7. Bending Stress

Tree blown over by strong wind. A knowledge of where the maximum bending stress will occurs give us an idea of why trees when blown over come off by the root and don't snap at the trunk.

7.1 Introduction

In the previous chapter we saw how loads acting on beams create internal bending moments and shear forces (otherwise known as stress resultants) within a beam. In this section the stresses and strains associated with those stress resultants are investigated. Knowledge of these stresses and strains allow a variety of loading conditions and support types to be analysed.
In this section only beams with a **longitudinal plane of symmetry** will be considered. The loading, support conditions, and member cross section are symmetric with respect to the longitudinal plane.

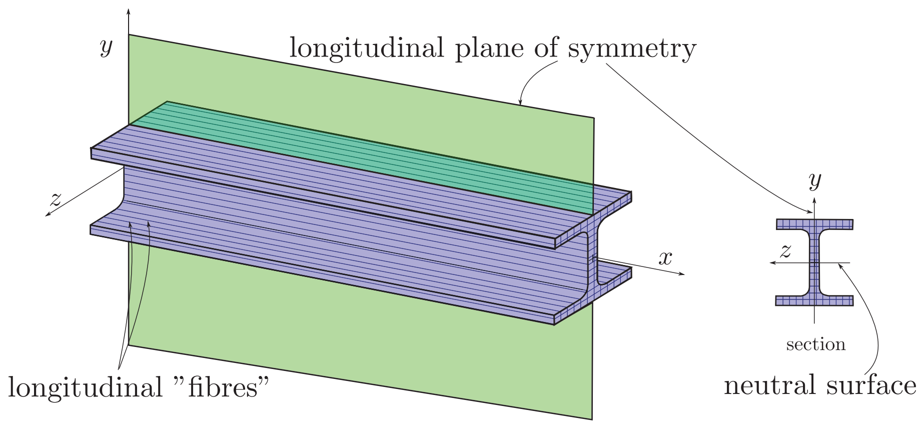

Figure 7.1: Longitudinal fibres shown with longitudinal plane of symmetry

The notion of longitudinal "fibres" is introduced which run parallel to the longitudinal axis of the beam. The usual coordinate axis scheme is the y-axis vertically upwards and the x-axis running along the longitudinal axis.

7.2 Flexural Strains

Consider a beam in **pure bending** where there is no transverse shear force, $\mathscr{V} = \frac{d\mathscr{M}}{dx} = 0$. On the top surface the longitudinal "fibres"are in compression and on the lower surface the longitudinal fibres are in tension. There is a surface which is not elongated or shortened and is call the **neutral surface**. The intersection of the neutral surface with any cross section of the beam is called the **neutral axis**.

