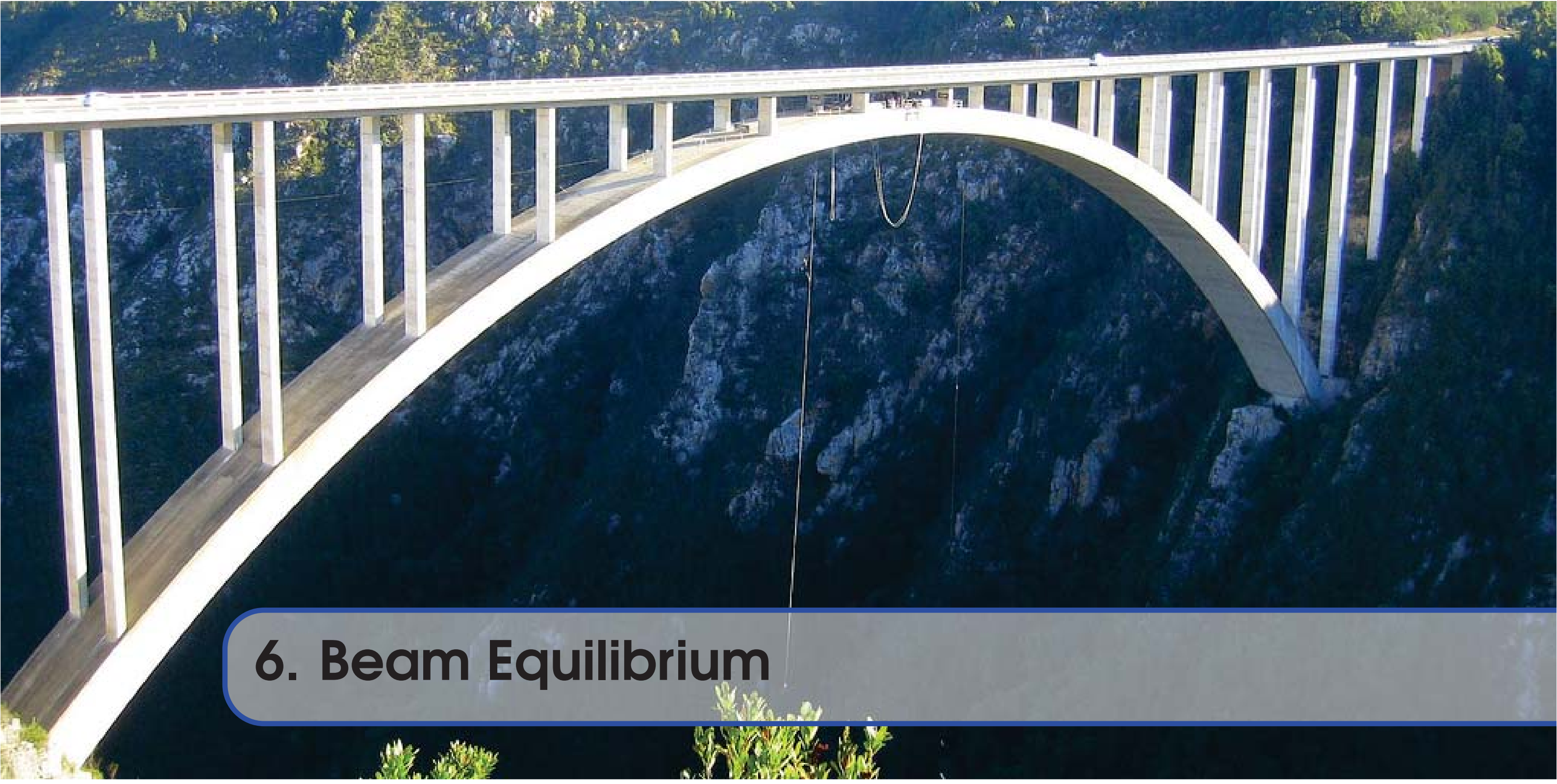


6. Beam Equilibrium

The van Stadens Bridge over the Fish River can be analysed with a knowledge of how beams are modelled with various loads

