4. Axial Loading

The portico at Jameson Hall has a overhang supported by columns under axial compression.

4.1 Introduction

Axial loading occurs along the longitudinal axis of a load bearing member. Examples include trusses, ropes, cables and columns. Axial stress differs from normal stress in that all axial stresses are normal stresses however not all normal stresses are axial stresses. You can get normal stress from bending and this will be discussed in a later chapter.

4.2 Saint-Venant's Principle

Consider a rectangular bar subject to a concentrated load P applied to the ends of the bar. Due to the loading there is an increased local stress concentration close to the force P. This stress becomes constant across the section as we move away from the load application point toward the midsection. Consider the largest dimension of the cross-section which is the width of the bar not the thickness. At a distance greater than the width from load point can the stress be considered uniform. See Figure 4.1 for the stress distributions at various cross-sections. This effect is independent of the distribution of the forces as long as the forces are statically equivalent. This is known as **Saint-Venant's Principle** [1].

Any loading can be replaced with another statically equivalent one for the purpose of simplifying the loading. This idea is also extended to fillets, grooves, holes or any feature that can cause a stress concentration. At a distance greater than the width from these features can the stress be considered uniform.

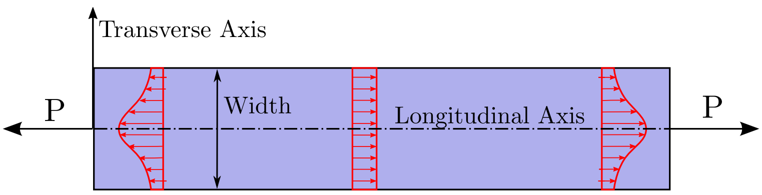

Figure 4.1: Rectangular bar subjected to concentrated force showing stress distributions at various cross sections.

[1] This is named after the French mathematician and engineer Adhémar Barré de Saint-Venant (1797 - 1886).

4.3 Changes in Lengths of Members under Axial Loads

When a member is under axial loading with an internal force $\mathscr{F}$, the axial strain is assumed to be constant. Assuming an isotropic, homogeneous material and a member which is prismatic (all cross-sections are the same) the equations 2.2, 2.4 and 2.6 combines to give:

$$\boxed{\delta = \frac{\mathscr{F}L}{AE}.} \tag{4.1}$$

If a member is subjected to axial loads at intermediate points (points other than at the ends) or the member consists of segment with different, areas and/or stiffnesses. The overall deflection can be determined as follows:

$$\boxed{\delta = \sum \frac{\mathscr{F}_i L_i}{A_i E_i}.} \tag{4.2}$$

Here $\mathscr{F}_i$, L_i, A_i and E_i are the internal force, the length, the cross sectional area, and the Young's Modulus of the i^{th} segment respectively.

■ **Example 4.1** Consider the stepped shaft shown in Figure 4.2. The shaft is manufactured from steel with a Young's Modulus of 200 GPa. The segment areas are $A_1 = 100\,\text{mm}^2$, $A_2 = 200\,\text{mm}^2$ and $A_3 = 150\,\text{mm}^2$. The applied forces are $P_1 = 10\,\text{kN}$, $P_2 = 15\,\text{kN}$ and $P_3 = 5\,\text{kN}$. The lengths are $L_1 = 240\,\text{mm}$, $L_2 = 200\,\text{mm}$ and $L_3 = 180\,\text{mm}$. Calculate the overall change in length.

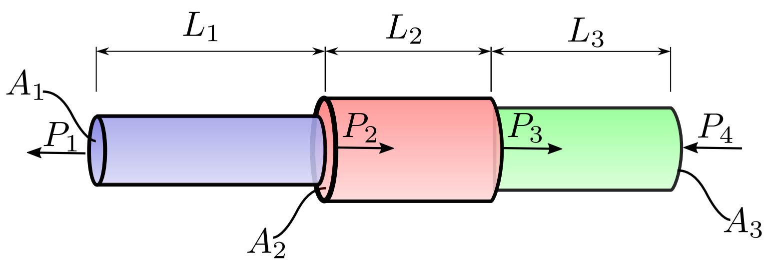

Figure 4.2: Elongation of an stepped shaft with 3 segments of varying diameters.