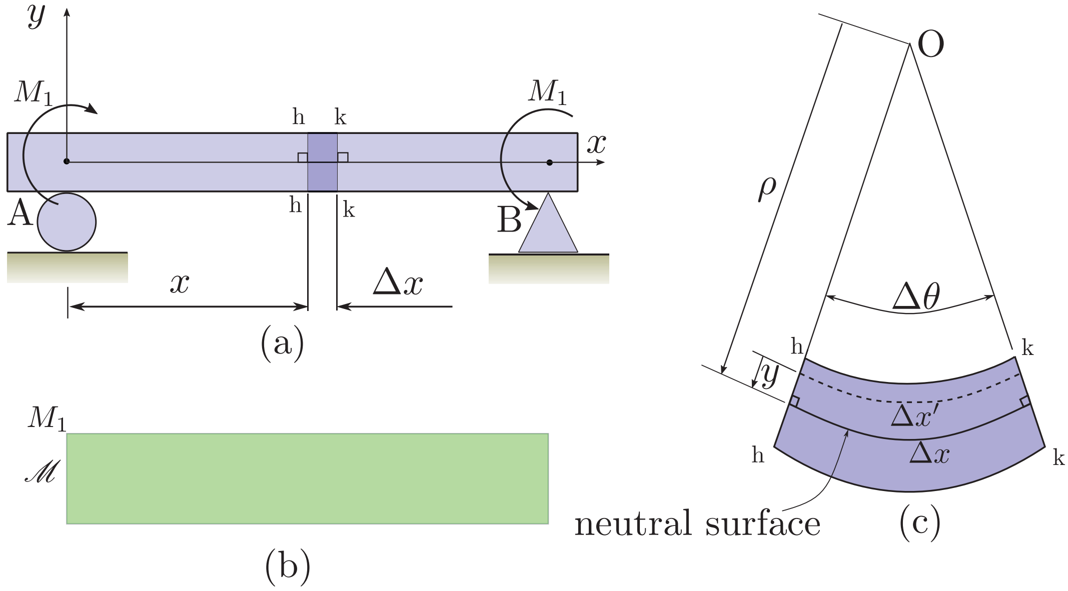

Figure 7.2: Beam in pure bending (a) with zero shear force and constant bending moment (b). The flexural deformation is shown in (c)

When subject to pure bending the beam deforms to the shape of a circular arc as can be seen in Figure 7.2(c). The centre of this arc is labelled O the **centre of curvature**. The radius of the arc is called the **radius of curvature** which is denoted with a Greek letter ρ (rho). Consider a longitudinal fibre a distance y above the neutral surface. It has length Δx before bending and $\Delta x'$ after bending. The strain can be related to shortening of Δx by the equation:

$$\varepsilon_x = \frac{\delta}{L} = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x}$$

If the interior angle is denoted $\Delta\theta$ then arc length Δx and $\Delta x'$ can be related to the radius of curvature ρ:

$$\varepsilon_x = \frac{\delta}{L} = \lim_{\Delta x \to 0} \frac{\Delta x' - \Delta x}{\Delta x} = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta - \rho\Delta\theta}{\rho\Delta\theta} \Rightarrow$$

$$\boxed{\varepsilon_x = -\frac{y}{\rho}} \tag{7.1}$$

The normal strain is proportional to the distance from the neutral axis. The signs indicate for a negative compressive strain a positive value of y will result and for a positive tensile value of strain a negative value will be given.
Curvature is denoted with the Greek letter κ (kappa):

$$\boxed{\kappa = \frac{1}{\rho}} \tag{7.2}$$

Radius and curvature are both positive if the centre of curvature O is above the beam or in positive y-direction. Conversely they are both negative if the center is below the beam or has a negative y-coordinate

7.3 Normal Stresses

The normal stress in a beam can be determined by considering the stress-strain relationship or Hooke's Law($\sigma = E\varepsilon$). An important assumption before this is that the strains are elastic and the material does not yield.
The variation of stress in a beam is then substituting Hooke's Law $\sigma = \varepsilon E$ in into equation (7.1):

$$\boxed{\sigma_x = -\frac{E}{\rho}y = -E\kappa y} \tag{7.3}$$

The stress therefore varies linearly with the distance from the neutral axis. This can be seen in Figure 7.3 with zero normal bending stress at the neutral axis.

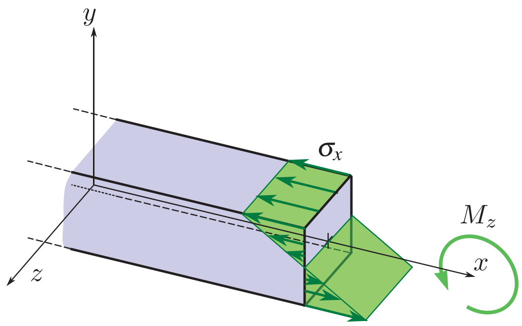

Figure 7.3: Normal stress distribution in a beam. The arrows pointing back at the beam indicate compression whereas the arrows pointing away from the beam indicate tension.

If a beam is subject to pure bending with no resultant force in the x-direction then the internal resultant moment acting about the z-axis can be related to the normal stresses in the x-direction and can be used to calculate the position of the neutral axis.

7.3.1 Neutral Axis Location

The neutral axis is parallel to the longitudinal axis of a beam or the x-axis in figure 7.3. There is no strain due to bending on the neutral axis.

