5. Torsion

Wind farm in Darling with old wind mill in foreground. Understanding torsion is key in the design of these structures.

5.1 Introduction

Torsion is twisting of a structure about its longitudinal axis due to a moment. These twisting moments are called **torques** and can cause rotation about the longitudinal axis. The most common structure to accomplish this is a transmission **shaft**. They are used to transmit power from one point to another along the shaft length. Only circular shafts will be analysed in this section.

5.2 Torsional Deformation

When applying a torque to a shaft it is assumed that circular cross sections remain circular and flat. This is illustrated in figure 5.1(b) In other words the cross sections remain rigid rotating through various angles relative to each other. Radial lines will stay straight and radial.
When applying torque to non-circular cross sections there is warping which distorts the shape of the cross section and causes it to become non-planar.

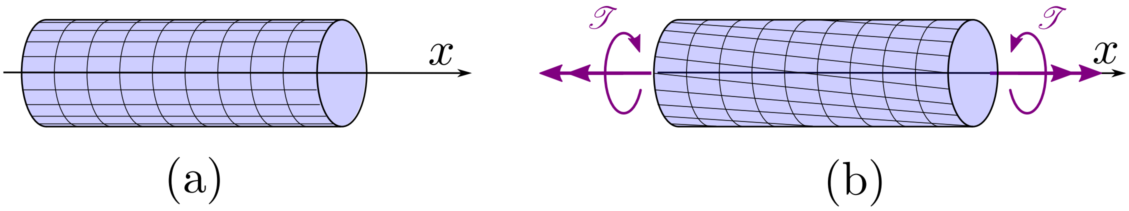

Figure 5.1: (a) Undeformed shaft and (b) deformed shaft showing two internal torque representations

Torsion can be represented pictorially using either a curved arrow or double headed arrow seen in figure 5.1(b). The direction sense for internal torque is defined by a right hand rule if your thumb points in the direction of the double headed arrow then your curled fingers would represent the direction of the applied torque. The Internal Torque $\mathscr{T}$ in figure 5.1(b) is considered positive.

Consider the red coloured rectangular element as seen in figure 5.2(a) of length L and radius c. A torque T is applied to a shaft. Since cross sections remain rigid as they rotate the rectangular element undergoes shear to become a parallelogram as seen figure 5.2(b). The shear strain of the element is γ. Line AB is the initial undeformed line on the surface of the shaft. It deforms to A′B and the angle ABA′ is the same as the shear angle γ. From geometry we know:

$$\phi\rho = \text{length of arc AA}' = \gamma L$$

here the shear γ and the angle of twist ϕ are expressed in radians and ρ is the distance from the centre to the point of interest A which becomes A′.

$$\boxed{\gamma = \frac{\rho\phi}{L}} \tag{5.1}$$

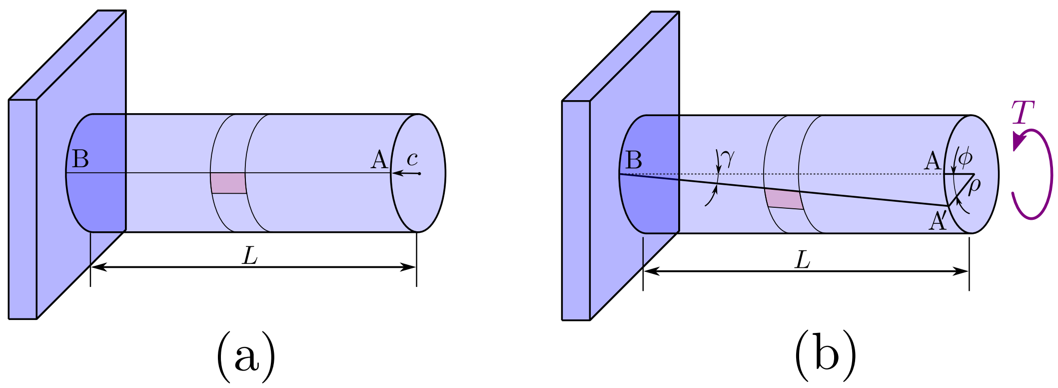

Figure 5.2: Undeformed shaft with clamped support shown in (a) and and deformed shaft under torsion(b).

The shear strain is maximum on the surface where $\rho = c$ so

$$\gamma_{\text{max}} = \frac{c\phi}{L}$$

eliminating the ϕ and L we get the following relation:

$$\boxed{\gamma = \frac{\rho}{c}\gamma_{\text{max}}} \tag{5.2}$$

The shear strain varies linearly from the centre of the shaft and is maximum at the surface.