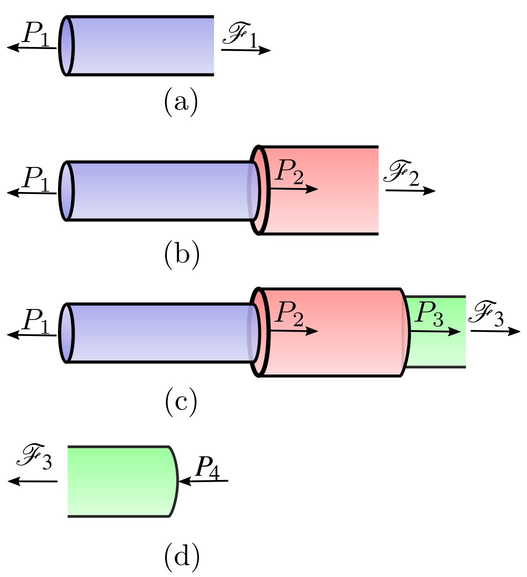

Figure 4.3: Construction of free body diagrams

To begin we draw free body diagrams for each section. Positive defined to the right and P_1 points to the left.

Figure 4.3(a)
$\sum F_x = 0 \Rightarrow \mathscr{F}_1 - P_1 = 0 \Rightarrow \mathscr{F}_1 = 10\,\text{kN}$

Figure 4.3(b)
$\sum F_x = 0 \Rightarrow \mathscr{F}_2 - P_1 + P_2 = 0$
$\Rightarrow \mathscr{F}_2 = 10 - 15 = -5\,\text{kN}$ compression

Figure 4.3(c)
$\sum F_x = 0 \Rightarrow \mathscr{F}_2 - P_1 + P_2 + P_3 = 0$
$\Rightarrow \mathscr{F}_3 = 10 - 15 - 5 = -10\,\text{kN}$ compression

To check Figure 4.3(d) First find P_4
$\sum F_x = 0 \Rightarrow -P_1 + P_2 + P_3 - P_4 = 0$
$\Rightarrow P_4 = -10 + 15 + 5 = 10\,\text{kN}$ right to left
$\sum F_x = 0 \Rightarrow -\mathscr{F}_3 - P_4 = 0$
$\Rightarrow \mathscr{F}_3 = -10\,\text{kN}$ compression

