An Extension of DeGroot

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Issues with DeGroot

- All nodes are "the same" (other than their connections)
- Everyone converges to the same opinion whenever the graph is strongly connected and aperiodic
  - (other outcomes are equally bland)
- Every node is totally susceptible -- if the group ends up close to 1, it doesn't matter that I started out close to 0
- Paradox: strong disagreements lead to large shifts in opinion!

How to fix this?

Famous model: Friedkin-Johnsen model

- Each node has an individual "stubbornness" parameter
- Stubbornness dictates how strongly I hold to my initial opinion
- Questions:
  - what do node opinions converge to?
  - What happens if someone is infinitely stubborn?

Friedkin-Johnsen model

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DeGroot: $x_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t-1)$ weighted avg of out-neighbor opinions.

Issue: no "memory" of initial opinion.

Modification: let new opinion be a weighted average of out-neighbor opinions and my initial opinion.

Friedkin-Johnsen Model:

Let $\lambda_i \in [0,1]$ be a "susceptibility" parameter.

$\lambda_i = 1 \to$ I only go on ideas from network.

$\lambda_i = 0 \to$ I am totally stubborn, hold fast to my initial opinion.

Then $x_i(t) = \underbrace{\left[\lambda_i \sum_{j=1}^{n} a_{ij} x_j(t-1)\right]}_{\text{network opinions}} + \underbrace{(1-\lambda_i) x_i(0)}_{\text{my original opinion}}$

Relationship w/ DeGroot?

DeGroot = FJ w/ $\lambda_i = 1 \ \forall i$.

1st, before we analyze/simulate, need a bit more matrix math.

How to write FJ with matrices?

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Need a new tool to put FJ in ~~matrix form~~.

Def: a square $n \times n$ matrix D with entries $\{d_{ij}\}$ is called a <u>diagonal</u> matrix if its only nonzero entries are on the <u>main diagonal</u>. That is, if $i \neq j$, $d_{ij} = 0$:

$$D = \begin{bmatrix} \star & 0 & \cdots & 0 \\ & \star & 0 & 0 \\ 0 & & & \vdots \\ 0 & 0 & \ddots & \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 \;\; 0 & \star \end{bmatrix}$$

we write $D = \text{diag}(d_1, d_2 \cdots d_n)$ to mean $\begin{pmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \\ & & & d_n \end{pmatrix}$

Q: what does a diagonal matrix <u>do</u>?

Let $D = \text{diag}(d_1, d_2, \cdots d_n)$, and let A be any $n \times n$ matrix.

What is $D \cdot A$?

$$\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ 0 & & \ddots & \\ & & & d_n \end{pmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \\ \vdots & & & \\ a_{n1} & & & a_{nn} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & \cdots & d_1 a_{1n} \\ d_2 a_{21} & d_2 a_{22} & \cdots & d_2 a_{2n} \\ \vdots & & & \vdots \\ d_n a_{n1} & \cdots & & d_n a_{nn} \end{bmatrix}$$

Fact: if $D = \text{diag}(d_1, d_2, \cdots d_n)$, then DA is the matrix A with each row i scaled by d_i.

Part one of FJ in matrices

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$$x_i(t) = \underbrace{\left[\lambda_i \sum_{j=1}^{n} a_{ij} x_j(t-1)\right]}_{\text{Focus here}} + (1-\lambda_i)x_i(0)$$

- recall: can write DeGroot as $x(t) = A\,x(t-1)$, where ith row of A is the out-edges of node i.
- what do we want to do to the out-edges of node i? scale them down by λ_i !
- How do we scale the rows of a matrix? multiply by diagonal matrix!

So: let $\Lambda = \text{diag}(\lambda_1, \lambda_2, \cdots \lambda_n)$.

