrams and solving the equilibrium equations. These structures are classified as **statically determinate**.

In other structures these equilibrium equations are not sufficient for determining all the forces. We require additional equations from the geometry of the deformation to solve them.
We can use the following five step procedure[4]:
Step 1: Express all the **equations of equilibrium** for the structure in term of the unknown forces.
Step 2: The **geometry of deformation** is evaluated in order to account for the interaction between members. The geometry of deformation requires that the deflections δ are correct.
Step 3: The relationship between the **force and the deformations** are expressed by using Equation 4.2.
Step 4: A **compatibility equation** is set up substituting the geometry-deformation equations of Step 2 into the force-deformation equations of Step 3.
Step 5: Solve the equations of equilibrium from Step 1 and the compatibility equations from Step 4 simultaneously.

■ **Example 4.2** A steel bar ABC of length 3 m is fixed between two unmoving rigid constraints. A load of 10 kN is applied at point B which is 2 m from point A. Assume a Young's Modulus of 200 GPa. The cross sectional area of the bar is 50 mm^2. What is the stress in section AB and the movement of point B?

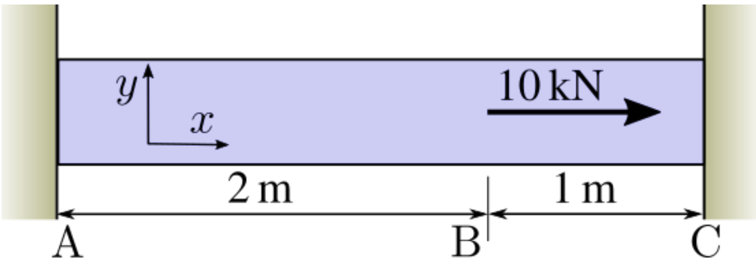


Figure 4.5: Steel Bar Clamped at each end

Solution

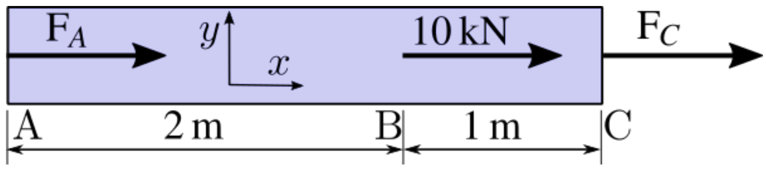


Figure 4.6: Free Body Diagram of steel bar

Step 1: Equations of equilibrium

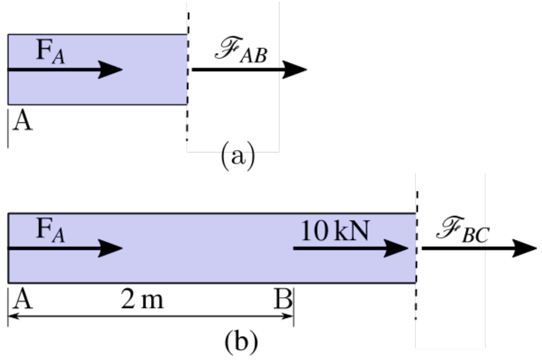


Figure 4.7: Sections through shaft

Positive defined to the right and F_A is assumed positive.

Figure 4.7(a)
$\sum F_x = 0 \Rightarrow F_A + \mathscr{F}_{AB} = 0 \Rightarrow F_A = -\mathscr{F}_{AB}$

Figure 4.7(b)
$\sum F_x = 0 \Rightarrow F_A + 10 + \mathscr{F}_{BC} = 0$
$\Rightarrow F_A = -10 + -\mathscr{F}_{BC}$

Eliminating F_A from equations above gives:

$$\mathscr{F}_{AB} = 10 + \mathscr{F}_{BC} \qquad (4.3)$$

Step 2: Geometry of Deformation
Since supports are unmoving Increase in section AB = Reduction in section BC

$$\delta_{AB} = -\delta_{BC} \qquad (4.4)$$

Step 3: Force - Deformations relationships
Using equation (4.2) $\delta = \frac{F_i L_i}{A_i E_i}$.

$$\delta_{AB} = \frac{\mathscr{F}_{AB} L_{AB}}{AE} \qquad \delta_{BC} = \frac{\mathscr{F}_{BC} L_{BC}}{AE} \qquad (4.5)$$

Step 4: Compatibility Equations
Combine equations (4.3), (4.4) and (4.5) to give $\dfrac{(10 + \mathscr{F}_{BC})\, L_{AB}}{AE} = -\dfrac{\mathscr{F}_{BC} L_{BC}}{AE}$

Step 5: Solve the equations
$(10 + \mathscr{F}_{BC})\, 2 = -\mathscr{F}_{BC} 1$
$\Rightarrow \mathscr{F}_{BC} = -6.666\,\text{kN}$
From equation (4.3) $\mathscr{F}_{AB} = 3.333\,\text{kN}$

The stress in section AB is (working in MPa, mm and N:
$\sigma_{AB} = \dfrac{\mathscr{F}_{AB}}{A} = \dfrac{3.333 \times 1000}{50} = 66.66\,\text{MPa tension}$

The movement of point B is
$\delta_{AB} = \dfrac{\mathscr{F}_{AB} L_{AB}}{AE} = \dfrac{(3.333 \times 1000)\,(2 \times 1000)}{(50)\,(200 \times 1000)} = 0.666\,\text{mm to the right}$

■

In the previous example the sections even though they had the same properties were end to end and are in **series**. When sections are inside each other they are in **coaxial**.

■ **Example 4.3** A steel outer tube is bonded to an inner brass shaft. A load of 24 kN is applied to this assembly which is fixed to the wall. The steel tube and brass shaft have moduli of 200 GPa and 100 GPa respectively. The outer diameter of the brass shaft is 18 mm which coincides with inner diameter of the steel tube. The steel tube's outer diameter is 21 mm.
Calculate the stresses in both the steel tube and brass shaft.

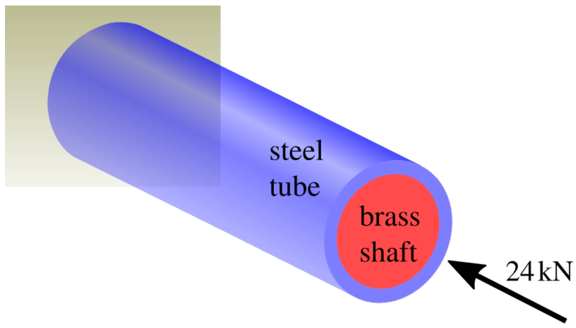


Figure 4.8: Steel tube encasing a brass shaft with load applied axially

Solution

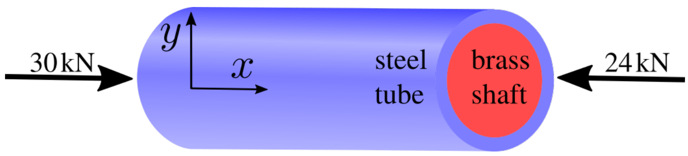


Figure 4.9: Free Body Diagram of steel bar

Step 1: Equations of equilibrium

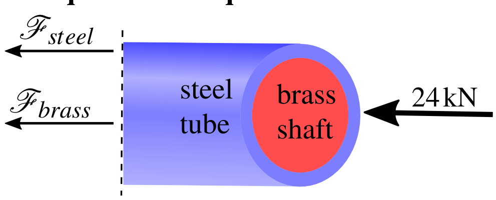


Figure 4.10: Section through shaft

Positive defined to the right and $\mathscr{F}_{steel}$ and $\mathscr{F}_{brass}$ are assumed positive.

From Figure 4.10:
$\sum F_x = 0 \Rightarrow -\mathscr{F}_{steel} - \mathscr{F}_{brass} - 24 = 0$

$$\Rightarrow \mathscr{F}_{steel} + \mathscr{F}_{brass} = -24 \qquad (4.6)$$

Step 2: Geometry of Deformation
Since steel and brass are bonded together their deformations are the same:

$$\delta_{steel} = \delta_{brass} \qquad (4.7)$$

Step 3: Force - Deformations relationships
Using equation (4.2) $\delta = \frac{F_i L_i}{A_i E_i}$.

$$\delta_{brass} = \frac{\mathscr{F}_{brass} L}{A_{brass} E_{brass}} \qquad \delta_{steel} = \frac{\mathscr{F}_{steel} L}{A_{steel} E_{steel}} \qquad (4.8)$$

Step 4: Compatibility Equations
Combine equations (4.6), (4.7) and (4.8) to give $\frac{\mathscr{F}_{brass} L}{A_{brass} E_{brass}} = \frac{(-\mathscr{F}_{brass} - 24) L}{A_{steel} E_{steel}}$

Step 5: Solve the equations

$$\frac{\mathscr{F}_{brass}L}{\frac{\pi 18^2}{4}100} = \frac{(-\mathscr{F}_{brass}-24)L}{\frac{\pi(21^2-18^2)}{4}200}$$

$\Rightarrow \mathscr{F}_{brass} = -2.811$ kN compression

From equation (4.6) $\mathscr{F}_{steel} = -21.189$ kN compression

The stress in steel and brass (working in MPa, mm and N):

$$\sigma_{steel} = \frac{-21.189}{\frac{\pi(21^2-18^2)}{4}} = -61.18\,\text{MPa compression}$$

$$\sigma_{steel} = \frac{-2.811}{\frac{\pi(18^2)}{4}} = -30.59\,\text{MPa compression}$$ ■

4.5 Temperature Effects

Most engineering materials when unrestrained contract when cooled and expand when heated. We characterise the size changes with the following equation:

$$\boxed{\varepsilon_T = \alpha \Delta T.} \tag{4.9}$$

Here ε_T is the strain due to thermal effects, Greek letter α (alpha) is the coefficient of thermal expansion in unit of °C^{-1} and ΔT is temperature change in °C. The sign convention is that positive temperature changes cause positive thermal strains which will cause expansion of the material.

If a object has both thermal changes and an applied mechanical load we have the total strain ε_{total} which is:

$$\boxed{\varepsilon_{total} = \varepsilon_\sigma + \varepsilon_T.} \tag{4.10}$$

If an axial member has length L the resulting deformation δ_T from a temperature change is:

$$\boxed{\delta_T = \varepsilon_T L = \alpha \Delta T L.} \tag{4.11}$$

If the strain is restricted so that $\varepsilon_T = 0$ a stress will develop in the axial member which is:

$$\boxed{\sigma_T = E\alpha\Delta T} \tag{4.12}$$

These ideas are illustrated in the following example:

■ **Example 4.4** A solid steel rod of diameter 20 mm is placed concentrically in a 5 mm thick aluminium tube of outer diameter 50 mm. The rod and the tube are of the same length and are welded to rigid end plates. If the temperature of the assembly is raised by 90 °C, determine the stresses in the rod and the tube. Ignore the thermal expansion of the rigid end plates. State whether the stresses are tensile or compressive. Use E = 207 GPa and $\alpha = 11 \times 10^{-6}/°C$ for steel and E = 70 GPa and $\alpha = 23 \times 10^{-6}/°C$ for aluminium.

Solution
Step 1: Equations of equilibrium
Aluminium tube and steel rod have forces opposite to each other:

$$F_{rod} = -F_{tube} \tag{4.13}$$

Step 2: Geometry of Deformation

$$\text{Strain in Rod} = \text{Strain in tube}$$
$$\varepsilon_{rod} = \varepsilon_{tube}$$

using equation (4.10)
$$\Rightarrow \varepsilon_{rod_T} + \varepsilon_{rod_\sigma} = \varepsilon_{tube_T} + \varepsilon_{tube_\sigma} \tag{4.14}$$

Here using equation (4.9)
$\varepsilon_{rod_T} = \alpha_{rod}\Delta T = \left(11 \times 10^{-6}\right)(90) = 9.90 \times 10^{-4}$
$\varepsilon_{tube_T} = \alpha_{tube}\Delta T = \left(23 \times 10^{-6}\right)(90) = 0.002\,07 > \varepsilon_{rod_T}$
Therefore the steel rod is in tension and the tube is in compression.

Step 3: Force - deformations relationships
The strains due to mechanical loading are from equation (4.2) $\delta = \frac{F_i L_i}{A_i E_i}$ and equation (2.4) $\varepsilon_{avg} = \frac{\delta}{L}$

$$\varepsilon_{rod_\sigma} = \frac{F_{rod}}{A_{rod}E_{steel}} \qquad \varepsilon_{tube_\sigma} = \frac{F_{tube}}{A_{tube}E_{aluminium}} \tag{4.15}$$

Step 4: Compatibility Equations
Substitute equation (4.15) in equation (4.14)
$$\frac{F_{rod}}{A_{rod}E_{steel}} - \frac{F_{tube}}{A_{tube}E_{aluminium}} = \varepsilon_{tube_T} - \varepsilon_{rod_T} = 0.0010800$$

Step 5: Solve the equations
$$\text{sub (1)} \Rightarrow F_{rod} = \frac{\left(\varepsilon_{tube_T} - \varepsilon_{rod_T}\right)}{\frac{1}{A_{tube}E_{aluminium}} + \frac{1}{A_{rod}E_{steel}}} = \frac{0.0010800}{\frac{1}{(25^2 - 20^2)\,\pi\,(70)} + \frac{1}{(10^2)\,\pi\,(207)}} = 30.348\,\text{kN}$$
$$\sigma_{rod} = \frac{F_{rod}}{A_{rod}} = \frac{30.348}{314.16} = 0.0966\,\text{GPa tension}$$
$$\sigma_{tube} = \frac{F_{tube}}{A_{tube}} = \frac{30.348}{706.86} = -0.042\,933\,\text{GPa compression}$$

■

5. Torsion

Wind farm in Darling with old wind mill in foreground. Understanding torsion is key in the design of these structures.

5.1 Introduction

Torsion is twisting of a structure about its longitudinal axis due to a moment. These twisting moments are called **torques** and can cause rotation about the longitudinal axis. The most common structure to accomplish this is a transmission **shaft**. They are used to transmit power from one point to another along the shaft length. Only circular shafts will be analysed in this section.

5.2 Torsional Deformation

When applying a torque to a shaft it is assumed that circular cross sections remain circular and flat. This is illustrated in figure 5.1(b) In other words the cross sections remain rigid rotating through various angles relative to each other. Radial lines will stay straight and radial.
When applying torque to non-circular cross sections there is warping which distorts the shape of the cross section and causes it to become non-planar.

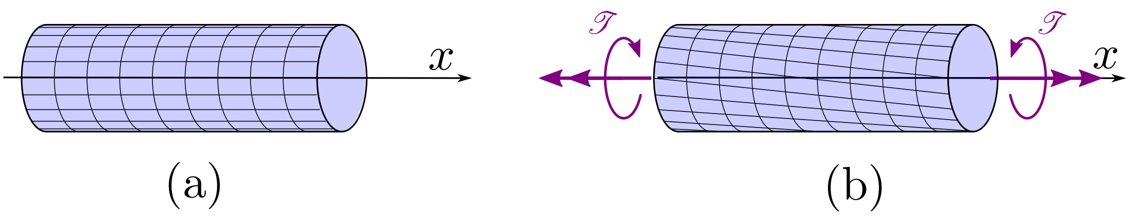


Figure 5.1: (a) Undeformed shaft and (b) deformed shaft showing two internal torque representations

Torsion can be represented pictorially using either a curved arrow or double headed arrow seen in figure 5.1(b). The direction sense for internal torque is defined by a right hand rule if your thumb points in the direction of the double headed arrow then your curled fingers would represent the direction of the applied torque. The Internal Torque $\mathscr{T}$ in figure 5.1(b) is considered positive.

Consider the red coloured rectangular element as seen in figure 5.2(a) of length L and radius c. A torque T is applied to a shaft. Since cross sections remain rigid as they rotate the rectangular element undergoes shear to become a parallelogram as seen figure 5.2(b). The shear strain of the element is γ. Line AB is the initial undeformed line on the surface of the shaft. It deforms to A′B and the angle ABA′ is the same as the shear angle γ. From geometry we know:

$$\phi\rho = \text{length of arc AA}' = \gamma L$$

here the shear γ and the angle of twist ϕ are expressed in radians and ρ is the distance from the centre to the point of interest A which becomes A′.

$$\boxed{\gamma = \frac{\rho\phi}{L}} \tag{5.1}$$

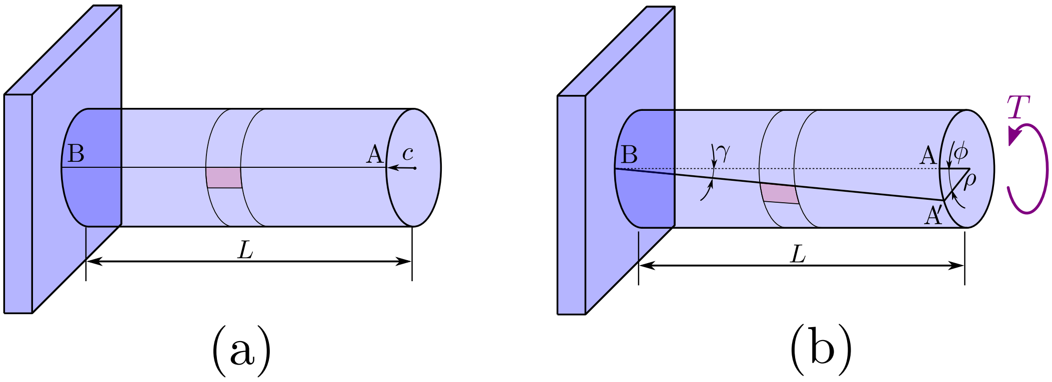


Figure 5.2: Undeformed shaft with clamped support shown in (a) and and deformed shaft under torsion(b).