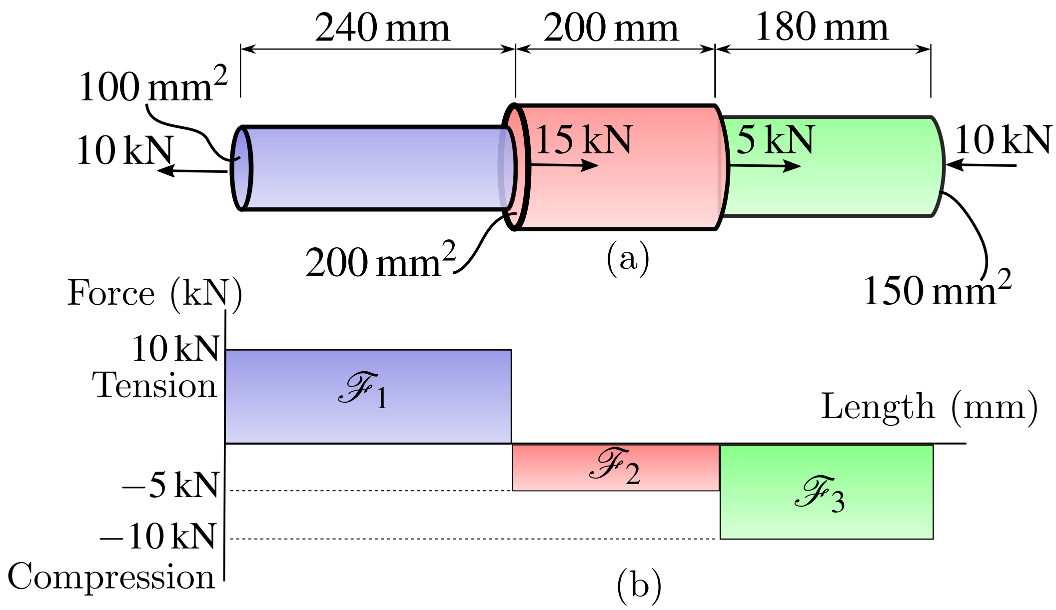

Figure 4.4: Illustration of (a) Compound bar with axial loading and (b) axial force diagram where P_i represents applied forces and F_i represents internal forces.

Now Calculate overall change in length
Units in kN, GPa and mm

$$\delta = \sum \frac{\mathscr{F}_i L_i}{A_i E_i} = \frac{(10)(240)}{(100)(200)} + \frac{(-5)(200)}{(200)(200)} + \frac{(-10)(180)}{(180)(200)} = 0.045\,\text{mm extension}$$ ■

4.4 Statically Indeterminate Axially Loaded Structures

In many structures all the reactions and loads can be determined by drawing free-body diagrams and solving the equilibrium equations. These structures are classified as **statically determinate**. In other structures these equilibrium equations are not sufficient for determining all the forces. We require additional equations from the geometry of the deformation to solve them.

We can use the following five step procedure[4]:

Step 1: Express all the **equations of equilibrium** for the structure in term of the unknown forces.

Step 2: The **geometry of deformation** is evaluated in order to account for the interaction between members. The geometry of deformation requires that the deflections δ are correct.

Step 3: The relationship between the **force and the deformations** are expressed by using Equation 4.2.

Step 4: A **compatibility equation** is set up substituting the geometry-deformation equations of Step 2 into the force-deformation equations of Step 3.

Step 5: Solve the equations of equilibrium from Step 1 and the compatibility equations from Step 4 simultaneously.

■ **Example 4.2** A central steel bolt with a diameter of 20 mm passes through a copper sleeve with 25 mm inside and 40 mm outside diameter. It has washers on both the nut and the bolt sides. The washers are rigid but can move when the nut is tightened. The nut is tightened until a stress of 12 MPa is set up in the steel. Calculate the stress in the copper sleeve. The whole assembly is then placed in a lathe and a cut is taken along half the length of the tube, removing the copper to a depth of 2.0 mm. Calculate the stress now existing in the steel and assume $E_{steel} = 2E_{copper}$.

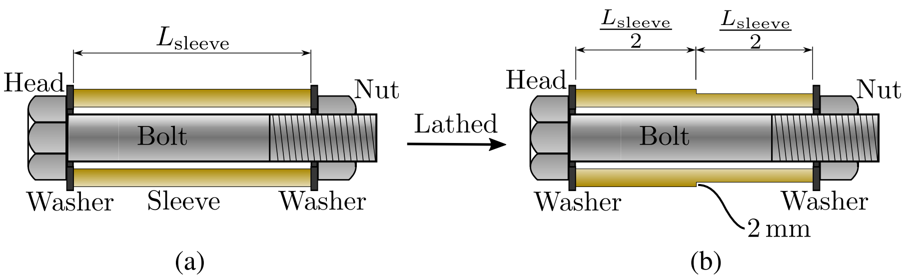

Figure 4.5: Bolt with sleeve (a) before being put on lathe and (b) after being put on lathe