6.1 Introduction

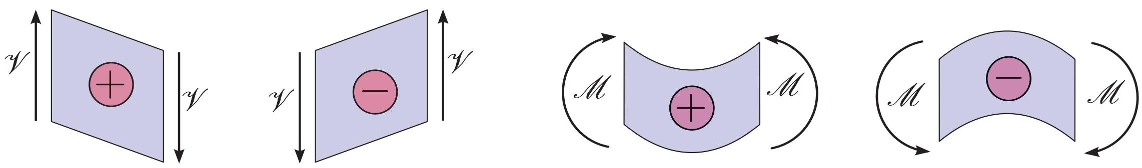
In the previous section we deal with loads that are directed along the axis of the structural members. In this section we deal with loads that are perpendicular to the structural members on which they act. The structural members are called **beams** and the loads are called transverse loads. Beams have one dimension, the length, much greater than either the width or the thickness.

6.2 Shear Force and Bending Moment Diagrams

In order to calculate the stresses due to bending in a beam, the internal shear forces, $\mathscr{V}$ and internal bending moments, $\mathscr{M}$ in the beam first have to be determined. The best way of doing this is by using bending moment and shear force diagrams.

6.2.1 Shear Force Sign and Bending Moment Conventions

The **deformation sign conventions** gives an indication of how a body deforms. In a previous section the axial force sign conventions indicated whether a body was in tension or compression. Bending deformation conventions are illustrated in figure 6.1.



(a) (b)

Figure 6.1: (a) Shear force sign convention and (b) moment sign convention shown on a beam element. A positive internal moment is a 'smile' and a negative internal moment is a 'frown'.

Positive internal shear force $\mathscr{V}$

- acts downward on the right-hand side of a beam.
- acts upward on the left-hand side of a beam.
- causes a beam element to rotate clockwise

Positive internal moment $\mathscr{M}$

- acts anticlockwise on the right-hand face of a beam.
- acts clockwise on the left-hand face of a beam.
- causes the beam element to go into compression on top and tension at the bottom

These sign conventions will be illustrated with examples in the following sections:

6.2.2 Concentrated Loads

■ **Example 6.1** Consider with a simply supported beam ABC with a concentrated or point load midway between A and C. The distance between A and C is L.

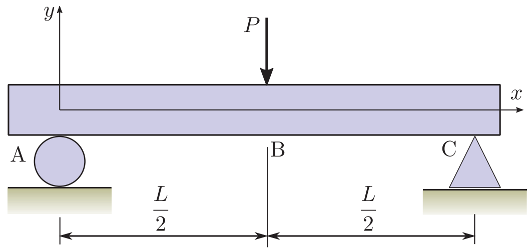


Figure 6.2: Simply supported beam with a concentrated load.

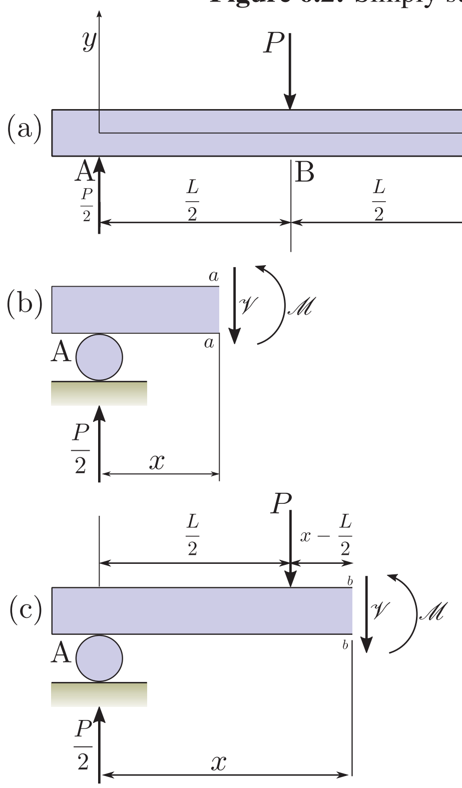


Figure 6.3: Depiction of (a) free body diagram (b) section view before concentrated load and (c) section view after concentrated load.

To start the solution we draw a free body diagram of the structure. This is shown in figure 6.3(a)

Support Reactions

$\Sigma M_A = 0 \Rightarrow C_y = \frac{1}{2}P$

$\Sigma F_y = 0 \Rightarrow A_y = \frac{1}{2}P$

Cut the diagram between places of interest and consider the interval between the cuts.