The shear strain is maximum on the surface where $\rho = c$ so

$$\gamma_{\max} = \frac{c\phi}{L}$$

eliminating the ϕ and L we get the following relation:

$$\boxed{\gamma = \frac{\rho}{c}\gamma_{\max}} \tag{5.2}$$

The shear strain varies linearly from the centre of the shaft and is maximum at the surface.

5.3 Shear Stress in Torsion

Here it is assumed that the stresses are linearly related to the strain and the stresses remain below the elastic limit. In other words Hooke's Law for shear in equation (2.12):

$$\tau = \gamma G \tag{5.3}$$

Here G is the Modulus of Rigidity and τ is the shear stress.
Multiply equation (5.2) by G and substitute equation (5.3) to give:

$$\boxed{\tau = \frac{\rho}{c}\tau_{\max}} \tag{5.4}$$

shows that the shear stress varies linearly with ρ from the centre to the surface of the shaft.

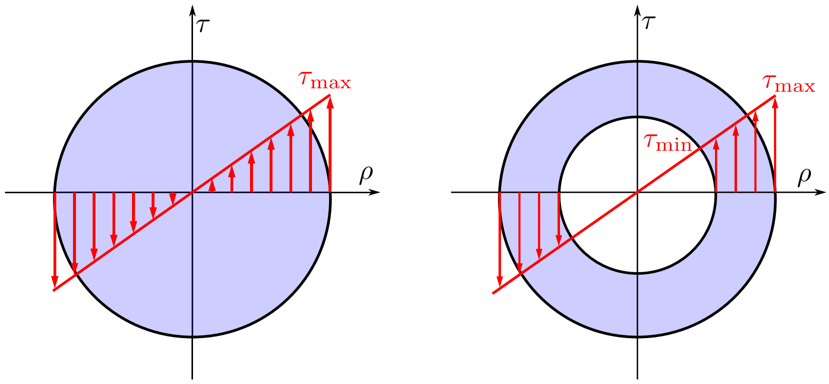


Figure 5.3: Linear distribution of shear stress in torsion in a solid shaft (a) and a hollow shaft (b).

Consider a differential element on the shaft with area dA in figure 5.4. A torque $\mathscr{T}$ acts on the shaft which has radius c.

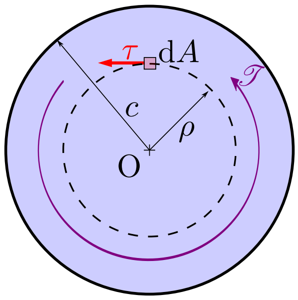


Figure 5.4: Determination of the resultant of the shear stresses acting on a cross section.

The resulting shear force acting on the element area is $d\mathscr{V} = \tau dA$. There is an moment acting on the element resulting from this shear force and by substitution of equation (5.4) we obtain:

$$d\mathscr{M} = \rho d\mathscr{V} = \rho \tau dA = \frac{\tau_{max}}{c}\rho^2 dA$$

The elemental moments are added over the area to give the torque $\mathscr{T}$ to satisfy equilibrium

$$\boxed{\mathscr{T} = \int_A d\mathscr{M} = \frac{\tau_{max}}{c}\int_A \rho^2 dA} \tag{5.5}$$

The parameters τ_{max} and c can be taken outside the integral sign since they are independent of dA. The integral is the **polar moment of inertia**, J:

$$\boxed{J = \int_A \rho^2 dA} \tag{5.6}$$

Equation (5.5) can be rearranged and equation (5.6) can be substituted to give:

$$\boxed{\tau_{max} = \frac{\mathscr{T}c}{J}} \tag{5.7}$$

For any shear along the radius of the shaft ρ

$$\boxed{\tau = \frac{\mathscr{T}\rho}{J}} \tag{5.8}$$

This equation is known as the **elastic torsion formula** of which equation (5.8) is a special case. It only applies to linear elastic, isotropic, homogeneous materials.

5.4 Polar Moment of Inertia

The polar moment of inertia is also known as the polar second moment of area and will be derived for a solid shaft.

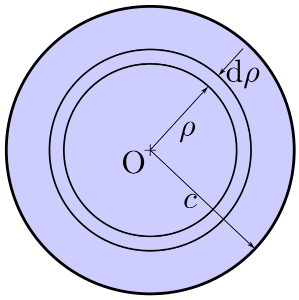


Figure 5.5: Derivation of polar moment of inertia for a solid shaft.

$$J = \int_A \rho^2 dA = \int_0^c \rho^2 2\pi\rho d\rho = \left[\frac{1}{4} 2\pi\rho^4\right]_0^c = \frac{\pi c^4}{2}$$

For a shaft diameter where $D = 2c$

$$\boxed{J = \frac{\pi c^4}{2} = \frac{\pi D^4}{32}} \tag{5.9}$$

Polar moment of inertia is a geometric property of the circular area. The SI units for J are m^4
For a hollow shaft the integration limits would be from the outer surface to the inner surface. If the outer surface has radius c_o and diameter D and the inner radius c_i with diameter d. The polar moment of inertia for a hollow shaft is:

$$\boxed{J = \frac{\pi\left(c_o^4 - c_i^4\right)}{2} = \frac{\pi\left(D^4 - d^4\right)}{32}} \tag{5.10}$$

5.5 Angle of Twist due to Torsion

The relationship between shear stress and torque along any radial component is given by equation (5.8):

$$\tau = \frac{\mathscr{T}\rho}{J}$$

By taking the product of shear strain and modulus of rigidity (G) the angle of twist, ϕ along the radius from equation (5.1) is:

$$\tau = G\gamma = G\frac{\rho\phi}{L} \tag{5.11}$$

Equating equation (5.8) and equation (5.11) to the radius $\frac{\tau}{\rho}$ gives:

$$\boxed{\frac{G\phi}{L} = \frac{\tau}{\rho} = \frac{\mathscr{T}}{J}} \tag{5.12}$$

These relationships are useful in dual specification problems where the angle of twist and shear stress are both limiting factors. See the following example.

■ **Example 5.1** The hollow steel shaft is 2 m long and has an outer diameter of 40 mm. A torque of 400 N m is applied to the shaft. Determine the smallest thickness of the shaft if the allowable shear stress is $\tau_{allow} = 140\,\text{MPa}$ and the shaft is restricted not to twist more than 0.05 rad. The shear modulus of the steel is 75 GPa.

Solution

First the limitation due to shear stress is calculated using equation (5.8) and equation (5.10). The maximum shear stress is on the outside of the shaft $c_o = \rho$

$$\tau = \frac{\mathcal{T}\rho}{J}$$

$$140 \times 10^6 = \frac{400 c_o}{\frac{1}{2}\pi\left(c_o^4 - c_i^4\right)} = \frac{400\,(0.02)}{\frac{1}{2}\pi\left((0.02)^4 - c_i^4\right)}$$

$$\Rightarrow c_i = 0.01875\,\text{m}$$

Now the angle of twist restriction is checked use part of equation (5.12) :

$$\frac{G\phi}{L} = \frac{\mathcal{T}}{J}$$

$$\frac{\left(75 \times 10^9\right)(0.05)}{2} = \frac{400}{\frac{1}{2}\pi\left(0.02^4 - c_i^4\right)}$$

$$\Rightarrow c_i = 0.01247\,\text{m}$$

Choose the smallest inner diameter will give the strongest shaft needed for the **angle of twist**.
The shaft thickness t required then is:
$t = c_o - c_i = 0.02 - 0.01247 = 0.00753\,\text{m}$ ■

The **torsional rigidity** is the product of the shear modulus and polar moment of inertia and represents the stiffness of the shaft to torsion:

$$\boxed{GJ = \frac{\mathcal{T}}{\phi/L}} \tag{5.13}$$

Equation (5.12) can be rearranged to give the angle of twist:

$$\boxed{\phi = \frac{\mathcal{T}L}{JG}} \tag{5.14}$$

If a torsion member is composed of different segments with different materials, different polar moments of inertia and/or different lengths with different torques on different segments, the angle of twist can be added algebraically as follows:

$$\boxed{\phi = \sum \frac{\mathcal{T}_i L_i}{J_i G_i}} \tag{5.15}$$

Here $\mathscr{T}_i, L_i, J_i$ and G_i are the internal torque, length, polar moment of inertia and shear modulus of the i^{th} segment respectively. The internal torque can be calculated by passing a section through a segment and drawing a free body diagram of the portion of the shaft on one side of the diagram.
For practical purposes all local discontinuities are ignored at connections where there are pulleys, gears or some coupling connection.

5.6 Power Transmitted By Shafts

An important usage of shafts is transmitting power from a motor or engine to a component. **Power** is the work done per unit time. The work transmitted W by a rotating shaft is the product of the applied torque T and the angle θ through which the shaft rotates:

$$W = T\theta$$

The power transmitted by a shaft subject to a constant torque T is:

$$P = \frac{dW}{dt} = T\frac{d\theta}{dt}$$

The angular velocity ω equals the rate of change angular displacement. The power transmitted is therefore:

$$\boxed{P = T\omega} \tag{5.16}$$

Here ω is measured in radians per second. The SI unit for torque is N m and power is the watt (W). For machinery the frequency is often expressed in hertz (Hz), where 1 Hz = 1 cycle/s. Since 1 cycle = 2π radians therefore $\omega = 2\pi f$ and equation (5.16) can be expressed as:

$$\boxed{P = 2\pi f T} \tag{5.17}$$

A very common unit of measure is rotational speed in revolutions per minute (rpm).

$$1\,\text{rpm} = \frac{1}{60}\text{s}^{-1} = \frac{1}{60}\text{Hz}$$

Then power can be written in terms of rpm n with:

$$\boxed{P = \frac{2\pi n T}{60}} \tag{5.18}$$

5.7 Statically Indeterminate Shafts under Torsion

Often to determine the stresses in a shafts, the torques can be determined from drawing free body diagrams and solving the equations of equilibrium. The solution to this problem is termed **statically determinate**.
For many other structures these equilibrium equations are not sufficient for determining all the torques. These problems are **statically indeterminate** and the geometry of the deformation is also needed to solve these problems.
A similar analysis to the section on axial loading in section 4.4 is required. The five step procedure is given as follows:
Step 1: Express all the **equations of equilibrium** for the structure in term of the unknown internal torques.

Step 2: The **geometry of deformation** is evaluated in order to account for the interaction between the torsion members.
Step 3: The relationship between the **torque and the deformations** are expressed by using equation (5.14).
Step 4: A **compatibility equation** is set up substituting the geometry-deformation equations of Step 2 into the torque-deformation equations of Step 3.
Step 5: Solve the equations of equilibrium from Step 1 and the compatibility equations from Step 4 simultaneously to calculate the unknown internal torques and deformations.
Typical statically indeterminate torsion member configurations are given below:

5.7.1 Coaxial Torsion Members

For coaxial torsion members one member is inside another, see figure 5.6. We assume they rotate with each other as one member.

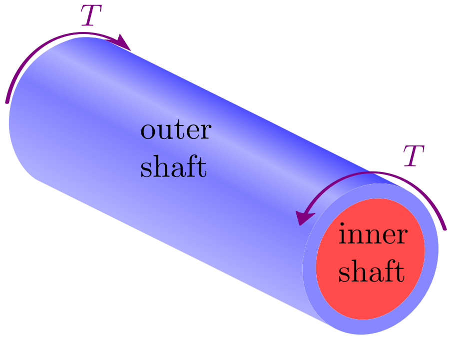


Figure 5.6: Torsion member coaxially connected.

The two shafts are bonded together so the angle of twist in each member is the same. However the total applied torque, T is shared between the two members.

$$\boxed{\phi_{\text{inner}} = \phi_{\text{outer}}} \tag{5.19}$$

and

$$\boxed{T = \mathscr{T}_{\text{inner}} + \mathscr{T}_{\text{outer}}} \tag{5.20}$$

The following example is an illustration of how this type of problem is done.

■ **Example 5.2** A steel outer tube is bonded to an inner brass shaft. A torque of 30 kN m is applied to this assembly which is fixed to the wall. The steel tube and brass shaft have moduli of 80 GPa and 40 GPa respectively. The outer diameter of the brass shaft is 18 mm which coincides with inner diameter of the steel tube. The steel tube's outer diameter is 21 mm.
Calculate the shear stresses in both the steel tube and brass shaft.

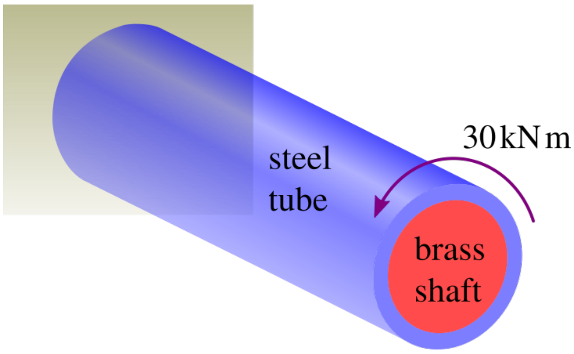


Figure 5.7: Steel tube encasing a brass shaft with a torque applied

Solution
Step 1: Equations of equilibrium
From equation (5.20)

$$30 \times 10^3 = \mathscr{T}_{\text{steel}} + \mathscr{T}_{\text{brass}} \tag{5.21}$$

Step 2: Geometry of Deformation
Since steel and brass are bonded together their deformations are the same and we use From equation (5.19)

$$\phi_{steel} = \phi_{brass} \tag{5.22}$$

Step 3: Force - Deformations relationships
Using equation (5.14) $\phi = \frac{\mathscr{T}L}{JG}$.

$$\phi_{brass} = \frac{\mathscr{T}_{brass}L}{J_{brass}G_{brass}} \qquad \phi_{steel} = \frac{\mathscr{T}_{steel}L}{J_{steel}G_{steel}} \tag{5.23}$$

Step 4: Compatibility Equations
Combine equations (5.21), (5.22) and (5.23) to give $\frac{\mathscr{T}_{brass}L}{J_{brass}G_{brass}} = \frac{\left(30 \times 10^3 - \mathscr{T}_{brass}\right)L}{J_{steel}G_{steel}}$

Step 5: Solve the equations
Using equation (5.10) and (5.9): $\frac{\mathscr{T}_{brass}L}{\frac{\pi 0.018^4}{32} 40 \times 10^9} = \frac{\left(30 \times 10^3 - \mathscr{T}_{brass}\right)L}{\frac{\pi(0.021^4 - 0.018^4)}{32} 80 \times 10^9}$

$\Rightarrow \mathscr{T}_{brass} = 3.514 \times 10^3$ N m
From equation (5.21) $\mathscr{T}_{steel} = 26.486 \times 10^3$ N m

Calculate the shear stress in steel and brass from equation (5.8)

$$\tau_{steel} = \frac{26.486 \times 10^3 \left(\frac{1}{2} 0.021\right)}{\frac{\pi}{32}\left(0.021^2 - 0.018^2\right)} = 6.424 \times 10^6 \,\text{Pa} = 6.424 \,\text{MPa}$$

$$\tau_{brass} = \frac{3.514 \times 10^3 \left(\frac{1}{2} 0.018\right)}{\frac{\pi}{32}\left(0.018^2\right)} = 2.753 \times 10^6 \,\text{Pa} = 2.753 \,\text{MPa}$$

■

5.7.2 Torsion Members in Series

Torsion members connected in series are connected end to end.

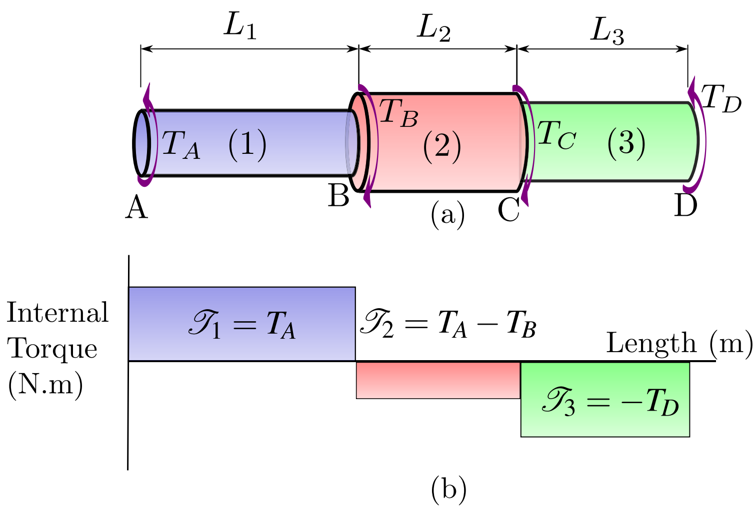


Figure 5.8: Torsion members connected in series

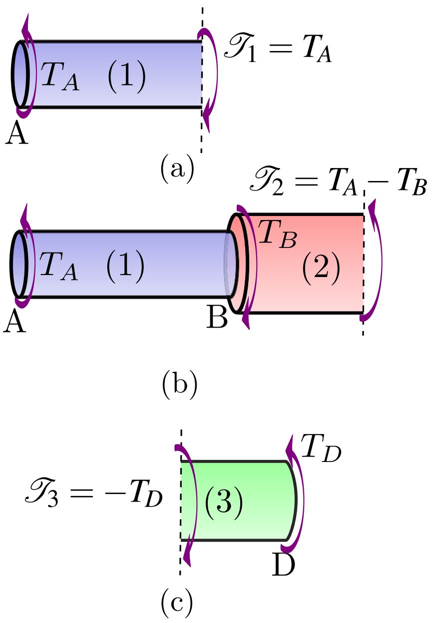


Figure 5.9: Free body diagrams of various sections of a member in torsion

To determine the internal torque in each section free body diagrams of each section would have to be considered, see figure 5.9.