Step 1: Equations of equilibrium
Before using lathe
Compression of copper = Tension in steel
$\sum F_x = 0 \Rightarrow F_{steel} + F_{copper} = 0 \Rightarrow \sigma_{steel}A_{steel} + \sigma_{copper}A_{copper} = 0$
$\Rightarrow \sigma_{copper} = \dfrac{\sigma_{steel}A_{steel}}{A_{copper}} = \dfrac{12\left(\pi 20^2/4\right)}{\pi\left(40^2-25^2\right)/4} = -4.923\,\text{MPa}$ compression

After using lathe forces in steel and both copper sections are the same:

$$\sigma_{steel-a}A_{steel} = -\sigma_{copper-r}A_{copper-r} = -\sigma_{copper-n}A_{copper} \tag{4.3}$$

Here $\sigma_{steel-a}$ is the steel bolt stress after lathing;
$\sigma_{copper-r}$ is the stress in the reduced copper section after lathing;
$A_{copper-r} = \pi\left((40-4)^2 - 25^2\right)/4$ is the area in the reduced copper section;
$\sigma_{copper-n}$ is the stress in the non-reduced copper section after lathing;
$A_{copper-n} = A_{copper} = \pi\left(40^2 - 25^2\right)/4$ is the area in the non-reduced copper section

Step 2: Geometry of Deformation
After using lathe the steel bolt loses tension and the copper sleeve becomes more compressed.

$$\text{Reduction in steel tension} = \text{Increase in copper compression}$$
$$\delta_{steel} - \delta_{steel-a} = \delta_{copper} - \delta_{copper-r} + \delta_{copper} - \delta_{copper-n} \tag{4.4}$$

Step 3: Force - Deformations relationships
Using equation (4.2) $\delta = \frac{F_iL_i}{A_iE_i}$ and equation (2.2) $\sigma_{avg} = \dfrac{F}{A}$ the following equations are obtained with appropriate subscript changes:

$$\begin{gathered}\delta_{steel} = \frac{\sigma_{steel}L_{steel}}{E_{steel}} \qquad \delta_{steel-a} = \frac{\sigma_{steel-a}L_{steel}}{E_{steel}} \qquad \delta_{copper} = \frac{\sigma_{copper}L_{copper}}{E_{copper}} \\ \delta_{copper-a} = \frac{\sigma_{copper-a}L_{copper}/2}{E_{copper}} \qquad \delta_{copper-n} = \frac{\sigma_{copper-n}L_{copper}/2}{E_{copper}}\end{gathered} \tag{4.5}$$

Step 4: Compatibility Equations
Substitute equations (4.5) into equation (4.4) to give
$$(\sigma_{steel} - \sigma_{steel-a})\frac{L_{steel}}{E_{steel}} = (\sigma_{copper} - \sigma_{copper-r})\frac{L_{copper}/2}{E_{copper}} + (\sigma_{copper} - \sigma_{copper-n})\frac{L_{copper}/2}{E_{copper}}$$

Step 5: Solve the equations
Assume $L_{copper} = L_{steel}$ and it is given $2E_{copper} = E_{steel}$. Substitute equation (4.3) and simplify to give:

$$\sigma_{steel} - \sigma_{steel-a} = \sigma_{copper} + \sigma_{steel-a}\frac{A_{steel}}{A_{copper-r}} + \sigma_{copper} + \sigma_{steel-a}\frac{A_{steel}}{A_{copper-n}}$$

$$\Rightarrow \sigma_{steel-a} = \frac{\sigma_{steel} - 2\sigma_{copper}}{\frac{A_{steel}}{A_{copper-r}} + \frac{A_{steel}}{A_{copper-n}} + 1} = 10.89\,\text{MPa}$$

