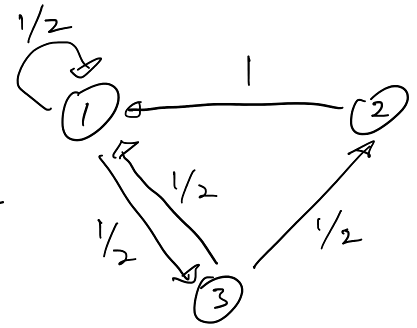
From last time

Monday, January 10, 2022   4:26 PM

Averaging opinion dynamics can be represented as multiplying a weighted adjacency matrix w/ a column vector of opinions!

rows: lists of trusts of each node!

$G =$



$$A = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

Idea: a_{ij} (the element in row i, column j of A) is the <u>trust</u> that i places on j.

Can think of this as an edge weight on the weighted, directed graph.

then if $x(t)$ is the column vector of opinions @ time t, we have that

$$x(t+1) = A\,x(t)$$

Example: if $x(t) = [1, 0, 0]^T$, what is $x(t+1)$?

$$x(t+1) = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/2(1) + 0(0) + 1/2(0) \\ 1(1) + 0(0) + 0(0) \\ 1/2(1) + 1/2(0) + 0(0) \end{bmatrix}}_{} = \begin{bmatrix} 1/2 \\ 1 \\ 1/2 \end{bmatrix}$$

column vector of inner products betw. rows of A and $x(t)$!

Row-stochasticity

Wednesday, March 2, 2022 9:50 AM

Observation: all rows of A sum to 1:
(how to do that w/ matrix mult?)

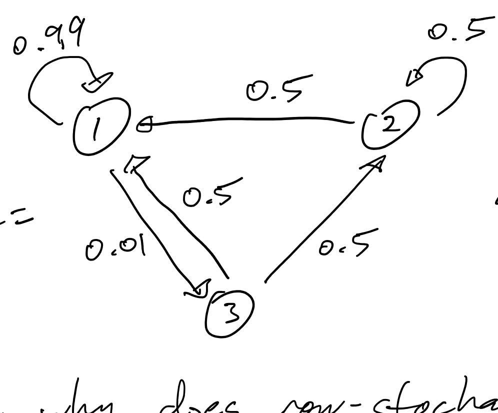
$$\begin{bmatrix} 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \\ 1/2 & 1/2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2(1) + 0(1) + 1/2(1) \\ 1(1) + 0(1) + 0(1) \\ 1/2(1) + 1/2(1) + 0(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

(aside: this means that the vector $\mathbb{1} := [1, 1, 1]^T$ is an eigenvector of A w/ eigenvalue of 1).

Def: a matrix A w/ all non-negative entries whose rows all sum to 1 is called a row-stochastic matrix.

Goal: extend the averaging opinion dynamics idea to arbitrary weighted directed graphs defined by row-stochastic matrices. (called DeGroot model)

Example:

$G =$ 

$$\longleftrightarrow A = \begin{bmatrix} 0.99 & 0 & 0.01 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

Q: why does row-stochastic matter? This models my opinion being a weighted average of others' opinions.
No "opinion is lost" over time.

Where do opinions go?

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Look @ how opinions spread:

$$\begin{bmatrix} 0.99 & 0 & 0.01 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.99 \\ 0.5 \\ 0.5 \end{bmatrix}$$

Q: what happens in the long run?

For averaging model, we know: if graph is strongly connected and aperiodic, dynamics converge to a common opinion that only depends on initial opinion.

what about in this more general DeGroot model?

Now we have more math, can ask Q more precisely.

$$x(t+1) = A x(t) \quad \text{and} \quad x(0) \text{ is given.}$$

Q: what is $\lim_{t \to \infty} x(t)$? (and does that limit exist?)

Start small: what is $x(1)$?

$$x(1) = \underbrace{A x(0)}.$$

what is $x(2)$? plug in

$$x(2) = \underbrace{A x(1)} = A(A x(0)) = A^2 x(0)$$

Matrix product

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Q: what do we mean by A^2?

Interpretation: A^2 is a n-by-n matrix that describes how opinions change over 2 time steps.

Q: How can we compute A^2?

Def: Let A and B be two $n \times n$ matrices depicted as

$$A = \begin{bmatrix} — & a^{1T} & — \\ — & a^{2T} & — \\ & \vdots & \\ — & a^{nT} & — \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} | & | & & | \\ b^1 & b^2 & \cdots & b^n \\ | & | & & | \end{bmatrix}.$$

Then the product of A and B, written AB, is an $n \times n$ matrix whose (i,j) entry is given by $a^{iT} b^j$:

$$AB = \begin{bmatrix} a^{1T}b^1 & a^{1T}b^2 & \cdots & a^{1T}b^n \\ a^{2T}b^1 & a^{2T}b^2 & \cdots & a^{2T}b^n \\ \vdots & & \ddots & \vdots \\ a^{nT}b^1 & a^{nT}b^2 & \cdots & a^{nT}b^n \end{bmatrix}$$

Example: compute $A^2 = AA$ from prev. example:

$$\begin{bmatrix} 0.99 & 0 & 0.01 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix} \begin{bmatrix} 0.99 & 0 & 0.01 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix} = \begin{bmatrix} 0.9851 & 0.005 & 0.0099 \\ 0.745 & 0.25 & 0.005 \\ 0.745 & 0.25 & 0.005 \end{bmatrix}$$

Interpret: after 2 timesteps, everybody cares about everybody's initial opinion!