↑ capital lambda

Then $x(t) = \underbrace{\Lambda A}_{\text{scaled adjacency matrix}} x(t-1) + \underbrace{??}_{\text{what goes here?}}$

<u>Def</u>: An <u>identy</u> matrix, written I, is a diagonal matrix with all 1's on the diagonal:

$$I = \text{diag}(1, 1, \cdots 1) \qquad I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Completing matrix math for FJ

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How to write $(1-\lambda_i)x_i(0)$ in matrix form?

$$\rightarrow \underbrace{(I-\Lambda)}_{\text{diag}(1-\lambda_1,\ 1-\lambda_2, \ldots, 1-\lambda_n)} \underbrace{x(0)}_{\text{column vector of initial opinions}}$$

So in total, FJ is

$$x(t) = \Lambda A x(t-1) + (I-\Lambda)x(0).$$

Q: how does this evolve over time?

$$x(1) = \Lambda A x(0) + (I-\Lambda)x(0).$$

$$x(2) = \Lambda A x(1) + (I-\Lambda)x(0).$$

$$= \Lambda A\left(\Lambda A x(0) + (I-\Lambda)x(0)\right) + (I-\Lambda)x(0)$$

$$= (\Lambda A)^2 x(0) + \Lambda A(I-\Lambda)x(0) + (I-\Lambda)x(0)$$

$$x(3) = (\Lambda A)^3 x(0) + (\Lambda A)^2(I-\Lambda)x(0) + \Lambda A(I-\Lambda)x(0) + (1-\Lambda)x(0)$$

$$x(k) = (\Lambda A)^k x(0) + \left[(\Lambda A)^{k-1} + (\Lambda A)^{k-2} + \cdots + I\right](I-\Lambda)x(0)$$

$$(\Lambda A)^k x(0) + \left[\sum_{i=0}^{k-1} (\Lambda A)^i\right](I-\Lambda)x(0)$$

What is limiting opinion?

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$$X(k) = \left[(\Lambda A)^k + \left[\sum_{i=0}^{k-1} (\Lambda A)^i \right] (I - \Lambda) \right] X(0)$$

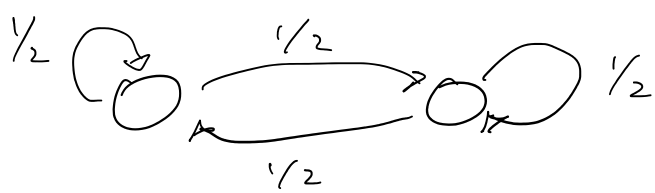
Q: what is $\lim_{k \to \infty} X(k)$?

A: when the limit exists, it is <u>still</u> a weighted avg of initial opinions. what kind of avg?

2 pieces: 1) $\lim_{k\to\infty} (\Lambda A)^k$.

2) $\left[\sum_{i=0}^{k-1} (\Lambda A)^i \right] (I - \Lambda)$

1): test this. $\Lambda = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}$ $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$



$\Lambda A = \begin{bmatrix} 1/4 & 1/4 \\ 1/8 & 1/8 \end{bmatrix}$ $(\Lambda A)^2 = \begin{bmatrix} 1/4 & 1/4 \\ 1/8 & 1/8 \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 \\ 1/8 & 1/8 \end{bmatrix} = \begin{bmatrix} 3/32 & 3/32 \\ 3/64 & 3/64 \end{bmatrix}$

Fact: if <u>any</u> $\lambda_i < 1$, then $\lim_{k\to\infty} (\Lambda A)^k = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$.

1) vanishes!

Equilibrium

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$$\lim_{k\to\infty} x(k) = \lim_{k\to\infty}\left[\cancelto{0}{(\Lambda A)^k} + \underbrace{\left[\sum_{i=0}^{k-1}(\Lambda A)^i\right]}_{?}(I-\Lambda)\right]x(0)$$

!

Q: what is $\displaystyle\lim_{k\to\infty}\left[\sum_{i=0}^{k-1}(\Lambda A)^i\right]$?

Magical fact: if A's graph is strongly connected and aperiodic, and at least one $\lambda_i \in (0,1)$, then

$$\lim_{k\to\infty}\left[\sum_{i=0}^{k-1}(\Lambda A)^i\right] = (I-\Lambda A)^{-1}$$

Not presenting proof here.

However, here is one way to see where this comes from:

Alternative Explanation: let x^* be equilibrium.

$$x^* = \Lambda A x^* + (I-\Lambda)x(0)$$

$$(I-\Lambda A)x^* = (I-\Lambda)x(0)$$

$$x^* = (I-\Lambda A)^{-1}(I-\Lambda)x(0) \quad \checkmark$$

Wrap up FJ

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FJ model is valuable for a few reasons:

- It builds on early models using an intuitive social phenomenon
- Its math is still relatively simple (linear dynamics)
- It has some empirical validation!