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4.5 Temperature Effects

Most engineering materials when unrestrained contract when cooled and expand when heated. We characterise the size changes with the following equation:

$$\boxed{\varepsilon_T = \alpha \Delta T.} \tag{4.6}$$

Here ε_T is the strain due to thermal effects, Greek letter α (alpha) is the coefficient of thermal expansion in unit of $°C^{-1}$ and ΔT is temperature change in °C. The sign convention is that positive temperature changes cause positive thermal strains which will cause expansion of the material.
If a object has both thermal changes and an applied mechanical load we have the total strain ε_{total} which is:

$$\boxed{\varepsilon_{total} = \varepsilon_\sigma + \varepsilon_T.} \tag{4.7}$$

If an axial member has length L the resulting deformation δ_T from a temperature change is:

$$\boxed{\delta_T = \varepsilon_T L = \alpha \Delta T L.} \tag{4.8}$$

If the strain is restricted so that $\varepsilon_T = 0$ a stress will develop in the axial member which is:

$$\boxed{\sigma_T = E\alpha\Delta T} \tag{4.9}$$

These ideas are illustrated in the following example:

■ **Example 4.3** A solid steel rod of diameter 20 mm is placed concentrically in a 5 mm thick aluminium tube of outer diameter 50 mm. The rod and the tube are of the same length and are welded to rigid end plates. If the temperature of the assembly is raised by 90 °C, determine the stresses in the rod and the tube. Ignore the thermal expansion of the rigid end plates. State whether the stresses are tensile or compressive. Use E = 207 GPa and $\alpha = 11 \times 10^{-6}/°C$ for steel and E = 70 GPa and $\alpha = 23 \times 10^{-6}/°C$ for aluminium.

Solution
Step 1: Equations of equilibrium
Aluminium tube and steel rod have forces opposite to each other:

$$F_{rod} = -F_{tube} \tag{4.10}$$

Step 2: Geometry of Deformation

$$\text{Strain in Rod} = \text{Strain in tube}$$
$$\varepsilon_{rod} = \varepsilon_{tube}$$

using equation (4.7)
$$\Rightarrow \varepsilon_{rod_T} + \varepsilon_{rod_\sigma} = \varepsilon_{tube_T} + \varepsilon_{tube_\sigma} \tag{4.11}$$

Here using equation (4.6)
$\varepsilon_{rod_T} = \alpha_{rod}\Delta T = \left(11\times10^{-6}\right)(90) = 9.90\times10^{-4}$
$\varepsilon_{tube_T} = \alpha_{tube}\Delta T = \left(23\times10^{-6}\right)(90) = 0.002\,07 > \varepsilon_{rod_T}$
Therefore the steel rod is in tension and the tube is in compression.

Step 3: Force - deformations relationships
The strains due to mechanical loading are from equation (4.2) $\delta = \frac{F_iL_i}{A_iE_i}$ and equation (2.4) $\varepsilon_{avg} = \frac{\delta}{L}$

$$\varepsilon_{rod_\sigma} = \frac{F_{rod}}{A_{rod}E_{steel}} \qquad \varepsilon_{tube_\sigma} = \frac{F_{tube}}{A_{tube}E_{aluminium}} \tag{4.12}$$

Step 4: Compatibility Equations
Substitute equation (4.12) in equation (4.11)
$\dfrac{F_{rod}}{A_{rod}E_{steel}} - \dfrac{F_{tube}}{A_{tube}E_{aluminium}} = \varepsilon_{tube_T} - \varepsilon_{rod_T} = 0.0010800$

Step 5: Solve the equations
sub (1) $\Rightarrow F_{rod} = \dfrac{\left(\varepsilon_{tube_T} - \varepsilon_{rod_T}\right)}{\dfrac{1}{A_{tube}E_{aluminium}} + \dfrac{1}{A_{rod}E_{steel}}} = \dfrac{0.0010800}{\dfrac{1}{(25^2-20^2)\,\pi\,(70)} + \dfrac{1}{(10^2)\,\pi\,(207)}} = 30.348\,\text{kN}$
$\sigma_{rod} = \dfrac{F_{rod}}{A_{rod}} = \dfrac{30.348}{314.16} = 0.0966\,\text{GPa tension}$
$\sigma_{tube} = \dfrac{F_{tube}}{A_{tube}} = \dfrac{30.348}{706.86} = -0.042\,933\,\text{GPa compression}$

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