The total angle of twist would be the sum of the angles of twist in each section:

$$\boxed{\phi_{\text{total}} = \sum \phi_i} \tag{5.24}$$

The difference to equation (5.15) is that not all the torques are necessarily known.

If both ends are *rigidly supported*, the total angle of twist would be zero.

$$\boxed{0 = \sum \phi_i} \tag{5.25}$$

■ **Example 5.3** A steel bar ABC of length 3 m is fixed between two unmoving rigid constraints. A load of 10 kN is applied at point B which is 2 m from point A. Assume a shear modulus of 80 GPa. The diameter of the bar is 10 mm. What is the maximum shear stress in section AB and the angle of twist of point B?

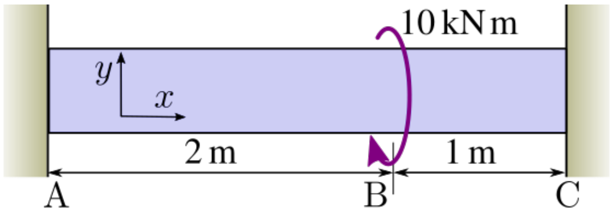


Figure 5.10: Steel Bar Clamped at each end

Solution

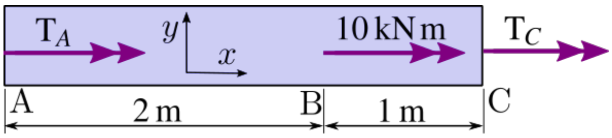


Figure 5.11: Free Body Diagram of steel bar using double arrow notation

Step 1: Equations of equilibrium

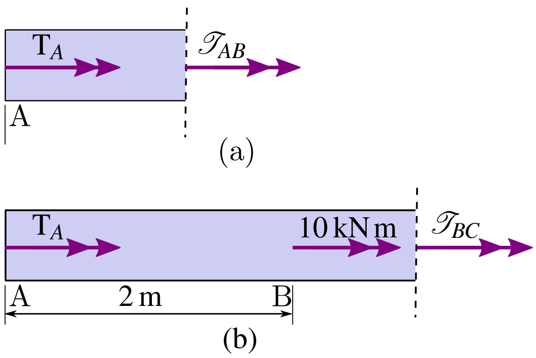


Figure 5.12: Sections through shaft

Positive defined to the right and F_A is assumed positive.

Figure 5.12(a)
$\sum M_x = 0 \Rightarrow T_A + \mathscr{T}_{AB} = 0 \Rightarrow T_A = -\mathscr{T}_{AB}$

Figure 5.12(b)
$\sum M_x = 0 \Rightarrow T_A + 10 + \mathscr{T}_{BC} = 0$
$\Rightarrow T_A = -10 + -\mathscr{T}_{BC}$

Eliminating T_A from equations above gives:

$$\mathscr{T}_{AB} = 10 + \mathscr{T}_{BC} \tag{5.26}$$

Step 2: Geometry of Deformation
Since supports are unmoving Twist in section AB = Negative twist in section BC

$$\phi_{AB} = -\phi_{BC} \tag{5.27}$$

Step 3: Force - Deformations relationships
Using equation (5.12) $\dfrac{G\phi}{L} = \dfrac{\mathscr{T}}{J}$.

$$\phi_{AB} = \frac{\mathscr{T}_{AB} L_{AB}}{GJ} \qquad \phi_{BC} = \frac{\mathscr{T}_{BC} L_{BC}}{GJ} \tag{5.28}$$

Step 4: Compatibility Equations
Combine equations (5.26), (5.27) and (5.28) to give $\dfrac{(10 + \mathscr{T}_{BC}) L_{AB}}{GJ} = -\dfrac{\mathscr{T}_{BC} L_{BC}}{GJ}$

Step 5: Solve the equations

$(10+\mathscr{T}_{BC})\,2 = -\mathscr{T}_{BC}1$

$\Rightarrow \mathscr{T}_{BC} = -6.666\,\text{kN m}$

From equation (5.26) $\mathscr{T}_{AB} = 3.333\,\text{kN m}$

The maximum shear stress in section AB is (working in MPa, mm and N):

$$\tau_{AB} = \frac{\mathscr{T}_{AB}\rho}{J} = \frac{(3.333\times 1000)\left(\frac{1}{2}10\right)}{\frac{\pi}{32}10^4} = 16.97\,\text{MPa tension}$$

The angle of twist of point B is

$$\phi_{AB} = \frac{\mathscr{T}_{AB}L_{AB}}{GJ} = \frac{(3.333\times 1000)\,(2\times 1000)}{(80\times 1000)\left(\frac{\pi}{32}10^4\right)} = 0.084\,88\,\text{rad}$$

■

6. Beam Equilibrium

The van Stadens Bridge over the Fish River can be analysed with a knowledge of how beams are modelled with various loads

6.1 Introduction

In the previous section we deal with loads that are directed along the axis of the structural members. In this section we deal with loads that are perpendicular to the structural members on which they act. The structural members are called **beams** and the loads are called transverse loads. Beams have one dimension, the length, much greater than either the width or the thickness.

6.2 Shear Force and Bending Moment Diagrams

In order to calculate the stresses due to bending in a beam, the internal shear forces, $\mathscr{V}$ and internal bending moments, $\mathscr{M}$ in the beam first have to be determined. The best way of doing this is by using bending moment and shear force diagrams.

6.2.1 Shear Force Sign and Bending Moment Conventions

The **deformation sign conventions** gives an indication of how a body deforms. In a previous section the axial force sign conventions indicated whether a body was in tension or compression. Bending deformation conventions are illustrated in figure 6.1.

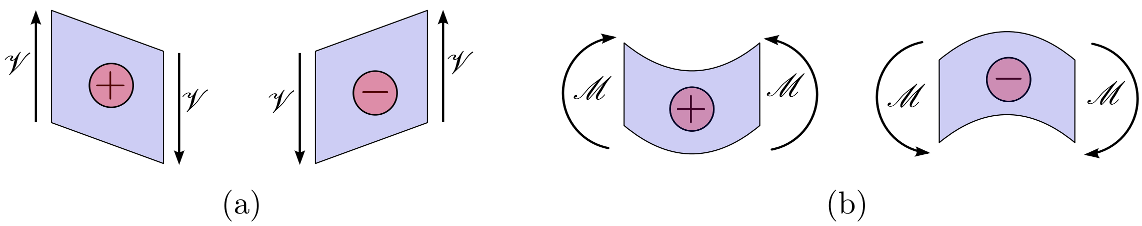


Figure 6.1: (a) Shear force sign convention and (b) moment sign convention shown on a beam element. A positive internal moment is a 'smile' and a negative internal moment is a 'frown'.

Positive internal shear force $\mathscr{V}$

- acts downward on the right-hand side of a beam.
- acts upward on the left-hand side of a beam.
- causes a beam element to rotate clockwise

Positive internal moment $\mathscr{M}$

- acts anticlockwise on the right-hand face of a beam.
- acts clockwise on the left-hand face of a beam.
- causes the beam element to go into compression on top and tension at the bottom

These sign conventions will be illustrated with examples in the following sections:

6.2.2 Concentrated Loads

■ **Example 6.1** Consider with a simply supported beam ABC with a concentrated or point load midway between A and C. The distance between A and C is L.

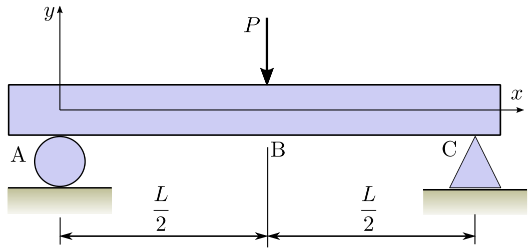


Figure 6.2: Simply supported beam with a concentrated load.

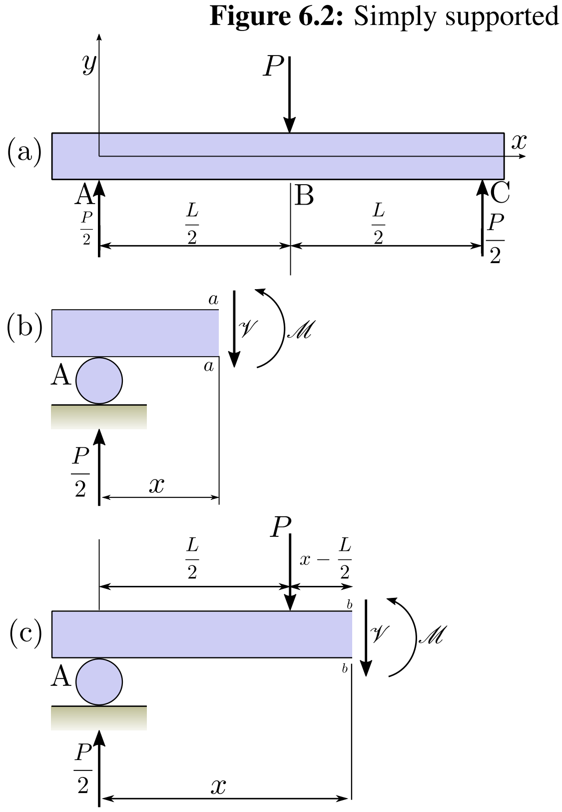


Figure 6.3: Depiction of (a) free body diagram (b) section view before concentrated load and (c) section view after concentrated load.

To start the solution we draw a free body diagram of the structure. This is shown in figure 6.3(a)

Support Reactions
$\Sigma M_A = 0 \Rightarrow C_y = \frac{1}{2}P$
$\Sigma F_y = 0 \Rightarrow A_y = \frac{1}{2}P$

Cut the diagram between places of interest and consider the interval between the cuts.

Interval $0 \le x < \frac{L}{2}$ (Section a-a), figure 6.3(b)
$\Sigma F_y = \frac{P}{2} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{P}{2}$
A positive shear force is assumed which acts downwards on the right-hand side of a beam. See figure 6.1(a) for the sign convention.

$\Sigma M_{a-a} = -\frac{P}{2}x + \mathscr{M} = 0 \Rightarrow \mathscr{M} = \frac{P}{2}x$
A positive moment is assumed force which acts anticlockwise on the right-hand face of a beam.

Interval $\frac{L}{2} \le x < L$ (Section b-b), figure 6.3(c)
$\Sigma F_y = \frac{P}{2} - P - \mathscr{V} = 0 \Rightarrow \mathscr{V} = -\frac{P}{2}$

$\Sigma M_{b-b} = P\left(x - \frac{L}{2}\right) - \frac{P}{2}x + \mathscr{M} = 0$
$\Rightarrow \mathscr{M} = -\frac{P}{2}x + \frac{PL}{2}$

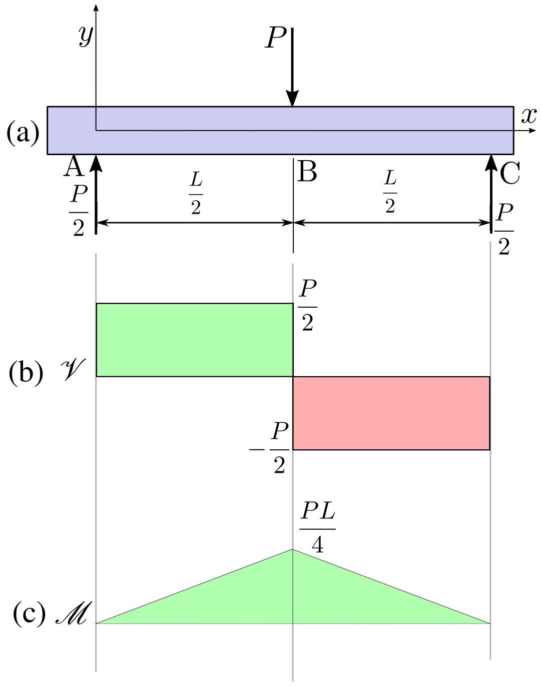


Figure 6.4: Construction of (b) shear force and (c) bending moment diagrams

Plotting the functions

Free body diagram replotted in Figure 6.4(a) repeated from figure 6.3(a)

Shear Force Diagram
On the interval $0 \leq x < \frac{L}{2}$ plot $\mathscr{V} = \frac{P}{2}$ and on the interval $\frac{L}{2} \leq x < L$ plot $\mathscr{V} = -\frac{P}{2}$ as shown in Figure 6.4(b).

Note the discontinuity at x = $\frac{L}{2}$ where the load P is located.

Bending Moment Diagram
On the interval $0 \leq x < \frac{L}{2}$ plot $\mathscr{M} = \frac{P}{2}x$ and on the interval $\frac{L}{2} \leq x < L$ plot $\mathscr{M} = -\frac{P}{2}x + \frac{PL}{2}$ as shown in Figure 6.4(c).

■

6.2.3 Applied Moments

■ **Example 6.2** Consider a simply supported beam ABC with a point moment midway between A and C. The distance between A and C is L.

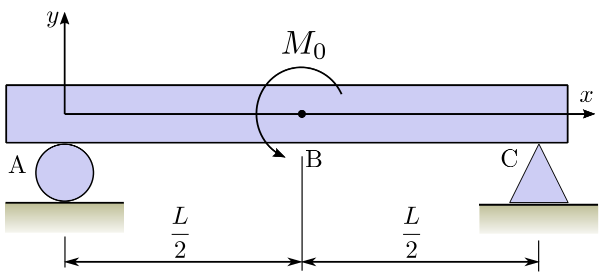


Figure 6.5: Simply supported beam with a point moment.

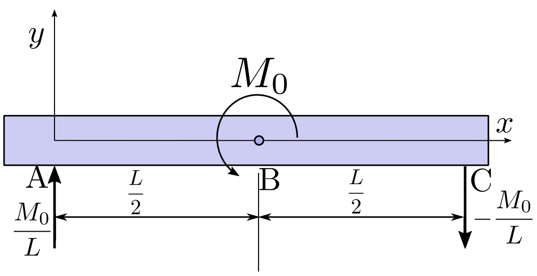


Figure 6.6: Free body diagram for simply supported structure with point moment midway

To start the a free body diagram of the structure is drawn. This is shown in figure 6.6

Determine reactions at supports:

$$\Sigma M_A = 0 \Rightarrow C_y = -\frac{M_0}{L}$$

$$\Sigma F_y = 0 \Rightarrow A_y = \frac{M_0}{L}$$

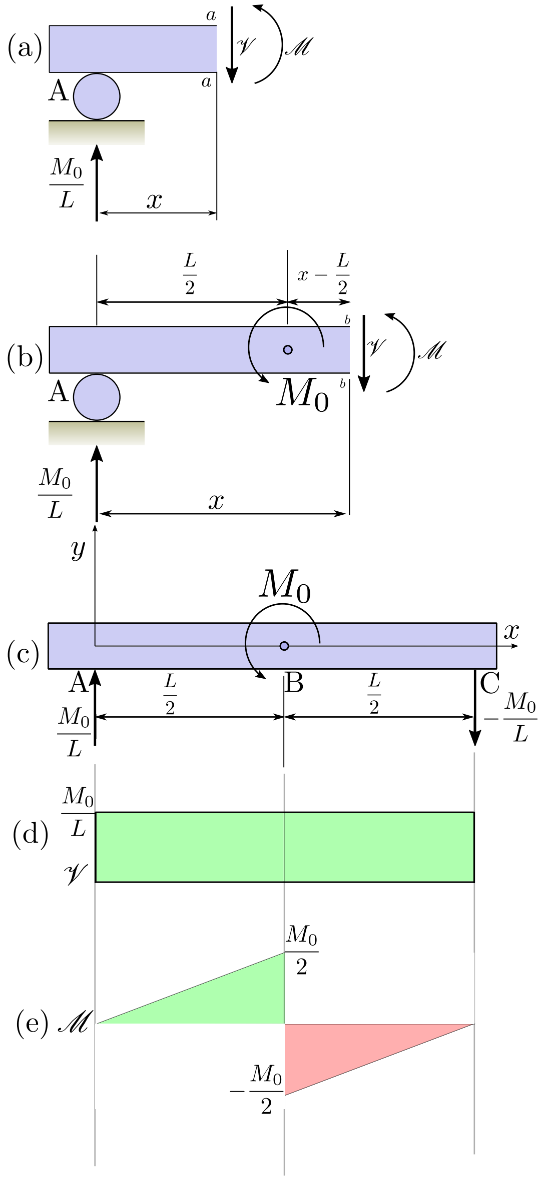


Figure 6.7: Construction of shear force and bending moment diagrams

Interval $0 \le x < \frac{L}{2}$ (Section a-a), figure 6.7(a)

$$\Sigma F_y = \frac{M_0}{L} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{M_0}{L}$$

A positive shear force is assumed which acts downwards on the right-hand side of a beam.

$$\Sigma M_{a-a} = -\frac{M_0}{L}x + \mathscr{M} = 0 \Rightarrow \mathscr{M} = \frac{M_0}{L}x$$

A positive moment is assumed force which acts anticlockwise on the right-hand face of a beam.