5.3 Shear Stress in Torsion

Here it is assumed that the stresses are linearly related to the strain and the stresses remain below the elastic limit. In other words Hooke's Law for shear in equation (2.12):

$$\tau = \gamma G \tag{5.3}$$

Here G is the Modulus of Rigidity and τ is the shear stress.
Multiply equation (5.2) by G and substitute equation (5.3) to give:

$$\boxed{\tau = \frac{\rho}{c}\tau_{\text{max}}} \tag{5.4}$$

shows that the shear stress varies linearly with ρ from the centre to the surface of the shaft.

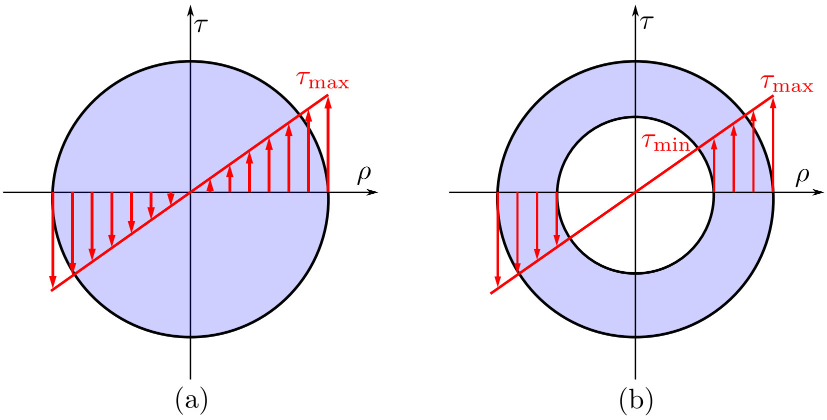

Figure 5.3: Linear distribution of shear stress in torsion in a solid shaft (a) and a hollow shaft (b).

Consider a differential element on the shaft with area dA in figure 5.4. A torque $\mathscr{T}$ acts on the shaft which has radius c.

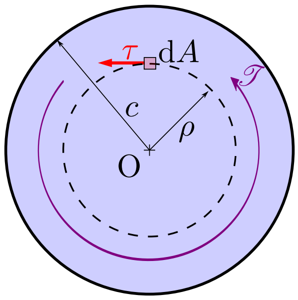

Figure 5.4: Determination of the resultant of the shear stresses acting on a cross section.

The resulting shear force acting on the element area is $dV = \tau dA$. There is an moment acting on the element resulting from this shear force and by substitution of equation (5.4) we obtain:

$$dM = \rho dV = \rho \tau dA = \frac{\tau_{\max}}{c}\rho^2 dA$$

The elemental moments are added over the area to give the torque $\mathscr{T}$ to satisfy equilibrium

$$\boxed{\mathscr{T} = \int_A dM = \frac{\tau_{\max}}{c}\int_A \rho^2 dA} \tag{5.5}$$

The parameters $\tau_{\max}$ and c can be taken outside the integral sign since they are independent of dA. The integral is the **polar moment of inertia**, J:

$$\boxed{J = \int_A \rho^2 dA} \tag{5.6}$$

Equation (5.5) can be rearranged and equation (5.6) can be substituted to give:

$$\boxed{\tau_{\max} = \frac{\mathscr{T}c}{J}} \tag{5.7}$$

For any shear along the radius of the shaft ρ

$$\boxed{\tau = \frac{\mathscr{T}\rho}{J}} \tag{5.8}$$

This equation is known as the **elastic torsion formula** of which equation (5.8) is a special case. It only applies to linear elastic, isotropic, homogeneous materials.

5.4 Polar Moment of Inertia

The polar moment of inertia is also known as the polar second moment of area and will be derived for a solid shaft.

