1^{∞}型结论运用

若$\lim\alpha(x)=0,\lim\beta(x)=\infty$,且$\lim\alpha(x)\beta(x)=A$,则有

$$\lim(1+\alpha(x))^{\beta(x)}=e^{A}$$

推导：借用重要极限的概念

$$\text{原式}=lim(1+\alpha(x))^{\beta(x)}=\lim[(1+\alpha)^{\frac{1}{\alpha}}]^{\alpha\beta}$$

$$\because\lim(1+\alpha)^{\frac{1}{\alpha}}=e\xRightarrow{\text{原式}\lim[(1+\alpha)^{\frac{1}{\alpha}}]^{\alpha\beta}}\lim e^{\alpha\beta}\xRightarrow{\text{又}\lim\alpha(x)\beta(x)=A}e^{A}$$

$$\therefore\lim(1+\alpha(x))^{\beta(x)}=e^{A}$$

使用方法 :

1.写成标准形式　原式 : $\lim[1+\alpha(x)]^{\beta(x)}$

2.求极限　$\lim\alpha(x)\beta(x)=A$

3.结果　　原式 $=e^{A}$

$eg1.\ \lim\limits_{n\to\infty}\frac{n^{n+1}}{(n+1)^{n}}\sin\frac{1}{n}$

思路：转变成$\lim(1+\alpha(x))^{\beta(x)}$的形式,且满足$\lim\alpha(x)=0,\lim\beta(x)=\infty.$

所以上述可以写为

$$\lim_{n\to\infty}n\cdot(\tfrac{n}{n+1})^{n}\cdot\sin\tfrac{1}{n}$$

$\because\lim\limits_{n\to\infty}(\frac{n}{n+1})^{n}=\lim\limits_{n\to\infty}(1+(-\frac{1}{n+1}))^{n}\xRightarrow{use\ conclusion}\lim\limits_{n\to\infty}-\frac{1}{n+1}\cdot n=-1$

$\therefore\lim\limits_{n\to\infty}(\frac{n}{n+1})^{n}=e^{-1}$

$and\ \lim\limits_{x\to\infty}n\cdot\sin\frac{1}{n}=\lim\limits_{x\to0}\frac{\sin n}{n}=1$

$\therefore\lim\limits_{n\to\infty}\frac{n^{n+1}}{(n+1)^{n}}\sin\frac{1}{n}=e^{-1}$

$eg2.\ \lim\limits_{x\to0^{+}}(\cos\sqrt{x})^{\frac{\pi}{x}}$

$=\lim\limits_{x\to0^{+}}((\cos\sqrt{x}-1)+1)^{\frac{\pi}{x}}$　　　accoring to above method

$\lim\alpha(x)\beta(x)=\lim\limits_{x\to0^{+}}(\cos\sqrt{x}-1)\cdot\frac{\pi}{x}=-\frac{\pi}{2}$　　　Taylor expansion or Replacement

$\therefore\lim\limits_{x\to0^{+}}(\cos\sqrt{x})^{\frac{\pi}{x}}=e^{-\frac{\pi}{2}}$

$eg3.\ \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^{n}$　　　$a,b,c>0$

$=\lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1+1)^{n}$

$\therefore\lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1)\cdot n$

$$= \lim_{x \to 0} \frac{1}{3}(\frac{a^n-1}{n} + \frac{b^n-1}{n} + \frac{c^n-1}{n})$$

$$= \frac{\ln a+\ln b+\ln c}{3} = \frac{1}{3}\ln abc$$

$$\therefore \lim_{x \to \infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^n = e^{\frac{1}{3}\ln abc} = \sqrt[3]{abc}$$