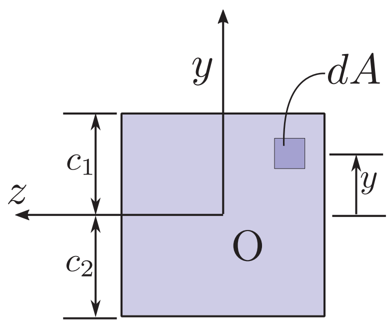

Figure 7.4: Normal stress distribution in a beam. The arrows pointing back at the beam indicate compression whereas the arrows pointing away from the beam indicate tension.

Consider a small element on a homogeneous beam which has area dA and is distance y from the the neutral surface in figure 7.4. The resultant force acting on this element is dF given by $\sigma_x dA$. In to order satisfy static equilibrium for the beam the sum of all these resultants should equal zero:

$$\sum F_x = \int dF = \int_A \sigma_x dA = 0$$

substitute equation (7.3) gives:

$$\sum F_x = \int_A \sigma_x dA = \int_A -\frac{E}{\rho} y dA = -\frac{E}{\rho}\int_A y dA = 0$$

For a solid material the elastic modulus E does not equal zero and if the radius of curvature was infinity the beam would not bend at all which implies:

$$\boxed{\int_A y dA = 0} \tag{7.4}$$

This means the **first moment of area** with respect to the z-axis must be zero. Recall from your course in statics that the distance of the centroid with respect to the x-axis includes the first moment of area:

$$\boxed{\bar{y} = \frac{\int_A y dA}{\int_A dA}} \tag{7.5}$$

Substitution of Equation (7.5) into Equation (7.4) shows that the distance from the neutral surface to the centroid must be zero. In other words the **neutral axis must pass through the centroid**.

7.3.2 Moment-Curvature Relationship

The second equation of equilibrium to be satisfied is the moment equilibrium. The internal bending moment M equals the moment resultant of the bending stresses σ_x acting over the cross section. The element of force $\sigma_x dA$ is positive when σ_x is positive and acts in the positive direction of the axis. Similarly $\sigma_x dA$ is negative when it acts in the negative $x-$direction and places the section in compression.

$$\sum M_z = -\int_A y\sigma_x dA - \mathscr{M} = 0$$

Subsitute equation (7.3) for σ_x gives

$$\mathscr{M} = -\int_A y\sigma_x dA = -\frac{E}{\rho}\int_A y^2 dA \tag{7.6}$$

The integral term in this equation is called the **second moment of inertia**

$$\boxed{I_z = \int_A y^2 dA} \tag{7.7}$$

The subscript z indicates the area moment of inertia is taken with respect to the z- axis. The integral term can be replaced by the moment of inertia in equation (7.6):

$$\boxed{\kappa = \frac{1}{\rho} = \frac{\mathscr{M}}{EI_z}} \tag{7.8}$$

This is called the moment-curvature relationship and show that the beam curvature is directly related to the bending moment and inversely related to the quantity EI_z which is called the **flexural rigidity**. The flexural rigidity is a measure of the bending resistance.

7.3.3 Flexure Formula

The relationship between the normal stress σ_x and the curvature from equation (7.3) can be substituted into the moment curvature relationship to give:

$$\boxed{\sigma_x = -\frac{\mathscr{M} y}{I_z}} \tag{7.9}$$

This is known as the **elastic flexure formula** or the **flexure formula**. The normal stresses produced by bending are called the **bending stresses** or **flexural stresses**.

7.4 Bending of Composite Beams

Beams that are composed two or more materials are called **composite beams**. Examples include wooden beams with steel reinforcing plates and concrete reinforced with steel bars. Using composite beams allows the stronger material to be used more efficiently to support loads.
The flexure formula, equation (7.9) was for homogeneous beams of one material. Modifications are required to make it applicable for composite beams. The composite beam will be changed into an equivalent cross section that consists of a single material. The dimensions of the beam with an equivalent cross section can then be used in the flexure formula to calculate the bending stresses.
The cross sections still remain plane during bending and the strains vary linearly through the beam cross section as expressed in equation (7.1)

$$\boxed{\varepsilon_x = -\frac{y}{\rho}} \tag{7.10}$$

Consider a beam section of two materials where material (2) is stiffer than material (1). In other words $E_2 > E_1$. The force transmitted by an area element dA is given by:

$$dF = \sigma_x dA = (E_2 \varepsilon_x)\, dydz$$

Here the stresses are transformed to strains with Hooke's Law. Now consider a transformed section which replaces Material (2) with Material (1). The distribution of strain should be the same in these cross sections so the y-dimension must be the same. The width in the z-dimension can be changed. Since Material (2) is stiffer, more of Material (1) will be required to replace Material (2).
Let the equivalent amount of Material (1) have area dA' with height dy and width ndz.

