1^∞型结论运用

$eg1. \lim\limits_{n\to\infty} \frac{n^{n+1}}{(n+1)^n}\sin\frac{1}{n}$

思路：转变成$\lim(1+\alpha(x))^{\beta(x)}$的形式,且满足$\lim\alpha(x)=0,\lim\beta(x)=\infty.$

所以上述可以写为

$$\lim\limits_{n\to\infty} n\cdot(\frac{n}{n+1})^n\cdot\sin\frac{1}{n}$$

$\because \lim\limits_{n\to\infty}(\frac{n}{n+1})^n = \lim\limits_{n\to\infty}(1+(-\frac{1}{n+1}))^n \xRightarrow{use\ conclusion} \lim\limits_{n\to\infty} -\frac{1}{n+1}\cdot n = -1$

$\therefore \lim\limits_{n\to\infty}(\frac{n}{n+1})^n = e^{-1}$

$and\ \lim\limits_{x\to\infty} n\cdot\sin\frac{1}{n} = \lim\limits_{x\to 0}\frac{\sin n}{n} = 1$

$\therefore \lim\limits_{n\to\infty}\frac{n^{n+1}}{(n+1)^n}\sin\frac{1}{n} = e^{-1}$

$eg2. \lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{\pi}{x}}$

$= \lim\limits_{x\to 0^+}((\cos\sqrt{x}-1)+1)^{\frac{\pi}{x}}$ accoring to above method

$\lim\alpha(x)\beta(x) = \lim\limits_{x\to 0^+}(\cos\sqrt{x}-1)\cdot\frac{\pi}{x} = -\frac{\pi}{2}$ Taylor expansion or Replacement

$\therefore \lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{\pi}{x}} = e^{-\frac{\pi}{2}}$

$eg3. \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^n \qquad a,b,c>0$

$= \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1+1)^n$

$\therefore \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1)\cdot n$

$= \lim\limits_{x\to 0}\frac{1}{3}(\frac{a^n-1}{n}+\frac{b^n-1}{n}+\frac{c^n-1}{n})$

$= \frac{\ln a+\ln b+\ln c}{3} = \frac{1}{3}\ln abc$

$\therefore \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^n = e^{\frac{1}{3}\ln abc} = \sqrt[3]{abc}$

$eg4. \lim\limits_{x\to\frac{\pi}{4}}(\tan x)^{\frac{1}{\cos x-\sin x}}$

$= \lim\limits_{x\to\frac{\pi}{4}}(\tan x-1+1)^{\frac{1}{\cos x-sinx}}$

$\therefore \lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\cdot\frac{1}{\cos x-\sin x} = \lim\limits_{x\to\frac{\pi}{4}}(\tan x-1)\frac{1}{\cos x(1-tanx)} = \lim\limits_{x\to\frac{\pi}{4}} -\frac{1}{cosx} = -\sqrt{2}$

$\therefore \lim\limits_{x\to\frac{\pi}{4}}(tanx)^{\frac{1}{cosx-sinx}} = e^{-\sqrt{2}}$

$eg5.$

Equivalent substitution

$eg6.\lim\limits_{x\to\infty}(\frac{x^2}{(x-a)(x+b)})^x$
$=\lim\limits_{x\to\infty}(\frac{x}{x-a}\cdot\frac{x}{x+b})^x=\lim\limits_{x\to\infty}(\frac{x}{x-a})^x\cdot(\frac{x}{x+b})^x$
$=\lim\limits_{x\to\infty}(1-\frac{a}{x})^{-x}\cdot(1+\frac{b}{x})^{-x}$ twice conclusion
$=\lim\limits_{x\to\infty}e^a\cdot e^{-b}=e^{a-b}$