Note also: A^2 is row-stochastic!

The limit

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Back to earlier Q:

$$x(t+1) = A\,x(t) \quad \text{and} \quad x(0) \text{ is given.}$$

Q: what is $\lim_{t \to \infty} x(t)$? (and does that limit exist?)

What is $x(t)$ for a generic t?

$$\begin{aligned} x(t) &= A\,x(t-1) \\ &= AA\,x(t-2) \\ &= AAA\,x(t-3) = A^3 x(t-3) \\ &\;\vdots \\ &= A^t x(t-t) = A^t x(0). \end{aligned}$$

So $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} A^t x(0) = \left(\lim_{t \to \infty} A^t\right) x(0)$

Can rephrase our Q: when does $\lim_{t \to \infty} A^t$ exist?

Fact: if A is an $n \times n$ row-stochastic matrix, then the limiting matrix $\bar{A} := \lim_{t \to \infty} A^t$ exists if and only if every component of the graph represented by A is aperiodic.

Proof idea: if some component of the graph has period T then the opinions of that component repeat every T time steps. That repeating behavior will appear in the powers of A!

What do limiting matrices look like?

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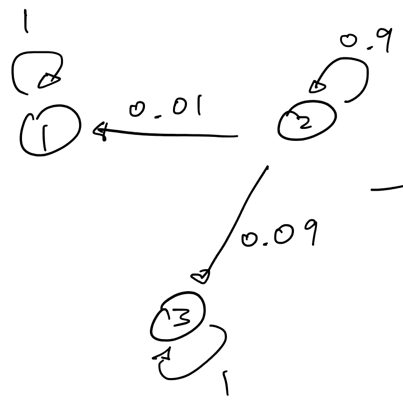
Q: what can $\bar{A} = \lim_{t \to \infty} A^t$ look like?

3 possibilities for aperiodic graphs:

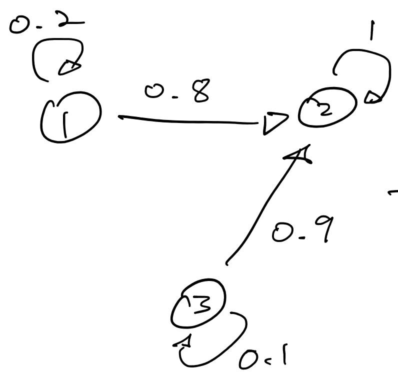
(a)



$$\rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 0.01 & 0.9 & 0.09 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0 & 0.9 \\ 0 & 0 & 1 \end{bmatrix}$$

node 1, 3 never change their opinions, node 2 adopts a mixture of 1, 3.

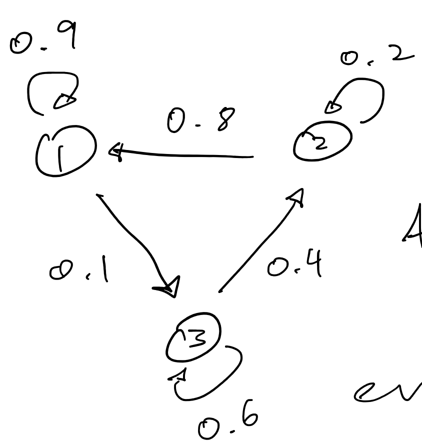
(b)



$$\rightarrow A = \begin{bmatrix} 0.2 & 0.8 & 0 \\ 0 & 1 & 0 \\ 0 & 0.9 & 0.1 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

node 1, 3 opinions are eventually forgotten, but all opinions are same in end!

(c)



$$A = \begin{bmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.2 & 0 \\ 0 & 0.4 & 0.6 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 8/11 & 1/11 & 2/11 \\ 8/11 & 1/11 & 2/11 \\ 8/11 & 1/11 & 2/11 \end{bmatrix}$$

everyone's limiting opinion is a specific weighted average of initial opinions!

Strongly connected and aperiodic

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Fact: In the DeGroot model, if the graph is strongly connected and aperiodic with row-stochastic adjacency matrix A, then the limiting matrix $\bar{A} = \lim_{t \to \infty} A^t$ can be depicted

$$\bar{A} = \begin{bmatrix} \text{---} & w^T & \text{---} \\ \text{---} & w^T & \text{---} \\ & \vdots & \\ \text{---} & w^T & \text{---} \end{bmatrix}$$

for some vector w.

Furthermore, every node's limiting opinion is given by $w^T x(0)$.

(Advanced: furthermore: w is a left-eigenvector of A w/ eigenvalue 1: $w^T = w^T A$)

Takeaway: aperiodic/strongly connected is powerful, and makes the DeGroot model mix node opinions into a weighted average of initial opinions.

Further reading:

Free textbook by UCSB Professor Francesco Bullo: http://motion.me.ucsb.edu/book-lns/
Perron-Frobenius theory (theory of matrices with nonnegative entries): https://en.wikipedia.org/wiki/Perron%E2%80%93Frobenius_theorem