$eg1.$

$$\lim_{x\to0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{e^{x^2}-1}$$

$\because f(x)=\sqrt{x}$泰勒展开后 $\rightarrow 1+\frac{1}{2}x-\frac{1}{8}x^2+o(x^2)$

$\therefore$ 原式 $=\lim_{x\to0}\frac{f(x)=\sqrt{1+x}-1-\frac{x}{2}}{x^2}=\lim_{x\to0}\frac{1+\frac{1}{2}(x+1)-\frac{1}{8}(x+1)^2+o(x^2)-1-\frac{1}{2}x}{x^2}=-\frac{1}{8}$

$eg2.$

$$\lim_{x\to0}\frac{e^x+\ln(1-x)-1}{x-\arctan x}$$

$\because f(x)=\ln(1+x)=0+x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3)$

$\because f(x)=e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+o(x^3)$

$\because f(x)=\arctan x=0+x+-\frac{x^3}{3}+o(x^3)$

$\therefore$ 原式 $=\frac{-\frac{1}{6}x^3}{\frac{x^3}{3}}=-\frac{1}{2}$

$eg3.$

$$\lim_{x\to0}\frac{(1+x)^{\frac{2}{x}}-e^2[1-\ln(1+x)]}{x}$$

$=\lim_{x\to0}\frac{(1+x)^{\frac{2}{x}}-e^2}{x}+\lim_{x\to0}\frac{e^2\ln(1+x)}{x}=\lim_{x\to0}\frac{e^2[e^{\frac{2}{x}\ln(1+x)-2}-1]}{x}+e^2$

$=e^2\cdot\lim_{x\to0}\frac{\frac{2}{x}\ln(1+x)-2}{x}+e^2=e^2(1+\lim_{x\to0}\frac{2\ln(1+x)-2x}{x^2})=e^2(1-1)=0$

$eg4.$

$$\lim_{x\to0}\frac{(1+x^2)(1-\cos2x)-2x^2}{x^4}$$ 注意此处的泰勒展开要到4阶，因为前面系数有个1

原式 $=\frac{4}{3}$

$eg5.$

$$\lim_{x\to0}\frac{\sqrt{1-x^2}sin^2x-\tan^2x}{x^2[\ln(1+x)]^2}$$

$\because f(x)=\tan x=x+\frac{x^3}{3}+o(x^3)$

$\because f(x)=\sin x=x-\frac{x^2}{6}+o(x^3)$

$\because f(x)=(1+x)^\alpha=1+\alpha x+\frac{\alpha(\alpha-1)}{2}x^2+o(x^2)$

$\therefore$ 原式 $=-\frac{3}{2}$

$eg6.$

$$\lim_{x\to0}\frac{(3+2\tan x)^x-3^x}{3\sin^2x+x^3\cos\frac{1}{x}}$$ 注意分母相加的话可以考虑谁对谁是高阶无穷小的情况

$\because \lim_{x\to0}\frac{x^3\cos\frac{1}{x}}{3sin^2x}=\lim_{x\to0}x\cos\frac{1}{x}=0$ $\quad\therefore x^3\cos\frac{1}{x}$是$3\sin^2x$的高阶无穷小

$\therefore x^3\cos\frac{1}{x}+3\sin^2x\backsim3\sin^2x$ $\quad$ 低阶 + 高阶 $\backsim$ 低阶

$\therefore\lim_{x\to0}\frac{3^x[(1+\frac{2}{3}\tan x)^x-1]}{3x^2}=\lim_{x\to0}3^x\cdot\lim_{x\to0}\frac{e^{x\ln(1+\frac{2}{3}\tan x)}-1}{3x^2}=\lim_{x\to0}\frac{x\ln(1+\frac{2}{3}\tan x)}{3x^2}=\frac{1}{3}\cdot$

$\lim_{x\to0}\frac{\ln(1+\frac{2}{3}\tan x)}{x}=\frac{2}{9}$

$eg7.$

$$\lim_{x\to2}\frac{\sqrt{5x-1}-\sqrt{2x+5}}{x^2-4}$$

$$=\lim_{x\to2}\frac{3(x-2)}{(x+2)(x-2)(\sqrt{5x-1}+\sqrt{2x+5})}=\lim_{x\to2}\frac{3}{(x+2)(\sqrt{5x-1}+\sqrt{2x+5})}=\frac{1}{8}$$

