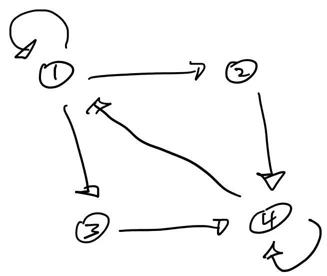
Recap of opinion dynamics

Monday, January 10, 2022 4:26 PM

Last time: averaging opinion dynamics.



ex:

$N_1 = \{1, 2, 3\}$

$N_2 = \{4\}$

$N_3 = \{4\}$

$N_4 = \{1, 4\}$

Idea: $x_i(t)$ is node i's opinion (between 0, 1) @ time t.

Nodes listen along edges

Notation: let N_i be the set of node i's "out-neighbors", or

$N_i = \{ j : (i,j) \in E \}$.

Then @ each time step, each node i updates its opinion to the average of the opinions in N_i.

Mathematically:

$$x_i(t+1) = \frac{1}{|N_i|} \sum_{j \in N_i} x_j(t).$$

$|N_i|$ = size of N_i.

Last time: we coded a simple simulator and the code was ugly.

Q: is there a clean way to make that process nicer?

A: Yes! it involves multiplying adjacency matrices.

Linear algebra recap

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Formula from last page:

$$x_i(t+1) = \frac{1}{|N_i|} \sum_{j \in N_i} x_j(t).$$

if $N_i = \{1, 2, 3\}$, then

$$= \frac{1}{3} x_1(t) + \frac{1}{3} x_2(t) + \frac{1}{3} x_3(t).$$

This is called a "linear combination" of the values x_1, x_2, x_3.

Def: Given a set of numbers $\{x_i\}_{i=1}^n = \{x_1, x_2 \cdots x_3\}$, a <u>linear combination</u> of $\{x_i\}_{i=1}^n$ is an expression $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum_{i=1}^n a_i x_i$, where the a_i coefficients are constant real #s.

Examples:

1. Average. The average of $\{x_i\}_{i=1}^n$ is given by $\frac{1}{n} \sum_{i=1}^n x_i$. What are a_i's? $a_i = \frac{1}{n}$ $\forall i$.

2. Sum. $\sum_{i=1}^n x_i$ is a linear combination of $\{x_i\}_{i=1}^n$ with $a_i = 1$ $\forall i$.

3. Single element. x_1 is a LC of $\{x_i\}_{i=1}^n$. $a_1 = 1$, $a_i = 0$ $\forall i \neq 1$.

Linear combinations vector shorthand

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Now for shorthand:

We like to call fixed-length lists of #s "vectors."

In physics, write $\vec{x} = \{x_i\}_{i=1}^{n}$.

In this class, even more compact: $x = \{x_i\}_{i=1}^{n}$.

x is the vector,

x_i is the ith element of vector x.

(sometimes vectors are written as bold letters)

Convention: always think of a vector as a <u>column</u> of #s.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

To write as row, apply the transpose operation:

$$x^T = [x_1, x_2, \cdots, x_n]$$

Inner product

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Often, we want to take 2 vectors, multiply elementwise, and sum the results.

Linear combination! Let a and x be vectors w/ n elements.

Want a compact way to write the LC of x w/ coefficients a: $\sum_{i=1}^{n} a_i x_i$.

This is called the "inner product" of a and x. Physics: "dot product."

Notation is very compact: $a^T x = \sum_{i=1}^{n} a_i x_i$

Picture: $\begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$.

Note: <u>not</u> $a x^T$. <u>Very</u> different thing! (we will <u>probably</u> never use in this class.)

"outer product"

Back to the graph

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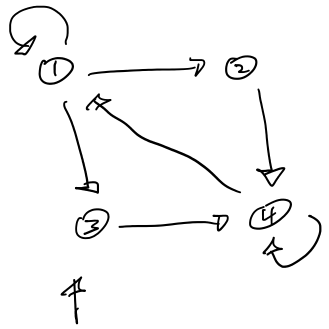

Represent neighbors as vectors:

a^i encodes N_i w/ 1's, 0's.

$$a^i_j = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else.} \end{cases}$$

ex:

$N_1 = \{1, 2, 3\} \longrightarrow a^1 = [1, 1, 1, 0]^T$

$N_2 = \{4\} \longrightarrow a^2 = [0, 0, 0, 1]^T$

$N_3 = \{4\} \longrightarrow a^3 = [0, 0, 0, 1]^T$

$N_4 = \{1, 4\} \longrightarrow a^4 = [1, 0, 0, 1]^T$.

Now: what is $x_1(t+1)$? $= \frac{1}{3}(a^1)^T x(t)$

$$= \frac{1}{3}[1, 1, 1, 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \frac{1}{3}(1 \cdot x_1 + 1 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4).$$

what is $x_2(t+1)$? $= (a^2)^T x(t)$.

$x_3(t+1) = (a^3)^T x(t)$

$x_4(t+1) = \frac{1}{2}(a^4)^T x(t)$.