Interval $0 \le x < \frac{L}{2}$ (Section a-a), figure 6.3(b)

$\Sigma F_y = \frac{P}{2} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{P}{2}$

A positive shear force is assumed which acts downwards on the right-hand side of a beam. See figure 6.1(a) for the sign convention.

$\Sigma M_{a-a} = -\frac{P}{2}x + \mathscr{M} = 0 \Rightarrow \mathscr{M} = \frac{P}{2}x$

A positive moment is assumed force which acts anticlockwise on the right-hand face of a beam.

Interval $\frac{L}{2} \le x < L$ (Section b-b), figure 6.3(c)

$\Sigma F_y = \frac{P}{2} - P - \mathscr{V} = 0 \Rightarrow \mathscr{V} = -\frac{P}{2}$

$\Sigma M_{b-b} = P\left(x - \frac{L}{2}\right) - \frac{P}{2}x + \mathscr{M} = 0$

$\Rightarrow \mathscr{M} = -\frac{P}{2}x + \frac{PL}{2}$

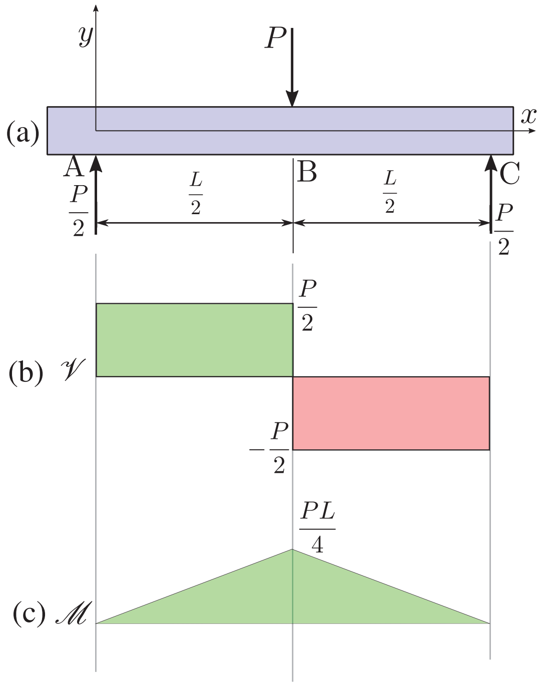


Figure 6.4: Construction of (b) shear force and (c) bending moment diagrams

Plotting the functions

Free body diagram replotted in Figure 6.4(a) repeated from figure 6.3(a)

Shear Force Diagram
On the interval $0 \leq x < \frac{L}{2}$ plot $\mathscr{V} = \frac{P}{2}$ and on the interval $\frac{L}{2} \leq x < L$ plot $\mathscr{V} = -\frac{P}{2}$ as shown in Figure 6.4(b).

Note the discontinuity at x = $\frac{L}{2}$ where the load P is located.

Bending Moment Diagram
On the interval $0 \leq x < \frac{L}{2}$ plot $\mathscr{M} = \frac{P}{2}x$ and on the interval $\frac{L}{2} \leq x < L$ plot $\mathscr{M} = -\frac{P}{2}x + \frac{PL}{2}$ as shown in Figure 6.4(c).

■

6.2.3 Applied Moments

■ **Example 6.2** Consider a simply supported beam ABC with a point moment midway between A and C. The distance between A and C is L.

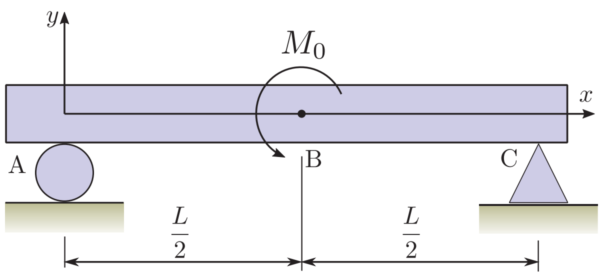


Figure 6.5: Simply supported beam with a point moment.