Interval $\frac{L}{2} \le x < L$ (Section b-b), figure 6.7(b)
Same as on section a-a since point moment does not affect load directly

$$\Sigma F_y = \frac{M_0}{L} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{M_0}{L}$$

$$\Sigma M_{b-b} = -\frac{M_0}{L}x + M_0 + \mathscr{M} = 0$$

$$\Rightarrow \mathscr{M} = \frac{M_0}{L}x - M_0$$

Plotting the functions

Free body diagram replotted in figure 6.7(c)

Shear Force Diagram

On the interval $0 \le x < \frac{L}{2}$ plot $V = \frac{M_0}{L}$ as shown in figure 6.7(d)

Bending Moment Diagram

On the interval $0 \le x < \frac{L}{2}$ plot $\mathscr{M} = \frac{M_0}{L}x$ and on the interval $\frac{L}{2} \le x < L$ plot $\mathscr{M} = \frac{M_0}{L}x - M_0$ as shown in Figure 6.7(e)

■

6.2.4 Uniformly Distributed Loads

■ **Example 6.3** A simply supported beam AB of length L has a distributed load w all the way between the supports.

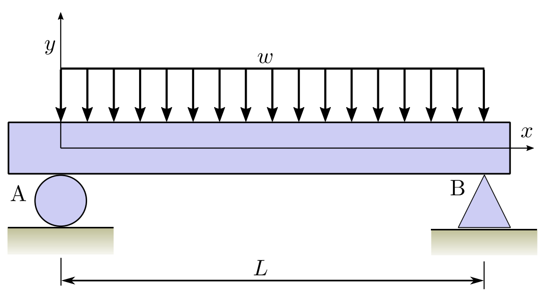


Figure 6.8: Uniformly distributed load of intensity w on simply supported beam.

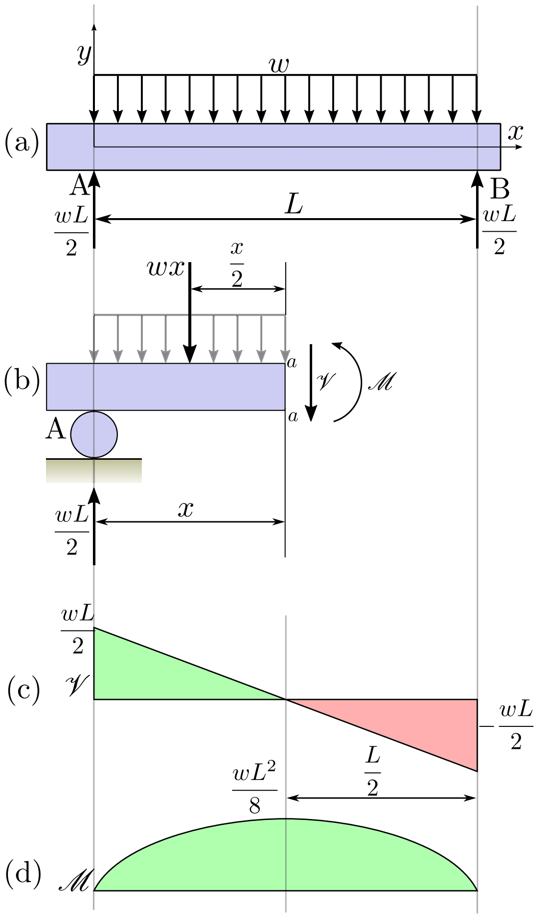


Figure 6.9: Construction of shear force and bending moment diagrams

Support Reactions
The beam is symmetrically loaded and supported therefore the load is shared equally in each support.

$\Sigma F_y = -wL + A_y + B_y = 0$

$\Rightarrow A_y = B_y = \dfrac{wL}{2}$

Interval $0 \leq x < L$ (Section a-a), figure 6.9(b)

$\Sigma F_y = \dfrac{wL}{2} - wx - \mathscr{V} = 0$

$$\Rightarrow \mathscr{V} = \frac{wL}{2} - wx \qquad \text{(a)}$$

$$\Sigma M_{a-a} = -\frac{wL}{2}x + wx\frac{x}{2} + \mathscr{M} = 0$$

$$\Rightarrow \mathscr{M} = \frac{wL}{2}x - \frac{w}{2}x^2 \qquad \text{(b)}$$

Plotting the functions

Shear force diagram plotted from equation (a) and shown in in figure 6.9(c)

Moment diagram plotted from equation (b) and shown in in figure 6.9(d)

Note that the maximum bending moment occurs when the shear force equals zero.

■

6.3 Relationships between Loads, Shear forces and Bending Moments

In the previous section shear and moment diagrams were constructed by expressing the bending moment $\mathscr{M}(x)$ and shear force $\mathscr{V}(x)$ as functions of the beam length and plotting these functions. When a beam has several loadings this method can be quite laborious. In this section a simpler method is discussed based on the differential relationships between load and shear and then shear and moment.

6.3.1 Distributed Loads

In figure 6.10(a) and (b) it is important to note that **all the directions are shown as positive** according to the deformation sign convention. We look at a small element of the beam of length Δx. The internal resultant shear $\mathscr{V} + \Delta\mathscr{V}$ and moment $\mathscr{M} + \Delta\mathscr{M}$ are slightly different on the right hand side from the left to satisfy equilibrium. The resultant force $w\Delta x$ replaces the distributed load acting on the length Δx at a distance $\frac{\Delta x}{2}$ from the right side. Applying the equations of equilibrium to this element we get:

$$\Sigma F_y = \mathscr{V} - (\mathscr{V} + \Delta\mathscr{V}) + w(x)\Delta x = 0$$
$$\Rightarrow \Delta\mathscr{V} = w(x)\Delta x$$

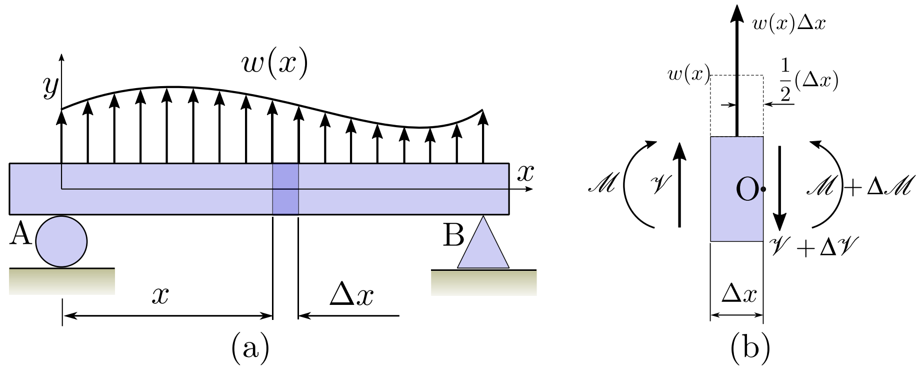


Figure 6.10: A beam subjected to distributed load (a) and a beam element showing internal shear forces and bending moments (b)

$$\Sigma M_O = -\mathscr{V}(\Delta x) - w(x)\Delta x\frac{\Delta x}{2} - \mathscr{M} + (\mathscr{M} + \Delta\mathscr{M}) = 0$$
$$\Rightarrow \Delta\mathscr{M} = \mathscr{V}(\Delta x) + w(x)\Delta x\frac{\Delta x}{2}$$

Dividing each term in each equation by Δx and taking the limit as $\Delta x \to 0$ gives:

$$\boxed{\frac{d\mathscr{V}}{dx} = w(x)} \tag{6.1}$$

$$\boxed{\frac{d\mathscr{M}}{dx} = \mathscr{V}(x)} \tag{6.2}$$

Equation (6.1) indicates that the load intensity $w(x)$ at any point x along a beam is numerically equal to slope of the shear force $\mathscr{V}(x)$ at that same point. In the same way Equation (6.2) shows the shear force $\mathscr{V}(x)$ at any point is equal to the slope of the internal $\mathscr{M}(x)$ at the same point. These equations can be used to quickly obtain the bending moment and shear force diagrams for a beam. Equations (6.1) can be rewritten in the form $dV = w(x)dx$. The term $w(x)dx$ represents the differential area under the distributed load diagram and can be integrated between any two limits x_1 and x_2 on the beam to give:

$$\boxed{\Delta\mathscr{V} = \int_{x_1}^{x_2} w(x)dx} \tag{6.3}$$

In a similar way Equation (6.2) can be rewritten in the form $d\mathscr{M} = \mathscr{V}(x)dx$ where the term $\mathscr{V}(x)dx$ represents the differential area under the shear force diagram. Integrating again between two limits x_1 and x_2 gives:

$$\boxed{\Delta\mathscr{M} = \int_{x_1}^{x_2} \mathscr{V}(x)dx} \tag{6.4}$$

6.3.2 Concentrated Loads

Consider a concentrated or point load and moment. Free body diagrams are shown for beam elements subject to a point load and point moment in Figure 6.11.

Looking at the force equilibrium for Figure 6.11(b) gives:

$$\Sigma F_y = \mathscr{V} + F\,(\mathscr{V} + \Delta\mathscr{V}) = 0 \Rightarrow$$

$$\boxed{\Delta\mathscr{V} = F} \tag{6.5}$$

At the location of a positive external load the shear force diagram is discontinuous. The **shear-force diagram jumps by an amount equal to the applied load , F**.

Now consider a beam element subject to a point load M_O as shown in Figure 6.11(a). Taking moments about point O gives:

$$\Sigma M_O = -\mathscr{V}(\Delta x) + M_O - \mathscr{M} + (\mathscr{M} + \Delta\mathscr{M}) = 0 \Rightarrow$$

Consider the limit as $\Delta x \to 0$ gives:

$$\boxed{\Delta\mathscr{M} = -M_O} \tag{6.6}$$

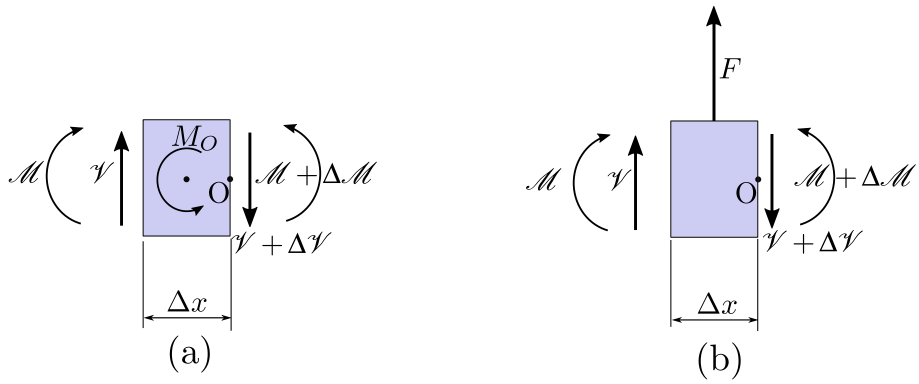


Figure 6.11: Beam elements of point moment (a) and concentrated load (b).

The ideas presented will be illustrated in the following example from Hibbeler[3].

■ **Example 6.4** A clamped beam has concentrated loads P L and $2L$ from the fixed end.

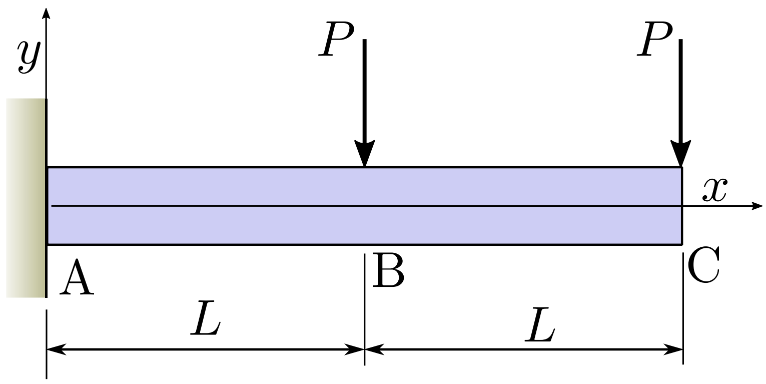


Figure 6.12: Cantilever beam with concentrated loads

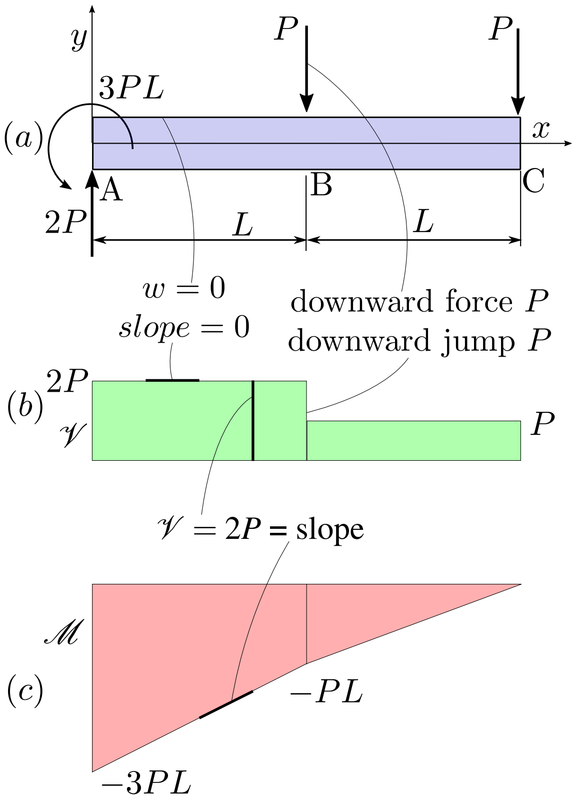


Figure 6.13: Construction of shear force and bending moment diagram

Support Reactions
Using force and moment equilibrium the reactions at A can be determined and are shown in Figure 6.13(a).

Shear Diagram
The shear at left end of the beam is plotted first, $2P$. It is positive by the deformation sign convention. The shear diagram on the extreme right end is positive P using the same convention. Since there is no distributed loading on the beam, the slope of the shear diagram is zero. The downward force P at the center of the beam causes the shear diagram to jump downward an amount P and can be seen in Figure 6.13(b).

Moment Diagram
This is shown in Figure 6.13(c). The moments at the left end of the beam is -3PL again negative because of the moment sign convention. Here the moment diagram consists of two sloping lines, one with a slope of +2P and the other with a slope of +P. The value of the moment in the center of the beam can be determined from the area under the shear diagram. If we choose the left half of the shear diagram:

$$\mathscr{M}|_{x=L} = \mathscr{M}|_{x=0} + \Delta\mathscr{M}$$

$$\Rightarrow \mathscr{M}|_{x=L} - 3PL + (2P)L = -PL$$

■

Another example is given here to illustrate how a uniformly distributed load can be dealt with.

■ **Example 6.5** a cantilver beam with a uniformly distributed load of length, L.

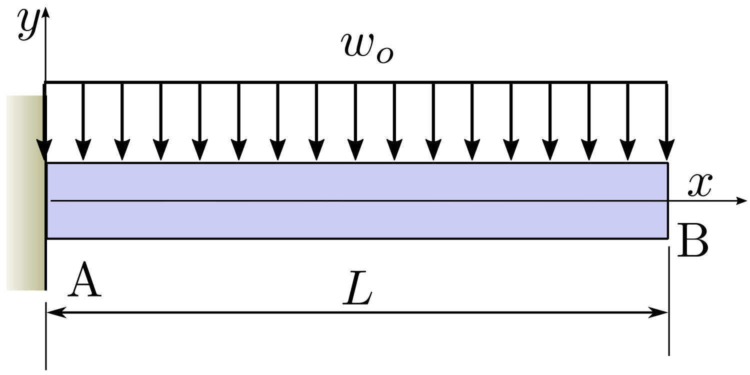


Figure 6.14: Cantilever beam with concentrated loads

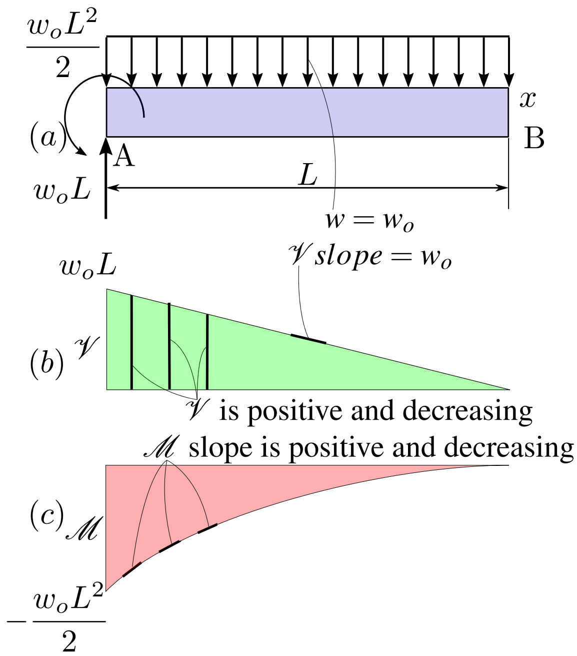


Figure 6.15: Construction of shear force and bending moment diagram

Support Reactions
Using force and moment equilibrium the reactions at A can be determined and are shown in Figure 6.15(a).

Shear Diagram
The shear at the left end is plotted first, $2w_oL$. It is positive by the deformation sign convention. At B the shear is zero. The slope of the shear diagram is $-w_o$ and can be seen in Figure 6.15(b).

Moment Diagram
This is shown in Figure 6.13(c). The moments at the left end of the beam is -$\frac{w_oL^2}{2}$ again negative because of the moment sign convention. At B the moment is zero. Various values of the shear at each point on the beam indicate the slope of the moment diagram at the point. Notice how this variation produces the curves shown.

■

7. Bending Stress

Tree blown over by strong wind. A knowledge of where the maximum bending stress will occurs give us an idea of why trees when blown over come off by the root and don't snap at the trunk.

7.1 Introduction

In the previous chapter we saw how loads acting on beams create internal bending moments and shear forces (otherwise known as stress resultants) within a beam. In this section the stresses and strains associated with those stress resultants are investigated. Knowledge of these stresses and strains allow a variety of loading conditions and support types to be analysed.