$$dF' = \sigma_x dA' = (E_1 \varepsilon_x)\, dy\, ndz$$

Since these two sections have to transmit the same amount of forces $dF = dF'$

$$(E_2 \varepsilon_x)\, dydz = (E_1 \varepsilon_x)\, dy\, ndz$$

It follows that:

$$\boxed{n = \frac{E_2}{E_1}} \tag{7.11}$$

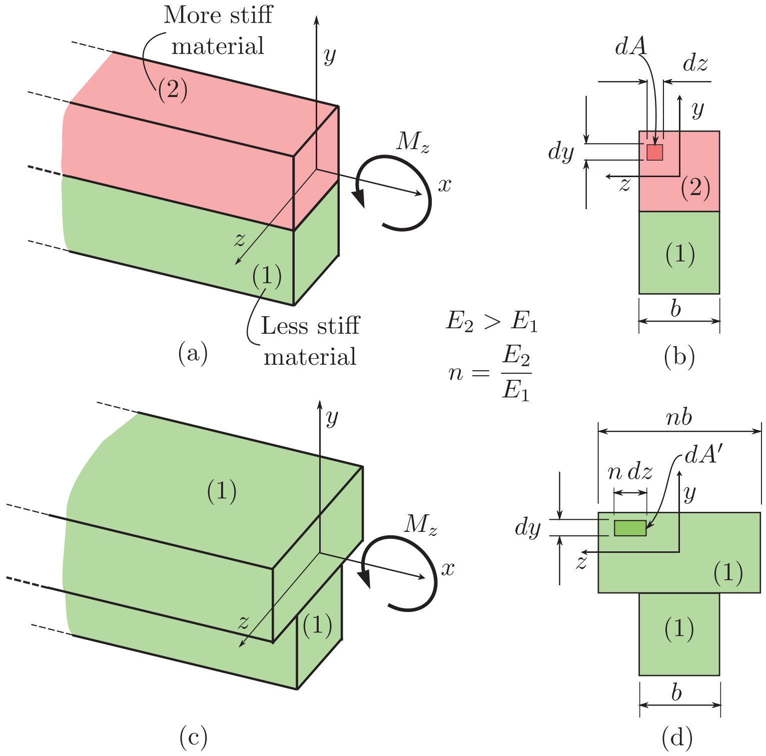

Figure 7.5: Illustration of bending stresses in composite beams (c) and (d) show how the more stiff Material (2) is transformed into the less stiff Material (1) keeping the strain distribution the same.

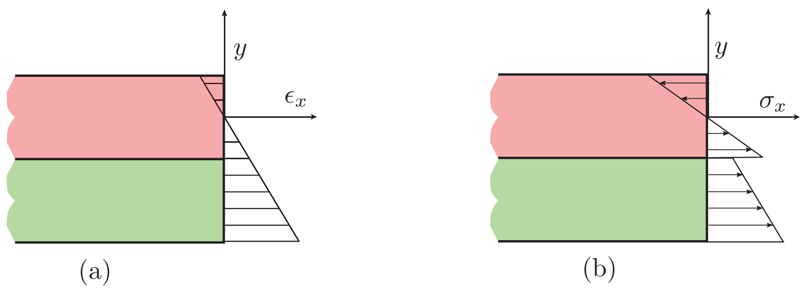

Figure 7.6: Distribution of strain (a) and stress (b) in a composite beam.