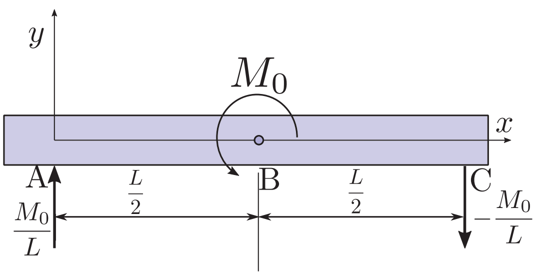


Figure 6.6: Free body diagram for simply supported structure with point moment midway

To start the a free body diagram of the structure is drawn. This is shown in figure 6.6

Determine reactions at supports:

$$\Sigma M_A = 0 \Rightarrow C_y = -\frac{M_0}{L}$$

$$\Sigma F_y = 0 \Rightarrow A_y = \frac{M_0}{L}$$

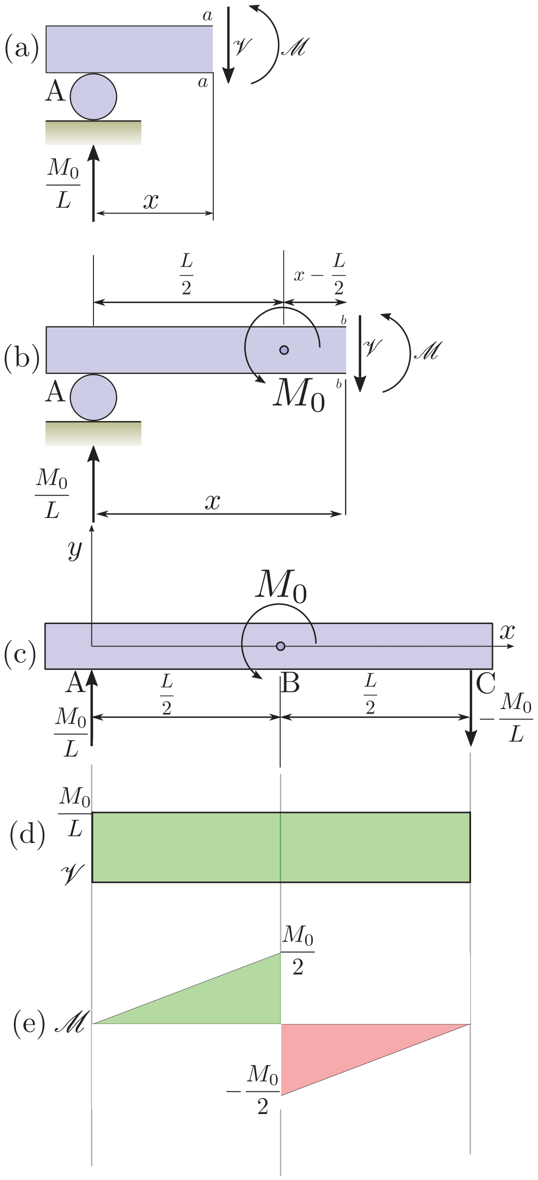


Figure 6.7: Construction of shear force and bending moment diagrams

Interval $0 \leq x < \frac{L}{2}$ (Section a-a), figure 6.7(a)

$$\Sigma F_y = \frac{M_0}{L} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{M_0}{L}$$

A positive shear force is assumed which acts downwards on the right-hand side of a beam.

$$\Sigma M_{a-a} = -\frac{M_0}{L}x + \mathscr{M} = 0 \Rightarrow \mathscr{M} = \frac{M_0}{L}x$$

A positive moment is assumed force which acts anticlockwise on the right-hand face of a beam.

Interval $\frac{L}{2} \leq x < L$ (Section b-b), figure 6.7(b)
Same as on section a-a since point moment does not affect load directly

$$\Sigma F_y = \frac{M_0}{L} - \mathscr{V} = 0 \Rightarrow \mathscr{V} = \frac{M_0}{L}$$

$$\Sigma M_{b-b} = -\frac{M_0}{L}x + M_0 + \mathscr{M} = 0$$

$$\Rightarrow \mathscr{M} = \frac{M_0}{L}x - M_0$$

Plotting the functions

Free body diagram replotted in figure 6.7(c)

Shear Force Diagram

On the interval $0 \leq x < \frac{L}{2}$ plot $V = \frac{M_0}{L}$ as shown in figure 6.7(d)

Bending Moment Diagram

On the interval $0 \leq x < \frac{L}{2}$ plot $\mathscr{M} = \frac{M_0}{L}x$ and on the interval $\frac{L}{2} \leq x < L$ plot $\mathscr{M} = \frac{M_0}{L}x - M_0$ as shown in Figure 6.7(e)

■

6.2.4 Uniformly Distributed Loads

■ **Example 6.3** A simply supported beam AB of length L has a distributed load w all the way between the supports.