In this section only beams with a **longitudinal plane of symmetry** will be considered. The loading, support conditions, and member cross section are symmetric with respect to the longitudinal plane.

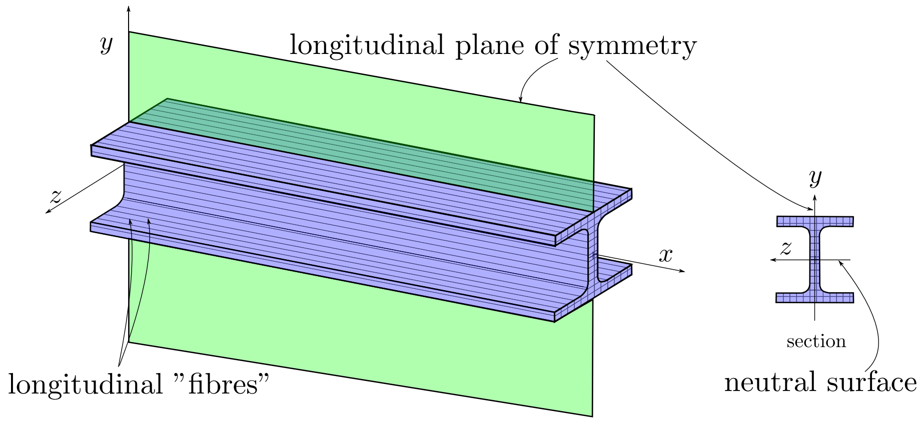


Figure 7.1: Longitudinal fibres shown with longitudinal plane of symmetry

The notion of longitudinal "fibres" is introduced which run parallel to the longitudinal axis of the beam. The usual coordinate axis scheme is the y-axis vertically upwards and the x-axis running along the longitudinal axis.

7.2 Flexural Strains

Consider a beam in **pure bending** where there is no transverse shear force, $\mathscr{V} = \frac{d\mathscr{M}}{dx} = 0$. On the top surface the longitudinal "fibres"are in compression and on the lower surface the longitudinal fibres are in tension. There is a surface which is not elongated or shortened and is call the **neutral surface**. The intersection of the neutral surface with any cross section of the beam is called the **neutral axis**.

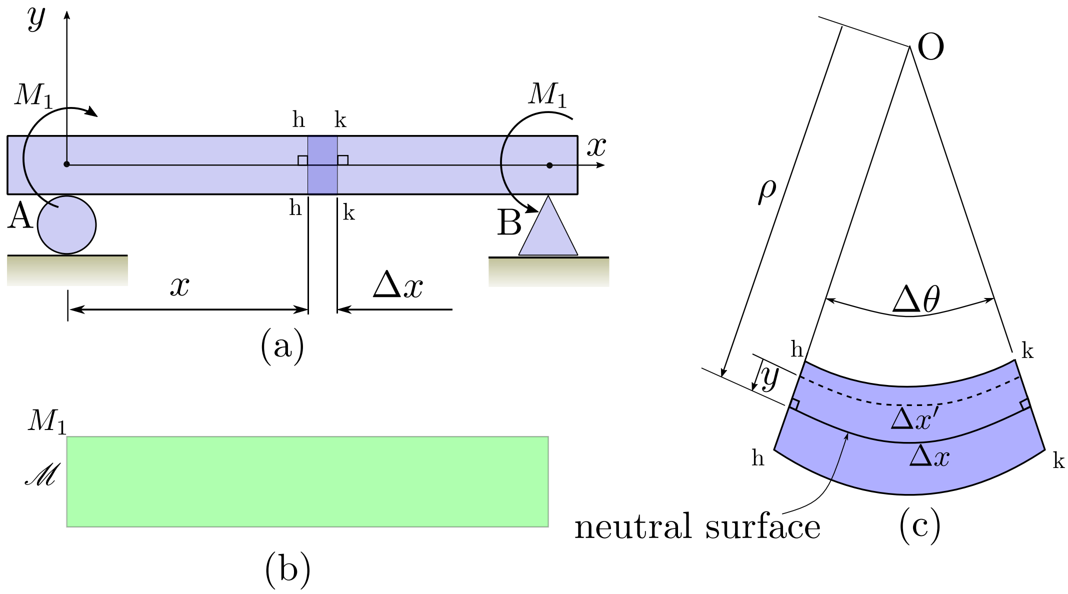


Figure 7.2: Beam in pure bending (a) with zero shear force and constant bending moment (b). The flexural deformation is shown in (c)

When subject to pure bending the beam deforms to the shape of a circular arc as can be seen in Figure 7.2(c). The centre of this arc is labelled O the **centre of curvature**. The radius of the arc is called the **radius of curvature** which is denoted with a Greek letter ρ (rho). Consider a longitudinal fibre a distance y above the neutral surface. It has length Δx before bending and $\Delta x'$ after bending. The strain can be related to shortening of Δx by the equation:

$$\varepsilon_x = \frac{\delta}{L} = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x}$$

If the interior angle is denoted $\Delta\theta$ then arc length Δx and $\Delta x'$ can be related to the radius of curvature ρ:

$$\varepsilon_x = \frac{\delta}{L} = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x} = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\,\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \Rightarrow$$

$$\boxed{\varepsilon_x = -\frac{y}{\rho}} \tag{7.1}$$

The normal strain is proportional to the distance from the neutral axis. The signs indicate for a negative compressive strain a positive value of y will result and for a positive tensile value of strain a negative value will be given.
Curvature is denoted with the Greek letter κ (kappa):

$$\boxed{\kappa = \frac{1}{\rho}} \tag{7.2}$$

Radius and curvature are both positive if the centre of curvature O is above the beam or in positive y-direction. Conversely they are both negative if the center is below the beam or has a negative y-coordinate

7.3 Normal Stresses

The normal stress in a beam can be determined by considering the stress-strain relationship or Hooke's Law($\sigma = E\varepsilon$). An important assumption before this is that the strains are elastic and the material does not yield.
The variation of stress in a beam is then substituting Hooke's Law $\sigma = \varepsilon E$ in into equation (7.1):

$$\boxed{\sigma_x = -\frac{E}{\rho} y = -E\kappa y} \tag{7.3}$$

The stress therefore varies linearly with the distance from the neutral axis. This can be seen in Figure 7.3 with zero normal bending stress at the neutral axis.

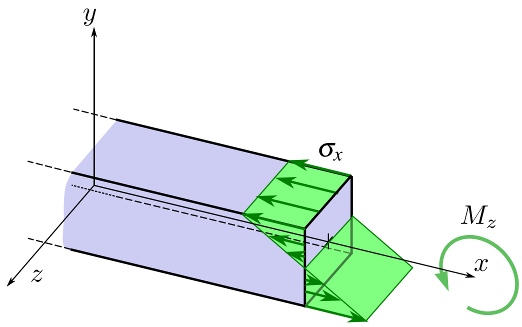


Figure 7.3: Normal stress distribution in a beam. The arrows pointing back at the beam indicate compression whereas the arrows pointing away from the beam indicate tension.

If a beam is subject to pure bending with no resultant force in the x-direction then the internal resultant moment acting about the z-axis can be related to the normal stresses in the x-direction and can be used to calculate the position of the neutral axis.

7.3.1 Neutral Axis Location

The neutral axis is parallel to the longitudinal axis of a beam or the x-axis in figure 7.3. There is no strain due to bending on the neutral axis.

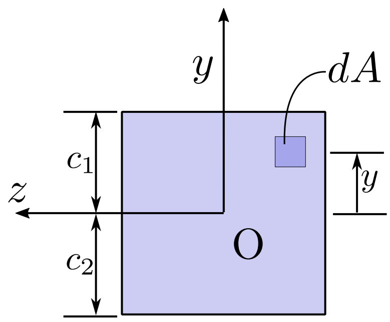


Figure 7.4: Normal stress distribution in a beam. The arrows pointing back at the beam indicate compression whereas the arrows pointing away from the beam indicate tension.

Consider a small element on a homogeneous beam which has area dA and is distance y from the the neutral surface in figure 7.4. The resultant force acting on this element is dF given by $\sigma_x dA$. In to order satisfy static equilibrium for the beam the sum of all these resultants should equal zero:

$$\sum F_x = \int dF = \int_A \sigma_x dA = 0$$

substitute equation (7.3) gives:

$$\sum F_x = \int_A \sigma_x dA = \int_A -\frac{E}{\rho} y dA = -\frac{E}{\rho}\int_A y dA = 0$$

For a solid material the elastic modulus E does not equal zero and if the radius of curvature was infinity the beam would not bend at all which implies:

$$\boxed{\int_A y dA = 0} \tag{7.4}$$

This means the **first moment of area** with respect to the z-axis must be zero. Recall from your course in statics that the distance of the centroid with respect to the x-axis includes the first moment of area:

$$\boxed{\bar{y} = \frac{\int_A y dA}{\int_A dA}} \tag{7.5}$$

Substitution of Equation (7.5) into Equation (7.4) shows that the distance from the neutral surface to the centroid must be zero. In other words the **neutral axis must pass through the centroid**.

7.3.2 Moment-Curvature Relationship

The second equation of equilibrium to be satisfied is the moment equilibrium. The internal bending moment M equals the moment resultant of the bending stresses σ_x acting over the cross section. The element of force $\sigma_x dA$ is positive when σ_x is positive and acts in the positive direction of the axis. Similarly $\sigma_x dA$ is negative when it acts in the negative $x-$direction and places the section in compression.

$$\sum M_z = -\int_A y\sigma_x dA - \mathscr{M} = 0$$

Subsitute equation (7.3) for σ_x gives

$$\mathscr{M} = -\int_A y\sigma_x dA = -\frac{E}{\rho}\int_A y^2 dA \tag{7.6}$$

The integral term in this equation is called the **second moment of area**

$$\boxed{I_z = \int_A y^2 dA} \tag{7.7}$$

The subscript z indicates the area moment of area is taken with respect to the z- axis. The integral term can be replaced by the moment of area in equation (7.6):

$$\boxed{\kappa = \frac{1}{\rho} = \frac{\mathscr{M}}{EI_z}} \tag{7.8}$$

This is called the moment-curvature relationship and show that the beam curvature is directly related to the bending moment and inversely related to the quantity EI_z which is called the **flexural rigidity**. The flexural rigidity is a measure of the bending resistance.

7.3.3 Flexure Formula

The relationship between the normal stress σ_x and the curvature from equation (7.3) can be substituted into the moment curvature relationship to give:

$$\boxed{\sigma_x = -\frac{\mathscr{M} y}{I_z}} \tag{7.9}$$

This is known as the **elastic flexure formula** or the **flexure formula**. The normal stresses produced by bending are called the **bending stresses** or **flexural stresses**.

7.4 Bending of Composite Beams

Beams that are composed two or more materials are called **composite beams**. Examples include wooden beams with steel reinforcing plates and concrete reinforced with steel bars. Using composite beams allows the stronger material to be used more efficiently to support loads.
The flexure formula, equation (7.9) was for homogeneous beams of one material. Modifications are required to make it applicable for composite beams. The composite beam will be changed into an equivalent cross section that consists of a single material. The dimensions of the beam with an equivalent cross section can then be used in the flexure formula to calculate the bending stresses.
The cross sections still remain plane during bending and the strains vary linearly through the beam cross section as expressed in equation (7.1)

$$\boxed{\varepsilon_x = -\frac{y}{\rho}} \tag{7.10}$$

Consider a beam section of two materials where material (2) is stiffer than material (1). In other words $E_2 > E_1$. The force transmitted by an area element dA is given by:

$$dF = \sigma_x dA = (E_2 \varepsilon_x)\, dydz$$

Here the stresses are transformed to strains with Hooke's Law. Now consider a transformed section which replaces Material (2) with Material (1). The distribution of strain should be the same in these cross sections so the y-dimension must be the same. The width in the z-dimension can be changed. Since Material (2) is stiffer, more of Material (1) will be required to replace Material (2).
Let the equivalent amount of Material (1) have area dA' with height dy and width ndz.

$$dF' = \sigma_x dA' = (E_1 \varepsilon_x)\, dy\, ndz$$

Since these two sections have to transmit the same amount of forces $dF = dF'$

$$(E_2 \varepsilon_x)\, dydz = (E_1 \varepsilon_x)\, dy\, ndz$$

It follows that:

$$\boxed{n = \frac{E_2}{E_1}} \tag{7.11}$$

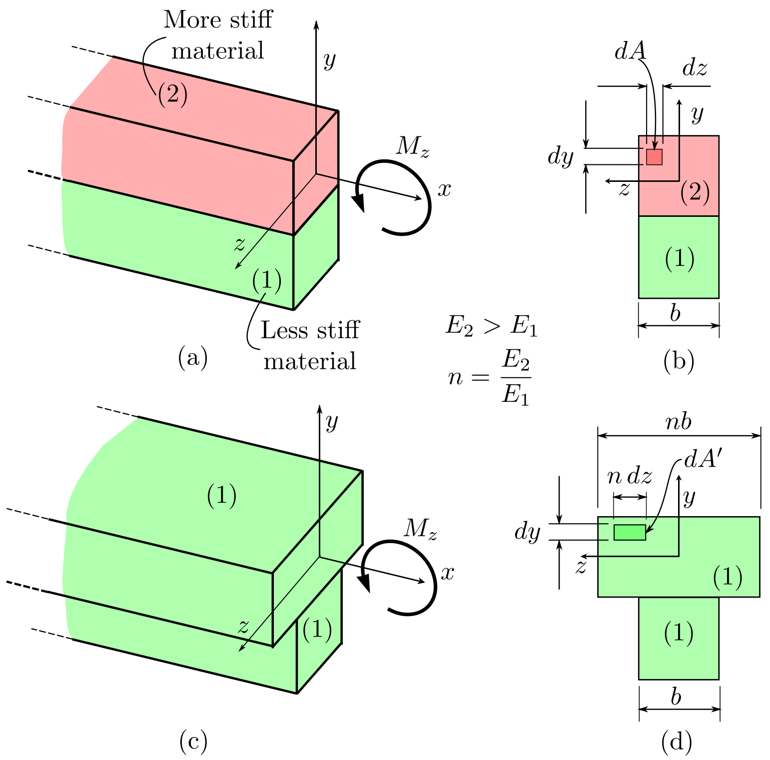


Figure 7.5: Illustration of bending stresses in composite beams (c) and (d) show how the more stiff Material (2) is transformed into the less stiff Material (1) keeping the strain distribution the same.

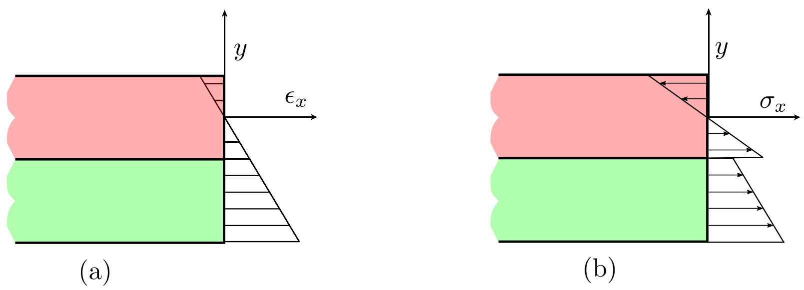


Figure 7.6: Distribution of strain (a) and stress (b) in a composite beam.

The ration n is called the **modular ratio**. The new material (1) is called the **transformed section**. The normal stresses can be expressed in terms of the radius of curvature ρ as:

$$\boxed{\sigma_{x1} = -\frac{E_1}{\rho}y \quad ; \quad \sigma_{x2} = -\frac{E_2}{\rho}y} \tag{7.12}$$

Here σ_{x1} and σ_{x2} are the stresses in Material (1) and Material (2) respectively.
Now lets consider the force equilibrium on the composite beam.

$$\sum F_x = \int_A \sigma_x dA = \int_{A_1} \sigma_{x1} dA + \int_{A_2} \sigma_{x2} dA = 0$$

Substitute equation (7.12) to give:

$$-\int_{A_1} \frac{E_1}{\rho} y dA - \int_{A_2} \frac{E_2}{\rho} y dA = 0$$

The curvature is the same and can be cancelled out. The modular ratio can be substituted from equation (7.11) to give:

$$E_1 \int_{A_1} y dA + E_1 \int_{A_2} yn\, dA = 0$$

The area of a transformed cross section (A_t) can be expressed as:

$$\int_{A_t} dA = \int_{A_1} dA + \int_{A_2} n\, dA = 0$$

Combining the previous two equations gives:

$$\boxed{\int_{A_t} y dA = 0} \tag{7.13}$$

This means the *neutral axis passes through the centroid of the transformed cross section* not the centroid of the original section.
The moment relationship for the beam of two materials is:

$$\mathscr{M} = -\int_A y\sigma_x dA = -\int_{A_1} y\sigma_x dA - \int_{A_2} y\sigma_x dA$$

Substitute equation (7.11) and equation (7.12) gives

$$\mathscr{M} = \frac{E_1}{\rho}\left(\int_{A_1} y^2 dA + \int_{A_2} ny^2 dA\right)$$

The moment of area of a transformed section I_t is defined as:

$$\boxed{I_t = \int_{A_t} y^2 dA_t = \int_{A_1} y^2 dA + \int_{A_2} ny^2 dA} \tag{7.14}$$

The moment curvature relationship is then:

$$\boxed{\mathscr{M} = \frac{E_1 I_t}{\rho}} \tag{7.15}$$

The stresses in Material (1) can be expressed as:

$$\boxed{\sigma_{x1} = -\frac{\mathscr{M} y}{I_t}} \tag{7.16}$$

and the stresses in Material (2) bearing in mind that the section in transformed

$$\boxed{\sigma_{x2} = -n\frac{\mathscr{M} y}{I_t}} \tag{7.17}$$

■ **Example 7.1** A beam has maximum moment, $\mathscr{M}_{\max} = 60\,\text{kN m}$. The cross section of the beam is a hollow box with wood flanges and steel side plates, as shown in figure 7.7. The wood flanges are 75 mm by 100 mm in cross section, and the steel plates are 300 mm deep. What is the required thickness t of the steel plates if the allowable stresses are 120 MPa for the steel and 6.5 MPa for the wood? (Assume that the moduli of elasticity for the steel and wood are 210 GPa and 10 GPa, respectively, and disregard the weight of the beam.)