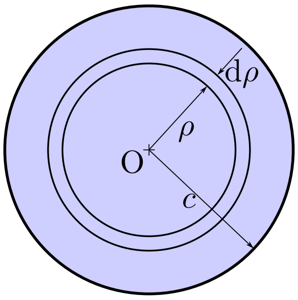

Figure 5.5: Derivation of polar moment of inertia for a solid shaft.

$$J = \int_A \rho^2 dA = \int_0^c \rho^2 2\pi\rho d\rho = \left[\frac{1}{4}2\pi\rho^4\right]_0^c = \frac{\pi c^4}{2}$$

For a shaft diameter where $D = 2c$

$$\boxed{J = \frac{\pi c^4}{2} = \frac{\pi D^4}{32}} \tag{5.9}$$

Polar moment of inertia is a geometric property of the circular area. The SI units for J are m^4
For a hollow shaft the integration limits would be from the outer surface to the inner surface. If the outer surface has radius c_o and diameter D and the inner radius c_i with diameter d. The polar moment of inertia for a hollow shaft is:

$$\boxed{J = \frac{\pi\left(c_o^4 - c_i^4\right)}{2} = \frac{\pi\left(D^4 - d^4\right)}{32}} \tag{5.10}$$

5.5 Angle of Twist due to Torsion

The relationship between shear stress and torque along any radial component is given by equation (5.8):

$$\tau = \frac{\mathscr{T}\rho}{J}$$

By taking the product of shear strain and modulus of rigidity (G) the angle of twist, ϕ along the radius from equation (5.1) is:

$$\tau = G\gamma = G\frac{\rho\phi}{L} \tag{5.11}$$

Equating equation (5.8) and equation (5.11) to the radius $\frac{\tau}{\rho}$ gives:

$$\boxed{\frac{G\phi}{L} = \frac{\tau}{\rho} = \frac{\mathscr{T}}{J}} \tag{5.12}$$

These relationships are useful in dual specification problems where the angle of twist and shear stress are both limiting factors.

The **torsional rigidity** is the product of the shear modulus and polar moment of inertia and represents the stiffness of the shaft to torsion:

$$\boxed{GJ = \frac{\mathscr{T}}{\phi/L}} \tag{5.13}$$

Equation (5.12) can be rearranged to give the angle of twist:

$$\boxed{\phi = \frac{\mathscr{T}L}{JG}} \tag{5.14}$$

If a torsion member is composed of different segments with different materials, different polar moments of inertia and/or different lengths with different torques on different segments, the angle of twist can be added algebraically as follows:

$$\boxed{\phi = \sum \frac{\mathscr{T}_i L_i}{J_i G_i}} \tag{5.15}$$

Here $\mathscr{T}_i, L_i, J_i$ and G_i are the internal torque, length, polar moment of inertia and shear modulus of the i^{th} segment respectively. The internal torque can be calculated by passing a section through a segment and drawing a free body diagram of the portion of the shaft on one side of the diagram. For practical purposes all local discontinuities are ignored at connections where there are pulleys, gears or some coupling connection.

5.6 Power Transmitted By Shafts

An important usage of shafts is transmitting power from a motor or engine to a component. **Power** is the work done per unit time. The work transmitted W by a rotating shaft is the product of the applied torque T and the angle ϕ through which the shaft rotates:

$$W = T\phi$$

The power transmitted by a shaft subject to a constant torque T is:

$$P = \frac{dW}{dt} = T\frac{d\phi}{dt}$$

The angular velocity ω equals the rate of change angular displacement. The power transmitted is therefore:

$$\boxed{P = T\omega} \tag{5.16}$$

Here ω is measured in radians per second. The SI unit for torque is N m and power is the watt (W). For machinery the frequency is often expressed in hertz (Hz), where 1 Hz = 1 cycle/s. Since 1 cycle = 2π radians therefore $\omega = 2\pi f$ and equation (5.16) can be expressed as:

$$\boxed{P = 2\pi f T} \tag{5.17}$$

A very common unit of measure is rotational speed in revolutions per minute (rpm).

$$1\,\mathrm{rpm} = \frac{1}{60}\mathrm{s}^{-1} = \frac{1}{60}\mathrm{Hz}$$

Then power can be written in terms of rpm n with:

$$\boxed{P = \frac{2\pi n T}{60}} \tag{5.18}$$

5.7 Statically Indeterminate Shafts under Torsion

Often to determine the stresses in a shafts, the torques can be determined from drawing free body diagrams and solving the equations of equilibrium. The solution to this problem is termed **statically determinate**.

For many other structures these equilibrium equations are not sufficient for determining all the torques. These problems are **statically indeterminate** and the geometry of the deformation is also needed to solve these problems.

A similar analysis to the section on axial loading in section 4.4 is required. The five step procedure is given as follows:

Step 1: Express all the **equations of equilibrium** for the structure in term of the unknown internal torques.