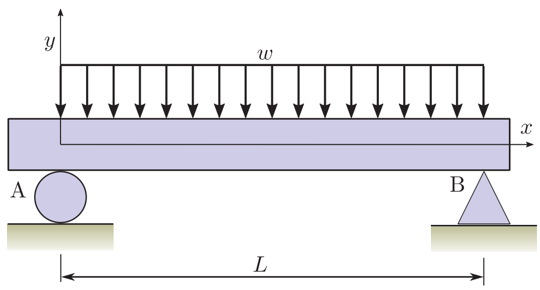


Figure 6.8: Uniformly distributed load of intensity w on simply supported beam.

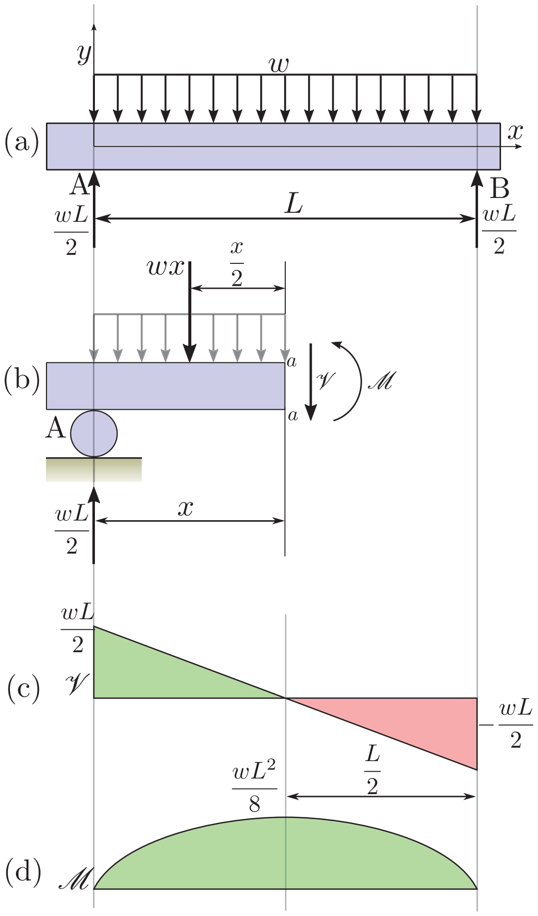


Figure 6.9: Construction of shear force and bending moment diagrams

Support Reactions
The beam is symmetrically loaded and supported therefore the load is shared equally in each support.

$\Sigma F_y = -wL + A_y + B_y = 0$

$\Rightarrow A_y = B_y = \dfrac{wL}{2}$

Interval $0 \le x < L$ (Section a-a), figure 6.9(b)

$\Sigma F_y = \dfrac{wL}{2} - wx - \mathscr{V} = 0$

$$\Rightarrow \mathscr{V} = \frac{wL}{2} - wx \qquad \text{(a)}$$

$$\Sigma M_{a-a} = -\frac{wL}{2}x + wx\frac{x}{2} + \mathscr{M} = 0$$

$$\Rightarrow \mathscr{M} = \frac{wL}{2}x - \frac{w}{2}x^2 \qquad \text{(b)}$$

Plotting the functions

Shear force diagram plotted from equation (a) and shown in in figure 6.9(c)

Moment diagram plotted from equation (b) and shown in in figure 6.9(d)

Note that the maximum bending moment occurs when the shear force equals zero.

■

6.3 Relationships between Loads, Shear forces and Bending Moments

In the previous section shear and moment diagrams were constructed by expressing the bending moment $\mathscr{M}(x)$ and shear force $\mathscr{V}(x)$ as functions of the beam length and plotting these functions. When a beam has several loadings this method can be quite laborious. In this section a simpler method is discussed based on the differential relationships between load and shear and then shear and moment.