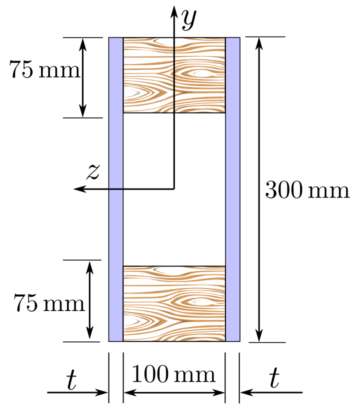


Figure 7.7: Composite beam section made of steel and wood.

Solution
Units mm, N, MPa
Maximum Moment = $\mathscr{M}_{max} = 60\,\text{kN m} = 60 \times 10^6\,\text{N mm}$

The wood moment of area:
$$\text{I}_{wood} = \frac{100 \times 300^3}{12} - \frac{100 \times 150^3}{12} = 196.8 \times 10^6\,\text{mm}^4$$

The required moment of area of the whole section if the wood was transformed to steel and is limited by the strength of the steel:
$$\text{I}_{eqsteel} = \frac{\mathscr{M}_{max} y_{steel}}{\sigma_{steel}} = \frac{60 \times 10^6\,(300/2)}{120} = 75.0 \times 10^6\,\text{mm}^4$$

The required moment of area of the whole section if steel was transformed to wood and is limited by the strength of the wood:
$$\text{I}_{eqwood} = \frac{\mathscr{M}_{max} y_{wood}}{\sigma_{wood}} = \frac{60 \times 10^6\,(300/2)}{6.5} = 1.384 \times 10^9\,\text{mm}^4$$

The required moment of area of the steel section if the wood was transformed to steel:
$$\text{I}_{steel-steel} = \text{I}_{eqsteel} - \frac{E_{wood}}{E_{steel}}\text{I}_{wood} = 65.625 \times 10^6\,\text{mm}^4$$

The required moment of area of the steel section if the steel was transformed to wood:
$$\text{I}_{steel-wood} = \left(I_{eqwood} - I_{wood}\right)\frac{E_{wood}}{E_{steel}} = 56.559 \times 10^6\,\text{mm}^4$$

Since the requirement of the moment of area of the steel is greater if the steel strength is used opposed to the wood strength, this is used to calculate the thickness.
$$\text{I}_{steel-steel} = \frac{2t \times 300^3}{12} \Rightarrow t = 14.583\,\text{mm}$$

■

7.5 Eccentric Loading

In a previous chapter loads acted through the centroid of the cross section and is called **centric loading**. This load creates a stress that is uniform over the cross section. If the load does not pass through the centroid of the section then there is an **eccentric axial load**. Additional bending stresses occur with the normal stresses.

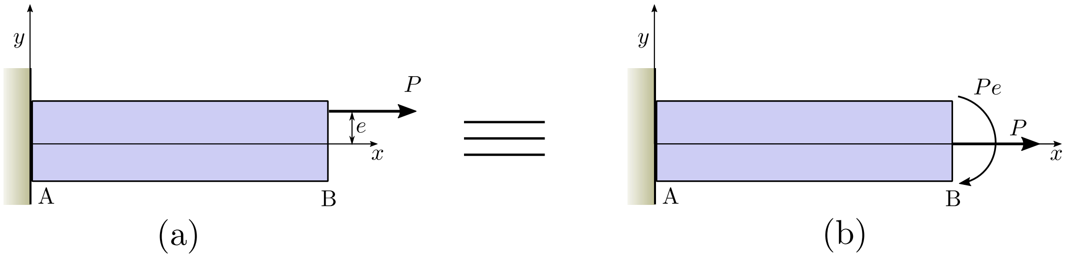


Figure 7.8: Cantilever beam with an eccentric load P and equivalent system

Consider an eccentric axial load P which acts a distance *e* (eccentricity of the load) away from the centroid as seen in figure 7.8. The eccentricity is measured in the positive y-direction.
The stresses are calculated by combining the normal stresses and the bending stresses:

$$\sigma_x = \frac{P}{A} - \frac{\mathscr{M}y}{I_z}$$

This system is statically equivalent to an axial force P through the centroid with an applied moment of magnitude Pe. Since the internal moment is the negative of the applied moment the expression is $-M = Pe$.

$$\boxed{\sigma_x = \frac{P}{A} + \frac{(Pe)\,y}{I_z}} \tag{7.18}$$

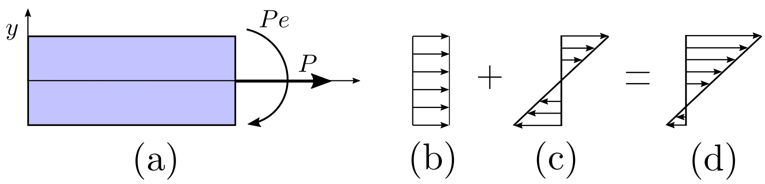


Figure 7.9: Stress distributions caused by an eccentric axial load.

In Figure 7.9 the applied moment and force is seen in (a). The uniform stress distribution is seen in (b) which is added to the stress from bending in (c) to give the complete distribution of stresses in (d).
The neutral axis will no longer be situated at the centre of the cross section. If the axial force is large enough relative to the bending stresses there will be no neutral axis on the structure. To obtain the location of the neutral axis set $\sigma_x = 0$ and solve for the distance from the centroid of the section.
4

8. Bending Deflection

An example of beam deflection seen here with the wing flex in a Boeing 787.

8.1 Introduction

In the previous chapter the stress in bending was calculated in order to design beams safely. In this chapter the deflection of beam structures will be analysed. This is also a consideration in design. Excessive deflection is seen in buildings with cracks in ceilings and walls, as well as visible sagging of floors. Some aircraft and other machine components can function incorrectly when large deflections cause unwanted vibration.

8.2 The Differential Equation of the Elastic Curve

When an initially straight beam is loaded and then behaves in an elastic manner the longitudinal centroidal axis of the beam becomes a curve this is called the **deflection** or **elastic curve** of the beam. The x- and y-axis are shown along with the deflection of the elastic curve v in Figure 8.1. The y-axis is positive upwards and indicates the coordinate within the beam cross section from the neutral axis. The deflection v is also positive upwards and is assumed quite small.

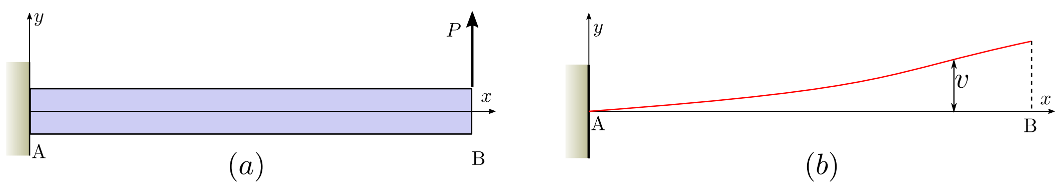


Figure 8.1: Cantilever beam with concentrated load in (a) and elastic curve shown in (b) with deflection.

In Figure 8.2 the point C on the elastic curve has a deflection v, a radius of curvature ρ and makes an angle θ with the x-axis. A point D is a small distance dx to the right of C and has deflection $v+dv$ and make an angle $\theta + d\theta$ with the horizontal. Since v is small, the angle θ is small and the length of the curve $ds \approx dx$. Angle θ is in radians so the arc length of the circle centred at O is:

$$\rho d\theta = ds \approx dx.$$

This is rearranged to give:

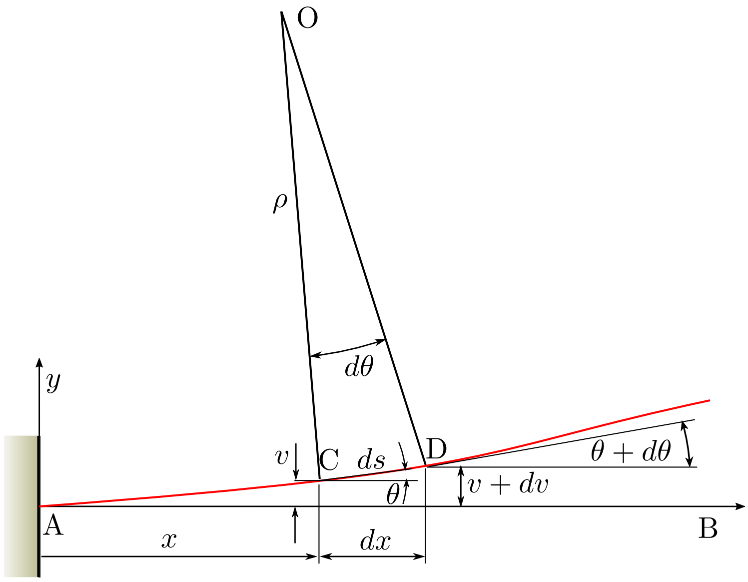


Figure 8.2: Elastic curve of a cantilever beam

$$\boxed{\frac{1}{\rho} = \frac{d\theta}{dx}} \tag{8.1}$$

The slope of the curve at C is $\frac{dv}{dx}$. This is related to the angle theta with:

$$\frac{dv}{dx} = \tan\theta \approx \theta$$

Taking the derivative of both sides, the following is obtained:

$$\boxed{\frac{d^2v}{dx^2} = \frac{d\theta}{dx}} \tag{8.2}$$

Combining equations (8.1), (8.2) and (7.8) the following relationship results:

$$\boxed{\frac{d^2v}{dx^2} = \frac{1}{\rho} = \frac{\mathscr{M}}{EI_z}} \tag{8.3}$$

This is usually written in the form:

$$\boxed{EI\frac{d^2v}{dx^2} = \mathscr{M}(x)} \tag{8.4}$$

The sign convention for internal moments and curvatures are the same and illustrated in Figure 8.3

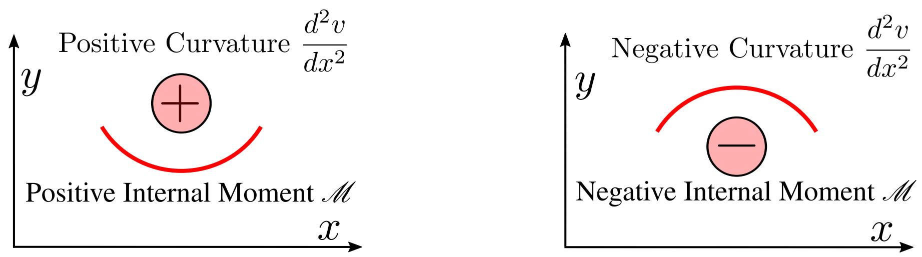


Figure 8.3: Sign convention of curvature and moment shown with elastic curve

8.3 Deflection and Slope by Double Integration

In order to calculate various physical quantities associated with beam bending, the following relationships are important:

For a deflection v the slope is $\frac{dv}{dx} = \theta$ of the elastic curve. The moment $\mathscr{M} = EI\frac{d^2v}{dx^2}$ from equation (8.4) can have further derivatives taken to give the relationship for shear $\mathscr{V}$:

$$\boxed{\mathscr{V} = EI\frac{d^3v}{dx^3}} \tag{8.5}$$

and furthermore the relationship for the distributed load, w.

$$\boxed{w = EI\frac{d^4v}{dx^4}} \tag{8.6}$$

The term EI is called the **flexural rigidity** of the beam and is assumed constant for equations (8.5) and (8.6) above. It is the resistance of a structure to bending.

To calculate the slope and deflection of beams as a function of distance x along the length: equations of the load and bending moment are first derived. These equations are repeatedly integrated to in a procedure called **double integration**. From the repeated integration, various constants are obtained which have to be evaluated. These are evaluated using the known conditions of the slope and deflection and can be grouped into three categories: boundary conditions, continuity conditions and symmetry conditions.

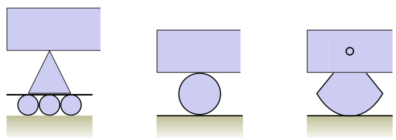
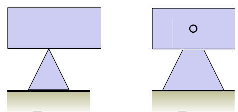
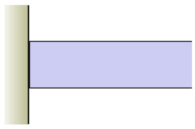
Boundary Conditions

When specific values of slope or deflections are known at points on a beam these are called **boundary conditions**. When the equation for M(x) is derived it is usually derived within specific regions of the beam eg $0 \leq x < a$ then the boundary conditions would refer to known values of slope or deflection at $x = 0$ and $x = a$. These boundaries for each region are not necessarily the bounds of the beam but rather the bounds of the region. A variety of boundary conditions are seen in Table 8.1. Each boundary condition solves one and only one constant of integration.

Continuity Conditions

Beams are often subject to discontinuous loading such as concentrated loads or uniform loads which can start and stop abruptly. It is then necessary to divide the beam up into different regions to account for the load discontinuities. For example if there is beam with a point load at $x = a$ then the shear along the beam cannot be represented with a single algebraic equation. Seperate equations will have to be derived to the left and right of the load. The beam is physically continuous so the equations to the left and the right should give the same deflections and slopes where they meet.

Table 8.1: Boundary Conditions for Beams

Support or connection	Boundary Condition
 Roller Roller Rocker	$v = 0$ $\mathscr{M} = 0$
 Pin Pin	$v = 0$ $\mathscr{M} = 0$
 Clamped, fixed or rigid	$v = 0$ $\dfrac{dv}{dx} = 0$
 Free End	$\mathscr{V} = 0$ $\mathscr{M} = 0$

Symmetry Conditions

The beam supports and the loading may be symmetric for a beam. These symmetry conditions will allow the value of beam slope to be determined at certain locations. For example the beam slope is zero at its midspan for a symmetric simply supported beam with a uniformly distributed load between its supports.

Procedure for Double Integration Method

The following procedure to solve for deflection and slope is recommended from Philpot [4].

1. **Sketch** supports, loads along with the axis. Sketch approximate shape of elastic curve especially the deflections/slopes at supports.
2. Determine the **support reactions**. This can be done using static equilibrium.
3. Consider **segment equilibrium**. For each segment draw a free body diagram showing all loads
4. **Integrate** the differential equation $EI\dfrac{d^2v}{dx^2}$ twice. The first time to determine the slope equation and then second time the deflection equation with two constants of integration.
5. List all **boundary, continuity and symmetry conditions.**
6. **Evaluate constants** using all the boundary conditions found above.
7. Determine the **elastic curve and slope equations** by substituting the constants found.
8. Determine the **deflections and slopes at points** with equations at specific points if required.

The procedure defined will be illustrated in the following example.

■ **Example 8.1** A cantilever beam is subject to a concentrated vertical load P at its free end A as shown in figure 8.4(a). Determine the elastic curve and the deflection and slope at the free end.

Assume EI is constant.

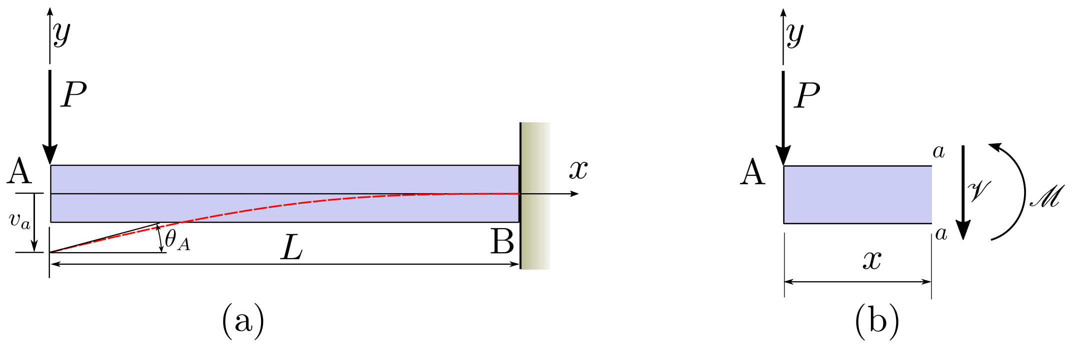


Figure 8.4: Clamped beam with concentrated load

Equilibrium
Take a section through the beam a distance x from the free end and draw a free body diagram shown in Figure 8.4(b). The equilibrium equation for the sum of moments through section a-a is:

$$\sum M_{a-a} = Px + \mathscr{M} = 0$$

The bending moment equation is then: $\mathscr{M} = -Px$. This is true for $0 \le x \le L$ in this beam. Substitute into equation(8.4) which gives:

$$EI\frac{d^2v}{dx^2} = -Px \tag{8.7}$$

Integration
Integrating equation (8.7) once gives the slope equation:

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \tag{8.8}$$

C_1 is a constant of integration and integrating equation (8.8) a second time gives:

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \tag{8.9}$$

Another constant of integration C_2 results which is evaluated using the boundary conditions.

Boundary Conditions
The deflection and slope is known at the right end. Since it is clamped $v = 0$ and $\frac{dv}{dx} = 0$ at $x = l$. At the free end neither the slope nor the deflection is known.