$eg8.$

$$\lim_{x\to0}\int_0^x\frac{\sin 2t}{\sqrt{4+t^2}\int_0^x(\sqrt{t+1}-1)dt}dt \qquad {\color{red}f(x)=\int_0^x f(t)dt} \quad 区分: x为自变量, t为积分变量$$

$$=\lim_{x\to0}\frac{1}{\int_0^x(\sqrt{t+1}-1)dt}\cdot\int_0^x\frac{\sin 2t}{\sqrt{4+t^2}}dt \qquad {\color{red}F(x)=\int_0^x g(x)f(t)dt=g(x)\int_0^x f(t)dt}$$

$$=\lim_{x\to0}\frac{\sin 2x}{\sqrt{4+t^2}(\sqrt{t+1}-1)}=\frac{1}{2}\cdot\frac{\sin 2x}{\sqrt{x+1}-1}=2$$

$eg9.$

$$\lim_{x\to\infty}e^{-x}(1+\tfrac{1}{x})^{x^2}$$

$$=\lim_{x\to\infty}e^{-x}\cdot e^{x^2\ln(1+\frac{1}{x})}=\lim_{x\to\infty}e^{x^2(\ln 1+\frac{1}{x})-x}$$

$$\because\lim_{x\to\infty}x^2(1+\ln\tfrac{1}{x})-x \qquad {\color{red}令t=\tfrac{1}{x}}$$

$$=\lim_{t\to0}\frac{\ln(1+t)-t}{t^2} \qquad {\color{red}f(x)=\ln(x+1)=x-\tfrac{x^2}{2}+\tfrac{x^3}{3}+o(x^2)}$$

$$=-\tfrac{1}{2}\ \therefore 原式=e^{-\frac{1}{2}}$$

$eg10.$

$$\lim_{x\to3^+}\frac{\cos x\ln(x-3)}{\ln(e^x-e^3)}$$

$$=\cos 3\cdot\lim_{x\to3^+}\frac{\ln(x-3)}{\ln(e^x-e^3)}=\lim_{x\to3^+}\frac{1}{e^x}\cdot\frac{e^x-e^3}{x-3}=\cos 3$$

$eg11.$

$$\lim_{x\to\infty}x^2\left(a^{\frac{1}{x}}+a^{-\frac{1}{x}}-2\right) \qquad 其中a>0$$

$${\color{red}令t=\tfrac{1}{x}}\ \therefore 原式=\lim_{t\to0}\frac{a^t+a^{-t}-2}{t^2} \qquad \text{type:}\frac{0}{0}$$

$$=\lim_{t\to0}\frac{\ln a(a^t-a^{-t})}{2t}$$

$$=\lim_{t\to0}\frac{\ln^2 a(a^t+a^{-t})}{2}=\ln^2 a$$

$eg12.$

$$\lim_{x\to0}\frac{1}{x}(\cot x-\tfrac{1}{x})$$

$$=\lim_{x\to0}\frac{x\cos x-\sin x}{x^2\sin x}=\frac{x(1-\frac{1}{2}x^2)-(x-\frac{x^3}{6})}{x^2\sin x}=-\frac{1}{3}$$

$eg13.$

$$\lim_{x\to+\infty}\sqrt{x^3+2x^2+1}-xe^{\frac{1}{x}}$$

$${\color{red}令t=\tfrac{1}{x}} \qquad \therefore 原式=\lim_{t\to0^+}\sqrt[3]{\frac{1+2t+t^3}{t^3}}-\frac{e^t}{t}$$

$$=\frac{\sqrt[3]{1+2t+t^3}-e^t}{t}$$

$$= \frac{1+\frac{1}{3}(2t+t^3+o(t^3))-(1+t+o(t))}{t}$$

$$= \lim_{t\to 0^+} \frac{\frac{2}{3}t-t+\frac{1}{3}t^3+o(t)}{t} = -\frac{1}{3}$$