6.3.1 Distributed Loads

In figure 6.10(a) and (b) it is important to note that **all the directions are shown as positive** according to the deformation sign convention. We look at a small element of the beam of length Δx. The internal resultant shear $\mathscr{V} + \Delta\mathscr{V}$ and moment $\mathscr{M} + \Delta\mathscr{M}$ are slightly different on the right hand side from the left to satisfy equilibrium. The resultant force $w\Delta x$ replaces the distributed load acting on the length Δx at a distance $\frac{\Delta x}{2}$ from the right side. Applying the equations of equilibrium to this element we get:

$$\Sigma F_y = \mathscr{V} - (\mathscr{V} + \Delta\mathscr{V}) + w(x)\Delta x = 0$$
$$\Rightarrow \Delta\mathscr{V} = w(x)\Delta x$$

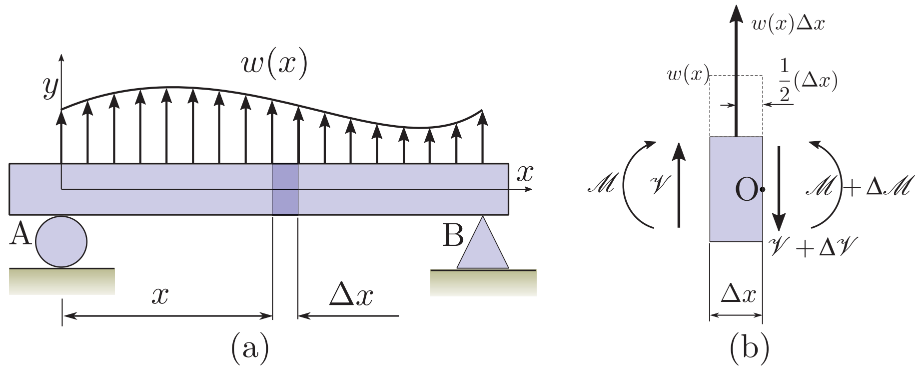


Figure 6.10: A beam subjected to distributed load (a) and a beam element showing internal shear forces and bending moments (b)

$$\Sigma M_O = -\mathscr{V}(\Delta x) - w(x)\Delta x\frac{\Delta x}{2} - \mathscr{M} + (\mathscr{M} + \Delta\mathscr{M}) = 0$$
$$\Rightarrow \Delta\mathscr{M} = \mathscr{V}(\Delta x) + w(x)\Delta x\frac{\Delta x}{2}$$

Dividing each term in each equation by Δx and taking the limit as $\Delta x \to 0$ gives:

$$\boxed{\frac{d\mathscr{V}}{dx} = w(x)} \tag{6.1}$$

$$\boxed{\frac{d\mathscr{M}}{dx} = \mathscr{V}(x)} \tag{6.2}$$

Equation (6.1) indicates that the load intensity $w(x)$ at any point x along a beam is numerically equal to slope of the shear force $\mathscr{V}(x)$ at that same point. In the same way Equation (6.2) shows the shear force $\mathscr{V}(x)$ at any point is equal to the slope of the internal $\mathscr{M}(x)$ at the same point. These equations can be used to quickly obtain the bending moment and shear force diagrams for a beam. Equations (6.1) can be rewritten in the form $dV = w(x)dx$. The term $w(x)dx$ represents the differential area under the distributed load diagram and can be integrated between any two limits x_1 and x_2 on the beam to give:

$$\boxed{\Delta\mathscr{V} = \int_{x_1}^{x_2} w(x)dx} \tag{6.3}$$

In a similar way Equation (6.2) can be rewritten in the form $d\mathscr{M} = \mathscr{V}(x)dx$ where the term $\mathscr{V}(x)dx$ represents the differential area under the shear force diagram. Integrating again between two limits x_1 and x_2 gives:

$$\boxed{\Delta\mathscr{M} = \int_{x_1}^{x_2} \mathscr{V}(x)dx} \tag{6.4}$$

6.3.2 Concentrated Loads

Consider a concentrated or point load and moment. Free body diagrams are shown for beam elements subject to a point load and point moment in Figure 6.11.

Looking at the force equilibrium for Figure 6.11(b) gives:

$$\Sigma F_y = \mathscr{V} + F\,(\mathscr{V} + \Delta\mathscr{V}) = 0 \Rightarrow$$

$$\boxed{\Delta\mathscr{V} = F} \tag{6.5}$$

At the location of a positive external load the shear force diagram is discontinuous. The **shear-force diagram jumps by an amount equal to the applied load , F**.

Now consider a beam element subject to a point load M_O as shown in Figure 6.11(a). Taking moments about point O gives:

$$\Sigma M_O = -\mathscr{V}(\Delta x) + M_O - \mathscr{M} + (\mathscr{M} + \Delta\mathscr{M}) = 0 \Rightarrow$$

Consider the limit as $\Delta x \to 0$ gives:

$$\boxed{\Delta\mathscr{M} = -M_O} \tag{6.6}$$

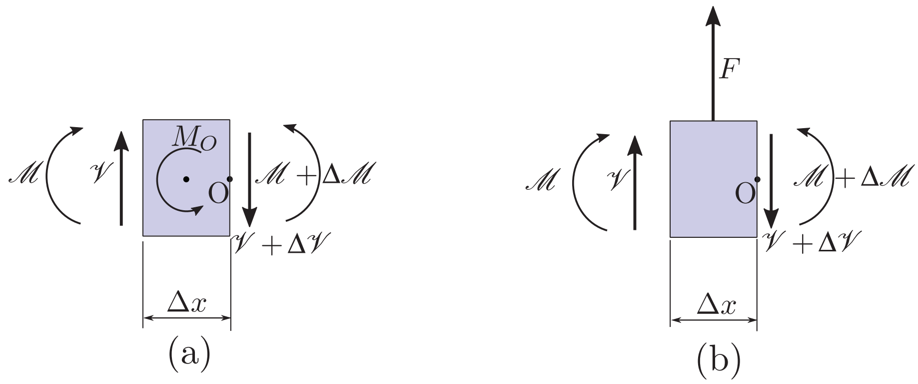


Figure 6.11: Beam elements of point moment (a) and concentrated load (b).

The ideas presented will be illustrated in the following example from Hibbeler[3].

■ **Example 6.4** A clamped beam has concentrated loads P L and $2L$ from the fixed end.

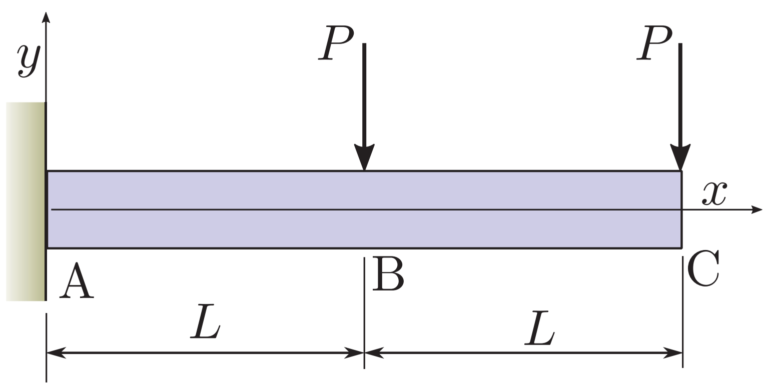


Figure 6.12: Cantilever beam with concentrated loads

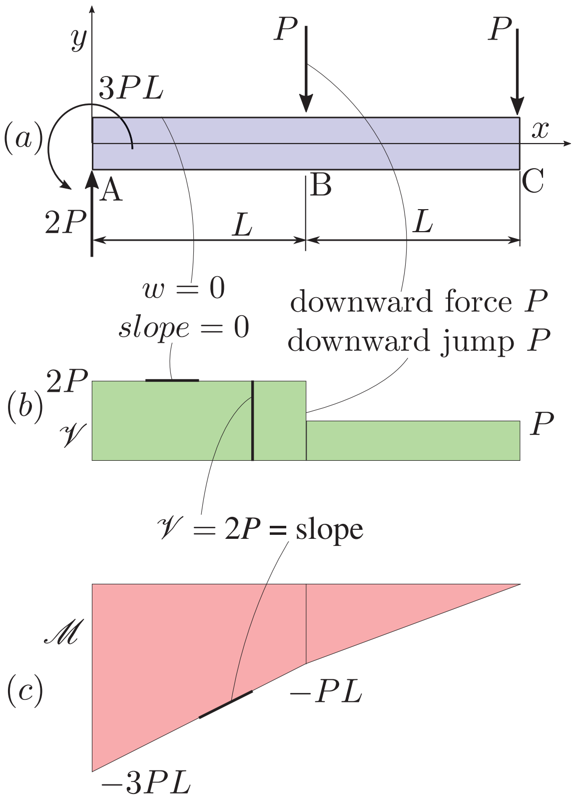


Figure 6.13: Construction of shear force and bending moment diagram

Support Reactions
Using force and moment equilibrium the reactions at A can be determined and are shown in Figure 6.13(a).

Shear Diagram
The shear at left end of the beam is plotted first, $2P$. It is positive by the deformation sign convention. The shear diagram on the extreme right end is positive P using the same convention. Since there is no distributed loading on the beam, the slope of the shear diagram is zero. The downward force P at the center of the beam causes the shear diagram to jump downward an amount P and can be seen in Figure 6.13(b).

Moment Diagram
This is shown in Figure 6.13(c). The moments at the left end of the beam is -3PL again negative because of the moment sign convention. Here the moment diagram consists of two sloping lines, one with a slope of +2P and the other with a slope of +P. The value of the moment in the center of the beam can be determined from the area under the shear diagram. If we choose the left half of the shear diagram:

$$\mathscr{M}|_{x=L} = \mathscr{M}|_{x=0} + \Delta\mathscr{M}$$

$$\Rightarrow \mathscr{M}|_{x=L} - 3PL + (2P)L = -PL$$

■

Another example is given here to illustrate how a uniformly distributed load can be dealt with.

■ **Example 6.5** a cantilver beam with a uniformly distributed load of length, L.

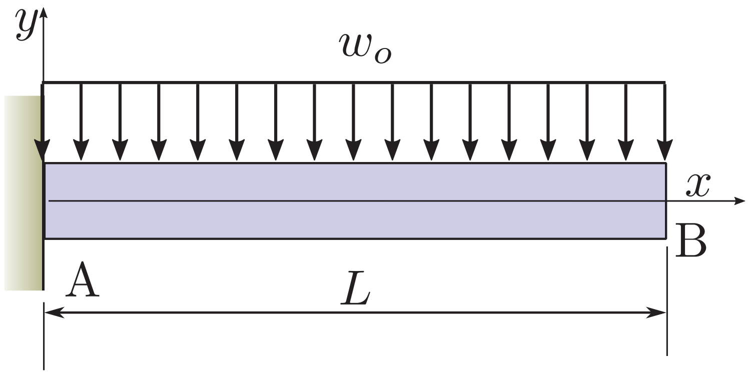


Figure 6.14: Cantilever beam with concentrated loads

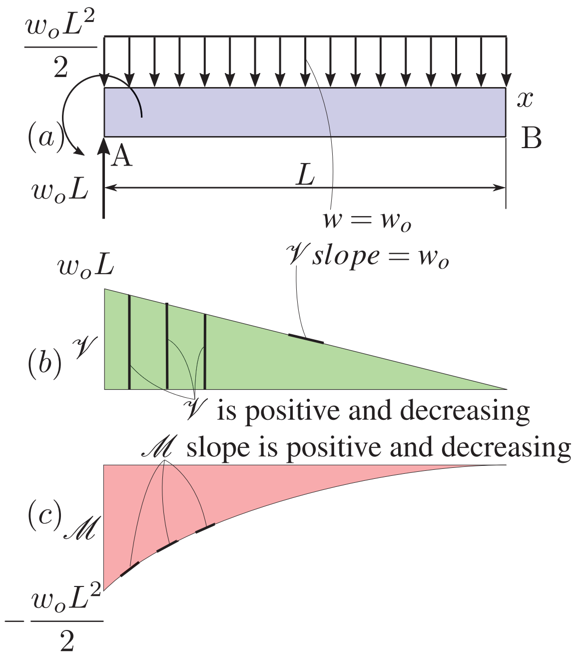


Figure 6.15: Construction of shear force and bending moment diagram

Support Reactions
Using force and moment equilibrium the reactions at A can be determined and are shown in Figure 6.15(a).

Shear Diagram
The shear at the left end is plotted first, $2w_oL$. It is positive by the deformation sign convention. At B the shear is zero. The slope of the shear diagram is $-w_o$ and can be seen in Figure 6.15(b).

Moment Diagram
This is shown in Figure 6.13(c). The moments at the left end of the beam is -$\frac{w_oL^2}{2}$ again negative because of the moment sign convention. At B the moment is zero. Various values of the shear at each point on the beam indicate the slope of the moment diagram at the point. Notice how this variation produces the curves shown.

■