To matrix multiplication

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Let $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$

Then $x(t+1) = \begin{bmatrix} \frac{1}{3} a^{1T} x(t) \\ a^{2T} x(t) \\ a^{3T} x(t) \\ \frac{1}{2} a^{4T} x(t) \end{bmatrix}$

Progress? Still very cumbersome.

2 main issues:

1. those pesky $1/3$, $1/2$ – feel inconsistent.
2. $x(t)$ is rewritten many times!

Tackle 1 first. Approach: redefine a's.

Let $a^1 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{bmatrix}$ $a^2 = a^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $a^4 = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$.

Just bake the fractions directly into the vectors!

Matrix multiplication

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Then $$x(t+1) = \begin{bmatrix} a^{1T} x(t) \\ a^{2T} x(t) \\ a^{3T} x(t) \\ a^{4T} x(t) \end{bmatrix}$$

Now to part 2: get rid of redundant $x(t)$.

Def: an n-by-n matrix is n row vectors of n linear combination coefficients stacked on top of each other. example:

$$A = \begin{bmatrix} \text{---} & a^{1T} & \text{---} \\ \text{---} & a^{2T} & \text{---} \\ \text{---} & a^{3T} & \text{---} \\ \text{---} & a^{4T} & \text{---} \end{bmatrix}$$

convention: use capital letters for matrices.

Def: the product of $n \times n$ matrix A with length-n vector x, written Ax, is a new length-n vector formed by inner products of the rows of A w/ x.

$$Ax = \begin{bmatrix} \text{---} & a^{1T} & \text{---} \\ \text{---} & a^{2T} & \text{---} \\ & \vdots & \\ \text{---} & a^{nT} & \text{---} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a^{1T} x \\ a^{2T} x \\ \vdots \\ a^{nT} x \end{bmatrix}$$

Matrix product for averaging opinion dynamics

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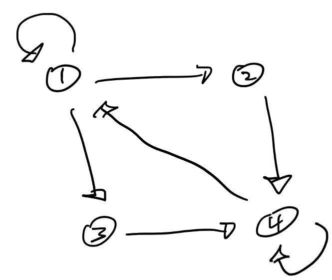
Q: what matrix works for our example graph?

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$



Fact: The above matrix A defines the averaging opinion dynamics for the above graph:

$$\underbrace{x(t+1)}_{} = A\, \underbrace{x(t)}_{}$$

Column vectors of opinions

$n \times n$ matrix describing graph edges.

Q: where have we seen such a thing?

adjacency matrix!

Fact: A is a weighted adjacency matrix. Let a_{ij} be the element in row i, column j. Then

$$a_{ij} = \begin{cases} \frac{1}{|N_i|} & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Wrap up

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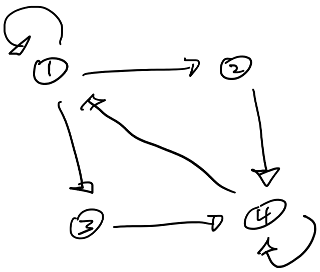
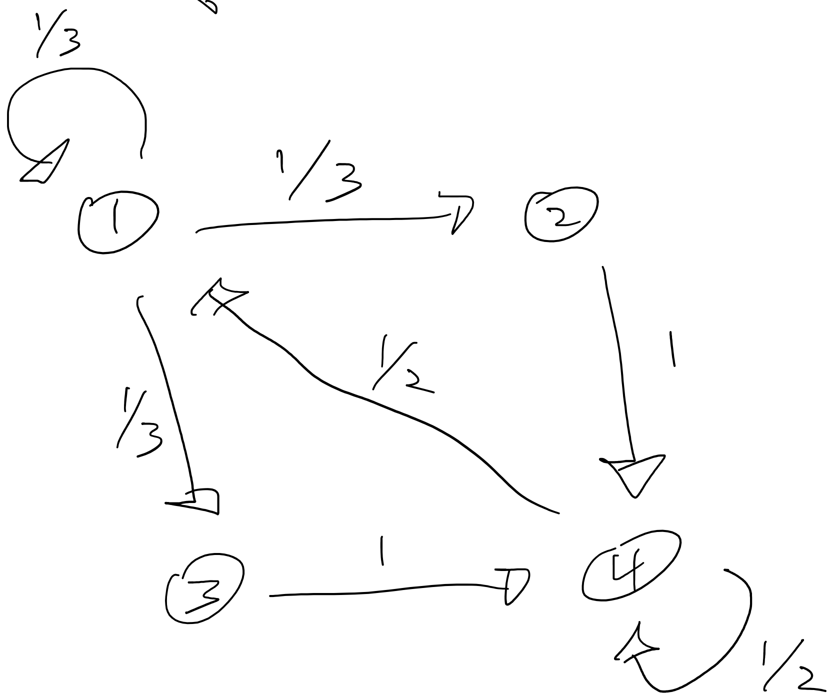
G = 

$$A = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

weighted graph:



Interpretation: the weight of edge (i, j) is the fraction of i's "attention" that it ascribes to node j.

Then $X(t+1) = AX(t)$ defines the opinion dynamics!

Fact: this attention doesn't have to represent merely an average. Ex: