8. Bending Deflection

An example of beam deflection seen here with the wing flex in a Boeing 787.

8.1 Introduction

In the previous chapter the stress in bending was calculated in order to design beams safely. In this chapter the deflection of beam structures will be analysed. This is also a consideration in design. Excessive deflection is seen in buildings with cracks in ceilings and walls, as well as visible sagging of floors. Some aircraft and other machine components can function incorrectly when large deflections cause unwanted vibration.

8.2 The Differential Equation of the Elastic Curve

When an initially straight beam is loaded and then behaves in an elastic manner the longitudinal centroidal axis of the beam becomes a curve this is called the **deflection** or **elastic curve** of the beam. The x- and y-axis are shown along with the deflection of the elastic curve v in Figure 8.1. The y-axis is positive upwards and indicates the coordinate within the beam cross section from the neutral axis. The deflection v is also positive upwards and is assumed quite small.

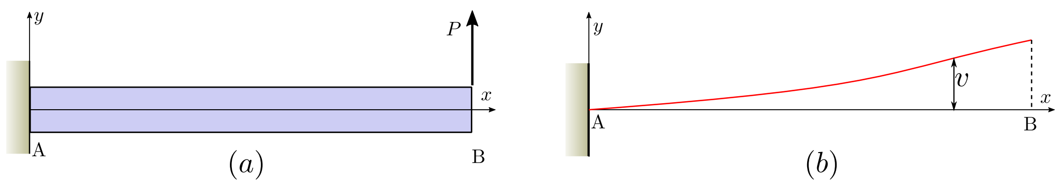


Figure 8.1: Cantilever beam with concentrated load in (a) and elastic curve shown in (b) with deflection.

In Figure 8.2 the point C on the elastic curve has a deflection v, a radius of curvature ρ and makes an angle θ with the x-axis. A point D is a small distance dx to the right of C and has deflection $v+dv$ and make an angle $\theta+d\theta$ with the horizontal. Since v is small, the angle θ is small and the length of the curve $ds \approx dx$. Angle θ is in radians so the arc length of the circle centred at O is:

$$\rho d\theta = ds \approx dx.$$

This is rearranged to give:

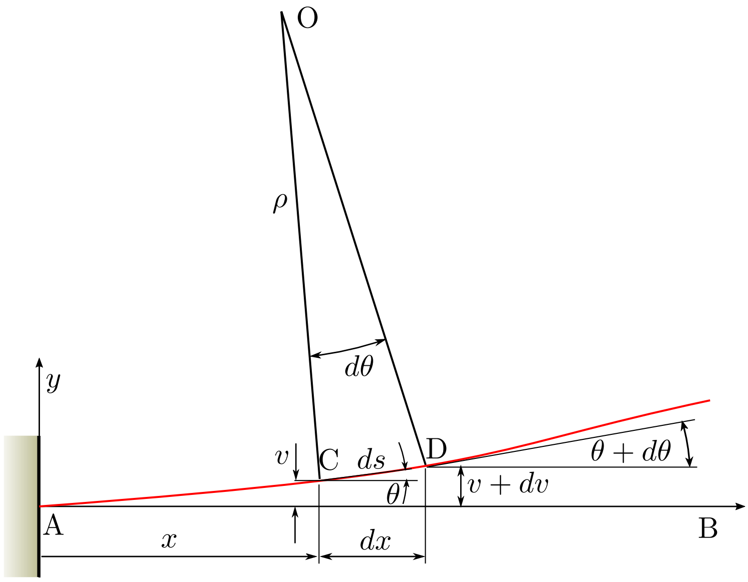


Figure 8.2: Elastic curve of a cantilever beam

$$\boxed{\frac{1}{\rho} = \frac{d\theta}{dx}} \tag{8.1}$$

The slope of the curve at C is $\dfrac{dv}{dx}$. This is related to the angle theta with:

$$\frac{dv}{dx} = \tan\theta \approx \theta$$

Taking the derivative of both sides, the following is obtained:

$$\boxed{\frac{d^2v}{dx^2} = \frac{d\theta}{dx}} \tag{8.2}$$

Combining equations (8.1), (8.2) and (7.8) the following relationship results:

$$\boxed{\frac{d^2v}{dx^2} = \frac{1}{\rho} = \frac{\mathscr{M}}{EI_z}} \tag{8.3}$$

This is usually written in the form:

$$\boxed{EI\frac{d^2v}{dx^2} = \mathscr{M}(x)} \tag{8.4}$$

The sign convention for internal moments and curvatures are the same and illustrated in Figure 8.3

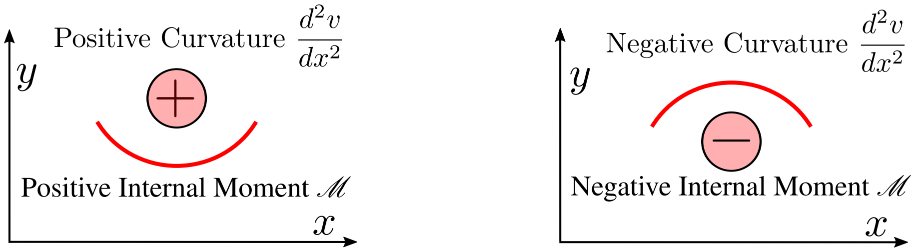


Figure 8.3: Sign convention of curvature and moment shown with elastic curve

8.3 Deflection and Slope by Double Integration

In order to calculate various physical quantities associated with beam bending, the following relationships are important:

For a deflection v the slope is $\frac{dv}{dx} = \theta$ of the elastic curve. The moment $\mathscr{M} = EI\frac{d^2v}{dx^2}$ from equation (8.4) can have further derivatives taken to give the relationship for shear $\mathscr{V}$:

$$\boxed{\mathscr{V} = EI\frac{d^3v}{dx^3}} \tag{8.5}$$

and furthermore the relationship for the distributed load, w.

$$\boxed{w = EI\frac{d^4v}{dx^4}} \tag{8.6}$$

The term EI is called the **flexural rigidity** of the beam and is assumed constant for equations (8.5) and (8.6) above. It is the resistance of a structure to bending.

To calculate the slope and deflection of beams as a function of distance x along the length: equations of the load and bending moment are first derived. These equations are repeatedly integrated to in a procedure called **double integration**. From the repeated integration, various constants are obtained which have to be evaluated. These are evaluated using the known conditions of the slope and deflection and can be grouped into three categories: boundary conditions, continuity conditions and symmetry conditions.

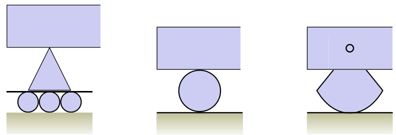
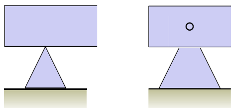
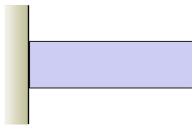
Boundary Conditions

When specific values of slope or deflections are known at points on a beam these are called **boundary conditions**. When the equation for M(x) is derived it is usually derived within specific regions of the beam eg $0 \leq x < a$ then the boundary conditions would refer to known values of slope or deflection at $x = 0$ and $x = a$. These boundaries for each region are not necessarily the bounds of the beam but rather the bounds of the region. A variety of boundary conditions are seen in Table 8.1. Each boundary condition solves one and only one constant of integration.

Continuity Conditions

Beams are often subject to discontinuous loading such as concentrated loads or uniform loads which can start and stop abruptly. It is then necessary to divide the beam up into different regions to account for the load discontinuities. For example if there is beam with a point load at $x = a$ then the shear along the beam cannot be represented with a single algebraic equation. Seperate equations will have to be derived to the left and right of the load. The beam is physically continuous so the equations to the left and the right should give the same deflections and slopes where they meet.

Table 8.1: Boundary Conditions for Beams

Support or connection	Boundary Condition
 Roller Roller Rocker	$v=0$ $\mathscr{M}=0$
 Pin Pin	$v=0$ $\mathscr{M}=0$
 Clamped, fixed or rigid	$v=0$ $\dfrac{dv}{dx}=0$
 Free End	$\mathscr{V}=0$ $\mathscr{M}=0$

Symmetry Conditions

The beam supports and the loading may be symmetric for a beam. These symmetry conditions will allow the value of beam slope to be determined at certain locations. For example the beam slope is zero at its midspan for a symmetric simply supported beam with a uniformly distributed load between its supports.

Procedure for Double Integration Method

The following procedure to solve for deflection and slope is recommended from Philpot [4].

1. **Sketch** supports, loads along with the axis. Sketch approximate shape of elastic curve especially the deflections/slopes at supports.
2. Determine the **support reactions**. This can be done using static equilibrium.
3. Consider **segment equilibrium**. For each segment draw a free body diagram showing all loads
4. **Integrate** the differential equation $EI\dfrac{d^2v}{dx^2}$ twice. The first time to determine the slope equation and then second time the deflection equation with two constants of integration.
5. List all **boundary, continuity and symmetry conditions.**
6. **Evaluate constants** using all the boundary conditions found above.
7. Determine the **elastic curve and slope equations** by substituting the constants found.
8. Determine the **deflections and slopes at points** with equations at specific points if required.

The procedure defined will be illustrated in the following example.

■ **Example 8.1** A cantilever beam is subject to a concentrated vertical load P at its free end A as shown in figure 8.4(a). Determine the elastic curve and the deflection and slope at the free end.

Assume EI is constant.

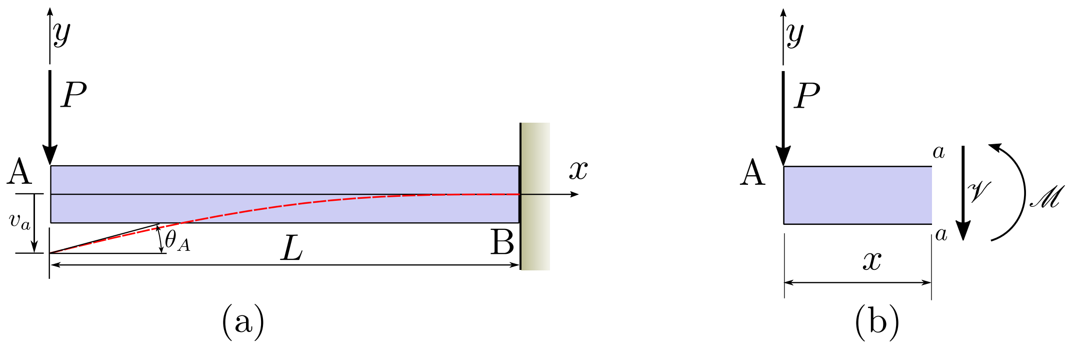


Figure 8.4: Clamped beam with concentrated load

Equilibrium
Take a section through the beam a distance x from the free end and draw a free body diagram shown in Figure 8.4(b). The equilibrium equation for the sum of moments through section a-a is:

$$\sum M_{a-a} = Px + \mathscr{M} = 0$$

The bending moment equation is then: $\mathscr{M} = -Px$. This is true for $0 \leq x \leq L$ in this beam. Substitute into equation(8.4) which gives:

$$EI\frac{d^2v}{dx^2} = -Px \tag{8.7}$$

Integration
Integrating equation (8.7) once gives the slope equation:

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + C_1 \tag{8.8}$$

C_1 is a constant of integration and integrating equation (8.8) a second time gives:

$$EIv = -\frac{Px^3}{6} + C_1x + C_2 \tag{8.9}$$

Another constant of integration C_2 results which is evaluated using the boundary conditions.

Boundary Conditions

The deflection and slope is known at the right end. Since it is clamped $v = 0$ and $\dfrac{dv}{dx} = 0$ at $x = l$. At the free end neither the slope nor the deflection is known.

Evaluate Constants
Substitute $\dfrac{dv}{dx} = 0$ at $y = l$ into equation (8.8) to give:

$$C_1 = \frac{PL^2}{2} \tag{8.10}$$

Then substitute $v = 0$ at $y = l$ into equation (8.9) to give:

$$C_2 = -\frac{PL^3}{3} \tag{8.11}$$

Elastic Curve and Slope Equations

Substitute equation (8.10) and (8.11) into equation (8.9). The elastic curve equation is:

$$EIv = -\frac{Px^3}{6} + \frac{PL^2}{2}x - \frac{PL^2}{3} \tag{8.12}$$

The slope equation is then:

$$EI\frac{dv}{dx} = -\frac{Px^2}{2} + \frac{PL^2}{2} \tag{8.13}$$

Deflections and Slopes at Specified Points

At x = 0: $v_A = -\frac{PL^3}{3EI}$ and $\theta_A = \frac{dv}{dx} = \frac{PL^2}{2EI}$

■

8.4 Discontinuity Functions

Double integration to obtain the deflection of the differential equation can be a tedious, lengthy procedure. Discontinuity functions or Macaulay functions can be used to obtain concise expressions for load, shear and moments for beams and greatly simplifies the determination of deflection in multi-interval deflection problems. Consider the following beam loaded with various concentrated loads.

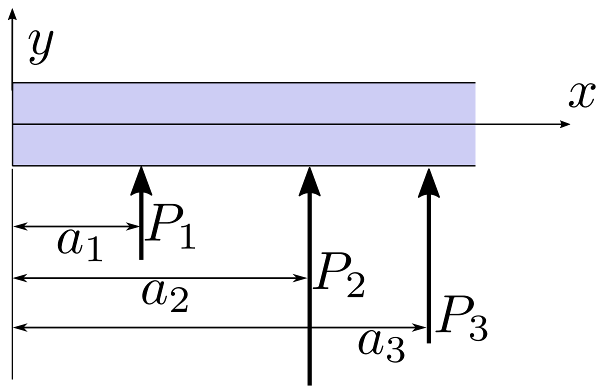


Figure 8.5: Loading of a beam section

The loading can be represented with the bending moment expression:

$$\boxed{\mathscr{M} = P_1\langle x-a_1\rangle + P_2\langle x-a_2\rangle + P_3\langle x-a_3\rangle + \ldots} \tag{8.14}$$

where the quantities $P_i\langle x-a_i\rangle$ represent the bending moments due to point loads, and the quantity $\langle x-a_i\rangle$ is a **Macaulay bracket** defined as:

$$\boxed{\langle x-a_i\rangle = \begin{cases} 0 & \text{if } x < a_i \\ x-a_i & \text{if } x \geq a_i \end{cases}} \tag{8.15}$$

Ordinarily, when integrating $P(x-a)$ we get:

$$\int P(x-a)\ dx = P\left[\frac{x^2}{2} - ax\right] + C$$

However, when integrating a Macaulay bracket, we get:

$$\boxed{\int P\langle x-a\rangle\ dx = P\frac{\langle x-a\rangle^2}{2} + C_m} \tag{8.16}$$

The Macaulay term is treated as single term of integration and C_m represents a constant of integration. This requires a more general definition of Macaulay brackets:

$$\boxed{\langle x-a_i\rangle^n = \begin{cases} 0 & \text{if } x < a_i \\ (x-a_i)^n & \text{if } x \geq a_i \end{cases} \quad \text{for n } = 0, 1, 2...} \tag{8.17}$$

The derivative of the bending moment equation (8.14) gives the shear expression:

$$\boxed{\frac{\mathrm{d}\mathscr{M}}{\mathrm{d}x} = \mathscr{V} = P_1\langle x-a_1\rangle^0 + P_2\langle x-a_2\rangle^0 + P_3\langle x-a_3\rangle^0 + \dots} \tag{8.18}$$

The angle bracket is interpreted with help of equation (8.17) and noting that that $(x-a_i)^0 = 1$
Integration of the bending moment equation (8.14) gives an expression for the slope:

$$\boxed{EI\frac{\mathrm{d}v}{\mathrm{d}x} = \int \mathscr{M}\,dx = \frac{P_1\langle x-a_1\rangle^2}{2} + \frac{P_2\langle x-a_2\rangle^2}{2} + \frac{P_3\langle x-a_3\rangle^2}{2} + C_1 + \dots} \tag{8.19}$$

Here C_1 is a constant from the Macaulay intergration. Further integration gives the deflection

$$\boxed{EIv = \frac{P_1\langle x-a_1\rangle^3}{6} + \frac{P_2\langle x-a_2\rangle^3}{6} + \frac{P_3\langle x-a_3\rangle^3}{6} + C_1x + C_2 \dots} \tag{8.20}$$

Consider a beam with moments applied as follows:

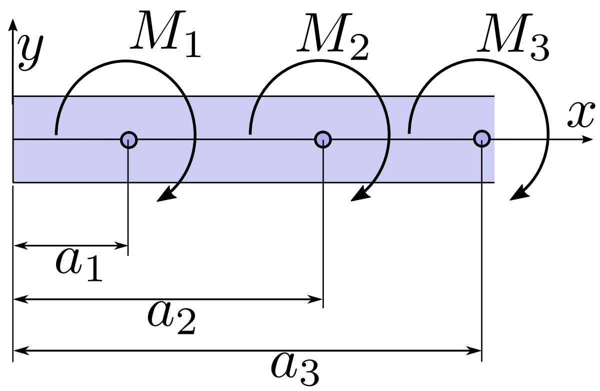


Figure 8.6: Moments applied to a beam section

The bending moment expression is as follows:

$$\mathscr{M} = M_1\langle x-a_1\rangle^0 + M_2\langle x-a_2\rangle^0 + M_3\langle x-a_3\rangle^0 + \dots \tag{8.21}$$

In a similar way to equation (8.19) and (8.20) we get slope and deflection equations.

Now consider a distributed load as follows:

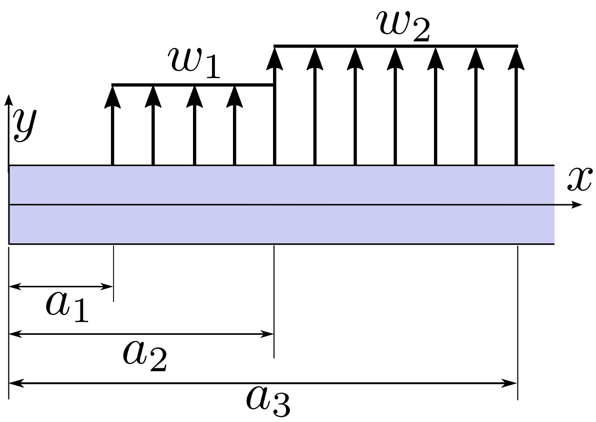


Figure 8.7: Uniformly distributed load applied to a beam section

The shear expression is:

$$\mathscr{V} = w_1\langle x-a_1\rangle + (w_2-w_1)\langle x-a_2\rangle - w_2\langle x-a_3\rangle + \dots \tag{8.22}$$

In a similar fashion to previously we can repeatedly integrate to first get the bending moment, then slope and deflection expressions.

■ **Example 8.2** A simply supported beam has loading as follows:

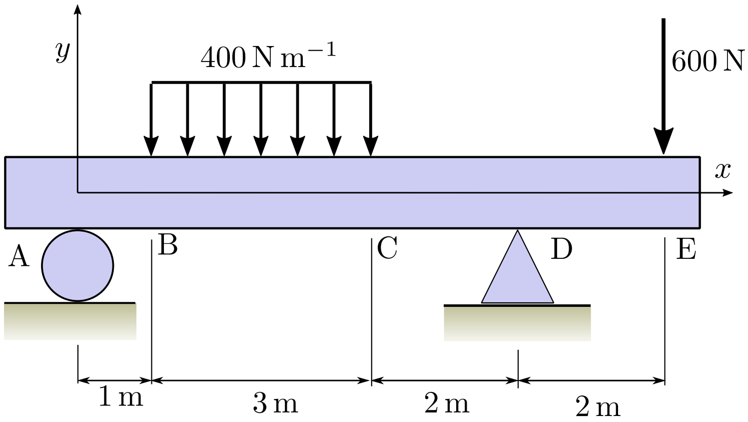


Figure 8.8: Concentrated and uniformly distributed load on 8 m beam

Support Reactions

$\sum M_A = -400 \times 3 \times 2.5 + D_y \times 6 - 600 \times 8 = 0 \qquad \Rightarrow D_y = 1300 \text{ N}$

$\sum F_y = A_y - 400 \times 3 + D_y - 600 = 0 \qquad \Rightarrow A_y = 500 \text{ N}$

Moment function

$$\mathscr{M}(x) = 500\langle x\rangle^1 - \frac{400\langle x-1\rangle^2}{2} + \frac{400\langle x-4\rangle^2}{2} + 1300\langle x-6\rangle^1$$

Slope function

$$EI\frac{\mathrm{dv}}{\mathrm{d}x} = \frac{500\langle x\rangle^2}{2} - \frac{400\langle x-1\rangle^3}{6} + \frac{400\langle x-4\rangle^3}{6} + \frac{1300\langle x-6\rangle^2}{2} + C_1$$

Deflection function

$$EIv = \frac{500\langle x\rangle^3}{6} - \frac{400\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6} + C_1x + C_2$$

Boundary conditions

$v = 0$ at $x = 0$ and $v = 0$ at $x = 6$ m

Evaluate Constants

$v = 0$ at $x = 0 \Rightarrow C_2 = 0$ and $v = 0$ at $x = 6 \Rightarrow$

$$EIv = \frac{500\langle 6\rangle^3}{6} - \frac{400\langle 6-1\rangle^4}{24} + \frac{400\langle 6-4\rangle^4}{24} + C_1 6 = 0$$

$$\Rightarrow C_1 = -\frac{3925}{3} = -1308 \text{ N.m}^2$$

Deflection function

$$EIv = \frac{500\langle x\rangle^3}{6} - \frac{400\langle x-1\rangle^4}{24} + \frac{400\langle x-4\rangle^4}{24} + \frac{1300\langle x-6\rangle^3}{6} - 1308x$$

Deflection at x = 3

$$v = \frac{1}{EI}\left(\frac{500\langle 3\rangle^3}{6} - \frac{400\langle 3-1\rangle^4}{24} - 1308(3)\right) = \frac{-1942}{EI}$$

Deflection at x = 8

$$v = \frac{1}{EI}\left(\frac{500\langle 8\rangle^3}{6} - \frac{400\langle 8-1\rangle^4}{24} - 1308x + \frac{400\langle 8-4\rangle^4}{24} + \frac{1300\langle 8-6\rangle^3}{6} - 1308(8)\right)$$

$$\Rightarrow v = \frac{-1817}{EI}$$

■

■ **Example 8.3** An initially straight and horizontal cantilever of uniform section and length L is rigidly built-in at one end and carries a uniformly distributed load of intensity w for a distance $L/2$ measured from the built-in end. The second moment of area is I and the modulus of elasticity E. Determine, in terms of w, L, E and I:

1. An expression for the slope of the cantilever at the end of the load.
2. The deflection at the free end.
3. The force in a vertical prop which is to be applied at the free end in order to restore this end to the same horizontal level as the built-in end.

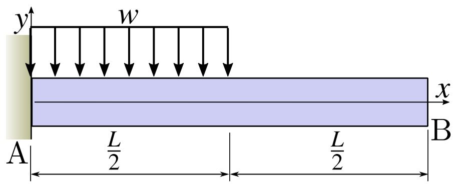


Figure 8.9: Clamped beam with uniform load over half the length

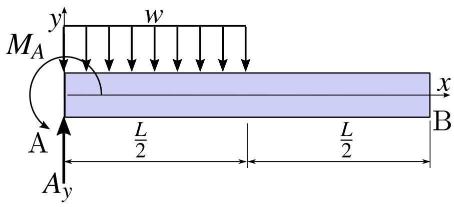


Figure 8.10: Free body diagram of beam

Moments $\sum M_A = M_A = -\dfrac{wL^2}{8}$

Resolve $\sum F_y = 0 \Rightarrow A_y = \dfrac{wL}{2}$

<u>Moment function</u> $M(x) = -\dfrac{wL^2\langle x\rangle^0}{8} + \dfrac{wL\langle x\rangle^1}{2} - \dfrac{w\langle x\rangle^2}{2} + \dfrac{w\langle x-\frac{L}{2}\rangle^2}{2}$

<u>Slope function</u> $EI\dfrac{\mathrm{dv}}{\mathrm{d}x} = -\dfrac{wL^2\langle x\rangle^1}{8} + \dfrac{wL\langle x\rangle^2}{4} - \dfrac{w\langle x\rangle^3}{6} + \dfrac{w\langle x-\frac{L}{2}\rangle^3}{6} + C_1$

<u>Deflection function</u> $EIv = -\dfrac{wL^2\langle x\rangle^2}{16} + \dfrac{wL\langle x\rangle^3}{12} - \dfrac{w\langle x\rangle^4}{24} + \dfrac{w\langle x-\frac{L}{2}\rangle^4}{24} + C_1x + C_2$

<u>Boundary conditions</u> $v = 0$ at $x = 0$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = 0$ at $x = 0$ m

<u>Evaluate Constants</u> $v = 0$ at $x = 0 \Rightarrow C_2 = 0$ and $\dfrac{\mathrm{dv}}{\mathrm{d}x} = 0$ at $x = 0$ m $\Rightarrow C_1 = 0$

1. <u>Slope at x = L/2</u>
$$\frac{\mathrm{dv}}{\mathrm{d}x} = \frac{1}{EI}\left(-\frac{wL^2\langle x\rangle^1}{8} + \frac{wL\langle x\rangle^2}{4} - \frac{w\langle x\rangle^3}{6} + \frac{w\langle x-\frac{L}{2}\rangle^3}{6}\right) = -\frac{wL^3}{48EI}$$
2. <u>Deflection at x = L</u>
$$v = \frac{1}{EI}\left(-\frac{wL^2\langle x\rangle^2}{16} + \frac{wL\langle x\rangle^3}{12} - \frac{w\langle x\rangle^4}{24} + \frac{w\langle x-\frac{L}{2}\rangle^4}{24}\right) = \frac{-7wl^4}{348EI}$$
3. See example 8.6
Using superposition $P = \dfrac{7wL}{128}$ upwards.

■

8.5 Moment Area Method

In this section a semigraphical technique is used to find deflections and angles of beams and is called the **moment area method**. This technique uses the relationship between the area of the bending moment diagram and the beam deflections and -angles. It may be quicker where only properties of one point are needed. This method is also more effective in dealing with beams where there is a change in cross section.

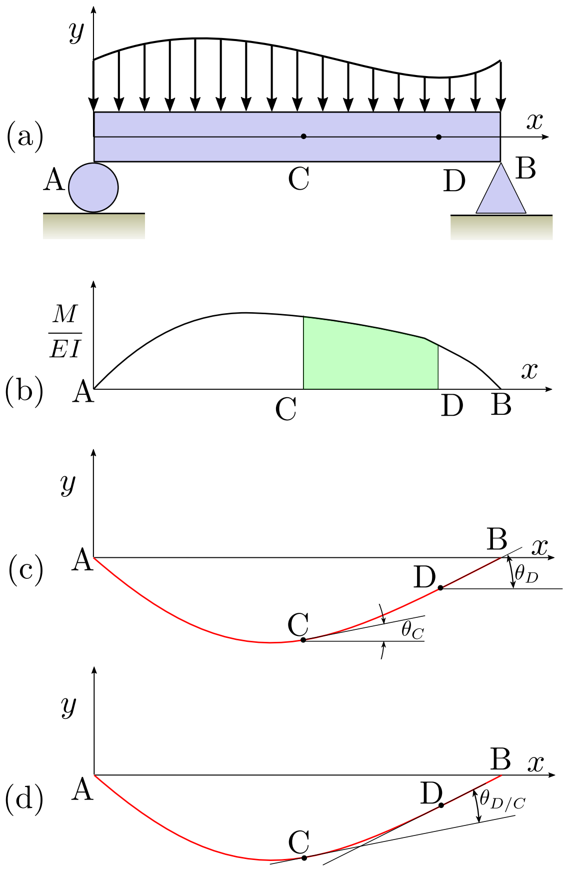


Figure 8.11: Finding the slope with the moment area method: (a) is the load diagram; (b) is the $\mathscr{M}/EI$ diagram; (c) elastic curve

Combine equations (8.2) and (8.3) to give the relationship between the change in angle and the bending moment:

$$\boxed{d\theta = \frac{\mathscr{M}}{EI}dx} \tag{8.23}$$

In figure 8.11(a) we see two points C and D on a elastic beam with a distributed load.

The difference in slope between and two points C and D can be expressed as:

$$\boxed{\theta_{D/C} = \theta_D - \theta_C = \int_C^D \frac{\mathscr{M}}{EI}dx} \tag{8.24}$$

where θ_C and θ_D are the slopes at C and D respectively shown in figure 8.11(c). The area under the $\frac{\mathscr{M}}{EI}$ diagram is shown in figure 8.11(b). The change in slope $\theta_{D/C}$ is shown in figure 8.11(c).

The angle $\theta_{D/C}$ and the area under the $\frac{\mathscr{M}}{EI}$ have the same sign. If the area under the $\frac{\mathscr{M}}{EI}$ diagram is positive then $\theta_{D/C}$ will be anti-clockwise.

This is called the **first moment area theorem**.

$$\boxed{\theta_{D/C} = \textbf{area of } \frac{\mathscr{M}}{EI} \textbf{ diagram between C and D}} \tag{8.25}$$

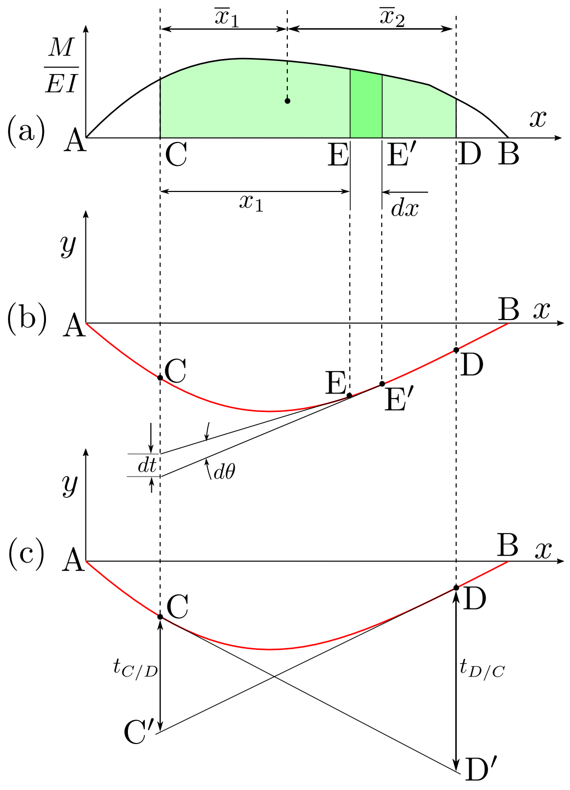


Figure 8.12: Finding deflection with the moment area method: (a) is the M/EI diagram; (b) and (c) are elastic curves

Consider two points E and E′ a distance dx apart between C and D see figure 8.12. Since the slope θ at E and the angle $d\theta$ formed by the tangents at E and E′ are small, we can approximate:

$$dt = x_1 d\theta = x_1 \frac{\mathscr{M}}{EI} dx$$

Integrate from C to D to get the vertical distance from C to the tangent at D which is at C′. This represents the tangential deviation of C with respect to D and is called $t_{C/D}$ which is:

$$\boxed{t_{C/D} = \int_D^C x_1 \frac{\mathscr{M}}{EI} dx} \tag{8.26}$$

The right hand side of equation (8.26) represent the first moment with respect to the area under the located under the $\frac{\mathscr{M}}{EI}$ diagram between C and D. Using equation (7.5), $\bar{y} = \frac{\int_A y dA}{\int_A dA}$ gives the relationship between the first moment and the centroid.

The right hand side represents the product of the area with the centroid. Therefore equation (8.26) can also be expressed as:

$$\boxed{t_{C/D} = \textbf{(area between C and D)}\ \bar{x}_1} \tag{8.27}$$

This is the **second moment area theorem**. In a similar fashion the tangential deviation of C with respect to D is:

$$\boxed{t_{D/C} = \textbf{(area between C and D)}\ \bar{x}_2} \tag{8.28}$$

Note finally that a point with positive tangential deviation is located above the tangent drawn from the other point as seen with figure 8.12(c)

■ **Example 8.4** Consider the following example with a concentrated load with a cantilever beam. Determine the deflection v_B and angle of rotation θ_B of the free end B of a cantilever beam AB with a concentrated load P at B. The beam has length L and constant flexural rigidity EI.

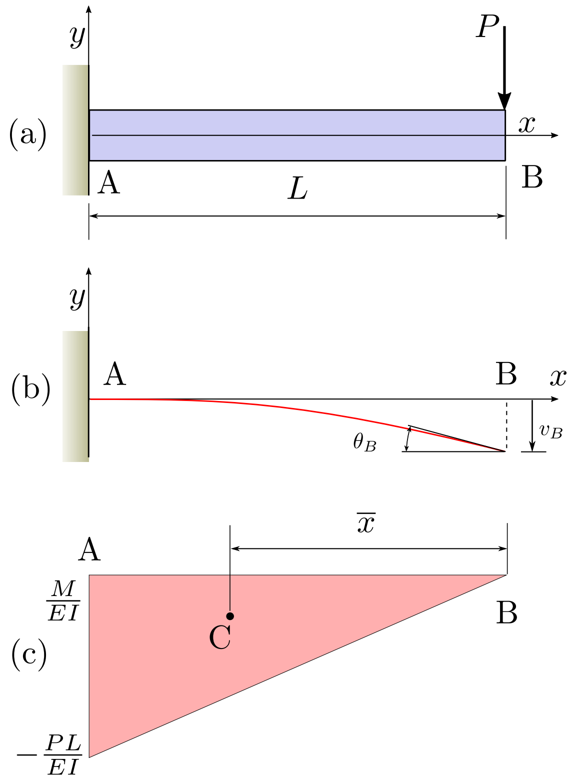


Figure 8.13: Cantilever beam with a concentrated load: (a) is the load diagram; (b) elastic curve; (c)is the $\frac{\mathcal{M}}{EI}$ diagram

Slope at B By inspection the angle of rotation is clockwise and the deflection is downwards.

The bending moment diagram is triangular with the internal moment at the support equal to $-PL$. The flexural rigidity is constant so the bending moment diagram has the same shape as the $\frac{\mathcal{M}}{EI}$ diagram as seen in figure 8.13(c).

The area of the $\frac{\mathcal{M}}{EI}$ diagram between the points A and B represents the angle $\theta_{A/B}$ between the tangents at those points. This area which we will call A_1 is evaluated as follows:

$$A_1 = -\frac{1}{2}(L)\left(\frac{PL}{2EI}\right) = -\frac{PL^2}{EI}$$

This is a negative area since it is below the x-axis. The relative rotation between points A and B from the first moment-area theorem is:

$$\theta_{B/A} = \theta_B - \theta_A = A_1 = -\frac{PL^2}{2EI}$$

The tangent to the curve at A is horizontal $\theta_A = 0$ we get:

$$\theta_B = -\frac{PL^2}{2EI}$$

This is a negative quantity so the angle is clockwise.

Deflection at B

The deflection v_B can be obtained from the second moment area theorem. Noting that the tangent to A is a horizontal along the x-axis.

$$v_B = t_{B/A} = A_1\overline{x} = -\frac{PL^2}{2EI}\left(\frac{2L}{3}\right) = -\frac{PL^3}{3EI}$$

This is a negative quantity so the deflection of B is downwards. ■

■ **Example 8.5** Consider example 8.3 and repeat using the moment area method. Diagram repeated below:

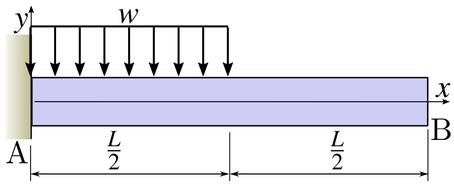
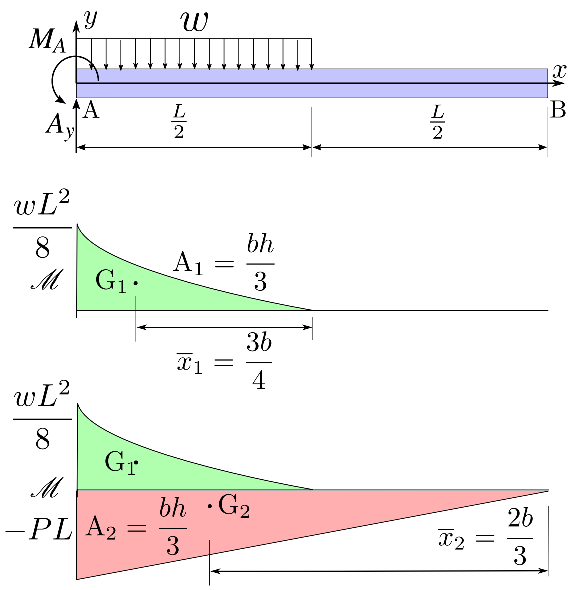


Figure 8.14: Clamped beam with uniform load over half the length



Moment at A = $M_A = -\frac{wL^2}{8}$

1. Since beam is clamped slope at A is zero, therefore:
   slope at x = $\frac{L}{2}$ is area under $\frac{M}{EI}$ diagram
   $= \frac{bh}{3EI} = \frac{wL^2}{8} \times \frac{L}{2} \times \frac{1}{3EI} = \frac{wL^3}{48EI}$
2. Since beam is clamped deflection at A is zero therefore:
   deflection at x = L is area under $\frac{M}{EI}$ diagram
   $\times$ centroid distance from B
   $= \frac{wL^3}{48EI} \times \left(\frac{3}{4}\frac{L}{2} + \frac{L}{2}\right) = \frac{7wL^4}{384EI}$
3. For the deflection to be zero, the product of the area and centroid distance for the concentrated and distributed loads is the same from B:
   $= \frac{PL^2}{2EI} \times \frac{2}{3}L = \frac{PL^3}{3EI} = \frac{7wL^4}{384EI}$
   $\Rightarrow P = \frac{7wL}{128}$ upwards