Evaluate Constants
Substitute $\frac{dv}{dx} = 0$ at $y = l$ into equation (8.8) to give:

$$C_1 = \frac{PL^2}{2} \tag{8.10}$$

Then substitute $v = 0$ at $y = l$ into equation (8.9) to give:

$$C_2 = -\frac{PL^3}{3} \tag{8.11}$$

Elastic Curve and Slope Equations

Substitute equation (8.10) and (8.11) into equation (8.9). The elastic curve equation is:

$$EIv = -\frac{Px^3}{6} + \frac{PL^2}{2}x - \frac{PL^2}{3} \tag{8.12}$$

The slope equation is then:

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + \frac{PL^2}{2} \tag{8.13}$$

Deflections and Slopes at Specified Points

At x = 0: $v_A = -\dfrac{PL^3}{3EI}$ and $\theta_A = \dfrac{dv}{dx} = \dfrac{PL^2}{2EI}$

■

8.4 Discontinuity Functions

Double integration to obtain the deflection of the differential equation can be a tedious, lengthy procedure. Discontinuity functions or Macaulay functions can be used to obtain concise expressions for load, shear and moments for beams and greatly simplifies the determination of deflection in multi-interval deflection problems. Consider the following beam loaded with various concentrated loads.

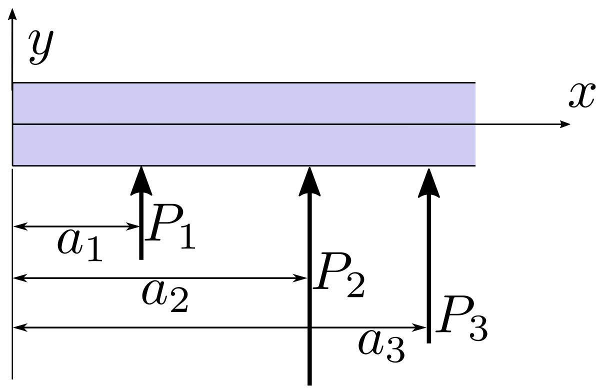


Figure 8.5: Loading of a beam section

The loading can be represented with the bending moment expression:

$$\boxed{\mathscr{M} = P_1\langle x-a_1\rangle + P_2\langle x-a_2\rangle + P_3\langle x-a_3\rangle + \dots} \tag{8.14}$$

where the quantities $P_i\langle x-a_i\rangle$ represent the bending moments due to point loads, and the quantity $\langle x-a_i\rangle$ is a **Macaulay bracket** defined as:

$$\boxed{\langle x-a_i\rangle = \begin{cases} 0 & \text{if } x < a_i \\ x-a_i & \text{if } x \geq a_i \end{cases}} \tag{8.15}$$

Ordinarily, when integrating $P(x-a)$ we get:

$$\int P(x-a)\ dx = P\left[\frac{x^2}{2} - ax\right] + C$$

However, when integrating a Macaulay bracket, we get:

$$\boxed{\int P\langle x-a\rangle\ dx = P\frac{\langle x-a\rangle^2}{2} + C_m} \tag{8.16}$$

The Macaulay term is treated as single term of integration and C_m represents a constant of integration. This requires a more general definition of Macaulay brackets:

$$\boxed{\langle x-a_i\rangle^n = \begin{cases} 0 & \text{if } x < a_i \\ (x-a_i)^n & \text{if } x \geq a_i \end{cases} \quad \text{for n } = 0, 1, 2...} \tag{8.17}$$

The derivative of the bending moment equation (8.14) gives the shear expression:

$$\boxed{\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}x} = \mathscr{V} = P_1\langle x-a_1\rangle^0 + P_2\langle x-a_2\rangle^0 + P_3\langle x-a_3\rangle^0 + \ldots} \tag{8.18}$$

The angle bracket is interpreted with help of equation (8.17) and noting that that $(x-a_i)^0 = 1$ Integration of the bending moment equation (8.14) gives an expression for the slope:

$$\boxed{EI\frac{\mathrm{d}v}{\mathrm{d}x} = \int \mathscr{M}\,dx = \frac{P_1\langle x-a_1\rangle^2}{2} + \frac{P_2\langle x-a_2\rangle^2}{2} + \frac{P_3\langle x-a_3\rangle^2}{2} + C_1 + \ldots} \tag{8.19}$$

Here C_1 is a constant from the Macaulay intergration. Further integration gives the deflection

$$\boxed{EIv = \frac{P_1\langle x-a_1\rangle^3}{6} + \frac{P_2\langle x-a_2\rangle^3}{6} + \frac{P_3\langle x-a_3\rangle^3}{6} + C_1x + C_2\ldots} \tag{8.20}$$

Consider a beam with moments applied as follows:

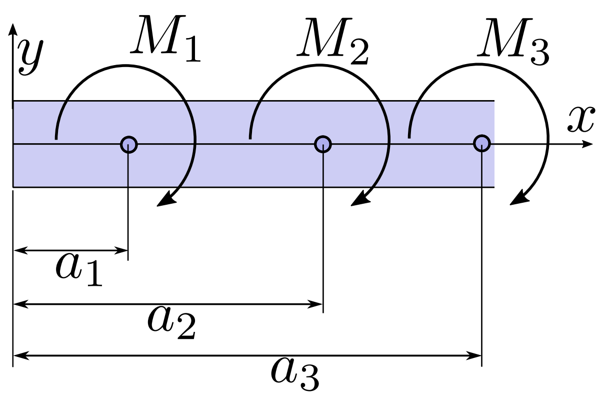


Figure 8.6: Moments applied to a beam section

The bending moment expression is as follows:

$$\boxed{\mathscr{M} = M_1\langle x-a_1\rangle^0 + M_2\langle x-a_2\rangle^0 + M_3\langle x-a_3\rangle^0 + \ldots} \tag{8.21}$$

In a similar way to equation (8.19) and (8.20) we get slope and deflection equations.

Now consider a distributed load as follows:

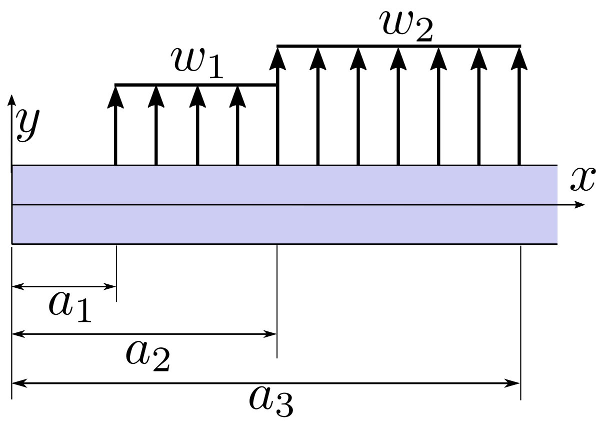


Figure 8.7: Uniformly distributed load applied to a beam section

The shear expression is:

$$\boxed{\mathscr{V} = w_1\langle x-a_1\rangle + (w_2-w_1)\langle x-a_2\rangle - w_2\langle x-a_3\rangle + \ldots} \tag{8.22}$$

In a similar fashion to previously we can repeatedly integrate to first get the bending moment, then slope and deflection expressions.

■ **Example 8.2** A simply supported beam has loading as follows:

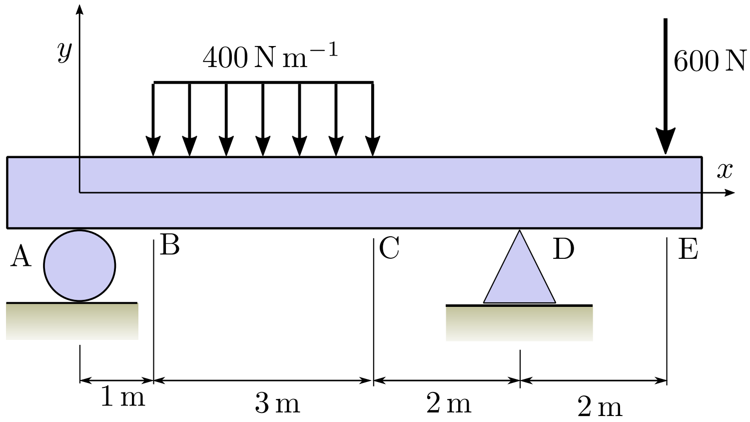


Figure 8.8: Concentrated and uniformly distributed load on 8 m beam

Support Reactions

$\sum M_A = -400\times 3\times 2.5 + D_y\times 6 - 600\times 8 = 0 \qquad \Rightarrow D_y = 1300\ \text{N}$

$\sum F_y = A_y - 400\times 3 + D_y - 600 = 0 \qquad \Rightarrow A_y = 500\ \text{N}$

Moment function

$$\mathscr{M}(x) = 500\langle x\rangle^1 - \frac{400\langle x-1\rangle^2}{2} + \frac{400\langle x-4\rangle^2}{2} + 1300\langle x-6\rangle^1$$

Slope function

$$EI\frac{\mathrm{dv}}{\mathrm{d}x} = \frac{500\langle x\rangle^2}{2} - \frac{400\langle x-1\rangle^3}{6} + \frac{400\langle x-4\rangle^3}{6} + \frac{1300\langle x-6\rangle^2}{2} + C_1$$

Deflection function

$$EIv = \frac{500\langle x\rangle^3}{6} - \frac{400\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6} + C_1x + C_2$$

Boundary conditions

$v = 0$ at $x = 0$ and $v = 0$ at $x = 6\ \text{m}$

Evaluate Constants

$v = 0$ at $x = 0 \Rightarrow C_2 = 0$ and $v = 0$ at $x = 6 \Rightarrow$

$$EIv = \frac{500\langle 6\rangle^3}{6} - \frac{400\langle 6-1\rangle^4}{24} + \frac{400\langle 6-4\rangle^4}{24} + C_1 6 = 0$$

$$\Rightarrow C_1 = -\frac{3925}{3} = -1308\ \text{N.m}^2$$

Deflection function

$$EIv = \frac{500\langle x\rangle^3}{6} - \frac{400\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6} - 1308x$$

Deflection at x = 3

$$v = \frac{1}{EI}\left(\frac{500\langle 3\rangle^3}{6} - \frac{400\langle 3-1\rangle^4}{24} - 1308(3)\right) = \frac{-1942}{EI}$$

Deflection at x = 8

$$v = \frac{1}{EI}\left(\frac{500\langle 8\rangle^3}{6} - \frac{400\langle 8-1\rangle^4}{24} - 1308x + \frac{400\langle 8-4\rangle^4}{24} + \frac{1300\langle 8-6\rangle^3}{6} - 1308(8)\right)$$

$$\Rightarrow v = \frac{-1817}{EI}$$

■

■ **Example 8.3** An initially straight and horizontal cantilever of uniform section and length L is rigidly built-in at one end and carries a uniformly distributed load of intensity w for a distance $L/2$ measured from the built-in end. The second moment of area is I and the modulus of elasticity E. Determine, in terms of w, L, E and I:

1. An expression for the slope of the cantilever at the end of the load.
2. The deflection at the free end.
3. The force in a vertical prop which is to be applied at the free end in order to restore this end to the same horizontal level as the built-in end.

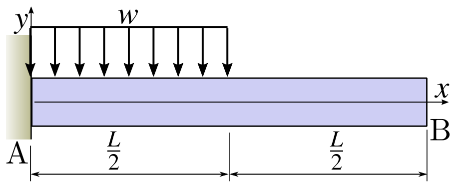


Figure 8.9: Clamped beam with uniform load over half the length

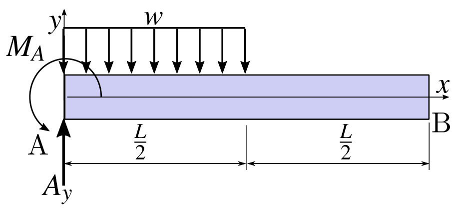


Figure 8.10: Free body diagram of beam

Moments $\sum M_A = M_A = -\dfrac{wL^2}{8}$

Resolve $\sum F_y = 0 \Rightarrow A_y = \dfrac{wL}{2}$

<u>Moment function</u> $M(x) = -\dfrac{wL^2\langle x\rangle^0}{8} + \dfrac{wL\langle x\rangle^1}{2} - \dfrac{w\langle x\rangle^2}{2} + \dfrac{w\langle x-\frac{L}{2}\rangle^2}{2}$

<u>Slope function</u> $EI\dfrac{\mathrm{d}v}{\mathrm{d}x} = -\dfrac{wL^2\langle x\rangle^1}{8} + \dfrac{wL\langle x\rangle^2}{4} - \dfrac{w\langle x\rangle^3}{6} + \dfrac{w\langle x-\frac{L}{2}\rangle^3}{6} + C_1$

<u>Deflection function</u> $EIv = -\dfrac{wL^2\langle x\rangle^2}{16} + \dfrac{wL\langle x\rangle^3}{12} - \dfrac{w\langle x\rangle^4}{24} + \dfrac{w\langle x-\frac{L}{2}\rangle^4}{24} + C_1x + C_2$

<u>Boundary conditions</u> $v = 0$ at $x = 0$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$ m

<u>Evaluate Constants</u> $v = 0$ at $x = 0 \Rightarrow C_2 = 0$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ at $x = 0$ m $\Rightarrow C_1 = 0$

1. <u>Slope at x = L/2</u>
$$\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1}{EI}\left(-\frac{wL^2\langle x\rangle^1}{8} + \frac{wL\langle x\rangle^2}{4} - \frac{w\langle x\rangle^3}{6} + \frac{w\langle x-\frac{L}{2}\rangle^3}{6}\right) = -\frac{wL^3}{48EI}$$
2. <u>Deflection at x = L</u>
$$v = \frac{1}{EI}\left(-\frac{wL^2\langle x\rangle^2}{16} + \frac{wL\langle x\rangle^3}{12} - \frac{w\langle x\rangle^4}{24} + \frac{w\langle x-\frac{L}{2}\rangle^4}{24}\right) = \frac{-7wl^4}{348EI}$$
3. See example 8.6
Using superposition $P = \dfrac{7wL}{128}$ upwards.

■

8.5 Moment Area Method

In this section a semigraphical technique is used to find deflections and angles of beams and is called the **moment area method**. This technique uses the relationship between the area of the bending moment diagram and the beam deflections and -angles. It may be quicker where only properties of one point are needed. This method is also more effective in dealing with beams where there is a change in cross section.

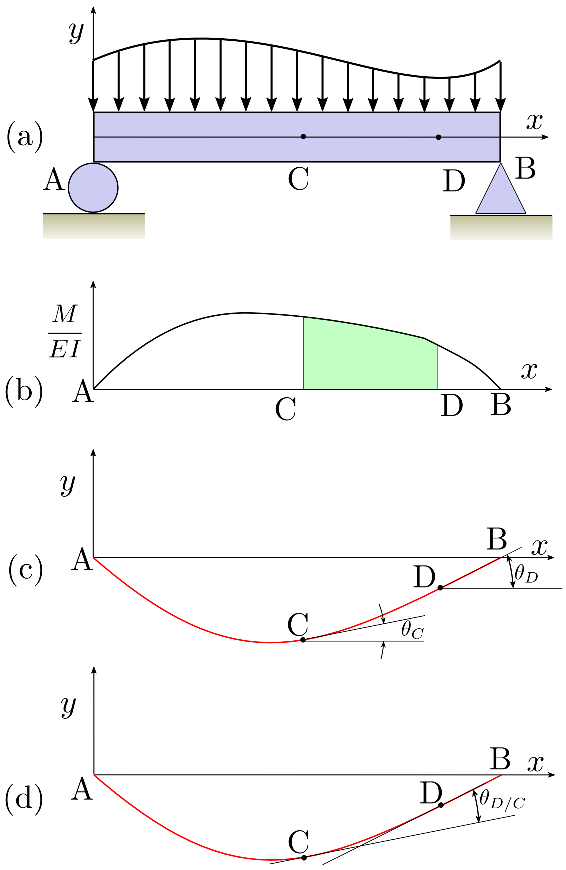


Figure 8.11: Finding the slope with the moment area method: (a) is the load diagram; (b) is the $\mathscr{M}/EI$ diagram; (c) elastic curve

Combine equations (8.2) and (8.3) to give the relationship between the change in angle and the bending moment:

$$\boxed{d\theta = \frac{\mathscr{M}}{EI}dx} \tag{8.23}$$

In figure 8.11(a) we see two points C and D on a elastic beam with a distributed load.

The difference in slope between and two points C and D can be expressed as:

$$\boxed{\theta_{D/C} = \theta_D - \theta_C = \int_C^D \frac{\mathscr{M}}{EI}dx} \tag{8.24}$$

where θ_C and θ_D are the slopes at C and D respectively shown in figure 8.11(c). The area under the $\frac{\mathscr{M}}{EI}$ diagram is shown in figure 8.11(b). The change in slope $\theta_{D/C}$ is shown in figure 8.11(c).

The angle $\theta_{D/C}$ and the area under the $\frac{\mathscr{M}}{EI}$ have the same sign. If the area under the $\frac{\mathscr{M}}{EI}$ diagram is positive then $\theta_{D/C}$ will be anti-clockwise.