The ration n is called the **modular ratio**. The new material (1) is called the **transformed section**. The normal stresses can be expressed in terms of the radius of curvature ρ as:

$$\boxed{\sigma_{x1} = -\frac{E_1}{\rho}y \quad ; \quad \sigma_{x2} = -\frac{E_2}{\rho}y} \tag{7.12}$$

Here σ_{x1} and σ_{x2} are the stresses in Material (1) and Material (2) respectively. Now lets consider the force equilibrium on the composite beam.

$$\sum F_x = \int_A \sigma_x dA = \int_{A_1} \sigma_{x1} dA + \int_{A_2} \sigma_{x2} dA = 0$$

Substitute equation (7.12) to give:

$$-\int_{A_1} \frac{E_1}{\rho} y dA - \int_{A_2} \frac{E_2}{\rho} y dA = 0$$

The curvature is the same and can be cancelled out. The modular ratio can be substituted from equation (7.11) to give:

$$E_1 \int_{A_1} y dA + E_1 \int_{A_2} yn\, dA = 0$$

The area of a transformed cross section (A_t) can be expressed as:

$$\int_{A_t} dA = \int_{A_1} dA + \int_{A_2} n\, dA = 0$$

Combining the previous two equations gives:

$$\boxed{\int_{A_t} y dA = 0} \tag{7.13}$$

This means the *neutral axis passes through the centroid of the transformed cross section* not the centroid of the original section.
The moment relationship for the beam of two materials is:

$$\mathscr{M} = -\int_{A} y\sigma_x dA = -\int_{A_1} y\sigma_x dA - \int_{A_2} y\sigma_x dA$$

Substitute equation (7.11) and equation (7.12) gives

$$\mathscr{M} = \frac{E_1}{\rho}\left(\int_{A_1} y^2 dA + \int_{A_2} ny^2 dA\right)$$

The moment of inertia of a transformed section I_t is defined as:

$$\boxed{I_t = \int_{A_t} y^2 dA_t = \int_{A_1} y^2 dA + \int_{A_2} ny^2 dA} \tag{7.14}$$

The moment curvature relationship is then:

$$\boxed{\mathscr{M} = \frac{E_1 I_t}{\rho}} \tag{7.15}$$

The stresses in Material (1) can be expressed as:

$$\boxed{\sigma_{x1} = -\frac{\mathscr{M} y}{I_t}} \tag{7.16}$$

and the stresses in Material (2) bearing in mind that the section in transformed

$$\boxed{\sigma_{x2} = -n\frac{\mathscr{M} y}{I_t}} \tag{7.17}$$

■ **Example 7.1** A beam has maximum moment, $\mathscr{M}_{max} = 60\,\text{kN m}$. The cross section of the beam is a hollow box with wood flanges and steel side plates, as shown in figure 7.7. The wood flanges are 75 mm by 100 mm in cross section, and the steel plates are 300 mm deep. What is the required thickness t of the steel plates if the allowable stresses are 120 MPa for the steel and 6.5 MPa for

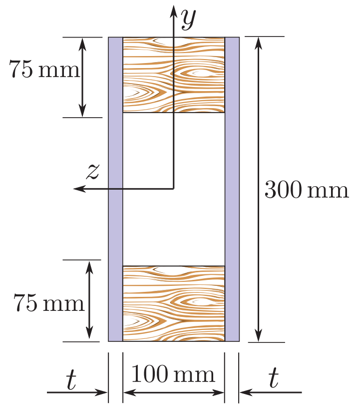

Figure 7.7: Composite beam section made of steel and wood.

the wood? (Assume that the moduli of elasticity for the steel and wood are 210 GPa and 10 GPa, respectively, and disregard the weight of the beam.)