■

8.6 Principle of Superposition

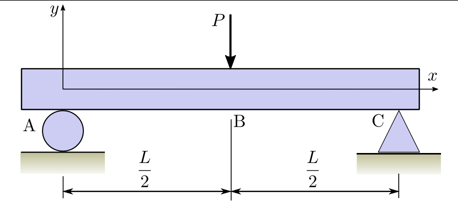
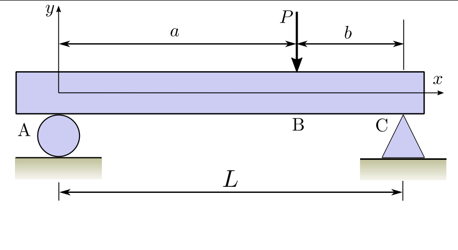
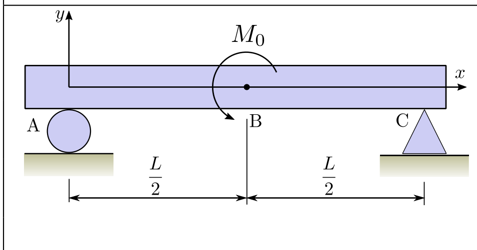
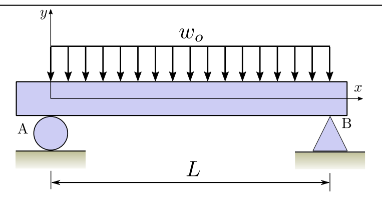
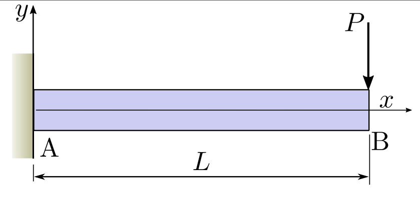
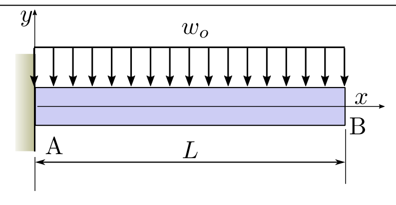
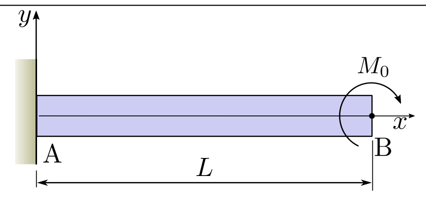
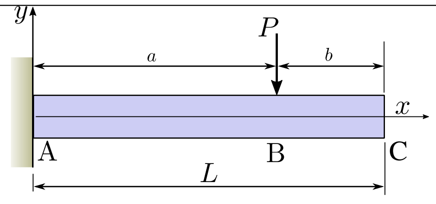
The **principle of superposition** states that the resultant effect of several load cases applied simultaneously may be treated as the algebraic sum of the load effects applied separately. Therefore quite complicated load cases may be solved easily by combining several simpler load cases. These load cases are usually tabulated in many engineering handbooks and a selection of them are stated in table 8.2 . The principle of superposition may only be applied because the following conditions are met: the load $w(x)$ is linearly related to the deflection $v(x)$ and the load does not change the geometry of the structure in other words deflections are small.

■ **Example 8.6** Consider example 8.3 repeat, using superposition.
Superposition tables from table 8.2

1. $\theta_{max} = \frac{-w\,(L/2)^3}{6EI} = \frac{-wL^3}{48EI}$
2. v_{max} =deflection of distributed load at L/2 + slope deflection
   $= \frac{-w\,(L/2)^4}{128EI} + \frac{-wL^3}{48EI} \times \frac{L}{2} = \frac{-7wL^4}{384EI}$
3. $v_{x=L} = \frac{PL^3}{3EI} + \frac{-7wL^4}{384EI} = 0 \Rightarrow P = \frac{7wL}{128}$ upwards

■

Table 8.2: Beam slopes and deflections

Beam	Slope	Deflection	Elastic Curve
	$\theta_{max} = \frac{-PL^2}{16EI}$	$v_{max} = \frac{-PL^3}{48EI}$	$v = \frac{-Px}{48EI}\left(3L^2 - 4x^2\right)$ $0 \leq x < \frac{L}{2}$
	$\theta_A = \frac{-Pab\,(L+b)}{6EIL}$ $\theta_C = \frac{Pab\,(L+a)}{6EIL}$	$v_B = \frac{-Pab}{6EIL}\left(L^2 - b^2 - a^2\right)$	$v = \frac{-Pbx}{6EIL}\left(L^2 - b^2 - x^2\right)$ $0 \leq x < a$
	$\theta_A = \frac{-M_0L}{24EI}$ $\theta_C = \frac{M_0L}{24EI}$	$v_B = 0$	$v = \frac{-M_0x}{24EIL}\left(L^2 - 4x^2\right)$ $0 \leq x < \frac{L}{2}$
	$\theta_{max} = \frac{-w_0L^3}{24EI}$	$v_{max} = \frac{-5w_0L^4}{384EI}$	$v = \frac{-w_0x}{24EI}\left(x^3 - 2Lx^2 + L^3\right)$
	$\theta_{max} = \frac{-PL^2}{2EI}$	$v_{max} = \frac{-PL^3}{3EI}$	$v = \frac{-Px^2}{6EI}(3L - x)$
	$\theta_{max} = \frac{-w_0L^3}{6EI}$	$v_{max} = \frac{-w_0L^4}{8EI}$	$v = \frac{-w_0x^2}{24EI}\left(x^2 - 4Lx + 6L^2\right)$
	$\theta_{max} = \frac{-M_0L}{EI}$	$v_{max} = \frac{-M_0L^2}{2EI}$	$v = \frac{-M_0x^2}{2EI}$
	$\theta_{max} = \frac{-Pa^2}{2EI}$	$v_{max} = \frac{-Pa^2}{6EI}(3L - a)$	$v = \frac{-Pa^2}{6EI}(3x - a)$ $0 \leq x < a$