Step 2: The **geometry of deformation** is evaluated in order to account for the interaction between the torsion members.

Step 3: The relationship between the **torque and the deformations** are expressed by using equation (5.14).

Step 4: A **compatibility equation** is set up substituting the geometry-deformation equations of Step 2 into the torque-deformation equations of Step 3.

Step 5: Solve the equations of equilibrium from Step 1 and the compatibility equations from Step 4 simultaneously to calculate the unknown internal torques and deformations.

Typical statically indeterminate torsion member configurations are given below:

5.7.1 Coaxial Torsion Members

For coaxial torsion members one member is inside another, see figure 5.6. We assume they rotate with each other as one member.

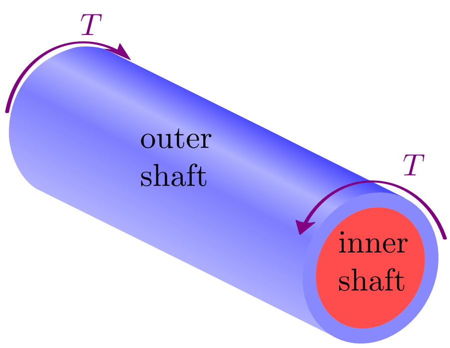

Figure 5.6: Torsion member coaxially connected.

The angle of twist in each member is the same however the total torque is shared between the two members.

$$\boxed{\phi_{\text{inner}} = \phi_{\text{outer}}} \tag{5.19}$$

and

$$\boxed{T = T_{\text{inner}} + T_{\text{outer}}} \tag{5.20}$$

5.7.2 Torsion Members in Series

Torsion members connected in series are connected end to end.

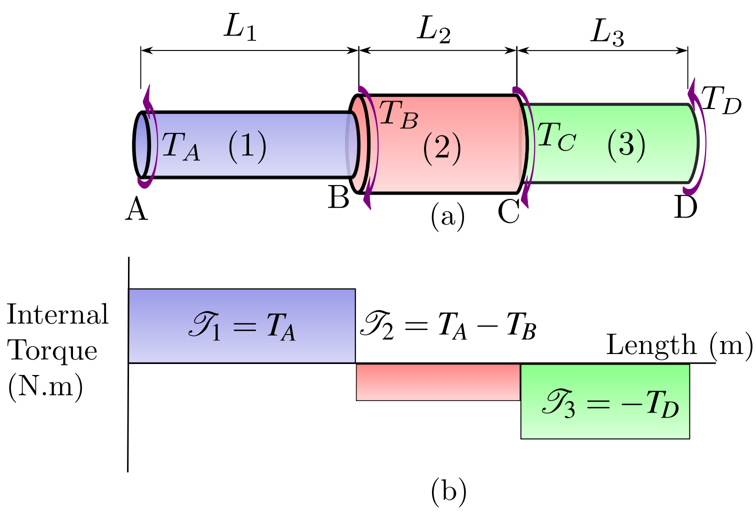

Figure 5.7: Torsion members connected in series

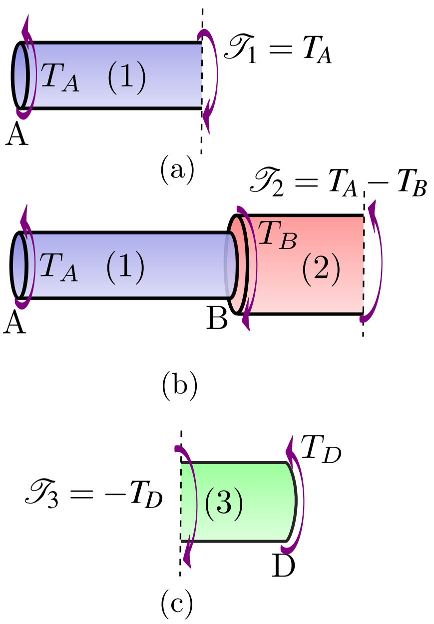

Figure 5.8: Free body diagrams of various sections of a member in torsion

To determine the internal torque in each section free body diagrams of each section would have to be considered, see figure 5.8.

The total angle of twist would be the sum of the angles of twist in each section:

$$\boxed{\phi_{\text{total}} = \sum \phi_i} \tag{5.21}$$

The difference to equation (5.15) is that not all the torques are necessarily known.

If both ends are *rigidly supported*, the total angle of twist would be zero.

$$\boxed{0 = \sum \phi_i} \tag{5.22}$$