1^{∞}型结论运用

若$\lim\alpha(x)=0,\lim\beta(x)=\infty$,且$\lim\alpha(x)\beta(x)=A$,则有

$$\lim(1+\alpha(x))^{\beta(x)}=e^A$$

推导：借用重要极限的概念

$$原式=lim(1+\alpha(x))^{\beta(x)}=\lim[(1+\alpha)^{\frac{1}{\alpha}}]^{\alpha\beta}$$

$$\because\lim(1+\alpha)^{\frac{1}{\alpha}}=e\xRightarrow{原式\lim[(1+\alpha)^{\frac{1}{\alpha}}]^{\alpha\beta}}\lim e^{\alpha\beta}\xRightarrow{又\lim\alpha(x)\beta(x)=A}e^A$$

$$\therefore\lim(1+\alpha(x))^{\beta(x)}=e^A$$

使用方法 :

1.写成标准形式　原式 : $\lim[1+\alpha(x)]^{\beta(x)}$

2.求极限　$\lim\alpha(x)\beta(x)=A$

3.结果　　原式 $=e^A$

$eg1.\ \lim\limits_{n\to\infty}\frac{n^{n+1}}{(n+1)^n}\sin\frac{1}{n}$

思路：转变成$\lim(1+\alpha(x))^{\beta(x)}$的形式,且满足$\lim\alpha(x)=0,\lim\beta(x)=\infty.$

所以上述可以写为

$$\lim_{n\to\infty}n\cdot(\frac{n}{n+1})^n\cdot\sin\frac{1}{n}$$

$$\because\lim_{n\to\infty}(\frac{n}{n+1})^n=\lim_{n\to\infty}(1+(-\frac{1}{n+1}))^n\xRightarrow{use\ conclusion}\lim_{n\to\infty}-\frac{1}{n+1}\cdot n=-1$$

$$\therefore\lim_{n\to\infty}(\frac{n}{n+1})^n=e^{-1}$$

$$and\ \lim_{x\to\infty}n\cdot\sin\frac{1}{n}=\lim_{x\to 0}\frac{\sin n}{n}=1$$

$$\therefore\lim_{n\to\infty}\frac{n^{n+1}}{(n+1)^n}\sin\frac{1}{n}=e^{-1}$$

$eg2.\ \lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{\pi}{x}}$

$=\lim\limits_{x\to 0^+}((\cos\sqrt{x}-1)+1)^{\frac{\pi}{x}}$　　　accoring to above method

$\lim\alpha(x)\beta(x)=\lim\limits_{x\to 0^+}(\cos\sqrt{x}-1)\cdot\frac{\pi}{x}=-\frac{\pi}{2}$　　Taylor expansion or Replacement

$\therefore\lim\limits_{x\to 0^+}(\cos\sqrt{x})^{\frac{\pi}{x}}=e^{-\frac{\pi}{2}}$

$eg3.\ \lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^n$　　$a,b,c>0$

$=\lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1+1)^n$

$\therefore\lim\limits_{x\to\infty}(\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3}-1)\cdot n$

$= \lim\limits_{x\to 0} \frac{1}{3}(\frac{a^n-1}{n} + \frac{b^n-1}{n} + \frac{c^n-1}{n})$
$= \frac{\ln a + \ln b + \ln c}{3} = \frac{1}{3}\ln abc$
$\therefore \lim\limits_{x\to\infty} (\frac{\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}}{3})^n = e^{\frac{1}{3}\ln abc} = \sqrt[3]{abc}$

$eg4. \lim\limits_{x\to\frac{\pi}{4}} (\tan x)^{\frac{1}{\cos x - \sin x}}$
$= \lim\limits_{x\to\frac{\pi}{4}} (\tan x - 1 + 1)^{\frac{1}{\cos x - sinx}}$
$\therefore \lim\limits_{x\to\frac{\pi}{4}} (\tan x - 1) \cdot \frac{1}{\cos x - \sin x} = \lim\limits_{x\to\frac{\pi}{4}} (\tan x - 1) \frac{1}{\cos x (1 - tanx)} = \lim\limits_{x\to\frac{\pi}{4}} -\frac{1}{cosx} = -\sqrt{2}$
$\therefore \lim\limits_{x\to\frac{\pi}{4}} (tanx)^{\frac{1}{cosx - sinx}} = e^{-\sqrt{2}}$

$eg5. \lim\limits_{x\to\infty} (\frac{x^2}{(x-a)+(x+b)})^x$
$= \lim\limits_{x\to\infty} (\frac{x^2}{(x-a)+(x+b)} - 1 + 1)^x$
$\therefore \lim\limits_{x\to\infty}$

$eg6. \lim\limits_{x\to\infty} (\frac{x^2}{(x-a)(x+b)})^x$
$= \lim\limits_{x\to\infty} (\frac{x}{x-a} \cdot \frac{x}{x+b})^x = \lim\limits_{x\to\infty} (\frac{x}{x-a})^x \cdot (\frac{x}{x+b})^x$
$= \lim\limits_{x\to\infty} (1 - \frac{a}{x})^{-x} \cdot (1 + \frac{b}{x})^{-x}$ twice conclusion
$= \lim\limits_{x\to\infty} e^a \cdot e^{-b} = e^{a-b}$

$eg7. \lim\limits_{x\to 0} \frac{\sqrt{1+f(x)\sin 2x} - 1}{e^{3x}-1} = 2, \quad so \lim\limits_{x\to 0} f(x) = ?$
$\because e^{3x} - 1 \backsim 3x,\ (1+x)^\alpha - 1 \backsim \alpha x$ if x-> 0
$tips:\ x\ could\ be\ a\ func,\ like\ f(x), but\ precondition: \lim\limits_{x\to 0} f(x) = 0$
$\therefore \sqrt{1+f(x)sin2x} - 1 \backsim \frac{1}{2}f(x)sin2x \backsim \frac{1}{2}f(x)2x$
$so \lim\limits_{x\to 0} \frac{\sqrt{1+f(x)\sin 2x} - 1}{e^{3x}-1} = \lim\limits_{x\to 0} \frac{\frac{1}{2}f(x)2x}{3x} = \lim\limits_{x\to 0} \frac{f(x)x}{3x} = \lim\limits_{x\to 0} \frac{f(x)}{3} = 2$
$\therefore \lim\limits_{x\to 0} f(x) = 6$

$eg8. \lim\limits_{x\to 0} \frac{\ln(\cos x)}{x^2}$
$= \lim\limits_{x\to 0} \frac{\ln([cosx-1]+1)}{x^2} = \lim\limits_{x\to 0} \frac{cosx-1}{x^2} = \lim\limits_{x\to 0} \frac{-\frac{1}{2}x^2}{x^2} = -\frac{1}{2}$ $Replacement:\ ln(f(x)+1) \backsim f(x) \quad if \lim\limits_{x\to 0} f(x) = 0$

$eg9. \lim\limits_{x\to 0} \frac{e - e^{cosx}}{\sqrt[3]{1+x^2} - 1}$
$= \lim\limits_{x\to 0} \frac{e(1-e^{cosx-1})}{\frac{1}{3}x^2} = e \cdot \lim\limits_{x\to 0} \frac{1-e^{-\frac{1}{2}x^2}}{\frac{1}{3}x^2} = e \cdot \lim\limits_{x\to 0} \frac{\frac{1}{2}x^2}{\frac{1}{3}x^2} = \frac{3e}{2}$
$\because (1+x^2)^{\frac{1}{3}} - 1 \backsim \frac{1}{3}x^2,\ e^x - 1 \backsim x,\ cosx - 1 \backsim \frac{1}{2}x^2$

$eg10. \lim\limits_{x\to 0}\frac{1}{x^3}[(\frac{2+cosx}{3})^x-1]$

$=\lim\limits_{x\to 0}\frac{1}{x^3}[xln(\frac{2+cosx}{3})]=\lim\limits_{x\to 0}\frac{ln([\frac{(2+cosx)}{3}-1]+1)}{x^2}=\lim\limits_{x\to 0}\frac{cosx-1}{3}\cdot\frac{1}{x^2}=-\frac{1}{6}$

$\because \alpha^x-1\backsim xlna,\ cosx-1\backsim -\frac{1}{2}x^2,\ ln(x+1)\backsim x$

$eg11. \lim\limits_{x\to 0}\frac{arcsinx-sinx}{arctanx-tanx}$

$=\lim\limits_{x\to 0}\frac{[arcsinx-x]-[sinx-x]}{[arctanx-x]-[tanx-x]}$

$eg12. \lim\limits_{x\to 0}\frac{(1-cosx)[x-ln(1+tanx)]}{\sin^4 x}$

$=\lim\limits_{x\to 0}\frac{\frac{1}{2}x^2[x-tanx]}{x^4}=\frac{\frac{1}{2}[x-tanx]}{x^2}$

$eg13. \lim\limits_{x\to 0}[\frac{1}{x}-(\frac{1}{x}-a)e^x]=1,\qquad a=?$

$\lim\limits_{x\to 0}[\frac{1-e^x}{x}+ae^x]=\lim\limits_{x\to 0}-1+a=1$

$\qquad\therefore a=2$

$eg14. \lim\limits_{x\to +\infty}[(ax+b)e^{\frac{1}{x}}-x]=2,\qquad a,b=?$

$=\lim\limits_{x\to 0}[(\frac{a}{x}+b)e^x-\frac{1}{x}]=\lim\limits_{x\to 0}[\frac{ae^x-1}{x}]+\lim\limits_{x\to 0}be^x=1+b=2$

$\therefore a=b=1$

$According\ to\ basic\ four\ algorithm\ rule,$

$if\ \ \lim f(x)g(x)=a\ \&\&\exists\ \lim f(x), there\ must\ be\ \exists\ \lim g(x)$

$eg15. \lim\limits_{x\to -\infty}\frac{\sqrt{4x^2+x-1}+x+1}{\sqrt{x^2+sinx}}$

$=\lim\limits_{x\to -\infty}\frac{-x\cdot(\sqrt{4+\frac{1}{x}-\frac{1}{x^2}}-1-\frac{1}{x})}{-x\cdot\sqrt{1+\frac{sinx}{x^2}}}=\lim\limits_{x\to -\infty}\frac{2-1-0}{1}=1$

$the\ key\ point\ is\ eliminating\ the\ infty\ factor$