Solution

Units mm, N, MPa

Maximum Moment at centre = $\mathscr{M}_{max} = A_y \times 3200/2 - 48 \times 3200^2/8 = 60 \times 10^6\,\text{N mm}$

$$\text{I}_{wood} = \frac{100 \times 300^3}{12} - \frac{100 \times 150^3}{12} = 196.8 \times 10^6\,\text{mm}^4$$

$$\text{I}_{eqsteel} = \frac{\mathscr{M}_{max} y_{steel}}{\sigma_{steel}} = \frac{60 \times 10^6\,(300/2)}{120} = 75.0 \times 10^6\,\text{mm}^4$$

$$\text{I}_{eqwood} = \frac{\mathscr{M}_{max} y_{wood}}{\sigma_{wood}} = \frac{60 \times 10^6\,(300/2)}{6.5} = 1.384 \times 10^9\,\text{mm}^4$$

$$\text{I}_{steel} = \text{I}_{eqsteel} - \frac{E_{wood}}{E_{steel}}\text{I}_{wood} = 65.625 \times 10^6\,\text{mm}^4$$

$$\text{I}_{steel-wood} = \left(I_{eqwood} - I_{wood}\right)\frac{E_{wood}}{E_{steel}} = 1.229 \times 10^8 \;\; mm^4$$

$$\text{I}_{steel-wood} = \frac{2t \times 300^3}{12} \Rightarrow t_{steel-wood} = 12.920\,\text{mm}$$

$$\text{I}_{steel} = \frac{2t \times 300^3}{12} \Rightarrow t_{steel} = 14.983\,\text{mm}$$ Steel strength limits - choose this one ■

7.5 Eccentric Loading

In a previous chapter loads acted through the centroid of the cross section and is called **centric loading**. This load creates a stress that is uniform over the cross section. If the load does not pass through the centroid of the section then there is an **eccentric axial load**. Additional bending stresses occur with the normal stresses.

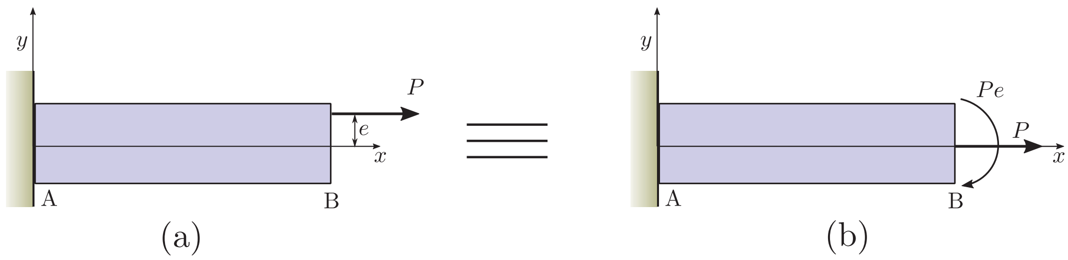

Figure 7.8: Cantilever beam with an eccentric load P and equivalent system

Consider an eccentric axial load P which acts a distance e (eccentricity of the load) away from the centroid as seen in figure 7.8. The eccentricity is measured in the positive y-direction.
The stresses are calculated by combining the normal stresses and the bending stresses:

$$\sigma_x = \frac{P}{A} - \frac{\mathscr{M} y}{I_z}$$

This system is statically equivalent to an axial force P through the centroid with an applied moment of magnitude Pe. Since the internal moment is the negative of the applied moment the expression is $-M = Pe$.

$$\boxed{\sigma_x = \frac{P}{A} + \frac{(Pe)\,y}{I_z}} \tag{7.18}$$

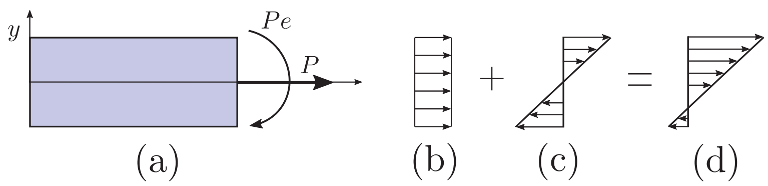

Figure 7.9: Stress distributions caused by an eccentric axial load.

In Figure 7.9 the applied moment and force is seen in (a). The uniform stress distribution is seen in (b) which is added to the stress from bending in (c) to give the complete distribution of stresses in (d).
The neutral axis will no longer be situated at the centre of the cross section. If the axial force is large enough relative to the bending stresses there will be no neutral axis on the structure. To obtain the location of the neutral axis set $\sigma_x = 0$ and solve for the distance from the centroid of the section.