$eg7.\lim\limits_{x\to0}\frac{\sqrt{1+f(x)\sin 2x}-1}{e^{3x}-1}=2,\quad so\ \lim\limits_{x\to0}f(x)=?$
$\because e^{3x}-1\backsim 3x,\ (1+x)^\alpha-1\backsim\alpha x$ if x-> 0
$tips:\ x\ could\ be\ a\ func,\ like\ f(x), but\ precondition:\lim\limits_{x\to0}f(x)=0$
$\therefore\sqrt{1+f(x)sin2x}-1\backsim\frac{1}{2}f(x)sin2x\backsim\frac{1}{2}f(x)2x$
$so\lim\limits_{x\to0}\frac{\sqrt{1+f(x)\sin 2x}-1}{e^{3x}-1}=\lim\limits_{x\to0}\frac{\frac{1}{2}f(x)2x}{3x}=\lim\limits_{x\to0}\frac{f(x)x}{3x}=\lim\limits_{x\to0}\frac{f(x)}{3}=2$
$\therefore\lim\limits_{x\to0}f(x)=6$

$eg8.\lim\limits_{x\to0}\frac{\ln(\cos x)}{x^2}$
$=\lim\limits_{x\to0}\frac{\ln([cosx-1]+1)}{x^2}=\lim\limits_{x\to0}\frac{cosx-1}{x^2}=\lim\limits_{x\to0}\frac{-\frac{1}{2}x^2}{x^2}=-\frac{1}{2}$ $Replacement:\ ln(f(x)+1)\backsim f(x)\quad if\ \lim\limits_{x\to0}f(x)=0$

$eg9.\lim\limits_{x\to0}\frac{e-e^{cosx}}{\sqrt[3]{1+x^2}-1}$
$=\lim\limits_{x\to0}\frac{e(1-e^{cosx-1})}{\frac{1}{3}x^2}=e\cdot\lim\limits_{x\to0}\frac{1-e^{-\frac{1}{2}x^2}}{\frac{1}{3}x^2}=e\cdot\lim\limits_{x\to0}\frac{\frac{1}{2}x^2}{\frac{1}{3}x^2}=\frac{3e}{2}$
$\because(1+x^2)^{\frac{1}{3}}-1\backsim\frac{1}{3}x^2,\ e^x-1\backsim x,\ cosx-1\backsim\frac{1}{2}x^2$

$eg10.\lim\limits_{x\to0}\frac{1}{x^3}[(\frac{2+cosx}{3})^x-1]$
$=\lim\limits_{x\to0}\frac{1}{x^3}[xln(\frac{2+cosx}{3})]=\lim\limits_{x\to0}\frac{ln([\frac{(2+cosx)}{3}-1]+1)}{x^2}=\lim\limits_{x\to0}\frac{cosx-1}{3}\cdot\frac{1}{x^2}=-\frac{1}{6}$
$\because\alpha^x-1\backsim xlna,\ cosx-1\backsim-\frac{1}{2}x^2,\ ln(x+1)\backsim x$

$eg11.\lim\limits_{x\to0}\frac{arcsinx-sinx}{arctanx-tanx}$
$=\lim\limits_{x\to0}\frac{[arcsinx-x]-[sinx-x]}{[arctanx-x]-[tanx-x]}$

$eg12.\lim\limits_{x\to0}\frac{(1-cosx)[x-ln(1+tanx)]}{\sin^4 x}$
$=\lim\limits_{x\to0}\frac{\frac{1}{2}x^2[x-tanx]}{x^4}=\frac{\frac{1}{2}[x-tanx]}{x^2}$

$eg13.\lim\limits_{x\to0}[\frac{1}{x}-(\frac{1}{x}-a)e^x]=1,\qquad a=?$
$\lim\limits_{x\to0}[\frac{1-e^x}{x}+ae^x]=\lim\limits_{x\to0}-1+a=1$

$\therefore a = 2$

$eg14. \lim\limits_{x\to+\infty}[(ax+b)e^{\frac{1}{x}}-x]=2, \qquad a,b=?$

$=\lim\limits_{x\to0}[(\frac{a}{x}+b)e^x-\frac{1}{x}]=\lim\limits_{x\to0}[\frac{ae^x-1}{x}]+\lim\limits_{x\to0}be^x=1+b=2$

$\therefore a=b=1$

$According\ to\ basic\ four\ algorithm\ rule,$

$if\ \ \lim f(x)g(x)=a\ \&\&\exists\ \lim f(x), there\ must\ be\ \exists\ \lim g(x)$

$eg15. \lim\limits_{x\to-\infty}\frac{\sqrt{4x^2+x-1}+x+1}{\sqrt{x^2+sinx}}$

$=\lim\limits_{x\to-\infty}\frac{-x\cdot(\sqrt{4+\frac{1}{x}-\frac{1}{x^2}}-1-\frac{1}{x})}{-x\cdot\sqrt{1+\frac{sinx}{x^2}}}=\lim\limits_{x\to-\infty}\frac{2-1-0}{1}=1$

$the\ key\ point\ is\ eliminating\ the\ infty\ factor$

Law of Robida

$eg16: \lim\limits_{x\to1}(1-x^2)\tan\frac{\pi}{2}x \qquad type: 0\cdot\infty$

$=\lim\limits_{x\to1}(x+1)(x-1)\cdot\frac{sin\frac{\pi}{2}x}{cos\frac{\pi}{2}x} \qquad \because \lim\limits_{x\to1}(x+1)=2, \lim\limits_{x\to1}sin(\frac{\pi}{2}x)=1.$

$=\lim\limits_{x\to1}\frac{x-1}{cos\frac{\pi}{2}x}=\lim\limits_{x\to1}\frac{1}{-\frac{\pi}{2}\cdot sin\frac{\pi}{2}x}=-\frac{2}{\pi}$

$eg17. \lim\limits_{x\to1}\frac{ln(cosx-1)}{1-sin\frac{\pi}{2}x} \qquad type: \frac{0}{0}$

$=\lim\limits_{x\to1}\frac{-\tan(x-1)}{-\frac{\pi}{2}cos\frac{\pi}{2}x}=\lim\limits_{x\to1}\frac{-(x-1)}{-\frac{\pi}{2}cos\frac{\pi}{2}x} \qquad if\ x-1\to0,\ have\ tan(x-1)\backsim x-1$

$=-\frac{2}{\pi}\lim\limits_{x\to1}\frac{1}{\frac{\pi}{2}\cdot sin\frac{\pi}{2}x}=-\frac{4}{\pi^2}$

$eg18. \lim\limits_{x\to+\infty}(x+\sqrt{1+x^2})^{\frac{1}{x}} \qquad tpye: \infty^0$

$=\lim\limits_{x\to+\infty}e^{\frac{1}{x}ln(x+\sqrt{1+x^2})}=\lim\limits_{x\to+\infty}e^{\frac{\infty}{\infty}}=\lim\limits_{x\to+\infty}e^{\frac{1}{\sqrt{1+x^2}}}=e^0=1$

Taylor Expansion

$eg19. \lim\limits_{x\to0}\frac{\cos x-e^{-\frac{x^2}{2}}}{x^4}$

$\because \lim\limits_{x\to0}\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4+o(x^4), \quad \lim\limits_{x\to0}e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+o(x^4)$

$\therefore \lim\limits_{x\to0}\frac{\cos x-e^{-\frac{x^2}{2}}}{x^4}=-\frac{1}{12}$

$eg20. \lim\limits_{x\to0}\frac{\ln(1+x)-(ax+bx^2)}{x^2}=2 \qquad a,b=?$

$\because \lim_{x\to 0} \ln(1+x) = 0 + x + -\frac{1}{2}x^2 + o(x^2)$
$\therefore a = 1, b = -\frac{5}{2}$

$eg21. \lim_{x\to 0} \frac{\sin 6x + xf(x)}{x^3} = 0 \qquad \lim_{x\to 0} \frac{6+f(x)}{x^2} = ?$
$\because \lim_{x\to 0} \sin x = x - \frac{1}{6}x^3 + o(x^3)$
$\therefore \lim_{x\to 0} \frac{\sin 6x + xf(x)}{x^3} = \lim_{x\to 0} \frac{x(6+f(x)) - 36x^3}{x^3} = 0 \Rightarrow \lim_{x\to 0} \frac{6+f(x)}{x^2} - 36 = 0$
$\therefore \lim_{x\to 0} \frac{6+f(x)}{x^2} = 36$