This is called the **first moment area theorem**.

$$\boxed{\theta_{D/C} = \textbf{area of } \frac{\mathscr{M}}{EI} \textbf{ diagram between C and D}} \tag{8.25}$$

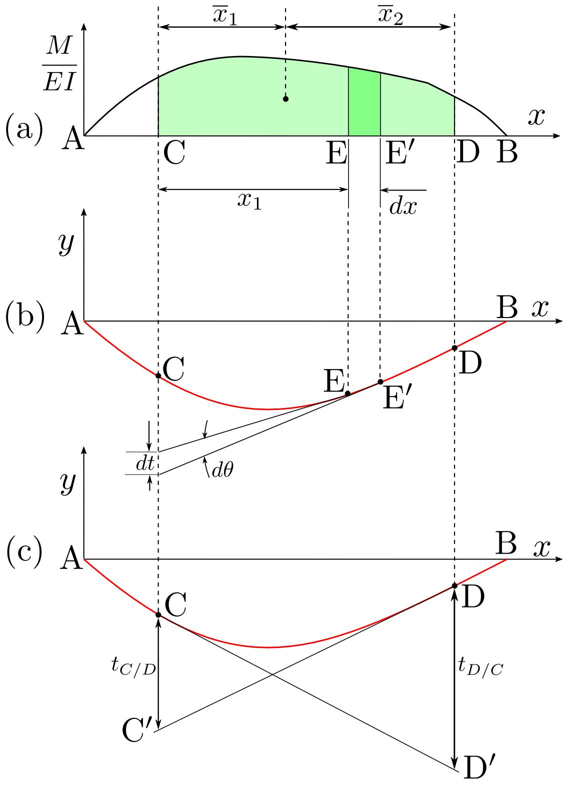


Figure 8.12: Finding deflection with the moment area method: (a) is the M/EI diagram; (b) and (c) are elastic curves

Consider two points E and E′ a distance dx apart between C and D see figure 8.12. Since the slope θ at E and the angle $d\theta$ formed by the tangents at E and E′ are small, we can approximate:

$$dt = x_1 d\theta = x_1 \frac{\mathscr{M}}{EI} dx$$

Integrate from C to D to get the vertical distance from C to the tangent at D which is at C′. This represents the tangential deviation of C with respect to D and is called $t_{C/D}$ which is:

$$\boxed{t_{C/D} = \int_D^C x_1 \frac{\mathscr{M}}{EI} dx} \tag{8.26}$$

The right hand side of equation (8.26) represent the first moment with respect to the area under the located under the $\frac{\mathscr{M}}{EI}$ diagram between C and D. Using equation (7.5), $\bar{y} = \frac{\int_A y dA}{\int_A dA}$ gives the relationship between the first moment and the centroid.

The right hand side represents the product of the area with the centroid. Therefore equation (8.26) can also be expressed as:

$$\boxed{t_{C/D} = \textbf{(area between C and D)}\ \bar{x}_1} \tag{8.27}$$

This is the **second moment area theorem**. In a similar fashion the tangential deviation of C with respect to D is:

$$\boxed{t_{D/C} = \textbf{(area between C and D)}\ \bar{x}_2} \tag{8.28}$$

Note finally that a point with positive tangential deviation is located above the tangent drawn from the other point as seen with figure 8.12(c)

■ **Example 8.4** Consider the following example with a concentrated load with a cantilever beam. Determine the deflection v_B and angle of rotation θ_B of the free end B of a cantilever beam AB with a concentrated load P at B. The beam has length L and constant flexural rigidity EI.

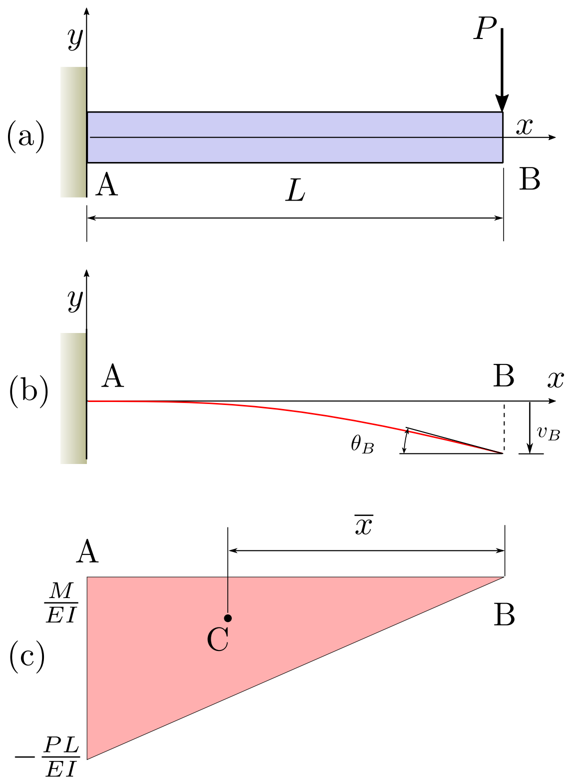


Figure 8.13: Cantilever beam with a concentrated load: (a) is the load diagram; (b) elastic curve; (c)is the $\frac{\mathscr{M}}{EI}$ diagram

Slope at B By inspection the angle of rotation is clockwise and the deflection is downwards.

The bending moment diagram is triangular with the internal moment at the support equal to $-PL$. The flexural rigidity is constant so the bending moment diagram has the same shape as the $\frac{\mathscr{M}}{EI}$ diagram as seen in figure 8.13(c).

The area of the $\frac{\mathscr{M}}{EI}$ diagram between the points A and B represents the angle $\theta_{A/B}$ between the tangents at those points. This area which we will call A_1 is evaluated as follows:

$$A_1 = -\frac{1}{2}(L)\left(\frac{PL}{2EI}\right) = -\frac{PL^2}{EI}$$

This is a negative area since it is below the x-axis. The relative rotation between points A and B from the first moment-area theorem is:

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = -\frac{PL^2}{2EI}$$

The tangent to the curve at A is horizontal $\theta_A = 0$ we get:

$$\theta_B = -\frac{PL^2}{2EI}$$

This is a negative quantity so the angle is clockwise.

Deflection at B
The deflection v_B can be obtained from the second moment area theorem. Noting that the tangent to A is a horizontal along the x-axis.

$$v_B = t_{B/A} = A_1\bar{x} = -\frac{PL^2}{2EI}\left(\frac{2L}{3}\right) = -\frac{PL^3}{3EI}$$

This is a negative quantity so the deflection of B is downwards. ■

■ **Example 8.5** Consider example 8.3 and repeat using the moment area method. Diagram repeated below:

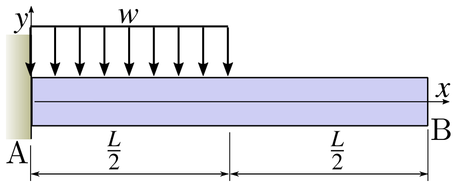
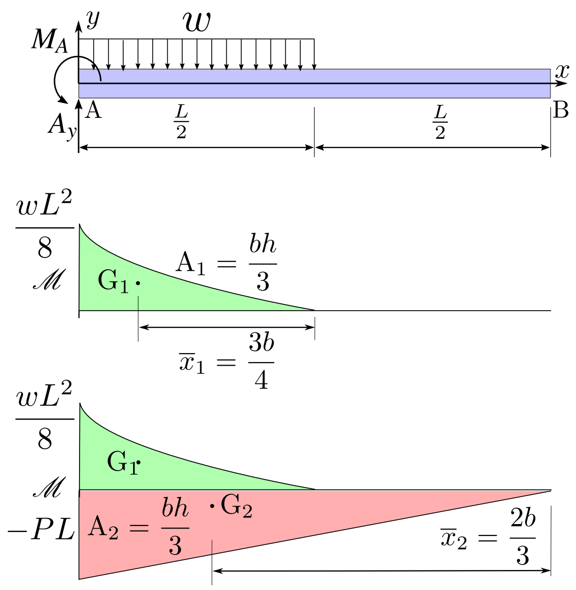


Figure 8.14: Clamped beam with uniform load over half the length



Moment at A = $M_A = -\frac{wL^2}{8}$

1. Since beam is clamped slope at A is zero, therefore:
   slope at x = $\frac{L}{2}$ is area under $\frac{M}{EI}$ diagram
   $= \frac{bh}{3EI} = \frac{wL^2}{8} \times \frac{L}{2} \times \frac{1}{3EI} = \frac{wL^3}{48EI}$
2. Since beam is clamped deflection at A is zero therefore:
   deflection at x = L is area under $\frac{M}{EI}$ diagram $\times$ centroid distance from B
   $= \frac{wL^3}{48EI} \times \left(\frac{3}{4}\frac{L}{2} + \frac{L}{2}\right) = \frac{7wL^4}{384EI}$
3. For the deflection to be zero, the product of the area and centroid distance for the concentrated and distributed loads is the same from B:
   $= \frac{PL^2}{2EI} \times \frac{2}{3}L = \frac{PL^3}{3EI} = \frac{7wL^4}{384EI}$
   $\Rightarrow P = \frac{7wL}{128}$ upwards

■

8.6 Principle of Superposition

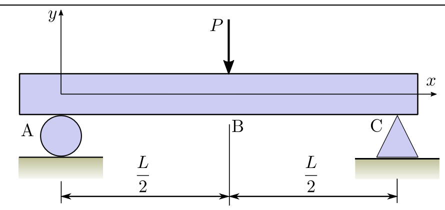
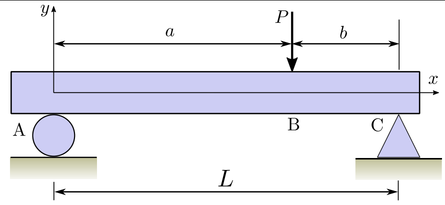
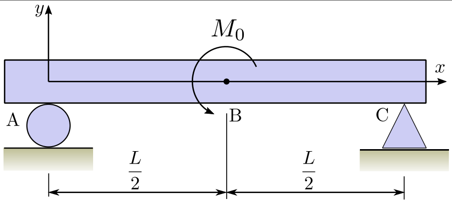
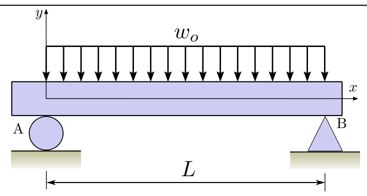
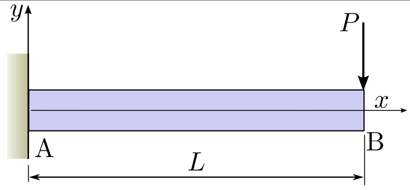
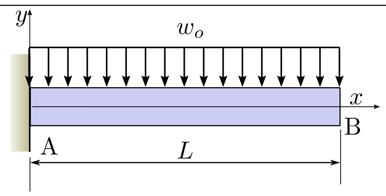
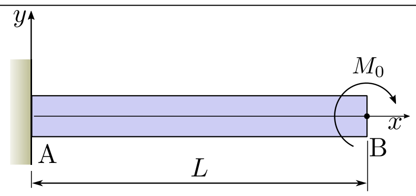
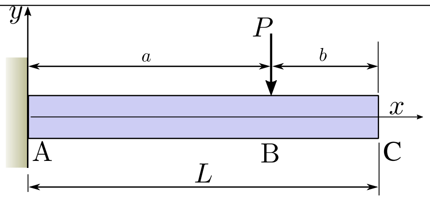
The **principle of superposition** states that the resultant effect of several load cases applied simultaneously may be treated as the algebraic sum of the load effects applied separately. Therefore quite complicated load cases may be solved easily by combining several simpler load cases. These load cases are usually tabulated in many engineering handbooks and a selection of them are stated in table 8.2 . The principle of superposition may only be applied because the following conditions are met: the load $w(x)$ is linearly related to the deflection $v(x)$ and the load does not change the geometry of the structure in other words deflections are small.

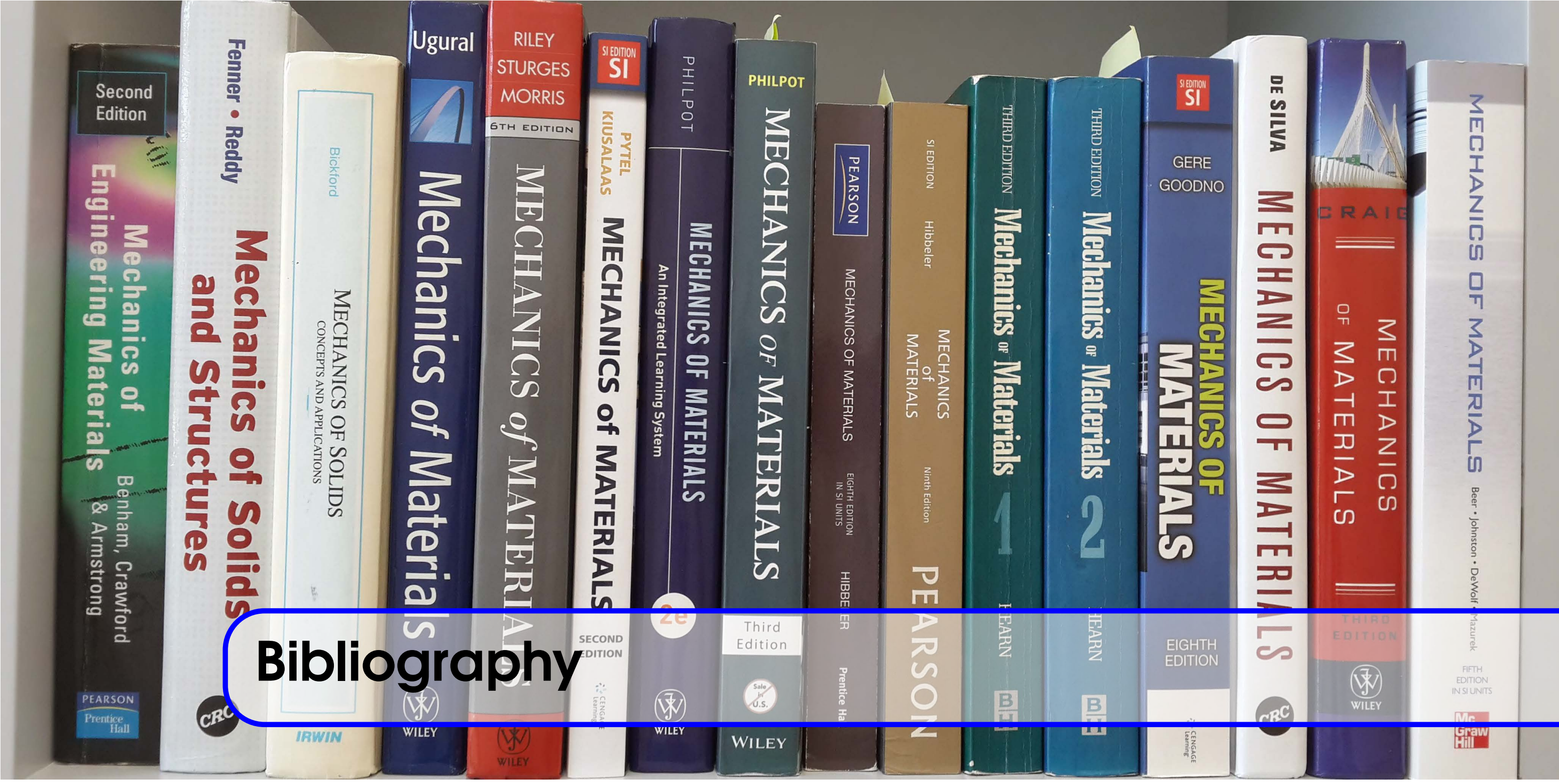
■ **Example 8.6** Consider example 8.3 repeat, using superposition.
Superposition tables from table 8.2

1. $\theta_{max} = \frac{-w(L/2)^3}{6EI} = \frac{-wL^3}{48EI}$
2. v_{max} =deflection of distributed load at L/2 + slope deflection
   $= \frac{-w(L/2)^4}{128EI} + \frac{-wL^3}{48EI} \times \frac{L}{2} = \frac{-7wL^4}{384EI}$
3. $v_{x=L} = \frac{PL^3}{3EI} + \frac{-7wL^4}{384EI} = 0 \Rightarrow P = \frac{7wL}{128}$ upwards

■

Table 8.2: Beam slopes and deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \dfrac{-PL^2}{16EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$v = \dfrac{-Px}{48EI}\left(3L^2 - 4x^2\right)$ $0 \leq x < \dfrac{L}{2}$
	$\theta_A = \dfrac{-Pab\,(L+b)}{6EIL}$ $\theta_C = \dfrac{Pab\,(L+a)}{6EIL}$	$v_B = \dfrac{-Pab}{6EIL}\left(L^2 - b^2 - a^2\right)$	$v = \dfrac{-Pbx}{6EIL}\left(L^2 - b^2 - x^2\right)$ $0 \leq x < a$
	$\theta_A = \dfrac{-M_0L}{24EI}$ $\theta_C = \dfrac{M_0L}{24EI}$	$v_B = 0$	$v = \dfrac{-M_0x}{24EIL}\left(L^2 - 4x^2\right)$ $0 \leq x < \dfrac{L}{2}$
	$\theta_{max} = \dfrac{-w_0L^3}{24EI}$	$v_{max} = \dfrac{-5w_0L^4}{384EI}$	$v = \dfrac{-w_0x}{24EI}\left(x^3 - 2Lx^2 + L^3\right)$
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \dfrac{-w_0L^3}{6EI}$	$v_{max} = \dfrac{-w_0L^4}{8EI}$	$v = \dfrac{-w_0x^2}{24EI}\left(x^2 - 4Lx + 6L^2\right)$
	$\theta_{max} = \dfrac{-M_0L}{EI}$	$v_{max} = \dfrac{-M_0L^2}{2EI}$	$v = \dfrac{-M_0x^2}{2EI}$
	$\theta_{max} = \dfrac{-Pa^2}{2EI}$	$v_{max} = \dfrac{-Pa^2}{6EI}(3L - a)$	$v = \dfrac{-Pa^2}{6EI}(3x - a)$ $0 \leq x < a$



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